Journal of Topology Advance Access published August 6, 2013

Journal of Topology (2013) Page 1 of 40
doi:10.1112/jtopol/jtt026

The K-theory of toric varieties in positive characteristic

G. Cortiñas, C. Haesemeyer, Mark E. Walker and C. Weibel

Abstract

We show that if X is a toric scheme over a regular ring containing a field of finite characteristic, then the direct limit of the K-groups of X taken over any infinite sequence of non-trivial dilations is homotopy invariant. This theorem was known in characteristic 0. The affine case of our result was conjectured by Gubeladze.

Contents

Introduction

Let X be a toric variety over a field k. For each integer $c \geqslant 2$, multiplication by c on the lattice associated to X induces an endomorphism θ_c on X, called a *dilation*. For example, if $k = \mathbb{Z}/p$, the endomorphism θ_p coincides with the Frobenius map. Locally, θ_c is determined by its action on affine space: $\theta_c(a_1, a_2, \ldots) = (a_1^c, a_2^c, \ldots)$. Each sequence $(c_1, c_2, \ldots)$ of integers $\geqslant 2$ yields a sequence of dilations on X and hence a sequence of endomorphisms θ_{c_i} of its K-theory $K_*(X)$ and its homotopy K-theory $KH_*(X)$. The *Dilation theorem* for K-theory states that the resulting 'dilated' K-theory and KH-theory agree on toric varieties:

$$\varinjlim_{\theta_c} K_*(X) \xrightarrow{\cong} \varinjlim_{\theta_c} KH_*(X). \tag{0.1}$$

When k has characteristic 0, this was proved by Gubeladze for affine toric varieties in [**16**], and in full generality by the authors in [**8**].

If R is a commutative regular k-algebra, we may consider (0.1) with X replaced by $X_R = X \times_k \operatorname{Spec}(R)$. The main result of this paper is that the Dilation theorem (0.1) holds for X_R, even at the level of spectra. This is proved in characteristic 0 in Theorem 7.5 and in finite characteristic in Theorem 8.3.

Received 12 July 2012; revised 21 February 2013.

2010 *Mathematics Subject Classification* 14F20, 19E08 (primary), 19D55 (secondary).

G. Cortiñas' research was supported by Conicet and partially supported by grants UBACyT W386, PIP 112-200801-00900 and MTM2007-64704 (Feder funds). C. Haesemeyer's research was partially supported by NSF grant DMS-0966821. Mark E. Walker's research was partially supported by NSF grant DMS-0966600. C. Weibel's research was supported by NSA and NSF grants.

Putting the affine cases of these results together, and taking the direct limit over finitely generated submonoids, we obtain Theorem 0.2 (8.4 below); it settles a conjecture of Gubeladze [**16**, 1.1] about monoid algebras $k[A]$ when k is any regular ring. The affine case in characteristic 0 was proved by Gubeladze [**17**].

THEOREM 0.2. *Let A be a cancellative, torsion-free commutative monoid with no non-trivial units. Then, for every sequence $\mathfrak{c} = (c_1, c_2, \ldots)$ of integers $\geqslant 2$ and every regular ring k containing a field, there is an isomorphism*

$$K_*(k) \xrightarrow{\cong} \varinjlim_{\theta_c} K_*(k[A]).$$

Our proof of the Dilation theorem follows the geometric approach used by the authors in [**8**]. That proof used the Chern character from K-theory to cyclic homology to reduce everything to a problem about the cohomology of Kähler differentials, which could be solved by explicit calculation using Danilov's sheaves $\tilde{\Omega}$.

In order to adapt the strategy of that proof, two main difficulties have to be overcome. First, resolution of singularities is not available in positive characteristic, and the usual combinatorial resolution of singularities for toric varieties is not good enough to control the effect on algebraic K-theory and similar invariants. Overcoming this difficulty makes it necessary to study the geometry of monoid schemes and their resolutions; this was done in the companion paper [**9**] using the Bierstone–Milman theorem. Secondly, the 'correct' Chern character to use in positive characteristic is the cyclotomic trace; its target is topological cyclic homology, which is much more complicated than cyclic homology. As a result, it is not sufficient (as in characteristic 0) to study the cohomology of Kähler differentials and Danilov's sheaves $\tilde{\Omega}$; instead, we need to study a homotopy theoretical version of Danilov's sheaves of differentials. The bulk of this paper is dedicated to that task.

From an aesthetic point of view, the proof contained in this paper and its companion [**9**] has the advantage that, at its heart, it is a theorem about monoid schemes—algebraic data (such as the base field) only enter the story in its very late stages. It is an intriguing question whether the original problem—the Dilation theorem—can be formulated (and proved) completely within the world of monoid schemes (or schemes over 'the field with one element'). Such a formulation would describe a kind of K-theory functor for monoid schemes, one which is not homotopy invariant. (All of the current candidates for such a K-theory are homotopy invariant in a strong way; dilations on affine space induce equivalences, so the analogs of (0.1) become trivial.)

Here is a more detailed overview of the content of this paper. Section 1 recalls various notions from the world of monoid schemes as developed in the companion paper [**9**]. The notions of a partially cancellative monoid (respectively, monoid scheme), seminormal monoid (respectively, monoid scheme) and the process of seminormalization are of special importance for this paper. We remark that toric monoids are torsion-free cancellative and normal so, in particular, they are pctf and seminormal. Section 2 recalls cyclic sets and their realizations as well as the cyclic bar construction on monoids, N^{cy}, as used by Bökstedt, Hsiang and Madsen [**3**] in their construction of topological cyclic homology. Section 3 studies the effect inverting a sequence of non-trivial dilations has on the cyclic nerve of a monoid. We introduce $\tilde{N}^{cy}$, a variant of the cyclic bar construction in Definition 3.1 that has better technical properties with regard to ideals of monoids but is equivalent to the usual cyclic bar construction after inverting a sequence of dilations.

In Section 4, we introduce the presheaf of $\mathbb{S}^1$-spaces $\tilde{\Omega}$ that is crucial for our proof of the main theorem. This presheaf, whose definition is based on the variant $\tilde{N}^{cy}$ of the cyclic bar construction, plays the role that Danilov's sheaves of differentials played in our proof of the dilation theorem in characteristic 0 in [8]. In Proposition 4.4, we show that $\tilde{\Omega}$ satisfies excision for ideals of monoids. It is here that working with seminormal monoids is critical. In Section 5, we build on our work in [9] and prove a technical result (Proposition 5.5) about recognizing when a presheaf of spectra satisfies *cdh* descent adapted to the presheaves we study in this paper.

In Section 6, we apply the results of Sections 4 and 5 to prove that presheaves of spectra obtained from smashing a (fixed) $\mathbb{S}^1$-spectrum with $\tilde{\Omega}$ and then taking fixed points for a finite subgroup of $\mathbb{S}^1$ satisfy *cdh* descent. The main theorem of Section 6 (Theorem 6.2) forms the technical core of this article; its proof is inspired by Danilov's proof that his sheaves satisfy *cdh* descent (cf. [10], although of course he does not formulate his result this way). As a consequence we conclude in Corollary 6.6 that the 'dilated' topological cyclic homology $\varinjlim_{\theta_c} TC^n(-;p)$ satisfies *cdh* descent.

In Section 7, we prove the Dilation theorem when k is a regular $\mathbb{Q}$-algebra. Finally, in Section 8, we combine Corollary 6.6 with the main result of [9] to prove our main theorem, the Dilation theorem 8.3 when k is a regular $\mathbb{F}_p$-algebra and X is a pctf monoid scheme. We conclude by establishing Gubeladze's conjecture (Theorem 0.2) in Corollary 8.4.

1. *Monoids*

In this section, we present the basic facts about monoids and monoid schemes that we shall need. We refer the reader to [9] for more details.

Unless otherwise stated, in this paper a *monoid* will mean a pointed abelian monoid; that is, an abelian monoid object in the category of pointed sets. More explicitly, it is a pointed set A equipped with a pairing $\mu : A \wedge A \to A$ that is associative and commutative and has an identity element.

We usually write the pairing μ of A as multiplication $\cdot$, in which case the identity element is written as 1 and the base point as 0, so that $1 \cdot a = a = a \cdot 1$ and $0 \cdot a = 0 = a \cdot 0$ for all $a \in A$. For example, if R is a commutative ring, then we have the underlying multiplicative monoid $(R, \cdot)$. If A is an unpointed abelian monoid (that is, an abelian monoid object in the category of sets), we write A_* for the pointed abelian monoid formed by adjoining a base point.

A monoid morphism $f : A \to B$ is a function preserving multiplication, with $f(0) = 0$, $f(1) = 1$. If B is the finite union of subsets Ab_i, $b_i \in B$, we say that f is *finite*. For a multiplicatively closed subset S of A (which may or may not contain 0), there is a localization morphism $A \to S^{-1}A$, where the monoid $S^{-1}A$ is defined as the set of fractions a/s with $a \in A$ and $s \in S$ with the usual equivalence relation.

Here is some standard terminology. An *ideal* I in a monoid A is a pointed subset such that $AI \subseteq I$. If $I \subset A$ is an ideal, A/I is the monoid obtained by collapsing I to 0. A proper ideal is *prime* if its complement $S = A \backslash \mathfrak{p}$ is multiplicatively closed. In this case, we write $S^{-1}A$ as $A_{\mathfrak{p}}$.

For a monoid A, we write $\mathrm{MSpec}(A)$ for the set of its prime ideals. We equip $\mathrm{MSpec}(A)$ with the 'Zariski' topology, whose closed subsets are those of the form

$$V(I) := \{\mathfrak{p} \in \mathrm{MSpec}(A) \mid I \subset \mathfrak{p}\},$$

where I is an ideal of A. In contrast with the analogous setting of rings, every monoid has a unique maximal ideal, given as the complement of the set of units. The set $\mathrm{MSpec}(A)$ comes equipped with a sheaf of monoids $\mathcal{A}$, whose stalk at $\mathfrak{p} \in \mathrm{MSpec}(A)$ is $A_{\mathfrak{p}}$. As explained below, $\mathrm{MSpec}(A)$ is an affine 'monoid scheme'.

Given an ideal I of A, define its radical to be

$$\sqrt{I} := \{a \in A \mid a^n \in I \text{ for some } n \geqslant 1\}.$$

An ideal I is *radical* if $I = \sqrt{I}$. It is easy to prove (using Zorn's Lemma) that $\sqrt{I}$ is the intersection of the prime ideals containing I. The *nilradical* of a monoid A is the radical of the zero ideal

$$\begin{aligned} \operatorname{nil}(A) &:= \sqrt{\{0\}} \\ &= \{a \in A \mid a^n = 0, \text{ for some } n \geqslant 1\}. \end{aligned} \tag{1.1}$$

Equivalently, $\operatorname{nil}(A)$ is the intersection of all the prime ideals of A. For general monoids, the passage from A to $A/\operatorname{nil}(A)$ is not as useful as the corresponding notion for commutative rings, so we introduce a slightly stronger notion.

A monoid A is said to be *reduced* if whenever $a^2 = b^2$ and $a^3 = b^3$ for some $a, b \in A$, then $a = b$. Equivalently, A is reduced if and only if whenever $a^n = b^n$ for all $n \gg 0$, we have $a = b$. For any A, write A_{red} for the monoid obtained by modding out A by the congruence relation $a \sim b$ if $a^n = b^n$ for all $n \gg 0$. It is clear that the canonical surjection $A \twoheadrightarrow A_{\text{red}}$ is universal among maps from A to reduced monoids. If A is reduced, then $\operatorname{nil}(A) = 0$ because $x^n = 0$ implies $x = 0$. The converse does not hold for all monoids, but we shall see that it holds for pc monoids (defined below) by Proposition 1.6. (In [**9**], we work with the weaker notion of reduced. The referenced proposition ensures that this is not a problem.)

1.1. *Cancellative and partially cancellative monoids*

A (pointed commutative) monoid C is *cancellative* if whenever $ac = bc$ for some $c \neq 0$, we have $a = b$. Equivalently, C is cancellative if $C \setminus \{0\}$ is an unpointed monoid that maps injectively to its group completion $(C \setminus \{0\})^+$. In this situation, the *pointed group completion* of C is

$$C^+ := ((C \setminus \{0\})^+)_*.$$

We say a monoid A is *torsion-free* if whenever $a^n = b^n$ for $a, b \in A$ and some $n \geqslant 1$, we have $a = b$. If C is cancellative, then C is torsion-free if and only if $C^+ \setminus \{0\}$ is a torsion-free abelian group.

Example 1.2. Every cancellative monoid C is reduced. Indeed, if $x^n = y^n$ for all $n \gg 0$, then $y^n y = y^{n+1} = x^{n+1} = x^n x = y^n x$ for all $n \gg 0$ and hence $x = y$.

If A is a cancellative monoid, the *normalization* of A, written as A_{nor}, is the submonoid of the pointed group completion A^+ of A consisting of all elements $\alpha \in A^+$ such that $\alpha^n \in A$ for some $n \geqslant 1$. We say A is *normal* if $A = A_{\text{nor}}$. It is easy to see (and proved in [**9**, 1.6.1]) that $\operatorname{MSpec}(A_{\text{nor}}) \to \operatorname{MSpec}(A)$ is a homeomorphism. The normalization of non-cancellative monoids is not defined.

Definition 1.3. A monoid A is *partially cancellative*, or *pc* for short, if A is isomorphic to C/I where C is a cancellative monoid and I is an ideal of C.

A monoid A is *partially cancellative and torsion-free*, or *pctf* for short, if A is isomorphic to C/I where C is a torsion-free cancellative monoid and I is an ideal of C.

Remark 1.4. The name *partially cancellative* is inspired by the following observation: If $ac = bc$ in a pc monoid, then either $ac = bc = 0$ or $a = b$.

To see this, suppose $A = C/I$ with C cancellative. Given $a, b, c \in C$, if $ac = bc$ holds in A, then either $ac, bc \in I$ or $ac = bc$ in C and hence $a = b$.

Proposition 1.5. *Let $\mathfrak{p}$ be a prime ideal in a pc monoid A. Then the following conditions are satisfied*:

(1) *$A/\mathfrak{p}$ is a cancellative monoid*;
(2) *if A is pctf, then $(A/\mathfrak{p})^+$ is a torsion-free pointed group*;
(3) *if A is cancellative and normal, then so is $A/\mathfrak{p}$.*

Proof. Say $A = C/I$ with C cancellative and $\mathfrak{p} = \mathfrak{p}'/I$. Then $A \setminus \mathfrak{p} = C \setminus \mathfrak{p}'$, so assertion (1) follows from the observation that, for $a, b, c \in C$ with $c \notin \mathfrak{p}'$, if $ac, bc \in \mathfrak{p}'$, then $a, b \in \mathfrak{p}'$. If in addition C^+ is torsion-free, suppose that $a^n = b^n$ in $A/\mathfrak{p}$. Then either $a^n = b^n$ in C or $a^n, b^n \in \mathfrak{p}$. In either case we must have $a = b$ in $A/\mathfrak{p}$ since $\mathfrak{p}$ is prime and C is torsion-free. This proves assertion (2). For (3), suppose that $a, b, c \in A \setminus \mathfrak{p}$ satisfy $(a/b)^n = c$ in $(A/\mathfrak{p})^+$. Then $a^n = b^n c$ in the cancellative monoid $A/\mathfrak{p}$, and hence in A. When A is normal, this implies that $a = bx$ for some $x \in A$; since $a = bx$ also holds in $A/\mathfrak{p}$, we have $a/b \in A/\mathfrak{p}$. □

Proposition 1.6. *Assume that A is a pc monoid*; *A is reduced if and only if* $\mathrm{nil}(A) = 0$. *Moreover*, $A_{red} = A/\mathrm{nil}(A)$.

Proof. Suppose $A = C/I$ with C cancellative, $\mathrm{nil}(A) = 0$ and $x^n = y^n$ for all $n \gg 0$. Then either $x^n = y^n = 0$ for all $n \gg 0$ or $x^n = y^n$ holds in C for all $n \gg 0$. In the former case, $x = y = 0$ since $\mathrm{nil}(A) = 0$; in the latter case $x = y$ holds in C and hence in A since C is reduced by Example 1.2. The final assertion is an immediate consequence. □

Remark 1.6.1. Propositions 1.5 and 1.6 fail when the monoid A is not pc, because the quotient monoid $A/\mathfrak{p}$ by a prime ideal $\mathfrak{p}$ need not be cancellative, or reduced.

Seminormal monoids. The notion of a seminormal monoid plays a central role in this paper. Many of our structural results have parallels in the theory of seminormal rings.

Definition 1.7. A monoid A is *seminormal*, provided that A is reduced and whenever $x, y \in A$ satisfy $x^3 = y^2$, there is a $z \in A$ such that $x = z^2, y = z^3$. Since A is reduced, such a z is unique.

Example 1.8. If M is an abelian group, then the associated pointed monoid M_* is seminormal: Given $x, y \in M_*$ with $x^3 = y^2$, either $x = y = 0$ or $x = z^2, y = z^3$ where $z = y/x$. If A is a submonoid of M_* and $x, y \in A$, then $z \in A_{\mathrm{nor}}$ because z^2 is in A. Thus normal monoids are seminormal.

Lemma 1.9. *If A is seminormal and S is a multiplicatively closed subset, then $S^{-1}A$ is seminormal.*

Proof. Say $x, y \in S^{-1}A$ satisfy $x^3 = y^2$. We may assume $x = a/s,\ y = b/s$ for $a, b \in A$, $s \in S$. Then $a^3 s^2 u = b^2 s^3 u$ for some $u \in S$ and so

$$(asu^2)^3 = a^3 s^2 u (su^5) = b^2 s^3 u (su^5) = (bs^2u^3)^2.$$

Since A is seminormal, there is a $c \in A$ with

$$asu^2 = c^2, \quad bs^2u^3 = c^3$$

and hence if we set $z = c/(su)$, we have

$$x = a/s = c^2/(s^2u^2) = z^2, \quad y = b/s = c^3/(u^3s^3) = z^3. \qquad \square$$

Lemma 1.10. *Suppose that A is a pc seminormal monoid and I is an ideal of A. The monoid A/I is seminormal if and only if I is a radical ideal.*

Proof. One implication is obvious, so suppose I is radical and $x^3 = y^2$ holds in A/I. Either $x^3 = y^2 = 0$ (in which case $x = y = 0$ since A/I is reduced by Proposition 1.6), or $x^3 = y^2$ holds in A. In this case, since A is seminormal, the equations $x = z^2$, $y = z^3$ hold in A for some $z \in A$, and hence we obtain such equations in A/I as well. $\square$

1.2. *Seminormalization*

For a monoid A, a *seminormalization* of A is a seminormal monoid B, together with a map of monoids $A \to B$ such that the induced map $A_{\text{red}} \to B$ is injective, and, for every $b \in B$, we have $b^n \in A_{\text{red}}$ for all $n \gg 0$.

Seminormalizations are universal with respect to maps from A to seminormal monoids, as we now show.

Lemma 1.11. *If $i : A \to B$ is a seminormalization of A and $f : A \to C$ is any map such that C is seminormal, there is a unique map $g : B \to C$ such that $g \circ i = f$. In particular, a seminormalization of a monoid is unique up to unique isomorphism.*

Proof. By the universal mapping property of A_{red}, we may assume that A is reduced, so that A may be regarded as a submonoid of B. Consider all pairs (B', h) with $A \subset B' \subset B$ and $h : B' \to C$ a map such that $h|_A = f$. This is an ordered set in the evident way and by Zorn's lemma there is a maximal element (B', h). It suffices to prove $B' = B$.

If not, we may find a $b \in B \setminus B'$ such that $b^2, b^3 \in B'$. Let $x = h(b^2), y = h(b^3)$, so that $x^3 = h(b^6) = y^2$. Since C is seminormal, there is a z with $x = z^2, y = z^3$. Consider $B'' = B'[b]$, the smallest submonoid of B containing B' and b. Every element of B'' may be written (non-uniquely) either as an element of B' or as yb with $y \in B'$.

Consider the set $B'b$ of all elements of B'' of the form $t = yb$ with $y \in B'$. We claim that, for each t, the element $h(y)z$ of C is independent of the choice of y. If $y_1b = y_2b$ for $y_1, y_2 \in B'$, then, for all $m \geqslant 2$,

$$(h(y_1)z)^m = h(y_1^m b^m) = h(y_2^m b^m) = (h(y_2)z)^m.$$

Since C is reduced, it follows that $h(y_1)z = h(y_2)z$, as claimed. Thus $h''(t) = h(y)z$ is a well-defined element of C. If $t \in B'$, then $h''(t)^n = h(y^n)z^n = h(yb)^n = h(t)^n$, whence $h''(t) = h(t)$. This shows that h extends to a function $h'' : B'' \to C$. A similar argument shows h'' is a homomorphism of monoids, contradicting the maximality of (B', h). Thus $B' = B$.

For the uniqueness, if $g_1, g_2 : B \to C$ are two such maps, then, for each $b \in B$, we have $b^n \in A$ for all $n \gg 0$ and hence $g_1(b)^n = f(b^n) = g_2(b)^n$ for all $n \gg 0$. Since C is reduced, $g_1 = g_2$. $\square$

If A has a seminormalization, then, by Lemma 1.11, we are justified in calling it *the* seminormalization of A, and we write it as A_{sn}. It is clear that the canonical map $A \to A_{\text{sn}}$ is an isomorphism if (and only if) A is seminormal, and hence that $A_{\text{sn}} \xrightarrow{\cong} (A_{\text{sn}})_{\text{sn}}$ is always an isomorphism.

Example 1.12. If A is cancellative with pointed group completion A^+, then

$$A_{\text{sn}} = \{f \in A^+ \mid \; f^n \in A \text{ for all } n \gg 0\}$$

(with the canonical inclusion $A \rightarrowtail A_{\text{sn}}$) is the seminormalization of A. This follows from Example 1.8, which shows that A_{nor} is seminormal. Thus $A \subseteq A_{\text{sn}} \subseteq A_{\text{nor}}$.

Lemma 1.13. *For any monoid A, if its seminormalization $A \to A_{\text{sn}}$ exists, then, for any multiplicatively closed subset S of A, the map $S^{-1}A \to S^{-1}A_{\text{sn}}$ is the seminormalization of $S^{-1}A$.*

Proof. We have $(S^{-1}A)_{\text{red}} \cong S^{-1}(A_{\text{red}})$ and the map $S^{-1}(A_{\text{red}}) \to S^{-1}A_{\text{sn}}$ is injective, since localization preserves injections. By Lemma 1.9, $S^{-1}(A_{\text{sn}})$ is seminormal. Finally, if $x \in S^{-1}(A_{\text{sn}})$, then $x^n \in S^{-1}A$ for all $n \gg 0$. ☐

Lemma 1.14. *Let A be a pc monoid. Assume that its seminormalization $A \to A_{\text{sn}}$ exists and let I be any ideal of A. Set $J = \sqrt{IA_{\text{sn}}}$. Then the induced map*

$$A/I \longrightarrow A_{\text{sn}}/J$$

is the seminormalization of A/I.

Proof. The map $A \to A_{\text{sn}}$ factors through A_{red} and $A/I \to A_{\text{sn}}/J$ factors through $(A/I)_{\text{red}} = A/\sqrt{I}$. These factorizations allow us to assume that A is reduced and I is radical.

We claim that $J \cap A = I$. Clearly $I \subseteq J \cap A$. If $a \in J \cap A$, then $a^n \in IA_{\text{sn}}$ for all large n. For any $b \in A_{\text{sn}}$ we have $b^n \in A$ for all $n \gg 0$. It follows that $a^m \in I$ for some m (and even for all $m \gg 0$). Since I is radical in A, we get $a \in I$.

Since $J \cap A = I$, the map $A/I \to A_{\text{sn}}/J$ is injective. Given $y \in A_{\text{sn}}/J$, it is clear that $y^n \in A/I$ for all $n \gg 0$. Finally, A_{sn}/J is seminormal by Lemma 1.10. ☐

Proposition 1.15. *If A is pc, then its seminormalization $A \to A_{\text{sn}}$ exists and A_{sn} is also a pc monoid. If A is pctf, then so is A_{sn}.*

Moreover, there is a functor $(-)_{\text{sn}}$ from the category of pc monoids to the category of seminormal pc monoids, sending a monoid to its seminormalization. The maps $A \to A_{\text{sn}}$ determine a natural transformation from the identity to $(-)_{\text{sn}}$.

Proof. By assumption $A = C/I$ for a cancellative monoid C. By Example 1.12, C_{sn} exists; by Lemma 1.14, A_{sn} exists and $A_{\text{sn}} = C_{\text{sn}}/J$ with $J = \sqrt{IC_{\text{sn}}}$. Since C_{sn} is cancellative, this shows that A_{sn} is pc. If C is also torsion-free, then C_{sn} is torsion-free and hence A_{sn} is pctf.

Choose, once and for all, a seminormalization $A \to A_{\text{sn}}$ for each pc monoid A. The universal mapping property (Lemma 1.11) shows that, given a morphism $f : A \to B$ of pc monoids, there is a unique map $f_{\text{sn}} : A_{\text{sn}} \to B_{\text{sn}}$ causing the evident square to commute. The assignments $A \mapsto A_{\text{sn}}$ and $f \mapsto f_{\text{sn}}$ determine a functor and the maps $A \to A_{\text{sn}}$ form a natural transformation, as claimed. ☐

Lemma 1.16. *For any pc monoid A, the map* $\operatorname{MSpec}(A_{\text{sn}}) \to \operatorname{MSpec}(A)$ *of affine monoid schemes is a homeomorphism on underlying topological spaces.*

Proof. Since $\mathrm{MSpec}(A) \cong \mathrm{MSpec}(A_{\mathrm{red}})$, we may assume A reduced. The function $\mathrm{MSpec}(A) \to \mathrm{MSpec}(A_{\mathrm{sn}})$ sending $\mathfrak{p}$ to $\tilde{\mathfrak{p}} := \{x \in A_{\mathrm{sn}} \mid x^n \in \mathfrak{p} \text{ for } n \gg 0\}$ is a continuous inverse of the canonical map $\mathrm{MSpec}(A_{\mathrm{sn}}) \to \mathrm{MSpec}(A)$. □

Recall from [**9**, after Remark 2.9.1] that finitely generated monoids are *noetherian*: they satisfy the ascending chain condition on ideals, and every ideal is finitely generated.

LEMMA 1.17. *If A is a finitely generated pc monoid, then $A \to A_{\mathrm{sn}}$ is a finite morphism.*

Proof. Suppose first that A is cancellative; by Cortiñas, Haesemeyer, Walker and Weibel [**9**, 6.3], $A \to A_{\mathrm{nor}}$ is finite, so A_{nor} is given as a finite union $\cup Ac_i$. Since A is finitely generated, so is every ideal of A, including $J_i = \{a \in A : ac_i \in A_{\mathrm{sn}}\}$. Writing J_i as a finite union $J_i = \cup Ab_{ij}$, we have $x_{ij} := b_{ij}c_i \in A_{\mathrm{sn}}$ and $A_{\mathrm{sn}} = \cup Ax_{ij}$. This proves that $A \to A_{\mathrm{sn}}$ is finite. (Cf. [**5**, 2.31].)

In the general case, write A as C/I with C cancellative; replacing C by a submonoid, we may assume that C is finitely generated (see [**9**, Proof of part (1) of 9.1]). By Lemma 1.14, $A_{\mathrm{sn}} = C_{\mathrm{sn}}/J$ for an ideal J. Since $C \to C_{\mathrm{sn}}$ is finite, $A \to A_{\mathrm{sn}}$ is also finite. □

The *conductor ideal* for an inclusion $A \subset B$ is the ideal I of A consisting of those $a \in A$ with $B \cdot a \subset A$. Observe that I is also an ideal of B. If $A \subset B$ is finite and $S \subset A$ is multiplicatively closed, the conductor of $S^{-1}A \subset S^{-1}B$ is $S^{-1}I$.

EXAMPLE 1.18. Let A denote the submonoid of the free monoid $\langle x, y\rangle$ generated by x^2, xy, y^2, x^2y, xy^2. Then A is seminormal and $A_{\mathrm{nor}} = \langle x, y\rangle$. The conductor ideal for $A \subset A_{\mathrm{nor}}$ is $I = \{x^m y^n : mn > 0\}$.

LEMMA 1.19. *Suppose that A is seminormal and cancellative. Then the conductor ideal I for the inclusion $A \rightarrowtail A_{\mathrm{nor}}$ is a radical ideal of both A and A_{nor}.*

Proof. It suffices to prove that I is radical in A_{nor}. Say $\alpha \in A_{\mathrm{nor}}$ and $\alpha^n \in I$. Since I is an ideal of A_{nor}, we have $\alpha^m \in I$ for all $m \geqslant n$ and hence, since A is seminormal, $\alpha \in A$. For each $\beta \in A_{\mathrm{nor}}$ and all $m \geqslant n$ we have

$$(\beta \cdot \alpha)^m = (\beta^m \alpha^{m-n})\alpha^n \in A,$$

since $\alpha^n \in I$. Since A is seminormal, we have $\beta \cdot \alpha \in A$, proving that $\alpha \in I$. □

REMARK 1.19.1. If A is a finitely generated and cancellative monoid, $A \subset A_{\mathrm{nor}}$ is finite; see [**9**, 6.3]. If I is the conductor and S is multiplicatively closed in A, it follows that $S^{-1}I$ is the conductor of $S^{-1}A \subset S^{-1}A_{\mathrm{nor}}$. Therefore the conductor defines a sheaf of ideals on $\mathrm{MSpec}(A)$.

Seminormal monoid schemes. By a *monoid scheme* we mean a topological space X, equipped with a sheaf of monoids $\mathcal{A}$, which can be covered by open subspaces isomorphic to $\mathrm{MSpec}(A)$ for some monoid A. We refer the reader to [**9**] for a more precise definition.

We now review the most relevant definitions. If $\mathcal{I}$ is a (quasi-coherent) sheaf of ideals of $\mathcal{A}$, there is a monoid subscheme Z whose topological space is $V(\mathcal{I})$ and whose sheaf of monoids is $\mathcal{A}/\mathcal{I}$. We call such a subscheme an *equivariant closed* subscheme. A monoid scheme X is of *finite type* if the underlying topological space is finite and every stalk $\mathcal{A}_x$ is a finitely generated monoid. Since finitely generated monoids are noetherian, monoid schemes of finite type have the descending chain condition on equivariant closed subschemes.

For technical reasons, we will need our monoid schemes to be *separated* in the sense of [**9**, 3.3]: X is separated if the diagonal $X \to X \times X$ is a closed immersion (see [**9**, 2.5]). This notion is parallel to its counterpart in algebraic geometry.

DEFINITION 1.20. A monoid scheme X is *pc* (respectively, *pctf*) if, for each x in X, the stalk monoid $\mathcal{A}_x$ is pc (respectively, pctf) in the sense of Definition 1.3. If $X = \mathrm{MSpec}(A)$, then X is pc if and only if A is pc by the proof of [**9**, 9.1]. We write $\mathcal{M}_{\mathrm{pc}}$ for the category consisting of separated pc monoid schemes of finite type, and $\mathcal{M}_{\mathrm{pctf}}$ for the full subcategory of separated pctf monoid schemes of finite type.

EXAMPLE 1.20.1. Recall from [**9**, 4.1] that a *toric monoid scheme* is a separated, connected, torsion-free, normal monoid scheme of finite type. Thus toric monoid schemes are in $\mathcal{M}_{\mathrm{pctf}}$.

In fact, a toric monoid scheme is the same thing as a connected normal scheme in $\mathcal{M}_{\mathrm{pctf}}$, since the stalks $\mathcal{A}_x$ of a normal pctf scheme are torsion-free by Proposition 1.5(2). We showed in [**9**, 4.4] that (up to isomorphism) any toric monoid scheme X uniquely determines a fan (N, Δ) and that X may be constructed from this fan.

In this paper, we shall also be interested in *seminormal* monoid schemes, by which we mean monoid schemes whose stalk monoids $\mathcal{A}_x$ are seminormal. Lemma 1.9 implies that an affine scheme $X = \mathrm{MSpec}(A)$ is seminormal if and only if A is seminormal. (A is the stalk at the unique closed point of X.) Lemma 1.16 implies that $\mathcal{A}_{\mathrm{sn}}$ is a sheaf of monoids on $X = \mathrm{MSpec}(A)$ and that $\mathrm{MSpec}(A_{\mathrm{sn}})$ is isomorphic to the monoid scheme $(\mathrm{MSpec}(A), \mathcal{A}_{\mathrm{sn}})$. It follows that for any pc monoid scheme $(X, \mathcal{A})$, there is a sheaf of monoids $\mathcal{A}_{\mathrm{sn}}$ and we define the seminormalization of X to be $(X, \mathcal{A}_{\mathrm{sn}})$. The sheaf map $\mathcal{A} \to \mathcal{A}_{\mathrm{sn}}$ induces a natural map $X_{\mathrm{sn}} \to X$.

PROPOSITION 1.21. *If $X = (X, A)$ is a pc monoid scheme, then $X_{\mathrm{sn}} := (X, \mathcal{A}_{\mathrm{sn}})$ is also a pc monoid scheme and every map from a seminormal monoid scheme to X factors uniquely through $X_{\mathrm{sn}} \to X$.*

Proof. Since being pc and the universal mapping property are local in X, we may assume that X is affine. In this case, the assertions follow from Proposition 1.15 and Lemma 1.11. □

Let $p : Y \to X$ be a morphism between monoid schemes of finite type. We say that p is *finite* if X can be covered by affine open subschemes U such that $p^{-1}(U) \to U$ is isomorphic to $\mathrm{MSpec}(B) \to \mathrm{MSpec}(A)$ for some finite monoid map $A \to B$. (See [**9**, 6.2].) We say that p is *birational* if there exists an open, dense subset U of X such that $p^{-1}(U)$ is dense in Y and the induced map $p^{-1}(U) \to U$ is an isomorphism. (See [**9**, 10.1].)

REMARK 1.21.1. If the conductor contains a non-zerodivisor $s \in B$, then $A \subset B$ is birational (as $A[1/s] = B[1/s]$). Conversely, if $\mathrm{nil}(B) = 0$ and $A \subset B$ is finite birational, then the conductor ideal contains a non-zerodivisor of both A and B. This is because if $\mathrm{MSpec}(B[1/s])$ is dense in $\mathrm{MSpec}(B)$, then s is a non-zerodivisor, and if $B = \cup A b_i$ and $b_i = a_i/s$, then $Bs \subset A$.

A monoid scheme is *reduced* if its stalks are reduced monoids (see [**9**, Section 2]).

Lemma 1.22. *If X belongs to $\mathcal{M}_{pctf}$ (respectively, $\mathcal{M}_{pc}$), then so does X_{sn} and the map $X_{\mathrm{sn}} \to X$ is a finite morphism. If in addition X is reduced, the map $X_{\mathrm{sn}} \to X$ is birational.*

Proof. The first assertion is immediate from Lemma 1.17 and Proposition 1.15. Assume now that X is reduced. The map $X_{\mathrm{sn}} \to X$ is a homeomorphism by Lemma 1.6. By Cortiñas, Haesemeyer, Walker and Weibel [**9**, 10.1], it suffices to show that this map induces an isomorphism on stalks at generic points. Since X is reduced, the stalk A at a generic point is the pointed monoid associated to an abelian group, and the map on generic stalks induced by $X_{\mathrm{sn}} \to X$ has the form $A \to A_{\mathrm{sn}}$, which is an isomorphism as A is seminormal. □

Lemma 1.23. *Let X be a pc monoid scheme with seminormalization $X_{\mathrm{sn}} \to X$, and let $Z \subset X$ be an equivariant closed subscheme of X. Set $Y := Z \times_X X_{\mathrm{sn}}$. Then $Y_{red} \to Z$ is the seminormalization of Z.*

Proof. This is a local question, so we may assume that $X = \mathrm{MSpec}(A)$, $Z = \mathrm{MSpec}(A/I)$ and $Y = \mathrm{MSpec}(A_{\mathrm{sn}}/J,)$, $J = IA_{\mathrm{sn}}$. But $Z_{\mathrm{sn}} = \mathrm{MSpec}(A_{\mathrm{sn}}/\sqrt{J})$ by Lemma 1.14. The result follows. □

2. *The cyclic bar construction*

2.1. *The cyclic bar construction for monoids*

Let A be a (pointed abelian) monoid. A two-sided pointed A-set is a pointed set B equipped with two pairings $A \wedge B \to B$ and $B \wedge A \to B$ such that $1b = b = b1$, $a_1(a_2b) = (a_1a_2)b$, $(ba_1)a_2 = b(a_1a_2)$ and $(a_1b)a_2 = a_1(ba_2)$ hold for all $a_1, a_2 \in A$, $b \in B$. The two-sided bar construction $N^{cy}(B, A)$ is the pointed simplicial set whose set of n-simplices is

$$N^{cy}(B, A)_n = B \wedge \overbrace{A \wedge \cdots \wedge A}^{n\,\text{factors}}$$

and whose face and degeneracy maps are just like those used to define the Hochschild complex of a ring.

The multiplication action of A on itself makes it a two-sided pointed A-set, and we write $N^{cy}(A)$ instead of $N^{cy}(A, A)$. The operations

$$t_n(a_0, \ldots, a_n) = (a_1, \ldots, a_n, a_0),$$

make $N^{cy}(A, A)$ into a pointed cyclic set (see [**25**, 9.6.2] and the next section for more details). Following Bökstedt–Hsiang–Madsen [**3**], we call $N^{cy}(A)$ the *cyclic nerve* of A.

For a pointed set X and a ring R, we let $R[X]$ denote the free R-module on X, modulo the summand indexed by the base point of X. If A is a pointed monoid, the free R-module $R[A]$ is a ring in the usual way, with multiplication given by the product rule for A. For any commutative ring R, the chain complex associated to the simplicial free R-module $R[N^{cy}(B, A)]$ is the usual Hochschild complex relative to the base ring R of the monoid ring $R[A]$ with coefficients in the $R[A]$-bimodule $R[B]$. In particular, the homology of $|N^{cy}(B, A)|$ with R coefficients is the Hochschild homology relative to R of $R[A]$ with coefficients in the $R[A]$-bimodule $R[B]$:

$$H_q(|N^{cy}(B, A)|, R) = HH_q(R[A]/R, R[B]). \tag{2.1}$$

Definition 2.2. Let B be a two-sided pointed A-set for which the two actions coincide (that is, $ab = ba$ for all $a \in A, b \in B$). For each $b \in B$, we define $N^{cy}(B, A; b)$ to be the pointed simplicial subset of $N^{cy}(B, A)$ consisting of those n-simplices $(b_0, a_1, \ldots, a_n)$ satisfying $b_0 \cdot \prod_i a_i = b$, together with the base point $(*, 0, \ldots, 0)$. Thus $N^{cy}(B, A)$ decomposes as a wedge of pointed simplicial sets

$$N^{cy}(B, A) = \bigvee_{b \in B} N^{cy}(B, A; b).$$

In particular, when $B = A$, we write $N^{cy}(A; a) = N^{cy}(A, A; a)$ and have $N^{cy}(A) = \bigvee_{a \in A} N^{cy}(A; a)$. Each subset $N^{cy}(A; a)$ is invariant under the operators t_n on $N^{cy}(A)$, so each $N^{cy}(A; a)$ is a cyclic subset of $N^{cy}(A)$. For example, if A has no zerodivisors, then each

$$N^{cy}(A; 0)_n = \left\{ (a_0, \ldots, a_n)\} \,\middle|\, \prod_{i=0}^{n} a_i = 0 \right\}$$

is a point for all n, that is, $N^{cy}(A; 0)$ is a point, regarded as a constant simplicial set.

Remark 2.2.1. If I is an ideal and $a \notin I$, then $N^{cy}(A; a) \xrightarrow{\cong} N^{cy}(A/I; a)$. This is immediate from the observation that if $(x_0, \ldots, x_n)$ is an n-simplex of $N^{cy}(A; a)$ and it is not the base point, then $x_0 \cdots x_n = a \notin I$ and hence none of $x_0, \ldots, x_n$ can be in I.

Observe that if $A = \bigwedge_{i \in I} A^i$ is a smash product of pointed monoids, then

$$N^{cy}(A) \cong \bigwedge_{i \in I} N^{cy}(A^i). \tag{2.3}$$

In particular, if $\{B^i\}_{i \in I}$ is a family of unpointed monoids, then $\bigwedge(B^i_*) = (\prod B^i)_*$ and

$$N^{cy}\left(\prod_i B^i\right)_* = \bigwedge_{i \in I} N^{cy}(B^i_*). \tag{2.4}$$

2.2. *Cyclic sets and $\mathbb{S}^1$-spaces*

We briefly review the basic theory of $\mathbb{S}^1$-spaces and cyclic sets. We refer the reader to [**3**], [**21**, 7.1], [**25**, 9.6] for additional information.

Let $\mathbb{S}^1$ denote the Lie group of complex numbers of norm 1. We will often identify $\mathbb{S}^1$ with the group $\mathbb{R}/\mathbb{Z}$ via the homeomorphism induced by sending $\theta \in \mathbb{R}$ to $e^{2\pi\theta i}$ in $\mathbb{S}^1$. The only closed subgroups of $\mathbb{S}^1$ are $\mathbb{S}^1$ itself and the cyclic group C_r of rth roots of unity for each $r \geqslant 1$.

An $\mathbb{S}^1$-*space* is a topological space equipped with a continuous action of $\mathbb{S}^1$. An equivariant map of $\mathbb{S}^1$-spaces $f : X \to Y$ is an $\mathbb{S}^1$-*weak equivalence* if $f : X^H \to Y^H$ is a weak equivalence in the usual sense for every closed subgroup H of $\mathbb{S}^1$ (that is, for $H = \mathbb{S}^1$ and $H = C_r, r \geqslant 1$). Let I denote the unit interval with trivial $\mathbb{S}^1$-action. Two equivariant maps $f_0, f_1 : X \to Y$ are $\mathbb{S}^1$-*homotopic* if there is an equivariant map $h : X \times I \to Y$ with $f_i = h|_{X \times \{i\}}$ for $i = 0, 1$. A map $f : X \to Y$ is an $\mathbb{S}^1$-*homotopy equivalence* if it admits an inverse up to $\mathbb{S}^1$-homotopy.

All of the above definitions have evident pointed variants.

A pointed $\mathbb{S}^1$-CW complex is a pointed $\mathbb{S}^1$-space built from pointed $\mathbb{S}^1$-cells of the form $(\mathbb{S}^1/H)_+ \wedge D^n$ with H as above. The following analog of the usual Whitehead theorem is useful; see [**20**, I.1].

Equivariant Whitehead Theorem 2.5. *A morphism between $\mathbb{S}^1$-CW complexes is an $\mathbb{S}^1$-homotopy equivalence if and only if it is an $\mathbb{S}^1$-weak equivalence.*

Let Λ denote the *cyclic category* with objects $[0], [1], [2], \ldots$ and whose morphisms are generated by the face and degeneracy maps as in the simplicial category along with an automorphism $\tau_n : [n] \to [n]$ of order $n+1$ for each $n \geqslant 0$. See, for example, [**21**, 6.1.1] or [**25**, 9.6.1] for a complete list of relations. A *cyclic set* is a contravariant functor from Λ to the category of sets. Since Λ contains the simplicial category as a subcategory, there is a forgetful functor from cyclic sets to simplicial sets.

If $X_\bullet$ is a cyclic set, then the geometric realization of its underlying simplicial set is an $\mathbb{S}^1$-space (see [**11**]); in fact, $|X_\bullet|$ is an $\mathbb{S}^1$-CW complex by [**13**, Theorem 1].

The representable functor $\operatorname{Hom}_\Lambda(-,[n])$ on the cyclic category Λ is a cyclic set, and hence a simplicial set, and there is a homeomorphism

$$|\operatorname{Hom}_\Lambda(-,[n])| \xrightarrow{\cong} \Lambda^n := \mathbb{S}^1 \times \Delta^n, \tag{2.6}$$

where $\Delta^n := \{(u_0,\dots,u_n) \mid u_i \in \mathbb{R}, u_i \geqslant 0, \sum_i u_i = 1\}$ is the standard topological n-simplex. (See [**3**, 1.6].) The collection of spaces $\Lambda^n, n \geqslant 0$, is a cocyclic $\mathbb{S}^1$-space; that is, a covariant functor from Λ to the category of $\mathbb{S}^1$-space. We choose the homeomorphism (2.6) as in [**18**, 7.2], so that, for the automorphism $\tau_n : [n] \to [n]$, we have

$$(\tau_n)_*(z; u_0,\dots,u_n) = (z\, e^{2\pi i/(n+1)}; u_1,\dots,u_n,u_0). \tag{2.7}$$

Note that Λ^n is an $\mathbb{S}^1 \times C_{[n]}$-space, where $C_{[n]} = C_{n+1}$ is the group of cyclic permutations of $[n]$. Explicitly, $\mathbb{S}^1$ acts (only) on the first component by translation, and the cyclic group acts (only) in the second component by permuting the vertices.

For a general cyclic set $X_\bullet$, we use the $\mathbb{S}^1$-spaces Λ^n to construct the coend

$$\int_{n\in\Lambda} X_n \times \Lambda^n := \coprod_n X_n \times \Lambda^n / \sim\,,$$

where $\sim$ is the equivalence relation generated by $(f^*x,t) \sim (x, f_*t)$ for f a morphism in Λ. As proved in [**13**, Theorem 1], this coend is canonically homeomorphic to the geometric realization of the simplicial set underlying $X_\bullet$. We are thus justified in defining the geometric realization of a cyclic set $X_\bullet$ as

$$|X_\bullet| := \int_{n\in\Lambda} X_n \times \Lambda^n.$$

Thus, $|X_\bullet|$ is an $\mathbb{S}^1$-space with the action of $\mathbb{S}^1$ determined by the action on the Λ^n of (2.6).

EXAMPLE 2.8. For any cyclic set $X_\bullet$, the subspace $|X_\bullet|^{\mathbb{S}^1} \subset |X_\bullet|$ is the discrete set of points given by the equalizer of the two maps $s_0, t_1 \circ s_0 : X_0 \rightrightarrows X_1$ (see [**12**, p. 145]). If A is a monoid, then $s_0, t_1 \circ s_0 : A = N^{cy}(A)_0 \to N^{cy}(A)_1 = A \wedge A$ are given by $a \mapsto a \wedge 1$ and $a \mapsto 1 \wedge a$, and thus we have $|N^{cy}(A)|^{\mathbb{S}^1} = \{0,1\} = S^0$ for any $A \neq 0$, with $|N^{cy}(A;a)|^{\mathbb{S}^1} = \{0\}$ for $a \neq 1$. In particular, the map between the cyclic nerves of non-zero monoids induced by a monoid homomorphism is an $\mathbb{S}^1$-homotopy equivalence if and only if it is a weak equivalence on C_r-fixed points for each $r \geqslant 1$.

EXAMPLE 2.9. Suppose that M is an abelian group, written additively for convenience, and let $N^{cy}_\circ(M)$ denote the unpointed version of the cyclic bar construction, so that an n-simplex of $N^{cy}_\circ(M)$ is an $m+1$-tuple $(m_0,\dots,m_n)$ of elements of M. Note that $N^{cy}(M_*) = N^{cy}_\circ(M)_*$ and we have an evident decomposition

$$N^{cy}_\circ(M) = \coprod_{m\in M} N^{cy}_\circ(M;m).$$

For each $m \in M$, the simplicial set $N^{cy}_\circ(M;m)$ is isomorphic to BM, the standard simplicial classifying space of M. The isomorphism sends $(m_0,\dots,m_n)$ to $(m_1,\dots,m_n)$ and its inverse sends $(m_1,\dots,m_n)$ to $(m - m_1 - \cdots - m_n, m_1,\dots,m_n)$. For each $m \in M$, this isomorphism induces a cyclic structure on BM given by

$$\tau_n(m_1, \ldots, m_n) = (m_2, \ldots, m_n, m - m_1 - \cdots - m_n).$$

This cyclic structure on BM depends on m of course, and we will describe the fixed-point sets of the geometric realization for finite subgroups of $\mathbb{S}^1$ in what follows.

There is a natural map from $|N^{cy}_\circ(M)|$ to the free loop space $\Lambda|BM|$, adjoint to the composition of the $\mathbb{S}^1$-action $\mathbb{S}^1 \times |N^{cy}_\circ(M)| \to |N^{cy}_\circ(M)|$ with the geometric realization of the map $N^{cy}_\circ(M) \to BM$ that is, on $N^{cy}_\circ(M; m)$, the isomorphism described above. It is proved in [**3**, Proposition 2.6] that this map induces a homotopy equivalence $|N^{cy}_\circ(M)|^{C_r} \to (\Lambda|BM|)^{C_r}$ for all $r \geqslant 1$.

Since M is abelian, there is a homotopy equivalence $\Lambda|BM| \to |BM| \times M$; the projection onto $|BM|$ is given by evaluation at 1, and the projection to $M = \pi_0(\Omega|BM|)$ is obtained by deforming a loop in $|BM|$ to a loop based at the base point and then taking its homotopy class (this is well-defined since M is abelian). It is easy to check that this map is actually an $\mathbb{S}^1$-homotopy equivalence for the action on $|BM| \times M$ that is, on the component $|BM| \times \{m\}$, induced by the composite map $\mathbb{S}^1 \times |BM| \xrightarrow{\cong} |B(\mathbb{Z} \times M)| \to |BM|$, where $B(\mathbb{Z} \times M) \to BM$ is given by applying B to the homomorphism $(n, x) \mapsto n \cdot m + x$.

Under the composite map $|N^{cy}_\circ(M)| \to \Lambda|BM| \to |BM| \times M$, the component $|N^{cy}_\circ(M; m)|$ is mapped to the component $|BM| \times \{m\}$ and it follows from [**3**, Proposition 2.6.] that the induced map $|N^{cy}_\circ(M; m)|^{C_r} \to (|BM| \times \{m\})^{C_r}$ (with action as described above) is a homotopy equivalence for all $r \geqslant 1$.

Now consider the case where M is a free abelian group of rank n and let $e \in M$. It follows from the preceding discussion that $|N^{cy}(M_*; e)|^{C_r}$ is homotopy equivalent to an n-dimensional torus with disjoint basepoint if e is divisible by r (that is, if $e = r \cdot e'$ for some $e' \in M$), and to a one-point space otherwise.

Example 2.9.1. More generally, for any (pointed) monoid A and any unit $a \in A$, we have

$$N^{cy}(A; a) = N^{cy}(U(A)_*; a) \cong B(U(A))_*,$$

where $U(A)$ is the group of units in A. This follows from Remark 2.2.1.

For each $r > 0$, the canonical group homomorphism

$$\rho_r : \mathbb{S}^1 \twoheadrightarrow \mathbb{S}^1/C_r \cong \mathbb{S}^1, \quad z \longmapsto z^r,$$

allows us to view an $\mathbb{S}^1$-space X as an $\mathbb{S}^1$-space $\rho_r^* X$ on which z acts as z^r. In particular, C_r acts trivially on $\rho_r^* X$.

Example 2.9.2. Consider the free monoid $\langle x \rangle$ on one generator. By Definition 2.2, $N^{cy}(\langle x \rangle)$ decomposes as $\vee_{r \geqslant 0} N^{cy}(\langle x \rangle; x^r)$. By inspection, we have $N^{cy}(\langle x \rangle; 1) = S^0$ and $N^{cy}(\langle x \rangle; x) = S^1_*$. For $r \geqslant 1$, we have an $\mathbb{S}^1$-equivariant homeomorphism

$$|N^{cy}(\langle x \rangle; x^r)| \xrightarrow{\sim} (\mathbb{S}^1 \times_{C_r} \Delta^{r-1})_*,$$

and the canonical map $|N^{cy}(\langle x \rangle; x^r)| \to |N^{cy}(\langle x^{\pm 1} \rangle; x^r)|$ is an $\mathbb{S}^1$-homotopy equivalence (see [**22**, 3.20–21] for example). That is, $|N^{cy}(\langle x \rangle; x^r)| \xrightarrow{\sim} (\mathbb{S}^1/C_r)_* = \rho_r^* \mathbb{S}^1$. In particular, we have an $\mathbb{S}^1$-equivariant deformation retraction

$$|N^{cy}(\langle x \rangle)| \xrightarrow{\sim} S^0 \vee \bigvee_{r \geqslant 1} (\mathbb{S}^1/C_r)_*.$$

By convention, we set $C_0 = \mathbb{S}^1$, so that $|N^{cy}(\langle x \rangle)| \xrightarrow{\sim} \bigvee_{r \geqslant 0} (\mathbb{S}^1/C_r)_*$.

We will need to understand the $\mathbb{S}^1$-homotopy type of the cyclic nerve of the free abelian pointed monoid $F_r := \langle x_1, \ldots, x_r \rangle$. Using (2.3) and (2.4), it suffices to consider the case $r = 1$, which is handled in Example 2.9.2. We record the result:

LEMMA 2.10. *For integers $e_1, \ldots, e_r \geqslant 0$, there is an $\mathbb{S}^1$-homotopy equivalence*

$$(\mathbb{S}^1/C_{e_1} \times \cdots \times \mathbb{S}^1/C_{e_r})_* \xrightarrow{\sim} |N^{cy}(F_r; x_1^{e_1} \cdots x_r^{e_r})|.$$

Moreover, the map

$$|N^{cy}(F_r; x_1^{e_1} \cdots x_r^{e_r})| \rightarrowtail |N^{cy}(\langle x_1^{\pm 1}, \ldots, x_r^{\pm 1} \rangle ; x_1^{e_1} \cdots x_r^{e_r})|$$

is an $\mathbb{S}^1$-homotopy equivalence.

The space $(\mathbb{S}^1/C_{e_1} \times \cdots \mathbb{S}^1/C_{e_r})_*$ in Lemma 2.10 is a pointed torus, whose dimension is the number of non-zero e_i. If $e_i > 0$ for all i, it is the space $(\mathbb{S}^1 \times \cdots \times \mathbb{S}^1)_*$ equipped with the $\mathbb{S}^1$-action given as

$$z \cdot (w_1, \ldots, w_r) = (z^{e_1} w_1, \ldots, z^{e_r} w_r).$$

EXAMPLE 2.11. Let A_* be the pointed monoid associated to the integer lattice points in the cone in $\mathbb{R}^2$ spanned by the vectors $v = (1, 0)$ and $w = (1, 3)$, and set $x = (1, 1)$, $y = (1, 2)$; these are irreducible elements in A. The element $a = (2, 3)$ satisfies $a = x + y = v + w$. Let B_* be the pointed monoid generated by x and y and let C_* be the pointed monoid generated by v and w. Then each of B_* and C_* is the pointed monoid associated to a free abelian monoid with two generators. Since a can be written as a non-trivial sum in only the two ways given above, we have

$$N^{cy}(A_*; a) = N^{cy}(B_*; a) \cup N^{cy}(C_*; a)$$

and

$$N^{cy}(B_*; a) \cap N^{cy}(C_*; a) = N^{cy}(D_*; a),$$

where D_* is the monoid generated freely by a. We see that $|N^{cy}(A_*; a)|$ is the pushout in the category of $\mathbb{S}^1$-spaces of

$$\begin{tikzcd} {\mathbb{S}^1}_* \arrow[r, "\mathrm{diag}"] \arrow[d, "\mathrm{diag}"'] & (\mathbb{S}^1 \times \mathbb{S}^1)_* \\ (\mathbb{S}^1 \times \mathbb{S}^1)_* & \end{tikzcd}$$

in which each map is the diagonal map $z \mapsto (z, z)$ and, for each copy of $\mathbb{S}^1 \times \mathbb{S}^1$, the group $\mathbb{S}^1$ acts diagonally: $z(t, u) = (zt, zu)$.

2.3. *Edge-wise subdivision*

We recall Segal's r-fold edge-wise subdivision $sd_r(X_\bullet)$ of a simplicial or cyclic set $X_\bullet$, as presented in [**3**]. If we think of a simplicial set as a contravariant functor from the category **Ord** of finite, totally ordered, non-empty sets to the category of sets, then $sd_r(X_\bullet)$ is the functor obtained from $X_\bullet : \mathbf{Ord} \to \mathbf{Sets}$ by precomposing with the endofunctor $(-)^{\amalg r} : \mathbf{Ord} \to \mathbf{Ord}$ sending A to $A^{\amalg r} := A \amalg \cdots \amalg A$, with r copies of A, indexed by $\{1, \ldots, r\}$. More formally, $A^{\amalg r}$ is the set $A \times \{1, \ldots, r\}$, with $(a, i) \leqslant (b, j)$ if $i < j$ or $i = j$ and $a \leqslant b$. One may also view $(-)^{\amalg r}$ as an endofunctor on the simplicial category, which is the full skeletal subcategory of **Ord** consisting of objects $[0], [1], \ldots$. Then $[n]^{\amalg r} = [r(n+1) - 1]$ and $sd_r(X_\bullet)_n = X_{r(n+1)-1}$.

Recall from [3, 1.1] that, for any simplicial set $X_\bullet$, there is a homeomorphism

$$D_r : |sd_r(X_\bullet)| \xrightarrow{\cong} |X_\bullet| \tag{2.12}$$

induced by the maps

$$(sd_r(X_\bullet))_n \times \Delta^n = X_{r(n+1)-1} \times \Delta^n \to X_{r(n+1)-1} \times \Delta^{r(n+1)-1}$$

sending $(x, \vec{u})$, with x in $sd_r(X_\bullet)_n = X_{r(n+1)-1}$ and $\vec{u} = (u_0, \ldots, u_n) \in \Delta^n$ for $n \geqslant 0$, to $(x, \vec{u}/r \oplus \cdots \oplus \vec{u}/r)$, with r copies of $\vec{u}/r$.

For $r \geqslant 1$, define Λ_r to be the category with the same objects as the cyclic category Λ and whose morphisms have the same generators, except that the relation on $\tau_n : [n] \to [n]$ is $\tau_n^{r(n+1)} = \mathrm{id}$. Note that $\Lambda_1 = \Lambda$. A Λ_r-set is a contravariant functor from Λ_r to the category of sets. As in the case $r = 1$, Λ_r contains the simplicial category as a subcategory and hence a Λ_r-set has a geometric realization. As observed in [3, (1.4)], the cyclic group C_r acts on Λ_r, with the generator acting on $[n]$ by τ_n^{n+1}. Thus C_r acts naturally on any Λ_r-set.

The geometric realization of the representable Λ_r-set $\mathrm{Hom}_{\Lambda_r}(-, [n])$ is homeomorphic to $\mathbb{R}/r\mathbb{Z} \times \Delta^n$, as shown in [3, 1.6]. Just as in the case $r = 1$, we endow the geometric realization of a Λ_r-set $X_\bullet$ with an action of the topological group $\mathbb{R}/r\mathbb{Z}$ by using the homeomorphism [3, 1.8]

$$|X_\bullet| \cong \int_n |\mathrm{Hom}_{\Lambda_r}(-, [n])| \times X_n.$$

Henceforth, we identify $\mathbb{R}/r\mathbb{Z}$ with $\mathbb{S}^1$ via $z \mapsto e^{2z\pi i/r}$ so that the geometric realization of a Λ_r-set is an $\mathbb{S}^1$-space. The following fact will be used:

Lemma 2.13. *For any Λ_r-set $X_\bullet$, the action of the subgroup C_r of $\mathbb{S}^1$ on $|X_\bullet|$ is induced by the natural action of C_r on the simplicial set $X_\bullet$, where the generator acts on X_n by τ_n^{n+1}.*

Proof. This is established on p. 469 of [3] (after Definition 1.5). □

Corollary 2.14. *If a is not an rth power in A, then $N^{cy}(A; a)^{C_r} = \{a\}$ and $|N^{cy}(A; a)|^{C_r}$ is a 1-point space.*

If $X_\bullet$ is a cyclic set, $sd_r(X_\bullet)$ is a Λ_r-set on which $\tau_n \in \mathrm{Hom}_{\Lambda_r}([n], [n])$ acts on $sd_r(X_\bullet)_n = X_q$ $(q = r(n+1) - 1)$ via the action of $\tau_q \in \mathrm{End}_\Lambda([q])$ on X_q. With our convention for viewing $|sd_r(X_\bullet)|$ as an $\mathbb{S}^1$-space, the homeomorphism $D_r : |sd_r X_\bullet| \xrightarrow{\cong} |X_\bullet|$ is $\mathbb{S}^1$-equivariant [3, 1.11].

In the special case $X_\bullet = N^{cy}(A)$, if tA^r denotes A^r with the twisted A^r-action

$$(a_1, \ldots, a_r) \cdot (x_1, \ldots, x_r) \cdot (b_1, \ldots, b_r) = (a_r x_1 b_1, a_1 x_2 b_2, \ldots, a_{r-1} x_r b_r),$$

then $sd_r(X_\bullet)$ is again a cyclic bar construction $N^{cy}(tA^r, A^r)$; see [3, 2.1]. It is useful to think of an n-simplex of $sd_r(N^{cy}(A))$ as an $r \times (n+1)$ matrix of entries in A. By Lemma 2.13, the action of C_r on $|sd_r(N^{cy}(A))|$ is induced by the simplicial action that permutes the rows of such matrices.

In order to study the effect of the dilation θ_r, we will need a comparison between $N^{cy}(A; a)$ and $N^{cy}(A; a^r)$, given in (2.16); our presentation is taken from the discussion in [3]. There is a natural diagonal map of simplicial sets,

$$\delta_r : N^{cy}(A) \longrightarrow sd_r(N^{cy}(A)),$$

which sends $(a_0, \ldots, a_n)$ to the $r \times (n+1)$ matrix with this row repeated r times. The map of associated $\mathbb{S}^1$-spaces

$$|\delta_r| : |N^{cy}(A)| \longrightarrow |sd_r(N^{cy}(A))|$$

is not $\mathbb{S}^1$-equivariant, but instead satisfies [**3**, (2.7)]:

$$|\delta_r|(z^r \cdot p) = z \cdot |\delta_r|(p) \quad \text{for all } p \in |N^{cy}(A)|,\ z \in \mathbb{S}^1.$$

Taking z in C_r, this shows that the image of $|\delta_r|$ is contained in the C_r-fixed points of $|sd_r(N^{cy}(A))|$. For all $r \geqslant 1$, the map δ_r determines an isomorphism of simplicial sets,

$$\delta_r : N^{cy}(A) \xrightarrow{\cong} sd_r(N^{cy}(A))^{C_r}.$$

As shown in [**3**, (2.3)], this determines a homeomorphism

$$|\delta_r| : |N^{cy}(A)| \xrightarrow{\cong} |sd_r(N^{cy}(A))^{C_r}| = |sd_r(N^{cy}(A))|^{C_r}.$$

Using the homeomorphism D_r of (2.12), define the endomorphism $\bar{\delta}_r$ of $|N^{cy}(A)|$ as

$$\bar{\delta}_r = D_r \circ \delta_r : |N^{cy}(A)| \xrightarrow{\delta_r} |sd_r N^{cy}(A)| \xrightarrow{D_r} |N^{cy}(A)|. \tag{2.15}$$

By what we have shown above, we have

$$\bar{\delta}_r(z^r \cdot x) = z \cdot \bar{\delta}_r(x) \quad \text{for all } x \in |N^{cy}(A)| \text{ and } z \in \mathbb{S}^1,$$

and $\bar{\delta}$ induces a homeomorphism

$$\bar{\delta}_r : |N^{cy}(A)| \xrightarrow{\cong} |N^{cy}(A)|^{C_r}.$$

In other words, $\bar{\delta}_r$ is an $\mathbb{S}^1$-equivariant map

$$\bar{\delta}_r : \rho_r^*|N^{cy}(A)| \longrightarrow |N^{cy}(A)|$$

that induces an $\mathbb{S}^1/C_r$-equivariant homeomorphism of C_r fixed point subspaces:

$$\bar{\delta}_r : \rho_r^*|N^{cy}(A)| \xrightarrow{\cong} |N^{cy}(A)|^{C_r}.$$

The map $\bar{\delta}_r$ sends the component indexed by $a \in A$ to that indexed by $a^r \in A$, and so restricts to a homeomorphism

$$\bar{\delta}_r : \bigvee_{a \in A, a^r = b} |N^{cy}(A; a)| \xrightarrow{\cong} |N^{cy}(A; b)|^{C_r},$$

for each $b \in A$. In particular, $|N^{cy}(A; b)|^{C_r} = \{*\}$ if $b \in A$ is not the rth power of any element of A. If A is the quotient of a cancellative and torsion-free monoid by a radical ideal, then we have a homeomorphism for each $a \in A$:

$$\bar{\delta}_r : |N^{cy}(A; a)| \xrightarrow{\cong} |N^{cy}(A; a^r)|^{C_r}. \tag{2.16}$$

Example 2.17. Consider the free commutative pointed monoid F_n on generators $x_1, \ldots, x_n$. From Lemma 2.10, we have the equivalence for each $(e_1, \ldots, e_n)$:

$$(\mathbb{S}^1/C_{e_1} \times \cdots \times S/C_{e_n})_* \xrightarrow{\sim} |N^{cy}(F_n; x_1^{e_1} \cdots x_n^{e_n})|,$$

where $\mathbb{S}^1/C_0$ is defined to be the trivial group. For any $r \geqslant 1$, the action of C_r on $\mathbb{S}^1/C_{e_i}$ is fixed-point free unless $r \mid e_i$, in which case C_r acts trivially. Thus the action of C_r on $|N^{cy}(F_n, x_1^{e_1} \cdots x_n^{e_n})|$ fixes only the basepoint unless $r \mid e_i$ for all i. If $r \mid e_i$ for all i, the action of C_r is trivial, and the homeomorphism

$$|N^{cy}(F_n; x_1^{e_1/r}, \ldots, x_n^{e_n/r})| \xrightarrow{\cong} |N^{cy}(F_n; x_1^{e_1}, \ldots, x_n^{e_n})|^{C_r} = |N^{cy}(F_n; x_1^{e_1}, \ldots, x_n^{e_n})|$$

of (2.16) corresponds, under the equivalences of Lemma 2.10, to the homeomorphism

$$(\mathbb{S}^1/C_{e_1/r} \times \cdots \mathbb{S}^1/C_{e_n/r})_* \xrightarrow{\cong} (\mathbb{S}^1/C_{e_1} \times \cdots \times S/C_{e_n})_*$$

whose inverse is the map that raises elements to the rth power.

3. *The comparison theorem*

3.1. *The dilated cyclic bar construction*

We now introduce a variant on N^{cy}. If a is an element of a monoid A, we write $A[1/a]$ for the monoid formed by adjoining an inverse of a; formally, $A[1/a] = S^{-1}A$ where $S = \{a^n \mid n \geqslant 0\}$.

Definition 3.1. For a monoid A, define

$$\tilde{N}^{cy}(A) = \bigvee_{a \in A} N^{cy}(A[1/a]; a),$$

where the element a in $N^{cy}(A[1/a]; a)$ refers to the element $a/1$ of $A[1/a]$. Thus an n-simplex of $\tilde{N}^{cy}(A)$ is given by $(a; \alpha_0, \dots, \alpha_n)$ where $\alpha_0, \dots, \alpha_n \in A[1/a]$, $a \in A$, and the equation $\alpha_0 \cdots \alpha_n = a/1$ holds in $A[1/a]$.

As we saw in Definition 2.2, $N^{cy}(A[1/a]; a)$ is a cyclic set for each a, so $\tilde{N}^{cy}(A)$ is a cyclic set. For each $a \in A$, $N^{cy}(A; a) \to N^{cy}(A[1/a]; a)$ is a morphism of cyclic sets, and hence there is a morphism of cyclic sets

$$N^{cy}(A) \longrightarrow \tilde{N}^{cy}(A)$$

that is natural for homomorphisms of monoids.

In nice enough situations, N^{cy} and $\tilde{N}^{cy}$ coincide up to equivalence:

Proposition 3.2. *If $A = (\mathbb{N}^r \times G)_*$ where G is an abelian group, then the natural map $|N^{cy}(A)| \to |\tilde{N}^{cy}(A)|$ induces an $\mathbb{S}^1$-homotopy equivalence.*

Proof. The result is immediate if $A = G_*$, and follows from Lemma 2.10 if $A = \mathbb{N}^r_*$. The general case follows from this using (2.3) and (2.4). ☐

The monoids A appearing in the proposition above with G finitely generated are those for which $\mathbb{Q}[A]$ is smooth. (If the group G has p-torsion, $k[A]$ will not be smooth for fields k of characteristic p.) We thus think of the failure of the map $|N^{cy}(A)| \to |\tilde{N}^{cy}(A)|$ to be an equivalence as measuring the lack of 'smoothness of a monoid'.

The functor $\tilde{N}^{cy}$ is better behaved than N^{cy} for 'pathological' monoids. An example of this behavior is the following result. Recall from (1.1) that $\operatorname{nil}(A)$ denotes the ideal of nilpotent elements in A.

Lemma 3.3. *The map $\tilde{N}^{cy}(A) \to \tilde{N}^{cy}(A/\operatorname{nil}(A))$ is an isomorphism.*

Proof. If $a \in A$ is nilpotent, then $A[1/a]$ is the trivial pointed monoid $\{0\}$, and $N^{cy}(A[1/a]; a) = N^{cy}(0)$ is the one-point space. It follows that

$$\tilde{N}^{cy}(A) = \bigvee_{a \in A \setminus \operatorname{nil}(A)} N^{cy}\left(A\left[\frac{1}{a}\right]; a\right).$$

We also have

$$\tilde{N}^{cy}(A/\operatorname{nil}(A)) = \bigvee_{a \in A \setminus \operatorname{nil}(A)} N^{cy}\left((A/\operatorname{nil}(A))\left[\frac{1}{a}\right]; a\right)$$

and so it suffices to prove that, for all $a \in A \setminus \mathrm{nil}(A)$, the map

$$N^{cy}\left(A\left[\frac{1}{a}\right];a\right) \longrightarrow N^{cy}\left((A/\mathrm{nil}(A))\left[\frac{1}{a}\right];a\right)$$

is an isomorphism. But this follows from Remark 2.2.1, since $(A/\mathrm{nil}(A))[1/a] = A[1/a]/\mathrm{nil}(A[1/a])$ and $a \notin \mathrm{nil}(A[1/a])$. □

Example 3.3.1. If B is the free monoid on x, it follows from Example 2.9.2 that $|\tilde{N}^{cy}(B)|$ is a bouquet of copies of $\mathbb{S}^1/C_r$ indexed by $r \geqslant 1$. If A is the submonoid $\langle x^2, x^3\rangle$, then $|\tilde{N}^{cy}(A)|$ is the sub-bouquet indexed by $r \geqslant 2$.

3.2. *Comparison Theorem*

The Comparison Theorem 3.6 concerns the map $N^{cy} \to \tilde{N}^{cy}$. Roughly, it says that the map becomes an equivariant homotopy equivalence upon inverting enough dilations. Intuitively, upon inverting dilations, every (partially cancellative) monoid resembles a smooth monoid.

For any positive integer c, the dilation

$$\theta_c : A \longrightarrow A,\ a \longmapsto a^c$$

defines a monoid endomorphism of a monoid A. If $c = p$ is prime, then the induced map on monoid rings $\theta_p : \mathbb{Z}/p[A] \to \mathbb{Z}/p[A]$ coincides with the usual Frobenius.

Now let $\mathfrak{c} = (c_1, c_2, \ldots)$ be an infinite list of integers with $c_i \geqslant 2$ for all i, and define $A^{\mathfrak{c}}$ to be the abelian monoid

$$A^{\mathfrak{c}} := \mathrm{colim}\{A \xrightarrow{\theta_{c_1}} A \xrightarrow{\theta_{c_2}} \cdots\}.$$

We think of this construction as a form of localization, since, for example, if $c_i = c$ for all i, then $(-)^{\mathfrak{c}}$ amounts to inverting the endomorphism θ_c. Similarly, if F is any functor on monoids, we define $F(A)^{\mathfrak{c}}$ by

$$F(A)^{\mathfrak{c}} = \mathrm{colim}\{F(A) \xrightarrow{\theta_{c_1}} F(A) \xrightarrow{\theta_{c_2}} \cdots\}, \tag{3.4}$$

provided this (sequential) colimit exists in the target category. If F commutes with sequential colimits, then clearly $F(A)^{\mathfrak{c}} = F(A^{\mathfrak{c}})$. For example, both N^{cy} and $\tilde{N}^{cy}$ commute with filtered colimits; this is easy for N^{cy} and straightforward for $\tilde{N}^{cy}$. Hence (3.4) defines the cyclic sets $N^{cy}(A)^{\mathfrak{c}} = N^{cy}(A^{\mathfrak{c}})$ and $\tilde{N}^{cy}(A)^{\mathfrak{c}} = \tilde{N}^{cy}(A^{\mathfrak{c}})$.

If $\rho : A \to A'$ is a homomorphism of pointed monoids, then we may regard A' as a two-sided pointed A-set (with the two actions coinciding) in the obvious way. Observe that raising elements to the power of c determines an endomorphism of the simplicial set $N^{cy}(A', A)$, which we will also write as θ_c. More generally, suppose that B is a pointed A-subset of A' that is closed under $b \mapsto b^c$. Then θ_c restricts to an endomorphism of simplicial sets on $N^{cy}(B, A)$.

Lemma 3.5. *Let A be a monoid and let $\mathfrak{c} = (c_1, c_2, \ldots)$ be an infinite sequence of integers with $c_i \geqslant 2$ for all i. Then the following conditions are satisfied*:

1. *$A^{\mathfrak{c}} \to (A_{\mathrm{red}})^{\mathfrak{c}}$ is an isomorphism*;
2. *if A is pc, then $A^{\mathfrak{c}} \to (A_{\mathrm{sn}})^{\mathfrak{c}}$ is an isomorphism.*

Proof. Since $\pi : A \to A_{\mathrm{red}}$ is a surjection, so is $A^{\mathfrak{c}} \to (A_{\mathrm{red}})^{\mathfrak{c}}$. Suppose that $a, b \in A$ agree in $(A_{\mathrm{red}})^{\mathfrak{c}}$. Then there is a $c = c_1 \cdots c_n$ so that $\pi(a^c) = \pi(b^c)$ in A_{red}. Thus either $a^c = b^c$ in A or else $a, b \in \mathrm{nil}(A)$ and $a^n = b^n = 0$ for $n \gg 0$. In either case, a and b agree in $A^{\mathfrak{c}}$.

Now assume that A is pc, so that A_{sn} exists. By (1), we may assume that A is reduced. By definition, $A \to A_{\text{sn}}$ is an injection; it follows that $A^{\mathfrak{c}} \to (A_{\text{sn}})^{\mathfrak{c}}$ is an injection. To see that it is a surjection, fix $t \in A_{\text{sn}}$. If $c = c_1 \cdots c_n$ and n is a suitably large integer, then $a = t^c$ is in A; in particular $a = \theta_c(t)$. Thus the image of t in $(A_{\text{sn}})^{\mathfrak{c}}$ is in $A^{\mathfrak{c}}$. $\square$

Theorem 3.6. *Let A be a pctf monoid and let $\mathfrak{c} = (c_1, c_2, \ldots)$ be an infinite sequence of integers with $c_i \geqslant 2$ for all i. Then the natural maps of cyclic sets*

$$N^{cy}(A)^{\mathfrak{c}} \longrightarrow \tilde{N}^{cy}(A)^{\mathfrak{c}} \xrightarrow{\cong} \tilde{N}^{cy}(A_{\text{sn}})^{\mathfrak{c}}$$

are $\mathbb{S}^1$-homotopy equivalences (on geometric realizations).

The proof of the Theorem 3.6 occupies the remainder of this section. We begin with a series of reductions. The first reduction is that we may assume A seminormal, since Lemma 3.5 implies that $N^{cy}(A^{\mathfrak{c}}) \cong N^{cy}((A_{\text{sn}})^{\mathfrak{c}})$ and $\tilde{N}^{cy}(A^{\mathfrak{c}}) \cong \tilde{N}^{cy}((A_{\text{sn}})^{\mathfrak{c}})$.

Next, observe that both cyclic sets are amalgamated sums indexed by $A^{\mathfrak{c}}$. An element of $A^{\mathfrak{c}}$ is given by an element $a \in A$ occurring at some stage in the colimit of $(A \xrightarrow{\theta_{c_1}} A \xrightarrow{\theta_{c_2}} \cdots)$. Dropping the first few terms from $\mathfrak{c}$ does not change the colimit $A^{\mathfrak{c}}$ used to index the terms. Thus, to prove the theorem, it suffices to show that, for all non-zero $a \in A$,

$$N^{cy}(A;a)^{\mathfrak{c}} \longrightarrow N^{cy}\left(A\left[\frac{1}{a}\right];a\right)^{\mathfrak{c}} = \tilde{N}^{cy}(A;a)^{\mathfrak{c}} \tag{3.6.1}$$

induces an $\mathbb{S}^1$-homotopy equivalence on geometric realizations. In this equation, $N^{cy}(A;a)^{\mathfrak{c}}$ is the colimit of

$$N^{cy}(A;a) \xrightarrow{\theta_{c_1}} N^{cy}(A;a^{c_1}) \xrightarrow{\theta_{c_2}} N^{cy}(A;a^{c_1c_2}) \longrightarrow \cdots$$

and similarly for $\tilde{N}^{cy}(A;a)^{\mathfrak{c}}$, with A replaced by $A[1/a]$.

We also claim that it suffices to assume that A is cancellative and torsion-free. Indeed, since a is not nilpotent, there is a prime ideal $\mathfrak{p}$ of A such that $a \notin \mathfrak{p}$. Since A is pctf, the monoid $A/\mathfrak{p}$ is cancellative and $(A/\mathfrak{p})^+$ is torsion-free by Proposition 1.5. Since $(A/\mathfrak{p})[1/a] \cong A[1/a]/\mathfrak{p}[1/a]$, the maps

$$N^{cy}(A;a) \longrightarrow N^{cy}(A/\mathfrak{p};a) \quad \text{and} \quad N^{cy}\left(A\left[\frac{1}{a}\right];a\right) \longrightarrow N^{cy}\left((A/\mathfrak{p})\left[\frac{1}{a}\right];a\right)$$

are isomorphisms by Remark 2.2.1. Our claim follows.

By the equivariant Whitehead Theorem 2.5, the map (3.6.1) is an $\mathbb{S}^1$-homotopy equivalence provided

$$|N^{cy}(A^{\mathfrak{c}};a)|^H = |N^{cy}(A;a)^{\mathfrak{c}}|^H \longrightarrow \left|N^{cy}\left(A\left[\frac{1}{a}\right];a\right)^{\mathfrak{c}}\right|^H = \left|N^{cy}\left(A\left[\frac{1}{a}\right]^{\mathfrak{c}};a\right)\right|^H \tag{3.6.2}$$

is a (non-equivariant) weak equivalence for each closed subgroup H of $\mathbb{S}^1$. By Example 2.8, (3.6.1) is a homotopy equivalence for $H = \mathbb{S}^1$ (because $a \neq 1$ in $A^{\mathfrak{c}}$ implies $a \neq 1$ in $A[1/a]^{\mathfrak{c}}$). Thus, to prove Theorem 3.6, it suffices to show that (3.6.2) is a weak equivalence for $H = C_r$, $r \geqslant 1$. In fact, as we next show, we may assume $r = 1$:

Lemma 3.7. *For any cancellative, torsion-free monoid A, if the map (3.6.1) is a weak equivalence for all non-zero a, then the map (3.6.2) is a weak equivalence for all non-zero a and all closed subgroups $H \subset \mathbb{S}^1$.*

Proof. Fix $H = C_r$. Since A is torsion-free, if $a = b^r$ for some $b \in A$, then b is unique and we have the natural homeomorphism (2.16) for A and $A[1/a]$. Then the assertion that (3.6.2) is a homotopy equivalence for a is equivalent to the assertion that the map

$$|N^{cy}(A; b)^{\mathfrak{c}}| \longrightarrow \left| N^{cy}\left(A\left[\frac{1}{b}\right]; b\right)^{\mathfrak{c}} \right| \tag{3.7.1}$$

is a (non-equivariant) homotopy equivalence, which is the hypothesis. More generally, if $a^{c_1 \cdots c_n} = b^r$ for some n and $b \in A$, then it suffices to prove (3.7.1) is a homotopy equivalence (since we can omit the first $n-1$ terms from each colimit); again, this is a special case of the hypothesis that (3.6.1) is a homotopy equivalence.

Finally, suppose that a^c is not the rth power of an element of A for any $c = c_1 \cdots c_n$ with $n \geqslant 1$. Then the source of (3.6.2) is a one-point space, by Corollary 2.14, and it suffices to prove that the same holds for the target. Again, by Corollary 2.14, it suffices to prove that a^c is not the rth power of an element of $A[1/a]$ for any such c. Say $a^c = u^r$ for some $u \in A[1/a]$. Writing $u = b/a^m$, for some $b \in A$ and $m \geqslant 0$, it follows that

$$u^{rm+1} = (u^r)^m u = a^{mc}\frac{b}{a^m} = a^{m(c-1)} b \in A.$$

Since r and $rm+1$ are relatively prime, there is a positive integer L (specifically, $L = mr(r-1)$) such that every $l \geqslant L$ is a non-negative integer linear combination of r and $mr+1$. It follows that $u^l \in A$ for all $l \geqslant L$. But then $a^{c_1 \cdots c_N} = (u^r)^{c_{n+1} \cdots c_N}$ in A for $N \gg 0$, which is a contradiction to the assumption that a^c is never an rth power in A. □

In summary, we have reduced the proof of Theorem 3.6 to proving that (3.6.1) is a (non-equivariant) weak equivalence when A is cancellative, seminormal and torsion-free and $a \neq 0$. Thus we need to prove that the map

$$\begin{aligned}&\operatorname{colim}\{|N^{cy}(A; a)| \xrightarrow{\theta_{c_1}} |N^{cy}(A; a^{c_1})| \xrightarrow{\theta_{c_2}} \cdots\}\\ &\longrightarrow \operatorname{colim}\left\{\left|N^{cy}\left(A\left[\frac{1}{a}\right]; a\right)\right| \xrightarrow{\theta_{c_1}} \left|N^{cy}\left(A\left[\frac{1}{a}\right]; a^{c_1}\right)\right| \xrightarrow{\theta_{c_2}} \cdots\right\}\end{aligned}$$

is a weak equivalence. As with any filtered system of simplicial sets, the colimits of these sequences of maps coincide with their homotopy colimits (up to homotopy equivalence) and these homotopy colimits are given by the mapping telescope construction; see [**4**, XII.3.5]. It thus suffices to prove that

$$\begin{aligned}&\operatorname{hocolim}\{|N^{cy}(A; a)| \xrightarrow{\theta_{c_1}} |N^{cy}(A; a^{c_1})| \xrightarrow{\theta_{c_2}} \cdots\}\\ &\longrightarrow \operatorname{hocolim}\left\{\left|N^{cy}\left(A\left[\frac{1}{a}\right]; a\right)\right| \xrightarrow{\theta_{c_1}} \left|N^{cy}\left(A\left[\frac{1}{a}\right]; a^{c_1}\right)\right| \xrightarrow{\theta_{c_2}} \cdots\right\}\end{aligned}$$

is a homotopy equivalence. The advantage of using homotopy colimits (that is, mapping telescopes) is that we may replace the θ_{c_i} with homotopy equivalent maps without affecting the homotopy colimit.

For a two-sided A-set B for which the two actions of A on B coincide, if we regard an n-simplex of $sd_r(N^{cy}(B, A))$ as an $r \times (n+1)$ matrix then taking products of the columns defines a simplicial map

$$\begin{gathered}\mu_r : sd_r(N^{cy}(B, A)) \longrightarrow N^{cy}(B, A),\\ \mu_r(b, a_{1,1}, \ldots, a_{1,n}, a_{2,0}, \ldots, a_{r,n}) = ((ba_{2,0} \cdots a_{r,0}), (a_{1,1} \cdots a_{r,1}), \ldots).\end{gathered}$$

For any simplicial set $X_\bullet$, define a continuous map on geometric realizations

$$\psi_r : |sd_r(X_\bullet)| \longrightarrow |X_\bullet|$$

by sending $(x, \vec{u})$ in $sd_r(X)_n \times \Delta^n$ to $(x, \vec{0} \oplus \cdots \oplus \vec{0} \oplus \vec{u}) = (d_0^{(r-1)(n+1)}(x), \vec{u})$ in $|X_\bullet|$. Here, $d_0^{(r-1)(n+1)}$ is the iterated simplicial face map. (ψ_r is the map called $d_{r,0}$ in the proof of [**3**, 2.5].) The map ψ_r is homotopic to the map D_r defined in (2.12) via the homotopy appearing in the proof of [**3**, 2.5]:

$$H_t(x, \vec{u}) = (x, (t/r)\vec{u} \oplus \cdots \oplus (t/r)\vec{u} \oplus (1 - t + t/r)\vec{u}).$$

Here is a picture summarizing the spaces and maps we have introduced so far:

$$\begin{tikzcd}
{|N^{cy}(B,A)|} \arrow[r, dashed, "|\delta_r|"] \arrow[dr, dashed, "\theta_r"] & {|sd_r N^{cy}(B,A)|} \arrow[r, shift left, "D_r \ \cong"] \arrow[r, shift right, "\psi_r"'] \arrow[d, "\mu_r"] & {|N^{cy}(B,A)|} \\
 & {|N^{cy}(B,A)|.} &
\end{tikzcd}$$

The dotted arrows are defined when $A = B$. One may easily verify that the triangle commutes when $A = B$. The map $\psi_r \circ |\delta_r|$ is induced by the simplicial map

$$\alpha_r : N^{cy}(A) \longrightarrow N^{cy}(A)$$

sending $(a_0, \ldots, a_n)$ to $(a_0^r (\prod_{i=1}^n a_i)^{r-1}, a_1, \ldots, a_n)$. In other words, for each m in A, the map

$$\alpha_r : N^{cy}(A, A; m) \longrightarrow N^{cy}(A, A; m^r)$$

is induced by multiplication by m^{r-1} on the first copy of A.

Even though δ_r is not defined unless $A = B$, the simplicial map

$$\alpha_r : N^{cy}(B, A; m) \longrightarrow N^{cy}(B, A; m^r)$$

is defined for any A-subset B of the group completion A^+ and any $m \in A$. In this situation, α_r is the map induced by the endomorphism $b \mapsto bm^{r-1}$ of B. We define the map

$$\theta'_r : |N^{cy}(B, A; m)| \longrightarrow |N^{cy}(B, A; m^r)|$$

to be the composition $\theta'_r = \mu_r \circ D_r^{-1} \circ \alpha_r$. When $B = A$, $\theta'_r = \mu_r \circ D_r^{-1} \circ \psi_r \circ \delta_r$ is homotopic to θ_r.

Since in any mapping telescope one may replace the maps by homotopic ones without affecting the homotopy type, to prove the theorem it suffices to show that

$$\begin{aligned}
&\operatorname{hocolim}\{|N^{cy}(A; m)| \xrightarrow{\theta'_{c_1}} |N^{cy}(A; m^{c_1})| \xrightarrow{\theta'_{c_2}} \cdots\} \\
&\quad \longrightarrow \operatorname{hocolim}\left\{\left|N^{cy}\left(A\left[\frac{1}{m}\right]; m\right)\right| \xrightarrow{\theta'_{c_1}} \left|N^{cy}\left(A\left[\frac{1}{m}\right]; m^{c_1}\right)\right| \xrightarrow{\theta'_{c_2}} \cdots\right\}
\end{aligned}$$

is a homotopy equivalence.

Lemma 3.8. *The natural inclusion map*

$$\left|N^{cy}\left(A\left[\frac{1}{m}\right]; A; m\right)\right| \rightarrowtail \left|N^{cy}\left(A\left[\frac{1}{m}\right]; A\left[\frac{1}{m}\right]; m\right)\right|$$

is a homotopy equivalence.

Proof. As observed in Example 2.9.1, the right-hand side is the classifying space of the group $U(A[1/m])$, since $N^{cy}(A[1/m]; m) = N^{cy}(U(A[1/m]), m)$. Similarly, if we let U_A denote $A \cap U(A[1/m])$, then the left-hand side is

$$N^{cy}\left(A\left[\frac{1}{m}\right], A; m\right) = N^{cy}\left(U\left(A\left[\frac{1}{m}\right]\right), U_A; m\right),$$

and the map $N^{cy}(U(A[1/m]), U_A; m) \to B(U_A)$ sending $(b, a_1, \ldots, a_n)$ to $(a_1, \ldots, a_n)$ is an isomorphism. This shows that the geometric realization $|N^{cy}(A[1/m], A; m)|$ is the classifying space of the monoid U_A, and the map in the lemma is the map on classifying spaces induced by the inclusion of U_A into $U(A[1/m])$. Since $U(A[1/m])$ is the group completion of U_A, Lemma 3.8 follows from the fact that the classifying space of an abelian monoid coincides (up to homotopy equivalence) with that of its group completion. □

We have thus reduced the proof of Theorem 3.6 to showing that the map from

$$\operatorname{hocolim}\{|N^{cy}(A; m)| \xrightarrow{\theta'_{c_1}} |N^{cy}(A; m^{c_1})| \xrightarrow{\theta'_{c_2}} \cdots\}$$

to

$$\operatorname{hocolim}\left\{\left|N^{cy}\left(A\left[\frac{1}{m}\right], A; m\right)\right| \xrightarrow{\theta'_{c_1}} \left|N^{cy}\left(A\left[\frac{1}{m}\right], A; m^{c_1}\right)\right| \xrightarrow{\theta'_{c_2}} \cdots\right\}$$

is a homotopy equivalence for all $m \in A$.

To do this, we form a filtration of $A[1/m]$ by A-subsets by defining, for each non-negative integer l, the A-set

$$B_l = \left\{b \in A\left[\frac{1}{m}\right] \mid bm^l \in A\right\}.$$

Observe that B_0 is A and $\bigcup_{l\geqslant 0} B_l$ is $A[1/m]$. We get a filtration of spaces

$$|N^{cy}(A; m)| = |N^{cy}(B_0, A; m)| \subset |N^{cy}(B_1, A; m)| \subset \cdots \subset \left|N^{cy}\left(A\left[\frac{1}{m}\right], A; m\right)\right|$$

with $|N^{cy}(A[1/m], A; m)| = \bigcup_{l\geqslant 0} |N^{cy}(B_l, A; m)|$. Moreover, the map θ'_r preserves this filtration, since each of the maps α_r, D_r and μ_r does. It therefore suffices to prove that the map from

$$\operatorname{hocolim}\{|N^{cy}(B_l, A; m)| \xrightarrow{\theta'_{c_1}} |N^{cy}(B_l, A; m^{c_1})| \xrightarrow{\theta'_{c_2}} \cdots\}$$

to

$$\operatorname{hocolim}\{|N^{cy}(B_{l+1}, A; m)| \xrightarrow{\theta'_{c_1}} |N^{cy}(B_{l+1}, A; m^{c_1})| \xrightarrow{\theta'_{c_2}} \cdots\}$$

is a homotopy equivalence for all $l \geqslant 0$. But this is clear: For any $r \geqslant 2$, the map

$$|N^{cy}(B_{l+1}, A; m)| \xrightarrow{\theta'_r} |N^{cy}(B_{l+1}, A; m^r)|$$

lands in the subspace $|N^{cy}(B_l, A; m^r)|$, since α_r sends the n-simplex $(b, a_1, \ldots, a_n)$ to $(bm^{r-1}, a_1, \ldots, a_n)$, and if $bm^{l+1} \in A$, then $bm^{r-1}m^l \in A$.

This completes the proof of the Comparison Theorem 3.6.

3.3. *Topological Hochschild homology*

We now interpret Theorem 3.6 in terms of topological Hochschild homology.

For a ring k, we write $TH(k)$ for the *topological Hochschild homology* spectrum of k. It is a cyclotomic spectrum in the sense of [**18**, 2.2]; see [**18**, 2.4] for details. For a prime p and integer $n \geqslant 1$, one defines the ordinary spectrum

$$TR^n(k; p) := TH(k)^{C_{p^{n-1}}}. \tag{3.9}$$

There are natural maps of ordinary spectra, called *Restriction* and *Frobenius*, of the form

$$\operatorname{Res}, F : TR^n(k; p) \longrightarrow TR^{n-1}(k, p).$$

The ordinary spectrum $TC^n(k;p)$ is the homotopy equalizer of the maps Res, F. The *topological cyclic homology at* p of the ring k is the pro-spectrum

$$\{TC^n(k;p)\} = \{\cdots \xrightarrow{\text{Res}} TC^3(k;p) \xrightarrow{\text{Res}} TC^2(k;p) \xrightarrow{\text{Res}} TC^1(k;p)\},$$

where the transition maps are given by *Res*.

We shall need the following important theorem of Hesselholt and Madsen [**18**, 7.1]. Recall that a map of $\mathbb{S}^1$-spectra $X \to Y$ is called an *$\mathcal{F}$-equivalence* if the induced map on fixed point spectra $X^C \to Y^C$ is a weak equivalence of (non-equivariant) spectra for all finite subgroups C of $\mathbb{S}^1$.

THEOREM 3.10 (Hesselholt–Madsen). *For any ring k and monoid A, there is a natural $\mathcal{F}$-equivalence*

$$TH(k) \wedge |N^{cy}(A)| \xrightarrow{\sim_{\mathcal{F}}} TH(k[A])$$

of cyclotomic spectra. In particular, we have an equivalence of ordinary spectra

$$TR^n(k[A],p) \sim (TH(k) \wedge |N^{cy}(A)|)^{C_{p^{n-1}}}$$

for all primes p and integers $n \geqslant 1$.

Here is an immediate consequence of the Comparison Theorem 3.6 and the Hesselholt–Madsen Theorem 3.10. The colimit $TR^n(k[A];p)^{\mathfrak{c}}$ is defined in (3.4).

COROLLARY 3.11. *Let k be a ring, let A be a pctf monoid, and let $\mathfrak{c} = (c_1, c_2, \ldots)$ be an infinite sequence of integers with $c_i \geqslant 2$ for all i. Then we have an equivalence of ordinary spectra, natural in k and A:*

$$TR^n(k[A];p)^{\mathfrak{c}} \xrightarrow{\sim} (TH(k) \wedge |\tilde{N}^{cy}(A)^{\mathfrak{c}}|)^{C_{p^{n-1}}}.$$

4. $\tilde{\Omega}$ *and affine excision*

The cyclic set $\tilde{N}^{cy}(A)$ is best behaved for seminormal monoids; see Example 3.3.1. Since we will need to consider non-seminormal monoids in this paper, we introduce $\tilde{\Omega}_A$, a slight modification of the functor $A \mapsto \tilde{N}^{cy}(A)$. Recall that A_{sn} denotes the seminormalization of A; if A is a pc monoid, then A_{sn} exists by Proposition 1.15.

DEFINITION 4.1. For a pc monoid A, define $\tilde{\Omega}_A$ to be the $\mathbb{S}^1$-space

$$\tilde{\Omega}_A = |\tilde{N}^{cy}(A_{\text{sn}})|.$$

Since $A \mapsto A_{\text{sn}}$ is a functor, the assignment $A \mapsto \tilde{\Omega}_A$ is also functorial. Moreover, there is a natural map $|N^{cy}(A)| \to \tilde{\Omega}_A$ of $\mathbb{S}^1$-spaces.

If I is an ideal in a pc monoid A, then $\tilde{\Omega}_{A/I} = |\tilde{N}^{cy}(A_{\text{sn}}/J)|$, where $J = \sqrt{IA_{\text{sn}}}$, because $A_{\text{sn}}/J \to (A/I)_{\text{sn}}$ is an isomorphism by Lemma 1.14. The quotient map $A \to A/I$ induces a map $\tilde{\Omega}_A \to \tilde{\Omega}_{A/I}$; we claim that this map is onto. To see this, we may assume A seminormal and I radical. Then, for each $a \notin I$ the monoid $A[1/a]$ is seminormal by Lemma 1.9, and $(A/I)[1/a] = (A[1/a])/(I[1/a])$. Thus the claim follows from the observation in Remark 2.2.1 that $N^{cy}(A;a) \overset{\cong}{\to} N^{cy}(A/I, a)$.

DEFINITION 4.2. Given an ideal I in a pc monoid A, we define $\tilde{\Omega}_{A,I}$ to be the fiber over the basepoint of $\tilde{\Omega}_{A/I}$ of the surjection $\tilde{\Omega}_A \twoheadrightarrow \tilde{\Omega}_{A/I}$ induced by the canonical map $A \twoheadrightarrow A/I$; by construction, $\tilde{\Omega}_{A,I}$ is the realization of a cyclic set, and hence an $\mathbb{S}^1$-space. Since $(A/I)_{\mathrm{sn}} = (A/\sqrt{I})_{\mathrm{sn}}$, we have $\tilde{\Omega}_{A,I} = \tilde{\Omega}_{A,\sqrt{I}}$.

LEMMA 4.3. *If A is a pc monoid and I is an ideal, then*

$$\tilde{\Omega}_{A,I} \rightarrowtail \tilde{\Omega}_A \twoheadrightarrow \tilde{\Omega}_{A/I}$$

is a split cofibration sequence of $\mathbb{S}^1$-spaces. In particular, this sequence determines a fibration sequence of $\mathbb{S}^1$-spectra upon smashing it with any $\mathbb{S}^1$-spectrum.

Proof. Set $J = \sqrt{IA_{\mathrm{sn}}}$. By the definition of $\tilde{N}^{cy}(A)$ and the above remarks, there is an $\mathbb{S}^1$-equivariant decomposition

$$\tilde{\Omega}_{A,I} = \bigvee_{a \in J} \left| N^{cy}\left(A_{\mathrm{sn}}\left[\frac{1}{a}\right]; a\right)\right|. \tag{4.3.1}$$

This immediately implies that $\tilde{\Omega}_A \cong \tilde{\Omega}_{A/I} \bigvee \tilde{\Omega}_{A,I}$ equivariantly. □

Our next result shows that $\tilde{\Omega}$ satisfies ‘affine excision’.

PROPOSITION 4.4. *Suppose that $f : A \to B$ is a homomorphism of seminormal pc monoids, $I \subset A$ and $I' \subset B$ are radical ideals such that f maps I bijectively onto I', and the induced map $A[1/a] \to B[1/f(a)]$ is an isomorphism for all $a \in I$. Then*

$$\begin{tikzcd} \tilde{\Omega}_A \arrow[r] \arrow[d] & \tilde{\Omega}_{A/I} \arrow[d] \\ \tilde{\Omega}_B \arrow[r] & \tilde{\Omega}_{B/I'} \end{tikzcd}$$

induces a homotopy cartesian square of $\mathbb{S}^1$-spectra upon smashing with any $\mathbb{S}^1$-spectrum.

Proof. It suffices to prove that the induced map $\tilde{\Omega}_{A,I} \to \tilde{\Omega}_{B,I'}$ is an isomorphism. This is immediate from the description (4.3.1) of these spaces and the assumptions. □

EXAMPLE 4.5. Suppose that I, J are radical ideals of a seminormal pc monoid A such that $I \cap J = 0$. Taking $B = A/J$ in Proposition 4.4, we see that

$$\begin{tikzcd} \tilde{\Omega}_A \arrow[r] \arrow[d] & \tilde{\Omega}_{A/I} \arrow[d] \\ \tilde{\Omega}_{A/J} \arrow[r] & \tilde{\Omega}_{A/(I \cup J)} \end{tikzcd}$$

induces a homotopy cartesian square of $\mathbb{S}^1$-spectra upon smashing with any $\mathbb{S}^1$-spectrum.

Corollary 4.6. *Suppose that A is cancellative and seminormal. Let $I \subset A$ be the conductor ideal of the inclusion $A \rightarrowtail A_{\mathrm{nor}}$. Then*

$$\begin{tikzcd}
\tilde{\Omega}_A \arrow[r] \arrow[d] & \tilde{\Omega}_{A/I} \arrow[d] \\
\tilde{\Omega}_{A_{\mathrm{nor}}} \arrow[r] & \tilde{\Omega}_{A_{\mathrm{nor}}/I}
\end{tikzcd}$$

induces a homotopy cartesian square of $\mathbb{S}^1$-spectra upon smashing it with any $\mathbb{S}^1$-spectrum.

Proof. By Lemma 1.19, I is a radical ideal of A and A_{nor}. For $a \in I$, the canonical map $A[1/a] \to A_{\mathrm{nor}}[1/a]$ is an isomorphism, since any element of $A_{\mathrm{nor}}[1/a]$ can be written as $x/a^n = (ax)/a^{n+1}$ and $ax \in A$. Now the result follows from Proposition 4.4. □

5. *Zariski and* cdh *descent*

In this section, we introduce the technique of descent. Functors on monoids can be promoted to presheaves on monoid schemes using Zariski descent, and *cdh* descent will be used to prove our main result, Theorem 8.3.

For any Grothendieck topology t on the category $\mathcal{M}_{\mathrm{pctf}}$, we may impose a model structure on the category of presheaves of spectra on $\mathcal{M}_{\mathrm{pctf}}$ as follows (see [**19**]): A morphism $\mathcal{F} \to \mathcal{F}'$ of presheaves is a cofibration provided $\mathcal{F}(X) \to \mathcal{F}'(X)$ is a cofibration of spectra for all X. A morphism is a weak equivalence, provided that, for all $n \in \mathbb{Z}$, the t-sheafification of the induced map of presheaves of abelian groups $\pi_n \mathcal{F} \to \pi_n \mathcal{F}'$ is an isomorphism:

$$\pi_n \mathcal{F}(-)^{\sim}_t \xrightarrow{\cong} \pi_n \mathcal{F}'(-)^{\sim}_t .$$

As usual, fibrations are determined by the lifting property with respect to trivial cofibrations. The model structure comes equipped with a functorial fibrant replacement functor, which we fix. Given a presheaf of spectra $\mathcal{F}$ on $\mathcal{M}_{\mathrm{pctf}}$, we write $\mathbb{H}_t(-, \mathcal{F})$ for the fibrant replacement of $\mathcal{F}$ for this model structure. Thus, $\mathbb{H}_t(-, \mathcal{F})$ is a fibrant object for this model structure and there is a natural cofibration and weak equivalence $\mathcal{F} \rightarrowtail \mathbb{H}_t(-, \mathcal{F})$. We say that $\mathcal{F}$ satisfies *t-descent* if $\mathcal{F}(X) \to \mathbb{H}_t(X, \mathcal{F})$ is weak equivalence of spectra for all $X \in \mathcal{M}_{\mathrm{pctf}}$.

The category of presheaves of spaces on a site is similarly equipped with a closed model structure; just replace 'spectra' by 'space' everywhere.

All of the Grothendieck topologies we are interested in will be determined by a *cd structure* [**23**], which is a specified family of commutative squares

$$\begin{tikzcd}
D \arrow[r, "f"'] \arrow[d, "q"] & Y \arrow[d, "p"'] \\
C \arrow[r, "e"] & X
\end{tikzcd} \tag{5.1}$$

in $\mathcal{M}_{\mathrm{pctf}}$. The associated topology is the minimal Grothendieck topology such that $\{e, p\}$ is a covering sieve for every square in the cd structure.

For example, the Zariski topology on $\mathcal{M}_{\mathrm{pctf}}$ may be defined as the topology associated to the *Zariski cd structure*, which by definition consists of all squares (5.1) for which Y, C form an open cover of X, p and e are the canonical inclusions, and $D = C \times_X Y = C \cap Y$. Affine monoid schemes admit no non-trivial covers in the Zariski topology, since MSpec A has a unique closed point.

LEMMA 5.2. *For any affine $U \in \mathcal{M}_{pctf}$, and any presheaf of spectra $\mathcal{F}$ on $\mathcal{M}_{pctf}$, the morphism $\mathcal{F}(U) \to \mathbb{H}_{zar}(U, \mathcal{F})$ is a weak homotopy equivalence.*

Proof. Let x be the unique closed point of U. Then U represents the stalk of the monoid scheme U at x. Since $\mathcal{F} \to \mathbb{H}_{\mathrm{zar}}(-, \mathcal{F})$ is a Zariski weak equivalence, the stalks of the map $\pi_q\mathcal{F} \to \pi_q\mathbb{H}_{\mathrm{zar}}(-, \mathcal{F})$ are isomorphisms. But the stalk at x is the map $\pi_q\mathcal{F}(U) \xrightarrow{\cong} \pi_q\mathbb{H}_{\mathrm{zar}}(U, \mathcal{F})$. Hence the map $F(U) \to \mathbb{H}_{\mathrm{zar}}(U, \mathcal{F})$ is a weak homotopy equivalence. □

REMARK 5.2.1. Lemma 5.2 also holds for presheaves of spaces, with the same proof.

We also have the *cdh* cd structure on $\mathcal{M}_{\mathrm{pctf}}$, which consists of the squares in the Zariski cd structure together with all *abstract blow-ups*. The latter refers to a square (5.1) such that $p : Y \to X$ is proper (in the sense of [**9**, 8.4]), e is the inclusion of an equivariant, closed subscheme C into X, $D = C \times_X Y$ and the induced map $p : Y \setminus D \to X \setminus C$ is an isomorphism. The associated topology is called the *cdh* topology on $\mathcal{M}_{\mathrm{pctf}}$.

Let $\mathcal{F}$ be a presheaf of spectra defined on $\mathcal{M}_{\mathrm{pctf}}$. We say that $\mathcal{F}$ satisfies the *Mayer–Vietoris property* for a square (5.1) if applying $\mathcal{F}$ to the square results in a homotopy cartesian square of spectra. If a cd structure is complete (see [**23**, 2.3 and 2.4]), regular (see [**23**, 2.10]) and bounded (see [**23**, 2.21]), then a presheaf $\mathcal{F}$ has the Mayer–Vietoris property for every square in the cd structure if and only if $\mathcal{F}$ satisfies descent for the associated Grothendieck topology.

EXAMPLE 5.3. (a) The Zariski cd structure is complete, regular and bounded. Thus $\mathcal{F}$ has Zariski descent if and only if it has the Mayer–Vietoris property for every square in the Zariski cd structure; see [**9**, Section 12]. Given any presheaf of spectra $\mathcal{F}$ on $\mathcal{M}_{\mathrm{pctf}}$, $\mathbb{H}_{\mathrm{zar}}(-, \mathcal{F})$ is a presheaf of spectra on $\mathcal{M}_{\mathrm{pctf}}$ which satisfies the Mayer–Vietoris property for every square in the Zariski cd structure.

(b) The *cdh* structure also is complete, regular and bounded; see [**9**, 12.9]. Thus a presheaf of spectra $\mathcal{F}$ satisfies *cdh* descent if and only if it has the Mayer–Vietoris property for every square in the *cdh* structure. Given any presheaf of spectra $\mathcal{F}$ on $\mathcal{M}_{\mathrm{pctf}}$, $\mathbb{H}_{\mathrm{cdh}}(-, \mathcal{F})$ is a presheaf of spectra which satisfies the Mayer–Vietoris property for every square in the *cdh* structure.

LEMMA 5.4. *Let $\{F_i\}_{i\in I}$ be a filtering system of presheaves of spectra on $\mathcal{M}_{pctf}$. Then the natural map of presheaves*

$$\operatorname{colim} \mathbb{H}_{zar}(-, F_i) \longrightarrow \mathbb{H}_{zar}(-, \operatorname{colim} F_i)$$

is an (object-wise) weak equivalence, and similarly for the cdh topology. If each of the presheaves F_i satisfies Zariski descent (respectively, cdh descent), then so does the presheaf $F = \operatorname{colim} F_i$.

Proof. The last sentence follows from the preceding one in light of the fact that filtering colimits of spectra commute with weak equivalences. That the fibrant replacement commutes (up to weak equivalence) with filtering colimits is a consequence of the fact that the Zariski and *cdh* sites are noetherian; see [**1**, Exp. VI, 2.11 and 5.2]. In *loc. cit.*, this is proved for ordinary cohomology of sheaves; however, since the Zariski and *cdh* sites have locally bounded cohomological dimension, the corresponding descent spectral sequences converge to give the desired result for the hypercohomology spectra $\mathbb{H}_{\mathrm{zar}}(-, F)$ and $\mathbb{H}_{\mathrm{cdh}}(-, F)$. □

We will need an alternative characterization of *cdh* descent, which is provided by the following proposition. Recall from [**9**, Section 10] that the underlying space of a monoid scheme of finite type is a finite poset, and that the *height* of a point x in a monoid scheme X is the largest integer n such that there exists a strictly decreasing chain $x = x_n > \cdots > x_0$ in the poset underlying X. Recall also [**9**, 6.4] that $Y \in \mathcal{M}_{\text{pctf}}$ is *smooth* if each stalk is the product of a free group of finite rank and a free monoid of finite rank.

PROPOSITION 5.5. *Suppose that $\mathcal{G}$ is a presheaf of spectra on $\mathcal{M}_{pctf}$ with $\mathcal{G}(\emptyset) \sim *$. Then $\mathcal{G}$ satisfies cdh descent if and only if it has the Mayer–Vietoris property for every commutative square* (5.1) *in $\mathcal{M}_{pctf}$ that satisfies one of the following conditions*:

(1) *$C = D = \emptyset$ and $Y \to X$ is the semi-normalization of X;*
(2) *C and Y form an open covering of X and $D = C \cap Y$;*
(3) *X is cancellative and seminormal, $Y \to X$ is the normalization of X, C is the subscheme cut out locally by the conductor ideals, and $D = C \times_Y X$;*
(4) *X is seminormal, C and Y are equivariant closed subschemes of X that form a closed covering of X, and $D = C \times_X Y$;*
(5) *X and Y are connected and normal, p is proper and birational, and C and D are the reduced, equivariant closed subschemes of X and Y determined by the closures of the height one points of each.*

Finally, if $\mathcal{G}$ has cdh descent and $\mathcal{G}(Y)$ is contractible for all smooth Y in $\mathcal{M}_{pctf}$, then $\mathcal{G}(X)$ is contractible for all X in $\mathcal{M}_{pctf}$.

Proof. The proof will proceed in a number of steps which we outline now. Step 1 shows that *cdh* descent for a sheaf $\mathcal{G}$ implies the Mayer–Vietoris property for the types of squares listed in the statement; the hard case is a square of type (1). Step 2 demonstrates that if $\mathcal{G}$ satisfies the Mayer–Vietoris property for squares of types (1) through (5), then it satisfies the Mayer–Vietoris property for a larger class of squares that includes in particular equivariant blow-ups of monoid schemes associated to fans; this step is the technical core of the proof. Step 3 proves that $\mathcal{G}$ satisfies *cdh* descent by induction on the dimension of X. In the process, the final assertion of the proposition will also be proved.

Step 1: We first assume that $\mathcal{G}$ satisfies *cdh* descent and show it has the Mayer–Vietoris property for each of the five types of squares. Squares of type (2), (3) and (4) belong to the cd structure defining the *cdh* topology and hence $\mathcal{G}$ has the Mayer–Vietoris property for each of them by Example 5.3. Additionally, the abstract blow-up square (5.1) with $C = X_{\text{red}}$ and $Y = D = \emptyset$ belongs to the *cdh* structure, so $\mathcal{G}(X) \to \mathcal{G}(X_{\text{red}})$ is a weak equivalence for all X. For X, Y, C, D as in a square of type (5), the cartesian square given by X, Y, C and $C \times_X Y$ is an abstract blow-up and hence, again by Example 5.3, $\mathcal{G}$ has the Mayer–Vietoris property for it. Since $D = (C \times_X Y)_{\text{red}}$, it follows that $\mathcal{G}$ has the Mayer–Vietoris for squares of type (5).

It remains to establish the Mayer–Vietoris property for squares satisfying (1). This is equivalent to showing that $\mathcal{G}(X) \to \mathcal{G}(X_{\text{sn}})$ is an equivalence for all X. We proceed by induction on the Krull dimension of X. Since we have seen that $\mathcal{G}(X) \to \mathcal{G}(X_{\text{red}})$ is an equivalence, we may assume that X is reduced.

If $\dim(X) = 0$, then $X = X_{\text{red}}$ is the disjoint union of the affine monoid schemes associated to abelian groups. Hence $X_{\text{sn}} = X$, and we are done.

For $\dim(X) > 0$, let C denote the reduced, closed equivariant subscheme of X determined by the closure of the (finitely many) height one points of X. By Lemma 1.22, the map $X_{\text{sn}} \to X$ is finite, and induces an isomorphism over $X \setminus C$. Since X, X_{sn}, C and $D = C \times_X X_{\text{sn}}$ form an abstract blow-up square, $\mathcal{G}$ has the Mayer–Vietoris property for this square, and it suffices to show that $\mathcal{G}(C) \to \mathcal{G}(D)$ is a weak equivalence. By Lemma 1.23, the map $D_{\text{red}} \to C$ is the seminormalization of C. Since $\dim(C) < \dim(X)$, the map $\mathcal{G}(C) \to \mathcal{G}(D_{\text{red}})$ is a weak equivalence by the induction hypothesis. Since $\mathcal{G}(D) \to \mathcal{G}(D_{\text{red}})$ is a weak equivalence, so is $\mathcal{G}(C) \to \mathcal{G}(D)$.

Step 2: Now we assume that $\mathcal{G}$ has the Mayer–Vietoris property for each of the five types of squares appearing in the statement and prove that $\mathcal{G}$ has the Mayer–Vietoris property for any square (5.1) of type (5′), which we define to be one that satisfies the following conditions:

(5′) *X and Y are connected and normal, C is any reduced equivariant closed subscheme, $D = C \times_X Y$, p is proper and the induced map $Y \setminus D \to X \setminus C$ is an isomorphism.*

To prove the Mayer–Vietoris property for such squares, we proceed by induction on the number of points of $X \setminus C$. If this number is 0, there is nothing to prove since $C = X$ and $D = Y$. If this number is 1, then $X \setminus C$ is the generic point of X and X, Y, C, D_{red} form a square of type (5); by assumption, $\mathcal{G}$ has the Mayer–Vietoris property for this square, and we are done since $D_{\text{red}} = C_{\text{sn}}$ by Lemma 1.23, and $\mathcal{G}(D) \simeq \mathcal{G}(D_{\text{red}}) \simeq \mathcal{G}(C)$ by (1).

In general, choose a maximal point η of $X \setminus C$ and let C' denote the reduced equivariant closed subscheme of X defined by the closure of the points of $C \cup \{\eta\}$. Setting $D' = C' \times_X Y$, we have the diagram

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d] & Y \arrow[d] \\
C \arrow[r] & C' \arrow[r] & X.
\end{tikzcd}$$

By the induction hypothesis, $\mathcal{G}$ has the Mayer–Vietoris property for the right-hand square. Thus it suffices to prove $\mathcal{G}$ has the Mayer–Vietoris property for the left-hand square. The scheme C' might be only partially cancellative, but it is always seminormal (by Lemma 1.23), η is a generic point of C' and each of the irreducible components of C' is normal (by Proposition 1.5). Let C'_1 be the irreducible component of C' with generic point η, and let C_2 denote the union of the other irreducible components of C'. Then we have closed coverings $C' = C'_1 \cup C_2$ and $C = C_1 \cup C_2$, where $C_1 = C \cap C'_1$. It follows from (4) that the right square and outer square in the following diagram are homotopy cartesian.

$$\begin{tikzcd}
\mathcal{G}(C') \arrow[r] \arrow[d] & \mathcal{G}(C) \arrow[r] \arrow[d] & \mathcal{G}(C_2) \arrow[d] \\
\mathcal{G}(C'_1) \arrow[r] & \mathcal{G}(C_1) \arrow[r] & \mathcal{G}(C_1 \cap C_2)
\end{tikzcd}$$

It follows that $\text{hofiber}(\mathcal{G}(C') \to \mathcal{G}(C)) \sim \text{hofiber}(\mathcal{G}(C'_1) \to \mathcal{G}(C_1))$.

Since $Y \setminus D \cong X \setminus C$, there is a unique point $\tilde{\eta}$ in D' mapping to η. If D'_1 is the irreducible component of D' with generic point $\tilde{\eta}$ and D_2 is the union of the other components, then we have similar closed coverings $D' = D'_1 \cup D_2$ and $D = D_1 \cup D_2$ where $D_1 = D \cap D'_1$. By the same argument, it follows from (4) that $\text{hofiber}(\mathcal{G}(D') \to \mathcal{G}(D)) \sim \text{hofiber}(\mathcal{G}(D'_1) \to \mathcal{G}(D_1))$. Hence it suffices to show that

$$\begin{tikzcd}
\mathcal{G}(C'_1) \arrow[r] \arrow[d] & \mathcal{G}(C_1) \arrow[d] \\
\mathcal{G}(D'_1) \arrow[r] & \mathcal{G}(D_1)
\end{tikzcd}$$

is homotopy cartesian; this follows from (5).

Step 3: We now assume that $\mathcal{G}$ has the Mayer–Vietoris property for all squares of type (1)–(4) and (5′). In particular, $\mathcal{G}$ has the Mayer–Vietoris property for all Zariski squares as well as all squares (5.1) in which X is smooth, C is an equivariant smooth closed subscheme and Y

is the blow-up of X along C (these are called *smooth blow-up squares*). Thus, by Cortiñas, Haesemeyer, Walker and Weibel [**9**, 12.12 and 12.13], we have

$$\mathcal{G}(Y) \xrightarrow{\sim} \mathbb{H}_{\mathrm{cdh}}(Y, \mathcal{G})$$

for all smooth Y in $\mathcal{M}_{\mathrm{pctf}}$. That is, the homotopy fiber $\mathcal{F}$ of the map $\mathcal{G} \to \mathbb{H}_{\mathrm{cdh}}(-, \mathcal{G})$ satisfies $\mathcal{F}(Y) \sim *$ for all smooth Y. Since $\mathcal{G}$ and (by what we have already proved) $\mathbb{H}_{\mathrm{cdh}}(-, \mathcal{G})$ have the Mayer–Vietoris property for squares of type (1)–(4) and (5′), so does $\mathcal{F}$.

Thus it suffices to prove that if $\mathcal{F}$ is a presheaf of spectra on $\mathcal{M}_{\mathrm{pctf}}$ that has the Mayer–Vietoris property for squares (1)–(4) and (5′), and is such that $\mathcal{F}(Y)$ is contractible for all smooth Y in $\mathcal{M}_{\mathrm{pctf}}$, then $\mathcal{F}(X)$ is contractible for all X in $\mathcal{M}_{\mathrm{pctf}}$. In light of what we already proved, this will also establish the final assertion of the proposition.

We proceed by induction on the dimension of X. Since $\mathcal{F}(X) \xrightarrow{\sim} \mathcal{F}(X_{\mathrm{sn}})$ by (1), we may assume X is seminormal. In particular, the closures of the minimal points of X form a closed covering of X by cancellative and seminormal equivariant closed subschemes (using Lemma 1.10 and Proposition 1.5). Using the Mayer–Vietoris property for squares of type (4), and the inductive assumption, we may assume that X itself is connected, seminormal and cancellative. Using the Mayer–Vietoris property for normalization squares of type (3), and the induction hypothesis, we may assume that X is also normal. By Example 1.20.1, X is the toric monoid scheme associated to a fan. In this case, by subdividing this fan, we may form an abstract blow-up square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & Y \arrow[d] \\ C \arrow[r] & X \end{tikzcd}$$

such that Y is smooth and C is a reduced, equivariant closed subscheme with $\dim(C) < \dim(X)$. Using the assumption that $\mathcal{F}(Y)$ is contractible, the inductive hypothesis that $\mathcal{F}(C)$ and $\mathcal{F}(D)$ are contractible and the Mayer–Vietoris property for squares of type (5′), we conclude that $\mathcal{F}(X)$ is contractible. □

It will be useful to weaken the hypotheses of Proposition 5.5 even further. Recall that each X in $\mathcal{M}_{\mathrm{pctf}}$ is separated, so that (by Cortiñas, Haesemeyer, Walker and Weibel [**9**, 3.8]) the intersection of affine open subschemes is again affine open.

LEMMA 5.6. *Suppose that $\mathcal{G}$ is a presheaf of spectra on $\mathcal{M}_{pctf}$ with $\mathcal{G}(\emptyset) \sim *$. If $\mathcal{G}$ satisfies Zariski descent, and $\mathcal{G}$ has the Mayer–Vietoris property for all squares of type* (1), (3), (4) *and* (5) *in Proposition* 5.5, *which satisfy the additional condition that X is affine, then $\mathcal{G}$ satisfies cdh descent.*

Proof. It suffices to prove that $\mathcal{G}$ has the Mayer–Vietoris property for all squares (5.1) of type (1) and (3)–(5) in Proposition 5.5. Let $\square_X$ denote any such square, and, for an open subscheme $U \subset X$, let $\square_U$ denote the pullback of $\square_X$ along $U \rightarrowtail X$. Write $\mathcal{G}(\square_U)$ for the result of applying $\mathcal{G}$ to this square and taking iterated homotopy fibers. The hypotheses imply that $\mathcal{G}(\square_U)$ is contractible for any affine open U, and we need to prove that $\mathcal{G}(\square_X)$ is contractible.

Since $\mathcal{G}$ has the Mayer–Vietoris property for Zariski squares, given $X = U \cup V$, the spectra $\mathcal{G}(\square_X)$, $\mathcal{G}(\square_U)$, $\mathcal{G}(\square_V)$, $\mathcal{G}(\square_{U\cap V})$ fit together to form a homotopy cartesian square. (This may be seen by consideration of the evident four-dimensional cube of spectra and properties of homotopy fibers.) This shows that $U \mapsto \mathcal{G}(\square_U)$ has the Mayer–Vietoris property on X_{zar} for open covers. Since $\mathcal{G}(\square_U) \sim *$ for all affine U, it follows that $\mathcal{G}(\square_X) \sim *$. □

6. *Descent for $\tilde{\Omega}$ and dilated TC*

The main goal of this section is to prove Corollary 6.6, that the functors $X \mapsto TC^n(X_k;p)^{\mathfrak{c}}$ satisfy *cdh* descent. We begin by considering the presheaf $\tilde{\Omega}$.

Descent for $\tilde{\Omega}$. In Definition 4.1, we introduced a covariant functor $\tilde{\Omega}$ from monoids to $\mathbb{S}^1$-spaces. We promote it to a contravariant functor from the category of monoid schemes to $\mathbb{S}^1$-spaces by the formula

$$(X,\mathcal{A}) \longmapsto \tilde{\Omega}_{\mathcal{A}(X)}.$$

By Remark 5.2.1, if $X = \mathrm{MSpec}(A)$ is affine, then the natural map

$$\tilde{\Omega}_A \longrightarrow \mathbb{H}_{\mathrm{zar}}(X,\tilde{\Omega}_{\mathcal{A}})$$

is a weak homotopy equivalence.

Definition 6.1. For an $\mathbb{S}^1$-spectrum T the $\mathbb{S}^1$-equivariant smash product $\tilde{\Omega}_{\mathcal{A}} \wedge T$ is a presheaf of $\mathbb{S}^1$-spectra on $\mathcal{M}_{\mathrm{pctf}}$. For integers $r \geqslant 1$, we write $\tilde{\Omega}^{T,r}$ for the presheaf of fixed-point spectra $U \mapsto (\tilde{\Omega}_{\mathcal{A}(U)} \wedge T)^{C_r}$ on $\mathcal{M}_{\mathrm{pctf}}$. Note that $\tilde{\Omega}^{T,1} = \tilde{\Omega}_{\mathcal{A}} \wedge T$. We consider the fibrant replacements $\mathbb{H}_{\mathrm{zar}}(-,\tilde{\Omega}^{T,r})$.

Remark 6.1.1. For any affine pctf scheme $X = \mathrm{MSpec}(A)$, Lemma 5.2 implies that $\tilde{\Omega}_A \wedge T \to \mathbb{H}_{\mathrm{zar}}(X,\tilde{\Omega}_{\mathcal{A}} \wedge T)$ and $(\tilde{\Omega}_A \wedge T)^{C_r} \to \mathbb{H}_{\mathrm{zar}}(X,\tilde{\Omega}^{T,r}_{\mathcal{A}})$ are weak equivalences.

Theorem 6.2. *For any $\mathbb{S}^1$-spectrum T and integer $r \geqslant 1$, the presheaf of spectra $\mathbb{H}_{\mathrm{zar}}(-,\tilde{\Omega}^{T,r})$ satisfies cdh descent on $\mathcal{M}_{pctf}$.*

Proof. It suffices to check that $\mathbb{H}_{\mathrm{zar}}(-,\tilde{\Omega}^{T,r})$ has the Mayer–Vietoris property for each of the five types of squares (5.1) listed in Proposition 5.5. Case (2) holds by construction, since $\mathbb{H}_{\mathrm{zar}}(-,\tilde{\Omega}^{T,r})$ satisfies Zariski descent. For squares of type (1), (3)–(5), we may assume that X is affine by Lemma 5.6; say $X = \mathrm{MSpec}\,A$. Case (1) is immediate, since X_{sn} and X are homeomorphic and $\tilde{\Omega}_A = \tilde{\Omega}_{A_{\mathrm{sn}}}$ by definition. In particular $\mathbb{H}_{\mathrm{zar}}((-)_{\mathrm{red}},\tilde{\Omega}^{T,r}) \sim \mathbb{H}_{\mathrm{zar}}(-,\tilde{\Omega}^{T,r})$, so we may restrict to proving the remaining cases when all schemes in each of the squares are reduced. Cases (3) and (4) follow from Corollary 4.6 and Example 4.5 using Lemma 5.2, since in these cases C, Y and D are affine when X is.

For case (5), assuming that X is affine does not simplify the argument, so we do not assume it. As pointed out in Example 1.20.1, the hypotheses imply that X is a toric monoid scheme, that is, that X is the monoid scheme associated to a lattice N and a fan Δ in $N_{\mathbb{R}}$ and Y is the monoid scheme associated to a subdivision $\tilde{\Delta}$ of Δ; see [9, 4.3, 8.16]. The underlying space of X is the poset of cones in the fan Δ, and $\{0\}$ is the (open) minimal point of X. Moreover, C, D are the reduced, equivariant closed subschemes associated to the closed subsets $\Delta \setminus \{0\}$ and $\tilde{\Delta} \setminus \{0\}$. As a matter of notation, for $\sigma \in \Delta$, write $U_\sigma \subset X$ for the corresponding affine open subscheme; by construction [9, 4.2], $\mathcal{A}_X(U_\sigma) = (\sigma^\vee \cap M)_*$, where $M := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$.

Following Danilov [10], our next goal, achieved in (6.2.2), is to understand the homotopy fiber of

$$\mathbb{H}_{\mathrm{zar}}(X,\tilde{\Omega}^{T,r}) \longrightarrow \mathbb{H}_{\mathrm{zar}}(C,\tilde{\Omega}^{T,r}).$$

Let $\mathcal{I} \subset \mathcal{A}_X$ denote the quasi-coherent sheaf of ideals cutting out C in X. We have $\mathcal{I}(U_\sigma) = \{m \in M \,|\, m > 0 \text{ on } \sigma \setminus \{0\}\}_*$. Using that $\mathcal{I}$ and $\mathcal{A}_X$ are sheaves, it follows that, for every open subscheme U of X, we have

$$\mathcal{A}_X(U) = \{m \in M \mid m \geqslant 0 \text{ on } |U|\}_*$$

and

$$\mathcal{I}(U) = \{m \in M \mid m > 0 \text{ on } |U| \setminus \{0\}\}_*,$$

where $|U|$ denotes the closed subset of $N_{\mathbb{R}}$ given as a union of the cones in the set $U \subset X = \Delta$. For example, if $U = \{0\}$, then $|U| = \{0\}$ and $\mathcal{I}(U) = \mathcal{A}(U) = M_*$.

Let X_{zar} be the category whose objects are the open monoid subschemes of X and whose morphisms are the inclusions. We have the presheaf of spectra on X_{zar},

$$U \longmapsto (\tilde{\Omega}_{\mathcal{A}_X(U), I(U)} \wedge T)^{C_r},$$

where $\tilde{\Omega}_{A,I}$ is defined in Definition 4.2, and we define the spectrum

$$\mathbb{H}_{\mathrm{zar}}((X,C), \tilde{\Omega}^{T,r}) := \mathbb{H}_{\mathrm{zar}}(X, U \longmapsto (\tilde{\Omega}_{\mathcal{A}_X(U),I(U)} \wedge T)^{C_r}).$$

There is a sequence of maps of presheaves of spectra on X_{zar} given on an open $U \subset X$ as

$$(\tilde{\Omega}_{\mathcal{A}_X(U),I(U)} \wedge T)^{C_r} \longrightarrow (\tilde{\Omega}_{\mathcal{A}_X(U)} \wedge T)^{C_r} \longrightarrow (\tilde{\Omega}_{\mathcal{A}_X(U)/I(U)} \wedge T)^{C_r}. \tag{6.2.1}$$

By Lemma 4.3, $\tilde{\Omega}_{\mathcal{A}_X,I} \to \tilde{\Omega}_{\mathcal{A}_X} \to \tilde{\Omega}_{\mathcal{A}_X/I}$ is a presheaf of cofibration sequences, and (6.2.1) is a fibration sequence for each U. Applying $\mathbb{H}_{\mathrm{zar}}(X,-)$ therefore yields a fibration sequence of spectra

$$\mathbb{H}_{\mathrm{zar}}((X,C), \tilde{\Omega}^{T,r}) \longrightarrow \mathbb{H}_{\mathrm{zar}}(X, \tilde{\Omega}^{T,r}) \longrightarrow \mathbb{H}_{\mathrm{zar}}(C, \tilde{\Omega}^{T,r}). \tag{6.2.2}$$

This fibration is natural in (X,C). Replacing (X,C) by (Y,D) gives an analogous fibration sequence and a map

$$\mathbb{H}_{\mathrm{zar}}((X,C), \tilde{\Omega}^{T,r}) \longrightarrow \mathbb{H}_{\mathrm{zar}}((Y,D), \tilde{\Omega}^{T,r}). \tag{6.2.3}$$

To prove Theorem 6.2, it remains to prove that the map (6.2.3) is a weak equivalence.

Our next goal, achieved in (6.2.5), is to decompose the map (6.2.3) into a wedge sum of maps indexed by M. By (4.3.1), we have a decomposition into presheaves of $\mathbb{S}^1$-spaces on X_{zar}:

$$\tilde{\Omega}_{\mathcal{A}_X(U),\mathcal{I}(U)} = \bigvee_{m \in \mathcal{I}(U)} |N^{cy}(\mathcal{A}_X(U)[-m], m)|. \tag{6.2.4}$$

(Because we are using additive notation, we write $A[-m]$ instead of $A[1/m]$.)

We claim that if $m \in \mathcal{I}(U) \setminus \{0\}$, then $\mathcal{A}_X(U)[-m] = M_*$. To see this, note that, for each $\sigma \in U$, we have $m > 0$ on $|\sigma| \setminus \{0\}$ and hence (provided $\sigma \neq \{0\}$), m lies in the interior of $\sigma^\vee$ and thus $(\sigma^\vee \cap M)[-m] = M$. That is, for each $l \in M$, there is a positive integer N_σ such that $l + N_\sigma m \in \sigma^\vee$. Since U consists of a finite number of cones, we may find a positive integer N such that $l + Nm \in \sigma^\vee$ for all $\sigma \in U$ and hence $l + Nm \in \mathcal{A}_X(U)$. This shows $l \in \mathcal{A}_X(U)[-m]$.

Using this and $|N^{cy}(\mathcal{A}_X(U), 0)| = *$, (6.2.4) becomes the $\mathbb{S}^1$-equivariant decomposition

$$\tilde{\Omega}_{\mathcal{A}_X(U),\mathcal{I}(U)} = \bigvee_{m \in M} G_m^N(U),$$

where G_m^N is the presheaf of $\mathbb{S}^1$-spaces on X_{zar} defined by

$$G_m^N(U) = \begin{cases} |N^{cy}(M_*; m)| & \text{if } m > 0 \text{ on } |U| \setminus \{0\}, \\ * & \text{otherwise.} \end{cases}$$

Since $(\bigvee G_m^N) \wedge T \xrightarrow{\sim} \bigvee (G_m^N \wedge T)$ and $(\bigvee G_m^N \wedge T)^{C_r} = \bigvee (G_m^N \wedge T)^{C_r}$, it follows that $\mathbb{H}_{\mathrm{zar}}((X,C), \tilde{\Omega}^{T,r}) = \mathbb{H}_{\mathrm{zar}}(X, \bigvee (G_m^N \wedge T)^{C_r})$. Set $G_m = (G_m^N \wedge T)^{C_r}$.

By Lemma 5.4, $\bigvee \mathbb{H}_{\mathrm{zar}}(-, G_m)$ satisfies Zariski descent and

$$\mathbb{H}_{\mathrm{zar}}((X,C), \tilde{\Omega}^{T,r}) = \mathbb{H}_{\mathrm{zar}}\left(X, \bigvee G_m\right) \xleftarrow{\sim} \bigvee_{m \in M} \mathbb{H}_{\mathrm{zar}}(X, G_m)$$

is a weak equivalence. Similarly, we have an equivalence

$$\mathbb{H}_{\mathrm{zar}}((Y,D),\tilde{\Omega}^{T,r}) = \mathbb{H}_{\mathrm{zar}}\left(X, \bigvee G_m\right) \xrightarrow{\sim} \bigvee_{m\in M} \mathbb{H}_{\mathrm{zar}}(Y, G_m).$$

The evident square commutes, so the map (6.2.3) decomposes as a wedge sum of maps

$$\mathbb{H}_{\mathrm{zar}}(X, G_m) \longrightarrow \mathbb{H}_{\mathrm{zar}}(Y, G_m). \tag{6.2.5}$$

Thus it suffices to show that, for all $m \in M$, the map (6.2.5) is an equivalence. This is done in Lemma 6.4 below, with $E = (|N^{cy}(M;m)| \wedge T)^{C_r}$. □

Before stating the lemma that was used in the above proof, we introduce a simple construction which we will use.

CONSTRUCTION 6.3. Fix a lattice N, an isomorphism $N \cong \mathbb{Z}^n$ and an element m of $\mathrm{Hom}(N,\mathbb{R})$. We set

$$C(m) := \{x \in N_{\mathbb{R}} \mid m(x) \leqslant 0, \|x\| \geqslant 1\} \subset N_{\mathbb{R}}.$$

For each fan Δ in $N_{\mathbb{R}}$, with underlying cone $|\Delta| \subseteq N_{\mathbb{R}}$, we let $K(\Delta)$ denote the quotient $K(\Delta) = |\Delta|/(C(m) \cap |\Delta|)$. By convention, if $|\Delta|$ is disjoint from $C(m)$, then $K(\Delta)$ is $|\Delta|$ with a disjoint basepoint adjoined. If Δ is affine, then $|\Delta|$ is a strongly convex rational polyhedral cone, so $K(\Delta)$ is contractible (because the intersections of both $|\Delta|$ and $C(m) \cap |\Delta|$ with any sphere are convex on the sphere). If Δ is the union of two fans Δ_1 and Δ_2, then $K(\Delta_1)$ and $K(\Delta_2)$ form a closed cover of $K(\Delta)$ whose intersection is $K(\Delta_1 \cap \Delta_2)$; one can even choose a CW structure on $|\Delta|$ so that the $|\Delta_i|$ and the $C(m) \cap |\Delta_i|$ are subcomplexes and the cover is cellular.

We now state and prove the lemma that was used in the proof of Theorem 6.2.

LEMMA 6.4. *Let $X = (\Delta, N)$ be a toric monoid scheme and let $Y = (\Delta', N)$ be another toric monoid scheme with the same lattice and such that Δ' is a refinement of Δ. Let $f : Y \to X$ be the associated morphism of monoid schemes* (*see* [**9**, 4.2]). *Fix a* (*non-equivariant*) *spectrum E and an element $m \in M = \mathrm{Hom}(N,\mathbb{R})$. Let G_m^E denote the presheaf of spectra on X_{zar}* (*and Y_{zar}, respectively*) *defined by*

$$U \longmapsto G_m^E(U) = \begin{cases} E & \text{if } m > 0 \text{ on } |U| \setminus \{0\}, \\ * & \text{otherwise.} \end{cases}$$

Then the map $f^* : \mathbb{H}_{\mathrm{zar}}(X, G_m^E) \to \mathbb{H}_{\mathrm{zar}}(Y, G_m^E)$ induced by f is a weak equivalence.

Proof. We give an explicit description (up to weak equivalence) of the presheaves $\mathbb{H}_{\mathrm{zar}}(-, G_m^E)$ whose global sections only depend on $|\Delta|$.

For U an open in X (or Y), we regard U as a fan in $N_{\mathbb{R}}$ and write $F(U)$ for $\mathrm{Maps}_*(K(U), E)$, where $K(U)$ is defined in Construction 6.3 and $\mathrm{Maps}_*(K, E)$ denotes the function spectrum associated to a pointed space K and a spectrum E. Because $K(U)$ is a contravariant functor of U, F is a presheaf on X, and on Y. Then we will prove the following assertions:

(a) $F(X) \to \mathbb{H}_{\mathrm{zar}}(X, F)$ is a weak equivalence, and similarly for Y;
(b) $\mathbb{H}_{\mathrm{zar}}(X, G_m^E) \xrightarrow{\sim} \mathbb{H}_{\mathrm{zar}}(X, F)$ is an equivalence of spectra, and similarly for Y;

(c) the following diagram commutes:

$$\begin{tikzcd}
\mathbb{H}_{\mathrm{zar}}(X, G_m^E) \arrow[r, "\sim"] \arrow[d, "f^*"] & \mathbb{H}_{\mathrm{zar}}(X, F) \arrow[d] \\
\mathbb{H}_{\mathrm{zar}}(Y, G_m^E) \arrow[r, "\sim"] & \mathbb{H}_{\mathrm{zar}}(Y, F).
\end{tikzcd}$$

Since $F(X) = F(Y)$ by the very definition of F, this will prove the lemma.

To establish assertion (a), it suffices (by Example 5.3) to show that the presheaf F has the Mayer–Vietoris property for open covers. If $U = V \cup W$ is an open cover, we saw in Construction 6.3 that $K(V)$ and $K(W)$ form a closed cellular cover of $K(U)$. Since $\mathrm{Maps}_*(-, E)$ has descent for closed cellular covers (by the homotopy extension property for embeddings of subcomplexes), we see that F has the Mayer–Vietoris property, establishing (a).

For (b), we define a morphism $G_m^E \to F$ of presheaves of spectra on X (or Y, respectively) by the following formula: when $C(m) \cap |U| \neq \emptyset$ so that $G_m^E(U) = *$, it is the inclusion of the basepoint; when $C(m) \cap |U| = \emptyset$, it is the map

$$G_m^E(U) = E = \mathrm{Maps}_*(S^0, E) \longrightarrow \mathrm{Maps}_*(|U|/\emptyset, E) = F(U)$$

induced by the continuous function $|U|/\emptyset \twoheadrightarrow 1/\emptyset = S^0$ that sends $|U|$ to 1.

We claim that $G_m^E(U) \to F(U)$ is a homotopy equivalence for all affine U in X_{zar}. Indeed, if $C(m) \cap |U| = \emptyset$, then the pointed space $K(U)$ is homotopy equivalent to S^0, and the claim is clear. On the other hand, if $C(m) \cap |U| \neq \emptyset$, then $G_m^E(U) = *$; when U is affine, the space $K(U)$ is contractible, as noted in Construction 6.3, and $F(U)$ is contractible. This establishes the claim.

The claim implies that $G_m^E \to F$ is (locally in the Zariski topology) a weak equivalence of presheaves of spectra, and hence

$$\mathbb{H}_{\mathrm{zar}}(X, G_m^E) \xrightarrow{\sim} \mathbb{H}_{\mathrm{zar}}(X, F)$$

is also a weak equivalence of spectra (and similarly for Y), which is assertion (b).

Note that the presheaves F and f_*F on X_{zar} are, in fact, equal. Assertion (c) is an immediate consequence of the functoriality of fibrant replacements. □

Descent for dilated TC. Given a commutative ring k, the k-realization X_k of a monoid scheme X is a scheme over $\mathrm{Spec}(k)$, constructed in [**9**, 5.3, 5.9]. It is functorial in X, and if $X = \mathrm{MSpec}\, A$ is affine, then $X_k = \mathrm{Spec}(k[A])$. If X is the toric monoid scheme associated to a fan (N, Δ) as in Example 1.20.1, and k is a field, then X_k is the usual toric k-variety associated to this fan.

Recall from (3.9) that $TR^n(-;p)$ is a covariant functor from commutative rings to $\mathbb{S}^1$-spectra. We promote this to a presheaf $TR^n(\mathcal{O};p)$ on schemes by sending $(Y, \mathcal{O}_Y)$ to $TR^n(\mathcal{O}_Y(Y);p)$. Following Geisser and Hesselholt [**15**, 3.3], we define the presheaf $TR^n(-;p)$ on schemes by

$$TR^n(Y;p) = \mathbb{H}_{\mathrm{zar}/k}(Y, TR^n(\mathcal{O};p)),$$

where $\mathbb{H}_{\mathrm{zar}/k}$ is the fibrant replacement for presheaves defined on schemes of finite type over k. By Geisser and Hesselholt [**15**, 3.3.3], $TR^n(S;p) \sim TR^n(\mathrm{Spec}\, S;p)$ for any ring S.

Composing with the k-realization functor turns $TR^n(-;p)$ into a presheaf on the category of monoid schemes. Given an open covering of a separated monoid scheme X by U and V, applying the k-realization functor yields an open covering of X_k by U_k and V_k, with $U_k \cap V_k = (U \cap V)_k$; see [**9**, 5.3]. Using Example 5.3(a), it follows that $TR^n(-;p)$ satisfies Zariski descent on $\mathcal{M}_{\mathrm{pctf}}$. Given an infinite sequence $\mathfrak{c}$ of integers $\geqslant 2$, the presheaf $TR^n(-;p)^{\mathfrak{c}}$ also satisfies Zariski descent by Lemma 5.4.

Theorem 6.5. *Let k be a ring and let $\mathfrak{c}$ be an infinite sequence of integers $c_1, c_2, \ldots$, with $c_i \geqslant 2$ for all i. Then, for all primes p and integers $n \geqslant 1$, the presheaf*

$$X \longmapsto TR^n(X_k; p)^{\mathfrak{c}}$$

satisfies cdh descent on $\mathcal{M}_{pctf}$.

Proof. Since $TR^n(-;p)^{\mathfrak{c}}$ satisfies Zariski descent on $\mathcal{M}_{\text{pctf}}$, the Hesselholt–Madsen Theorem 3.10 implies that there is an equivalence

$$\begin{aligned} TR^n(X_k;p)^{\mathfrak{c}} &\sim \mathbb{H}_{\text{zar}}(X, TR^n(-;p)^{\mathfrak{c}}) \\ &\sim_{\mathcal{F}} \mathbb{H}_{\text{zar}}(X, U \longmapsto (TH(k) \wedge |N^{cy}(\mathcal{A}_X(U))|^{\mathfrak{c}})^{C_{p^{n-1}}}). \end{aligned}$$

Here we have used the fact that the filtered colimit $(-)^{\mathfrak{c}}$ commutes with finite limits such as $(-)^{C_r}$ and with smashing with a spectrum T. By Theorem 3.6 and Definition 4.1, we have a natural equivalence of $\mathbb{S}^1$-spectra $|N^{cy}(A)|^{\mathfrak{c}} \simeq |\tilde{N}^{cy}(A)_{\text{sn}}|^{\mathfrak{c}} = \tilde{\Omega}^{\mathfrak{c}}_A$. Replacing A by $\mathcal{A}_X(U)$, smashing with $TH(k)$, taking C_r-fixed points with $r = p^n$ and then applying $\mathbb{H}_{\text{zar}}$, we obtain the equivalence

$$TR^n(X_k;p)^{\mathfrak{c}} \sim \mathbb{H}_{\text{zar}}(X, (TH(k) \wedge \tilde{\Omega}^{\mathfrak{c}}_{\mathcal{A}}))^{C_r} \cong \mathbb{H}_{\text{zar}}(X, (\tilde{\Omega}^{TH(k),r})^{\mathfrak{c}}).$$

(The final $\cong$ uses Definition 6.1.) By Theorem 6.2, $\mathbb{H}_{\text{zar}}(-, \tilde{\Omega}^{T,r})$ satisfies *cdh* descent. The result now follows from Lemma 5.4. □

Corollary 6.6. *For any ring k and integer $n \geqslant 1$, the spectrum-valued functor*

$$X \longmapsto TC^n(X_k;p)^{\mathfrak{c}}$$

satisfies cdh descent on $\mathcal{M}_{pctf}$.

Proof. Recall that $TC^n(X_k;p)$ is the homotopy equalizer of the two maps

$$TR^n(X_k;p) \rightrightarrows TR^{n-1}(X_k;p)$$

given by restriction and Frobenius. We may identify $TC^n(X_k;p)^{\mathfrak{c}}$ as either the colimit of the sequence of endomorphisms of the spectra $TC^n(X_k;p)$ by the map θ_{c_i} or as the homotopy equalizer of the induced maps from $TR^n(X_k;p)^{\mathfrak{c}}$ to $TR^{n-1}(X_k;p)^{\mathfrak{c}}$. Since a homotopy pullback of presheaves satisfying *cdh* descent satisfies *cdh* descent, the assertion now follows from Theorem 6.5. □

7. *The Dilation theorem in characteristic* 0

In our previous paper [8], we proved that the canonical map

$$\mathcal{K}(X_k)^{\mathfrak{c}} \longrightarrow \mathcal{K}H(X_k)^{\mathfrak{c}}$$

is a weak equivalence of spectra whenever k is a field of characteristic 0 and X is a toric monoid scheme (so that X_k is a toric variety). In this section, by using the results developed in this paper, we extend this result slightly to include all X in $\mathcal{M}_{\text{pctf}}$. (Our result here also applies to regular rings k containing $\mathbb{Q}$.) In the next section, we will prove this result in the more difficult case when the characteristic of k is positive.

For a commutative ring k, we shall write H_k for the Eilenberg–Mac Lane spectrum associated to k. Given a k-algebra R, we shall write $\text{HH}.(R/k)$ for the generalized Eilenberg–Mac Lane spectrum associated to the standard Hochschild complex for the k-algebra R. Thus $\text{HH}.(-/k)$ is a covariant functor from k-algebras to spectra such that $\pi_n(\text{HH}.(R/k))$ is $HH_n(R/k)$, the

nth Hochschild homology group of the k-algebra R. Recall from (2.1) that, for any monoid A and ring k, we have a natural weak equivalence of spectra

$$N^{cy}(A) \wedge H_k \sim k[N^{cy}(A)] = \mathrm{HH}.(k[A]/k),$$

and a natural isomorphism $H_q(N^{cy}(A), k) \cong HH_q(k[A]/k)$.

As $N^{cy}(A) \wedge H_k$ is a functor from monoids to spectra, we may promote it to a functor from monoid schemes to spectra by sending $(X, \mathcal{A})$ to $N^{cy}(\mathcal{A}(X)) \wedge H_k$. Let us write this functor as $\mathrm{HH}.(k[\mathcal{A}])$. Taking fibrant replacements for the Zariski topology, we obtain the presheaf of spectra

$$X \longmapsto \mathbb{H}_{\mathrm{zar}}(X, \mathrm{HH}.(k[\mathcal{A}]))$$

defined on monoid schemes. If $X = \mathrm{MSpec}(A)$ is an affine monoid scheme, the above remarks show there is a natural weak equivalence of spectra

$$\mathrm{HH}.(k[A]/k) \xrightarrow{\sim} \mathbb{H}_{\mathrm{zar}}(X, \mathrm{HH}.(k[\mathcal{A}])).$$

As in (3.4), given a sequence $\mathfrak{c} = \{c_1, c_2, \ldots\}$ of integers with $c_i \geqslant 2$ for all i, taking colimits yields the presheaf of spectra $\mathbb{H}_{\mathrm{zar}}(-, \mathrm{HH}.(k[\mathcal{A}]))^{\mathfrak{c}}$. For any X in $\mathcal{M}_{\mathrm{pctf}}$, we have

$$\mathbb{H}_{\mathrm{zar}}(X, \mathrm{HH}.(k[\mathcal{A}]))^{\mathfrak{c}} \xrightarrow{\sim} \mathbb{H}_{\mathrm{zar}/k}(X_k, \mathrm{HH}.(-/k))^{\mathfrak{c}}, \tag{7.1}$$

where $\mathbb{H}_{\mathrm{zar}/k}$ denotes fibrant replacement for the Zariski topology on the category of schemes of finite type over k.

Theorem 7.2. *If k is a regular ring containing $\mathbb{Q}$ and $\mathfrak{c} = \{c_1, c_2, \ldots\}$ is a sequence of integers with $c_i \geqslant 2$ for all i, then the presheaves of spectra $\mathbb{H}_{\mathrm{zar}}(-, \mathrm{HH}.(k[\mathcal{A}]))^{\mathfrak{c}}$ and $\mathbb{H}_{\mathrm{zar}/k}(-_k, \mathrm{HH}.(-/k))^{\mathfrak{c}}$ satisfy cdh descent on $\mathcal{M}_{\mathrm{pctf}}$.*

Proof. By Theorem 3.6, it suffices to prove that the analogously defined presheaf defined using $\tilde{N}^{cy}$ in place of N^{cy} satisfies *cdh* descent. This holds by Theorem 6.2, letting T be the Eilenberg–Mac Lane spectrum H_k, regarded as a $\mathbb{S}^1$-spectrum with trivial action, so that $\tilde{\Omega}^{T,0}(\mathrm{MSpec}(A))$ is $|\tilde{N}^{cy}(A_{\mathrm{sn}})| \wedge H_k$. □

Remark 7.2.1. An equivariant spectrum is indexed by finite sub-representations of an $\mathbb{S}^1$-universe U, while an ordinary spectrum is indexed by finite-dimensional subspaces of $U^{\mathbb{S}^1}$. Thus, in order to regard an ordinary spectrum H as an equivariant spectrum with trivial action, one needs to extend the indexing family. This is accomplished by the left adjoint of the forgetful functor from equivariant spectra or ordinary spectra. We shall not dwell on this standard construction.

Now let $\mathbb{H}_{\mathrm{cdh}/k}$ denote fibrant replacement for the *cdh* topology on schemes of finite type over k. We are interested in the canonical map

$$\mathbb{H}_{\mathrm{zar}/k}(X_k, \mathrm{HH}.(-/\mathbb{Q}))^{\mathfrak{c}} \longrightarrow \mathbb{H}_{\mathrm{cdh}/k}(X_k, \mathrm{HH}.(-/\mathbb{Q}))^{\mathfrak{c}}. \tag{7.3}$$

Proposition 7.4. *If $k = \mathbb{Q}$, (7.3) is a weak equivalence for all X in $\mathcal{M}_{\mathrm{pctf}}$.*

Proof. Let us write $\mathcal{F}(X)$ for the source of (7.3), regarding $\mathcal{F}$ as a presheaf on $\mathcal{M}_{\mathrm{pctf}}$. Using (7.1), Theorem 7.2 shows that $\mathcal{F}$ satisfies *cdh* descent on $\mathcal{M}_{\mathrm{pctf}}$. Since $\mathbb{Q}$-realization sends *cdh* squares in $\mathcal{M}_{\mathrm{pctf}}$ to *cdh* squares of schemes of finite type over $\mathbb{Q}$, the target of (7.3) also satisfies *cdh* descent as a presheaf on $\mathcal{M}_{\mathrm{pctf}}$.

Since monoid schemes are locally smooth and affine for the *cdh* topology by Cortiñas, Haesemeyer, Walker and Weibel [**9**, 11.1], we may assume that X is smooth and affine. In this case, $X_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$ by [**9**, 6.4–5]. (In fact, X is a finite product of copies of $\mathbb{A}^1$ and $\mathbb{A}^1 - \{0\}$.) Moreover, $HH.(X_{\mathbb{Q}}/\mathbb{Q}))^{\mathfrak{c}} \xrightarrow{\sim} \mathcal{F}(X)$ is a weak equivalence.

There is a notion of *scdh* descent for presheaves on $Sm/\mathbb{Q}$, and if a presheaf G satisfies *scdh* descent, then $\mathbb{H}_{\mathrm{cdh}}(-,G) \simeq \mathbb{H}_{\mathrm{scdh}}(-,G)$ (see the argument preceding Theorem 2.4 in [**7**]). It is proved in [**6**, 2.9, 2.10, 3.9] that $\mathbb{H}_{\mathrm{zar}/\mathbb{Q}}(-, HH.(-/\mathbb{Q}))$ satisfies *scdh* descent on $Sm/\mathbb{Q}$. Since $X_{\mathbb{Q}}$ is smooth affine, this implies that the maps

$$HH.(X_{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\sim} \mathbb{H}_{\mathrm{cdh}/\mathbb{Q}}(X_{\mathbb{Q}}, HH.(-/\mathbb{Q})) \xrightarrow{\sim} \mathbb{H}_{\mathrm{scdh}/\mathbb{Q}}(X_{\mathbb{Q}}, HH.(-/\mathbb{Q})) \tag{7.4.1}$$

are weak equivalences. Therefore $HH.(X_{\mathbb{Q}}/\mathbb{Q})^{\mathfrak{c}}$ is weakly equivalent to the target of (7.3), as required. □

THEOREM 7.5. *If k is a regular ring containing $\mathbb{Q}$ and $\mathfrak{c} = \{c_1, c_2, \ldots\}$ is a sequence of integers with $c_i \geqslant 2$ for all i, then the canonical map*

$$\mathcal{K}(X_k)^{\mathfrak{c}} \longrightarrow \mathcal{K}H(X_k)^{\mathfrak{c}}$$

is a weak equivalence for all $X \in \mathcal{M}_{pctf}$. In particular, if A is a cancellative and torsion-free monoid with no non-trivial units, then $K_*(k[A])^{\mathfrak{c}} \cong K_*(k)$ and

$$\mathcal{K}(k[A])^{\mathfrak{c}} \sim \mathcal{K}(k).$$

When k is a field with $\mathrm{char}(k) = 0$, this theorem was proved in [**8**]. The isomorphism $K_*(k[A])^{\mathfrak{c}} \cong K_*(k)$ when k is regular is due to Gubeladze [**17**].

Proof. The proofs of Corollary 6.8 and Theorem 6.9 of our previous paper [**8**] apply verbatim to show that $\mathcal{K}(X_k)^{\mathfrak{c}} \to \mathcal{K}H(X_k)^{\mathfrak{c}}$ is a weak equivalence if the canonical map (7.3) is a weak equivalence of spectra. The proof that this implies that $\mathcal{K}(k[A])^{\mathfrak{c}} \sim \mathcal{K}(k)$ is given in [**8**, 6.10]; a shorter proof is given in Corollary 8.4.

The source of (7.3) may be understood using the following device. By the Künneth formula, we have a natural weak equivalence

$$HH.(R/\mathbb{Q}) \wedge HH.(k/\mathbb{Q}) \xrightarrow{\sim} HH.(R \otimes_{\mathbb{Q}} k/\mathbb{Q})$$

for any $\mathbb{Q}$-algebra R. Applying this locally, we see that the canonical map

$$\mathbb{H}_{\mathrm{zar}/\mathbb{Q}}(Y, HH.(-/\mathbb{Q}))^{\mathfrak{c}} \wedge HH.(k/\mathbb{Q}) \xrightarrow{\sim} \mathbb{H}_{\mathrm{zar}/k}(Y_k, HH.(-/\mathbb{Q}))^{\mathfrak{c}}$$

is a weak equivalence for every noetherian scheme Y over $\mathbb{Q}$, including $Y = X_{\mathbb{Q}}$.

Since k is a filtered union of smooth $\mathbb{Q}$-algebras of finite type (by Popescu's theorem), and K-theory commutes with filtered colimits, we may also assume that k is finitely generated smooth over $\mathbb{Q}$. Since the base change $- \otimes_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\, k$ sends *cdh* squares to *cdh* squares, an arrow μ exists making the following diagram commutative up to homotopy:

$$\begin{tikzcd}
\mathbb{H}_{\mathrm{zar}/\mathbb{Q}}(X_{\mathbb{Q}}, HH.(-/\mathbb{Q}))^{\mathfrak{c}} \wedge HH.(k/\mathbb{Q}) \arrow[r, "\simeq"] \arrow[d, "7.4"', "\simeq"] & \mathbb{H}_{\mathrm{zar}/k}(X_k, HH.(-/\mathbb{Q}))^{\mathfrak{c}} \arrow[d, "(7.3)"] \\
\mathbb{H}_{\mathrm{cdh}/\mathbb{Q}}(X_{\mathbb{Q}}, HH.(-/\mathbb{Q}))^{\mathfrak{c}} \wedge HH.(k/\mathbb{Q}) \arrow[r, "\mu"] & \mathbb{H}_{\mathrm{cdh}/k}(X_k, HH.(-/\mathbb{Q}))^{\mathfrak{c}}.
\end{tikzcd}$$

The left arrow is a weak equivalence by Proposition 7.4, and the top arrow is a weak equivalence by the above remarks. Thus, it suffices to show that the bottom arrow μ is a weak equivalence.

Since k-realization sends *cdh* squares in $\mathcal{M}_{\mathrm{pctf}}$ to *cdh* squares of schemes of finite type over k, both the source and target of μ satisfy *cdh* descent on $\mathcal{M}_{\mathrm{pctf}}$. Since every X in $\mathcal{M}_{\mathrm{pctf}}$ is locally isomorphic in the *cdh* topology to a smooth affine monoid scheme, we may assume that X is such a scheme. Because k is smooth over $\mathbb{Q}$, the map

$$\mathrm{HH}_{.}(X_k/\mathbb{Q}) \longrightarrow \mathbb{H}_{\mathrm{cdh}/k}(X_k, \mathrm{HH}_{.}(-/\mathbb{Q}))$$

is a weak equivalence. We saw in (7.4.1) that $\mathrm{HH}_{.}(X_{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{H}_{\mathrm{cdh}/\mathbb{Q}}(X_{\mathbb{Q}}, \mathrm{HH}_{.}(-/\mathbb{Q}))$ is a weak equivalence, so μ is weakly equivalent to the map

$$\mathrm{HH}_{.}(X_{\mathbb{Q}}/\mathbb{Q})^{\mathfrak{c}} \wedge \mathrm{HH}_{.}(k/\mathbb{Q}) \longrightarrow \mathrm{HH}_{.}(X_k/\mathbb{Q})^{\mathfrak{c}}.$$

Since $X_{\mathbb{Q}}$ is affine, this is a weak equivalence by the Künneth formula. ☐

8. *The Dilation theorem*

We recall some basic facts about pro-objects from [2]. A *pro-abelian group* is a sequence of homomorphisms of abelian groups indexed by the positive integers,

$$\cdots \longrightarrow A^3 \longrightarrow A^2 \longrightarrow A^1,$$

or, in other words, it is a contravariant functor from $\mathbb{N}$ to the category of abelian groups, where $\mathbb{N}$ is the ordered set of positive integers viewed as a category in the standard way. We typically write such a pro-abelian group as $\{A^n\}$. The category pro-**Ab** has as objects all pro-abelian groups and morphisms defined by

$$\mathrm{Hom}_{\text{pro-}\mathbf{Ab}}(\{A^n\}, \{B^n\}) = \varprojlim_{m\in\mathbb{N}} \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_{\mathbf{Ab}}(A^n, B^m).$$

The category pro-**Ab** is an abelian category.

A *strict morphism* of pro-abelian groups will refer to a natural transformation of functors from $\mathbb{N}$ to abelian groups. The collection of pro-abelian groups with arrows defined by strict morphisms is the abelian category $\mathbf{Ab}^{\mathbb{N}}$ of contravariant functors from $\mathbb{N}$ to abelian groups. The evident functor from $\mathbf{Ab}^{\mathbb{N}}$ to pro-**Ab** is exact and preserves all finite limits and colimits [2, A.4.1]. In particular, if we consider the morphism in pro-**Ab** associated to a strict morphism, its kernel and cokernel are also given degree-wise.

Lemma 8.1. *A pro-abelian group $\{A^n\}$ is isomorphic to zero in* pro-**Ab** *if and only if, for all m, there is an $n \geqslant m$ such that $A^n \to A^m$ is the zero map.*

Proof. For each m, let $g_m \in \varinjlim_n \mathrm{Hom}_{\mathbf{Ab}}(A^n, A^m)$ denote the image of $A^m \xrightarrow{\mathrm{id}} A^m$ under the canonical map $\mathrm{Hom}_{\mathbf{Ab}}(A^m, A^m) \to \varinjlim_n \mathrm{Hom}_{\mathbf{Ab}}(A^n, A^m)$. Then the identity map of $\{A^n\}$ is represented by the element $(g_m)_{m\in\mathbb{N}}$ of the inverse limit of the $\{\varinjlim_n \mathrm{Hom}_{\mathbf{Ab}}(A^n, A^m)\}$. Clearly, $\{A^n\} \cong 0$ if and only if the identity map and the zero map coincide in $\mathrm{Hom}_{\text{pro-}\mathbf{Ab}}(\{A^n\}, \{A^n\})$. This is equivalent to the condition that all the g_m are zero. On the other hand, $g_m = 0$ in $\varinjlim_n \mathrm{Hom}_{\mathbf{Ab}}(A^n, A^m)$ if and only if there exists an $n \geqslant m$ so that $A^n \to A^m$ is the zero map. ☐

Lemma 8.2. *If a strict map $\{f^n : A^n \to B^n\}$ is a monomorphism in* pro-**Ab**, *and $\{A^n\}$ is a constant pro-abelian group, then f^n is an injection for all $n \gg 0$.*

Proof. Let $C^n = \ker(f^n)$. As noted above, $\{C^n\}$ is the kernel of $\{f^n\}$ in pro-**Ab** and hence it is the zero object of this category. By Lemma 8.1, this means that, for all m, there exists an

$n \geqslant m$ such that $C^n \to C^m$ is the zero map. But $\{A^n\}$ is constant and $C^n \subset A^n$ for all n and hence each map $C^n \to C^m$ is injective. It follows that $C^n = 0$ for all $n \gg 0$. □

A *pro-spectrum* is a contravariant functor from $\mathbb{N}$ to the category of spectra, that is, it is a sequence of maps of spectra of the form

$$\cdots \longrightarrow E^2 \longrightarrow E^1.$$

A strict map of pro-spectra is a natural transformation of such functors. We say that a strict map of pro-spectra $\{E^n\} \to \{F^n\}$ is a *weak equivalence* if, for each $q \in \mathbb{Z}$, the induced (strict) map of pro-abelian groups $\{\pi_q E^n\} \to \{\pi_q F^n\}$ is an isomorphism in the category pro-**Ab**. A commutative square of pro-spectra and strict maps,

$$\begin{tikzcd}
\{E^n\} \arrow[r] \arrow[d] & \{F^n\} \arrow[d] \\
\{G^n\} \arrow[r] & \{H^n\}
\end{tikzcd}$$

is said to be *homotopy cartesian* if the induced map of pro-spectra

$$\{E^n\} \longrightarrow \{\operatorname{holim}(G^n \longrightarrow H^n \longleftarrow F^n)\}$$

is a weak equivalence.

We arrive at the main theorem of this paper:

THEOREM 8.3. *Let $\mathfrak{c} = \{c_1, c_2, \ldots\}$ be a sequence of integers with $c_i \geqslant 2$ for all $i \geqslant 1$ and let k be a regular ring containing a field. Then, for any X in $\mathcal{M}_{\mathrm{pctf}}$, the map*

$$\mathcal{K}(X_k)^{\mathfrak{c}} \longrightarrow \mathcal{K}H(X_k)^{\mathfrak{c}}$$

is an equivalence.

Proof. If $\operatorname{char}(k) = 0$, this was proved in Theorem 7.5. Assume $\operatorname{char}(k) = p > 0$. The usual transfer argument, involving prime-to-p field extensions, allows us to assume that k contains an infinite field.

In more detail, let F be a field contained in k and suppose that F is finite of characteristic p (note that F is in particular perfect). Using Popescu's theorem and the fact that K-theory commutes with direct limits, we may assume that k is a smooth F-algebra. Consider the functor $\mathcal{T}$ on algebraic extensions of F that assigns to a field E the kernel (or cokernel) of the homomorphism on stable homotopy groups $\pi_s\mathcal{K}(X_R)^{\mathfrak{c}} \to \pi_s\mathcal{K}H(X_R)^{\mathfrak{c}}$, where $R = k \otimes_F E$. Observe that R is smooth over E and hence regular; thus our assumption says that $\mathcal{T}(E) = 0$ for any infinite extension E of F. The functor $\mathcal{T}$ has transfers for finite field extensions (because the K-theory functor has such transfers); that is, if E'/E is a finite extension of fields of degree d, then there is a transfer homomorphism $t : \mathcal{T}(E') \to \mathcal{T}(E)$ such that the composite of t with the base-change map $\mathcal{T}(E) \to \mathcal{T}(E')$ is multiplication by d on $\mathcal{T}(E)$. Now let $\alpha \in \mathcal{T}(F)$. Choose an infinite pro-ℓ extension E/F where ℓ is a prime; by assumption, $\mathcal{T}(E) = 0$. Since $\mathcal{T}$ also commutes with colimits, we can find a finite subextension E_1 of F of degree a power of ℓ such that the base change of α to $\mathcal{T}(E_1)$ vanishes. By the transfer property, this means that α is ℓ-power torsion. Repeating the argument with a distinct prime q, we see that α is also q-power torsion; but this implies that $\alpha = 0$.

If $\mathcal{F}$ is a functor defined on the category of k-schemes of finite type, we will interpret $\mathcal{F}$ in this proof as a functor on $\mathcal{M}_{\mathrm{pctf}}$ by precomposing with the k-realization functor. If $\Box$ is any square in $\mathcal{M}_{\mathrm{pctf}}$, write $\mathcal{F}(\Box)$ for the iterated homotopy fiber of the commutative square of spectra obtained by applying $\mathcal{F}$ to the diagram $\Box$.

With this notation, we claim $\mathcal{K}(-)^{\mathfrak{c}}$ satisfies *cdh* descent on $\mathcal{M}_{\mathrm{pctf}}$. To show this, we need to prove that $\pi_q\mathcal{K}(\square)^{\mathfrak{c}} = 0$ for all $q \in \mathbb{Z}$, where $\square$ is either a Zariski square or an abstract blow-up square.

Fix an integer q. By Cortiñas, Haesemeyer, Walker and Weibel [9, Proposition 14.7], we have an isomorphism in pro-**Ab**:

$$\{\pi_q\mathcal{K}(\square)\} \longrightarrow \{\pi_q TC^n(\square; p)\}.$$

Lemma 8.2 shows that $\pi_q\mathcal{K}(\square) \to \pi_q TC^n(\square; p)$ is injective for all $n \geqslant n(q)$, where $n(q)$ depends on q. Thus $\pi_q\mathcal{K}(\square)^{\mathfrak{c}} \to \pi_q TC^n(\square; p)^{\mathfrak{c}}$ is also injective for all $n \geqslant n(q)$. But, for any n, $X \mapsto TC^n(X_k; p)^{\mathfrak{c}}$ satisfies *cdh* descent by Corollary 6.6, and hence $\pi_q TC^n(\square; p)^{\mathfrak{c}} = 0$. Therefore $\pi_q\mathcal{K}(\square)^{\mathfrak{c}} = 0$, that is, $\mathcal{K}(-)^{\mathfrak{c}}$ satisfies *cdh* descent.

By Cortiñas, Haesemeyer, Walker and Weibel [9, 14.5], $\mathcal{K}H$ satisfies *cdh* descent on $\mathcal{M}_{\mathrm{pctf}}$ and hence so does $\mathcal{K}H^{\mathfrak{c}}$, by Lemma 5.4. It follows that the homotopy fiber $\mathcal{G}$ of $\mathcal{K}(-)^{\mathfrak{c}} \to \mathcal{K}H(-)^{\mathfrak{c}}$ also satisfies *cdh* descent on $\mathcal{M}_{\mathrm{pctf}}$. Suppose that k contains a field F. For every smooth Y in $\mathcal{M}_{\mathrm{pctf}}$, Y_F is smooth over F and hence Y_k is smooth over k (see [9, 6.5]); this implies that $\mathcal{K}(Y_k) \to \mathcal{K}H(Y_k)$ is a weak equivalence, and hence that $\mathcal{G}(Y) \sim *$. By Proposition 5.5, $\mathcal{G}(X) \sim *$ and hence $\mathcal{K}(X_k)^{\mathfrak{c}} \sim \mathcal{K}H(X_k)^{\mathfrak{c}}$ for all $X \in \mathcal{M}_{\mathrm{pctf}}$. ☐

The special case $X = \mathrm{MSpec}(A)$ affirms Gubeladze's dilation conjecture [16, 1.1] for $k[A]$:

Corollary 8.4. *Let $\mathfrak{c} = \{c_1, c_2, \cdots\}$ be a sequence of integers $\geqslant 2$ and let k be a regular ring containing a field. If A is any cancellative, torsion-free monoid with no non-trivial units, then*

$$\mathcal{K}(k[A])^{\mathfrak{c}} \sim \mathcal{K}(k).$$

Proof. Since A is the direct limit of its finitely generated submonoids, we may assume that A is finitely generated, cancellative and torsion-free, with no non-trivial units. The Separation Theorem [14, p. 13] gives a group homomorphism $p : A^+ \to \mathbb{Z}$ with $p(A \setminus \{0\}) > 0$ and $p(-A) \leqslant 0$; using p, the ring $k[A]$ admits a grading by the natural numbers with $k[A]_0 = k$. By Weibel [24, Theorem 1.2(iv)], it follows that $KH_*(k[A]) \cong K_*(k)$ via the canonical map and that the action of θ_c on $KH_*(k[A])$ is trivial. Taking $X = \mathrm{MSpec}(A)$, the assertion follows from Theorem 8.3. ☐

References

1. M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Tome 2 (Springer, Berlin, 1972). Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics 270.
2. M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Mathematics 100 (Springer, Berlin, 1969).
3. M. Bökstedt, W.-C. Hsiang and I. Madsen, 'The cyclotomic trace and algebraic K-theory of spaces', *Invent. Math.* 111 (1993) 465–539.
4. A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics 304 (Springer, Berlin, 1972).
5. C. Chu, O. Lorscheid and R. Santhanam, 'Sheaves and K-theory for $\mathbb{F}_1$-schemes', *Adv. Math.* 229 (2012) 2239–2286.
6. G. Cortiñas, C. Haesemeyer, M. Schlichting and C. Weibel, 'Cyclic homology, *cdh*-cohomology and negative K-theory', *Ann. Math.* 167 (2008) 549–563.
7. G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, 'K-regularity, *cdh*-fibrant Hochschild homology, and a conjecture of Vorst', *J. AMS* 21 (2008) 547–561.
8. G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, 'The K-theory of toric varieties', *Trans. AMS* 361 (2009) 3325–3341.
9. G. Cortiñas, C. Haesemeyer, M. Walker and C. Weibel, 'Toric Varieties, Monoid Schemes and *cdh* descent', *J. Reine Angew. Math.* (2013) doi:10.1515/crelle-2012-0123.

10. V. Danilov, *de Rham complex on toroidal variety*, Lecture Notes in Mathematics 1479 (Springer, Berlin, 1991) 26–38.
11. W. G. Dwyer, M. J. Hopkins and D. M. Kan, 'The homotopy theory of cyclic sets', *Trans. AMS* 291 (1985) 281–289.
12. G. Dunn, 'Dihedral and quaternionic homology and mapping spaces', *K-theory* 3 (1989) 141–161.
13. Z. Fiedorowicz and W. Gajda, 'The S^1-CW decomposition of the geometric realization of a cyclic set', *Fund. Math.* 145 (1994) 91–100.
14. W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies 131 (Princeton University Press, Princeton, NJ, 1993).
15. T. Geisser and L. Hesselholt, 'Topological cyclic homology of schemes', *Algebraic K-theory* (Seattle, WA, 1997), Proceedings of Symposia in Pure Mathematics 67 (American Mathematical Society, Providence, RI, 1999) 41–87.
16. J. Gubeladze, 'The nilpotence conjecture in K-theory of toric varieties', *Invent. Math.* 160 (2005) 173–216.
17. J. Gubeladze, 'Global coefficient ring in the nilpotence conjecture', *Trans. AMS* 136 (2008) 499–503.
18. L. Hesselholt and I. Madsen, 'On the K-theory of finite algebras over Witt vectors of perfect fields', *Topology* 36 (1997) 29–101.
19. J. F. Jardine, 'Stable homotopy theory of simplicial presheaves', *Canad. J. Math.* 39 (1987) 733–747.
20. L. G. Lewis, J. P. May and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics 1213 (Springer, Berlin, 1986).
21. J. L. Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften 301 (Springer, Berlin, 1992).
22. J. Rognes, 'Topological logarithmic structures', *New topological contexts for Galois theory and algebraic geometry* (BIRS 2008), Geometry & Topology Monographs 16 (Geom. Topol. Publ, Coventry, 2009) 401–544.
23. V. Voevodsky, 'Homotopy theory of simplicial sheaves in completely decomposable topologies', *J. Pure Appl. Algebra* 214 (2010) 1384–1398.
24. C. A. Weibel, 'Homotopy algebraic K-theory', *Algebraic K-theory and algebraic number theory (Honolulu 1987)*, Contemporary Mathematics 83 (American Mathematical Society, Providence, RI, 1989) 461–488.
25. C. A. Weibel, *An introduction to homological algebra* (Cambridge University Press, Cambridge, 1994).

G. Cortiñas
Dept. Matemática-Inst. Santaló, FCEyN
Universidad de Buenos Aires
Ciudad Universitaria
1428 Buenos Aires
Argentina

gcorti@dm.uba.ar

C. Haesemeyer
Department of Mathematics
University of California
Los Angeles, CA 90095
USA

chh@math.ucla.edu

Mark E. Walker
Department of Mathematics
University of Nebraska - Lincoln
Lincoln, NE 68588
USA

mwalker5@math.unl.edu

C. Weibel
Department of Mathematics
Rutgers University
New Brunswick, NJ 08901
USA

weibel@math.rutgers.edu