Asociación Argentina de Mecánica Computacional

Mecánica Computacional Vol XXXII, págs. 213-224 (artículo completo)
Carlos G. García Garino, Aníbal E. Mirasso, Mario A. Storti, Miguel E. Tornello (Eds.)
Mendoza, Argentina, 19-22 Noviembre 2013

A SMALL STRAIN TENSOR FOR THE GEOMETRICALLY EXACT THIN-WALLED COMPOSITE BEAM

C. Martín Saravia *, L. Joaquín Saravia *, Víctor H. Cortínez*

** Centro de Investigaciones en Mecánica Teórica y Aplicada, CONICET-Universidad Tecnológica Nacional, Facultad Regional Bahía Blanca, 11 de Abril 461, 8000 Bahía Blanca, Argentina.*

Keywords: Composite Beams; Finite Elements; Finite Rotations; Thin-walled beams.

Abstract. This work presents the derivation of a small strain tensor compatible with the geometrically exact kinematics of the thin walled composite beam theory. The formulation is based on the expression of the Green strain tensor in terms of a pure linear strain plus a pure nonlinear strain measure obtained through the decomposition of the deformation gradient. The discrete small strain measures are expressed in terms of the current director and displacement fields and its derivatives in terms of very simple relations; they result to be objective under rigid body motion and independent of the integration path. The formulation is consistent in the sense that both the strain measures and the constitutive relations are valid for small strains.

1 INTRODUCTION

The use composite beam formulations for modeling structural components is still a common practice; the structural behavior of several modern machines such as wind turbines, satellites and modern buildings is generally predicted using composite beam formulations.

Several approaches that deal with the thin-walled composite beam problem can be found in the literature; generally they are derived from Vlasov's theory. It is often found that the derivations lead to both geometrical and constitutive inconsistencies. Despite being vastly investigated, still a large amount of efforts are being done by researchers from all over the world to improve the thin-walled beam theory; thus we consider that it is worth to develop a consistent geometrically exact formulation for composite thin-beams.

The thin–walled beam formulation is due to Vlasov (Vlasov, 1961), it has survived fifty years without significant changes. One of the principal extensions of the theory was the introduction of the constitutive modeling of composite materials; in this direction, Prof. Librescu was probably who developed the majority of the composite material aspects (Librescu, 2006). As a common aspect, most of the thin-walled formulations that can be found in the literature begin with the assumption of a displacement field, which is then introduced into the Green strain expression to obtain the strain measures in terms of the kinematic variables and its derivatives. Almost exclusively, the kinematic variables are considered as three displacements and three rotations per node; a warping kinematic variable can be found as well.

A careful revision of the thin walled beam literature shows that at least one of the following four inconsistencies can be found in almost every work treating thin walled beams: i) the displacements field is said to describe moderate or large kinematical changes while the rotation variables are treated as vectors, ii) a linear or second order nonlinear displacement field is introduced into an exact large strain expression, iii) terms of the Green strain regarded as nonlinear strain measures are eliminated while the objectivity of the resulting "linear" strain measures is lost and iv) the kinematic part of the theory admits large strains while the constitutive law is only valid for small strains.

If, for example, the developments by Librescu et al. in (Librescu, 2006) are carefully analyzed, it can be seen that they suffer from inconsistencies *i*, *ii* and *iv*. Also the works by Pi et al. (Pi and Bradford, 2001; Pi and Bradford, 2001; Pi, Bradford et al., 2005) suffer from inconsistencies *i* and *iii*. In (Pi and Bradford, 2001; Pi and Bradford, 2001), the rotation matrix is second order accurate and its components are treated as vector functions; also, some non pure strain higher order terms of the Green strain measure are eliminated and then the objectivity is lost. In (Pi, Bradford et al., 2005) an exact rotation matrix is used, but the elimination of non pure strain terms leads again to a loss of objectivity of the formulation; also, the rotation matrix is said to belong to the Special Orthogonal Group (SO3) while it is linearized as it belongs to a vector space. The theories developed by Cortínez, Piován and Machado in works (Cortínez and Piovan, 2002; Machado and Cortínez, 2005; Cortínez and Piovan, 2006; Machado and Cortínez, 2007; Piovan and Cortínez, 2007) for the study of the dynamic stability, vibration, buckling and postbuckling of both open and cross section composite TWB suffer from inconsistencies *i*, *ii* and *iv*. The displacement field is assumed to be small or moderate while it is introduced into an expression of large; indeed the constitutive law is only valid for small strains.

Saravia et. al. (Saravia, Machado et al., 2011; Saravia, Machado et al., 2012) presented Eulerian and Lagrangian geometrically exact formulations for thin-walled composite beams using a parametrization in terms of director vectors, which suffer from inconsistency *iv*. This formulation can describe geometrical and strain changes of arbitrary magnitude consistently;

however, the constitutive law is only valid for small strains. The presence of inconsistency *iv* is also the case of most geometrically exact formulations developed for isotropic beams, see for instance (Simo, 1985; Simo and Vu-Quoc, 1986; Cardona and Geradin, 1988; Ibrahimbegovic, 1997).

The mentioned formulations are only examples of the vast amount of works that present the mentioned inconsistencies. Although it can be arguable if the errors that arise from them has influence for practical purposes, the uncertainty about the limit of application of these theories strongly motivates the development of a consistent approach in which an assessment of validity is not needed. In this context, this paper presents the development of a large deformation-small strain formulation for composite thin walled composite beams. The discrete small strain measures are expressed in terms of the current director and displacement fields and its derivatives; the obtained relations are remarkably simple and do not involve derivatives of the reference triads. Also, they result to be objective under rigid body motion and independent of the integration path.

2 KINEMATICS

The kinematic description of the beam is extracted from the relations between two states of a beam, an undeformed reference state (denoted as $\boldsymbol{\mathcal{B}}_0$) and a deformed state (denoted as $\mathcal{B}$), as it is shown in Fig. 1. Being $\boldsymbol{a}_i$ a spatial frame of reference, we define two orthonormal frames: a reference frame $\boldsymbol{E}_i$ and a current frame $\boldsymbol{e}_i$.

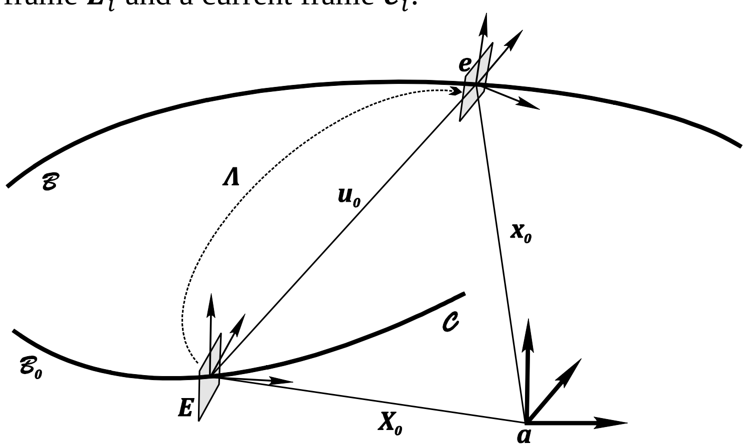

Figure 1. 3D beam.

The displacement of a point in the deformed beam measured with respect to the undeformed reference state can be expressed in the global coordinate system $\boldsymbol{a}_i$ in terms of a vector $\boldsymbol{u} = (u_1, u_2, u_3)$.

The current frame e_i is a function of a running length coordinate along the reference line of the beam, denoted as x, and is fixed to the beam cross-section. For convenience, we choose the reference curve $\mathcal{C}$ to be the locus of cross-sectional inertia centroids. The origin of e_i is located on the reference line of the beam and is called pole. The cross-section of the beam is arbitrary and initially normal to the reference line.

The relations between the orthonormal frames are given by the linear transformations:

$$\boldsymbol{E}_i = \boldsymbol{\Lambda_0}(x)\boldsymbol{a}_i, \qquad \boldsymbol{e}_i = \boldsymbol{\Lambda}(x)\boldsymbol{E}_i, \tag{1}$$

Where $\boldsymbol{\Lambda}_0(x)$ and $\boldsymbol{\Lambda}_0(x)$ are two-point tensor fields $\in \mathrm{SO}(3)$; the special orthogonal (Lie) group. Thus, it is satisfied that $\boldsymbol{\Lambda}_0{}^T\boldsymbol{\Lambda}_0 = \boldsymbol{I},\ \boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \boldsymbol{I}$. We will consider that the beam element is straight, so we set $\boldsymbol{\Lambda}_0 = \boldsymbol{I}$.

Recalling the relations (1), we can express the position vectors of a point in the beam in the undeformed and deformed configuration respectively as:

$$\boldsymbol{X}(s, X_2, X_3) = \boldsymbol{X}_0(x) + \sum_{i=2}^{3} X_i\,\boldsymbol{E}_i, \qquad \boldsymbol{x}(s, X_2, X_3, t) = \boldsymbol{x}_0(s,t) + \sum_{i=2}^{3} X_i\,\boldsymbol{e}_i. \tag{2}$$

Where in both equations the first term stands for the position of the pole and the second term stands for the position of a point in the cross section relative to the pole. Note that x is the running length coordinate and $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$ are cross section coordinates. At this point we note that since the present formulation is thought to be used for modeling high aspect ratio composite beams, the warping displacement is not included. As it is widely known, for such type of beams the warping effect is negligible (Hodges, 2006).

Also, it is possible to express the displacement field as:

$$\boldsymbol{u}(s, X_2, X_3, t) = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{u}_0(s,t) + (\boldsymbol{\Lambda} - \mathbf{I}) \sum_{2}^{3} X_i\,\boldsymbol{E}_i, \tag{3}$$

where u_0 represents the displacement of the kinematic center of reduction, i.e. the pole.

The nonlinear manifold of 3D rotation transformations $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ (belonging to the special orthogonal Lie Group SO(3)) is obtained mathematically by means of a trigonometric form in terms of the Cartesian rotation vector (Cardona and Geradin, 1988). The rotation tensor component form can be written as:

$$\boldsymbol{\Lambda} = \sum_{i,j=1}^{3} \Lambda_{ij}\boldsymbol{E}_i \otimes \boldsymbol{E}_j, \tag{4}$$

where the components $\boldsymbol{\Lambda}_{ij}$ can be obtained as

$$\Lambda_{ij} = \boldsymbol{E}_i \cdot \boldsymbol{\Lambda}\boldsymbol{E}_j = \boldsymbol{E}_i \cdot \boldsymbol{e}_j; \tag{5}$$

thus, it is possible to express the rotation tensor as:

$$\boldsymbol{\Lambda} = \sum_{i,j=1}^{3} \left(\boldsymbol{E}_i \cdot \boldsymbol{e}_j\right) \boldsymbol{E}_i \otimes \boldsymbol{E}_j. \tag{6}$$

Now, using the tensor product property $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = (\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}$, we can obtain:

$$\boldsymbol{\Lambda} = \sum_{i,j=1}^{3} (\boldsymbol{E}_i \otimes \boldsymbol{E}_i)\, \boldsymbol{e}_j \otimes \boldsymbol{E}_j = \sum_{j=1}^{3} \boldsymbol{I}\, \boldsymbol{e}_j \otimes \boldsymbol{E}_j, \tag{7}$$

Finally, with summation from 1 to 3 implicitly assumed, we can obtain the following expression for the rotation tensor:

$$\boldsymbol{\Lambda} = \boldsymbol{e}_j \otimes \boldsymbol{E}_j, \tag{8}$$

which will be a very useful expression for the derivation of a pure vectorial measure of the Green strain.

3 THE SMALL STRAIN TENSOR

3.1 The Green strain measure

The main motivation to develop a large deformations-small strain theory is to give consistency to the constitutive formulation of the geometrically exact composite thin-walled beam theory (Saravia, Machado et al., 2011; Saravia, Machado et al., 2012); since the constitutive equations are only valid for small strains, it is important to derive a strain tensor that is consistent with this assumption.

As it was stated, most of the geometrically exact beam formulations presented in the literature assume a linear elastic constitutive law which is valid only for small strains, but the constitutive equation are fed with a large strain deformation tensor.

It is not trivial to transform a large strain tensor in a small strain tensor without losing its objectivity under rigid body motions (Auricchio, Carotenuto et al., 2008). The Green strain tensor is commonly written in three different forms:

$$\boldsymbol{E} = \tfrac{1}{2}\left(\boldsymbol{x}_{,i}\cdot\boldsymbol{x}_{,j} - \boldsymbol{X}_{,i}\cdot\boldsymbol{X}_{,j}\right), \tag{9}$$

$$\boldsymbol{E} = \tfrac{1}{2}((\nabla_{\boldsymbol{X}}\otimes\boldsymbol{u})^{S} + (\nabla_{\boldsymbol{X}}\otimes\boldsymbol{u})^{T}\nabla_{\boldsymbol{X}}\otimes\boldsymbol{u}), \tag{10}$$

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{F}^{T}\boldsymbol{F} - \boldsymbol{I}), \tag{11}$$

where the displacement gradient is:

$$\nabla_{\boldsymbol{X}}\otimes\boldsymbol{u} = \frac{\partial\boldsymbol{u}}{\partial\boldsymbol{X}} \tag{12}$$

It must be noted that none of these forms can be directly seen as a linear plus a nonlinear pure strain measure, so elimination of any term in the above expressions does not guarantees that the resulting formulation will be objective. It must be noted that although Eq. (10) is sometimes understood as a linear plus a nonlinear component of strain, the gradient of the displacement field is not objective under rigid body motion, and thus it is not a pure measure of strain, i.e. it contains information related to both strain and kinematics.

3.2 The deformation gradient

In order to obtain a small strain measure without losing the capability of describing a large deformation behavior it is necessary to derive a linear pure strain measure from one of the expressions of the Green strain. The expression of the Green strain in terms of the deformation gradient, i.e. Eq. (11), has resulted useful for deriving a pure strain measure (Auricchio, Carotenuto et al., 2008).

The deformation gradient is a two point tensor given by the derivatives of the current positions with respect to the reference configuration as:

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}}\otimes\boldsymbol{x} = \frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}}; \tag{13}$$

then it relates quantities in the current configuration with quantities in the reference configuration. Eventually, we could also write the deformation gradient as:

$$\boldsymbol{F} = f_{ij}\boldsymbol{e}_i\otimes\boldsymbol{E}_j, \qquad f_{ij} = \frac{\partial\bar{x}_i}{\partial X_j}. \tag{14}$$

In order to exploit the above expression it should be necessary to express the current position vector as $\boldsymbol{x} = \bar{\boldsymbol{x}}_i \boldsymbol{e}_i$, but this is not convenient since the translational part of $\boldsymbol{x}$, i.e. $\boldsymbol{x}_0$, is naturally expressed as $\boldsymbol{x}_0 = x_{0i}\boldsymbol{E}_i$. Push forwarding $\boldsymbol{x}_0$ to express it in the current frame is would complicate the derivation since the rotation vector would appear explicitly; then, it is convenient to avoid thinking of $\boldsymbol{e}_i$ as a current reference frame and consider it as just a triad attached to the cross section. Then, the expression of the deformation gradient can be written as:

$$\boldsymbol{F} = F_{ij}\boldsymbol{E}_i \otimes \boldsymbol{E}_j, \qquad F_{ij} = \frac{\partial x_i}{\partial X_j}; \tag{15}$$

being $x_i = \boldsymbol{x} \cdot \boldsymbol{E}_i$ it is possible to operate on the deformation gradient as:

$$\boldsymbol{F} = \frac{\partial x_i}{\partial X_j}\boldsymbol{E}_i \otimes \boldsymbol{E}_j = \frac{\partial (x_i \boldsymbol{E}_i)}{\partial X_j} \otimes \boldsymbol{E}_j = \frac{\partial \boldsymbol{x}}{\partial X_j} \otimes \boldsymbol{E}_j, \tag{16}$$

and find a suitable explicit expression of the deformation gradient tensor.

Now, the materials derivatives of the position vector can be easily obtained as:

$$\begin{aligned} \frac{\partial \boldsymbol{x}}{\partial X_1} &= \boldsymbol{x}_{,s} = \boldsymbol{x}_0' + X_2 \boldsymbol{e}_2' + X_3 \boldsymbol{e}_3', \\ \frac{\partial \boldsymbol{x}}{\partial X_2} &= \boldsymbol{x}_{,X_2} = \boldsymbol{e}_2, \\ \frac{\partial \boldsymbol{x}}{\partial X_3} &= \boldsymbol{x}_{,X_3} = \boldsymbol{e}_2. \end{aligned} \tag{17}$$

Then we can insert these tangent vectors in Eq. (16) and obtain a pure vectorial expression for the deformation gradient as:

$$\boldsymbol{F} = \left(\boldsymbol{x}_0' + X_\alpha \boldsymbol{e}_\alpha'\right) \otimes \boldsymbol{E}_1 + \boldsymbol{e}_\alpha \otimes \boldsymbol{E}_\alpha, \tag{18}$$

where implicit summation over $\alpha = 2{,}3$ has been assumed.

It must be emphasized that Eq. (18) contains all the necessary information to describe the finite deformation-finite strain behavior of the beam.

3.3 The small Green strain tensor

Recalling the distributive property of the tensor product we can write the deformation gradient expression in Eq. (18) as:

$$\boldsymbol{F} = \left(\boldsymbol{x}_0' + X_\alpha \boldsymbol{e}_\alpha'\right) \otimes \boldsymbol{E}_1 + \left(\boldsymbol{e}_j \otimes \boldsymbol{E}_j - \boldsymbol{e}_1 \otimes \boldsymbol{E}_1\right). \tag{19}$$

Rearranging some terms and recalling Eq. (8) we can write:

$$\boldsymbol{F} = \boldsymbol{\Lambda} + \left(\boldsymbol{x}_0' - \boldsymbol{e}_1 + X_\alpha \boldsymbol{e}_\alpha'\right) \otimes \boldsymbol{E}_1. \tag{20}$$

From the above expression we define the pure current strain vector ϵ as:

$$\boldsymbol{\epsilon} = \boldsymbol{x}_0' - \boldsymbol{e}_1 + X_\alpha \boldsymbol{e}_\alpha', \tag{21}$$

what permits to write the deformation gradient as:

$$\boldsymbol{F} = \boldsymbol{\Lambda} + \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1. \tag{22}$$

The last expression has a interesting meaning since the deformation gradient can now be seen as a pure “rigid” rotation imposed by $\boldsymbol{\Lambda}$ plus a pure deformation measured by $\boldsymbol{\epsilon}$; this is remarkable since for a finite deformation-finite strain problems the strain measures are not commonly written in terms of a sum of rigid body motion plus a straining motion. It must be noted that this assertion as well as the derivation of the pure strain measure is in accordance with the theoretical developments in (Auricchio, Carotenuto et al., 2008).

Recalling Eq. (11) we can now write the Green strain tensor as:

$$\boldsymbol{E} = \tfrac{1}{2}[(\boldsymbol{\Lambda} + \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1)^T (\boldsymbol{\Lambda} + \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1) + \boldsymbol{I}]. \tag{23}$$

This expression can be expanded to give:

$$\boldsymbol{E} = \tfrac{1}{2}[\boldsymbol{\Lambda}^T \boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T (\boldsymbol{\epsilon} \otimes \boldsymbol{E}_1) + (\boldsymbol{E}_1 \otimes \boldsymbol{\epsilon})\, \boldsymbol{\Lambda} + (\boldsymbol{E}_1 \otimes \boldsymbol{\epsilon})(\boldsymbol{\epsilon} \otimes \boldsymbol{E}_1) + \boldsymbol{I}], \tag{24}$$

where we have used the property $(\boldsymbol{a} \otimes \boldsymbol{b})^T = (\boldsymbol{b} \otimes \boldsymbol{a})$. Exploiting the facts that $\boldsymbol{A}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{A}\boldsymbol{a}) \otimes \boldsymbol{b}$ and $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{A} = \boldsymbol{a} \otimes (\boldsymbol{A}^T \boldsymbol{b})$ then we can simplify the above expression to:

$$\begin{aligned} \boldsymbol{E} &= \tfrac{1}{2}[\boldsymbol{\Lambda}^T \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \boldsymbol{\Lambda}^T \boldsymbol{\epsilon} + (\boldsymbol{E}_1 \otimes \boldsymbol{\epsilon})(\boldsymbol{\epsilon} \otimes \boldsymbol{E}_1)] \\ &= \tfrac{1}{2}\big[\boldsymbol{\Lambda}^T \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \boldsymbol{\Lambda}^T \boldsymbol{\epsilon} + \big((\boldsymbol{E}_1 \otimes \boldsymbol{\epsilon})\boldsymbol{\epsilon}\big) \otimes \boldsymbol{E}_1\big]. \end{aligned} \tag{25}$$

The last term can be rearranged if we consider that $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a}$ and then:

$$\boldsymbol{E} = \tfrac{1}{2}[\boldsymbol{\Lambda}^T \boldsymbol{\epsilon} \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \boldsymbol{\Lambda}^T \boldsymbol{\epsilon} + (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon})\boldsymbol{E}_1 \otimes \boldsymbol{E}_1] \tag{26}$$

Being $\boldsymbol{\Lambda} = \boldsymbol{e}_j \otimes \boldsymbol{E}_j$, we can see that:

$$\boldsymbol{E} = \tfrac{1}{2}\left[\left(\left(\boldsymbol{e}_j \otimes \boldsymbol{E}_j\right)^T \boldsymbol{\epsilon}\right) \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \left(\left(\boldsymbol{e}_j \otimes \boldsymbol{E}_j\right)^T \boldsymbol{\epsilon}\right) + (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon})\boldsymbol{E}_1 \otimes \boldsymbol{E}_1\right] \tag{27}$$

Again, using $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a}$ on the first and second terms:

$$\boldsymbol{E} = \tfrac{1}{2}\big[\big(\boldsymbol{\epsilon} \cdot \boldsymbol{e}_j\big)\boldsymbol{E}_j \otimes \boldsymbol{E}_1 + \boldsymbol{E}_1 \otimes \boldsymbol{E}_j\big(\boldsymbol{\epsilon} \cdot \boldsymbol{e}_j\big) + (\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon})\boldsymbol{E}_1 \otimes \boldsymbol{E}_1\big] \tag{28}$$

From the above equation we see that the last term is a pure nonlinear strain measure; so if it is desired to develop a large deformation-small strain formulation this term can be dropped. Thus, the matrix form of the small Green strain tensor is given by:

$$\overline{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{\epsilon} \cdot \boldsymbol{e}_1 & \frac{1}{2}\boldsymbol{\epsilon} \cdot \boldsymbol{e}_2 & \frac{1}{2}\boldsymbol{\epsilon} \cdot \boldsymbol{e}_3 \\ \frac{1}{2}\boldsymbol{\epsilon} \cdot \boldsymbol{e}_2 & 0 & 0 \\ \frac{1}{2}\boldsymbol{\epsilon} \cdot \boldsymbol{e}_3 & 0 & 0 \end{bmatrix}. \tag{29}$$

We can write the explicit vector form of the small Green strain tensor as:

$$\overline{\boldsymbol{E}} = \begin{bmatrix} \boldsymbol{x}_0' \cdot \boldsymbol{e}_1 - 1 + X_\alpha \boldsymbol{e}_\alpha' \cdot \boldsymbol{e}_1 \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_2 + X_3 \boldsymbol{e}_3' \cdot \boldsymbol{e}_2 \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_3 + X_2 \boldsymbol{e}_2' \cdot \boldsymbol{e}_3 \end{bmatrix}. \tag{30}$$

The cross section geometric terms can be extracted from the above expression and then it is possible to write $\overline{\boldsymbol{E}}$ as:

$$\overline{\boldsymbol{E}} = \boldsymbol{D}\,\boldsymbol{\varepsilon}, \tag{31}$$

where the cross-sectional transformation matrix is:

$$\boldsymbol{D} = \begin{bmatrix} 1 & X_3 & X_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -X_3 \\ 0 & 0 & 0 & 0 & 1 & X_2 \end{bmatrix}. \tag{32}$$

And the generalized small strain vector needs to be defined as:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \epsilon \\ \kappa_2 \\ \kappa_3 \\ \gamma_2 \\ \gamma_3 \\ \kappa_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0' \cdot \boldsymbol{e}_1 - 1 \\ \boldsymbol{e}_1 \cdot \boldsymbol{e}_3' \\ \boldsymbol{e}_1 \cdot \boldsymbol{e}_2' \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_2 \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_3 \end{bmatrix}. \tag{33}$$

At this point it must be emphasized that the expression of the generalized large strain vector has a different expression:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \epsilon \\ \kappa_2 \\ \kappa_3 \\ \gamma_2 \\ \gamma_3 \\ \kappa_1 \\ \chi_2 \\ \chi_3 \\ \chi_{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\boldsymbol{x}_0' \cdot \boldsymbol{x}_0' - 1) \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_3' \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_2' \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_2 \\ \boldsymbol{x}_0' \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_3 \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_2' \\ \boldsymbol{e}_3' \cdot \boldsymbol{e}_3' \\ \boldsymbol{e}_2' \cdot \boldsymbol{e}_3' \end{bmatrix}. \tag{34}$$

As expected, not only the number of generalized strains is reduced from nine to six, but also the expressions of the individual components are not the same, except for the shear strains.

3.4 The virtual generalized small strain vector

As shown in (Saravia, Machado et al., 2011; Saravia, Machado et al., 2012; Saravia, Machado et al., 2013), the variational equilibrium of the geometrically exact thin-walled beam formulation is expressed in terms of the variation of the generalized strain vector; which requires the obtention of the variation of the director field and its derivatives.

In the present formulation the variation of the Green strain gives:

$$\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \delta\boldsymbol{x}_0' \cdot \boldsymbol{e}_1 + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_1 \\ \delta\boldsymbol{e}_1 \cdot \boldsymbol{e}_3' + \boldsymbol{e}_1 \cdot \delta\boldsymbol{e}_3' \\ \delta\boldsymbol{e}_1 \cdot \boldsymbol{e}_2' + \boldsymbol{e}_1 \cdot \delta\boldsymbol{e}_2' \\ \delta\boldsymbol{x}_0' \cdot \boldsymbol{e}_2 + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_2 \\ \delta\boldsymbol{x}_0' \cdot \boldsymbol{e}_3 + \boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_3 + \boldsymbol{e}_2' \cdot \delta\boldsymbol{e}_3 \end{bmatrix}. \tag{35}$$

In matrix form we can write the same expression as:

$$\delta\boldsymbol{\varepsilon} = \mathbb{H}\;\delta\boldsymbol{\varphi}. \tag{36}$$

where

$$\mathbb{H} = \begin{bmatrix} \boldsymbol{e}_1^T & \mathbf{0} & \boldsymbol{x}_0' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{e}_3'^{\,T} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_1^T \\ \mathbf{0} & \mathbf{0} & \boldsymbol{e}_2'^{\,T} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_1^T & \mathbf{0} \\ \boldsymbol{e}_2^T & \mathbf{0} & \mathbf{0} & \boldsymbol{x}_0'^{\,T} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{e}_3^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{x}_0'^{\,T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_2'^{\,T} & \boldsymbol{e}_3^T & \mathbf{0} \end{bmatrix}, \quad \delta\boldsymbol{\varphi} = \begin{bmatrix} \delta\boldsymbol{u}_0' \\ \delta\boldsymbol{\theta} \\ \delta\boldsymbol{e}_1 \\ \delta\boldsymbol{e}_2 \\ \delta\boldsymbol{e}_3 \\ \delta\boldsymbol{e}_2' \\ \delta\boldsymbol{e}_3' \end{bmatrix}. \tag{37}$$

The derivation of the geometrical stiffness terms requires the obtention of the linearization of the virtual generalized small strain vector, which gives:

$$\Delta\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \delta\boldsymbol{x}_0' \cdot \Delta\boldsymbol{e}_1 + \Delta\boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_1 + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_1 \\ \Delta\delta\boldsymbol{e}_1 \cdot \boldsymbol{e}_3' + \delta\boldsymbol{e}_1 \cdot \Delta\boldsymbol{e}_3' + \Delta\boldsymbol{e}_1 \cdot \delta\boldsymbol{e}_3' + \boldsymbol{e}_1 \cdot \Delta\delta\boldsymbol{e}_3' \\ \Delta\delta\boldsymbol{e}_1 \cdot \boldsymbol{e}_2' + \delta\boldsymbol{e}_1 \cdot \Delta\boldsymbol{e}_2' + \Delta\boldsymbol{e}_1 \cdot \delta\boldsymbol{e}_2' + \boldsymbol{e}_1 \cdot \Delta\delta\boldsymbol{e}_2' \\ \delta\boldsymbol{x}_0' \cdot \Delta\boldsymbol{e}_2 + \Delta\boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_2 + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_2 \\ \delta\boldsymbol{x}_0' \cdot \Delta\boldsymbol{e}_3 + \Delta\boldsymbol{x}_0' \cdot \delta\boldsymbol{e}_3 + \boldsymbol{x}_0' \cdot \Delta\delta\boldsymbol{e}_3 \\ \Delta\delta\boldsymbol{e}_2' \cdot \boldsymbol{e}_3 + \delta\boldsymbol{e}_2' \cdot \Delta\boldsymbol{e}_3 + \Delta\boldsymbol{e}_2' \cdot \delta\boldsymbol{e}_3 + \boldsymbol{e}_2' \cdot \Delta\delta\boldsymbol{e}_3 \end{bmatrix}. \tag{38}$$

Having derived the expressions for the virtual strains it is now possible to obtain the variational equilibrium equations of the beam. For the sake of shortness we skip this derivation and present s brief assessment of the results given by the implementation of the large deformation-small strain formulation.

4 NUMERICAL TESTS

In order to evaluate the accuracy of the present formulation we show the evolution of the displacement field and the generalized strains of a curved cantilever beam subjected to a tip oblique load $\mathrm{P} = \{4.0 \times 10^4, -4.0 \times 10^4, 8.0 \times 10^4\}$. The curved beam has a reference configuration defined by a 45º circular segment lying in the *xy* plane with radius $R = 100$, see Figure 2.

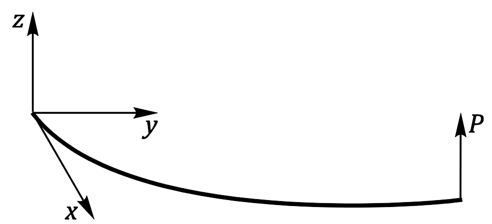

Figure 2. 45º Cantilever beam.

The material properties of the composite beam are listed in Table 1; the cross section of the beam is boxed with sides of length $b = h = 1$ and thickness $t = 0.1$.

E_{11}	E_{22}	G_{12}	G_{23}	ν_{12}
45.0×10^9	12.0×10^9	5.5×10^9	5.5×10^9	0.3

Table 1– Material properties of EFG-Epoxy layers.

The load generates a large deformation behavior, thus the ability of the large deformation-small strain formulation can be addressed by a close comparison with the large deformation-large strain formulation presented in (Saravia, Machado et al., 2012). As it can be seen from the evolution of the displacement field at the beam tip (see Figure 3), the present formulation behaves very well for the large deformation case.

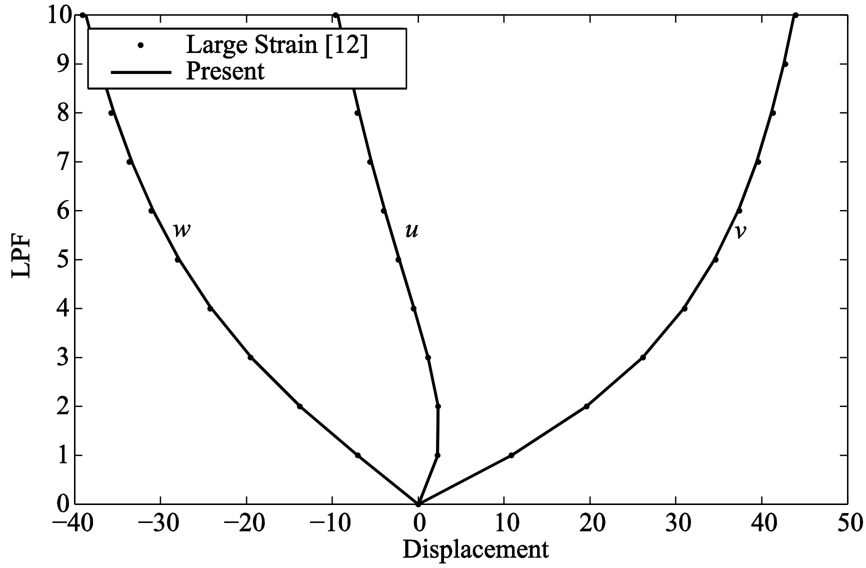

Figure 3. 45º Cantilever Beam – Evolution of displacements

The Figures 4 and 5 show the progression of the generalized small strain components. It can be observed that the present formulation behaves relatively well still for moderate deformation cases.

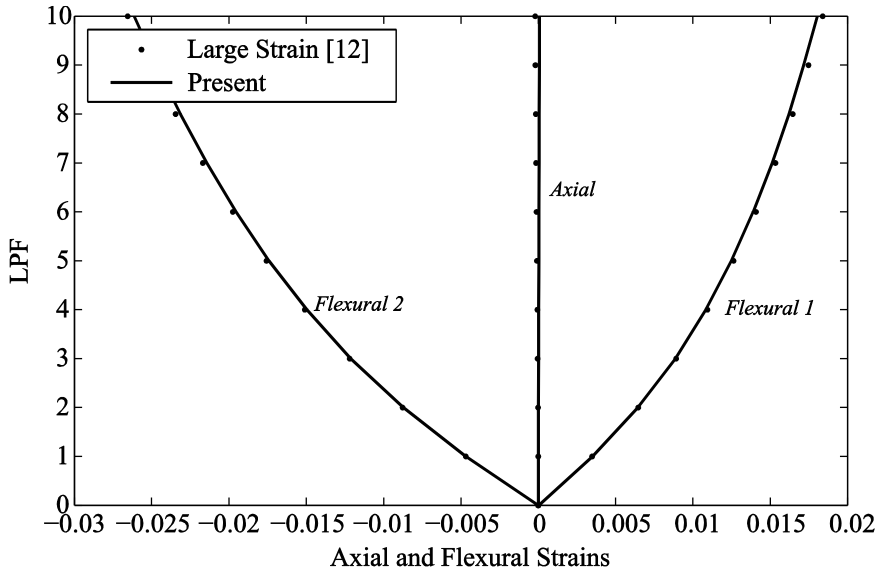

Figure 4. 45º Cantilever Beam – Evolution of strains

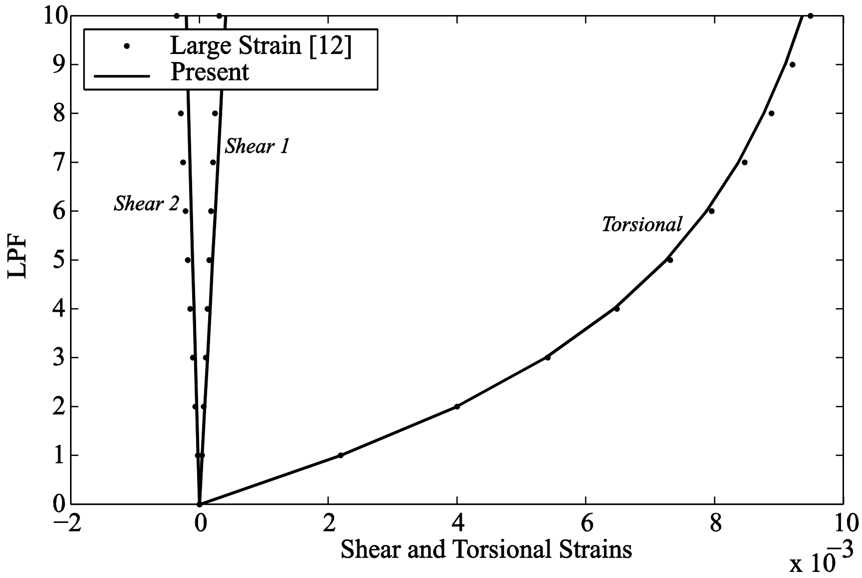

Figure 5. 45º Cantilever Beam – Evolution of strains

5 CONCLUSIONS

A formulation for the strain measure for studying large deformation-small strain cases of thin walled composite beams has been derived. The development was carried out with base in the geometrically exact thin-walled composite beam theory formulated previously by the authors. The formulation results in a simpler implementation; the evaluation of the element stiffness matrix is considerably cheaper in terms of computational cost.

The preliminary results show the new strain measure can successfully handle situations where both displacements and rotations are large. Although not presented in this work, the new discrete strain measures can be proved to be objective under observer transformations.

6 ACKNOWLEDGEMENTS

The authors wish to acknowledge the supports from Secretaría de Ciencia y Tecnología of Universidad Tecnológica Nacional and CONICET.

REFERENCES

Auricchio, F., P. Carotenuto and A. Reali. On the geometrically exact beam model: A consistent, effective and simple derivation from three-dimensional finite-elasticity. *International Journal of Solids and Structures* 45: 4766-4781, 2008.

Cardona, A. and M. Geradin. A beam finite element non-linear theory with finite rotations. *International Journal for Numerical Methods in Engineering* 26: 2403-2438, 1988.

Cortínez, V. H. and M. T. Piovan. Vibration and buckling of composite thin-walled beams with shear deformability. *Journal of Sound and Vibration*: 701–723, 2002.

Cortínez, V. H. and M. T. Piovan. Stability of composite thin-walled beams with shear deformability *Computers and Structures* 842006.

Hodges, D. H. *Nonlinear Composite Beam Theory*. Virginia, American Institute of Aeronautics and Astronautics, Inc., 2006.

Ibrahimbegovic, A. On the choice of finite rotation parameters. *Computer Methods in Applied Mechanics and Engineering* 149: 49-71, 1997.

Librescu, L. *Thin-Walled Composite Beams*. Dordrecht, Springer, 2006.

Machado, S. P. and V. H. Cortínez. Non-linear model for stability of thin-walled composite beams with shear deformation. *Thin-Walled Structures* 43: 1615-1645, 2005.

Machado, S. P. and V. H. Cortínez. Free vibration of thin-walled composite beams with static initial stresses and deformations. *Engineering Structures* 29: 372-382, 2007.

Pi, Y.-L. and M. A. Bradford. Effects of approximations in analyses of beams of open thin-walled cross-section - part I: Flexural-torsional stability. *International Journal for Numerical Methods in Engineering* 51: 757-772, 2001.

Pi, Y.-L., M. A. Bradford and B. Uy. Nonlinear analysis of members curved in space with warping and Wagner effects. *International Journal of Solids and Structures* 42: 3147-3169, 2005.

Pi, Y. L. and M. A. Bradford. Effects of approximations in analyses of beams of open thin-walled cross-section—part II: 3-D non-linear behaviour. *International Journal for Numerical Methods in Engineering* 51: 773-790, 2001.

Piovan, M. T. and V. H. Cortínez. Mechanics of thin-walled curved beams made of composite materials, allowing for shear deformability. *Thin-Walled Structures* 45: 759-789, 2007.

Saravia, C. M., S. P. Machado and V. H. Cortínez. A Geometrically Exact Nonlinear Finite Element for Composite Closed Section Thin-Walled Beams. *Computer and Structures* 89: 2337-2351, 2011.

Saravia, C. M., S. P. Machado and V. H. Cortínez. A composite beam finite element for multibody dynamics: Application to large wind turbine modeling. *Engineering Structures* 56: 1164-1176, 2013.

Saravia, M. C., S. P. Machado and V. H. Cortínez. A consistent total Lagrangian finite element for composite closed section thin walled beams. *Thin-Walled Structures* 52: 102-116, 2012.

Simo, J. C. A finite strain beam formulation. The three-dimensional dynamic problem. Part I. *Computer Methods in Applied Mechanics and Engineering* 49: 55-70, 1985.

Simo, J. C. and L. Vu-Quoc. A three-dimensional finite-strain rod model. part II: Computational aspects. *Computer Methods in Applied Mechanics and Engineering* 58: 79-116, 1986.

Vlasov, V. Z. *Thin-Walled Elastic Beams*. Jerusalem, Israel, Israel Program for Science Translation, 1961.