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Constructing q-cyclic Games with Unique Prefixed Equilibrium

Abstract: In this work we construct a wide family of q-cyclic n-person games (Marchi and Quintas (1983)) with unique prefixed Nash equilibrium points. We extend the constructions given for bimatrix games by Marchi and Quintas (1987) and Quintas (1988 a). We prove the uniqueness of equilibrium for a wide family of completely mixed q-cyclic game and also for a family of not completely mixed q-cyclic game, with each players having n strategies, being only m (m<n) of them active strategies.

Keywords: Nash Equilibrium; Uniqueness.

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1 Introduction

The concept of equilibrium introduced by Nash (1951) is considered a landmark in non-cooperative game theory. The set of Nash equilibrium points is non-empty for any finite game if mixed strategies are allowed (Nash (1951)).

However in general there is a multiplicity of equilibrium. It is an important problem to decide which equilibrium is taken as a solution of the game. If the players cannot communicate each player might choose an equilibrium strategy and the resulting play might not be an equilibrium[1]. Even if they can communicate, it still remains a serious problem because the utilities can be quite different from one equilibrium point to another. This problem does not arise if the equilibrium is unique

Many studies have been done on uniqueness of Nash equilibrium points. On one hand it was studied some sufficient conditions to guarantee uniqueness (Gale y Nikaido (1965))[2]. it has been also studied under what conditions it is possible to construct games with predetermined unique equilibrium predetermined. Constructions of games with prefixed unique equilibrium have been done for bimatrix games by Raghavan (1970). He proved that, if the equilibrium points of a game are completely mixed then the matrix of each player is square and the equilibrium is unique. Millham (1972) proved that a necessary and sufficient condition for the existence of a game with unique prefixed equilibrium points is that the equilibrium be completely mixed. Kreps (1974) gave uniqueness conditions when the equilibrium point is not completely mixed. Heuer (1979), extended and complemented these results and obtained the uniqueness of the equilibrium point within the class of mixed strategies whose non zero components are the same for all the players. Quintas (1988 a) b)) extended this result constructing a wide family of bimatrix games with unique equilibrium point. Marchi and Quintas (1987) also studied games with prefixed values and unique equilibrium points. Quintas, Marchi, Giunta and Alaniz (1991) extended this construction for other family of games with unique equilibrium points.

In this work we extend the above mentioned constructions, presenting a wide family of q-cyclic n-person games with unique equilibrium. We give some definitions and basic results in Section 2. We built the payoff matrices of a family of n-person games in Section 3 and we prove the uniqueness of Nash Equilibrium in Section 4. We extend the constructions of games with unique equilibrium when it is non-completely mixed in Section 5. We also include some concluding remarks in Section 6.

2 Definitions and Previous Results

Here we give some definitions, notations and review some results which will be used in this paper.

Definition 2.1 (Marchi - Quintas (1983)). Let $\Gamma = \{\Sigma_1, \Sigma_2, \ldots\ldots, \Sigma_n, A_1, A_2, \ldots\ldots\ldots, A_n\}$, be a finite n-person q-cyclic game in normal form, were Σ_i is the set of pure strategies for player i. Let A_i be the utility function of player i, with i=1,...n. The definition of the function A_i is given by: $A_i(\sigma_1, \ldots\ldots, \sigma_i, \ldots\ldots\ldots, \sigma_n) = B_i(\sigma_i, \sigma_{q(i)})$ with $\sigma_i \in \Sigma_i$ were the function q is that: $q(i) \neq i$ and $|q^{-1}(i)| = 1$ were $|.|$ stands for the cardinality of respective set .

In this work we consider games where:, $q(i) = i+1$ mod. n , we take $j = q(i)$ and each player has m strategies, thus $|\Sigma_i| = m$, for i=1,.....,n

Definition 2.2: A mixed strategy for player i is a probability distribution over the pure strategies $\Sigma_i = \{\sigma^i_1, \sigma^i_2, \ldots\ldots\ldots, \sigma^i_m\}$. That is a vector: $x_i = (x_i(\sigma^i_1), x_i(\sigma^i_2) \ldots\ldots, x_i(\sigma^i_m)) = (x^i_1, x^i_2, \ldots\ldots x^i_m)$ were x^i_t is the probability of player I uses his strategies $\sigma^i_t \in \Sigma_i$ with t = 1,2,.....,m.

[1] This problem does not appear in two person zero sum games because for this type of games the equilibrium could not be unique, but the equilibrium strategies are interchangeable and the equilibrium payoff is unique.

[2] There is also a vast bibliography on refinements of the Nash Equilibrium. Some refinements reduce the multiplicity of equilibrium.

Definition 2.3: Let be $\tilde{\Sigma}_i$ the set of mixed strategies for the player i .

$$\tilde{\Sigma}_i = \left\{ x_i : \sum_{t=1}^{t=m} x_t^i = 1 \quad with \quad x_t^i \geq 0, t = 1,2,\ldots.., m \right\}$$

$x = (x_1, x_2, \ldots\ldots x_n) \in \prod_{i=1}^{i=n} \tilde{\Sigma}_i$ is a n-tuple of mixed strategies for the n players and we denote $(x^*_{N-\{i\}}, x_i) = (x^*_1, \ldots\ldots.., x^*_{i-1}, x_i, x^*_{i+1}, \ldots., x^+_n)$

Definition 2.4 (Marchi - Quintas (1983)). The expected utility function E_i for each player i, in the q-cyclic game is defined as follows:

$E_i(x) = F_i(x_i, x_{q(i)})$ where F_i is the expected utility function of B_i.

Thus, we have $E_i(x) = F_i(x_i, x_j) = \sum_{l=1}^{l=m} \sum_{t=1}^{t=m} a_{tl}^{ij} x_l^j x_t^i$ with i=1, 2,......,n

We will indistinctly denote it by: $F_i(r, x_j) = F_i(e_r, x_j)$, where e_r is a m-tuple with one in the place r and zero in the other places.

Definition 2.5: (Nash (1951)). A $x^* = (x^*_1, x^*_2, \ldots\ldots x^*_n) \in \prod_{t=1}^{t=n} \tilde{\Sigma}_t$ n-tuple is a Nash equilibrium if and only if $E_i(\overset{*}{x}) \geq E_i(\overset{*}{x}_{N-\{i\}}, x_i)$ for each $x_i \in \tilde{\Sigma}_i$; and for each i=1,… ,n.

We will use the following characterization on Nash Equilibrium

Definition 2.6: The set of all the pure strategies, that are best reply against $x = (x_1, x_2, \ldots\ldots x_n) \in \prod_{t=1}^{t=n} \tilde{\Sigma}_t$, is defined as follows:

$$J_i(x_{q(i)}) = \left\{ \sigma_r^i \in \Sigma_i : F_i(\sigma_r^i, x_j) \geq F_i(\sigma_t^i, x_{q(i)}) \text{ for each } \sigma_t^i \in \Sigma_i \text{ with } t = 1,2,\ldots.,m \right\}$$

We will use the following characterization of equilibrium points:

Theorem 1: (Marchi - Quintas (1983)). $x = (x_1, x_2, \ldots\ldots x_n) \in \prod_{i=1}^{i=n} \tilde{\Sigma}_i$ is a Nash equilibrium, if and only if, $S(x_i) \subseteq J_i(x_{q(i)})$, for each i = 1,2,....., n. where $S(x_i) = \{ \sigma_s^i \in \Sigma_j : x_s^i > 0 \text{ with } s = 1,2,\ldots, m \}$ is the support of the mixed strategy x_i.

Definition 2.7: $x = (x_1, x_2, \ldots\ldots x_n) \in \prod_{i=1}^{i=n} \tilde{\Sigma}_i$ is completely mixed if $S(x_j) = \Sigma_j$. In this case, we say each player has all the strategies active.

Definition 2.8: Let $x^* = (x^*_1, x^*_2, \ldots\ldots x^*_n) \in \prod_{t=1}^{t=n} \tilde{\Sigma}_t$ be a Nash Equilibrium of $\tilde{\Gamma}$. We say that $v_1, v_2, \ldots\ldots, v_n$, where $v_i = E_i(x^*)$, i=1,2,....,n., are expected values associated to $x^* = (x^*_1, x^*_2, \ldots\ldots x^*_n) \in \prod_{t=1}^{t=n} \tilde{\Sigma}_t$

3 Construction of the payoff matrices of the players

We will present a general form of n-person q-cyclic games with prefixed equilibrium points on the mixed extension.

We will construct the payoff matrices A_i with i=1,2,...,n for each player i, and we will study under what conditions on the expected utility function E_i there is a unique equilibrium.

Let us consider an arbitrary point, $x = (x_1, x_2, \ldots\ldots x_n) \in \prod_{i=1}^{i=n} \tilde{\Sigma}_i$

It is $(x_1, x_2, \ldots\ldots.., x_n) = ((x_1^1, x_2^1, \ldots\ldots.., x_m^1), (x_1^2, x_2^2, \ldots\ldots.., x_m^2) \ldots\ldots\ldots\ldots\ldots (x_1^n, x_2^n, \ldots\ldots.., x_m^n))$ with $\sum_{t=1}^{t=m} x_t^i = 1$ for each i=1,2,....,n , $x_t^i > 0$ for each t=1,2,....,m and i=1,2,.....,n

Thus, $|S(x_i)| = m$ for i=1,2,....n.

We choose arbitrary non zero values ν_i with i=1,.......,n. They will be the expected payoffs of the game

The construction extends that presented for bimatrix games by Quintas (1988 a)). It consists in giving conditions in order that the region of the simplex $\tilde{\sum}_j$ limited for a: predetermined vertex, the above prefixed point x and some points chosen on the faces of the simplex, have a unique maximizing hyperplane.

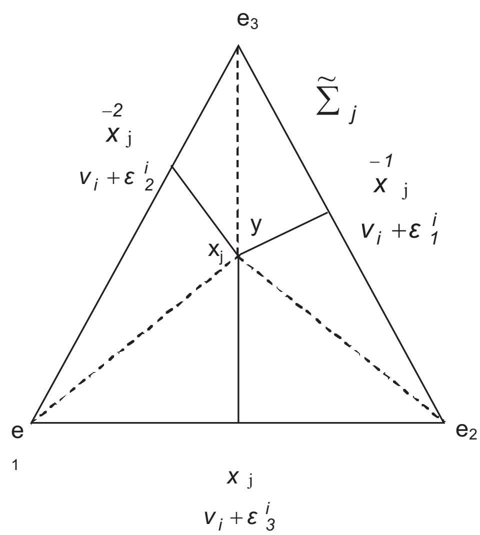
Thus, we take into consideration the prefixed point $x_j = \left(x_1^j, x_2^j, \ldots\ldots x_m^j\right) \in \Sigma_j$ and the simplex´s vertexes e_s , having one in the place s and zero in the other places.

We choose s points having the following form:
$\overset{-s}{x}_j = (\overset{-j}{x}_1, \overset{-j}{x}_2, \ldots\ldots, \overset{-j}{x}_{s-1}, 0, \overset{-j}{x}_{s+1}, \ldots\ldots, \overset{-j}{x}_m)$ with s=1,2,......m

The point $\overset{-s}{x}_j$ is on a face of simplex $\tilde{\sum}_j$, and we obtain it by extending the segment between e_s and x_j, until it reaches the opposite face to the corresponding vertex.

This is made in order to obtain a polyhedral partition of the simplex $\tilde{\sum}_j$, having as extreme points: the m vertices of simplex $\tilde{\sum}_j$, the m points $\overset{-s}{x}_j$ and the prefixed point x_j . (Marchi and Quintas (1987) studied characterizations of these points on some n-person games).

The geometric idea laying behind the construction of the payoff matrixes consists in analyzing which is the "maximizing hyperplane" in each subset of the simplex partition. We want to have a unique "maximizing hyperplane" in each region (see next figure when $\left|\Sigma_j\right| = 3$)



In order to obtain $\overset{-s}{x}_j$ we use the following equation:

$$e_s + \lambda^s \left(x_j - e_s\right) = \overset{-s}{x}_j$$

As the s-th component of $\overset{-s}{x}_j$ is zero, then we have: $1 + \lambda_s \left(x_s^j - 1\right) = 0$

Thus $\lambda^s = \dfrac{1}{1 - x_s^j} > 0$, and it follows that: $\overset{-s}{x}_t = \begin{cases} \dfrac{x_t^j}{1 - x_t^j} > 0 & \text{for each } t \neq s \\ 0 & \text{for each } t = s \end{cases}$

Thus: $\overset{-s}{x}_j = \left(\dfrac{x_1^j}{1 - x_s^j}, \dfrac{x_2^j}{1 - x_s^j}, \ldots\ldots, \dfrac{x_{s-1}^j}{1 - x_s^j}, 0, \dfrac{x_{s+1}^j}{1 - x_s^j}, \ldots\ldots, \dfrac{x_m^j}{1 - x_s^j}\right)$

As $1 - x_s^j = \sum_{t \neq s} x_t^j$ we have: $\overset{-s}{x}_j = \dfrac{1}{\sum_{t \neq s} x_t^j} \left(x_1^j, x_2^j, \ldots\ldots, x_{s-1}^j, 0, x_{s+1}^j, \ldots\ldots, x_m^j\right)$

For each vertex $e_r \in \tilde{\sum_i}$ and for each xj $\in \tilde{\sum_j}$, by definition 2.6 we obtain:

$$F_i(r, x_j) = \sum_{l=1}^{l=m} a_{rl}^{ij}\, x_l^j = A_i^r\, x_j^t,$$

where A_i^r is the r-th row of the matrix A_i of player i and x_j^t is the transposed of x_j.

On the hyperplane $F_i(r, x_j)$ we require the following properties:

- It should take the value v_i on $x_j = (x_1^j, x_2^j, \ldots\ldots x_m^j) \in \Sigma_j$, . That is:

$$F_i(r, x_j) = \sum_{l=1}^{l=m} a_{rl}^{ij}\, x_l^j = A_i^r\, x_j^t = v_i$$

- In each point $\overset{-s}{x}_j$ with s= 1, 2,.....,n, it should take the value $v_i + \varepsilon_s^i$, with $\varepsilon_s^i > 0$. Namely.

$$F_1\left(r, \overset{-s}{x}_j\right) = \sum_{\substack{l=1\\ l\neq s}}^{l=m} a_{rl}^{ij}\, x_l^j = v_i + \varepsilon_r^i$$

Then $\dfrac{1}{1-x_s^j} \sum_{\substack{l=1\\ l\neq s}}^{l=m} a_{rl}^{ij} = v_i + {}_s^i$

Now we introduce a bijective function f_{ji} *in order to complete the* definition of player i payoff matrix.

$f_{ji} : \Sigma_j \to \Sigma_i$, $f_{ji}(s) = r$ such that $F_i(r, x_j) = v_i$

(where r is the index of the corresponding maximizing hyperplane)

This implies that:

$F_i(r, s) = a_{rs}^{ij} > a_{ts}^{ij} = F_i(t, s)$ for each $t \in \tilde{\sum_i}$

Thus, by definition 2.4 we have: $F_i(r, s) = a_{rs}^{ij}$ and by theorem 1 we obtain: $J_i(e_s) = \{f_{ji}(s)\}$. In this way in each vertex of $\sum_j$ there is a unique maximizing hyperplane and f_{ji} distributes the different hyperplanes on the different vertices.

We also prescribe that each t, such as $f_{ji}(t) \neq r$, the $f_{ji}(t)$-hyperplane, "passes underneath" the $f_{ji}(s)$-hyiperplane at each point $\overset{-s}{x}_j$. Moreover we ask it takes the values $v_i + \varepsilon_s^i$.

Thus, for each r $\neq f_{ji}(t)$.

$$\sum_{l=1}^{l=m} a_{rl}^{ij}\, \overset{-s}{x}_l = \frac{1}{1-x_s^j} \sum_{\substack{l=1\\ l\neq s}}^{l=m} a_{rl}^{ij} x_l^j = v_i + \varepsilon_s^i$$

$$>$$

$$\sum_{l=1}^{l=m} a_{f_{ji}(s)l}^{ij}\, \overset{-s}{x}_l = \frac{1}{1-x_s^j} \sum_{\substack{l=1\\ l\neq s}}^{l=m} a_{f_{ji}(s)l}^{ij} x_l^j$$

Thus for r=1,2,...,m in the point $x_j = (x_1^j, x_2^j, \ldots\ldots\ldots, x_m^j)$ we prescribe that all the hyperplanes take the same value v_i. This is, for r =1,2,...,n,

$$\sum_{l=1}^{l=m} a_{rl}^{ij} x_l^j = v_i$$

And for each t such that $f_{ji}(t) \neq$ r

$$\frac{1}{1-x_t^j}\left[\left(\sum_{l=1}^{l=3} a_{rs}^{ij}\, x_l^j\right) - a_{rt}^{ij} x_t^j\right] = v_i + \varepsilon_t^i$$

Then, we have the following system:

$$\begin{cases} \sum_{l=1}^{l=m} a_{rl}^{ij} x_l^j = v_i \\ y \quad \forall t : f_{ji}(t) \neq q \\ \frac{1}{1 - x_t^j}\left[\left(\sum_{l=1}^{l=m} a_{rl}^{ij} x_l^j\right) - a_{rt}^{ij} x_t^j\right] = v_i + \varepsilon_t^i \end{cases}$$

Solving it, we have:

$$a_{rs}^{ij} = \begin{cases} v_i + \frac{1}{x_s^j} \sum_{f_{ji}(t) \neq s} \varepsilon_t^i (1 - x_t^j) & \text{for } f_{ji}(s) = r \\ v_i - \frac{(1 - x_s^j)\varepsilon_s^i}{x_s^j} & \text{for } f_{ji}(s) \neq r \end{cases}$$

Remark 1:

- As $0 < x_s^j < 1$ the coefficients are well defined.
- As $f_{ji}(s) = r$, then $a_{rs}^{ij} > v_i$.
- As $f_{ji}(s) \neq r$; then $a_{rs}^{ij} < v_i$.

These two inequalities agree with the geometric idea leading the whole construction.

- The payoff matrix A_i is non singular, its determinant is:

$$det(A_i) = \left(\sum_{s=1}^{s=m} (1 - x_s^i)\, \varepsilon_s^i\right)^{n-1} \frac{v_i}{x_1 . x_2 . x_3 \ldots\ldots x_n}$$

And as ε_s^i are arbitrary positives numbers, $\sum_{s=1}^{s=m} (1 - x_s^i)\, \varepsilon_s^i > 0$. Furthermore, v_i is not null, then $det(A_i) \neq 0$.

4 Existence and Uniqueness of Nash Equilibrium

We check here that the point $x = (x_1, x_2, \ldots\ldots x_n) \in \prod_{i=1}^{i=n} \tilde{\Sigma}_i$, with $x_s^j > 0$ for s= 1,..., m , is a Nash Equilibrium. It is so, because it fulfills the inclusions given, in Theorem 1

$S(x_i) \subseteq J_i(x_{q(i)})$, for each i = 1,2,....., n

As all the hyperplanes of player i take the same value v_i in the point x_j , and taking into account that $\sum_{l=1}^{l=m} a_{rl}^{ij} x_l^j = v_i$; then $J_i(x_j) = \{1, 2, \ldots\ldots, n\} = S(x_i)$.

Thus, the construction presented in the previous section guarantees the existence of a completely mixed Nash Equilibrium $(x_1, x_2, \ldots\ldots, x_n)$ for a q- cyclic game Γ with payoff matrix A_i for player i.

Theorem 2: Given

$(x_1, x_2, \ldots\ldots, x_n) = ((x_1^1, x_2^1, \ldots\ldots x_m^1), (x_1^2, x_2^2, \ldots\ldots x_m^2) \ldots\ldots\ldots\ldots (x_1^n, x_2^n, \ldots\ldots x_m^n))$

with $\sum_{t=1}^{t=m} x_t^i = 1$ for each i=1,2,....,n with $x_t^i > 0$ for each t=1,2,....,m and for each i=1,2,.....,n; and values nonzero $v_1, v_2, \ldots, v_n$. And given the functions $f_{ji} : \Sigma_j \to \Sigma_i$, with j= i+1 mod.n and positives numbers ε_s^i with i=1, 2, ,....,n.

There exists a q-cyclic game Γ having $(x_1, x_2, \ldots\ldots, x_n)$ as a completely mixed Nash Equilibrium with payoff values $v_1, v_2, \ldots, v_n$

We will prove that the family of games we constructed in the previous section has $(x_1, x_2,......x_n) \in \prod_{t=1}^{t=n} \tilde{\Sigma}_t$ as unique Nash Equilibrium.

Given the payoff matrices $A_1, A_2,.......A_n$ for the corresponding players, we choose functions f_{ji} and fulfilling that for each $(r', r'') \in \Sigma_j x \Sigma_j$:

If $\Phi \circ f_{ji}(r') = r''$ then $\Phi \circ f_{ji}(r'') \neq r'$ (1)

with $\Phi : \Sigma_i \rightarrow \Sigma_j$ resulting of the compositions of functions of the type f_{ji}.

Remark 2:

- These conditions are similar to those given for Bimatrix games by Quintas (1988 b),
- In a 3-person game, player 1 plays with player 2, and condition (1) takes the form:

  For each $(j', j'') \in \Sigma_2 x \Sigma_2$:

  If $\Phi \circ f_{21}(j') = j''$ then $\Phi \circ f_{21}(j'') \neq j'$

  In this case $\Phi = f_{32} \circ f_{13}$.

- The functions $f_{ji}(r) = r$ and $\Phi(r) = r - 1 \mod. m$ fulfill condition (1).

  Moreover, if we choose positive number ε_s^i we have that:

  $$\sum_{s=1}^{s=m} (1 - x_s^j) \varepsilon_s^i > 0 \quad (2)$$

This implies that the payoff matrix of each player i is non singular.

We will use the following notation:

- $(x_j)^t$ is the transposed vector of x_j.
- $V_i = (v_i \; v_i.........v_i)^t$ is a (mx1) matrix with v in all entries.
- A_i^j is the j-th row of player i payoff matrix

Theorem 3:

Given a q-cyclic game Γ constructed as in Theorem 2, having $(x_1, x_2,......., x_n)$ as a completely mixed Nash Equilibrium, with non zero values $v_1, v_2,...,v_n$, and given functions f_{ji} fulfilling condition (1) and given positive numbers ε_s^i with i=1, 2, ,....,n fulfilling condition (2) , then, $(x_1, x_2,......., x_n)$ is the unique Nash Equilibrium of the game Γ.

Proof:

In order to prove the uniqueness, we assume that there exists another Nash Equilibrium $(\gamma_1,\gamma_2,.......,\gamma_n)$ with values u_i , thus we have the expected utilities $E_i(\gamma_1,\gamma_2,.......,\gamma_n) = u_i$ for i=1,2,...n.

We will consider the following cases:

- Case 1: $(\gamma_1,\gamma_2,.......,\gamma_n)$ is completely mixed

As it is a Nash Equilibrium it fulfills the systems:

$$A_i(x_j)^t = V_i \qquad\qquad A_i(y_j)^t = U_i$$

Multiplying each equation of system a) by u_i , and multiplying each equation of system b) v_i and subtracting one system from the other we obtain:

$A_i(u_i x_j - v_i y_j)^t = 0$ being 0 the (mx1)-order null matrix.

The matrices A_i are non singular, because $v_i \neq 0$ and ε_p^i fulfills condition (2) (see remark 1), Thus the lineal homogeneous systems $A_i(u_i x_j - v_i y_j)^t = 0$ has unique solution, namely the trivial. $u_i x_j - v_i y_j = 0$, It implies that, $u_i x_s^j = v_i y_s^j$ with s=1,2,.......m.

Summing up over s we obtain: $\sum_{s=1}^{s=m} u_1 x_s^j = \sum_{s=1}^{s=m} v_1 y_s^j$

As x_j and y_j are probability vectors, then $u_i = v_i$, therefore: $A_i(x_j)^t = A_i(y_j)^t$ thus $A_i(x_j - y_j)^t = 0$ and as A_i is non singular then the system has unique solution $x_j - y_j = 0$, and thus $x_j = y_j$ for each $i=1,2,.....,n$

- Case 2: The point $(y_1, y_2,......., y_n)$ fulfills that:

$S(y_i)=S(x_i)$ with $i \in \{1,2,.......,n\}$, excepting for some $j \in \{1,2,.......,n\}$, such that $S(y_j) \subseteq S(x_j)$ with $j \neq i$

We assume that $y_j = \left(y_1^j, y_2^j, ..., y_k^j, ..., y_{m-1}^i, 0\right)$ and $f_{ji}(k) = k$

Let $k \in S(x_j) - S(y_j)$,

$$A_i^{f_{32}(k)}(y_j)^t = v_i \sum_{s \in S(y_j)} y_s^j - \sum_{s \in S(y_j)} \frac{\varepsilon_s^i(1 - x_s^i)}{x_s^i} y_s^j$$

Let $r \in S(y_i)$

$$A_i^{f_{ji}(r)}(y_j)^t = v_i \sum_{s \in S(y_j)} y_s^j - \sum_{\substack{s \in S(y_j) \\ s \neq r}} \frac{\varepsilon_s^i(1 - x_s^i)}{x_s^i} y_s^j + \frac{y_r^j}{x_r^j} \sum_{\substack{s=1 \\ s \neq r}}^{s=m} \varepsilon_s^i(1 - x_s^j)$$

Subtracting $A_i^{f_{ji}(r)}(y_j)^t$ y $A_i^{f_{ji}(k)}(y_j)^t$ we obtain:

$$v_i \sum_{s \in S(y_j)} y_s^j - \sum_{\substack{s \in S(y_j) \\ s \neq r}} \frac{\varepsilon_s^i(1 - x_s^i)}{x_s^i} y_s^j + \frac{y_r^j}{x_r^j} \sum_{\substack{s=1 \\ s \neq r}}^{s=m} \varepsilon_s^i(1 - x_s^j) - \left(v_i \sum_{s \in S(y_j)} y_s^j - \sum_{s \in S(y_j)} \frac{\varepsilon_s^i(1 - x_s^i)}{x_s^i} y_s^j \right),$$

That is:

$$A_i^{f_{ji}(r)}(y_j)^t - A_i^{f_{ji}(k)}(y_j)^t = \frac{y_r^j}{x_r^j} \sum_{\substack{s=1 \\ s \neq r}}^{s=m} \varepsilon_s^i(1 - x_s^j) + \frac{\varepsilon_r^i(1 - x_r^i)}{x_r^i} y_r^j = \frac{y_r^j}{x_r^j} \sum_{s=1}^{s=m} \varepsilon_s^i(1 - x_s^j)$$

We have: $\sum_{s=1}^{s=m} \varepsilon_s^i(1 - x_s^j) > 0$ y $r \in S(y_j)$

$A_i^{f_{ji}(r)}(y_j)^t - A_i^{f_{ji}(k)}(y_j)^t > 0$, implies that $f_{ji}(k) \notin J_i(y_j)$.

Thus $(y_1, y_2,......., y_n)$ is an Equilibrium point, because: $S(y_i) \subseteq J_i(y_j)$ (Theorem 1), then $f_{ji}(k) \notin S(y_i)$, and in consequence $y_{f_{ji}(k)}^j = 0$., but that is impossible because by hypothesis, we had that $S(y_i) = S(x_i)$, and thus $(x_1, x_2,......., x_n)$ is a point completely mixed Equilibrium, which implies that $|S(y_i)| = |S(x_i)| = m$.

In consequence, $S(y_j) = S(x_j)$ for all $j=1,2,...., n$ and then by case 1, $x_j = y_j$ therefore $(x_1, x_2,......., x_n)$ is unique equilibrium.

- Case 3: The point $(y_1, y_2,........, y_n)$ fulfills that:

$$S(x_i) \subseteq S(y_i) \quad \text{for all} \quad i = 1, 2, ..., n$$

Let $k \in S(x_n) - S(y_n)$. In this case, we suppose $k = m$ $f_{n(n-1)}(k) = k$,

In case 2, we obtained $y_{f_{n(n-1)}(k)}^{n-1} = 0$, thus $y_m^{n-1} = 0$.

Let $j \in S(x_{n-1}) - S(y_{n-1})$, by hypothesis we had $S(y_{n-1}) \subseteq S(x_{n-1})$, in this case $j = m$ satisfies that condition.

Let $f_{(n-1)(n-2)}(j) = j+1$ mod.m,

$$A_{n-2}^{f_{(n-1)(n-2)}(j)}(y_{n-1})^t = v_{n-2} \sum_{s \in S(y_{n-1}')} y_s^{n-1} - \sum_{s \in S(y_{n-1})} \frac{\varepsilon_s^{n-2}(1 - x^{n-1})}{y_s} y_s^{n-1}$$

Let $r \in S(y')$,

$$A_{n-2}^{f_{(n-1)(n-2)}(r)}(y_{n-1})^t = v_{n-2} \sum_{s \in S(y_{n-1})} y_s^{n-1} - \sum_{s \in S(y_{n-1})} \frac{\varepsilon_s^{n-2}(1 - x_s^{n-1})}{x_s^{n-1}} y_s^{n-1} + \frac{y_r^{n-1}}{x_r^{n-1}} \sum_{\substack{s=1 \\ s \neq r}}^{s=m} \varepsilon_s^{n-2}(1 - x_s^{n-1})$$

By subtracting $A_{n-2}^{f_{(n-1)(n-2)}(r)}(y_{n-1})^t$ from $A_{n-2}^{f_{(n-1)(n-2)}(j)}(y_{n-1})^t$ we obtain, as in the previous case that: $A_{n-2}^{f_{(n-1)(n-2)}(r)}(y_{n-1})^t > A_{n-2}^{f_{(n-1)(n-2)}(j)}(y_{n-1})^t$, then $f_{(n-1)(n-2)}(j) \notin J_{n-2}(y_{n-1})$.

As we have that $(y_1, y_2, \ldots\ldots, y_n)$ is an equilibrium $S(y_{n-2}) \subseteq J_{n-2}(y_{n-1})$ (Theorem 1), then $f_{(n-1)(n-2)}(j) \notin S(y_{n-2})$, and in consequence, $y^{n-2}_{f_{(n-1)(n-2)}(j)} = 0$, that is $y_1^{n-2} = 0$.

Let $i \in S(x_1) - S(y_1)$, and we consider $f_{1n}(i) = i$

Working as in the previous cases we obtain that $f_{1n}(i) \notin J_n(y_1)$.

Again as $(y_1, y_2, \ldots\ldots, y_n)$ is an equilibrium $S(y_n) \subseteq J_n(y_1)$ (Theorem 1), then $f_{1n}(i) \notin S(y_n)$, and in consequence: $y^n_{f_{1n}(i)} = 0$, that is is $y_1^n = 0$.

Now choosing $k \in S(x_n) - S(y_n)$, such that: $L(k) = f_{1n}(f_{21}(\varphi(f_{n(n-1)}(k)))) \in S(y_n)$,

The existence of such k, is guarantied by (1).

For k= m, $L(m) = 1$

$$A_n^{L(k)}(y_1^{\,'})^t = v_n \sum_{s \in S(y_1)} y_s^1 - \sum_{s \in S(y_1)} \frac{\varepsilon_s^n (1 - x_s^1)}{x_s^1} y_s^1$$

and for each r such that $h(r) = f_{21}(\varphi(f_{n(n-1)}(r))) \in S(y_1)$, in consequence, $\Phi(r) \neq 1$

$$A_n^{L(r)}(y_1)^t = v_n \sum_{s \in S(y_1)} y_s^1 - \sum_{\substack{s \in S(y_1) \\ s \neq h(r)}} \frac{\varepsilon_s^n (1 - x_s^1)}{x_s^1} y_s^1 + \frac{y^1_{h(r)}}{x^1_{h(r)}} \sum_{\substack{s=1 \\ s \neq h(r)}}^{s=m} \varepsilon_s^n (1 - x_s^1) \text{ i}$$

Subtracting $A_n^{L(r)}(y_1)^t$ from $A_n^{L(k)}(y_1)^t$, we obtain:

$$A_n^{L(r)}(y_1)^t - A_n^{L(k)}(y_1)^t = \frac{y^1_{h(r)}}{x^1_{h(r)}} \sum_{s=1}^{s=m} \varepsilon_s^n (1 - x_s^1)$$

This is positive because $h(r) \in S(y_1)$ and $\sum_{s=1}^{s=m} \varepsilon_s^n (1 - x_s^1) > 0$

Thus, $A_n^{L(r)}(y_1)^t > A_n^{L(k)}(y_1)^t$ (3)

However, k is such that $L(k) \in S(y_1)$, and as $(y_1, y_2, \ldots\ldots, y_n)$ is an Equilibrium, then $S(y_n) \subseteq J_n(y_1)$ (Theorem 1), we have

$A_n^{L(r)}(y_1)^t \leq A_n^{L(k)}(y_1)^t$. (4)

But the inequalities (3) and (4) are incompatible, unless $y^1_{h(r)} = 0$.

If this occurs the vector y_1 is the null, and this is an absurd, thern there exist no other equilibrium of the game Γ. It completes the proof.

5 Constructing Matrices of n- Players Cyclic Game of with Unique Points non Completely Mixed Equilibrium

In this section we consider cyclic games having the following characteristics: Each player have m_i strategies, $|\Sigma_i| \leq m_i$ with i=1, 2,........, n, and each player have m strategies actives.

Without loss of generality we can consider that the positives components of each vector x_i i=1,2, ..., n, are the first m for each player.

Let $v_1, v_2, \ldots\ldots, v_n$, be n arbitrary nonzero values and let $(x_1, x_2, \ldots\ldots, x_n)$ be a n- tupla of probability vectors were each vector x_i is equal to:

$$x_i = (x_i(\sigma_1^i), x_i(\sigma_2^i) \ldots\ldots, x_i(\sigma^i_{m_i})) = (x_1^i, x_2^i, \ldots\ldots x^i_{m_i}).$$

We will construct matrices $A_1, A_2, \ldots, A_n$, with the following form:

$$A_i = \begin{pmatrix} a^{ij}_{11} & --- & a^{ij}_{1k} & --- & a^{ij}_{1m_j} \\ | & | & | & | & | \\ a^{ij}_{k1} & --- & a^{ij}_{kk} & --- & a^{ij}_{km_j} \\ | & | & | & | & | \\ a^{ij}_{m_i 1} & --- & a^{ij}_{m_i k} & --- & a^{ij}_{m_i m_j} \end{pmatrix}$$

for i=1,2,...n ; j= i+1 mod. n

We denote with the following submatrices of size mxm:

$$A_i^m = \begin{pmatrix} a^{ij}_{11} & --- & a^{ij}_{1m} \\ | & | & | \\ a^{ij}_{m1} & --- & a^{ij}_{mm} \end{pmatrix}$$

We define the elements of the submatrices for i=1,2,...n as follows:

$$a^{ij}_{rs} = \begin{cases} v_i + \dfrac{1}{x^j_s} \displaystyle\sum_{f_{ji}(t)\neq r} \varepsilon^i_t \left(1 - x^j_t\right) & \text{for } f_{ji}(s) = r \\ v_i - \dfrac{\left(1 - x^j_s\right)\varepsilon^i_s}{x^j_s} & \text{for } f_{ji}(s) \neq r \end{cases}$$

here with p=1,...,m are positive numbers

We choose bijective functions f_{ji} y , such that:

$f_{ji}: S(x_j) \to S(x_i)$ y $\Phi: S(x_i) \to S(x_j)$

fulfilling condition (1) given in section 4.

We split the construction in 3 different areas as shown

<table>
<tr><td>I</td><td>II</td></tr>
<tr><td colspan="2">III</td></tr>
</table>

- In I it places the submatrix A_i^m
- In II a submatrix of size m x (m_j-m), were the elements are chosen arbitrarily.
- In III a submatrix of size (m_i – m) x m_j, we chose q rows of A_i with $m < q \leq m_i$

. We denote ir with A_i^q, and it fulfills that $A_i^q \left(x_j\right)^t = v_i^q$ for a suitable v_i^q with $v_i^q < v_i$ for i=1,.....n

We denote with:

- $\tilde{x}_i$ is the vector $\tilde{x}_i = \left(x^i_1, x^i_2, \ldots, x^i_m\right)$ consisting of the first m components of x_i.
- $\Sigma_i^m = \{1, 2, \ldots, m\} = S(x_i)$ with i=1,2,....,n.
- $\overset{m}{\Gamma} = \left\{\Sigma_1^m, \Sigma_2^m, ,,,, \Sigma_n^m,\ A_1^m, A_2^m, \ldots, A_n^m\right\}$ is the n-person game,

Now we make the verification that $\left(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\right)$ is the unique Nash Equilibrium, of $\overset{m}{\Gamma}$, and $v_1, v_2, \ldots, v_n$ are the values of the game $\overset{m}{\Gamma}$ (Definition 2.8).

In view of above construction the matrix A_i have the following properties: if we can choose the q-th row A_i^q with $m < q \leq m_i$, and:

$A_i^q \left(x_j\right)^t = v_i^q$ for suitable v_i^q, fulfilling that: $v_i^q < v_i$ i=1,.....n .

Therefore , it verifies that:

$A_i \left(x_j\right)^t = V_i$ for $V_i = \left(v_i, v_i, \ldots, v_i, v_i^{m+1}, \ldots, v_i^{m_i}\right)^t$

with i = 1, 2,....., n.

In consequence, $(x_1,x_2,.....,x_n)$ is an equilibrium of the game $\tilde{\Gamma}$.

Just at the realized in the previous section, now it exists a q- cyclic game Γ no completely mixed, with payoff matrix A_i and values $v_1, v_2,...,v_n$ having $(x_1, x_2,......., x_n)$ as equilibrium of the game $\tilde{\Gamma}$. Therefore, it was proved the following theorem:

Theorem 4: *Give a n-tupla:*

$(x_1, x_2,........, x_n) = ((x_1^1, x_2^1,........x_m^1, 0,......,0),(x_1^2, x_2^2,........,x_m^2, 0.......0)..............,(x_1^n, x_2^n,........x_{m,}^n, 0,.....,0))$

with $\sum_{t=1}^{t=m} x_t^i = 1$ *for each* $i=1,2,....,n$ *with* $x_t^i > 0$ *for each* $t=1,2,....,m$ *and each* $i=1,2,.....,n$; *where each vector* x_i, *has mi positive components with* $i=1, 2,...,n$ *and* $v_1, v_2,...,v_n$ *are non zero values Let* f_{ji} *and* Φ *be functions fulfilling condition (1) , were* ε_s^i, *with* $i= 1, 2,.., n$ *.are positive numbers.*

Then there exists a q-cyclic game Γ, *having* $(x_1, x_2,......., x_n)$ *as unique non completely mixed equilibrium, and being* $v_1, v_2,...,v_n$ *are the corresponding values of the game.*

Remark 3: In the way we construct the matrices $A_1, A_2,......, A_n$ the new rows we included are strictly dominates, then $(x_1, x_2,........, x_n)$ *will* still be a unique equilibrium.

5.1 Uniqueness of the Point of Equilibrium of Nash

Theorem 5: Give a q-cyclic game Γ constructed as in Theorem 1, then $(x_1, x_2,......., x_n)$ is the unique equilibrium of the game.

Proof: In order to prove the uniqueness we assume that $(y_1, y_2,.......,y_n)$ is another equilibrium.

Taking into account conditions (2), (3) and Theorem 2, it is sufficient to prove the case when:

$S(x) = S(y)$

As $(y_1, y_2,.......,y_n)$ is an equilibrium, it fulfills that: $A_i(y_j)^t = U_i$, where $U_i = (u_i, u_i,......., u_i, u_i^{m+1},........, u_i^{m_i})^t$, for suitable u_i^q such that: $u_i^q < u_i$ with $m < q \leq m_i$, $i = 1,2,.....,n$,

Then:

$$A_i^m\left(\tilde{y}_j\right)^t = (u_i, u_i,..........,u_i)^t \text{ With } i=1,2,...n$$

Where $\tilde{y}_j$ is the corresponding truncated vector of y with respect to its m first components.

As $(x_1, x2,.......,x_n)$ is also an equilibrium, then: $A_i(x_j)^t = V_i$ where $V_i = (v_i, v_i,........, v_i, v_q^{m+1},........, v_i^{m_i})^t$, for suitable v_i^q, such that: $v_i^q < v_i$ for $m < q \leq m_i$ with $i=1,2,.....,n$,

Then

$$A_i^m\left(\tilde{x}_j\right)^t = (v_i, v_i,..........,v_i) \text{ with } i= 1,..., n.$$

where $\tilde{x}_j$ is the corresponding truncated vector of x_j

Choosing functions fji and Φ fulfilling condition (1); and positive numbers ε_p^i with $p=1,...,m$, then, we are in the hypothesis of Theorem 2. Therefore the point $(x_1, x_2,........, x_n)$ is the unique equilibrium.

6 Concluding remarks

In this work we presented a wide family of q-cyclic n-person games with unique prefixed Nash equilibrium points. We extend the constructions given for bimatrix games by Marchi and Quintas (1987) and Quintas (1988 a). We also proved the uniqueness of equilibrium for a wide family of completely mixed q-cyclic game and also for a family of not completely mixed q-cyclic game, with each players having n strategies, being only m ($m<n$) of them active strategies.

This is a step in extending results from 2 person games to n-person games. We might have expected that the procedure used here could also serve to generate games with unique equilibrium in Polymeric Games (Yanovskaya E. B (1968)). These games are an extension of q-cyclic games, but unfortunately the technique we used, didn't provide unique equilibrium in Polymeric Games and it would require the use of a different technique.

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