Different creation-destruction operators' ordering, quasi-probabilities, and Mandel parameter

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In this work we provide an introductory discussion to quasi-probabilities in quantum optics and how to use them for evaluating the Mandel parameter.

Keywords: Quantum distributions; phase space; Mandel parameter.

En este trabajo proveemos una discusión introductoria sobre cuasi-probabilidades en óptica cuántica y las usamos para evaluar el parámetro de Mandel.

Descriptores: Distribuciones cuánticas; espacio de fases; parámetro de Mandel.

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1. Introduction

The focus of this article is the notion of quasi-probability. Why? because this is an important concept for quantum optics (among other fields). What is a quasi-probability distribution (QPD)? It is a mathematical construction that, although resembling a probability distribution, does not exactly fulfill the Kolmogorov's axioms on which probability theory is founded [1], more specifically, the third one may be violated [1]. Quasi-probabilities exhibit general features of ordinary probabilities. They yield expectation values with respect to the weights of the quasi-distribution. However, let us reiterate, they violate the third probability postulate [1], in the sense that regions integrated under them do NOT represent probabilities of mutually exclusive states. In some cases, quasi-probability distributions exhibit zones of negative probability density. QPDs often arise in the phase space representation of quantum mechanics, customarily employed in quantum optics, time-frequency analysis, etc.

It is well known that the dynamics of a quantum system is determined by a master equation. One faces an equation of motion for the density operator $\hat{\rho}$, expressed via a complete orthonormal basis. It is always possible to cast the density operator in a diagonal manner, provided that an overcomplete basis be used [2]. The most famous such basis is that of coherent states $|\alpha\rangle$ [3]. One writes [2]

$$\hat{\rho} = \int \frac{d^2\alpha}{\pi}\, f(\alpha,\alpha^*)\, |\alpha\rangle\langle\alpha|. \quad (1)$$

We see that a central role is assigned to the ordinary function f, that becomes endowed with the features of a quasi-probability distribution. In particular, one has $\mathrm{d}^2\alpha/\pi = \mathrm{d}x\mathrm{d}p/2\pi\hbar$. The system evolves as prescribed by the evolution of the quasi-probability distribution function. Coherent states, right eigenstates of the annihilation operator $\hat{a}$, serve as the overcomplete basis in such a build-up [2, 3].

In quantum optics f is called the function P. One speaks of the P-representation. There exists two other important representations, known as the Q- and W- ones. There exists a family of different representations, each connected to a different ordering of the underlying creation and destruction operators $\hat{a}$ and $\hat{a}^\dagger$. Historically, the first of these is the Wigner quasi-probability distribution W [4], related to symmetric operator ordering. In quantum optics the particle number operator is naturally expressed in normal order and, in the pertinent scenario, the associated representation of the phase space distribution is the Glauber–Sudarshan P one [3]. In addition to these two (W and P), one often encounters other quasi-probability distributions emerging in alternative representations of the phase space distribution [5]. A quite popular representation is the Husimi Q one [6-9], employed when operators are in anti-normal order.

In this paper we wish illustrate how these orderings are tackled so as to find the W, P, and Q representations for the important instance of the Harmonic Oscillator (HO) of angular frequency ω in the canonical ensemble formulation. In such a scenario one deals with three functions associated to the W, Q, and P representations that are simple Gaussians and the treatment becomes entirely analytical, a very convenient didactic feature. The HO is a really important system that yields insights usually having a wide impact. Thus, the HO constitutes much more than a mere simple example. Nowadays, it is of particular interest for the dynamics of bosonic or fermionic atoms contained in magnetic traps [10-12] as well as for any system that exhibits an equidistant level spacing in the vicinity of the ground state, like nuclei or Luttinger liquids. For the sake of concreteness we focus attention on a physical important quantity, a noise indicator, called the Mandel parameter.

So as to accomplish our didactic purposes, this communication is organized as follows. In Sec. 2. we review some notions about the Mandel parameter. In Sec. 3. we discuss the formulation of the Mandel parameter in terms of the $P-$function for a thermal state. In Sec. 4. we reproduce the same ideas via $Q-$function, and in Sec. 5. we recalculate these results in terms of the Wigner function. Finally, some conclusions are drawn in Sec. 6.

2. Mandel parameter

Since the very beginnings of quantum mechanics there has been interest in gauging how non-classical a quantum system can be. Several questions can be asked in this respect. For instance, how closely does the probability distribution for a state after a given series of measurements resemble that of a classical system? Or, how much is a quantum state perturbed by a measurement on the system?

A useful parameter to characterize non-classicality in quantum optics is the so-called Mandel parameter, introduced by Mandel in Ref. 13. It is defined as

$$\mathcal{Q}_M = \frac{(\Delta\hat{n})^2}{\langle\hat{n}\rangle} - 1 \equiv \mathcal{F} - 1, \tag{2}$$

and intimately linked to the normalized variance, which is also denominates the quantum Fano factor $\mathcal{F}{=}(\Delta\hat{n})^2/\langle\hat{n}\rangle$ [15] of the photon distribution, with $(\Delta\hat{n})^2 = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2$, and the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$. One must take note that

- For $\mathcal{F} < 1$ ($\mathcal{Q}_M \leq 0$), emitted light is referred to as sub-Poissonian since it has photo-count noise smaller than that of coherent (ideal laser) light with the same intensity ($\mathcal{F} = 1;\ \mathcal{Q}_M = 0$), whereas
- for $\mathcal{F} > 1$, ($\mathcal{Q}_M > 0$) the light is called super-Poissonian, exhibiting photo-count noise higher than the coherent-light noise.

If for the photon number operator $\hat{n}$ the fluctuations in $\hat{n}$ disappear, the Mandel parameter becomes $\mathcal{Q}_M = -1$ ($\mathcal{F} = 0$). We pass now to our central issue, the evaluation of the Mandel parameter for our three quasi-probability instances, *i.e.*, P, Q, and W.

3. Mandel Parameter via the $P-$function

The most general density operator is just a superposition of projection operators, known as the Glauber-Sudarshan $P-$representation [14]. One has

$$\hat{\rho} = \int \frac{\mathrm{d}^2\alpha}{\pi}\, P(\alpha,\alpha^*)\, |\alpha\rangle\langle\alpha|, \tag{3}$$

where the function $P(\alpha,\alpha^*)$ plays the role of a probability density for the distribution of values of α over the complex plane. A coherent state $|\alpha\rangle$ is a specific kind of quantum state, the one that most resembles a classical state. It is applicable to the quantum harmonic oscillator, the electromagnetic field, etc., and describes a maximal kind of coherence and a classical kind of behavior. The states $|\alpha\rangle$ are normalized, *i.e.*, $\langle\alpha|\alpha\rangle = 1$, and they provide us with a resolution of the identity operator

$$\int \frac{\mathrm{d}^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha| = 1, \tag{4}$$

which is a completeness relation for the coherent states [3]. The standard coherent states $|\alpha\rangle$ for the harmonic oscillator are eigenstates of the annihilation operator $\hat{a}$, with complex eigenvalues α, which satisfy $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ [3]. Also, $\hat{a}$, $a^\dagger$, and $\hat{n}$ fulfills the commutation relations

$$[\hat{a},\hat{a}^\dagger] = 1, \tag{5a}$$

$$[\hat{n},\hat{a}] = -\hat{a}, \tag{5b}$$

$$[\hat{n},\hat{a}^\dagger] = \hat{a}^\dagger. \tag{5c}$$

As stated above, $P(\alpha)$ is a quasi-probability distribution function because it can have negative values and strong singularities, especially when the density operator corresponds to a nonclassical state with sub-Poisson photon statistics (see Ref. 16 and references therein). When this function tends to vary little over large ranges of the parameter α, the nonorthogonality of the coherent states will make little difference, and $P(\alpha)$ can be interpreted as a probability distribution [3]. The normalization property of the density operator requires that $P(\alpha)$ obey the normalization condition [3]

$$\mathrm{Tr}\ \hat{\rho} = \int \frac{\mathrm{d}^2\alpha}{\pi}\, P(\alpha,\alpha^*) = 1. \tag{6}$$

Accordingly, the expectation value of an observable $\hat{A}$ is given by [16]

$$\langle\hat{A}\rangle = \mathrm{Tr}(\hat{\rho}\hat{A}) = \int \frac{\mathrm{d}^2\alpha}{\pi}\, P(\alpha,\alpha^*)\, \langle\alpha|\hat{A}|\alpha\rangle. \tag{7}$$

In this context, *the average particle-number* –an important quantity for our present considerations– acquires a simple form that, according to Eq. (3), can be cast in the fashion [3]

$$\begin{aligned}\langle\hat{n}\rangle &= \mathrm{Tr}(\hat{\rho}\hat{a}^\dagger\hat{a}) = \langle\hat{a}^\dagger\hat{a}\rangle_P \\ &= \int \frac{\mathrm{d}^2\alpha}{\pi}\, P(\alpha,\alpha^*)\, |\alpha|^2 = \langle|\alpha|^2\rangle_P,\end{aligned} \tag{8}$$

indicating that the average photon number is the mean squared absolute value of the amplitude α. Note that $\langle\,\rangle_P$ is the average with respect to $P(\alpha)$.

In particular, for a thermal state for which the density operator is of the form prescribed by the canonical ensemble's representation we have

$$\hat{\rho} = (1 - e^{-\beta\hbar\omega})\, e^{-\beta\hbar\omega\hat{a}^\dagger\hat{a}} \tag{9}$$

and the corresponding $P-$function becomes

$$P(\alpha,\alpha^*) = \frac{1}{\langle\hat{n}\rangle}\,\exp\left(-\frac{|\alpha|^2}{\langle\hat{n}\rangle}\right), \tag{10}$$

while the average particle-number is given by [3]

$$\langle\hat{n}\rangle = \langle|\alpha|^2\rangle_P = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}. \tag{11}$$

Now, for all normal-ordered operator-averages we have

$$\langle\hat{a}^{\dagger r}\hat{a}^s\rangle_P = \int\frac{\mathrm{d}^2\alpha}{\pi}\,P(\alpha,\alpha^*)\,\alpha^{*r}\alpha^s. \tag{12}$$

In particular, for $r = s = 2$, and taking into account the commutation relations between $\hat{a}$ and $\hat{a}^\dagger$, we find

$$\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle_P = \langle(\hat{a}^\dagger\hat{a})^2\rangle_P - \langle\hat{a}^\dagger\hat{a}\rangle_P \equiv \langle\hat{n}^2\rangle - \langle\hat{n}\rangle, \tag{13}$$

where

$$\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle_P = \int\frac{\mathrm{d}^2\alpha}{\pi}\,P(\alpha,\alpha^*)\,|\alpha|^4 = \langle|\alpha|^4\rangle_P. \tag{14}$$

Considering Eqs. (13) and (14), we realize that one can write

$$\langle\hat{n}^2\rangle = \langle|\alpha|^2\rangle_P + \langle\hat{n}\rangle. \tag{15}$$

In accordance with Eqs. (11) and (15), the Mandel parameter now becomes, via the $P-$function,

$$\mathcal{Q}_M = \frac{\langle|\alpha|^4\rangle_P - \langle|\alpha|^2\rangle_P^2}{\langle|\alpha|^2\rangle_P} = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}, \tag{16}$$

a temperature dependent expression, where the statistical averages in phase space are computed utilizing $P(\alpha)$ as a weight function

$$\langle|\alpha|^s\rangle_P = 2\int_0^\infty \mathrm{d}\alpha\,|\alpha|^{s+1}\,P(\alpha,\alpha^*), \tag{17}$$

with $s = 2, 4$.

4. Mandel parameter via the $Q-$function

A closely related phase space distribution is obtained by taking the diagonal matrix element of density operator $\hat{\rho}$

$$Q(\alpha,\alpha^*) = \langle\alpha|\hat{\rho}|\alpha\rangle, \tag{18}$$

which it has all the properties of a classical probability distribution [18]. For a thermal state, the $Q-$function is the gaussian quantity given by

$$Q(\alpha,\alpha^*) = \frac{1}{1+\langle\hat{n}\rangle}\,\exp\left(-\frac{|\alpha|^2}{1+\langle\hat{n}\rangle}\right). \tag{19}$$

The $Q-$representation gives operator averages in antinormal order, so in this case the antinormal-ordered average becomes [18]

$$\langle\hat{a}^s\hat{a}^{\dagger r}\rangle_Q = \int\frac{\mathrm{d}^2\alpha}{\pi}\,Q(\alpha,\alpha^*)\,\alpha^{*r}\alpha^s, \tag{20}$$

where $\langle\,\rangle_Q$ denotes the average with respect to $Q(\alpha,\alpha^*)$. Taking $r = s = 2$ we have

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle_Q = \int\frac{\mathrm{d}^2\alpha}{\pi}\,Q(\alpha,\alpha^*)\,|\alpha|^4 = \langle|\alpha|^4\rangle_Q. \tag{21}$$

Considering the identity $\hat{a}^2\hat{a}^{\dagger 2} = \hat{n}^2 + 3\hat{n} + 2$ and Eq. (21), it is easy to show that

$$\langle\hat{n}^2\rangle = \langle|\alpha|^4\rangle_Q - 3\langle\hat{n}\rangle - 2. \tag{22}$$

Also, for $s = r = 1$, Eq. (20) reduces to

$$\langle\hat{a}\hat{a}^\dagger\rangle_Q = \int\frac{\mathrm{d}^2\alpha}{\pi}\,Q(\alpha,\alpha^*)\,|\alpha|^2 = \langle|\alpha|^2\rangle_Q, \tag{23}$$

and we have

$$\langle\hat{n}\rangle = \langle|\alpha|^2\rangle_Q - 1. \tag{24}$$

Finally, from Eqs. (22) and (24) the Mandel parameter in terms of averages of Q turns out to be

$$\begin{aligned}\mathcal{Q}_M &= \frac{\langle|\alpha|^4\rangle_Q - 2\langle|\alpha|^2\rangle_Q - \langle|\alpha|^2\rangle_Q^2 + 1}{\langle|\alpha|^2\rangle_Q - 1}\\ &= \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}},\end{aligned} \tag{25}$$

and coincides with the pertinent expression obtained via the P-representation. Here, the means values are calculated using

$$\langle|\alpha|^s\rangle_Q = 2\int_0^\infty \mathrm{d}\alpha\,|\alpha|^{s+1}\,Q(\alpha,\alpha^*), \tag{26}$$

with $s = 2, 4$, with $Q(\alpha,\alpha^*)$ given by (19).

5. Mandel parameter via the Wigner function

The Wigner function can be obtained from the $P-$function from the relation [17]

$$W(\alpha,\alpha^*) = 2\int\frac{\mathrm{d}^2 z}{\pi}\,P(z,z^*)\,\exp(-2|\alpha-z|^2), \tag{27}$$

such that for a thermal state this function becomes

$$W(\alpha,\alpha^*) = \frac{1}{\langle\hat{n}\rangle+1/2}\exp\left(-\frac{|\alpha|^2}{\langle\hat{n}\rangle+1/2}\right), \tag{28}$$

with $1/(\langle\hat{n}\rangle + 1/2) = 2\tanh(\beta\hbar\omega/2)$. The symmetric ordered operator used in this Wigner representation is, in this case,

$$\langle(\hat{a}^{\dagger r}\hat{a}^s)_S\rangle_W = \int\frac{\mathrm{d}^2\alpha}{\pi}\,W(\alpha,\alpha^*)\,\alpha^{*r}\alpha^s, \tag{29}$$

where $(\hat{a}^{\dagger r}\hat{a}^s)_S$ denotes the symmetric operator product of the creation operators and annihilation ones [18], and $\langle\,\rangle_W$ indicates the average with respect to $W(\alpha,\alpha^*)$. Thus, for $r=s=2$ we get

$$\langle(\hat{a}^{\dagger 2}\hat{a}^2)_S\rangle_W=\int\frac{\mathrm{d}^2\alpha}{\pi}\,W(\alpha,\alpha^*)\,|\alpha|^4=\frac{1}{6}\big(\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle+\langle\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\rangle + \langle\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger\rangle+\langle\hat{a}\hat{a}^{\dagger 2}\hat{a}\rangle+\langle\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\rangle+\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle\big), \tag{30}$$

implying, after to use the commutation relations (5), that

$$\langle\hat{n}^2\rangle=\langle|\alpha|^4\rangle_W-\langle\hat{n}\rangle-\frac{1}{2}. \tag{31}$$

Also, for $r=s=1$ we get,

$$\langle(\hat{a}^\dagger\hat{a})_S\rangle_W=\int\frac{\mathrm{d}^2\alpha}{\pi}\,W(\alpha,\alpha^*)\,|\alpha|^2 =\frac{1}{2}\big(\langle\hat{a}a^\dagger\rangle+\langle\hat{a}^\dagger\hat{a}\rangle\big), \tag{32}$$

yielding

$$\langle\hat{n}\rangle=\langle|\alpha|^2\rangle_W-\frac{1}{2}. \tag{33}$$

From Eqs. (31) and (33) we obtain now the Mandel parameter in the fashion

$$\mathcal{Q}_M=\frac{\langle|\alpha|^4\rangle_W-\langle|\alpha|^2\rangle_W^2-\langle|\alpha|^2\rangle_W+1/4}{\langle|\alpha|^2\rangle_W-1/2} =\frac{1-\tanh(\beta\hbar\omega/2)}{2\tanh(\beta\hbar\omega/2)}=\frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}, \tag{34}$$

where we have considered the mean values of $|\alpha|^2$ and $|\alpha|^4$ according to

$$\langle|\alpha|^s\rangle_W=2\int_0^\infty \mathrm{d}\alpha\,|\alpha|^{s+1}\,W(\alpha,\alpha^*), \tag{35}$$

with $s=2,4$, being $W(\alpha,\alpha^*)$ a statistical wight function given by (28).

Interestingly enough, the Mandel parameter for a thermal state is the same in the three representations we are discussing in this work.

6. Conclusions

We have shown into some detail how to proceed to evaluate the quasi-probability distributions corresponding, respectively, to the P-, Q- and Wigner representations in the case of a thermal state for the harmonic oscillator. Using them we have computed the noise-factor called the Mandel parameter. We have seen that it is the same quantity, independent on the nature of the quasi-probability distribution. It is hoped that these reflections may help students in their efforts to navigate quantum optics waters.

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