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Operator norm inequalities in semi-Hilbertian spaces

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ABSTRACT

In this work we extend Cordes inequality, McIntosh inequality and CPR-inequality for the operator seminorm defined by a positive semidefinite bounded linear operator A.

Introduction

This paper is devoted to the study of the following operator norm inequalities when an additional seminorm is consider on a complex Hilbert space $\mathcal{H}$:

(I) If $V, W \in L(\mathcal{H})$ are semidefinite positive then $\|W^tV^t\| \leqslant \|WV\|^t$ for every $t \in [0, 1]$;
(II) If $V, W, X \in L(\mathcal{H})$ then $\|WW^*X + XVV^*\| \geqslant 2\|W^*XV\|$;
(III) If $S, R \in L(\mathcal{H})$ are invertible then $\|SXR^{-1} + (S^*)^{-1}XR^*\| \geqslant 2\|X\|$ for every $X \in L(\mathcal{H})$.

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Here, $L(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$, T^* denotes the adjoint operator of $T \in L(\mathcal{H})$ and $\|\cdot\|$ denotes the operator uniform norm. Inequality (I) is due to Cordes [8] (see also the paper by Furuta [13] for another proof). Inequality (II) is due to McIntosh [17] and it is known as the arithmetic–geometric-mean inequality. Different proofs of this property and its extension for every unitarily invariant norm can be found in [4,5,15]. Finally, Corach et al. [7] gave the first proof of inequality (III) for $S = R$ invertible and selfadjoint operators, which is known as CPR-inequality. Later, Kittaneh [16] proved the nonsymmetric version of it valid for every unitarily invariant norm, for all $X \in L(\mathcal{H})$ and all invertible $S, R \in L(\mathcal{H})$. See [1] for several equivalent expressions of inequality (III).

The main goal of this article is to study these properties if we consider an additional seminorm $\|\cdot\|_A$, defined by means of a positive semidefinite operator $A \in L(\mathcal{H})$ by $\|\xi\|_A^2 = \langle \xi, \xi\rangle_A = \langle A\xi, \xi\rangle$, $\xi \in \mathcal{H}$, and we replace the operator norm in inequalities (I), (II) and (III) by the quantity

$$\|T\|_A = \sup\{\|T\xi\|_A : \|\xi\|_A = 1\}.$$

The extension of these properties is not trivial since many difficulties arise. For instance, it may happen that $\|T\|_A = \infty$ for some $T \in L(\mathcal{H})$. In addition, not every operator admits an adjoint operator for the semi-inner product $\langle\ ,\ \rangle_A$.

The contents of the paper are the following. In Section 1 we set up notation, terminology and we describe the preliminary material on operators which are bounded for the A-seminorm. In Section 2 we study the concept of an A-positive operator and we extend Cordes inequality for the seminorm in matter. In Section 3 we generalize the arithmetic–geometric-mean inequality for this seminorm and, as a consequence, we obtain different extensions of the CPR-inequality. At the end of this section we describe the classes of operators which satisfy these extensions.

1. Preliminaries

Along this work $\mathcal{H}$ denotes a complex Hilbert space with inner product $\langle\ ,\ \rangle$. $L(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$, $L(\mathcal{H})^+$ is the cone of positive (semidefinite) operators of $L(\mathcal{H})$, i.e., $L(\mathcal{H})^+ := \{T \in L(\mathcal{H}) : \langle T\xi, \xi\rangle \geqslant 0\ \forall \xi \in \mathcal{H}\}$ and $L_{cr}(\mathcal{H})$ is the subset of $L(\mathcal{H})$ of all operators with closed range. For every $T \in L(\mathcal{H})$ its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint operator by T^*. In addition, if $T_1, T_2 \in L(\mathcal{H})$ then $T_1 \geqslant T_2$ means that $T_1 - T_2 \in L(\mathcal{H})^+$. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$. On the other hand, $T^\dagger$ stands for the Moore–Penrose inverse of $T \in L(\mathcal{H})$. Recall that $T^\dagger$ is the unique linear mapping from $\mathcal{D}(T^\dagger) = R(T) \oplus R(T)^\perp$ to $\mathcal{H}$ which satisfies the four "Moore–Penrose equations":

$$TXT = T, \quad XTX = X, \quad XT = P_{\overline{R(T^*)}}, \quad \text{and} \quad TX = P_{\overline{R(T)}}|_{\mathcal{D}(T^\dagger)}.$$

In general, $T^\dagger \notin L(\mathcal{H})$. Indeed, $T^\dagger \in L(\mathcal{H})$ if and only if $T \in L(\mathcal{H})$ has closed range [18]. On the other hand, given $T, C \in L(\mathcal{H})$ such that $R(C) \subseteq R(T)$ then it holds $T^\dagger C \in L(\mathcal{H})$ even if $T^\dagger$ is not bounded.

Given $A \in L(\mathcal{H})^+$, the functional

$$\langle\ ,\ \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad \langle \xi, \eta\rangle_A := \langle A\xi, \eta\rangle$$

is a semi-inner product on $\mathcal{H}$. By $\|\cdot\|_A$ we denote the seminorm induced by $\langle\ ,\ \rangle_A$, i.e., $\|\xi\|_A = \langle \xi, \xi\rangle_A^{1/2}$. Observe that $\|\xi\|_A = 0$ if and only if $\xi \in N(A)$. Then $\|\cdot\|_A$ is a norm if and only if $A \in L(\mathcal{H})^+$ is an injective operator. Moreover, $\langle\ ,\ \rangle_A$ induces a seminorm on a certain subset of $L(\mathcal{H})$, namely, on the subset of all $T \in L(\mathcal{H})$ for which there exists a constant $c > 0$ such that $\|T\xi\|_A \leqslant c\|\xi\|_A$ for every $\xi \in \mathcal{H}$. In such case it holds

$$\|T\|_A = \sup_{\xi \notin N(A)} \frac{\|T\xi\|_A}{\|\xi\|_A} < \infty.$$

We denote

$$L_{A^{1/2}}(\mathcal{H}) = \{T \in L(\mathcal{H}) : \|T\xi\|_A \leqslant c\|\xi\|_A \text{ for every } \xi \in \mathcal{H}\}.$$

It is easy to see that $L_{A^{1/2}}(\mathcal{H})$ is a subalgebra of $L(\mathcal{H})$. In [2] we study some properties of the operator seminorm $\|\cdot\|_A$. One of them shows the relationship between the A-seminorm and the operator uniform norm as follows: if $T \in L_{A^{1/2}}(\mathcal{H})$ then $A^{1/2}T(A^{1/2})^\dagger$ is a bounded operator on $\mathcal{D}((A^{1/2})^\dagger)$. Moreover, it holds

$$\|T\|_A = \|A^{1/2}T(A^{1/2})^\dagger\| = \|\overline{A^{1/2}T(A^{1/2})^\dagger}\| = \|(A^{1/2})^\dagger T^* A^{1/2}\|,$$

where $\overline{A^{1/2}T(A^{1/2})^\dagger}$ denotes the unique bounded linear extension of $A^{1/2}T(A^{1/2})^\dagger$ to $L(\mathcal{H})$.

Given $T \in L(\mathcal{H})$, an operator $W \in L(\mathcal{H})$ is called an ***A*-adjoint** of T if

$$\langle T\xi, \eta\rangle_A = \langle \xi, W\eta\rangle_A \quad \text{for every } \xi, \eta \in \mathcal{H},$$

or, which is equivalent, if W satisfies the equation $AW = T^*A$. The operator T is called ***A*-selfadjoint** if $AT = T^*A$. The existence of an A-adjoint operator is not guaranteed. Observe that T admits an A-adjoint operator if and only if the equation $AX = T^*A$ has solution. This kind of equations can be studied applying the next theorem due to Douglas (for its proof see [10] or [11]).

Theorem 1. *Let $B, C \in L(\mathcal{H})$. The following conditions are equivalent*:

1. $R(C) \subseteq R(B)$.
2. *There exists a positive number λ such that $CC^* \leqslant \lambda BB^*$.*
3. *There exists $D \in L(\mathcal{H})$ such that $BD = C$.*

If one of these conditions holds then there exists a unique operator $E \in L(\mathcal{H})$ such that $BE = C$ and $R(E) \subseteq \overline{R(B^*)}$.

Therefore, if we denote by $L_A(\mathcal{H})$ the subalgebra of $L(\mathcal{H})$ of all operators which admit an A-adjoint operator then

$$L_A(\mathcal{H}) = \{T \in L(\mathcal{H}) : T^*R(A) \subseteq R(A)\}.$$

Furthermore, applying Douglas theorem we can see that

$$L_{A^{1/2}}(\mathcal{H}) = \{T \in L(\mathcal{H}) : T^*R(A^{1/2}) \subseteq R(A^{1/2})\}.$$

In [14, Theorem 5.1], the following relationship between the above sets is proved:

$$L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H}).$$

Moreover, it can be checked that the equality holds if and only if A has closed range.

If an operator equation $BX = C$ has solution then it is easy to see that the distinguished solution of Douglas theorem is given by $B^\dagger C$. Therefore, given $T \in L_A(\mathcal{H})$, if we denote by $T^\sharp$ the unique A-adjoint operator of T whose range is included in $\overline{R(A)}$ then

$$T^\sharp = A^\dagger T^* A.$$

Note that if W is an A-adjoint of T then $W = T^\sharp + Z$, with $Z \in L(\mathcal{H})$ such that $R(Z) \subseteq N(A)$. In the next proposition we collect some properties of $T^\sharp$ which we shall use along this work. For its proof see [2,3].

Proposition 1.1. *Let $T \in L_A(\mathcal{H})$. Then*:

1. $T^\sharp \in L_A(\mathcal{H})$, $(T^\sharp)^\sharp = P_{\overline{R(A)}} T P_{\overline{R(A)}}$ *and* $((T^\sharp)^\sharp)^\sharp = T^\sharp$.
2. *If $W \in L_A(\mathcal{H})$ then $TW \in L_A(\mathcal{H})$ and $(TW)^\sharp = W^\sharp T^\sharp$.*
3. $\|T\|_A = \|T^\sharp\|_A = \|T^\sharp T\|_A^{1/2}$.
4. *$\|W\|_A = \|T^\sharp\|_A$ for every $W \in L(\mathcal{H})$ which is an A-adjoint of T.*

2. Cordes inequality for the *A*-seminorm

Cordes inequality [8] states that if W, V are bounded positive operators then

$$\|W^t V^t\| \leqslant \|WV\|^t \tag{1}$$

for every $t \in [0, 1]$. Furuta [13] gave an alternative proof of (1) and he proved that this inequality is equivalent to the well-known Löwner-Heinz inequality:

if $0 \leqslant W \leqslant V$ then $W^t \leqslant V^t$ for every $t \in [0, 1]$.

This section is devoted to obtain a version of the well-known Cordes inequality for the operator seminorm $\| \cdot \|_A$. In order to extend (1) we prove the following two technical lemmas. In the sequel we say that $T \in L(\mathcal{H})$ is an ***A*-positive** operator if $AT \in L(\mathcal{H})^+$.

Lemma 2.1. *Let $A \in L(\mathcal{H})^+$ and $T \in L(\mathcal{H})$. The following assertions are equivalent*:

1. *T is an A-positive operator*;
2. *$T \in L_{A^{1/2}}(\mathcal{H})$ and $\overline{A^{1/2}T(A^{1/2})^\dagger} \in L(\mathcal{H})^+$.*

Proof. If $AT \in L(\mathcal{H})^+$ then $AT = T^*A$ and so $T \in L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H})$. Then $A^{1/2}T(A^{1/2})^\dagger = (A^{1/2})^\dagger T^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}$ is a bounded positive operator on $\mathcal{D}((A^{1/2})^\dagger)$. Therefore, $\overline{A^{1/2}T(A^{1/2})^\dagger} \in L(\mathcal{H})^+$. On the contrary, if $\overline{A^{1/2}T(A^{1/2})^\dagger} \in L(\mathcal{H})^+$ then $(A^{1/2})^\dagger T^* A^{1/2} \in L(\mathcal{H})^+$. Hence we get $A^{1/2}(A^{1/2})^\dagger T^* A^{1/2} A^{1/2} = P_{\overline{R(A)}}|_{\mathcal{D}((A^{1/2})^\dagger)} T^*A = T^*A \in L(\mathcal{H})^+$. So T is an A-positive operator. □

Lemma 2.2. *Let $A, T \in L(\mathcal{H})^+$. The following assertions are equivalent*:

1. *T is an A-positive operator*;
2. *T is an $A^{1/2}$-positive operator*.

Proof. If $T \in L(\mathcal{H})^+$ is an A-positive operator then $AT = TA$. So, $A^nT = TA^n$ for every $n \in \mathbb{N}$. Thus, $p(A)T = Tp(A)$ for every polynomial p. Now, consider $f(t) = t^{1/2}$. Then there exists a sequence of polynomials $\{p_n\}$ such that $p_n(t) \underset{n\longrightarrow\infty}{\longrightarrow} f(t)$ uniformly. So, $p_n(A) \underset{n\rightarrow\infty}{\longrightarrow} f(A) = A^{1/2}$. As a consequence we get that $A^{1/2}T = TA^{1/2}$ and so T is an $A^{1/2}$-positive operator. Conversely, if $T \in L(\mathcal{H})^+$ is an $A^{1/2}$-positive operator then $A^{1/2}T = TA^{1/2}$. Therefore $AT = A^{1/2}TA^{1/2}$ is a positive operator. So, T is A-positive. □

The next proposition is a restricted version of Cordes inequality for the A-seminorm.

Proposition 2.3. *Let $A, V, W \in L(\mathcal{H})^+$. If V and W are A-positive operators then*

$$\|W^{1/2}V^{1/2}\|_A \leqslant \|WV\|_A^{1/2}.$$

Proof. First note that since $W \in L(\mathcal{H})^+$ is an A-positive operator then, by Lemma 2.2, the operator $W^{1/2}$ is A-positive too. So, $W, W^{1/2} \in L_{A^{1/2}}(\mathcal{H})$ and, by Lemma 2.1, we get that $(A^{1/2})^\dagger WA^{1/2} \in L(\mathcal{H})^+$ and $(A^{1/2})^\dagger W^{1/2}A^{1/2} \in L(\mathcal{H})^+$. Now, observe that $((A^{1/2})^\dagger WA^{1/2})^{1/2} = (A^{1/2})^\dagger W^{1/2}A^{1/2}$. The same remarks hold for the operator V. Then we get,

$$\begin{aligned}
\|W^{1/2}V^{1/2}\|_A &= \|A^{1/2}W^{1/2}V^{1/2}(A^{1/2})^\dagger\| \\
&= \|(A^{1/2})^\dagger V^{1/2}A^{1/2}(A^{1/2})^\dagger W^{1/2}A^{1/2}\| \\
&= \|((A^{1/2})^\dagger VA^{1/2})^{1/2}((A^{1/2})^\dagger WA^{1/2})^{1/2}\| \\
&\leqslant \|(A^{1/2})^\dagger VA^{1/2}(A^{1/2})^\dagger WA^{1/2}\|^{1/2} \\
&= \|WV\|_A^{1/2};
\end{aligned}$$

where the inequality holds by Cordes inequality for $t = \frac{1}{2}$. □

In the following result we present a generalization of Cordes inequality for the A-seminorm. In the proof, the concept of spectral radius of a bounded linear operator appears. Remember that, given $T \in L(\mathcal{H})$, the **spectral radius** of T is the number

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|;$$

where $\sigma(T)$ denotes the spectrum of T. In addition, it holds that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. From this we get, $r(T) \leqslant \|T\|$. On the other hand, if $T = T^*$ then $r(T) = \|T\|$ and for every $T, S \in L(\mathcal{H})$ it holds $r(TS) = r(ST)$. For a proof of the above facts the reader is referred to the books of Reed and Simon [19], Conway [6] and Davidson [9]. The proof of the next theorem follows the idea of Fujii and Furuta [12].

Theorem 2.4. *Let $A, V, W \in L(\mathcal{H})^+$. If V and W are A-positive operators then for every $t \in [0,1]$ it holds*

$$\|W^t V^t\|_A \leqslant \|WV\|_A^t. \tag{2}$$

Proof. Note that since $W \in L(\mathcal{H})^+$ is an A-positive operator then, a similar argument to that of the proof of Lemma 2.2 shows that W^t is A-positive for every $t \in [0,1]$. Now, we claim that it is sufficient to prove the inequality (2) in a dense subset $\mathcal{D}$ of $[0,1]$. In fact, let $t_0 \in [0,1]$. Then, there exists a sequence $\{t_k\} \subseteq \mathcal{D}$ such that $t_k \underset{k\to\infty}{\longrightarrow} t_0$. So, $V^{t_k}W^{t_k} \underset{k\to\infty}{\longrightarrow} V^{t_0}W^{t_0}$. On the other hand, since W^t and V^t are A-positive for every $t \in [0,1]$ then, by Lemma 2.2, we get $A^{1/2}V^tW^t = V^tW^tA^{1/2}$ for every $t \in [0,1]$. In consequence, $\|W^{t_k}V^{t_k}\|_A \underset{k\to\infty}{\longrightarrow} \|W^{t_0}V^{t_0}\|_A$. Indeed,

$$\begin{aligned}
\left|\|W^{t_k}V^{t_k}\|_A - \|W^{t_0}V^{t_0}\|_A\right| &= \left|\|(A^{1/2})^\dagger V^{t_k}W^{t_k}A^{1/2}\| - \|(A^{1/2})^\dagger V^{t_0}W^{t_0}A^{1/2}\|\right| \\
&\leqslant \|(A^{1/2})^\dagger \left(V^{t_k}W^{t_k} - V^{t_0}W^{t_0}\right) A^{1/2}\| \\
&= \|(A^{1/2})^\dagger A^{1/2} \left(V^{t_k}W^{t_k} - V^{t_0}W^{t_0}\right)\| \\
&\leqslant \|V^{t_k}W^{t_k} - V^{t_0}W^{t_0}\| \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore, if the inequality (2) holds for every $t \in \mathcal{D}$ then

$$\|W^{t_0}V^{t_0}\|_A = \lim_{k\to\infty} \|W^{t_k}V^{t_k}\|_A \leqslant \lim_{k\to\infty} \|WV\|_A^{t_k} = \|WV\|_A^{t_0}.$$

Now consider $\mathcal{D} = \left\{\frac{m}{2^n};\ m = 1, \ldots, 2^n, n \in \mathbb{N}\right\}$ which is a dense subset of $[0,1]$. Note that the inequality (2) holds for $t = 0,\ t = \frac{1}{2}$ and $t = 1$. Therefore, to prove that it holds for every element of $\mathcal{D}$ it is sufficient to show that if $\|W^sV^s\|_A \leqslant \|WV\|_A^s$ and $\|W^tV^t\|_A \leqslant \|WV\|_A^t$ for $s, t \in \mathcal{D}$ then $\|W^rV^r\|_A \leqslant \|WV\|_A^r$ for $r = \frac{s+t}{2}$. Now, since $AW^rV^r = W^rV^rA$ then

$$\overline{A^{1/2}W^rV^r(A^{1/2})^\dagger} = (A^{1/2})^\dagger W^rV^rA^{1/2}. \tag{3}$$

On the other hand, since $W^rV^{2r}W^r \in L(\mathcal{H})^+$ and $AW^rV^{2r}W^r = W^rV^{2r}W^rA$ then $AW^rV^{2r}W^r$ is positive and so, by Lemma 2.1,

$$\overline{A^{1/2}W^rV^{2r}W^r(A^{1/2})^\dagger} = (A^{1/2})^\dagger W^rV^{2r}W^rA^{1/2} \tag{4}$$

is positive too. Now, from equalities (3) and (4) we get

$$\begin{aligned}
\|W^rV^r\|_A^2 &= \|A^{1/2}W^rV^r(A^{1/2})^\dagger\|^2 \\
&= \|\overline{A^{1/2}W^rV^r(A^{1/2})^\dagger}(A^{1/2})^\dagger(W^rV^r)^*A^{1/2}\| \\
&= \|(A^{1/2})^\dagger W^rV^rA^{1/2}(A^{1/2})^\dagger(W^rV^r)^*A^{1/2}\| \\
&= \|(A^{1/2})^\dagger W^rV^{2r}W^rA^{1/2}\| \\
&= r((A^{1/2})^\dagger W^rV^{2r}W^rA^{1/2}).
\end{aligned}$$

On the other hand, as $W^sV^sA = AW^sV^s$ then $(V^sW^s)^\sharp = P_{\overline{R(A)}}W^sV^s$. Therefore

$$\|V^sW^s\|_A = \|W^sV^s\|_A.$$

Now, by properties of spectral radius and by the fact that W^r and V^{2r} belong to $L_{A^{1/2}}(\mathcal{H})$ we get

$$\begin{aligned} r((A^{1/2})^\dagger W^rV^{2r}W^rA^{1/2}) &= r((A^{1/2})^\dagger W^rA^{1/2}(A^{1/2})^\dagger V^{2r}A^{1/2}(A^{1/2})^\dagger W^rA^{1/2}) \\ &= r((A^{1/2})^\dagger V^{2r}A^{1/2}(A^{1/2})^\dagger W^{2r}A^{1/2}) \\ &= r((A^{1/2})^\dagger V^tW^tA^{1/2}(A^{1/2})^\dagger W^sV^sA^{1/2}) \\ &\leqslant \|W^tV^t\|_A\|V^sW^s\|_A = \|W^tV^t\|_A\|W^sV^s\|_A \\ &\leqslant \|WV\|_A^{t+s} = \|WV\|_A^{2r}. \end{aligned}$$

Therefore, the proof is complete. □

3. The arithmetic–geometric-mean inequality for the *A*-seminorm

We begin this section by presenting the following operator form of the so-called "arithmetic–geometric-mean inequality"

$$\|WW^*X + XVV^*\| \geqslant 2\|W^*XV\|,$$

valid for any $V, W, X \in L(\mathcal{H})$. The above inequality is due to McIntosh [17] and it also holds for every unitarily invariant norm (see [5,15]). But here, we only shall deal with the version of McIntosh's inequality for the operator uniform norm. In the following result we generalize the arithmetic–geometric-mean inequality for the operator seminorm induced by $A \in L(\mathcal{H})^+$.

Proposition 3.1. *Let $V, W \in L_A(\mathcal{H})$ and $X \in L_{A^{1/2}}(\mathcal{H})$. The following inequalities hold and they are equivalent*:

1. $\|W^\sharp WX + XV^\sharp V\|_A \geqslant 2\|WXV^\sharp\|_A$;
2. $\|WW^\sharp X + XV^\sharp V\|_A \geqslant 2\|W^\sharp XV^\sharp\|_A$;
3. $\|WW^\sharp X + XVV^\sharp\|_A \geqslant 2\|W^\sharp XV\|_A$.

Proof. First let us prove that the inequality of item 1 holds. Note that $A^{1/2}W(A^{1/2})^\dagger$, $A^{1/2}V(A^{1/2})^\dagger$ and $A^{1/2}X(A^{1/2})^\dagger$ are bounded operators on $\mathcal{D}((A^{1/2})^\dagger)$. Now, it holds

$$\begin{aligned} \|W^\sharp WX + XV^\sharp V\|_A &= \|A^{1/2}A^\dagger W^*AWX(A^{1/2})^\dagger + A^{1/2}XA^\dagger V^*AV(A^{1/2})^\dagger\| \\ &\geqslant 2\|\overline{A^{1/2}W(A^{1/2})^\dagger A^{1/2}X(A^{1/2})^\dagger}(A^{1/2})^\dagger V^*A^{1/2}\| \\ &\geqslant 2\|\overline{A^{1/2}W(A^{1/2})^\dagger A^{1/2}X(A^{1/2})^\dagger}(A^{1/2})^\dagger V^*A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}\| \\ &= 2\|\overline{A^{1/2}W(A^{1/2})^\dagger A^{1/2}X(A^{1/2})^\dagger(A^{1/2})^\dagger V^*A(A^{1/2})^\dagger}\| \\ &= 2\|A^{1/2}WXA^\dagger V^*A(A^{1/2})^\dagger\| \\ &= 2\|WXV^\sharp\|_A; \end{aligned}$$

where the first inequality holds by the arithmetic–geometric-mean inequality. So item 1 holds.

1 → 2. Observe that

$$\begin{aligned} \|WW^\sharp X + XV^\sharp V\|_A &= \|P_{\overline{R(A)}}WP_{\overline{R(A)}}W^\sharp X + XV^\sharp V\|_A \\ &= \|(W^\sharp)^\sharp W^\sharp X + XV^\sharp V\|_A \\ &\geqslant 2\|W^\sharp XV^\sharp\|_A, \end{aligned}$$

where the inequality holds by item 1. Then item 2 is obtained. Employing a similar argument to that used above we prove implications 2 → 3 and 3 → 1. □

3.1. CPR-type-inequalities for the A-seminorm

In this subsection we obtain a Corach–Porta–Recht (CPR) type inequality for the A-operator seminorm. The CPR-inequality [7] asserts that if $S, X \in L(\mathcal{H})$ with S invertible and selfadjoint then

$$\|SXS^{-1} + S^{-1}XS\| \geqslant 2\|X\|.$$

Later, Kittaneh [16] proved it for general invertible $R, S \in L(\mathcal{H})$, $X \in L(\mathcal{H})$ and unitarily invariants norms in $L(\mathcal{H})$, that is

$$|||SXR^{-1} + (S^*)^{-1}XR^*||| \geqslant 2|||X|||. \tag{5}$$

He proved this inequality by showing that it is equivalent to the arithmetic–geometric-mean inequality. Following the same lines of the Kittaneh's proof, the inequality (5) can be extended to the case S, R injective operators in $L_{cr}(\mathcal{H})$. In such case, for every $X \in L(\mathcal{H})$ and every unitarily invariant norm it holds

$$|||SXR^{\dagger} + (S^*)^{\dagger}XR^*||| \geqslant 2|||X|||. \tag{6}$$

Remark 3.2. *If S or R is not an injective operator then inequality (6) is false, in general. In fact, let $\mathcal{H} = \mathbb{R}^2$. Now take $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $R = I$ (the identity operator) and $X = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$. It is easy to check that $S^{\dagger} = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$. Now, observe that $\|SX + S^{\dagger}X\|^2 = \left\|\begin{pmatrix} 5/8 & 0 \\ 5/8 & 0 \end{pmatrix}\right\|^2 = \frac{50}{64}$. Therefore $\|SX + S^{\dagger}X\| = \sqrt{\frac{50}{64}} < 1 = 2\|X\|$.*

In the next result we generalize the CPR-inequality for the A-seminorm in two different ways. The proof follows the idea used in [16, Corollary 1].

Theorem 3.3. *Let $X \in L_{A^{1/2}}(\mathcal{H})$ and $S \in L_{cr}(\mathcal{H})$ an injective operator such that $S, S^{\dagger} \in L_A(\mathcal{H})$. Then the following assertions hold:*

1. *If $R \in L_{cr}(\mathcal{H})$ is an injective operator such that $R, R^{\dagger} \in L_A(\mathcal{H})$ then:*
   $$\|SXR^{\dagger} + (S^{\dagger})^{\sharp}XR^{\sharp}\|_A \geqslant 2\|X\|_A.$$
2. *If R is a surjective operator such that $R, R^{\dagger} \in L_A(\mathcal{H})$ then:*
   $$\|SX(R^{\dagger})^{\sharp} + (S^{\dagger})^{\sharp}XR\|_A \geqslant 2\|X\|_A.$$

Proof. It is well-known that $S \in L_{cr}(\mathcal{H})$ if and only if $S^* \in L_{cr}(\mathcal{H})$. Therefore $S^{\dagger}, (S^*)^{\dagger} \in L(\mathcal{H})$ and $(S^{\dagger})^* = (S^*)^{\dagger}$. Now, as $S \in L_{cr}(\mathcal{H})$ is an injective operator such that $S, S^{\dagger} \in L_A(\mathcal{H})$ then $S^{\sharp}(S^{\dagger})^{\sharp} = P_{\overline{R(A)}}$.

1. Since R is injective then $R^{\dagger}R = I$. Thus
   $$\begin{aligned}\|SXR^{\dagger} + (S^{\dagger})^{\sharp}XR^{\sharp}\|_A &= \|SS^{\sharp}(S^{\dagger})^{\sharp}XR^{\dagger} + (S^{\dagger})^{\sharp}XR^{\dagger}RR^{\sharp}\|_A \\ &\geqslant 2\|S^{\sharp}(S^{\dagger})^{\sharp}XR^{\dagger}R\|_A = 2\|P_{\overline{R(A)}}X\|_A \\ &= 2\|X\|_A,\end{aligned}$$
   where the inequality holds by item 3 in Proposition 3.1.
2. Since R is surjective then $R^{\dagger} \in L(\mathcal{H})$ and $RR^{\dagger} = I$. So $(R^{\dagger})^{\sharp}R^{\sharp} = P_{\overline{R(A)}}$. Now,
   $$\begin{aligned}\|SX(R^{\dagger})^{\sharp} + (S^{\dagger})^{\sharp}XR\|_A &= \|SS^{\sharp}(S^{\dagger})^{\sharp}X(R^{\dagger})^{\sharp} + (S^{\dagger})^{\sharp}X(R^{\dagger})^{\sharp}R^{\sharp}R\|_A \\ &\geqslant 2\|S^{\sharp}(S^{\dagger})^{\sharp}X(R^{\dagger})^{\sharp}R^{\sharp}\|_A = 2\|P_{\overline{R(A)}}XP_{\overline{R(A)}}\|_A \\ &= 2\|X\|_A,\end{aligned}$$

where the inequality holds by item 2 in Proposition 3.1. □

In the sequel we study the sets of operators which satisfy Theorem 3.3, namely,

$$\Delta = \{T \in L_{cr}(\mathcal{H}) : \ T \text{ is injective and } T, T^\dagger \in L_A(\mathcal{H})\}$$

and

$$\Sigma = \{T \in L(\mathcal{H}) : \ T \text{ is surjective and } T, T^\dagger \in L_A(\mathcal{H})\}.$$

The description of Δ and Σ will be done by means of the matrix representation of operators of $L(\mathcal{H})$ induced by the decomposition $\mathcal{H} = N(A)^\perp \oplus N(A)$. In such case, $A \in L(\mathcal{H})^+$ has the representation

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

where $a \in L(N(A)^\perp)^+$ and $N(a) = \{0\}$.

Proposition 3.4. *Let $T \in L_{cr}(\mathcal{H})$ and $A \in L(\mathcal{H})^+$ with the matrix representation (7). Then the following assertions are equivalent:*

1. $T \in \Delta$;
2. $T = \begin{pmatrix} t_1 & 0 \\ t_3 & t_4 \end{pmatrix}$; *where $t_1 \in L_{cr}(N(A)^\perp)$ is injective, $t_4 \in L_{cr}(N(A))$ is injective, $R(t_1^* a) \subseteq R(a)$ and $R((t_1^\dagger)^* a) \subseteq R(a)$.*

Proof. 1 → 2. Consider the following matrix representations of T and $T^\dagger$ under the decomposition $\mathcal{H} = N(A)^\perp \oplus N(A)$,

$$T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \quad \text{and} \quad T^\dagger = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}.$$

Since $T, T^\dagger \in L_A(\mathcal{H})$ and $N(a) = \{0\}$ then $t_2 = 0$, $r_2 = 0$, $R(t_1^* a) \subseteq R(a)$ and $R(r_1^* a) \subseteq R(a)$. Now, as $T^\dagger T = I$ then $r_1 t_1$ and $r_4 t_4$ are the identity operator on $N(A)^\perp$ and $N(A)$, respectively. So t_1 and t_4 are injective operators. Furthermore, since $TT^\dagger$ is selfadjoint then $r_1 = (t_1)^\dagger$ and $r_4 = (t_4)^\dagger$. Therefore $t_1 \in L_{cr}(N(A)^\perp)$, $t_4 \in L_{cr}(N(A))$ and $R((t_1^\dagger)^* a) \subseteq R(a)$.

2 → 1. Since $T = \begin{pmatrix} t_1 & 0 \\ t_3 & t_4 \end{pmatrix}$ and $R(t_1^* a) \subseteq R(a)$ then $R(T^* A) \subseteq R(A)$ and so $T \in L_A(\mathcal{H})$. On the other hand, since t_1 and t_4 are injective operators then T is injective. As, in addition, t_1 and t_4 have closed range then it is easy to check that $T^\dagger = \begin{pmatrix} t_1^\dagger & 0 \\ -t_4^\dagger t_3 t_1^\dagger & t_4^\dagger \end{pmatrix}$. Furthermore, as $R((t_1^\dagger)^* a) \subseteq R(a)$ then $R((T^\dagger)^* A) \subseteq R(A)$. Therefore $T^\dagger \in L_A(\mathcal{H})$ and so $T \in \Delta$. □

Proposition 3.5. *Let $T \in L(\mathcal{H})$ and $A \in L(\mathcal{H})^+$ with the matrix representation (7). The following assertions are equivalent:*

1. $T \in \Sigma$;
2. $T = \begin{pmatrix} t_1 & 0 \\ t_3 & t_4 \end{pmatrix}$; *where $t_1 \in L(N(A)^\perp)$ is surjective, $t_4 \in L(N(A))$ is surjective, $R(t_1^* a) \subseteq R(a)$, $R((t_1^\dagger)^* a) \subseteq R(a)$ and $R(t_3^*) \subseteq R(t_1^*)$.*

Proof. 1 → 2. Consider the following matrix representations of T and $T^\dagger$ under the decomposition $\mathcal{H} = N(A)^\perp \oplus N(A)$,

$$T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \quad \text{and} \quad T^\dagger = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}.$$

Since $T, T^\dagger \in L_A(\mathcal{H})$ and $N(a) = \{0\}$ then $t_2 = 0,\ r_2 = 0,\ R(t_1^* a) \subseteq R(a)$ and $R(r_1^* a) \subseteq R(a)$. Now, as $TT^\dagger = I$ then $t_1 r_1$ and $t_4 r_4$ are the identity operator on $N(A)^\perp$ and $N(A)$, respectively. So t_1 and t_4 are surjective operators. Furthermore, since $T^\dagger T$ is a selfadjoint projection then $r_1 = (t_1)^\dagger$ and $t_3^* r_4^* = -t_1^* r_3^*$. So $R((t_1^\dagger)^* a) \subseteq R(a)$ and $R(t_3^*) = R(t_3^* r_4^*) \subseteq R(t_1^*)$.

$2 \to 1$. Since $T = \begin{pmatrix} t_1 & 0 \\ t_3 & t_4 \end{pmatrix}$ and $R(t_1^* a) \subseteq R(a)$ then $T \in L_A(\mathcal{H})$. On the other hand, since t_1 and t_4 are surjective operators and $R(t_3^*) \subseteq R(t_1^*)$ then it is easy to check that $T^\dagger = \begin{pmatrix} t_1^\dagger & 0 \\ -t_4^\dagger t_3 t_1^\dagger & t_4^\dagger \end{pmatrix}$ and, as $R((t_1^\dagger)^* a) \subseteq R(a)$ then $T^\dagger \in L_A(\mathcal{H})$. Therefore, as $TT^\dagger = I$, the operator T is surjective and then $T \in \Sigma$. □

Remark 3.6

1. *Given $T \in \Delta$ then $T^\dagger \in \Delta$ if and only if $T \in Gl(\mathcal{H})$.*
2. *Given $T \in \Sigma$ then $T^\dagger \in \Sigma$ if and only if $T \in Gl(\mathcal{H})$.*

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