THE BANACH IDEAL OF $\mathcal{A}$-COMPACT OPERATORS AND RELATED APPROXIMATION PROPERTIES

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Abstract. We use the notion of $\mathcal{A}$-compact sets (determined by an operator ideal $\mathcal{A}$), introduced by Carl and Stephani (1984), to show that many known results of certain approximation properties and several ideals of compact operators can be systematically studied under this framework. For Banach operator ideals $\mathcal{A}$, we introduce a way to measure the size of $\mathcal{A}$-compact sets and use it to give a norm on $\mathcal{K}_{\mathcal{A}}$, the ideal of $\mathcal{A}$-compact operators. Then, we study two types of approximation properties determined by $\mathcal{A}$-compact sets. We focus our attention on an approximation property which makes use of the norm defined on $\mathcal{K}_{\mathcal{A}}$. This notion fits the definition of the $\mathcal{A}$-approximation property, recently introduced by Oja (2012), with $\mathcal{K}_{\mathcal{A}}$ instead of $\mathcal{A}$. We exemplify the power of the Carl-Stephani theory and the geometric structure introduced here by appealing to some recent developments on p-compactness.

Introduction

Recall that a Banach space has the approximation property if the identity map can be uniformly approximated by finite rank operators on compact sets. This property, due to Grothendieck, has several reformulations, see Grothendieck's *Memoir* [15]. Reinforced by the fact that there are Banach spaces which lack the approximation property (the first example given by Enflo [13]), important variants of this property have emerged and were intensively studied, see [4, 11, 18, 21] and references therein. In particular, there is a recent inclination to study approximation properties related to (Banach) operator ideals, as it can be seen for instance in [1, 5, 7, 9, 14, 16, 17, 19, 22, 23, 29, 30].

The main purpose of this article is to undertake the study of a general method to understand a wide class of approximation properties and different ideals of compact operators which can be equally modeled once the system of compact sets has been chosen. To this end, we use notion of $\mathcal{A}$-compactness, introduced by Carl and Stephani [3], which is determined by an operator ideal $\mathcal{A}$ and can be seen as a refinement of the concept of compactness. The system of $\mathcal{A}$-compact sets induces in a natural way the class of $\mathcal{A}$-compact operators,

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consisting of all continuous linear operators mapping bounded sets into $\mathcal{A}$-compact sets. This ideal, which we denote by $\mathcal{K}_{\mathcal{A}}$, was also introduced an studied in [3]. However, the authors do not emphasize their study from an isometric point of view. In Section 1, we add a geometric structure to the Carl-Stephani theory by introducing a way to measure the size of $\mathcal{A}$-compact sets, denoted by $m_{\mathcal{A}}$. Our prototype is the definition given for the size of p-compact sets in [16] and studied later in [14]. We examine the class of $\mathcal{A}$-compact sets in a Banach space E and show that the definition can be reformulated considering only operators in $\mathcal{A}(\ell_1; E)$. The class of p-compact sets fits in this framework for the ideal $\mathcal{N}^p$ of right p-nuclear operators. This fact and the notion of $\mathcal{A}$-null sequences [3] allow us to give another proof of a question posed in [8] and solved by Oja in her recent work [24].

In Section 2 we use $m_{\mathcal{A}}$ to endow $\mathcal{K}_{\mathcal{A}}$ with a norm $\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}$, under which it is a Banach operator ideal. Then, we show that the main factorization result of [3] concerning $\mathcal{K}_{\mathcal{A}}$ is, in fact, an isometric identity. We use our characterization of $\mathcal{A}$-compact sets via ℓ_1, to prove that $\mathcal{K}_{\mathcal{A}}$ is regular. As a consequence, we show that a subset is $\mathcal{A}$-compact with equal size regardless it is considered as a set of a Banach space E or as a set of its bidual E''.

The system of $\mathcal{A}$-compact sets leads naturally to two types of approximation properties which are considered in Section 3. The first one is rather standard and is defined by requiring the identity map to be uniformly approximable by finite rank operators on $\mathcal{A}$-compact sets. We prove that a Banach space E enjoys this property if and only if, for any Banach space F, the set of finite rank operators from F to E is norm dense in $\mathcal{K}_{\mathcal{A}}(F; E)$. We call the latter property the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property. For the second one, the norm $\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}$ is considered instead of the supremum norm. In this case, we show that our definition coincides with that of $\mathcal{K}_{\mathcal{A}}$-approximation property of Oja [23]. With the particular case of $\mathcal{N}^p$, on the one hand, we cover the notion of p-approximation property introduced by Sinha and Karn [29] and studied by many authors in the last years, see for instance [1, 5, 7, 16]. On the other hand, we cover the κ_p-approximation property defined by Delgado, Piñeiro and Serrano [9] and studied later in [14, 16, 23]. Also, we address the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property in terms of a modified ϵ-product of Schwartz.

Throughout this paper E and F denote Banach spaces, E' and B_E denote the topological dual and the closed unit ball of E, respectively. A Banach operator ideal is denoted by $(\mathcal{A}, \|.\|_{\mathcal{A}})$. When the norm $\|\cdot\|_{\mathcal{A}}$ is understood or when we work with an operator ideal, we simply write $\mathcal{A}$. We denote by $\mathcal{L}, \mathcal{F}, \overline{\mathcal{F}}$ and $\mathcal{K}$ the linear operator ideals of bounded, finite rank, approximable and compact operators, respectively; all considered with the supremum norm. For $x' \in E'$ and $y \in F$, the 1-rank operator from E to F, $x \mapsto x'(x)y$ is denoted by $x' \otimes y$.

To illustrate our results, we appeal to the ideals $\mathcal{N}^p$ of right p-nuclear operators and $\mathcal{K}_p$ of p-compact operators. To give a brief description of these spaces, we need some definitions.

As usual, fixed $1 \leq p < \infty$, $\ell_p(E)$ and $\ell_p^w(E)$ denote the spaces of p-summable and weakly p-summable sequences in E, respectively. For $p = \infty$, we use the spaces $c_0(E)$ and $c_0^w(E)$ of null and weakly null sequences of E, respectively. All these are Banach spaces endowed with their natural norms. A mapping T belongs to $\mathcal{N}^p(E;F)$ if there exist sequences $(x_n')_n \in \ell_{p'}^w(E')$ and $(y_n)_n \in \ell_p(F)$, $\frac{1}{p} + \frac{1}{p'} = 1$ ($\ell_{p'} = c_0$ if $p = 1$), such that $T = \sum_{n=1}^\infty x_n' \otimes y_n$ and $\nu^p(T) = \inf\{\|(x_n')_n\|_{\ell_{p'}^w(E')} \|(y_n)_n\|_{\ell_p(F)} \colon T = \sum_{n=1}^\infty x_n' \otimes y_n\}$ is the right p-nuclear norm of T. Following [29], a subset K of E is relatively p-compact, $1 \leq p \leq \infty$, if there exists a sequence $(x_n)_n \subset \ell_p(E)$ so that $K \subset p\text{-co}\{x_n\}$, where $p\text{-co}\{x_n\} = \{\sum_{n=1}^\infty \alpha_n x_n \colon (\alpha_n)_n \in B_{\ell_{p'}}\}$ is called the p-convex hull of $(x_n)_n$ and $\frac{1}{p} + \frac{1}{p'} = 1$ ($\ell_{p'} = c_0$ if $p = 1$). With $p = \infty$, we have the relatively compact sets and the absolutely convex hull of $(x_n)_n$, denoted here by $\text{co}\{x_n\}$. A mapping T is in $\mathcal{K}_p(E;F)$ if it maps bounded sets into relatively p-compact sets and the p-compact norm of T is $\kappa_p(T) = \inf\{\|(y_n)_n\|_p \colon T(B_E) \subset p\text{-co}\{y_n\}\}$, see [10, 29].

The definitions and notation used regarding operator ideals can be found in the monograph by Defant and Floret [6]. For operator ideals we also refer the reader to the books of Pietsch [25], of Diestel, Jarchow and Tonge [12] and of Ryan [27]. For approximation properties, we refer the reader to the books of Lindenstrauss and Tzafriri [18] and of Diestel, Fourie and Swart [11]. See also [6, 27], the surveys [4] and [21] and references therein.

1. On compact sets and operator ideals

Fix an operator ideal $\mathcal{A}$. Following [3], a subset K of E is said to be relatively $\mathcal{A}$-compact if there exist a Banach space X, an operator $T \in \mathcal{A}(X;E)$ and a compact set $M \subset X$ such that $K \subset T(M)$. A sequence $(x_n)_n \subset E$ is $\mathcal{A}$-convergent to zero if there exist a Banach space X and $T \in \mathcal{A}(X;E)$ with the following property: given $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $x_n \in \varepsilon T(B_X)$ for all $n \geq n_\varepsilon$. Carl and Stephani gave a handy characterization of $\mathcal{A}$-null sequences [3, Lemma 1.2].

Lemma 1.1. (Carl-Stephani) *Let E be a Banach space and $\mathcal{A}$ an operator ideal. A sequence $(x_n)_n \subset E$ is $\mathcal{A}$-null if and only if there exist a Banach space X, an operator $T \in \mathcal{A}(X;E)$ and a null sequence $(y_n)_n \subset X$ such that $x_n = T(y_n)$ for all n.*

Now, we present a characterization of $\mathcal{A}$-compactness which is a combination of [3, Lemma 1.1] and [3, Theorem 1.1].

Theorem 1.2. (Carl-Stephani) *Let E be a Banach space, K a subset of E and $\mathcal{A}$ an operator ideal. The following are equivalent.*

(i) *K is relatively $\mathcal{A}$-compact.*

(ii) *There exist a Banach space X and an operator $T \in \mathcal{A}(X;E)$ such that for every $\varepsilon > 0$ there are finitely many elements $z_i^\varepsilon \in E$, $1 \leq i \leq k_\varepsilon$ realizing a covering of K: $K \subset \bigcup_{i=1}^{k_\varepsilon} \{z_i^\varepsilon + \varepsilon T(B_X)\}$.*

(iii) *There exists an $\mathcal{A}$-null sequence $(x_n)_n \subset E$ such that $K \subset \text{co}\{x_n\}$.*

The following remark is the key to see that the theory of p-compactness perfectly embodies into the Carl-Stephani theory.

Remark 1.3. Let $1 \leq p < \infty$. The proof of [14, Proposition 2.9] shows that given a sequence $(x_n)_n \in \ell_p(E)$, there is an operator $T \in \mathcal{N}^p(\ell_{p'}; E)$ such that $p\text{-co}\{x_n\} = T(M)$ with $M \subset B_{\ell_{p'}}$ relatively compact. Moreover, for fixed $\varepsilon > 0$, T may be chosen to satisfy

$$\|(x_n)_n\|_p \leq \|T\|_{\mathcal{N}^p} \leq \|(x_n)_n\|_p + \varepsilon.$$

Note that if $p = \infty$ and $(x_n)_n \in c_0(E)$, the obtained operator is in $\overline{\mathcal{F}}(\ell_1; E)$.

Notice that compact sets are $\overline{\mathcal{F}}$-sets or $\mathcal{K}$-sets and p-compact sets are $\mathcal{N}^p$-compact sets. Also, by [3, p. 79], p-compact sets are $\mathcal{K}_p$-compact sets. In [8], Delgado and Piñeiro define p-null sequences, $p \geq 1$, as follows. A sequence $(x_n)_n$ in a Banach space E is p-null if, given $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $(z_k)_k \in \varepsilon B_{\ell_p(E)}$ such that $x_n \in p\text{-co}\{z_k\}$ for all $n \geq n_0$. In [8, Theorem 2.5], p-compact sets are characterized as those which are contained in the absolutely convex hull of a p-null sequence. Then, the authors prove, under certain hypothesis on the Banach space E, that a sequence is p-null if and only if it is norm convergent to zero and relatively p-compact [8, Proposition 2.6]. Also, they wonder if the result remains true for arbitrary Banach spaces. An affirmative answer was recently given by Oja [24, Theorem 4.3]. In [24] the author describes the space of p-null sequences as a tensor product via the Chevet-Saphar tensor norm and, as an application, the result is obtained. Here, we show that the Delgado-Piñeiro-Oja result is an immediate consequence of the next two propositions.

Proposition 1.4. *Let $\mathcal{A}$ be an operator ideal and E a Banach space. A sequence $(x_n)_n \subset E$ is $\mathcal{A}$-null if and only if $(x_n)_n$ is relatively $\mathcal{A}$-compact and norm convergent to zero.*

Proof. Thanks to Theorem 1.2, only the "if" part requires a proof. Take a Banach space X, $T \in \mathcal{A}(X; E)$ and a compact set $M \subset X$ such that $\{x_n\}_n \subset T(M)$. Consider the quotient map $q\colon X \to X/\ker(T)$ and the injective operator $\widetilde{T}$ such that $T = \widetilde{T} \circ q$. Then, $(x_n)_n$ is a norm null sequence in $\widetilde{T}(q(M))$ with $q(M)$ compact. By standard arguments, there is a norm null sequence $(y_n)_n$ in X such that $x_n = T(y_n)$. An application of Lemma 1.1 completes the proof. □

Notice that a sequence $(z_k)_k$ as in the definition of a p-null sequence might be chosen independently of ε as the following result shows.

Proposition 1.5. *Let E be a Banach space, $(x_n)_n$ a sequence in E and $p \geq 1$. Then, $(x_n)_n$ is p-null if and only if there exists a sequence $(z_k)_k \in B_{\ell_p(E)}$ such that for any $\varepsilon > 0$ there exists n_0 with $x_n \in \varepsilon p\text{-co}\{z_k\}$ for all $n \geq n_0$. As a consequence, $\mathcal{N}^p$-null and p-null sequences coincide.*

Proof. We only show the "only if" part. Suppose that $(x_n)_n$ is p-null and find $(z_k)_k$ as in the statement. By [8, Definition 2.1], we may find a strictly increasing sequence n_j, $j = 1, 2, \ldots$, and sequences $(z_k^j)_k \in B_{\ell_p(E)}$ such that $x_n \in K_j = \frac{1}{4^j} p\text{-co}\{z_k^j\}$ for all $n \geq n_j$. Proceeding as in [1, Theorem 1], we may find a p-compact set K such that $2^j K_j \subset K$ for all j. Therefore, there exists $(z_k)_k \in B_{\ell_p(E)}$ such that $K \subset p\text{-co}\{z_k\}$ and $x_n \in \frac{1}{2^j} p\text{-co}\{z_k\}$ for all $n \geq n_j$. Then, the result follows. □

Corollary 1.6. (Delgado-Piñeiro-Oja) *Let E be a Banach space and $p \geq 1$. A sequence $(x_n)_n \subset E$ is p-null if and only if $(x_n)_n$ is relatively p-compact and norm convergent to zero.*

Now, we introduce a way to measure the size of relatively $\mathcal{A}$-compact sets. Our definition is inspired by the one given for p-compact sets in [16] and studied later in [14]. Fix an operator ideal $\mathcal{A}$ and a norm α on $\mathcal{A}$. For a relatively $\mathcal{A}$-compact set $K \subset E$, we define

$$m_{\mathcal{A},\alpha}(K; E) = \inf\{\alpha(T) \colon K \subset T(M),\ T \in \mathcal{A}(X; E) \text{ and } M \subset B_X\},$$

where the infimum is taken considering all Banach spaces X, all operators $T \in \mathcal{A}(X; E)$ and all compact sets $M \subset B_X$ for which the inclusion $K \subset T(M)$ holds.

If $K \subset E$ is not $\mathcal{A}$-compact, $m_{\mathcal{A},\alpha}(K; E) = \infty$. As it happens with the size of p-compact sets, see [14, Section 2], there are some properties which derive directly from the definition of $m_{\mathcal{A},\alpha}$. For instance, $m_{\mathcal{A},\alpha}(K; E) = m_{\mathcal{A},\alpha}(\text{co}\{K\}; E)$.

For simplicity, if $(\mathcal{A}, \|.\|_{\mathcal{A}})$ is a Banach operator ideal, we write $m_{\mathcal{A}}$ instead of $m_{\mathcal{A},\|\cdot\|_{\mathcal{A}}}$. Since $\|T\| \leq \|T\|_{\mathcal{A}}$, $\sup_{x \in K} \|x\| \leq m_{\mathcal{A}}(K; E)$. Moreover, if $\mathcal{B}$ is a Banach operator ideal such that $\mathcal{A} \subset \mathcal{B}$, a set $K \subset E$ is $\mathcal{B}$-compact whenever it is $\mathcal{A}$-compact and we have $m_{\mathcal{B}}(K; E) \leq m_{\mathcal{A}}(K; E)$.

Remark 1.7. Let $1 \leq p < \infty$. By Remark 1.3, if $K \subset E$ is p-compact then

$$m_{\mathcal{N}^p}(K; E) = \inf\{\|(x_n)\|_p \colon K \subset p\text{-co}\{x_n\}\}.$$

Analogously, if $K \subset E$ is compact then $m_{\mathcal{K}}(K; E) = \sup\{\|x\| \colon x \in K\}$.

Note that $m_{\mathcal{N}^p}$ covers exactly the size of p-compact sets (see [16] or [14, Definition 2.1]). Also, note that if F is a Banach space containing E as a closed subspace, a set $K \subset E$ is $\mathcal{A}$-compact as a subset of F whenever it is $\mathcal{A}$-compact as a subset of E, and $m_{\mathcal{A}}(K; F) \leq m_{\mathcal{A}}(K; E)$. However, $m_{\mathcal{A}}$ may depend on the space the sets are considered, as it is shown in [14, Corollary 3.5]. The particular case when $F = E''$, for which the size is preserved, is considered in Corollary 2.3.

The next result shows that the definition of $\mathcal{A}$-compact sets (and therefore the size $m_{\mathcal{A}}$) can be reformulated considering only operators in $\mathcal{A}(\ell_1; E)$.

Proposition 1.8. *Let E be a Banach space, K a subset of E and $\mathcal{A}$ a Banach operator ideal. The following are equivalent.*

(i) *K is relatively $\mathcal{A}$-compact.*

(ii) *There exist a Banach space X, operators $T \in \mathcal{A}(X;E)$ and $S \in \overline{\mathcal{F}}(\ell_1;X)$ and a relatively compact set $M \subset B_{\ell_1}$ such that $K \subset T \circ S(M)$. Moreover,*

$$m_{\mathcal{A}}(K;E) = \inf\{\|T\|_{\mathcal{A}}\|S\| \colon K \subset T \circ S(M) \text{ and } M \subset B_{\ell_1}\},$$

where the infimum is taken over all Banach spaces X, operators T and S and sets M as above.

(iii) *There exist an operator $T \in \mathcal{A}(\ell_1;E)$ and a relatively compact set $M \subset B_{\ell_1}$ such that $K \subset T(M)$. Also,*

$$m_{\mathcal{A}}(K;E) = \inf\{\|T\|_{\mathcal{A}} \colon K \subset T(M), \quad T \in \mathcal{A}(\ell_1;E) \quad \text{and} \quad M \subset B_{\ell_1}\},$$

where the infimum is taken over all operators $T \in \mathcal{A}(\ell_1;E)$ and all relatively compact sets $M \subset B_{\ell_1}$ such that $K \subset T(M)$.

Proof. Suppose $K \subset E$ is relatively $\mathcal{A}$-compact. Given $\varepsilon > 0$, take a Banach space X, a compact set $L \subset B_X$ and $T \in \mathcal{A}(X;E)$ such that $K \subset T(L)$ and $\|T\|_{\mathcal{A}} \leq m_{\mathcal{A}}(K;E) + \varepsilon$. Since $L \subset B_X$ is compact, we may find an approximable operator $S \colon \ell_1 \to X$ and a compact set $M \subset B_{\ell_1}$ such that $L \subset S(M)$. Moreover, S may be chosen to satisfy $\|S\| \leq 1+\varepsilon$. Then, $K \subset T(L) \subset T \circ S(M)$ and

$$m_{\mathcal{A}}(K;E) \leq \|T\|_{\mathcal{A}}\|S\| \leq \|T\|_{\mathcal{A}}(1+\varepsilon) \leq (m_{\mathcal{A}}(K;E)+\varepsilon)(1+\varepsilon).$$

Then (ii) follows from (i). It is clear that (ii) implies (iii) which implies (i), and the proof is complete. □

Corollary 1.9. *Let E be a Banach space, K a subset of E and $\mathcal{A}$ a Banach operator ideal. Then, K is relatively $\mathcal{A}$-compact if and only if K is relatively $\mathcal{A} \circ \overline{\mathcal{F}}$-compact and $m_{\mathcal{A}}(K;E) = m_{\mathcal{A}\circ\overline{\mathcal{F}}}(K;E)$.*

Proof. Every relatively $\mathcal{A}\circ\overline{\mathcal{F}}$-compact set is relatively $\mathcal{A}$-compact and $m_{\mathcal{A}}(K;E) \leq m_{\mathcal{A}\circ\overline{\mathcal{F}}}(K;E)$. The other implication is given by item (ii) of the above proposition, which combined with item (iii) gives $m_{\mathcal{A}}(K;E) = m_{\mathcal{A}\circ\overline{\mathcal{F}}}(K;E)$. □

2. The ideal of $\mathcal{A}$-compact operators

Hereinafter we use the procedures: $\mathcal{A} \to \mathcal{A}^{sur}, \mathcal{A} \to \mathcal{A}^{reg}$ and $\mathcal{A} \to \mathcal{A}^{d}$, which are given for Banach operator ideals as follows. The surjective hull $\mathcal{A}^{sur}$ of $\mathcal{A}$ is the class of $T \in \mathcal{L}(E;F)$ such that $T \circ q_E$ belongs to $\mathcal{A}$ where $q_E \colon \ell_1(B_E) \twoheadrightarrow E$ is the canonical surjection and $\|T\|_{\mathcal{A}^{sur}} = \|T \circ q_E\|_{\mathcal{A}}$. The regular hull $\mathcal{A}^{reg}$ of $\mathcal{A}$ is the class of $T \in \mathcal{L}(E;F)$ such that $J_F \circ T \in \mathcal{A}(E;F'')$ and $\|T\|_{\mathcal{A}^{reg}} = \|J_F \circ T\|_{\mathcal{A}}$, where $J_F \colon F \to F''$ is the canonical inclusion. It is said that $\mathcal{A}$ is surjective or regular if, respectively, $\mathcal{A} = \mathcal{A}^{sur}$ or $\mathcal{A} = \mathcal{A}^{reg}$ isometrically.

Also, denoting by T' the adjoint of an operator T, the dual ideal $\mathcal{A}^d$ of $\mathcal{A}$ is the class of operators $T \in \mathcal{L}$ such that $T' \in \mathcal{A}$ and $\|T\|_{\mathcal{A}^d} = \|T'\|_{\mathcal{A}}$.

Associated to the concept of $\mathcal{A}$-compact sets, Carl and Stephani [3] define and study the notion of $\mathcal{A}$-compact operators, which generalizes compact operators. An operator $T \in \mathcal{L}(E; F)$ is said to be $\mathcal{A}$-compact if $T(B_E)$ is a relatively $\mathcal{A}$-compact set in F [3, Definition 2]. We denote by $\mathcal{K}_{\mathcal{A}}$ the space of all $\mathcal{A}$-compact operators. When $\mathcal{A}$ is a Banach operator ideal, $\mathcal{K}_{\mathcal{A}}$ becomes a Banach operator ideal if for any $T \in \mathcal{K}_{\mathcal{A}}(E; F)$ one defines

$$\|T\|_{\mathcal{K}_{\mathcal{A}}} = m_{\mathcal{A}}(T(B_E); F).$$

Carl and Stephani describe the operator ideal $\mathcal{K}_{\mathcal{A}}$ in terms of $\mathcal{A}^{sur}$ via the identities: $\mathcal{K}_{\mathcal{A}} = (\mathcal{A} \circ \mathcal{K})^{sur} = \mathcal{A}^{sur} \circ \mathcal{K}$ [3, Theorem 2.1]. From this, the authors get that $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{K}_{\mathcal{A}}}$, and the process only may produce a new operator ideal the first time it is applied. The geometric structure introduced via $\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}$ fits in the Carl-Stephani theory turning both identities into isometries. Also, with Corollary 1.9 we obtain a slight modification as follows.

Proposition 2.1. *Let $\mathcal{A}$ be a Banach operator ideal. Then, the isometric identity holds*

$$\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A} \circ \overline{\mathcal{F}}} = (\mathcal{A} \circ \overline{\mathcal{F}})^{sur}.$$

Since $m_{\mathcal{N}^p}(T(B_E); F) = \kappa_p(T)$, the ideal $\mathcal{K}_p$ of p-compact operators coincides isometrically with $\mathcal{K}_{\mathcal{N}^p}$ and $\mathcal{K}_{\mathcal{K}_p}$.

Theorem 2.2. *Let $\mathcal{A}$ be a Banach operator ideal. Then, the isometric identity holds*

$$\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}}^{reg}.$$

Proof. Suppose we have proved that $\mathcal{A}$ and $\mathcal{A}^{reg}$ produce the same system of compact sets and therefore, $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}^{reg}}$. By the above, the isometric identity $\mathcal{K}_{\mathcal{A}} = \mathcal{A}^{sur} \circ \mathcal{K}$ holds. An application of [20, Corollary 2.1] then shows that $\mathcal{K}_{\mathcal{A}}^{reg} = \mathcal{K}_{\mathcal{A}^{reg}}$, which would complete the proof.

Since $\mathcal{A} \subset \mathcal{A}^{reg}$, it only remains to show that $\mathcal{A}^{reg}$-compact sets are $\mathcal{A}$-compact. Let E be a Banach space and K an $\mathcal{A}^{reg}$-compact set of E. Given $\varepsilon > 0$, by Proposition 1.8, we may find $T \in \mathcal{A}^{reg}(\ell_1; E)$ and M a compact set in $\subset B_{\ell_1}$ such that $K \subset T(M)$ with $\|T\|_{\mathcal{A}^{reg}} \leq (1+\varepsilon) m_{\mathcal{A}^{reg}}(K; E)$. By [27, Lemma 4.11], there are a compact set $L \subset B_{\ell_1}$, a Banach space F and an injective compact operator $S \in \mathcal{L}(F; \ell_1)$ such that $M \subset L = S(B_F)$ and $S^{-1}(M)$ is compact. In addition, we may choose $L = \{x \in \ell_1 \colon \|x - \pi_n(x)\| \leq \gamma_n, n \geq 1\}$, where $(\gamma_n)_n$ belongs to B_{c_0} and $\pi_n \colon \ell_1 \to \ell_1$ is the canonical projection to the first n coordinates. Now, we use the principle of local reflexivity for the finite dimensional subspaces $W_n = J_E \circ T \circ \pi_n \circ S(F)$ and find a sequence of operators $R_n \in \mathcal{L}(W_n; E)$ such that $\|R_n\| \leq 1 + \varepsilon$ and $R_n \circ J_E \circ T \circ \pi_n \circ S = T \circ \pi_n \circ S$ for all n. Since $W_n \subset W_m$ for all $m \geq n$, straightforward calculations show that $(T \circ \pi_n \circ S)_n$ is a Cauchy sequence in $\mathcal{A}$. Also,

$(T \circ \pi_n \circ S)_n$ is convergent to $T \circ S$ in $\mathcal{L}$. Then, $T \circ S$ belongs to $\mathcal{A}$. Since S is injective, $K \subset T \circ S(S^{-1}(M))$ which shows that K is $\mathcal{A}$-compact.

The isometry follows from the inequality

$$m_{\mathcal{A}}(K; E) \leq \|T\|_{\mathcal{A}^{reg}} \leq (1+\varepsilon) m_{\mathcal{A}^{reg}}(K; E). \quad \square$$

As an immediate consequence, we have the following results.

Corollary 2.3. *Let E be a Banach space, K a subset of E and $\mathcal{A}$ a Banach operator ideal. Then, K is relatively $\mathcal{A}$-compact if and only if $K \subset E''$ is relatively $\mathcal{A}$-compact. Moreover, $m_{\mathcal{A}}(K; E) = m_{\mathcal{A}}(K; E'')$.*

Corollary 2.4. *Let $\mathcal{A}$ be a Banach operator ideal. Then, the isometric identity holds*

$$\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}}^{dd}.$$

For p-compactness, the above identity was obtained in [10, Corollary 3.6], see [14, Corollary 2.6] and [26, Proposition 8] for the isometry. Also, Theorem 2.2 corresponds with [14, Theorem 2.5] and [26, Theorem 5]. Finally, Corollary 2.3 was shown in [10, Corollary 3.6], and the equality of the sizes appears in [14, Theorem 2.4].

We finish this section with a characterization of an $\mathcal{A}$-compact operator in terms of the continuity and compactness of its adjoint. The next result is well known for compact operators and was studied in the polynomial and holomorphic setting in [2]. We denote by $E'_{\mathcal{A}}$ the dual space of E considered with the topology of uniform convergence on $\mathcal{A}$-compact sets. As usual, E'_c denotes the dual space of E with the topology of uniform convergence on compact sets and K° denotes the polar set of a set K.

Proposition 2.5. *Let E and F be Banach spaces, $T \in \mathcal{L}(E; F)$ and $\mathcal{A}$ an operator ideal. The following statements are equivalent.*

(i) *$T \in \mathcal{K}_{\mathcal{A}}(E; F)$.*
(ii) *$T' : F'_{\mathcal{A}} \to E'$ is continuous.*
(iii) *$T' : F'_{\mathcal{A}} \to E'_c$ is compact.*
(iv) *$T' : F'_{\mathcal{A}} \to E'_{\mathcal{B}}$ is compact for any Banach operator ideal $\mathcal{B}$.*
(v) *There exists a Banach operator ideal $\mathcal{B}$ such that $T' : F'_{\mathcal{A}} \to E'_{\mathcal{B}}$ is compact.*
(vi) *$T' : F'_{\mathcal{A}} \to E'_{w^*}$ is compact.*

Proof. Suppose (i) holds, then $\overline{T(B_E)} = K$ is $\mathcal{A}$-compact and K° is a neighborhood in $F'_{\mathcal{A}}$. Thus, for $y' \in K^\circ$ we have that $\|T'(y')\| = \sup_{x \in B_E} |T'(y')(x)| \leq 1$, proving (ii).

Now, suppose (ii) holds. Then, there exists a relatively $\mathcal{A}$-compact set $K \subset F$ such that $T'(K^\circ)$ is equicontinuous in E' which, by the Ascoli theorem, is relatively compact in E'_c, obtaining (iii). Since $Id : E'_c \to E'_{\mathcal{B}}$ is continuous for any Banach operator ideal $\mathcal{B}$, (iii) implies (iv). That (iv) implies (v) and (v) implies (vi) are clear. It remains to show that

(vi) implies (i). Let L be a w^*-compact set of E' (hence $\|.\|$-bounded) and K an absolutely convex and $\mathcal{A}$-compact set of F such that $T'(K^\circ) \subset L$. As $T'(K^\circ)$ is $\|.\|$-bounded, there exists $c > 0$ such that $|T'(y')(x)| \leq c$ for every $y' \in K^\circ$ and $x \in B_E$. Therefore, $T(B_E) \subset cK$ which ends the proof. □

3. Approximation properties given by operator ideals

The approximation property of a Banach space E means that $Id \in \overline{\mathcal{F}(E;E)}^{\tau}$, where τ is the topology of uniform convergence on compact sets. A standard extension is obtained by considering the topology of uniform convergence on $\mathcal{A}$-compact sets $\tau_{\mathcal{A}}$ and requiring the identity map to satisfy $Id \in \overline{\mathcal{F}(E;E)}^{\tau_{\mathcal{A}}}$. However, for a Banach operator ideal $\mathcal{A}$ it seems to be more appropriate to consider the size of $\mathcal{A}$-compact sets and look at the convergence on $\mathcal{A}$-compact sets under $m_{\mathcal{A}}$. In order to formalize the latter idea, we introduce on $\mathcal{L}(E;F)$ the topology $\tau_{s\mathcal{A}}$ of strong uniform convergence on $\mathcal{A}$-compact sets, which is given by the seminorms

$$q_K(T) = m_{\mathcal{A}}(T(K);F),$$

where K ranges over all $\mathcal{A}$-compact sets of E.

It is easy to check that $\tau_{\mathcal{A}}$ and $\tau_{s\mathcal{A}}$ are locally convex topologies. Regarding $\tau_{s\mathcal{A}}$, we propose to study the approximation property for which $Id \in \overline{\mathcal{F}(E;E)}^{\tau_{s\mathcal{A}}}$.

Recall that a Banach space E has the approximation property if and only if for any Banach space F, $\mathcal{F}(F;E)$ is $\|\cdot\|$-dense in $\mathcal{K}(F;E)$ (see, e.g. [27, Proposition 4.12]). For the Carl-Stephani theory, this result can be extended as follows.

Proposition 3.1. *Let E be a Banach space and $\mathcal{A}$ a Banach operator ideal. The following are equivalent.*

(i) $Id \in \overline{\mathcal{F}(E;E)}^{\tau_{s\mathcal{A}}}$.
(ii) *For any Banach space F, $\mathcal{F} \circ \mathcal{K}_{\mathcal{A}}(F;E)$ is $\|.\|_{\mathcal{K}_{\mathcal{A}}}$-dense in $\mathcal{K}_{\mathcal{A}}(F;E)$.*
(iii) *For any Banach space F, $\mathcal{F}(F;E)$ is $\|.\|_{\mathcal{K}_{\mathcal{A}}}$-dense in $\mathcal{K}_{\mathcal{A}}(F;E)$.*

Proposition 3.2. *Let E be a Banach space and $\mathcal{A}$ an operator ideal. The following are equivalent.*

(i) $Id \in \overline{\mathcal{F}(E;E)}^{\tau_{\mathcal{A}}}$.
(ii) *For any Banach space F, $\mathcal{F} \circ \mathcal{K}_{\mathcal{A}}(F;E)$ is $\|.\|$-dense in $\mathcal{K}_{\mathcal{A}}(F;E)$.*
(iii) *For any Banach space F, $\mathcal{F}(F;E)$ is $\|.\|$-dense in $\mathcal{K}_{\mathcal{A}}(F;E)$.*

Proofs of Propositions 3.1 and 3.2 essentially follow their classical prototype [27, Proposition 4.12], basing on the following result, which holds by the proof of [27, Lemma 4.11]. Let E be a Banach space and $\mathcal{A}$ a Banach operator ideal. Suppose K is a convex, balanced and $\mathcal{A}$-compact set of E. Then, there exist a Banach space F and an injective operator $T \in \mathcal{K}_{\mathcal{A}}(F;E)$ such that $K \subset T(B_F)$ and $T^{-1}(K) \subset F$ is compact.

In [23], Oja introduces the concept of $\mathcal{A}$-approximation property as the property enjoyed by Banach spaces E such that $\mathcal{F}(F;E)$ is $\|\cdot\|_{\mathcal{A}}$-dense in $\mathcal{A}(F,E)$, for every Banach space F. Thus, a space satisfying any of the equivalences of Proposition 3.1 is said to have the $\mathcal{K}_{\mathcal{A}}$-approximation property. On the other hand, if $\mathcal{A}$ is an operator ideal and α is a norm on $\mathcal{A}$, in [16], the authors say that E has the $(\mathcal{A},\alpha)$-approximation property if $\mathcal{F}(F;E)$ is α-dense in $\mathcal{A}(F,E)$ for all Banach spaces F. When α is the operator norm in $\mathcal{L}$, we say that E has the $\mathcal{A}$-uniform approximation property instead of saying that E has the $(\mathcal{A},\|\cdot\|)$-approximation property. Thus, a space satisfying any of the equivalences of Proposition 3.2 is said to have the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property. Notice that $\mathcal{N}^p$ covers the p-approximation property [29, Definition 6.1] and the κ_p-approximation property [9, Definition 1.1].

For any operator ideal $\mathcal{A}$, the approximation property implies the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property and the converse is not true, as it can be deduced from [29, Theorem 6.4]. The $\mathcal{K}_{\mathcal{A}}$-approximation property is strictly stronger than the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property (to see this, combine [29, Theorem 6.4] and [9, Theorem 2.4]). Also, if $\mathcal{A}$ and $\mathcal{B}$ are two Banach operator ideals and $\mathcal{A} \subset \mathcal{B}$, the $\mathcal{K}_{\mathcal{B}}$-uniform approximation property implies the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property. Nonetheless, a Banach space may have the $\mathcal{K}_{\mathcal{B}}$-approximation property and fail to have the $\mathcal{K}_{\mathcal{A}}$-approximation property (to see this combine [9, Corollary 3.6] and [9, Theorem 2.4]).

We do not know if the approximation property implies or not the $\mathcal{K}_{\mathcal{A}}$-approximation property. However, the bounded approximation property yields a positive result. Recall that a Banach space E has the bounded approximation property if $Id \in \overline{\mathcal{F}(E;E) \cap \lambda B_{\mathcal{L}(E;E)}}^{\tau}$, for some $\lambda \geq 1$.

Proposition 3.3. *Let E be a Banach space with the λ-bounded approximation property, and let $\mathcal{A}$ be a Banach operator ideal. Then E has the $\mathcal{K}_{\mathcal{A}}$-approximation property.*

Proof. Let K be an $\mathcal{A}$-compact set of E. By Proposition 1.8, take $T \in \mathcal{A}(\ell_1;E)$ and a compact set $M \subset B_{\ell_1}$ such that $K \subset T(M)$. As in Proposition 2.2, we may find a Banach space F, an injective compact operator $S \in \mathcal{L}(F;\ell_1)$ and a compact set $L = \{x \in \ell_1 \colon \|x - \pi_n(x)\| \leq \gamma_n, n \geq 1\}$ with $(\gamma_n)_n$ in B_{c_0} such that $M \subset L = S(B_F)$ and $S^{-1}(M)$ is compact. Consider the finite dimensional subspaces $W_n = T \circ \pi_n \circ S(F)$. By [6, Proposition 16.9], for each n, there exists $R_n \in \mathcal{F}(E;E)$ such that $\|R_n\| \leq 2\lambda$ and $R_n \circ T \circ \pi_n \circ S = T \circ \pi_n \circ S$. Then we have

$$
\begin{aligned}
m_{\mathcal{A}}((R_n - Id)(K); E) &\leq \|(R_n - Id_E)\circ T\circ S\|_{\mathcal{A}} \\
&\leq \|R_n\circ T\circ S - R_n\circ T\circ \pi_n\circ S\|_{\mathcal{A}} + \|T\circ\pi_n\circ S - T\circ S\|_{\mathcal{A}} \\
&\leq (\|R_n\| + 1)\|T\|_{\mathcal{A}}\|S - \pi_n\circ S\| \\
&\leq (2\lambda + 1)\|T\|_{\mathcal{A}}|\gamma_n|.
\end{aligned}
$$

Since $(\gamma_n)_n$ belongs to c_0, the result follows. □

Recall that the minimal kernel of $\mathcal{A}$ is the composition ideal $\mathcal{A}^{min} = \overline{\mathcal{F}}\circ\mathcal{A}\circ\overline{\mathcal{F}}$. Now we restrict to the class of right-accessible Banach operator ideals (those satisfying $\mathcal{A}^{min} = \mathcal{A}\circ\overline{\mathcal{F}}$) and show that the classic approximation property implies the $\mathcal{K}_{\mathcal{A}}$-approximation property. We need the following result.

Proposition 3.4. *Let E be a Banach space and $\mathcal{A}$ a right-accessible Banach operator ideal. Then, E has the $\mathcal{K}_{\mathcal{A}}$-approximation property if and only if $\mathcal{K}_{\mathcal{A}}(F;E) = \mathcal{K}_{\mathcal{A}}^{min}(F;E)$ for all Banach spaces F.*

Proof. Since $\mathcal{A}$ is right-accessible, combining [6, Ex. 21.1] and [6, Proposition 21.4] we get that $\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}$ and $\|\cdot\|_{\mathcal{K}_{\mathcal{A}}^{min}}$ coincide over $\mathcal{F}(F;E)$. Thus, the result follows by a direct application of Proposition 3.1. □

The above proposition covers the characterization of the κ_p-approximation property in terms of the ideals $\mathcal{K}_p$ and $\mathcal{K}_p^{min}$, see the comments after [14, Proposition 3.9]. Also, the next result generalizes [14, Proposition 3.10].

Proposition 3.5. *Let E be a Banach space and $\mathcal{A}$ be a right-accessible Banach operator ideal. If E has the approximation property, then E has the $\mathcal{K}_{\mathcal{A}}$-approximation property.*

Proof. If E has the approximation property, by [6, Proposition 25.11],

$$
(\mathcal{K}_{\mathcal{A}}^{min})^{sur}(F;E) = (\mathcal{K}_{\mathcal{A}}^{sur})^{min}(F;E) = \mathcal{K}_{\mathcal{A}}^{min}(F;E),
$$

for every Banach operator ideal $\mathcal{A}$ and for every Banach space F. Now we show that if $\mathcal{A}$ is right-accessible, $\mathcal{K}_{\mathcal{A}} = (\mathcal{A}^{min})^{sur} = (\mathcal{K}_{\mathcal{A}}^{min})^{sur}$ isometrically. Indeed, the first isometry follows from Proposition 2.1. For the second one, apply [6, Ex. 21.1] to show that $\mathcal{K}_{\mathcal{A}}$ is right-accessible. The claim follows from the fact that $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{K}_{\mathcal{A}}}$ isometrically.

Finally, since $\mathcal{K}_{\mathcal{A}}(F;E) = \mathcal{K}_{\mathcal{A}}^{min}(F;E)$ for every Banach space F, an application of Proposition 3.4 completes the proof. □

It is well known (see, e.g. [27, Proposition 4.12]) that the dual E' of a Banach space E has the approximation property if and only if for any Banach space F, $\mathcal{F}(E;F)$ is $\|\cdot\|$-dense in $\mathcal{K}(E;F)$. Note that $\mathcal{K}^d = \mathcal{K}$. We characterize the $\mathcal{K}_{\mathcal{A}}$-uniform and the $\mathcal{K}_{\mathcal{A}}$-approximation properties on E' via the ideal $\mathcal{K}_{\mathcal{A}}^d$, which is not surprising at the light of the results obtained by Delgado, Oja, Piñeiro and Serrano in the p-compact setting [7, Theorem 2.8] and [9, Theorem 2.3]. We need the following lemma.

Lemma 3.6. *Let E and F be Banach spaces and $\mathcal{A}$ a Banach operator ideal, then the set $E \otimes F$ is $\tau_{s\mathcal{A}}$-dense in $\mathcal{F}(E'; F)$.*

Proof. Since $m_{\mathcal{A}}(x'' \otimes y(K); F) = \sup_{x' \in K} |x''(x')| \|y\|$ for any bounded set $K \subset E'$, any $x'' \in E''$ and any $y \in F$, the result follows by a direct application of the Alaoglu theorem. $\square$

Proposition 3.7. *Let E be a Banach space and $\mathcal{A}$ a Banach operator ideal. The following are equivalent.*

(i) *E' has the $\mathcal{K}_{\mathcal{A}}$-approximation property.*
(ii) *For any Banach space F, $\mathcal{F}(E; F)$ is $\| \cdot \|_{\mathcal{K}^d_{\mathcal{A}}}$- dense in $\mathcal{K}^d_{\mathcal{A}}(E; F)$.*

Proof. If (i) holds, fix $\varepsilon > 0$ and take $T \in \mathcal{K}^d_{\mathcal{A}}(E; F)$. Since $T' \in \mathcal{K}_{\mathcal{A}}(F'; E')$ and E' has the $\mathcal{K}_{\mathcal{A}}$-approximation property, by Lemma 3.6, there exists $S \in \mathcal{F}(E; F)$ such that $\|T - S\|_{\mathcal{K}^d_{\mathcal{A}}} = \|S' - T'\|_{\mathcal{K}_{\mathcal{A}}} \leq \varepsilon$ which gives (ii).

For the converse, take $T \in \mathcal{K}_{\mathcal{A}}(F; E')$. By Corollary 2.4, $T' \circ J_E \in \mathcal{K}^d_{\mathcal{A}}(E; F')$. Fix $\varepsilon > 0$, by hypothesis, there exists $S \in \mathcal{F}(E; F')$ such that $\|S - T' \circ J_E\|_{\mathcal{K}^d_{\mathcal{A}}} \leq \varepsilon$. Then,

$$\|S' \circ J_F - T\|_{\mathcal{K}_{\mathcal{A}}} \leq \|S' - (T' \circ J_E)'\|_{\mathcal{K}_{\mathcal{A}}} = \|S - T' \circ J_E\|_{\mathcal{K}^d_{\mathcal{A}}} \leq \varepsilon,$$

and the result follows by Proposition 3.1. $\square$

Analogously, E' has $\mathcal{K}_{\mathcal{A}}$-uniform approximation property if and only if $\mathcal{F}(E; F)$ is $\| \cdot \|$-dense in $\mathcal{K}^d_{\mathcal{A}}(E; F)$, for any Banach space F.

A Banach space E has the approximation property if and only if E'_c has the approximation property [28, Exposé 14]. Aron, Maestre and Rueda show the analogous result for the p-approximation property [1, Theorem 4.6]. Here, we present a generalization of these results.

Proposition 3.8. *Let E be a Banach space and $\mathcal{A}$ an operator ideal. Then, E has the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property if and only if $E'_{\mathcal{A}}$ has the approximation property.*

Proof. The locally convex space $E'_{\mathcal{A}}$ has the approximation property if and only if for any $\varepsilon > 0$, any $\mathcal{A}$-compact set $K \subset E$ and any relatively compact set $M \subset E'_{\mathcal{A}}$, there exists $S \in \mathcal{F}(E; E)$ such that

$$|(S' - Id)(x')(x)| \leq \varepsilon \quad \text{for all } x' \in M, \ x \in K. \tag{1}$$

The continuity of the identity map $E'_c \to E'_{\mathcal{A}} \to (E', w^*)$ says that relatively compact sets in $E'_{\mathcal{A}}$ coincide with $\|.\|$-bounded sets. Then, $E'_{\mathcal{A}}$ has the approximation property if and only if in (1) M is replaced by $B_{E'}$ which is equivalent to say that E has the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property. $\square$

Finally we give a reformulation of the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property in terms of the ϵ-product of Schwartz. A Banach space E has the approximation property if and only if $E \otimes F$ is dense in $\mathcal{L}_\epsilon(E'_c; F)$ for every locally convex space F [28, Exposé 14]. We denote by $\mathcal{L}_\epsilon(E'_{\mathcal{A}}; F)$ the space of all linear continuous maps from $E'_{\mathcal{A}}$ to a locally convex space F, endowed with the topology of uniform convergence on all equicontinuous sets of E'. The proof of the next proposition is standard and we omit it.

Proposition 3.9. *Let E be a Banach space and $\mathcal{A}$ an operator ideal. The following statements are equivalent.*

(i) *E has the $\mathcal{K}_{\mathcal{A}}$-uniform approximation property.*
(ii) *$E \otimes F$ is dense in $\mathcal{L}_\epsilon(E'_{\mathcal{A}}; F)$ for any locally convex space F.*
(iii) *$E \otimes E'$ is dense in $\mathcal{L}_\epsilon(E'_{\mathcal{A}}; E'_{\mathcal{A}})$.*

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References

[1] R. Aron, M. Maestre, P. Rueda. *p-compact holomorphic mappings*, RACSAM 104 (2) (2010), 353–364.
[2] R. Aron, M. Schottenloher. *Compact holomorphic mappings and the approximation property*, J. Funct. Anal. 21 (1976), 7–30.
[3] B. Carl, I. Stephani. *On A-compact operators, generalized entropy numbers and entropy ideals*, Math. Nachr. 199 (1984), 77–95.
[4] P. G. Casazza. *Approximation properties.* Handbook of the geometry of Banach spaces, Vol. I, 271–316. North-Holland, Amsterdam, 2001.
[5] Y. S. Choi, J. M. Kim. *The dual space of $(\mathcal{L}(X,Y);\tau_p)$ and the p-approximation property*, J. Funct. Anal. 259 (2010), 2437–2454.
[6] A. Defant, K. Floret. Tensor norms and operators ideal, North Holland Publishing Co., Amsterdam, 1993.
[7] J. M. Delgado, E. Oja, C. Piñeiro, E. Serrano. *The p-approximation property in terms of density of finite rank operators*, J. Math. Anal. Appl. 354 (2009), 159–164.
[8] J. M. Delgado, C. Piñeiro. *p-convergent sequences and Banach spaces in which p-compact sets are q-compact*, Proc. Amer. Math. Soc. 139 (3) (2011), 957–967.
[9] J. M. Delgado, C. Piñeiro, E. Serrano. *Density of finite rank operators in the Banach space of p-compact operators*, J. Math. Anal. Appl. 370 (2010), 498–505.
[10] J. M. Delgado, C. Piñeiro, E. Serrano. *Operators whose adjoints are quasi p-nuclear*, Studia Math. (3) 197 (2010), 291–304.

[11] J. Diestel, J. H. Fourie, J. Swart. The metric theory of tensor products, Grothendieck's résumé revisited, American Mathematical Society, Providence, RI, 2008.

[12] J. Diestel, H. Jarchow, A. Tonge. Absolutely summing operators, Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.

[13] P. Enflo. *A counterexample to the approximation problem in Banach spaces*, Acta Math. 130 (1) (1973), 309–317.

[14] D. Galicer, S. Lassalle, P. Turco. *The ideal of p-compact operators: a tensor product approach*, Studia Math. 211 (3) (2012), 269–286.

[15] A. Grothendieck. *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paolo 8 (1956), 1–79.

[16] S. Lassalle, P. Turco. *On p-compact mappings and the p-approximation property*, J. Math. Anal. Appl. 389 (2) (2012), 1204–1221.

[17] A. Lima, V. Lima, E. Oja. *Bounded approximation properties via integral and nuclear operators*, Proc. Amer. Math. Soc. 138 (2010), 287–297.

[18] J. Lindenstrauss, L. Tzafriri. Classical Banach Spaces I, Vol. 92, Springer-Verlag, Berlin, New York, 1977.

[19] A. Lissitsin, K. Mikkor, E. Oja. *Approximation properties defined by spaces of operators and approximability in operator topologies*, Ill. J. Math. 52 (2) (2008), 563–582.

[20] F. Oertel. *Compositions of operator ideals and their regular hulls*, Acta Univ. Carolin. Math. Phys. 36 (2) (1995), 69–72.

[21] E. Oja. *On bounded approximation properties of Banach spaces*, Banach Algebras 2009, Vol. 91, 219–231. Banach Center Publications, Polish Acad. Sci. Inst. Math., Warsaw, 2010.

[22] E. Oja. *Inner and outer inequalities with applications to approximation properties*, Trans. Amer. Math. Soc. 363 (2011), 5827–5846.

[23] E. Oja. *A remark on the approximation of p -compact operators by finite-rank operators*, J. Math. Anal. Appl. 387 (2) (2012), 949–952.

[24] E. Oja. *Grothendieck's nuclear operator theorem revisited with an application to p-null sequences*, J. Funct. Anal. 263 (9) (2012), 2876–2892.

[25] A. Pietsch. Operators ideals, North Holland Publishing Company, Amsterdam/New York/Oxford, 1980.

[26] A. Pietsch. *The ideal of p-compact operators and its maximal hull*, To appear in Proc. Amer. Math. Soc.

[27] R. Ryan. Introduction to Tensor products on Banach Spaces, Springer, London, 2002.

[28] L. Schwartz. *Produits tensoriels topologiques d'espaces vectoriels topologiques*, Exp. 14, Séminaire 1953/54, Inst. Henri Poincaré.

[29] D. P. Sinha, A. K. Karn. *Compact operators whose adjoints factor trough subspaces of* ℓ_p, Studia Math. 150 (2002), 17–33.

[30] D. P. Sinha, A. K. Karn. *Compact operators which factor trough subspaces of* ℓ_p, Math. Nachr. 281 (2008), 412–423.

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