Classical Limit of Non-Integrable Systems

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Self-induced decoherence formalism and the corresponding classical limit are extended from quantum integrable systems to non-integrable ones.

1 Introduction

Decoherence was initially considered to be produced by *destructive interference* [1]. Later the strategy changed and decoherence was explained as caused by the interaction with an environment [2], but this approach is not conclusive because:

i.- The environment cannot always be defined, e. g. in closed system like the universe.

ii.-There is not a clear definition of the "cut" between the proper system and its environment.

iii.- The definition of the *pointer basis* is not simple.

So we need a new and complete theory: *The self-induced approach* [3], based in a new version of destructive interference, which will be explained in this talk in its version for non-integrable systems. The essential idea is that this interference is embodied in Riemann-Lebesgue theorem where it is proved that if $f(\nu)\epsilon\mathbb{L}_1$ then

$$\lim_{t\to\infty}\int_{-a}^{a} f(\nu)e^{-i\frac{\nu t}{\hbar}}dt = 0$$

If we use this formula in the case when $\nu = \omega - \omega'$, where ω, ω' are the indices of the density operator $\widehat{\rho}$, in such a way that $\nu = 0$ corresponds to the diagonal, we obtain a *catastrophe*, since all *diagonal* and *not diagonal* terms would disappear. But, if $f(\nu) = A\delta(\nu) + f_1(\nu)$, where now $f_1(\nu)\epsilon\mathbb{L}_1$, we have

$$\lim_{t\to\infty}\int_{-a}^{a} f(\nu)e^{-i\frac{\nu t}{\hbar}}dt = A$$

and the diagonal terms $\nu = 0$ remain while the off-diagonal ones vanish. This is the trick we will use below.

2 Weyl-Wigner-Moyal mapping

Let $\mathcal{M} = \mathcal{M}_{\in(\mathcal{N}+\infty)} \equiv \mathbb{R}^{\in(\mathcal{N}+\infty)}$ be the phase space. The functions over $\mathcal{M}$ will be called $f(\phi)$, where ϕ symbolizes the coordinates of $\mathcal{M}$

$$\phi^a = (q^1, ..., q^{N+1}, p_q^1, ..., p_q^{N+1})$$

Then the Wigner transform reads

$$symb\widehat{f} \doteq f(\phi) = \int \langle q+\Delta|\widehat{f}|q-\Delta\rangle e^{i\frac{p\Delta}{\hbar}} d^{N+1}\Delta$$

where $\widehat{f}\epsilon\widehat{\mathcal{A}}$ and $f(\phi)\epsilon\mathcal{A}$ where $\widehat{\mathcal{A}}$ is the quantum algebra and the classical one is $\mathcal{A}$. We can also introduce the star product

$$symb(\widehat{f}\widehat{g}) = symb\widehat{f} * symb\widehat{g} = (f*g)(\phi),$$

$$(f*g)(\phi) = f(\phi)\exp\left(-\frac{i\hbar}{2}\overleftarrow{\partial}_a\omega^{ab}\overrightarrow{\partial}_b\right)g(\phi)$$

and the *Moyal bracket*, which is the symbol corresponding to the commutator

$$\{f,g\}_{mb} = \frac{1}{i\hbar}(f*g - g*f) = symb\left(\frac{1}{i\hbar}[f,g]\right)$$

so we have

$$(f*g)(\phi) = f(\phi)g(\phi) + 0(\hbar)\ ,\ \{f,g\}_{mb} = \{f,g\}_{pb} + 0(\hbar^2) \quad (1)$$

To obtain the inverse $symb^{-1}$ we will use the *symmetrical* or *Weyl* ordering prescription, namely

$$symb^{-1}[q^i(\phi)p^j(\phi)] = \frac{1}{2}\left(\widehat{q}^i\widehat{p}^j + \widehat{p}^j\widehat{q}^i\right)$$

Then we have an isomorphism between the quantum algebra $\widehat{\mathcal{A}}$ and the classical one $\mathcal{A}$

$$symb^{-1}: \mathcal{A}\to\widehat{\mathcal{A}}\ ,\quad \int\dagger\Updownarrow\lfloor : \widehat{\mathcal{A}}\to\mathcal{A}$$

The mapping so defined is the *Weyl-Wigner-Moyal symbol*.

For the state we have

$$\rho(\phi) = symb\widehat{\rho} = (2\pi\hbar)^{-N-1}symb_{(\text{for operators})}\widehat{\rho}$$

and it turns out that

$$(\widehat{\rho}|\widehat{O}) = (symb\widehat{\rho}|symb\widehat{O}) = \int d\phi^{2(N+1)}\rho(\phi)O(\phi) \quad (2)$$

Namely the definition $\widehat{\rho}\epsilon\,\widehat{\mathcal{A}'}$, as afunctional on $\widehat{\mathcal{A}}$, is equal to the definition $symb\rho\epsilon\,\mathcal{A}'$, as afunctional on $\mathcal{A}$.

3 Decoherence in non integrable systems

3.1 Local CSCO.

a.- When our quantum system is endowed with a CSCO of $N+1$ observables, containing $\widehat{H}$, the underlying classical system is *integrable*. In fact, let $N+1-$CSCO be $\{\widehat{H},\widehat{O}_1,...,\widehat{O}\ _N\}$ the Moyal brackets of these quantities are

$$\{O_I(\phi),O_J(\phi)\}_{mb}=symb\left(\frac{1}{i\hbar}[\widehat{O}_I,\widehat{O}_J]\right)=0$$

where $I,J,...=0,1,...,N$ and $\widehat{H}=\widehat{O}_0$. Then when $\hbar\to0$ from Eq.(1) we know that

$$\{O_I(\phi),O_J(\phi)\}_{pb}=0 \qquad (3)$$

then as $H(\phi)=O_0(\phi)$ the set $\{O_I(\phi)\}$ is a complete set of $N+1$ constants of the motion in involution, globally defined over all $\mathcal{M}$, and therefore the system is integrable. q. e. d.

b.- If this is not the case $N+1$ constants of the motion in involution $\{H,O_1,...,O\ _N\}$ *always exist locally*, as can be shown integrating the system of equations (3). Then, if $\phi_i\epsilon\mathcal{M}$ there is *maximal domain of integration* $\mathcal{D}_{\phi_\rangle}$ *around* $\phi_i\epsilon\mathcal{M}$ where these constants are defined. In this case the system in *non-integrable*. Moreover we can repeat the procedure with the system

$$\{O_I(\phi),O_J(\phi)\}_{mb}=0 \qquad (4)$$

Then we can extend the definition of the constant $\{H,O_1,...,O\ _N\}$, defined in each $\mathcal{D}_{\phi_\rangle}$, outside $\mathcal{D}_{\phi_\rangle}$ as null functions. Their Weyl transforms $\{\widehat{H},\widehat{O}_1,...,\widehat{O}\ _N\}$ can be considered as a local $N+1$-CSCOs related each one with a domain $\mathcal{D}_{\phi_\rangle}$ that we will call $\{\widehat{H},\widehat{O}_{1\phi_i},...,\widehat{O}\ _{N\phi_i}\}$ (we consider that $\widehat{H}$ is always globally defined).

c.-We also can define an *ad hoc positive partition of the identity*

$$1=I(\phi)=\sum_i I_{\phi_i}(\phi)$$

where $I_{\phi_i}(\phi)$ is the *characteristic* function or *index* function, i.e.:

$$I_{\phi_i}(\phi)=\begin{cases}1\text{ if }\phi\epsilon D_{\phi_i}\\0\text{ if }\phi\notin D_{\phi_i}\end{cases}$$

where the domains $D_{\phi_i}\subset\mathcal{D}_{\phi_\rangle}$ $D_{\phi_i}\cap D_{\phi_j}=\emptyset$. Then $\sum_i I_{\phi_i}(\phi)=1$. Then we can define $A_{\phi_i}(\phi)=A(\phi)I_{\phi_i}(\phi)$ and

$$A(\phi)=\sum_i A_{\phi_i}(\phi)$$

and using $symb^{-1}$

$$\widehat{A}=\sum_i\widehat{A}_{\phi_i}$$

We can further decompose

$$\widehat{A}_{\phi_i}=\sum_j A_{j\phi_i}|j\rangle_{\phi_i}\langle j|_{\phi_i} \qquad (5)$$

where the $|j\rangle_{\phi_i}$ are the corresponding eigenvectors of the local $N+1-$CSCO of $D_{\phi_i}\subset\mathcal{D}_{\phi_\rangle}$ where a local $N+1$-CSCO is defined.. So

$$\widehat{A}=\sum_{ij}A_{j\phi_i}|j\rangle_{\phi_i}\langle j|_{\phi_i}$$

all over $\mathcal{M}$. It can be proved that for $i\neq k$ it is

$$\langle j|_{\phi_i}|j\rangle_{\phi_k}=0$$

so the last decomposition is orthonormal, thus decomposition (5) generalizes the usual eigen-decomposition of integrable system to the non-integrable case. We will use this decomposition below.

3.2 Decoherence in the energy.

a.- Let us define *in each* D_{ϕ_i} a local $N+1-$CSCO $\{\widehat{H}$ $,\widehat{O_{\phi_i}}\}$ (as we have said we consider that $\widehat{H}$ is always globally defined) as

$$\widehat{H}=\int_0^\infty\omega\sum_{im}|\omega,m\rangle_{\phi_i}\langle\omega,m|_{\phi_i}d\omega,$$

$$\widehat{O_{\phi_iI}}=\int_0^\infty\sum_m O_{m_{I\phi_i}}|\omega,m\rangle_{\phi_i}\langle\omega,m|_{\phi_i}d\omega$$

where we have used decomposition (5). The energy spectrum is $0\leq\omega<\infty$ and $m_{I\phi_i}=\{m_{1\phi_i},...,m_{N\phi_i}\},m_{I\phi_i}\epsilon\mathbb{N}$. Therefore

$$\widehat{H}|\omega,m\rangle_{\phi_i}=\omega|\omega,m\rangle_{\phi_i}\ ,\ \widehat{O_{\phi_iI}}|\omega,m\rangle_{\phi_i}=O_{m_{I\phi_i}}|\omega,m\rangle_{\phi_i}$$

where, from the orthonormality of the eigenvector and Eq.(5), we have

$$\langle\omega,m|_{\phi_i}|\omega',m'\rangle_{\phi_j}=\delta(\omega-\omega')\delta_{mm'}\delta_{ij}$$

b.- A generic observable, in the orthonormal basis just defined, reads:

$$\widehat{O}=\sum_{imm'}\int_0^\infty\int_0^\infty d\omega d\omega'\widetilde{O}(\omega,\omega')_{\phi_imm'}|\omega,m\rangle_{\phi_i}\langle\omega',m'|_{\phi_i}$$

where $\widetilde{O}(\omega,\omega')_{\phi_imm'}$ is a generic *kernel* or *distribution* in ω,ω'. As explained in the introduction, the simplest choice to solve our problem is the van Hove choice [4].

$$\widetilde{O}(\omega,\omega')_{\phi_imm'}=O(\omega)_{\phi_imm'}\delta(\omega-\omega')+O(\omega,\omega')_{\phi_imm'} \qquad (6)$$

where we have a *singular* and a *regular* term, so called because the first one contains a Dirac delta and in the second one the $O(\omega,\omega')_{\phi_imm'}$ are ordinary functions of the real variables ω and ω'. As we will see these two parts appear in every formulae below. So our operators belong to an algebra $\widehat{\mathcal{A}}$ and they read

$$\widehat{O}=\sum_{imm'}\int_0^\infty d\omega O(\omega)_{\phi_imm'}|\omega,m\rangle_{\phi_i}\langle\omega,m'|_{\phi_i}+$$

$$\sum_{imm'}\int_0^\infty\int_0^\infty d\omega d\omega' O(\omega,\omega')_{\phi_i mm'}|\omega,m\rangle_{\phi_i}\langle\omega',m'|_{\phi_i}$$

The *observables* are the self adjoint $O^\dagger = O$ operators. These observables belong to a space $\widehat{\mathcal{O}} \subset \widehat{\mathcal{A}}$. This space has the *basis* $\{|\omega,m,m')_{\phi_i}, |\omega,\omega',m,m')_{\phi_i}\}$ defined as:

$$|\omega,m,m')_{\phi_i} \doteq |\omega,m\rangle_{\phi_i}\langle\omega,m'|_{\phi_i} \ ,$$

$$|\omega,\omega',m,m')_{\phi_i} \doteq |\omega,m\rangle_{\phi_i}\langle\omega',m'|_{\phi_i}$$

c.- Let us define the quantum states $\widehat{\rho}\in\widehat{\mathcal{S}}\subset\widehat{\mathcal{O}}'$, where $\widehat{\mathcal{S}}$ is a convex set. The basis of $\widehat{\mathcal{O}}'$ is $\{(\omega,mm'|_{\phi_i}, (\omega\omega',mm'|_{\phi_i}\}$ and its vectors are defined as functionals by the equations:

$$(\omega,m,m'|_{\phi_i}|\eta,n,n')_{\phi_j} = \delta(\omega-\eta)\delta_{mn}\delta_{m'n'}\delta_{ij} \ ,$$

$$(\omega,\omega',m,m'|_{\phi_i}|\eta,\eta',n,n')_{\phi_j} =$$
$$\delta(\omega-\eta)\delta(\omega'-\eta')\delta_{mn}\delta_{m'n'}\delta_{ij} \ ,$$

and all others $(.|.)$ are zero. Then, a generic quantum state reads:

$$\widehat{\rho} = \sum_{imm'}\int_0^\infty d\omega\overline{\rho(\omega)}_{\phi_i mm'}(\omega,mm'|_{\phi_i} +$$

$$\sum_{imm'}\int_0^\infty d\omega\int_0^\infty d\omega'\overline{\rho(\omega,\omega')}_{\phi_i mm'}(\omega\omega',mm'|_{\phi_i}$$

We require that:

$$\overline{\rho(\omega,\omega')}_{\phi_i mm'} = \rho(\omega',\omega)_{\phi_i m'm},$$

$$\rho(\omega,\omega)_{\phi_i mm} \geq 0,$$

$$(\widehat{\rho}|\widehat{I}) = \sum_{im}\int_0^\infty d\omega\rho(\omega)_{\phi_i} = 1, \tag{7}$$

where $\widehat{I} = \int_0^\infty d\omega\sum_{im}|\omega,m\rangle_{\phi_i}\langle\omega,m|_{\phi_i}$ is the identity operator. Then, in fact, $\widehat{\rho}\in\widehat{\mathcal{S}}$, where $\widehat{\mathcal{S}}$ is a convex set, and we have

$$\langle\widehat{O}\rangle_{\widehat{\rho}(t)} = (\widehat{\rho}(t)|\widehat{O}) = \sum_{imm'}\int_0^\infty d\omega\overline{\rho(\omega)}_{\phi_i mm'}O(\omega)_{\phi_i mm'} +$$

$$\sum_{imm'}\int_0^\infty d\omega\int_0^\infty d\omega' \ \overline{\rho(\omega,\omega')}_{\phi_i mm'}$$
$$\times \ e^{i(\omega-\omega')t/\hbar}O(\omega,\omega')_{\phi_i mm'} \tag{8}$$

If we now take the limit $t\to\infty$ and use the Riemann-Lebesgue theorem, being $O(\omega,\omega')$ and $\overline{\rho(\omega,\omega')}_{\phi_i mm'}$ regular (namely $'\overline{\rho(\omega,\omega')}_{\phi_i mm'}O(\omega,\omega')\epsilon\mathbb{L}_1$ in the variable $\nu = \omega-\omega'$), we arrive to

$$\lim_{t\to\infty}\langle\widehat{O}\rangle_{\widehat{\rho}(t)} = (\widehat{\rho}_*|\widehat{O}) =$$

$$\sum_{imm'}\int_0^\infty d\omega\overline{\rho(\omega)}_{\phi_i mm'}O(\omega)_{\phi_i mm'}$$

or to the *weak limit*

$$W\lim_{t\to\infty}\widehat{\rho}(t) = \widehat{\rho}_* = \sum_{imm'}\int_0^\infty d\omega\overline{\rho(\omega)}_{\phi_i mm'}(\omega,m,m'|_{\phi_i}$$

where only the diagonal-singular terms remain showing that the *system has decohered* in the energy.

Remarks

i.- It looks like that decoherence takes place without a coarse-graining, or an environment. It is not so, the van Hove choice (6) and the mean value (8) are a restriction of the information as effective as the coarse-graining is to produce decoherence.

ii.-Theoretically decoherence takes place at $t\to\infty$. Nevertheless, for atomic interactions, the *characteristic decoherence time* is $t_D = 10^{-15}$s [5]. For macroscopic systems this time is even smaller (e.g., 10^{-38}s). Models with two characteristic times (decoherence and relaxation) can also be considered [6].

3.3 Decoherence in the other variables.

By a change of basis we can diagonalize the $\overline{\rho(\omega)}_{\phi_i mm'}$ in m and m':

$$\rho(\omega)_{\phi_i mm'} \to \rho(\omega)_{\phi_i pp'} = \rho_{\phi_i}(\omega)_p\,\delta_{pp'}.$$

in a new basis orthonormal $\{|\omega,p\rangle_{\phi_i}\}$. Therefore $\rho_{\phi_i}(\omega)_p\,\delta_{pp'}$.is now diagonal in all its coordinates in a *final local pointer basis in each* D_{ϕ_i}, which, in the case of the observables is $\{\ |\omega,p,p')_{\phi_i}, |\omega,\omega',p,p')_{\phi_i}\}$ (i. e. essentially $\{|\omega',p'\rangle_{\phi_i}\}$), so in this pointer basis we have obtained a *boolean quantum mechanics with no interference terms* and we have the weak limit:

$$W\lim_{t\to\infty}\widehat{\rho}(t) = \widehat{\rho}_* = \sum_{ip}\int_0^\infty d\omega\overline{\rho_{\phi_i}(\omega)}_p(\omega,p,p|_{\phi_i}$$

or in the case of $\widehat{P}$ with continuous spectra:

$$W\lim_{t\to\infty}\widehat{\rho}(t) = \widehat{\rho}_* =$$

$$\sum_i\int_0^\infty d\omega\int_{p\epsilon D_{\phi_i}} dp^N\overline{\rho(\omega)_{\phi_i}}(\omega,p,p|_{\phi_i} \tag{9}$$

the only case that we will consider below.

4 The classical statistical limit

a.- Let us now take into account the Wigner transforms. *There is no problem for regular operators* which are considered in the standard theory. Moreover these operators are irrelevant since they disappear after decoherence.

b.- So we must only consider the singular ones as

$$\widehat{O}_S = \sum_i \int_{p\epsilon D_{\phi_i}} dp^N \int_0^\infty O_{\phi_i}(\omega,p)|\omega,p\rangle_{\phi_i}\langle\omega,p|_{\phi_i} d\omega$$

where now the $\widehat{P}$ have continuous spectra. So

$$\widehat{O}_S = \sum_i O_{\phi_i}(\widehat{H},\widehat{P_{\phi_i}}) = \sum_i \widehat{O}_{S\phi_i}$$

But $\widehat{H}, \widehat{P_{\phi_i}}$ commute thus

$$symb\widehat{O}_S = O_S(\phi) = \sum_i O_{\phi_i}(H(\phi), P_{\phi_i}(\phi)) + 0(\hbar^2)$$

and if $O_{\phi_i}(\omega,p) = \delta(\omega-\omega')\delta(p-p')$ we have

$$symb|\omega',p'\rangle_{\phi_i}\langle\omega',p'|_{\phi_i} = \delta(H(\phi)-\omega')(P_{\phi_i}(\phi)-p)$$

(*really up to* $0(\hbar^2)$, but for the sake of simplicity we will eliminate these symbols from now on).

Let us now consider the singular dual, the $symb\widehat{\rho}_S$ as the functional on $\mathcal{M}$ that must satisfy Eq.(2) that now reads

$$(symb\widehat{\rho}_S|symb\widehat{O_S}) = (\widehat{\rho}_S|\widehat{O}_S)$$

Then we define a density function $\rho_S(\phi) = symb\widehat{\rho}_S = \sum_i \rho_{\phi_i S}(\phi)$ such that

$$\sum_i \int d\phi^{2(N+1)} \rho_{\phi_i S}(\phi) O_{\phi_i S}(\phi) =$$

$$\sum_i \int_{p\epsilon D_{\phi_i}} \int_0^\infty \rho_{\phi i}(\omega,p,)O_{\phi_i}(\omega,p) d\omega dp^N \qquad (10)$$

$\widehat{\rho_S}$, is constant of the motion, so $\rho_{\phi_i}(\phi) = f(H(\phi), P_{\phi_i}(\phi))$. Then we *locally define at* D_{ϕ_i} the local action-angle variables $(\theta^0,\theta^1,...,\theta^N, J^0_{\phi_i}, J^1_{\phi_i},...,J^N_{\phi_i})$, where $J^0_{\phi_i}$, $J^1_{\phi_i},...,J^N_{\phi_i}$ would just be $H, P_{\phi_i 1},...,P_{\phi_i N}$ and we make the *canonical transformation* $\phi^a \longrightarrow \theta^0_{\phi_i},\theta^1_{\phi_i},...,\theta^N_{\phi_i}, H, P_{\phi_i 1},...,P_{\phi_i N}$ so that

$$d\phi^{2(N+1)} = dq^{(N+1)}dp^{(N+1)} = d\theta^{(N+1)}_{\phi_i} dH dP^N_{\phi_i}$$

Now we will integrate of the functions $f(H,P_{\phi_i}) = f(H,P_{\phi_i},...,P_{\phi_i})$ using the new variables.

$$\int_{D_{\phi_i}} d\phi^{2N+2} f(H,P_{\phi_i}) = \int_{D_{\phi_i}} d\theta^{N+1}_{\phi_i} dH dP^N_{\phi_i} f(H,P_{\phi_i})$$

$$= \int_{D\phi_i} dH dP^N_{\phi_i} C_{\phi_i}(H,P_{\phi_i}) f(H,P_{\phi_i})$$

where we have integrated the angular variables $\theta^0_{\phi_i},\theta^1_{\phi_i},...,\theta^N_{\phi_i}$, obtaining the *configuration volume* $C_{\phi_i}(H,P_{\phi_i})$ of the portion of the hypersurface defined by $(H = const., P_{\phi_i} = const.)$ and contained in D_{ϕ_i}. So Eq.(10) reads

$$\sum_i \int_{p\epsilon D_{\phi_i}} \int_0^\infty \rho_{\phi i}(\omega,p,)O_{\phi_i}(\omega,p) d\omega dp^N =$$

$$\sum_i \int dH dP^N_{\phi_i} C_{\phi_i}(H,P_{\phi_i}) \rho_{\phi_i S}(H,P_{\phi_i}) O_{\phi_i S}(H,P_{\phi_i})$$

for any $O_{\phi_i}(\omega,p)$ so $\rho_{S\phi_i}(H,P) = \frac{1}{C_{\phi_i}}\rho_{\phi_i}(H,P)$ for $\phi\epsilon\mathcal{D}_{\phi_\rangle}$ and

$$\rho_S(\phi) = \rho_*(\phi) = \sum_i \frac{\rho_{\phi_i}(H(\phi), P_{\phi_i}(\phi))}{C_{\phi_i}(H,P_{\phi_i})}$$

Putting $\rho_{\phi_i}(\omega,p) = \delta(\omega-\omega')\delta^N(p-p')$ for some i and all other $\rho_{\phi_j}(\omega,p) = 0$ for $j \neq i$, we have

$$symb(\omega',p',(\phi)|_{\phi_i} = \frac{\delta(H(\phi)-\omega')\,\delta^{(N)}\left(P(\phi)-p'_{\phi_i}\right)}{C_{\phi_i}(H,P_{\phi_i})}$$

c.- Moreover the $symb$ of Eq.(9) reads

$$\rho_S(\phi) = \rho_*(\phi) = \sum_i \int_{p\epsilon D_{\phi_i}} dp\times$$

$$\int_0^\infty d\omega \rho_{\phi_i}(\omega,p) \frac{\delta(H(\phi)-\omega)\,\delta^{(N)}(P(\phi)-p_{\phi_i})}{C_{\phi_i}(H,P_{\phi_i})} \qquad (11)$$

So we have obtained a decomposition of $\rho_*(\phi) = \rho_S(\phi)$ in classical hypersurfaces ($H = \omega$, $P_{\phi_i}(\phi) = p_{\phi_i}$), containing *chaotic trajectories* (since the system is not integrable), summed with different weight coefficients $\rho_{\phi_i}(\omega,p)/C_{\phi_i}(H,P_{\phi_i})$.

d.- Finally only after decoherence the positive definite diagonal-singular part remains and from Eqs.(7) and (11) we see that

$$\rho_{\phi_i}(\omega,p) \geq 0 \Rightarrow \rho_*(\phi) \geq 0$$

so the *classical statistical limit* is obtained.

5 The classical limit

The classical limit can be decomposed into the following processes

$$Quantum\ Mechanics - \text{(decohence)} \longrightarrow$$

$$Boolean\ Quantum\ Mechanics - \text{(symb and } \hbar \to 0\text{)} \longrightarrow$$

$$Classical\ Statistical\ Mechanics - \text{(choice of a trajectory)}$$

$$\longrightarrow Classical\ Mechanics$$

where the first two have been explained. It only remains the last one: For $\tau(\phi) = \theta^0_{\phi_i}(\phi)$ and at any fixed t we have

$$\sum_i \int_{D_{\phi_i}} \delta(\tau(\phi) - \tau_0 - \omega t)\delta(\theta_{\phi_i}(\phi) - \theta_{\phi_i 0} - p_{\phi_i} t) d\tau_0 d\theta_{\phi_i 0} = 1$$

then we can include this 1 in decomposition (11) and we obtain

$$\rho_*(\phi) = \sum_i \int \frac{\rho_{\phi_i}(\omega, p_{\phi_i})}{C(\omega, p_{\phi_i})} \delta(H(\phi) - \omega)\delta(P_{\phi_i} - p_{\phi_i}) \times$$

$$\delta(\tau(\phi) - \tau_0 - \omega t)\delta(\theta_{\phi_i}(\phi) - \theta_{\phi_i 0} - p_{\phi_i} t) d\omega d^N p_{\phi_i} d\tau_0 d\theta_{\phi_i 0}$$

namely a sum of *classical chaotic trajectories* satisfying:

$$H(\phi) = \omega \ , \ \ \tau(\phi) = \tau_0 + \omega t) \ ,$$

$$P_{\phi_i} = p_{\phi_i} \ , \ \ \theta_{\phi_i}(\phi) = \theta_{\phi_i 0} + p_{\phi_i} t$$

weighted by $\frac{\rho_{\phi_i}(\omega, p_{\phi_i})}{C(\omega, p_{\phi_i})}$,where we can choose any one of them. In this way the classical limit is completed, in fact we have found the classical limit of a quantum system since we have obtained the classical trajectories, so the *correspondence principle* is also obtained as a theorem.

6 Conclusion

i.- We have defined the classical limit in the non-integrable case.

ii.- Essentially, we have presented a *minimal formalism for quantum chaos* [7].

iii.- We have deduced the correspondence principle.

References

[1] N.G. van Kampen, Physica **XX**, 603 (1954). A. Daneri, A. Loinger, and G.M. Prosperi, Nucl. Phys. **33**, 297 (1962).

[2] J.P. Paz and W. Zurek, "Environment-induced decoherence and the transition from classical to quantum", arXiv: quant-ph/0010011 (2000).

[3] M. Castagnino and R. Laura, Phys. Rev., A**62**, 022107 (2000). M. Castagnino, Physica A **335**, 511 (2004). M. Castagnino and O. Lombardi, Stud. Phil. and Hist. Mod. Phys. **35**, 73 (2004). M. Castagnino and O. Lombardi, "Self-Induced decoherence and the classical limit of quantum mechanics", *Phil. of Sci.*, in press (2004).

[4] L. van Hove, Physica **21**, 517 (1955); **23**, 441 (1957); **25**, 268 (1959).

[5] M. Castagnino, M. Gadella, R. Liotta, and R. Id Betan, Journ. Phys. A**34**, 10067 (2001). M. Castagnino, R. Laura, R. Liotta, and R. Id Betan, Journ. Phys. A**35**, 6055 (2002).

[6] M. Castagnino, "The master and Fokker-Plank equations of the self-induced approach", in preparation (2004).

[7] M. Castagnino, "Quantum chaos according to the self-induced approach", in preparation (2004).