STABLE SOLUTIONS OF EQUATIONS WITH A QUADRATIC GRADIENT TERM

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ABSTRACT. We consider positive solutions to a non-variational family of equations of the form

$$-\Delta u - b(x)|\nabla u|^2 = \lambda g(u) \text{ in } \Omega,$$

where $\lambda \geq 0$, $b(x)$ is a given function, g is an increasing nonlinearity with $g(0) > 0$ and $\Omega \in \mathbb{R}^n$ is a bounded smooth domain. We introduce the definition of stability for nonvariational problems and establish existence and regularity results for stable solutions. These results generalize the clasical results obtained when $b(x) = b$ is a constant function making the problem variational after a suitable transformation.

1. INTRODUCTION

In this paper we are interested in the existence and qualitative properties of positive solutions to equations of the form

$$-\Delta u - b(x)|\nabla u|^2 = \lambda g(u)$$

in a bounded smooth domain Ω of $\mathbb{R}^n$, for $\lambda \geq 0$, $b = b(x)$ a given function and g an increasing nonlinearity with $g(0) > 0$. This type of equations arise in different contexts from physics to stochastic processes. Equations with the quadratic gradient term $-\Delta - b(x)|\nabla u|^2$ appear in relation to different contexts within the literature. If $b = b(x)$ is constant and positive, the equation can be thought as the stationary part of the parabolic equation $u_t - \epsilon\Delta u = |\nabla u|^2$ which in turn may be seen as the viscosity approximation, as ϵ tends to 0^+, of Hamilton-Jacobi equations from stochastic control theory [40]. In [37] the same equation (known in this context as Kardar-Parisi-Zhang equation) arises related to the physical theory of growth and roughening on surfaces. Also classical are the existence results for equations involving a quadratic gradient term and such that $b = b(u)$ (see for instance [38, 39]). For more on such equations with $b = b(u)$ see for example [1].

If the coefficient function b is constant, the above equation can be transformed, using the Hopf-Cole transformation, into the equation $-\Delta v = \lambda f(v)$, where f satisfies the same hypothesis as g. This simpler equation for v appears in many different contexts and has been extensively studied. This family of equations includes, for example, the Gelfand problem, where $f(v) = e^v$ with zero Dirichlet boundary conditions on the boundary of $\Omega = B_1$, the unit ball. Some first results concerning this problem involved the construction of explicit radial solutions in dimensions 2 and 3, and in the special case where $\lambda = 2$ and $n = 3$ it was established that there are infinitely many solutions.

The natural question that arises regarding the equation for v is the study of the solutions (λ, v), their existence and properties. The classical existence result says there exists a finite

Key words and phrases. Non-variational problems, Stable solutions.
2010 *Mathematics Subject Classification* 35B35, 35J61.
Supported by CONICET PIP625 Res. 960/12, ANPCyT PICT-2012-0153 and UBACYT X117.

extremal parameter λ^* such that for $\lambda > \lambda^*$ there exist no bounded solutions v, whereas for $0 < \lambda < \lambda^*$ there exists a minimal (i.e., smallest) bounded solution v_λ. Moreover, the branch $\{v_\lambda\}$ is increasing in λ and each solution v_λ is stable. A more delicate problem is the study of the increasing limit $v^* = \lim_{\lambda \uparrow \lambda^*} v_\lambda$, which turns out to be a weak solution of the problem with parameter λ^*. However v^* may be either bounded or singular, depending on the domain Ω and the nonlinearity f.

In the case where Ω is the unit ball of $\mathbb{R}^n$ and $f(v) = e^v$, Joseph and Lundgren [36] completely described the existence and regularity of solutions in terms of λ. Their result also applies to the other classical model, that is, when $f(v) = (1+v)^p$ and $p > 1$. For general domains Crandall and Rabinowitz [23] and Mignot and Puel [44] gave sufficient conditions for the extremal solution v^* to be classical, when the nonlinearity f is either exponential or power like. Regarding more general f, namely f convex, nonnegative and asymptotically linear, the existence of an extremal parameter was known. However, the case where $\lambda = \lambda^*$ was first studied by Mironescu and Radulescu in [57]. They establish two different scenarios, depending whether f obeyed the monotone case or the non-monotone case (see [57] for appropiate definitions). These cases implied non existence of extremal solution and uniqueness of extremal solution respectively. Brezis and Vázquez [16] raised the question of studying when is the extremal solution bounded for general f convex, depending on the dimension n and the domain Ω. For $n \leq 3$ Nedev [45] proved that v^* is bounded for any domain Ω. More recently in [17] Cabré proves that the extremal solution is bounded if the domain Ω is convex and $n \leq 4$. Then, Villegas [50] established a similar result for convex f rather than convex Ω. For higher dimensions the only known result so far for general f concerns radial solutions. Namely, Cabré and Capella [19] prove that if Ω is the unit ball and $n \leq 9$ then v^* is bounded for every f. Boundedness for general domains and $n \leq 9$ is still open. Note that, for $n \geq 10$ there exist unbounded solutions.

Assuming some extra conditions on f, on a recent paper by Cabré and Sanchón [24], L^∞ bounds were obtained for dimensions $n = 5$ and $n = 6$. They also obtain very interesting L^p estimates for $f'(v)$.

In the case where $\Omega = \mathbb{R}^N$ and f is a general convex non-decreasing functions, Dupaigne and Farina [55] established Liouville type results for stable solutions.

Other lines of research have included equations that can be written as $-\Delta u = a(x)g(u) + b(x)$ where g is a continuous nondecreasing non-negative function satisfying some growth conditions at infinity. The power functions $g(u) = u^p$ for $p > 1$ appear as the natural example. Under some conditions on a and b Beriz and Cabré [52] establish non-existence of weak solutions. If instead we assume g is non-increasing and unbounded near the origin and furthermore we consider $b = b(u) = \lambda f(u)$ where f is positive, non-decreasing and such that $f(s)/s$ in non-increasing, converging to some m, then Cirstea, Ghergu and Radulescu established in [53] that if $m = 0$ there is uniqueness of solution (for some range of λ depending on a) whereas if $m > 0$ then there exists an extremal parameter λ^* which is related to m and the first Dirichlet eigenvalue of the Laplacian in Ω. For a more comprehensive reading we recommend the book Singular Elliptic Problems [56] by Ghergu and Radulescu. In another direction we point out the results of Dávila and Dupaigne [54] regarding the clasical Gelfand problem but stated in domain obtained by a perturbation of a ball. They establish existence of singular solutions for dimensions $N \geq 4$ which correspond to the extremal solution in case $N \geq 11$.

In this paper we derive results similar to the classical ones but for the case where $b(x)$ is non-constant, and therefore the problem is not variational. Although there is no energy functional associated to our problem, and hence there is no quadratic form, we are still interested in "stable" solutions. To define stability of a solution to a non-variational problem we will use a different condition than the one used in the variational setting (see section 2).

The paper is organized as follows. In section 2 we introduce the definition of stability for non-variational problems. In section 3, for some special nonlinearities g, we derive the stability of the classical solutions. In addition, for the class of stable solutions, we prove new regularity results involving conditions on the function $b(x)$ and the dimension n.

In the following section we establish an existence theorem in terms of λ. The result is similar to the one in the classical context with $b \equiv 0$. Namely we prove the existence of an extremal parameter λ^* such that for $\lambda > \lambda^*$ there is no solution, whereas for $0 < \lambda < \lambda^*$ there is a minimal classical solution u_λ. Moreover, for $g(u) = e^u$ and some dimensions n we are able to prove that the minimal classical solutions u_λ are stable and that the extremal function $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$ is a weak solution for $\lambda = \lambda^*$. As before, in the case where $b(x) = b$ is constant, the existence result coincides with the classical one.

Finally in the last section we establish some sufficient conditions for stable solutions u to be in $H^1(\Omega)$. Once again we consider two cases separately: b positive and b negative. On the one hand, we have the case where $b(x) = b > 0$ is constant and positive. In this setting we are able to prove an $H^1(\Omega)$ result following the similar technique of the classical case (see Brezis-Vázquez [16]) which requires an extra condition on g. We note here that via the Hopf-Cole transformation, one could use the classical result to obtain a condition for e^u to be in $H^1(\Omega)$. This would imply, of course, that u is also in $H^1(\Omega)$ but this gives a much stronger condition on g than the more optimal one that we prove. On the other hand, using different techniques, namely truncations as introduced by Boccardo [9], we prove the $H^1(\Omega)$ result for every solution (not necessarily stable) with $b(x)$ strictly negative, and any L^1 nonlinearity g.

Acknowledgement: I would like to express my gratitude to Prof. Xavier Cabré for proposing the problem studied in this paper and for the endless discussions we held regarding the results obtained.

2. Preliminaries

We are interested in nonnegative solutions of the problem

$$\begin{cases} -\Delta u - b(x)|\nabla u|^2 = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $g : [0, +\infty) \to \mathbb{R}$ is a given nonlinearity, $\lambda \geq 0$ is a parameter and $b = b(x)$ is a bounded function. The properties of the solutions u of (2.1) will depend on the coefficient function b and hence we will distinguish different cases.

As was discussed in hte introduction, in the classical variational setting, the appropiate class of solutions to work with is that of stable solutions, in the sense that the second variation of the energy functional asociated to the equation is positive. In our current setting, equation (2.1) is not, in general, of variational nature and therefore we need to define what we mean by stable solutions. In order to define stability for a wider family of problems we use the linearized equation.

Definition 2.1. *Let u be a classical solution of problem* (2.1). *We say that u is stable if there exists a function $\phi \in W^{2,p}(\Omega)$ for some $p > n$ such that $\phi > 0$ in $\overline{\Omega}$ and*

$$-\Delta\phi - 2b(x)\nabla u\nabla\phi \geq \lambda g'(u)\phi \quad \text{in } \Omega. \tag{2.2}$$

In the variational setting (for example (2.1) with $b \equiv 0$), the existence of such a supersolution ϕ, positive in $\overline{\Omega}$, is equivalent to saying that u is stable in the variational sense, i.e., the quadratic form defined defined obtained from the second variation of the energy functional is positive for every test function $\xi \not\equiv 0$. Equivalently, the first eigenvalue of the linearized problem is positive (see [15]).

However, for a general function b, problem (2.1) is not self-adjoint and therefore we bypass this difficulty by considering the existence of ϕ instead of working with the quadratic form, which makes no sense (or does not exist) for non self-adjoint problems.

3. Case $b(x) \equiv b$ is constant

In this section we consider the case where the coefficient function b is constant, that is, we study nonnegative solutions to

$$\left\{ \begin{array}{rcll} -\Delta u - b|\nabla u|^2 &=& \lambda g(u) & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.3}$$

This problem can be easily transformed into a classical semilinear elliptic equation for a new function v and a new nonlinearity $f = f(v)$ that depends on g. Since the transformation, called Hopf-Cole transformation, depends on the sign of the constant b, we will treat both cases separately in the next two subsections. The nonlinearities f that arise in these two cases are quite different. Nevertheless, if $g(u) = e^{\beta u}$ for some constant β, the classical regularity results for v and our regularity results for u (that we later generalize to $b = b(x)$) agree regardless of the sign of b.

3.1. Case $b = ctt > 0$.

Let b be constant and positive, that is, $b(x) \equiv b > 0$. In this special case we can perform the Hopf-Cole transformation $v = e^{bu} - 1$. The new nonnegative function v satisfies

$$\left\{ \begin{array}{rcll} -\Delta v &=& \lambda b(v+1)g\left(\frac{1}{b}\log(v+1)\right) & \text{in } \Omega \\ v &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.4}$$

We will denote by f the nonlinearity appearing on the right-hand side of the equation above, that is, v satisfies

$$-\Delta v = \lambda f(v) \quad \text{in } \Omega, \text{ where } f(v) = b(v+1)g\left(\frac{1}{b}\log(v+1)\right). \tag{3.5}$$

A first example is the one we obtain letting $g(u) = e^{e^{bu}-1-bu}$, i.e., considering the equation

$$-\Delta u - b|\nabla u|^2 = \lambda e^{e^{bu}-1-bu}.$$

For this choice of g the equation for v becomes $-\Delta v = \lambda b e^v$, the classical exponential nonlinearity. We know (see [23]) that every stable weak solution satisfies $v \in L^\infty(\Omega)$ if $n \leq 9$.

Another example is the one we obtain letting $g(u) = e^{\beta u}$ for some constant $\beta > 0$, i.e.,

$$\left\{ \begin{array}{rcll} -\Delta u - b|\nabla u|^2 &=& \lambda e^{\beta u} & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{3.6}$$

Then, the equation for v becomes $-\Delta v = \lambda b(v+1)^p$, where $p = 1 + \frac{\beta}{b} > 1$. This is the classical power nonlinearity case for v. For this equation it is known (see [16]) that, if v is a $H_0^1(\Omega)$ semi-stable solution then

$$v \in L^\infty(\Omega) \quad \text{if} \quad \begin{cases} n \leq 10 & \text{or} \\ n > 10 & \text{and} \quad p < \dfrac{n - 2\sqrt{n-1}}{n - 4 - 2\sqrt{n-1}}, \end{cases}$$

that is, if $n \leq 10$ or $10 < n < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$. In our case, for $p = 1 + \beta/b$ we have that

$$v \in L^\infty(\Omega) \quad \text{if} \quad n \leq 10 \ \text{ or } \ 10 < n < 6 + 4\frac{b}{\beta} + 4\sqrt{1 + \frac{b}{\beta}}. \tag{3.7}$$

Note that our stability assumption on u, that is, the existence of a function ϕ, positive in $\overline{\Omega}$, satisfying (2.2) is equivalent to the existence of a function $\psi = e^{bu}\phi$, positive in $\overline{\Omega}$, satisfying

$$-\Delta\psi \geq \lambda f'(v)\psi,$$

where $v = e^{bu} - 1$ and f is given by (3.5), which is in turn equivalent to the stability of v.

Now, since $v = e^{bu} - 1$ we may conclude that, for every stable classical solution u of (3.6),

$$u \in L^\infty(\Omega) \quad \text{if} \quad n \leq 10 \ \text{ or } \ 10 < n < 6 + 4\frac{b}{\beta} + 4\sqrt{1 + \frac{b}{\beta}},$$

and that this is a uniform L^∞ estimate for all stable solutions (as the one for v in (3.7)). In particular this establishes a uniform bound for all minimal solutions u_λ and therefore yields a sufficient condition for the extremal weak solution u^* to be in $L^\infty(\Omega)$. That is, we have the following

Proposition 3.1. *Let $b > 0$ and u a positive classical stable solution to*

$$\begin{cases} -\Delta u - b|\nabla u|^2 = \lambda e^{\beta u} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter. Then

$$||u||_{L^\infty(\Omega)} \leq C \quad \textit{if} \quad n \leq 10 \ \textit{ or } \ 10 < n < 6 + 4\frac{b}{\beta} + 4\sqrt{1 + \frac{b}{\beta}},$$

where C is a constant depending only on n, b, β and Ω (in particular is independent of λ).

Let us now prove directly this result, in the case where $\beta > b/8$, by using the equation for u and the fact that we are assuming u stable. As we will see, for such β, we reach the same optimal result. The motivation for the following calculations is that in the case where $b(x)$ is non-constant, we are forced to work with the equation for u, since there is, in principle, no transformation to a classical semilinear problem without terms involving the square of the gradient.

We begin by establishing a technical lemma.

Lemma 3.1. *Let $b > 0$ and u be a positive classical solution to*

$$\begin{cases} -\Delta u - b|\nabla u|^2 = \lambda e^{\beta u} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and $\beta > 0$. For $\gamma \in \mathbb{N}$ satisfying $\gamma \geq 2$, we have

$$\int_\Omega |\nabla u|^2 e^{\gamma bu}(e^{bu}-1)^{2\alpha-2}dx \leq \frac{\lambda}{b(2\alpha+\gamma-3)}\int_\Omega e^{(\beta+(2\alpha+\gamma-2)b)u}dx + \lambda L_\gamma, \tag{3.8}$$

where $\alpha > \frac{3-\gamma}{2}$ is a parameter and L_γ is a linear combination of the $\gamma-2$ integrals $\int_\Omega e^{(\beta+(2\alpha+k)b)u}dx$, $k = 0, 1, \dots, \gamma-3$, with coefficients depending only on b, α and γ.

Proof. Let $2\alpha > 1$ and $\gamma \geq 2$ an integer. We have,

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{\gamma bu}(e^{bu}-1)^{2\alpha-2} &= \int_\Omega \nabla u e^{(\gamma-1)bu}\nabla u e^{bu}(e^{bu}-1)^{2\alpha-2}\\ &= \int_\Omega \nabla u e^{(\gamma-1)bu}\frac{\nabla(e^{bu}-1)^{2\alpha-1}}{b(2\alpha-1)}\\ &= \int_\Omega \frac{e^{(\gamma-1)bu}(e^{bu}-1)^{2\alpha-1}}{b(2\alpha-1)}\left(-\Delta u-(\gamma-1)b|\nabla u|^2\right).\end{aligned}$$

Using the equation for u we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{\gamma bu}(e^{bu}-1)^{2\alpha-2} &= \int_\Omega \frac{e^{(\gamma-1)bu}(e^{bu}-1)^{2\alpha-1}}{b(2\alpha-1)}\left(\lambda e^{\beta u}-(\gamma-2)b|\nabla u|^2\right)\\ &\leq \frac{\lambda}{b(2\alpha-1)}\int_\Omega e^{(\beta+(2\alpha+\gamma-2)b)u}-\\ -\frac{\gamma-2}{2\alpha-1}\int_\Omega |\nabla u|^2 e^{\gamma bu}(e^{bu}-1)^{2\alpha-2} &+ \frac{\gamma-2}{2\alpha-1}\int_\Omega |\nabla u|^2 e^{(\gamma-1)bu}(e^{bu}-1)^{2\alpha-2}.\end{aligned}$$

This yields, adding the left hand side to the second term on the right hand side, and since $2\alpha+\gamma-3 > 0$,

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{\gamma bu}(e^{bu}-1)^{2\alpha-2} &\leq \frac{\lambda}{b(2\alpha+\gamma-3)}\int_\Omega e^{(\beta+(2\alpha+\gamma-2)b)u} +\\ &+\frac{\gamma-2}{2\alpha+\gamma-3}\int_\Omega |\nabla u|^2 e^{(\gamma-1)bu}(e^{bu}-1)^{2\alpha-2}.\end{aligned} \tag{3.9}$$

If $\gamma = 2$ the second term on the right hand side of (3.9) is zero and we conclude (3.8) (as desired) with $L_\gamma = 0$. Otherwise, for $\gamma \in \mathbb{N}$, $\gamma \geq 3$, we may repeat the computations above with γ replaced by $\gamma - 1$ to obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{(\gamma-1)bu}(e^{bu}-1)^{2\alpha-2} &\leq \frac{\lambda}{b(2\alpha+\gamma-4)}\int_\Omega e^{(\beta+(2\alpha+\gamma-3)b)u} +\\ &+\frac{\gamma-3}{2\alpha+\gamma-4}\int_\Omega |\nabla u|^2 e^{(\gamma-2)bu}(e^{bu}-1)^{2\alpha-2}.\end{aligned}$$

Note that the left hand side is the second integral on the right hand side of (3.9), which remains to be controlled. Note also that the exponent of the exponential function in the first integral on the right hand side decreases on each iteration. We may continue this process until we are left with the integral of $|\nabla u|^2 e^{2bu}(e^{bu}-1)^{2\alpha-2}$. For this integral,

$$\int_\Omega |\nabla u|^2 e^{2bu}(e^{bu}-1)^{2\alpha-2} \leq \frac{\lambda}{b(2\alpha-1)}\int_\Omega e^{(\beta+2\alpha b)u}.$$

Hence,

$$\int_\Omega |\nabla u|^2 e^{\gamma bu}(e^{bu}-1)^{2\alpha-2}dx \leq \frac{\lambda}{b(2\alpha+\gamma-3)}\int_\Omega e^{(\beta+(2\alpha+\gamma-2)b)u}dx + \lambda L_\gamma,$$

where L_γ is a linear combination of the $\gamma - 2$ integrals $\int_\Omega e^{(\beta+(2\alpha+k)b)u}dx$, $k = 0, 1, \ldots, \gamma - 3$, with coefficients depending only on b, α and γ. $\square$

Next we use the assumption that u is stable according to Definition 2.1, that is, there exists a positive function ϕ in $\overline{\Omega}$ such that

$$-\Delta\phi - 2b\nabla u\nabla\phi \geq \lambda\beta e^{\beta u}\phi.$$

We multiply the previous inequality by $(e^{bu} - 1)^{2\alpha}e^{2bu}/\phi$ for $\alpha > 0$ and integrate by parts to obtain

$$\begin{aligned}
\lambda\beta\int_\Omega e^{(\beta+2b)u}(e^{bu}-1)^{2\alpha} \quad &\leq \\
&\leq \int_\Omega \nabla\phi\nabla\left(\frac{e^{2bu}(e^{bu}-1)^{2\alpha}}{\phi}\right) - \int_\Omega 2b\frac{\nabla u\nabla\phi}{\phi}e^{2bu}(e^{bu}-1)^{2\alpha} \\
&= -\int_\Omega \frac{|\nabla\phi|^2}{\phi^2}e^{2bu}(e^{bu}-1)^{2\alpha} + \int_\Omega 2b\frac{\nabla u\nabla\phi}{\phi}e^{2bu}(e^{bu}-1)^{2\alpha} + \\
&+\int_\Omega 2\alpha b\frac{\nabla u\nabla\phi}{\phi}e^{bu}e^{2bu}(e^{bu}-1)^{2\alpha-1} - \int_\Omega 2b\frac{\nabla u\nabla\phi}{\phi}e^{2bu}(e^{bu}-1)^{2\alpha} \\
&= -\int_\Omega \frac{|\nabla\phi|^2}{\phi^2}e^{2bu}(e^{bu}-1)^{2\alpha} + \int_\Omega 2\alpha b\frac{\nabla u\nabla\phi}{\phi}e^{2bu}e^{bu}(e^{bu}-1)^{2\alpha-1} \\
&\leq -\int_\Omega \frac{|\nabla\phi|^2}{\phi^2}e^{2bu}(e^{bu}-1)^{2\alpha} + \alpha^2b^2\int_\Omega |\nabla u|^2e^{4bu}(e^{bu}-1)^{2\alpha-2} + \\
&\qquad + \int_\Omega \frac{|\nabla\phi|^2}{\phi^2}e^{2bu}(e^{bu}-1)^{2\alpha} \\
&= \alpha^2b^2\int_\Omega |\nabla u|^2e^{4bu}(e^{bu}-1)^{2\alpha-2}.
\end{aligned} \tag{3.10}$$

Using Lemma 3.1 with $\gamma = 4$ we get, if $\alpha > 0$,

$$\int_\Omega |\nabla u|^2e^{4bu}(e^{bu}-1)^{2\alpha-2}dx \leq \frac{\lambda}{b(2\alpha+1)}\int_\Omega e^{(\beta+(2\alpha+2)b)u}dx + \lambda L_4,$$

where L_4 is a linear combination of the integrals $\int_\Omega e^{(\beta+2\alpha b)u}dx$ and $\int_\Omega e^{(\beta+(2\alpha+1)b)u}dx$ with coefficients depending only on b and α.

Therefore, replacing $(e^{bu} - 1)^{2\alpha}$ by $e^{2\alpha bu}$ on the left hand side of (3.10) and combining all the remaining terms with L_4 from above and denoting it by L, we obtain

$$\lambda\beta\int_\Omega e^{(\beta+(2\alpha+2)b)u} \leq \lambda\frac{\alpha^2b}{2\alpha+1}\int_\Omega e^{(\beta+(2\alpha+2)b)u} + \lambda L. \tag{3.11}$$

Note that L represents a linear combination of integrals involving the exponential function $e^{\delta u}$ with exponent $\delta < \beta + (2\alpha + 2)b$. Such terms can be absorbed into the left hand side of (3.11). In fact, if $0 < a_1 < a_2$ then, for every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that $e^{a_1u} \leq \epsilon e^{a_2u} + C_\epsilon$ for all $u \in (0, +\infty)$. Hence, for every δ as above and every ϵ there exists $C_{\epsilon,\delta}$ such that

$$\int_\Omega e^{\delta u} \leq \epsilon\int_\Omega e^{(\beta+(2\alpha+2)b)u} + C_{\epsilon,\delta}|\Omega|$$

Thus, if $\alpha > 0$ satisfies

$$\frac{\alpha^2 b}{2\alpha+1} < \beta,$$

then $e^{(\beta+(2\alpha+2)b)u} \in L^1(\Omega)$. Solving for α we get

$$\alpha < \frac{\beta + \sqrt{\beta(\beta+b)}}{b}. \tag{3.12}$$

Therefore

$$e^{\beta u} \in L^q(\Omega) \quad \text{for} \quad 1 + 3\frac{b}{\beta} < q = 2\frac{(\alpha+1)b}{\beta} + 1 < 3 + 2\frac{b}{\beta} + 2\sqrt{1+\frac{b}{\beta}}.$$

Note that the function $v = e^{bu} - 1$ defined at the beginning of this section is thus in $L^{r_1 = q\beta/b}(\Omega)$ and v satisfies $-\Delta v = \lambda b(v+1)^p$ where $p = 1 + \beta/b$. Therefore $(v+1)^p \in L^{r_1/p}$ and hence, using the equation for v, we have that $v \in W^{2,r_1/p}$. Since $W^{2,r} \subset L^s$ if $1/s = 1/r - 2/n$ we get that $v \in L^s$ for $s = (nr_1)/(pn - 2r_1)$. If $s > r_1$, that is, $n < 2q$, then v is bounded by an iterative procedure. Hence, $v \in L^\infty(\Omega)$ if

$$n < 6 + 4\frac{b}{\beta} + 4\sqrt{1+\frac{b}{\beta}},$$

as we already knew from (3.7). This was totally expected since both results are achieved using equivalent assumptions.

Finally, as an example, consider the case $\beta = 1$ and $b \equiv 1$ in the expression above. The equation for u becomes

$$-\Delta u - |\nabla u|^2 = e^u.$$

The stable solutions u of this equation satisfy

$$u \in L^\infty(\Omega) \quad \text{if} \quad n < 10 + 4\sqrt{2}, \quad \text{that is } n \leq 15,$$

with a uniform L^∞ bound as in Proposition 3.1.

For another example let $\beta = 1$ and b tend to 0. The equation becomes

$$-\Delta u = e^u$$

and the result above yields $u \in L^\infty(\Omega)$ if and only if $n < 10$, which coincides with the result of [23].

Remark 3.1. We note here that by perturbing the equation $-\Delta u = e^u$ with a quadratic gradient term we actually obtain more regularity for stable solutions u.

3.2. **Case $b = ctt < 0$.** In this case we use a modified Hopf-Cole transformation $v = 1 - e^{bu}$. If u is bounded, the new function v is positive and bounded by 1, that is, $0 < v < 1$ for u bounded. We note here that $v = 1$ corresponds to $u = \infty$. Moreover, v satisfies

$$\left\{\begin{array}{rcll} -\Delta v &=& \lambda|b|(1-v)g\left(\frac{1}{b}\log(1-v)\right) & \text{in } \Omega \\ v &\geq& 0 & \text{in } \Omega \\ v &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{3.13}$$

We will again denote by f the nonlinearity appearing on the right-hand side of the equation above, that is, v satisfies

$$-\Delta v = \lambda f(v) \text{ in } \Omega, \text{ where } f(v) = |b|(1-v)g\left(\frac{1}{b}\log(1-v)\right).$$

Considering the same typical example as in the previous section, we let $g(u) = e^{\beta u}$ for some constant $\beta > 0$,

$$-\Delta u - b|\nabla u|^2 = e^{\beta u}.$$

The equation for v becomes

$$-\Delta v = \lambda |b|(1-v)^p,$$

where $p = 1 + \frac{\beta}{b}$. If $\beta > -b = |b|$ then $p < 0$. This is the case studied by Mignot and Puel [44] and more recently by Esposito in [29]. They prove that stable solutions v satisfy

$$v < 1 \text{ in } \Omega \quad \text{if} \quad n < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}},$$

with a bound for v away from 1, uniform in v. In our case, for $p = 1 + \beta/b$ and $\beta > -b$ we have that

$$v < 1 \text{ in } \Omega \quad \text{if} \quad n < 6 + 4\frac{b}{\beta} + 4\sqrt{1 + \frac{b}{\beta}}.$$

That is,

Proposition 3.2. *Let $b < 0$ be a constant, $\beta > -b$ and u a positive classical stable solution to*

$$\begin{cases} -\Delta u - b|\nabla u|^2 = \lambda e^{\beta u} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ a parameter. Then

$$||u||_{L^\infty(\Omega)} \leq C \quad \text{if} \quad n < 6 + 4\frac{b}{\beta} + 4\sqrt{1 + \frac{b}{\beta}},$$

where C is a constant depending only on n, b, β and Ω (in particular is independent of λ).

It is a nontrivial fact to note that, since $b < 0$, this result yields less regularity for stable solutions than the one obtained for $b > 0$ (recall Proposition 3.1). Note that the condition on the exponent β is more restrictive than the one we have in the case of $b > 0$. For example, if we consider the case $b = -1$, that is, if u satisfies the equation

$$-\Delta u + |\nabla u|^2 = e^{\beta u},$$

we find the assumption $\beta > 1$ in Proposition 3.2, which means that we can not apply the previous result to the equation $-\Delta u + |\nabla u|^2 = e^u$. Nevertheless, for this particular case, the equation for v would be a linear Poisson equation $-\Delta v = \lambda |b|$, and therefore u would be bounded for all dimensions.

4. General $b(x)$

In this section we study the case of a general bounded function $b = b(x)$. Let u be a positive solution to the equation

$$-\Delta u - b(x)|\nabla u|^2 = \lambda g(u) \tag{4.14}$$

with Dirichlet boundary conditions. We denote by $\underline{b}$ and $\overline{b}$ the infimum and the supremum of $b(x)$ respectively, that is,

$$\underline{b} \le b(x) \le \overline{b} \quad \text{for every } x \in \Omega.$$

Equation (4.14) can no longer be transformed into a classical one, and there are no known regularity results for stable solutions. Following the computations we introduced in the previous sections we will study this equation directly, only with the assumptions on u, that is, u is stable as defined in Definition 2.1.

We consider the special case where $g(u) = e^{\beta u}$. Then, there exists $\phi > 0$ in $\overline{\Omega}$ such that

$$-\Delta\phi - 2b(x)\nabla u \nabla\phi \ge \lambda e^{\beta u}\phi. \tag{4.15}$$

The first result that we prove is the following.

Proposition 4.1. *Let $b = b(x)$ be a bounded function such that $\underline{b} \le b(x) \le \overline{b}$ for some constants $\underline{b}$ and $\overline{b}$ with $\overline{b} > 0$ and u a positive classical stable solution to*

$$\begin{cases} -\Delta u - b(x)|\nabla u|^2 &= \lambda e^{\beta u} \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $\lambda > 0$ is a parameter. Then, for every positive constants δ and η with $\delta^2 + \eta^2 \le 1$, if $(\overline{b} - \underline{b}) < \frac{\delta^2}{\eta^2}(\eta^2 - \frac{\overline{b}}{8})$,

$$||e^u||_{L^q(\Omega)} \le C \quad \text{for} \quad q < \beta + 2\beta\eta^2 + 2\overline{b} + 2\sqrt{\beta\eta^2(\beta\eta^2 + \overline{b}) - 2\overline{b}(\overline{b} - \underline{b})\frac{\eta^2}{\delta^2}},$$

where C depends only on n, b and Ω (in particular is independent of λ).

We note that we have made no assumptions on the sign of the function $b(x)$. In fact, the only condition we have on $b(x)$ is the oscillation condition involving $\underline{b}$ and $\overline{b}$, $(\overline{b} - \underline{b}) < \frac{\delta^2}{\eta^2}(\eta^2 - \frac{\overline{b}}{8})$. This condition guarantees that the expression inside the square root above is nonnegative.

In the case that $b > 0$ is constant we have that $b \equiv \overline{b} = \underline{b} > 0$ and hence $\overline{b} - \underline{b} = 0$ and we may choose $\eta \uparrow 1$ and $\delta \downarrow 0$ to obtain

$$e^{\beta u} \in L^q(\Omega), \quad \text{for} \quad q < 3 + 2\frac{b}{\beta} + 2\sqrt{\frac{b}{\beta} + 1},$$

if $b < 8$, which coincides with the result of Proposition 3.1.

The same regularity result still holds if b has oscillation of order ϵ, since we may choose $\delta^2 = 2\overline{b}\sqrt{\overline{b} - \underline{b}}$ which is again of order ϵ and therefore we can let η tend to 1 and δ tend to 0.

As before, we begin by establishing the following estimate:

Lemma 4.1. *Let $b(x) \le \overline{b}$ with $\overline{b} > 0$ and u be a positive solution to*

$$\begin{cases} -\Delta u - b(x)|\nabla u|^2 &= \lambda e^{\beta u} \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and $\beta > 0$. For $\gamma \in \mathbb{N}$ satisfying $\gamma \geq 2$ have

$$\int_\Omega |\nabla u|^2 e^{\gamma \bar{b} u}(e^{\bar{b}u} - 1)^{2\alpha-2} dx \leq \frac{\lambda}{b(2\alpha+\gamma-3)} \int_\Omega e^{(\beta+(2\alpha+\gamma-2)\bar{b})u} dx + \lambda L_\gamma, \tag{4.16}$$

where $\alpha > \frac{1}{2}$ is a parameter and L_γ is a linear combination of the $\gamma - 2$ integrals $\int_\Omega e^{(\beta+(2\alpha+k)b)u} dx$, $k = 0, 1, \ldots, \gamma - 3$, with coefficients depending only on b, α and γ.

Proof. Let $2\alpha > 1$ and $\gamma \geq 2$ an integer. We have,

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{\gamma \bar{b} u}(e^{\bar{b}u} - 1)^{2\alpha-2} &= \int_\Omega \nabla u e^{(\gamma-1)\bar{b}u} \nabla u e^{\bar{b}u}(e^{\bar{b}u} - 1)^{2\alpha-2} \\ &= \int_\Omega \nabla u e^{(\gamma-1)\bar{b}u} \frac{\nabla (e^{\bar{b}u} - 1)^{2\alpha-1}}{b(2\alpha-1)} \\ &= \int_\Omega \frac{e^{(\gamma-1)\bar{b}u}(e^{\bar{b}u} - 1)^{2\alpha-1}}{\bar{b}(2\alpha-1)} \left(-\Delta u - (\gamma-1)\bar{b}|\nabla u|^2\right).\end{aligned}$$

Using the equation for u and the fact that $b(x) \leq \bar{b}$ with $\bar{b} > 0$ we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{\gamma \bar{b} u}(e^{\bar{b}u} - 1)^{2\alpha-2} &= \int_\Omega \frac{e^{(\gamma-1)\bar{b}u}(e^{\bar{b}u} - 1)^{2\alpha-1}}{\bar{b}(2\alpha-1)} \left(\lambda e^{\beta u} - (\gamma-2)\bar{b}|\nabla u|^2\right) + \\ &\quad + \int_\Omega \frac{e^{(\gamma-1)\bar{b}u}(e^{\bar{b}u} - 1)^{2\alpha-1}}{\bar{b}(2\alpha-1)} (b - \bar{b})|\nabla u|^2 \\ &\leq \frac{\lambda}{\bar{b}(2\alpha-1)} \int_\Omega e^{(\beta+(2\alpha+\gamma-2)\bar{b})u} - \\ - \frac{\gamma-2}{2\alpha-1} \int_\Omega |\nabla u|^2 e^{\gamma \bar{b} u}&(e^{\bar{b}u} - 1)^{2\alpha-2} + \frac{\gamma-2}{2\alpha-1} \int_\Omega |\nabla u|^2 e^{(\gamma-1)\bar{b}u}(e^{\bar{b}u} - 1)^{2\alpha-2}.\end{aligned}$$

This yields, adding the left hand side to the second term on the right hand side, and since $2\alpha + \gamma - 3 > 0$,

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{\gamma \bar{b} u}(e^{\bar{b}u} - 1)^{2\alpha-2} \quad &\leq \quad \frac{\lambda}{\bar{b}(2\alpha+\gamma-3)} \int_\Omega e^{(\beta+(2\alpha+\gamma-2)\bar{b})u} + \\ &\quad + \frac{\gamma-2}{2\alpha+\gamma-3} \int_\Omega |\nabla u|^2 e^{(\gamma-1)\bar{b}u}(e^{\bar{b}u} - 1)^{2\alpha-2}.\end{aligned} \tag{4.17}$$

If $\gamma = 2$ the second term on the right hand side of (4.17) is zero and we conclude (4.16) (as desired) with $L = 0$. Otherwise, for $\gamma \in \mathbb{N}$, $\gamma \geq 3$, we may repeat the computations above with γ replaced by $\gamma - 1$ to obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2 e^{(\gamma-1)\bar{b}u}(e^{\bar{b}u} - 1)^{2\alpha-2} \quad &\leq \quad \frac{\lambda}{\bar{b}(2\alpha+\gamma-4)} \int_\Omega e^{(\beta+(2\alpha+\gamma-3)\bar{b})u} + \\ &\quad + \frac{\gamma-3}{2\alpha+\gamma-4} \int_\Omega |\nabla u|^2 e^{(\gamma-2)\bar{b}u}(e^{\bar{b}u} - 1)^{2\alpha-2}.\end{aligned}$$

Note that the left hand side is the second integral on the right hand side of (4.17), which remains to be controlled. Note also that the exponent of the exponential function in the first integral on the right hand side decreases on each iteration. We may continue this process until we are left with the integral of $|\nabla u|^2 e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha-2}$. For this integral,

$$\int_\Omega |\nabla u|^2 e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha-2} \leq \frac{\lambda}{\bar{b}(2\alpha-1)}\int_\Omega e^{(1+2\alpha\bar{b})u}.$$

Hence,

$$\int_\Omega |\nabla u|^2 e^{\gamma\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha-2}dx \leq \frac{\lambda}{\bar{b}(2\alpha+\gamma-3)}\int_\Omega e^{(\beta+(2\alpha+\gamma-2)\bar{b})u}dx + \lambda L_\gamma,$$

where L_γ is a linear combination of the $\gamma-2$ integrals $\int_\Omega e^{(\beta+(2\alpha+k)b)u}dx$ for $k=0,1,\dots,\gamma-3$, with coefficients depending only on b,α and γ. $\square$

We now prove the proposition. We follow the computations as in the proof of Proposition 3.1.

Proof. The assumption we have made on u is that it is stable according to Definition 2.1, that is, there exists a positive function ϕ in $\overline{\Omega}$ such that

$$-\Delta\phi - 2b(x)\nabla u\nabla\phi \geq \lambda\beta e^{\beta u}\phi.$$

We multiply the previous inequality by $(e^{\bar{b}u}-1)^{2\alpha}e^{2\bar{b}u}/\phi$ for $\alpha>0$ and integrate by parts to obtain

$$\begin{aligned}
\lambda\beta\int_\Omega e^{(\beta+2\bar{b})u}(e^{\bar{b}u}-1)^{2\alpha} \quad &\leq \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.18)\\
&\leq \int_\Omega \nabla\phi\nabla\left(\frac{e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha}}{\phi}\right) - \int_\Omega 2b\frac{\nabla u\nabla\phi}{\phi}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha}\\
&= -\int_\Omega \frac{|\nabla\phi|^2}{\phi^2}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha} + \int_\Omega 2\bar{b}\frac{\nabla u\nabla\phi}{\phi}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha} +\\
&\quad+\int_\Omega 2\alpha\bar{b}\frac{\nabla u\nabla\phi}{\phi}e^{\bar{b}u}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha-1} - \int_\Omega 2b\frac{\nabla u\nabla\phi}{\phi}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha}\\
&= -\int_\Omega \frac{|\nabla\phi|^2}{\phi^2}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha} + \int_\Omega 2\alpha\bar{b}\frac{\nabla u\nabla\phi}{\phi}e^{2\bar{b}u}e^{\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha-1} +\\
&\qquad\qquad + \int_\Omega 2(\bar{b}-b)\frac{\nabla u\nabla\phi}{\phi}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha}\\
&\leq (\delta^2+\eta^2-1)\int_\Omega \frac{|\nabla\phi|^2}{\phi^2}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha} +\\
&\quad+\frac{2\bar{b}(\bar{b}-\underline{b})}{\delta^2}\int_\Omega |\nabla u|^2 e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha} + \frac{\alpha^2\bar{b}^2}{\eta^2}\int_\Omega |\nabla u|^2 e^{4\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha-2}\\
&= (\delta^2+\eta^2-1)\int_\Omega \frac{|\nabla\phi|^2}{\phi^2}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha} +\\
&\qquad + \left(\frac{2\bar{b}(\bar{b}-\underline{b})}{\delta^2}+\frac{\alpha^2\bar{b}^2}{\eta^2}\right)\int_\Omega |\nabla u|^2 e^{4\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha-2},
\end{aligned}$$

where $\delta>0$ and $\eta>0$ are constants. Using Lemma 4.1 with $\gamma=4$ we get, for $\alpha>0$,

$$\int |\nabla u|^2 e^{4\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha-2}dx \leq \frac{\lambda}{\bar{b}(2\alpha+1)}\int e^{(1+(2\alpha+2)\bar{b})u}dx + \lambda L_4,$$

where L_4 is a linear combination of the two integrals $\int_\Omega e^{(\beta+(2\alpha)b)u}dx$ and $\int_\Omega e^{(\beta+(2\alpha+1)b)u}dx$ with coefficients depending only on b and α.

Therefore, replacing $(e^{\bar{b}u}-1)^{2\alpha}$ by $e^{2\alpha\bar{b}u}$ on the left hand side of (3.10) and combining all the remaining terms with L_4 from above and denoting it by L, we obtain

$$\lambda\beta\int e^{(\beta+(2\alpha+2)\bar{b})u} \leq (\delta^2+\eta^2-1)\int\frac{|\nabla\phi|^2}{\phi^2}e^{2\bar{b}u}(e^{\bar{b}u}-1)^{2\alpha}+ + \left(\frac{2\bar{b}(\bar{b}-\underline{b})}{\delta^2}+\frac{\alpha^2\bar{b}^2}{\eta^2}\right)\frac{\lambda}{\bar{b}(2\alpha+1)}\int e^{(\beta+(2\alpha+2)\bar{b})u}+\lambda L. \tag{4.19}$$

Note that L represents a linear combination of integrals involving the exponential function $e^{\delta u}$ with exponent $\delta<\beta+(2\alpha+2)\bar{b}$. Such terms can be absorbed into the left hand side of (4.19). In fact, if $0<a_1<a_2$ then, for every $\epsilon>0$ there exists a constant $C_\epsilon>0$ such that $e^{a_1u}\leq\epsilon e^{a_2u}+C_\epsilon$ for all $u\in(0,+\infty)$. Hence, for every δ as above and every ϵ there exists $C_{\epsilon,\delta}$ such that

$$\int_\Omega e^{\delta u}\leq\epsilon\int_\Omega e^{(\beta+(2\alpha+2)\bar{b})u}+C_{\epsilon,\delta}|\Omega|$$

Thus, if $\delta^2+\eta^2\leq 1$ and $\alpha>1/2$ satisfies

$$\left(\frac{2\bar{b}(\bar{b}-\underline{b})}{\delta^2}+\frac{\alpha^2\bar{b}^2}{\eta^2}\right)\frac{1}{\bar{b}(2\alpha+1)}<\beta,$$

then $e^{(\beta+(2\alpha+2)\bar{b})u}\in L^1(\Omega)$. Solving for α we get

$$\frac{1}{2}<\alpha<\frac{\beta\eta^2+\sqrt{\beta\eta^2(\beta\eta^2+\bar{b})-2\bar{b}(\bar{b}-\underline{b})\frac{\eta^2}{\delta^2}}}{\bar{b}}. \tag{4.20}$$

This inequality is satisfied for some α since $(\bar{b}-\underline{b})<\frac{\delta^2}{\eta^2}(\eta^2-\frac{\bar{b}}{8})$. Therefore

$$||e^u||_{L^q(\Omega)}\leq C \quad \text{for} \quad q<\beta+2\beta\eta^2+2\bar{b}+2\sqrt{\beta\eta^2(\beta\eta^2+\bar{b})-2\bar{b}(\bar{b}-\underline{b})\frac{\eta^2}{\delta^2}}, \tag{4.21}$$

where C is independent of λ. □

Remark 4.1. *We note that we can perform all the computations if we assume only that there exists a function ϕ_ϵ, positive in $\overline{\Omega}$, such that*

$$-\Delta\phi_\epsilon-2b(x)\nabla u\nabla\phi_\epsilon\geq(\lambda-\epsilon)e^u\phi_\epsilon,$$

for some small $\epsilon>0$. Letting ϵ tend to 0 we obtain the result above with the constant C independent of ϵ.

5. Further regularity for $b(x)\geq 0$

In the case where the function b is non-negative, we can reach further regularity and prove a similar result to the one where b is constant, even though we are not able to transform the equation into a classical one. Using a well chosen Hopf-Cole transformation we can construct a subsolution of the classical equation with a power nonlinearity. Using a bootstrap argument and Proposition 4.1 this is enough to conclude about the regularity of u.

Proposition 5.1. *Let $b(x) \geq 0$ and $0 \leq \underline{b} \leq b(x) \leq \overline{b}$ in Ω for some constants $\underline{b}$ and $\overline{b}$, and u a positive classical stable solution of*

$$\begin{cases} -\Delta u - b(x)|\nabla u|^2 &= \lambda e^u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

and $\lambda > 0$ a parameter. For every positive constants δ and η with $\delta^2 + \eta^2 \leq 1$, let $(\overline{b} - \underline{b}) < \frac{\delta^2}{\eta^2}(\eta^2 - \frac{\overline{b}}{8})$ and $n < 2 + 4\eta^2 + 4\overline{b} + 4\sqrt{\eta^2(\eta^2+\overline{b}) - 2\overline{b}(\overline{b}-\underline{b})\frac{\eta^2}{\delta^2}}$. Then, $||u||_{L^\infty(\Omega)} \leq C$, where C depends only on n, b and Ω.

Proof. Consider the Hopf-Cole transformation $v = e^{\overline{b}u} - 1$. We have that

$$\begin{aligned} -\Delta v &= \overline{b}e^{\overline{b}u}\left(-\Delta u - \overline{b}|\nabla u|^2\right) \\ &= \overline{b}e^{\overline{b}u}\left(\lambda e^u + (b(x) - \overline{b})|\nabla u|^2\right) \\ &\leq \lambda \overline{b} e^{\overline{b}u} e^u \\ &= \lambda \overline{b}(v+1)^{\frac{\overline{b}+1}{\overline{b}}} \end{aligned}$$

which means that v is a positive subsolution of the classical equation, i.e.,

$$-\Delta v \leq \lambda \overline{b}(v+1)^p \quad \text{in } \Omega,$$

for $p = (\overline{b}+1)/\overline{b}$. Let w be the solution to the linear problem

$$\begin{cases} -\Delta w &= \lambda \overline{b}(v+1)^p & \text{in } \Omega \\ w &= v & \text{in } \partial\Omega. \end{cases}$$

Then, trivially $-\Delta v \leq -\Delta w$ and hence, by the maximum principle

$$0 \leq v \leq w.$$

If $v \in L^s(\Omega)$ then, using the equation for w we get that $w \in W^{2,s/p}(\Omega) \subset L^r(\Omega)$ for $r = (ns)/(np - 2q)$. Now, $r > s$ if $n < 2s/(p-1)$. Therefore, by a bootstrap argument, w and hence v is in $L^\infty(\Omega)$ if $n < 2s/(p-1)$, that is, $n < 2\overline{b}s$.

From the previous section we know that $e^u \in L^q(\Omega)$ for q given by (4.21). Given the definition of v we get that $v \in L^{q/\overline{b}}(\Omega)$, i.e., we can replace $s = q/\overline{b}$ in the discussion above. Thus we obtain that v and hence u are in $L^\infty(\Omega)$ if $n < 2q$, that is,

$$u \in L^\infty(\Omega) \quad \text{if} \quad n < 2 + 4\eta^2 + 4\overline{b} + 4\sqrt{\eta^2(\eta^2+\overline{b}) - 2\overline{b}(\overline{b}-\underline{b})\frac{\eta^2}{\delta^2}},$$

with $\delta^2 + \eta^2 \leq 1$. $\square$

6. Existence for $b(x) \geq 0$

In this section we prove an existence theorem, in terms of λ, of solutions to the problem

$$\begin{cases} -\Delta u - b(x)|\nabla u|^2 &= \lambda g(u) & \text{in } \Omega \\ u &\geq 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{6.22}$$

where g is a nonlinearity with assumptions to be detailed later, $b(x) \geq 0$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with $n \geq 2$.

If $b(x) = b$ is constant, using the Hopf-Cole transformation we reach an equation for v of the form

$$\begin{cases} -\Delta v &= \lambda f(v) \quad \text{in } \Omega \\ \quad v &= 0 \quad\quad\ \ \text{in } \partial\Omega. \end{cases} \tag{6.23}$$

where $f(v) = b(v+1)g\left(\frac{1}{b}\ln(v+1)\right)$. Equations of the type (6.23) have been extensively studied. Under the following conditions on f:

$$f \text{ is } C^1, \text{ convex, nondecreasing }, \ f(0) > 0 \text{ and } \lim_{v\to+\infty} \frac{f(v)}{v} = +\infty, \tag{6.24}$$

there exists a finite parameter $\lambda^* > 0$ such that, for $\lambda > \lambda^*$ there is no bounded solution to (6.23). On the other hand, for $0 < \lambda < \lambda^*$ there exists a minimal bounded solution v_λ, where minimal means smallest.

These conditions hold for f if we assume that g satisfies:

$$g \text{ is } C^1, \text{convex, nondecreasing }, \ g(0) > 0 \text{ and } \lim_{u\to+\infty} g(u) = +\infty. \tag{6.25}$$

In the general case for a non-negative function b we prove the following theorem.

Theorem 6.1. *Let $b = b(x) \geq 0$ be a $C^\alpha(\overline{\Omega})$ function defined in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and g be a nondecreasing C^1 function with $g(0) > 0$ and $\lim_{u\to+\infty} \frac{g(u)}{u} = +\infty$. Then, there exists a parameter $0 < \lambda^* < \infty$ such that:*
(a) *If $\lambda > \lambda^*$ then there is no classical solution of* (6.22)*.*
(b) *If $0 \leq \lambda < \lambda^*$ then there exists a minimal classical solution u_λ of* (6.22)*. Moreover, $u_\lambda < u_\mu$ if $\lambda < \mu < \lambda^*$.*

*In addition, if $g(u) = e^u$ and for every positive constants δ and η with $\delta^2 + \eta^2 \leq 1$, the function b satisfies $0 \leq \underline{b} \leq b \leq \overline{b}$ in Ω for constants $\underline{b}$ and $\overline{b}$ such that $(\overline{b} - \underline{b}) < \frac{\delta^2}{\eta^2}(\eta^2 - \frac{\overline{b}}{8})$ and $n < 2 + 4\eta^2 + 4\overline{b} + 4\sqrt{\eta^2(\eta^2 + \overline{b}) - 2\overline{b}(\overline{b} - \underline{b})\frac{\eta^2}{\delta^2}}$, then u_λ is semi-stable. Moreover, the limit $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is a weak solution of (*6.22*) for $\lambda = \lambda^*$. That is, it satisfies*

$$-\int_\Omega u^*\Delta\xi - \int_\Omega b(x)|\nabla u^*|^2\xi = \lambda^* \int_\Omega e^{u^*}\xi,$$

for every $\xi \in C^2(\overline{\Omega})$ with $\xi = 0$ on $\partial\Omega$. In addition, the estimates of Proposition 5.1 *apply to u^*.*

Proof. First, we prove that there is no classical solution for large λ. Let u_λ be a bounded solution corresponding to λ. Then, since $b \geq 0$ this function u_λ is a supersolution of the classical problem

$$\begin{cases} -\Delta u &\geq \lambda g(u) \quad \text{in } \Omega \\ \quad u &= 0 \quad\quad\quad \text{on } \partial\Omega. \end{cases}$$

Since $g(0) > 0$, $\underline{u} = 0$ is a strict subsolution for every $\lambda > 0$. This would imply the existence of a classical solution corresponding to λ between 0 and our supersolution u_λ. We know this is only possible for λ smaller than a finite extremal parameter, hence the same applies for the solutions to our problem (6.22).

Next, we prove the existence of a classical solution of (6.22) for small λ. For general λ, the existence of a bounded supersolution implies the existence of a minimal (smallest) classical solution u_λ. This solution is obtained by monotone iteration starting from 0. That is, u_λ is the increasing limit of u_m where the functions u_m are defined as $u_0 \equiv 0$ and, for $m \geq 1$

$$\left\{ \begin{array}{rcll} -\Delta u_m - b(x)|\nabla u_m|^2 &=& \lambda g(u_{m-1}) & \text{in } \Omega \\ u_m &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{6.26}$$

The equation for u_m may be written as

$$-\Delta u_m = F(x, u_m, \nabla u_m)$$

where F satisfies

$$|F(x, u_m, \xi)| \leq K(1 + |\xi|^2),$$

for some constant K, since b is a bounded function and, at step m, the function u_{m-1} is known and bounded. For this equation and under such conditions on F we have existence of solution $u_m \in W^{2,p}(\Omega)$ for every $p > 1$ (see [7]) and this implies, for p large, that $u_m \in C^{1,\alpha}(\overline{\Omega})$. Moreover $C^{1,\alpha}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$.

We will prove by induction that this sequence u_m is increasing. For $m = 1$ we have that

$$-\Delta u_1 - b(x)|\nabla u_1|^2 = \lambda g(0) > 0 = -\Delta u_0 - b(x)|\nabla u_0|^2,$$

which implies, for $b(x) \geq 0$ and since $u_0 \equiv 0$,

$$-\Delta u_1 > -\Delta u_0.$$

By the classical maximum principle we have $u_1 \geq u_0$.

Now assume $u_m \geq u_{m-1}$. Then,

$$\begin{array}{rcl} -\Delta u_{m+1} - b(x)|\nabla u_{m+1}|^2 &=& \lambda g(u_m) \\ &\geq& \lambda g(u_{m-1}) \\ &=& -\Delta u_m - b(x)|\nabla u_m|^2, \end{array}$$

where we have used that g is nondecreasing. Let $w = u_{m+1} - u_m$. From the previous inequality we derive an inequality satisfied by w. Namely,

$$-\Delta w - \vec{B}(x) \cdot \nabla w \geq 0,$$

where $\vec{B}(x) = b(x)\nabla(u_{m+1} + u_m)$. By the maximum principle we have that $w \geq 0$, that is,

$$u_{m+1} \geq u_m.$$

Therefore we have constructed an increasing sequence u_m.

Let now $\overline{u}$ be the solution of

$$\left\{ \begin{array}{rl} -\Delta \overline{u} - b(x)|\nabla \overline{u}|^2 = 1 & \text{in } B_1 \\ \overline{u} = 0 & \text{on } \partial B_1. \end{array} \right. \tag{6.27}$$

This function $\overline{u}$ is a bounded supersolution of (6.22) for small λ, whenever $\lambda g(\max \overline{u}) < 1$.

Using induction and the maximum principle as above we can prove that the sequence is bounded by $\overline{u}$, i.e.,

$$u_0 \leq u_1 \leq \cdots \leq u_m \leq u_{m+1} \leq \cdots \leq \overline{u}.$$

This implies there exists a limit,

$$u_\lambda := \lim_{m \to \infty} u_m,$$

and moreover, u_λ is a solution to (6.22). In fact, since $u_m \in W^{2,p}(\Omega)$ we get that, for p large, $u_m \in C^{1,\alpha}(\overline{\Omega})$. Using the equation and the fact that $b \in C^\alpha(\overline{\Omega})$, we get that $u_m \in C^{2,\alpha}(\overline{\Omega})$ and hence converges to a solution of (6.22).

The extremal parameter λ^* is now defined as the supremum of all $\lambda > 0$ for which (6.22) admits a classical solution. Hence, both $0 < \lambda^* < \infty$ and part (a) of the proposition holds.

(b) Next, if $\lambda < \lambda^*$ there exists μ with $\lambda < \mu < \lambda^*$ and such that (6.22) admits a classical solution u_μ. Since $g > 0$, u_μ is a bounded supersolution of (6.22), and hence the same monotone iteration argument used above shows that (6.22) admits a classical solution u_λ with $u_\lambda \le u$. In addition, we have shown that u_λ is smaller than any classical supersolution of (6.22). It follows that u_λ is minimal (i.e., the smallest solution) and that $u_\lambda < u_\mu$.

Consider now the case where $g(u) = e^u$, and assume that for every positive constants δ and η with $\delta^2 + \eta^2 \le 1$ we have

$$(\bar{b} - \underline{b}) < \frac{\delta^2}{\eta^2}(\eta^2 - \frac{\bar{b}}{8}) \text{ and } n < 2 + 4\eta^2 + 4\bar{b} + 4\sqrt{\eta^2(\eta^2 + \bar{b}) - 2\bar{b}(\bar{b} - \underline{b})\frac{\eta^2}{\delta^2}}.$$

First we prove that u_λ is semi-stable, meaning by semi-stable that the first eigenvalue λ_1 of the linearized operator L_λ is non-negative. That is,

$$\lambda_1(L_\lambda) \ge 0 \quad \text{where } L_\lambda := -\Delta - 2b(x)\nabla u_\lambda \nabla - \lambda e^{u_\lambda}.$$

We have seen that u_λ forms an increasing sequence with respect to λ. For $\delta > 0$ let $v_\delta = u_{\lambda+\delta} - u_\lambda > 0$. Using the equations for $u_{\lambda+\delta}$ and u_λ we have that v_δ satisfies $-\Delta v_\delta - 2b(x)\nabla\left(\frac{u_{\lambda+\delta}+u_\lambda}{2}\right)\nabla v_\delta - \lambda e^{u_\lambda} v_\delta > 0$, where η is between u_λ and $u_{\lambda+\delta}$ and we have used that $\delta > 0$. Therefore, if we define the linear operator

$$L_{\lambda,\delta} := -\Delta - 2b(x)\nabla\left(\frac{u_{\lambda+\delta}+u_\lambda}{2}\right)\nabla - \lambda e^{u_\lambda},$$

we have that, at v_δ,

$$L_{\lambda,\delta} v_\delta = -\Delta v_\delta - 2b(x)\nabla\left(\frac{u_{\lambda+\delta}+u_\lambda}{2}\right)\nabla v_\delta - \lambda e^{u_\lambda} v_\delta > 0 \text{ in } \Omega,$$

and thus v_δ is a strict supersolution positive in Ω of $L_{\lambda,\delta} = 0$ in Ω and hence $\lambda_1(L_{\lambda,\delta}) > 0$.

Now we pass to the limit in δ and obtain that $\lambda_1(L_\lambda) \ge 0$, that is, u_λ is semi-stable as defined above. This can be done using Propositions 2.1 and 5.1 of [15] which establishes that, for bounded coefficients, λ_1 is Lispchitz continuous with respect to both the first and the zeroth order coefficients.

For every $\epsilon > 0$, since $\lambda_1(L_\lambda) \ge 0$ we have that $\lambda_1(L_\lambda - \epsilon) > 0$. This implies there exists a function ϕ_ϵ, positive in $\overline{\Omega}$, as in Remark 4.1. Hence we have that $||u_\lambda||_{L^\infty(\Omega)} \le C$, where C is independent of λ.

Under the same conditions as above, we can establish that the limiting function $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is a weak solution to (6.22) with $\lambda = \lambda^*$. Just use the weak formulation for u_λ and the fact that $u_\lambda \in L^\infty(\Omega)$, so that we can take limits in λ and obtain that u^* is a weak solution. Therefore using the L^∞ uniform bound on u_λ we have $||u^*||_{L^\infty} \le C$. □

7. H^1 REGULARITY

In this section we study the H^1 regularity of positive solutions to the equation

$$-\Delta u - b(x)|\nabla u|^2 = \lambda g(u) \quad \text{in } \Omega, \tag{7.28}$$

such that $u \equiv 0$ on $\partial\Omega$, $\Omega \subset \mathbb{R}^n$ is a bounded domain, $g \geq 0$ and $g' > 0$ in Ω.

We consider two cases.

Case 1: $b(x) = b > 0$ is constant

In this setting one can use the Hopf-Cole transformation and study the resulting equation for the new function v. Then, using the results of [16], we have that v is in H^1 if it is stable and the nonlinearity f satisfies

$$\liminf_{s\to\infty} \frac{f'(s)s}{f(s)} > 1. \tag{7.29}$$

This condition could be rewritten in terms of the nonlinearity $g(u)$ allowing us to conclude that e^u, and hence u, is in H^1. It is natural to expect that such assumptions on g will be too restrictive, since they give a condition for e^u, and not just u, to be in H^1. In what follows we study directly the problem for u and find the natural conditions to impose on g.

Proposition 7.1. *Let $b > 0$ be a constant and u a positive classical solution of the problem $-\Delta u - b|\nabla u|^2 = \lambda g(u)$ with zero Dirichlet boundary conditions, $g \geq 0$ and $g' > 0$ in Ω and $\lambda > 0$ a parameter. Assume that u is a stable solution Then, if*

$$\liminf_{s\to\infty} \frac{g'(s)(e^{bs}-1)}{bg(s)} > 1,$$

we have that $||u|||_{H^1(\Omega)} \leq C$ where C is independent of λ.

To better understand the above condition on g, let us consider the case where equality holds, i.e.,

$$\frac{g'(s)(e^{bs}-1)}{bg(s)} = 1.$$

Integrating we get $\log g(s) = \log(e^{bs}-1) - bs + C$ for some constant C and hence,

$$g(s) = C(1 - e^{-bs}).$$

Recall that $b > 0$ so this means that g is bounded.

As we mentioned before, this condition on g is less restrictive than the one imposed via f. In fact, if $g(u) = e^u$ then u is in H^1 by the previous theorem. However, if we pass to the equation for $v = e^{bu} - 1$ we have that $-\Delta v = \lambda f(v)$ with $f(v) = b(v+1)^p$, $p = 1 + 1/b$ and f does not satisfy condition (7.29) of [16].

Proof. Since u is stable there exists a positive function ϕ on $\overline{\Omega}$ such that

$$-\Delta\phi - 2b\nabla u\nabla\phi \geq \lambda g'(u)\phi.$$

Multiplying by $(e^{bu}-1)^2/\phi$ and integrate in Ω.

$$\begin{aligned}
\lambda\int_\Omega g'(u)(e^{bu}-1)^2 &\le \int_\Omega -\frac{|\nabla\phi|^2}{\phi^2}(e^{bu}-1)^2+\int_\Omega 2b\frac{\nabla\phi}{\phi}\nabla u e^{bu}(e^{bu}-1)-\\
&\quad -\int_\Omega 2b\frac{\nabla\phi}{\phi}\nabla u(e^{bu}-1)^2\\
&= \int_\Omega -\frac{|\nabla\phi|^2}{\phi^2}(e^{bu}-1)^2+\int_\Omega 2b\frac{\nabla\phi}{\phi}\nabla u(e^{bu}-1)\\
&\le \int_\Omega b^2|\nabla u|^2.
\end{aligned}$$

On the other hand, multiplying (7.28) by $e^{bu}-1$ and integrating we get

$$\begin{aligned}
\lambda\int_\Omega g(u)(e^{bu}-1) &= \int_\Omega \nabla u\nabla(e^{bu}-1)-b|\nabla u|^2(e^{bu}-1)\\
&= b\int_\Omega|\nabla u|^2.
\end{aligned}\tag{7.30}$$

Thus, we have that

$$\lambda\int_\Omega g'(u)(e^{bu}-1)^2\le\lambda b\int_\Omega g(u)(e^{bu}-1).\tag{7.31}$$

Assume that

$$bg(s)(e^{bs}-1)\le\delta g'(s)(e^{bs}-1)^2+C\tag{7.32}$$

for some constant $\delta<1$ and some constant C.

Then, from (7.31) we get that

$$\int_\Omega g'(u)(e^{bu}-1)^2\le C,$$

which implies both

$$\int_\Omega g(u)(e^{bu}-1)\le C\qquad\text{and, by (7.30),}\int_\Omega|\nabla u|^2\le C.$$

Now, going back to (7.32) we see that, for s small it is always possible to find δ and C. The problem occurs when s tends to infinity (that is, when u is unbounded). It is easy to see that (7.32) holds if

$$\liminf_{s\to\infty}\frac{g'(s)(e^{bs}-1)}{bg(s)}\ge\frac{1}{\delta}>1.\tag{7.33}$$

□

Case 2: $b(x)\le-\epsilon<0$

This case is, in some sense, more general than the previous one since we do not need to assume that u is a stable solution. The proof uses a technique due to Boccardo (see [9]) involving truncations. For a function u we define the truncation T_1u as

$$T_1u=\begin{cases}1, & u>1\\ u, & |u|\le1\\ -1, & u<-1.\end{cases}\tag{7.34}$$

We have $\nabla T_1u=\nabla u$ where $|u|\le1$ and $\nabla T_1=0$ otherwise.

Proposition 7.2. *Let $b(x) \leq -\epsilon < 0$ for some $\epsilon > 0$ and u a positive classical solution to the problem $-\Delta u - b(x)|\nabla u|^2 = \lambda g(u)$ with zero Dirichlet boundary conditions, $\lambda > 0$ a parameter, and assume that $g(u) \in L^1(\Omega)$. Then, $||u||_{H^1(\Omega)} \leq C$, where C is independent of λ.*

Proof. We multiply equation (7.28) by $T_1 u$ and integrate by parts

$$\int_\Omega \nabla u \nabla T_1 u - \int_\Omega b(x)|\nabla u|^2 T_1 u = \lambda \int_\Omega g(u) T_1 u.$$

Given the definition of $T_1 u$ this yields

$$\int_{\{|u|\leq 1\}} |\nabla u|^2 = \lambda \int_\Omega g(u) T_1 u + \int_\Omega b(x) T_1 u |\nabla u|^2.$$

Since u is assumed to be positive, $b(x) \leq -\epsilon < 0$ for some $\epsilon > 0$ and $0 \leq T_1 u \leq 1$ we get

$$\int_{\{u\leq 1\}} |\nabla u|^2 + \epsilon \int_{\{u>1\}} |\nabla u|^2 \leq \lambda \int_\Omega |g(u)|.$$

□

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