A NOTE ON THE HOMOTOPY TYPE OF THE ALEXANDER DUAL

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Abstract. We investigate the homotopy type of the Alexander dual of a simplicial complex. It is known that in general the homotopy type of K does not determine the homotopy type of its dual K^*. We construct for each finitely presented group G, a simply connected simplicial complex K such that $\pi_1(K^*) = G$ and study sufficient conditions on K for K^* to have the homotopy type of a sphere. We extend the simplicial Alexander duality to the more general context of reduced lattices and relate this construction with Bier spheres using deleted joins of lattices. Finally we introduce an alternative dual, in the context of reduced lattices, with the same homotopy type as the Alexander dual but smaller and simpler to compute.

1. Introduction

Let A be a compact and locally contractible proper subspace of S^n. The classical Alexander duality theorem asserts that the reduced homology groups $H_i(S^n - A)$ are isomorphic to the reduced cohomology groups $H^{n-i-1}(A)$ (see for example [9, Thm. 3.44]). The combinatorial (or simplicial) Alexander duality is a special case of the classical duality: if K is a finite simplicial complex and K^* is the Alexander dual with respect to a ground set $V \supseteq K^0$, then for any i

$$H_i(K) \cong H^{n-i-3}(K^*).$$

Here K^0 denotes the set of vertices (i.e. 0-simplices) of K and n is the size of V. A nice and simple proof of the combinatorial Alexander duality can be found in [5]. An alternative proof of this combinatorial duality can be found in Barr's article [3].

In these notes we relate the homotopy type of K with that of K^*. It is known that, even though the homology of K determines the homology of K^* (and vice versa), the homotopy type of K does not determine the homotopy type of K^* (see for example [8]). We show that for any finitely presented group G, one can find a simply connected complex K such that its Alexander dual, with respect to some ground set V, has fundamental group isomorphic to G. In the same direction, we exhibit an example of a complex with the homotopy type of a sphere whose dual is not homotopy equivalent to a sphere. If K simplicially collapses to the boundary of a simplex, it can be shown that K^* is homotopy equivalent to a sphere. We exhibit a proof of this result using the nerve of the dual. We also use the nerve to find an easy-to-check sufficient condition for a complex to simplicially collapse to the boundary of a simplex.

In the last section of these notes we extend the duality to the context of reduced lattices. A reduced lattice is a finite poset with the property that any subset which is bounded below has an infimum. Any finite simplicial complex can be seen as a reduced lattice by

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means of its face poset. We define the Alexander dual for reduced lattices and show that the duality theorem remains valid in this context. When the poset is the face poset of a simplicial complex, the construction coincides with the simplicial one. We also extend the notion of *deleted join* of simplicial complexes to the context of reduced lattices. Following Bier's construction of spheres [4, 7, 11], we show that the deleted join $X *_D X^*$ of any reduced lattice X with its Alexander dual X^* produces a Bier sphere (by taking the atom crosscut complex of $X *_D X^*$) and a polyhedron homotopy equivalent to a sphere (by taking the order complex of $X *_D X^*$).

At the end of these notes we propose an alternative notion of dual $d(X)$, for any reduced lattice X, which also satisfies the Alexander duality. In the same way that the nerve of a simplicial complex is in general smaller than the original complex, the dual $d(X)$ is a smaller homotopy model of X^* and it is simpler to compute.

2. The homotopy type of the Alexander dual

Let K be finite simplicial complex and let V be a set which contains the set K^0 of 0-simplices of K. The Alexander dual of K (with respect to the fixed set V) is the simplicial complex

$$K^* = \{\sigma \subset V,\ \sigma^c \notin K\}.$$

Here $\sigma^c = V \setminus \sigma$, the complement of σ in V. It is clear that $K^{**} = K$. Note that the set V is implicit in the definition of the dual.

The simplicial Alexander dual K^* allows us to investigate the homology of K but in general the homotopy type of K^* does not determine the homotopy type of K. Moreover, the fundamental group of K^* does not provide any information about the fundamental group of K. In fact, one can prove the following.

Proposition 2.1. *For any given finitely presented group G, there exists a connected compact simplicial complex K such that $\pi_1(K) = G$ and such that its Alexander dual K^* with respect to any $V \supseteq K^0$ is simply connected.*

Proof. Since G is finitely presented, there exists a connected 2-dimensional finite simplicial complex K such that $\pi_1(K) = G$. We can suppose without loss of generality that K has more than six vertices. The dual of K, with respect to any $V \supseteq K^0$ contains the whole 2-skeleton of the simplex spanned by V, since the complement of any subset of three elements of V is not a simplex in K, by a cardinality argument. It follows that K^* is simply connected. □

Corollary 2.2. *For any finitely presented group G there is a simply connected complex whose dual, with respect to some V, has fundamental group isomorphic to G.*

In the same direction, the following example shows two homotopy equivalent simplicial complexes K, L such that $K^0 = L^0 = V$ and such that their duals K^*, L^* (with respect to V) are not homotopy equivalent.

Example 2.3. Let M be a triangulation of the Poincaré homology 3-sphere and let S be the boundary of a 4-simplex whose vertices are contained in the set $V = M^0$. Similarly as in the proof of Proposition 2.1, since any triangulation M of the homology 3-sphere has more than 7 vertices and M and S are 3-dimensional, their duals $K = M^*$ and $L = S^*$ (with respect to V) are simply connected. Since K and L have the homology of a sphere S^r, it follows that they are in fact homotopy equivalent. Moreover, $K^0 = L^0 = V$ and their duals are respectively M and S, which are not homotopy equivalent.

In particular, the last example shows that the dual of a complex which is homotopy equivalent to a sphere need not be homotopy equivalent to a sphere. The next lemma shows that, when we restrict ourselves to simplicial collapses, the homotopy type of the dual is preserved. We refer the reader to [6] for the basic notions on simplicial collapses and expansions and simple homotopy types. As usual, we will denote an elementary simplicial collapse by $K \overset{e}{\searrow} L$ and, in general, $K \searrow L$ will denote a simplicial collapse.

Lemma 2.4 (cf. Dong [8])**.** *Let L be a subcomplex of K and let V be a set containing K^0. Then $K \overset{e}{\searrow} L$ if and only if $K^* \overset{e}{\nearrow} L^*$. Consequently, if $K \searrow L$, then $K^* \nearrow L^*$.*

Proof. Note that if $L = K \backslash \{\tau, \sigma\}$ with τ a free face of σ, then $K^* = L^* \backslash \{\sigma^c, \tau^c\}$ with σ^c a free face of τ^c. □

Recall that the nerve $N(K)$ of a simplicial complex K is the complex whose vertices are the maximal simplices (=facets) of K and the simplices are the subsets of facets with non-empty intersection. It is well-known that $N(K)$ is homotopy equivalent to K.

Lemma 2.5. *Let $\dot{\tau}$ be the boundary of a simplex and let V be a set such that $\tau^0 \subsetneq V$. Then $(\dot{\tau})^*$ is homotopy equivalent to the sphere S^{n-1}, where $n = \#V - \#\tau^0$.*

Proof. If $n = 1$, $V = \tau^0 \cup \{v\}$ and $(\dot{\tau})^*$ is the disjoint union of the simplex τ and the vertex v. Then $(\dot{\tau})^*$ is homotopy equivalent to S^0.

In general, if $V = \tau^0 \cup \{v_1, \ldots, v_n\}$, $(\dot{\tau})^*$ has $n+1$ maximal simplices, namely the simplices η_i with vertex sets $\tau^0 \cup \{v_1, \ldots, \hat{v}_i, \ldots, v_n\}$, for $i = 1, \ldots, n$, and η_{n+1} whose vertex set is $\{v_1, \ldots, v_n\}$. The intersection of all these simplices is empty but any other intersection is non-empty. Then the nerve of $(\dot{\tau})^*$ is the boundary of the n-simplex and therefore $(\dot{\tau})^*$ is homotopy equivalent to S^{n-1}. □

Corollary 2.6. *If K collapses to the boundary of a simplex, then K^* is homotopy equivalent to a sphere.*

We can use the nerve of the complex to find an easy-to-check sufficient condition for a complex to collapse to the boundary of a simplex. Note that in many cases, the nerve of a complex K is much smaller than K. Moreover, in [2] it is proved that any complex K *strong collapses* to the square-nerve $N^2(K) = N(N(K))$. In particular, $K \searrow N^2(K)$. The strong collapses are easier to handle than the usual collapses. The concrete definition is the following.

Definition 2.7. Let K be a complex and let $v \in K$ be a vertex. We denote by $K \smallsetminus v$ the full subcomplex of K spanned by the vertices different from v (the *deletion* of the vertex v). We say that there is an *elementary strong collapse* from K to $K \smallsetminus v$ if the link of the vertex $lk(v, K)$ is a simplicial cone (i.e. there is some vertex v' which is contained in every maximal simplex of $lk(v, K)$). In this case we say that v is *dominated* (by v') and we denote $K \overset{e}{\searrow\!\!\searrow} K \smallsetminus v$. There is a *strong collapse* from a complex K to a subcomplex L if there exists a sequence of elementary strong collapses that starts in K and ends in L. In this case we write $K \searrow\!\!\searrow L$.

It is easy to see that $K \searrow\!\!\searrow L$ implies $K \searrow L$. We refer the reader to [2] for a comprehensive exposition on strong collapsibility and its relationship with simplicial collapsibility. The following lemma shows that this kind of collapses behaves well with respect to the nerve construction.

Lemma 2.8. *If $K \searrow\!\!\searrow L$, then $N(K) \searrow\!\!\searrow N(L)$.*

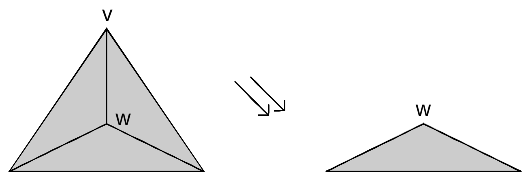

FIGURE 1. An elementary strong collapse.

Proof. We may suppose that $K \searrow\!\!\searrow^e L$, i.e. $L = K \setminus \{v\}$ with the vertex v dominated by w. Consider the simplicial map $f : N(L) \to N(K)$ defined in the vertices of $N(L)$ by

$$f(\sigma) = \begin{cases} \sigma & \text{if} \quad \sigma \in N(K) \\ \\ v\sigma & \text{if} \quad \sigma \notin N(K) \end{cases}$$

It is easy to see that $\bigcap \sigma_i \neq \phi$ if and only if $\bigcap f(\sigma_i) \neq \phi$. Therefore we only need to prove that $N(K) \searrow\!\!\searrow f(N(L))$. By [2, Lemma 3.3], it suffices to check that every vertex $\gamma \in N(K) \setminus f(N(L)$ is dominated by a vertex of $f(N(L))$.

Let γ be a vertex in $N(K) \setminus f(N(L))$. Since $\gamma \notin f(N(L))$, then $\gamma = v\gamma'$ with γ' not maximal in L. Therefore there exists $\tau \in L$ a maximal simplex with $\gamma' \subsetneq \tau$. We will show that γ is dominated by τ in $N(K)$.

Let $\{\sigma_0, ..., \sigma_l\} \in lk(\gamma, N(K))$ (i.e. $\cap\sigma_i \bigcap \gamma \neq \phi$). We need to prove that $\cap\sigma_i \bigcap \tau \neq \phi$. If $v \in \cap\sigma_i \bigcap \gamma$, then $v \in \sigma_i$. Since w dominates v and σ_i is maximal in K, we conclude that $w \in \sigma_i$ and therefore $w \in \cap\sigma_i \bigcap \tau$. If $v \notin \cap\sigma_i \bigcap \gamma$, then $\cap\sigma_i \bigcap \gamma \subseteq \gamma'$. Since $\gamma' \subsetneq \tau$, it follows that $\cap\sigma_i \bigcap \tau \neq \phi$

□

Note that in general the previous lemma is not true for simplicial collapses.

Corollary 2.9. *Let K be a simplicial complex such that $N(K) \searrow\!\!\searrow \dot{\sigma}$, where $\dot{\sigma}$ is the boundary of a simplex. Then $K \searrow \dot{\sigma}$, and therefore K^* is homotopy equivalent to a sphere.*

Proof. By Lemma 2.8, $N(N(K)) \searrow\!\!\searrow N(\dot{\sigma}) = \dot{\sigma}$ and by [2, Proposition 3.4], $K \searrow\!\!\searrow N^2(K)$. It follows that $K \searrow\!\!\searrow \dot{\sigma}$ and, in particular, $K \searrow \dot{\sigma}$. □

Naturality. Let $K \subseteq L$ be a subcomplex and let V be a set containing L^0. Note that, by construction, $L^* \subseteq K^*$. In [3] it is proved that the isomorphisms $H_i(K) \cong H^{n-i-3}(K^*)$ can be taken to be natural with respect to inclusions. Concretely, for any pair $K \subseteq L$ there is a commutative diagram

$$\begin{tikzcd} H_i(K) \arrow[r, "\cong"] \arrow[d, "i_*"] & H^{n-i-3}(K^*) \arrow[d, "j^*"] \\ H_i(L) \arrow[r, "\cong"] & H^{n-i-3}(L^*) \end{tikzcd}$$

where the horizontal maps are isomorphisms given by the duality and the vertical maps are the ones induced by the inclusions.

The following example shows that even if the inclusion $i : K \to L$ is a homotopy equivalence, the induced inclusion $j : L^* \to K^*$ might not be one.

Example 2.10. Let T be an acyclic and non contractible compact 2-complex (see [9, Example 2.38]) and let S be a single vertex of $V = T^0$. Take $K = T^* \subset L = S^*$. Note that the inclusion $i : K \to L$ is a homotopy equivalence since both are contractible (acyclic and simply connected) but the inclusion $j : S = L^* \to T = K^*$ is not.

3. The duality in terms of reduced lattices

Definition 3.1. A finite poset X is called a *reduced lattice* if every lower bounded set of X has an infimum.

Equivalently, a poset is a reduced lattice if and only if it is obtained from a finite lattice by deleting the maximum and the minimum. Note that if X is a reduced lattice, every upper bounded set has a supremum. For example, the face poset $\mathcal{X}(K)$ of any finite simplicial complex K is a reduced lattice.

Definition 3.2. Given a reduced lattice X, we denote by $m(X)$ the set of its minimal elements and by $T(X)$ the simplicial complex whose vertex set is $m(X)$ and whose simplices are the subsets of $m(X)$ which are bounded above.

Note that $T(X)$ is in fact the *atom crosscut complex* of X (see [10]). A similar construction appears also in [1, Section 9.2] under the name of $\mathcal{L}$-complex. In fact, it is easy to see that $T(X) = \mathcal{L}(X^{op})$, the $\mathcal{L}$-complex of the opposite of X.

Remark 3.3. It is clear that $T(\mathcal{X}(K)) = K$ for any finite simplicial complex K. Moreover, by [1, Section 9.2], for any reduced lattice X, the complex $T(X)$ is homotopy equivalent to the standard order complex $\mathcal{K}(X)$ whose simplices are the non-empty chains of X (see also [10]).

Definition 3.4. Given a reduced lattice X and a set V such that $m(X) \subseteq V$, we define its Alexander dual X^* as the reduced lattice $\mathcal{X}(T(X)^*)$. Here $T(X)^*$ denotes the Alexander dual of the simplicial complex $T(X)$ with respect to the ground set V.

By Remark 3.3, the simplicial Alexander duality immediately extends to this context as follows.

Proposition 3.5. *Given a reduced lattice X and a set V such that $m(X) \subseteq V$, then for any i*

$$H_i(X) \cong H^{n-i-3}(X^*),$$

where $n = \#V$.

The (co)homology of a poset X is the (co)homology of its associated order complex $\mathcal{K}(X)$. It is known that a finite poset is essentially a finite topological space (see [1, 2]) and therefore this result can be used to investigate the topology of finite spaces. However we don't adopt in this paper the finite space point of view. The topology of the posets is formulated here in terms of the topology of their order complexes.

Remark 3.6. Since $K = T(\mathcal{X}(K))$, this version of the duality extends the simplicial version. Note also that in general $X^{**} \neq X$, unless $X = \mathcal{X}(K)$ for some simplicial complex K. In fact, $X^{**} = \mathcal{X}(T(X))$.

Example 3.7. Figure 2 shows a reduced lattice X, which is not the face poset of a complex, and its dual X^*.

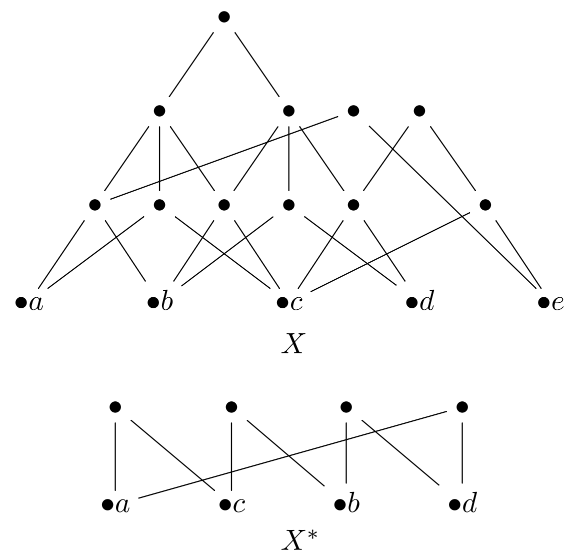

FIGURE 2. A reduced lattice and its dual.

Deleted joins of reduced lattices and Bier spheres. Recall that $m(X)$ denotes the set of minimal elements (atoms) of a reduced lattice X. Given a set V and two reduced lattices X, Y such that $m(X), m(Y) \subseteq V$, we define the *deleted join* $X *_D Y$ as follows. Denote by X_+ (resp. Y_+) the poset obtained from X (resp. from Y) by adding a minimum 0_X (resp. 0_Y). Given any $x \in X$ we denote by $m(x)$ the set of minimal elements of X which are less than or equal to x, by convention we set $m(0_X) = m(0_Y) = \emptyset$. We define

$$X *_D Y = \{(x, y) \in X_+ \times Y_+ : \ m(x) \cap m(y) = \emptyset\} - \{(0_X, 0_Y)\}$$

with partial order given by $(x, y) \le (x', y')$ in $X *_D Y$ if $x \le x'$ in X_+ and $y \le y'$ in Y_+.

Note that $X *_D Y$ is obtained from the disjoint union $X \coprod Y$ by adding an element $x * y$ for any pair $(x, y) \in X \times Y$ such that $m(x) \cap m(y) = \emptyset$.

By construction, if K and L are subcomplexes of a simplex with vertex set V, then

$$\mathcal{X}(K *_\Delta L) = \mathcal{X}(K) *_D \mathcal{X}(L)$$

(with respect to the same vertex set V). Here $K *_\Delta L$ denotes the deleted join of the complexes (see [11]).

Proposition 3.8. *Let X and Y be reduced lattices with $m(X), m(Y) \subseteq V$. Then $X *_D Y$ is a reduced lattice.*

Proof. Let $(x, y), (x', y') \in X *_D Y$ such that the set $\{(x, y), (x', y')\}$ is lower bounded. We have to see that it has infimum $(x, y) \wedge (x', y')$. Note that if $x = 0_X$ or $x' = 0_X$ then $y \ne 0_Y$ and $y' \ne 0_Y$. This implies that $(x \wedge x', y \wedge y') \ne (0_X, 0_Y)$. Moreover $m(x \wedge x') \cap m(y \wedge y') = \emptyset$ since (x, y) and (x', y') are in $X *_D Y$. Then $(x, y) \wedge (x', y') = (x \wedge x', y \wedge y')$. $\square$

Since $X *_D Y$ is a reduced lattice, we can consider its atom crosscut complex $T(X *_D Y)$.

Let $\sigma = \tau \cup \nu$, with $\tau \subseteq m(X)$ and $\nu \subseteq m(Y)$. Note that

$$\begin{aligned}\sigma \in T(X *_D Y) &\iff \sigma \text{ is upper bounded in } X *_D Y \\ &\iff \tau \cap \nu = \emptyset \text{ and } \tau \text{ is upper bounded in } X \text{ and } \nu \text{ is upper bounded in } Y \\ &\iff \sigma \in T(X) *_\Delta T(Y)\end{aligned}$$

This proves the following

Proposition 3.9. $T(X *_D Y) = T(X) *_\Delta T(Y)$.

Corollary 3.10. *For any reduced lattice X, $T(X *_D X^*)$ is a (Bier) sphere. The order complex $\mathcal{K}(X *_D X^*)$ is homotopy equivalent to a sphere.*

Proof. Both statements follow from the previous proposition, Remark 3.3 and the construction of Bier spheres (see [4, 7]). □

Thus any reduced lattice X produces a sphere by taking the atom crosscut complex $T(X *_D X^*)$ of the deleted join with its Alexander dual, and it produces also a complex homotopy equivalent to a sphere by taking the order complex $\mathcal{K}(X *_D X^*)$. Note that $\mathcal{K}(X *_D X^*)$ need not be a sphere, as the following example shows.

Example 3.11. Figure 3 shows a reduced lattice X with vertex set $V = \{1, 2, 3\}$ whose Alexander dual X^* consists of a single point, and the deleted join $X *_D X^*$. Note that the order complex of the deleted join is homotopy equivalent, but not homeomorphic, to S^1.

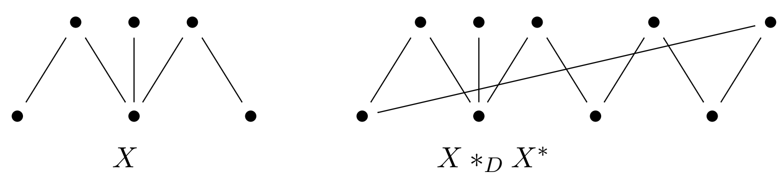

FIGURE 3. A deleted join of a reduced lattice with its dual whose order complex is not a sphere.

An alternative and simpler notion of dual. Given a reduced lattice X, one can define an alternative dual $d(X)$. The advantage of this alternative construction is that it is in general much smaller than the Alexander dual X^* (even for the face posets of simplicial complexes). It is also convenient since it can be defined and handled completely in the context of reduced lattices without need of computing the atom crosscut complex $T(X)$.

Definition 3.12. Given a reduced lattice X and a set V such that $m(X) \subseteq V$, we define its alternative dual $d(X)$ as the poset whose elements are the maximal subsets $A \subset V$ such that their complements $V \backslash A$ are not upper bounded subsets of $m(X)$, and all their non-empty intersections. The order is given by inclusion.

Note that $d(X)$ is also a reduced lattice and it is not necessarily the face poset of a simplicial complex, even if $X = \mathcal{X}(K)$ for some K.

Proposition 3.13. *Let X be a reduced lattice. Then $T(d(X))$ is homotopy equivalent to $T(X^*)$. In particular the alternative dual $d(X)$ satisfies the Alexander duality*

$$H_i(X) \cong H^{n-i-3}(d(X)),$$

where $n = \#V$.

Proof. Consider the opposite poset $d(X)^{op}$ of the alternative dual of X. It is not hard to see that its atom crosscut complex $T(d(X)^{op})$ coincides with the nerve $N(T(X^*))$ of the atom crosscut complex of the dual X^*. Since the nerve of any complex K is homotopy equivalent to K, then $T(d(X)^{op})$ and $T(X^*)$ are homotopy equivalent. It follows that $T(d(X))$ and $T(X^*)$ have the same homotopy type. □

In the same way that the nerve of a simplicial complex is in general smaller than the original complex, the alternative dual $d(X)$ is a much smaller homotopy model of X^*. In fact one can prove that $d(X)$ is a strong deformation retract of X^* when they are viewed as finite topological spaces.

Example 3.14. The Alexander dual of a single point $K = \{w\}$ with respect to a set V of 4 points is the leftmost simplicial complex of Figure 1 and its face posets has 13 points. The alternative dual is the opposite poset of the face poset of the nerve of K^* and has 7 points.

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