2D Born-Infeld electrostatic fields

Rafael Ferraro*
Instituto de Astronomía y Física del Espacio, Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina, and Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina
(Dated: September 18, 2003)

The electrostatic configurations of the Born-Infeld field in the 2-dimensional Euclidean plane are obtained by means of a non-analytical complex mapping which captures the structure of equipotential and field lines. The electrostatic field reaches the Born-Infeld limit value when the field lines become tangent to an epicycloid around the origin. The total energy by unit of length remains finite.

PACS numbers: 03.50.Kk

In 1934 Born and Infeld [1] proposed a non-linear electromagnetism that modifies the behavior of the Maxwell theory in the regime of strong fields. The aim was to formulate a theory where the self-energy of a point-like charge is finite, and thus open the possibility of conceiving a charged particle as a part of the field instead of an external source of it. The basic idea was to impose a finite limit value b to a purely electrostatic field. This could be achieved by reproducing the way the velocity of a particle remains lower than c when the classical Lagrangian $L = (1/2)\, m\, \dot{q}^2$ is replaced by the relativistic Lagrangian $L = -mc^2\sqrt{1-\dot{q}^2\, c^{-2}}$. This means that the Maxwell Lagrangian $L_M = -\sqrt{-g}\,(8\pi c)^{-1}(B^2 - E^2)$ should be replaced by the Lagrangian $L_B = -\sqrt{-g}\,(4\pi c)^{-1} b^2\sqrt{1+(B^2-E^2)\, b^{-2}}$ [2]. In order that the energy goes to zero when the field goes to zero, a "rest" energy $\sqrt{-g}\,(4\pi c)^{-1} b^2$ should be subtracted from L_B, without affecting the dynamical equations. Nevertheless, Born and Infeld followed Einstein by judging that the Lagrangian should combine the metric g_{ij} and the electromagnetic field $F_{ij} = \partial_i A_j - \partial_j A_i$ as the symmetric and antisymmetric parts of a unique field $b\, g_{ij} + F_{ij}$. The Born-Infeld Lagrangian density is

$$L_{BI}[A_k] = -\tfrac{1}{4\pi c}\sqrt{|\det(b\, g_{ij}+F_{ij})|} + \sqrt{-g}\,\tfrac{b^2}{4\pi c} = \sqrt{-g}\,\tfrac{b^2}{4\pi c}\left(1-\sqrt{1+\tfrac{2S}{b^2}-\tfrac{P^2}{b^4}}\right), \quad (1)$$

where S and P are the scalar and pseudoscalar field invariants,

$$S = \frac{1}{4} F_{ij}F^{ij} = \frac{1}{2}(B^2-E^2),$$
$$P = \frac{1}{8}\sqrt{-g}\,\varepsilon_{ijkl}\, F^{kl}F^{ij} = \frac{1}{4}\, {}^*F_{ij}F^{ij} = \mathbf{E}\cdot\mathbf{B} \quad (2)$$

ε_{ijkl} being the Levi-Civita symbol whose components are ± 1 depending on $(ijkl)$ is an even or odd permutation of (0123). The Maxwell Lagrangian is recovered from the Born-Infeld Lagrangian when $b \to \infty$. The field equations derived from the Born-Infeld Lagrangian (1) are

$$\partial_j\left(\sqrt{-g}\,\mathcal{F}^{ij}\right) = 0 \quad (3)$$

where $\mathcal{F}_{ij}$ is the tensor

$$\mathcal{F}_{ij} = \frac{F_{ij} - b^{-2}\, P\, {}^*F_{ij}}{\sqrt{1+\frac{2S}{b^2}-\frac{P^2}{b^4}}} \quad (4)$$

Since the field is an exact 2-form ($F = dA$), the identities $dF = 0$ must be added to the Euler-Lagrange equations (3).

The energy-momentum tensor results (the metric signature is $(+---)$)

$$T_{ij} = \frac{2c}{\sqrt{-g}}\frac{\partial L_{BI}}{\partial g^{ij}} = -\frac{1}{4\pi}\, F_{ik}\,\mathcal{F}_j{}^k - \frac{b^2}{4\pi}\, g_{ij}\left(1-\sqrt{1+\tfrac{2S}{b^2}-\tfrac{P^2}{b^4}}\right) \quad (5)$$

To get the Born-Infeld charge we solve Eq.(3) for an isotropic electrostatic field $F = E(r)\, dt\wedge dr$. Since $\sqrt{-g} = r^2\sin\theta$ in spherical coordinates, the solution is

$$\mathcal{F} = q\, r^{-2}\, dt\wedge dr, \quad F = b\left(\tfrac{b^2 r^4}{q^2}+1\right)^{-\frac{1}{2}} dt\wedge dr,$$
$$T_0{}^0 = \tfrac{b^2}{4\pi}\sqrt{1+\tfrac{q^2}{b^2 r^4}} - \tfrac{b^2}{4\pi},$$
$$U = \int_0^\infty T^{00}\, 4\pi r^2 dr = \tfrac{1}{6}\sqrt{\tfrac{q^3 b}{\pi}}\,\Gamma(\tfrac{1}{4})^2 \quad (6)$$

While F and $\mathcal{F}$ are equal in Maxwell theory, they differ in the Born-Infeld theory. As a result, only one of them diverges: $\mathcal{F}$ diverges at $r_o = 0$, but $F(r_o) = b\, dt\wedge dr$. This characteristic moderates the divergence of the energy-momentum tensor, and leads to a finite energy U.

Another nice example is the axial magnetostatic field $F = B(\rho)\, dz\wedge d\rho$. Since $\sqrt{-g} = \rho$ in cylindrical coordinates, then the solution of Eq.(3) is

$$\mathcal{F} = \frac{2I}{c\rho}\, dz\wedge d\rho, \quad F = b\left(\tfrac{b^2\rho^2 c^2}{4I^2}-1\right)^{-\frac{1}{2}} dz\wedge d\rho,$$
$$T_0{}^0 = \tfrac{b^2}{4\pi}\left(1-\tfrac{4I^2}{b^2\rho^2c^2}\right)^{-\frac{1}{2}} - \tfrac{b^2}{4\pi} \quad (7)$$

Now F diverges at $\rho_o = \frac{2\,I}{c\,b}$, but $\mathcal{F}(\rho_o) = b\; dz \wedge d\rho$. Although the energy is not finite in this case, as a consequence of the extended character of the source, however the integral $\int T^{00}\, 2\pi\rho\, d\rho$ remains finite at $\rho = \rho_o$.

In the last decades there was a renewal of interest in the Born-Infeld theory because it emerges in the low energy limit of string theories [3, 4, 5, 6, 7, 8]. Maxwell and Born-Infeld theories have proved to be the sole theories for the massless spin 1 field having causal propagation [9, 10] and absence of birefringence [11, 12]. However the essential features of field configurations other than the kind above considered are hardly known, due to the problem of dealing with the non-linear equations involved in the theory. Here we are going to introduce a procedure that works for Born-Infeld electrostatic fields lying in the 2-dimensional plane. This procedure extends the method of using analytic complex functions to get solutions of the Laplace equation in 2 dimensions. As it is well known, if $w(\mathbf{z}) = u(x,y) + i\, v(x,y)$ is an analytical function in the complex plane, then $u(x,y)$ and $v(x,y)$ solve the Laplace equation. Analytic functions generate conformal mappings $\mathbf{z} = f(w)$ in the Euclidean plane; in fact, $dx^2 + dy^2 = dz\; dz^* = f'\, f'^* dw\, dw^* = |f'(w)|^2\,(du^2 + dv^2)$, so u, v are orthogonal coordinates dilating distances without changing the shapes of infinitesimal figures. If we regard $u(x,y)$ as the electrostatic potential, then the coordinate lines $u(x,y) = const.$ are equipotential and the coordinate lines $v(x,y) = const.$ are field lines.

Although the Born-Infeld electrostatic potential is not a solution of the Laplace equation, a modified version of the complex mapping can be still applied to get the structure of the Born-Infeld electrostatic field in 2 dimensions. The substitute mapping must generate orthogonal coordinates u, v –the field lines are orthogonal to the equipotential surfaces–, but it will distort the infinitesimal shapes. So let us try with

$$\mathbf{z} = f(w) + \frac{g(w^*)}{4\; b^2} \tag{8}$$

where f and g are analytic functions of their respective arguments (in the sense that $df/dw^* = 0$ and $dg/dw = 0$). Besides, let us choose

$$f'(w) g'(w^*)^* = 1 \tag{9}$$

Then

$$\begin{aligned} d\mathbf{z} &= f'(w)\; dw + \frac{g'(w^*)}{4b^2}\; dw^* \\ &= f'(w)\; dw + \frac{1}{4b^2 f'(w)^*}\; dw^* \\ &= f'(w)\Big[\Big(1 + \frac{|f'(w)|^{-2}}{4b^2}\Big)\; du \\ &\quad + i\;\Big(1 - \frac{|f'(w)|^{-2}}{4b^2}\Big)\; dv\Big] \end{aligned} \tag{10}$$

and

$$\begin{aligned} dx^2 \; + dy^2 = d\mathbf{z}\; d\mathbf{z}^* = |f'(w)|^2 \Big[\Big(1 + \frac{|f'(w)|^{-2}}{4b^2}\Big)^2 du^2 \\ + \;\; \Big(1 - \frac{|f'(w)|^{-2}}{4b^2}\Big)^2 dv^2\Big] = -g_{uu}\, du^2 - g_{vv}\, dv^2 \end{aligned} \tag{11}$$

So the mapping (8)-(9) effectively generates orthogonal coordinates u, v in the Euclidean plane. Now we will prove that $u(x,y)$ is the electrostatic potential of a Born-Infeld field, i.e. the exact 2-form $F = du \wedge dt$ solves the Born-Infeld equations. Since $2S$ results to be g^{uu}, then

$$\sqrt{-g}\; \mathcal{F}^{t\,u} \;\; = -\frac{\sqrt{g_{uu} g_{vv}}\; g^{uu}}{\sqrt{1 + \frac{g^{uu}}{b^2}}} = \frac{\sqrt{g^{uu} g_{vv}}}{\sqrt{1 + \frac{g^{uu}}{b^2}}} = 1 \tag{12}$$

for all $f'(w)$, and the Eq.(3) is fulfilled. The cartesian components of the electric field, $E_x = -\partial u/\partial x$, $E_y = -\partial u/\partial y$, are obtained by inverting the Jacobian matrix of the coordinate transformation (8)-(9),

$$\begin{aligned} \frac{\partial x}{\partial u} &= \Big(1 + \frac{|f'(w)|^{-2}}{4b^2}\Big)\; \mathrm{Re} f'(w) \\ \frac{\partial y}{\partial u} &= \Big(1 + \frac{|f'(w)|^{-2}}{4b^2}\Big)\; \mathrm{Im} f'(w) \\ \frac{\partial x}{\partial v} &= -\Big(1 - \frac{|f'(w)|^{-2}}{4b^2}\Big)\; \mathrm{Im} f'(w) \\ \frac{\partial y}{\partial v} &= \Big(1 - \frac{|f'(w)|^{-2}}{4b^2}\Big)\; \mathrm{Re} f'(w) \end{aligned} \tag{13}$$

The inverse matrix is

$$\begin{aligned} -\, E_x = \frac{\partial u}{\partial x} &= \frac{4b^2\; \mathrm{Re} f'(w)}{1 + 4b^2\, |f'(w)|^2} \\ \frac{\partial v}{\partial x} &= \frac{4b^2\; \mathrm{Im} f'(w)}{1 - 4b^2\, |f'(w)|^2} \\ -E_y = \frac{\partial u}{\partial y} &= \frac{4b^2\; \mathrm{Im} f'(w)}{1 + 4b^2\, |f'(w)|^2} \\ \frac{\partial v}{\partial y} &= -\frac{4b^2\; \mathrm{Re} f'(w)}{1 - 4b^2\, |f'(w)|^2} \end{aligned} \tag{14}$$

and the field is

$$\mathbf{E} = E_x + iE_y = -\frac{4b^2 f'(w)}{1 + 4b^2\, |f'(w)|^2} \tag{15}$$

In spite of the simple appearance of this result for electrostatic fields in 2D, it must be remarked that the difficulty lies in writing w at $f'(w)$ as a function of $(x,\, y)$, because this implies solving the Eq.(8).

Special interest deserve the periodic non-isotropic solutions. In Maxwell theory these are $u_M(x,y) = A\, \rho^{-n} \cos n\varphi$, $n \in \mathrm{N}$, and come from the mapping $\mathbf{z} = f(w) = A\, w^{-\frac{1}{n}}$, where A is a constant. Consequently, for the Born-Infeld field we will use the mapping

$$\mathbf{z} = \frac{A}{w^{\frac{1}{n}}} - \frac{w^{*\;\frac{1}{n}+2}}{4\frac{1}{n}(\frac{1}{n}+2)\; A\, b^2} \tag{16}$$

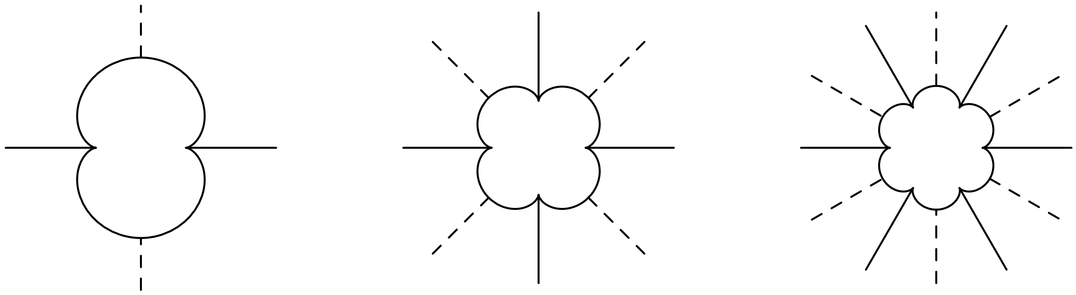

FIG. 1: Epicycloids with 2, 4 and 6 cusps. Lines $u = 0$ (dashed lines) and $v = 0$.

In order to check the periodicity of this mapping, let us note that the lines $u = 0$ (i.e. $w = i\,v$) are the points where $i^{1/n}\,\mathbf{z}$ is real. Thus the lines $u = 0$ coincide with those of the Maxwell field, so guaranteeing the periodicity of the mapping (the same can be said about the lines $v = 0$). Our first task is to find the places where the field reaches the limit value b. According to Eq.(15), this happens where $|f'(w)| = (2\,b)^{-1}$. Therefore

$$\frac{1}{2\,b} = |f'(w)| = \frac{A}{n}\,|w|^{-\frac{1}{n}-1} \qquad (17)$$

This equation corresponds to a circle in coordinates u, v, which can be parameterized as $w(\tau) = (\frac{2}{n}\,A\,b)^{\frac{n}{n+1}}\exp(-in\tau)$. By replacing this parametrized curve in the expression for $\mathbf{z}$, we get parametric equations in cartesian coordinates:

$$\begin{aligned}\mathbf{z}(\tau) = x(\tau) + i\,y\ \ (\tau) &= A\left(\frac{2}{n}\,A\,b\right)^{-\frac{1}{n+1}}\Big(\exp(i\,\tau) \\ &- \frac{1}{1+2n}\exp[i\,(1+2n)\tau]\Big) \qquad (18)\end{aligned}$$

This curve is a $2n$-cusped epicycloid, which is represented in Fig. 1 together with the lines $u = 0$, $v = 0$. The cusps are the points where $\tau = k\pi/n$, $k \in$ Z; then $\mathbf{z}_k = \frac{2n}{1+2n}\,A\,(\frac{2}{n}\,A\,b)^{-\frac{1}{n+1}}\exp(i\,k\pi/n)$. $w = u + i\,v$ is real at the cusps; so $|u|$ reaches there its maximum value $(\frac{2}{n}\,A\,b)^{\frac{n}{n+1}}$. The cusps should not be regarded as point-like charges because the field lines do not converge on the cusps. Instead, the field lines tangentially reach the epicycloid. In fact, on the one hand the complex field $\mathbf{E}$ on the epicycloid is $\mathbf{E}(\tau) = -2\,b^2\,f'(w(\tau)) = -2\,b^2\,\frac{A}{n}\,w(\tau)^{-\frac{1}{n}-1} = -b\,\exp[i\,(1+n)\tau]$. On the other hand the vector tangent to the epicycloid is $d\mathbf{z}/d\tau = i\,A\,(\frac{2}{n}\,A\,b)^{-\frac{1}{n+1}}\,(\exp(i\,\tau) - \exp[i\,(1+2n)\tau]) = 2\,A\,(\frac{2}{n}\,A\,b)^{-\frac{1}{n+1}}\,\sin n\tau\,\exp[i\,(1+n)\tau]$, which is parallel to $\mathbf{E}(\tau)$.

In Fig. 2 we show the main features of the 2D Born-Infeld electrostatic field for the case $n = 1$. In Maxwell context, this case corresponds to a pair of infinitely close parallel opposite uniform line charges, and A is the separation distance times the linear density of charge. The 2-cusped epicycloid is a nephroid. The mapping is

$$\begin{aligned} x &= \mathrm{Re}\big[A w^*\big(\frac{1}{ww^*} - \frac{w^{*\,2}}{12\,A^2b^2}\big)\big] \\ &= \frac{A\,u}{u^2+v^2} - u\,\frac{u^2-3\,v^2}{12\,Ab^2} \\ y &= \mathrm{I}m\big[A w^*\big(\frac{1}{ww^*} - \frac{w^{*\,2}}{12\,A^2b^2}\big)\big] \\ &= -\frac{A\,v}{u^2+v^2} - v\,\frac{v^2-3\,u^2}{12\,Ab^2} \qquad (19)\end{aligned}$$

Since the coordinate lines $v = v_o = const.$ are field lines, then the equations $x = x(u, v_o)$, $y = y(u, v_o)$ are parametric equations for the field lines and can be numerically plotted. The equipotential lines are plotted in the same way.

The energy of the field is the integral of T^{00} outside the epicycloid. This integral gets its simplest form when it is expressed in terms of coordinates u, v because the integration region is a circle of radius $(\frac{2}{n}\,A\,b)^{\frac{n}{n+1}}$ (remember the epicycloid is a circle in coordinates u, v; the field lives inside the circle because u, v goes to zero at the infinity). Since the volumen is $\sqrt{g_{uu}g_{vv}} = -g_{uu}\sqrt{1+\frac{g^{uu}}{b^2}} = E^{-2}\sqrt{1-\frac{E^2}{b^2}}$, then

$$\begin{aligned} U &= \int T^{\,00}\,\sqrt{g_{uu}\,g_{vv}}\;du\,dv\,dz \\ &= \frac{1}{4\pi}\int\frac{b^2}{E^2}\left(1-\sqrt{1-\frac{E^2}{b^2}}\right)du\,dv\,dz \\ &= \frac{1}{8\,\pi}\int\left(1+\frac{|f'(w)|^{-2}}{4b^2}\right)du\,dv\,dz \\ &= \frac{1}{8\,\pi}\int\left(1+\frac{n^2\,(u^2+v^2)^{1+\frac{1}{n}}}{4\,A^2b^2}\right)du\,dv\,dz \qquad (20)\end{aligned}$$

and the result is

$$U = \frac{1}{8}\,\frac{1+3n}{1+2n}\,\left(\frac{2}{n}\,A\,b\right)^{\frac{2\,n}{n+1}}\int dz \qquad (21)$$

Although it is nice to find that 2D electrostatic Born-Infeld fields have a finite energy by unit of length, some

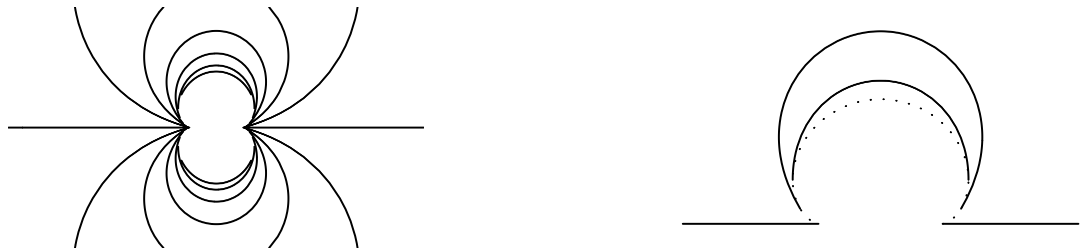

FIG. 2: Born-Infeld field lines for $f(w) \propto w^{-1}$. Behavior of the field near the nephroid.

other features of these fields seem to be less pleasant. The Euler-Lagrange equations break down on the epicycloid because the tensor $\mathcal{F}$ diverges there, which prevents us integrating the equations beyond the epicycloid; essentially the same thing happens to the magnetostatic field of Eq.(7). The field (6) of a point-like charge is not devoid of problems since it is perplexing the finite value at the origin of its isotropic vector field (Hoffmann and Infeld proposed a modification of the Born-Infeld Lagrangian to avoid this behavior [13]). Perhaps there is nothing wrong with these features, but they only invite us to consider non trivial combinations of electrostatic and magnetostatic fields as meaningful static solutions. Since the theory is non linear, the static solutions with both types of field do not reduce to a mere superposition. It would be enjoyable that a point-like charge get a completely satisfactory 3D monopolar electrostatic field once its Born-Infeld field includes the dipolar magnetostatic component.

This work was partially supported by Universidad de Buenos Aires (Proy. UBACYT X805)

* Member of Carrera del Investigador Científico (CONICET, Argentina).

Electronic address: ferraro@iafe.uba.ar

[1] M. Born and L. Infeld, Proc. R. Soc. (London) A **144** (1934), 425.
[2] M. Born, Proc. R. Soc. (London) A **143** (1934), 410.
[3] Fradkin y A. A. Tseytlin, Phys. Lett. B **163** (1985), 123.
[4] E. Bergeshoeff, E. Sezgin, C. N. Pope y P. K. Townsend, Phys. Lett. B **188** (1987), 70.
[5] R. R. Metsaev, M. A. Rahmanov y A. A. Tseytlin, Phys. Lett. B **193** (1987), 207.
[6] A. A. Tseytlin, Nuc. Phys. B **501** (1997), 41.
[7] J.P. Gauntlett, J.Gomis and P.K. Townsend, JHEP **01** (1998), 003.
[8] D. Brecher and M.J. Perry, Nucl. Phys. B **527** (1998), 121.
[9] J. Plebanski, *Lectures on non linear electrodynamics* , Nordita Lecture Notes, 1968.
[10] S. Deser y R. Puzalowski, J. Phys. A **13** (1980), 2501.
[11] G. Boillat, J. Math. Phys. **11** (1970), 941.
[12] I. Bialynicki-Birula, in "J.Lopuszanski's Festchrift, Quantum Theory of Particles and Fields", eds. B.Jancewicz and J.Lukierski, World Scientific (Singapore), 1983.
[13] B. Hoffmann and L. Infeld, Phys. Rev. **51** (1937), 765.