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EXAMPLES OF HOMOGENEOUS MANIFOLDS WITH UNIFORMLY BOUNDED METRIC PROJECTION

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ABSTRACT. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ. Denote by $L^p(\mathcal{M})_{sh}$ the skew-Hermitian part of the non-commutative L^p space associated with $(\mathcal{M},\tau)$. Let $1<p<\infty$, $z\in L^p(\mathcal{M})_{sh}$ and $\mathcal{S}$ be a real closed subspace of $L^p(\mathcal{M})_{sh}$. The metric projection $Q: L^p(\mathcal{M})_{sh}\longrightarrow \mathcal{S}$ is defined for every $z\in L^p(\mathcal{M})_{sh}$ as the unique operator $Q(z)\in\mathcal{S}$ such that $\|z-Q(z)\|_p=\min_{y\in\mathcal{S}}\|z-y\|_p$.

We show the relation between metric projection and metric geometry of homogeneous spaces of the unitary group $\mathcal{U}_{\mathcal{M}}$ of $\mathcal{M}$, endowed with a Finsler quotient metric induced by the p-norms of τ, $\|x\|_p=\tau(|x|^p)^{1/p}$, p an even integer. The problem of finding minimal curves in such homogeneous spaces leads to the notion of uniformly bounded metric projection. Then we show examples of metric projections of this type.

1. Introduction

The aim of this work is to make further comments and give complete proofs of the statements established in [1] on uniformly bounded metric projections and minimal curves in homogeneous spaces.

Let $\mathcal{M}$ be a finite von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful normal trace τ. Denote by $\tilde{\mathcal{M}}$ the set of densely-defined closed operators affiliated with $\mathcal{M}$, which becomes a Hausdorff topological $*$-algebra equipped with the measure topology (see [8], [9]). For any positive Hermitian operator $x\in\tilde{\mathcal{M}}$, we put

$$\tau(x)=\sup_{n\geq 1}\tau\Big(\int_0^n \lambda\, de_\lambda\Big),$$

where $x=\int_0^\infty \lambda\, de_\lambda$ is the spectral representation of x. For $1\leq p<\infty$ the non-commutative L^p space associated with $(\mathcal{M},\tau)$ is defined by

$$L^p(\mathcal{M})=\{\, x\in\tilde{\mathcal{M}}\,:\,\tau(|x|^p)<\infty\,\},$$

and the p-norm is given by

$$\|x\|_p=\tau(|x|^p)^{1/p},\ \ x\in L^p(\mathcal{M}).$$

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It turns out that $L^p(\mathcal{M})$ is a Banach space with the p-norm satisfying the expected properties such as Hölder inequality and duality. Moreover, the completion of $\mathcal{M}$ with the p-norm naturally identifies with $L^p(\mathcal{M})$. In the case $p = 2$, we have that $L^2(\mathcal{M})$ is a Hilbert space with the inner product $\langle x, y \rangle = \tau(xy^*)$. We will use the notation $L^p(\mathcal{M})_h$ (resp. $L^p(\mathcal{M})_{sh}$) to indicate the real Banach space of Hermitian (resp. skew-Hermitian) operators of $L^p(\mathcal{M})$.

Let $\mathcal{S}$ be a real closed subspace of $L^p(\mathcal{M})_{sh}$. It can be derived from Clarkson's inequalities for $L^p(\mathcal{M})_{sh}$ that this space is uniformly convex for $1 < p < \infty$ (see [5]). Then for each $x \in L^p(\mathcal{M})_{sh}$, the distance of x to $\mathcal{S}$ is attained in an unique operator $Q_{p,\mathcal{S}}(x) \in \mathcal{S}$, that is

$$\|x - Q_{p,\mathcal{S}}(x)\|_p = \inf_{y \in \mathcal{S}} \|x - y\|_p \,.$$

The map $Q_{p,\mathcal{S}} : L^p(\mathcal{M})_{sh} \longrightarrow \mathcal{S}$, where $Q_{p,\mathcal{S}}(x)$ is the best approximation of x in $\mathcal{S}$, is usually known as the metric projection. In the sequel, once the p and the subspace are clear, we will write only Q.

Let $\mathcal{M}_h$ (resp. $\mathcal{M}_{sh}$) denote the Hermitian (resp. skew-Hermitian) part of $\mathcal{M}$. Fix $\mathcal{S}$ a real subspace of $\mathcal{M}_{sh}$ closed in the uniform topology. Let Q be the metric projection of $L^p(\mathcal{M})_{sh}$ onto $\overline{\mathcal{S}}^p$. Notice that the best approximation of a bounded operator might be an unbounded operator of $L^p(\mathcal{M})_{sh}$. We say that Q preserves bounded operators if $Q(\mathcal{M}_{sh}) \subseteq \mathcal{S}$.

Let $\| \,.\, \|$ denote the uniform norm of $\mathcal{M}$. If the metric projection preserves bounded operators, we can go further and ask if Q is *uniformly bounded*. This means that there exists a constant $K_{p,\mathcal{S}} > 0$ such that

$$\|Q(x)\| \leq K_{p,\mathcal{S}} \|x\|, \quad x \in \mathcal{M}_{sh}.$$

We will show that uniformly bounded metric projections appear in the metric theory of homogeneous spaces of the unitary group of $\mathcal{M}$.

Let us describe the contents of the article. In Section 2 we exhibit the relevance of uniformly bounded metric projections in the setting of homogeneous spaces of the unitary group of $\mathcal{M}$. It turns out that the definition of uniformly bounded metric projections arises as a sufficient condition to state partial results on minimal curves in these homogeneous spaces. In Section 3 we give examples of uniformly bounded metric projections defined by homogeneous spaces.

2. Metric projection and homogeneous spaces

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ. It is a well known fact that the unitary group $\mathcal{U}_{\mathcal{M}}$ of $\mathcal{M}$ is a Banach-Lie group with the topology given by the uniform norm. The Lie algebra can be identified with the real Banach space $\mathcal{M}_{sh}$ of skew-Hermitian operators of $\mathcal{M}$. By a homogeneous space of the group $\mathcal{U}_{\mathcal{M}}$ we mean a set $\mathcal{O}$ where the unitary group acts transitively. Furthermore, we assume that for each $x \in \mathcal{O}$, the isotropy group

$$G_x = \{\, u \in \mathcal{U}_{\mathcal{M}} \,:\, u \cdot x = x \,\}$$

is a Banach-Lie subgroup of $\mathcal{U}_{\mathcal{M}}$. Therefore $\mathcal{O} \cong \mathcal{U}_{\mathcal{M}}/G_x$ has a unique Banach manifold structure such that the maps

$$\pi_x : \mathcal{U}_{\mathcal{M}} \longrightarrow \mathcal{O}, \;\; \pi_x(u) = u \cdot x$$

are analytic submersions. A good reference for these facts and others on homogeneous spaces is [11]. We denote the differential at the identity of the analytic map π_x by

$$(\pi_x)_{*1} : \mathcal{M}_{sh} \longrightarrow (T\mathcal{O})_x,$$

which is a surjective map, whose split kernel equals the Lie algebra $\mathcal{G}_x$ of the isotropy group at x. Examples of homogeneous spaces are unitary orbits of: normal operators, states, conditional expectations, $*$-homomorphisms and spectral measures. A detailed treatment of these examples can be found in [4, p. 98] and the references therein.

In the article [1], E. Andruchow, G. Larotonda and the author studied metric geometry of homogeneous spaces endowed with a Finsler quotient metric induced by the p-norms of τ. To be precise, for each $x \in \mathcal{O}$, the norm of a tangent vector $X \in (T\mathcal{O})_x$ is given by

$$\|X\|_{x,p} = \inf\{ \, \|z - y\|_p \, : \, y \in \mathcal{G}_x \, \},$$

where $z \in \mathcal{M}_{sh}$ satisfies $(\pi_x)_{*1}(z) = X$. In other words, the norm of a tangent vector X is the Banach quotient norm of one lifting z in $L^p(\mathcal{M})_{sh}/\overline{\mathcal{G}_x}^{p}$.

We can measure the length of a smooth curve γ in $\mathcal{O}$ by

$$L_{\mathcal{O},p}(\gamma) = \int_0^1 \|\dot{\gamma}\|_{\gamma,p} \, dt.$$

Then there is a rectifiable distance in $\mathcal{O}$, namely

$$d_{\mathcal{O},p}(x,y) = \inf\{ \, L_{\mathcal{O},p}(\gamma) \, : \, \gamma \subseteq \mathcal{O}, \, \gamma(0) = x, \, \gamma(1) = y \, \},$$

where the curves considered are piecewise smooth. It can be shown that $(\mathcal{O}, d_{\mathcal{O},p})$ is a complete metric space when the isotropy group G_x is closed in the p-norm. Moreover, $d_{\mathcal{O},p}$ defines the quotient topology in $\mathcal{O}$.

On the other hand, if Γ is a piecewise smooth curve in $\mathcal{U}_{\mathcal{M}}$ we can measure its length by

$$L_p(\Gamma) = \int_0^1 \|\dot{\Gamma}\|_p \, dt.$$

In the usual fashion we have another rectifiable distance d_p associated with this length functional. Let p be an even integer, $u_0, u_1, u_2 \in \mathcal{U}_{\mathcal{M}}$ and Δ a geodesic in $\mathcal{U}_{\mathcal{M}}$ joining u_1 and u_2. It is proved in [2] that the function

$$f(t) = d_p(u_0, \Delta(t)), \;\; t \in [0,1],$$

is strictly convex if $\|u_i - u_j\| < \frac{1}{2}\sqrt{2 - \sqrt{2}}$. Recall that the geodesics are curves of the form $\Delta(t) = u_2 e^{tz}$, $z \in \mathcal{M}_{ah}$, and have minimal length provided that $\|z\| \leq \pi$.

In the theory of finite dimensional Riemannian manifolds the completeness with the rectifiable distance implies the existence of curves of minimal length joining any pair of points. This is no longer true in Hilbert-Riemann manifolds (see [3],

[7]), essentially due to the absence of local compactness. In our case, the key argument to find minimal curves is to lift curves to the unitary group $\mathcal{U}_{\mathcal{M}}$ and use the convexity result.

The problem of finding minimal curves is closely related to the metric projection. Intuitively, it is natural to seek for the direction with minimal norm to find a minimal curve. For p an even integer denote by Q the metric projection onto $\overline{\mathcal{G}_x}^p$. Let $X \in (T\mathcal{O})_x$ and z be a lifting of X. Then, we have that

$$\|X\|_{x,p} = \|z - Q(z)\|_p .$$

So $z - Q(z)$ is the direction with minimal norm that we are looking for, and the curve in $\mathcal{O}$ given by

$$\delta(t) = e^{t(z-Q(z))} \cdot x, \;\; t \in [0,1],$$

is our prime candidate for minimal curve.

Recall that since G_x is a Banach-Lie subgroup of $\mathcal{U}_{\mathcal{M}}$ for each $x \in \mathcal{O}$, then there exists a closed linear supplement $\mathcal{F}_x \subseteq \mathcal{M}_{sh}$ such that $\mathcal{M}_{sh} = \mathcal{G}_x \oplus \mathcal{F}_x$. Moreover, the exponential map at $x \in \mathcal{O}$, defined by $\mathcal{F}_x \longrightarrow \mathcal{O}$, $z \mapsto e^z \cdot x$, is a local diffeomorphism at the origin. Then, when r is small enough and Q is uniformly bounded, the following set

$$U^r_{\mathcal{O}} = \{e^{z-Q(z)} \cdot x : z \in \mathcal{F}_x, \, \|z\| < r\}$$

has as exponents operators with small uniform norm and minimal p-norm. Hence any curve in $U^r_{\mathcal{O}}$ lifts to a curve in $\mathcal{U}_{\mathcal{M}}$ with uniform norm small enough to guarantee that the convexity result holds. Now we state the main theorem of [1] about minimality of curves.

Theorem 2.1. *Let p be an even integer. Assume that there exists a constant $K > 0$ satisfying $\|Q(z)\| \leq K\|z\|$ for all $z \in \mathcal{M}_{sh}$. Then for any $y \in U^r_{\mathcal{O}}$ there exists $z \in \mathcal{M}_{sh}$ such that $e^z \cdot x = y$ and*

$$\delta(t) = e^{tz} \cdot x, \quad t \in [0,1],$$

is shorter than any other piecewise smooth curve $\gamma \subset \mathcal{O}$ joining x to y, provided that $\gamma \subset U^r_{\mathcal{O}}$. Moreover, the curve δ is unique in the sense that if $\gamma \subset U^r_{\mathcal{O}}$ is another piecewise smooth curve joining x to y of length $\|z\|_p$ then $\gamma(t) = e^{tz} \cdot x$.

We point out that the set $U^r_{\mathcal{O}}$ might not be open in the quotient topology of $\mathcal{O}$. Also notice that the curve δ is minimal among non-wandering curves, i.e. curves such that do not leave the set $U^r_{\mathcal{O}}$.

3. Examples

In this section we will show examples of uniformly bounded metric projections. In view of Theorem 2.1, we will restrict to the case p an even integer.

We start recalling some general facts on metric projections. Let Q denote the metric projection from $L^p(\mathcal{M})_{sh}$ onto a closed subspace $\mathcal{S}$. The name of metric projection comes from the many properties that Q shares with an orthogonal projection. For example, it is easily shown that $Q(\lambda x) = \lambda(x)$, for all $x \in L^p(\mathcal{M})_{sh}$

and $\lambda \in \mathbb{R}$, $\mathcal{S} = Q(L^p(\mathcal{M})_{sh}) = (I-Q)^{-1}(0)$ and

$$Q^2 = Q\,, \quad (I-Q)^2 = I - Q\,, \quad (I-Q)\circ Q = Q\circ(I-Q) = 0.$$

Moreover, any $x \in L^p(\mathcal{M})_{sh}$ admits a unique decomposition as

$$x = (I-Q)(x) + Q(x),$$

where $(I-Q)(x) \in Q^{-1}(0)$ and $x \in S$. It is worth noting that Q is not linear in general for $p \neq 2$ (it is the linear orthogonal projection for $p = 2$). However, it does hold the following

$$Q(x+y) = Q(x) + y, \quad x \in L^p(\mathcal{M})_{sh}, \quad y \in \mathcal{S}.$$

Another interesting property is that Q is continuous with the p-norm since $L^p(\mathcal{M})_{sh}$ is uniformly smooth (see [6]).

The following remark shows an example of a metric projection that is not uniformly bounded. Furthermore, this projection does not preserve bounded operators.

Remark 3.1. Our finite von Neumann algebra is $L^\infty([0,1])$ with the Lebesgue measure. Pick a function $f \in L^2([0,1])$ such that $\|f\|_2 = 1$ and $f \notin L^\infty([0,1])$. Consider the following commutative Lie algebra:

$$\mathcal{S} = \{h \in L^\infty([0,1]) \,:\, \langle h, f\rangle = 0\} = \{\, f\,\}^\perp \cap L^\infty([0,1]).$$

Then the metric projection in the 2-norm, which is the orthogonal projection onto $\overline{\mathcal{S}}^2$, can be computed:

$$Q : L^2([0,1]) \longrightarrow \overline{\mathcal{S}}^2, \quad Q(h) = h - \langle h, f\rangle\, f.$$

Since f is an unbounded function it is clear that any bounded function h is mapped to an unbounded function $Q(h)$.

3.1. **Subalgebra of the center.** The first example is about a von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{M})$, where $\mathcal{Z}(\mathcal{M})$ is the center of $\mathcal{M}$. We will see that the metric projection $Q : L^p(\mathcal{M})_{sh} \longrightarrow L^p(\mathcal{N})_{sh}$ is uniformly bounded for all p even.

Lemma 3.2. *Let $p \geq 2$ an even number. Let $x, y \in L^p(\mathcal{M})$ satisfying $x \geq 0$, $y = y^*$ and $xy = yx$. Then*

$$\|x - y^+\|_p \leq \|x - y\|_p\,,$$

where $y = y^+ - y^-$ is the Jordan decomposition.

Proof. Let e denote the spectral projection of y corresponding to the interval $[\,0,\infty]$. First note the following fact,

$$(x-y^+)^p(1-e) = \sum_{k=0}^{p} \binom{p}{k} x^k(-y^+)^{p-k}(1-e) = x^p(1-e).$$

Analogously we can prove that $(x-y)^p e = (x-y^+)^p e$ and $(x-y)^p(1-e) = (x+y^-)^p(1-e)$. Therefore we have

$$\begin{aligned} \|x-y^+\|_p^p &= \tau((x-y^+)^p e) + \tau((x-y^+)^p(1-e)) \\ &= \tau((x-y^+)^p e) + \tau(x^p(1-e)) \\ &\leq \tau((x-y^+)^p e) + \tau((x+y^-)^p(1-e)) \qquad (1) \\ &= \tau((x-y^+)^p e) + \tau((x-y)^p(1-e)) = \|x-y\|_p^p \,. \end{aligned}$$

In the inequality (1) we use that $(x+y^-)^p$ is essentially the sum of polynomials of the form $x^{p-k}(y^-)^k$. Since the operators involved commute, it follows that each polynomial is a positive operator of $L^p(\mathcal{M})$, so $\tau(x^{p-k}(y^-)^k) \geq 0$, which proves the inequality. □

Remark 3.3. The previous lemma still holds for $p = 2$ without the requirement that the operators x, y commute. Indeed, note that

$$\|x-y^+\|_2^2 \leq \|x-y^+\|_2^2 + 2\tau(xy^-) + \tau((y^-)^2) = \|x-y\|_2^2,$$

and our claim is proved.

Remark 3.4. The above inequality is not true in our setting for $p > 2$, if we remove the hypothesis that x and y commute. For instance let $p = 4$ and consider the following matrices

$$x = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad y = \begin{pmatrix} 100 & 0 \\ 0 & -1 \end{pmatrix}.$$

The eigenvalues of $x - y^+$ are $\lambda_1 = 3.00990002$ and $\lambda_2 = -98.00990002$. Thus, we have $\|x-y^+\|_4^4 = 92274175$. On the other hand, $x-y$ has as eigenvalues $\mu_1 = 4.009802979$ and $\mu_2 = -98.00980298$. Then, $\|x-y\|_4^4 = 92273986$.

Proposition 3.5. *Let p an even integer and $\mathcal{N}$ a von Neumann subalgebra of $\mathcal{Z}(\mathcal{M})$. Then the metric projection $Q : L^p(\mathcal{M})_{sh} \longrightarrow L^p(\mathcal{N})_{sh}$ satisfies*

$$\|Q(z)\| \leq 3\|z\|, \;\; z \in \mathcal{M}_{sh}.$$

Proof. We need to consider the metric projection Q_0 of $L^p(\mathcal{M})$ onto $L^p(\mathcal{N})$, that is the unique continuous map $Q_0 : L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{N})$ satisfying

$$\|z - Q_0(z)\|_p \leq \|z-y\|_p,$$

for all $y \in L^p(\mathcal{N})$. We claim that Q_0 is uniformly bounded in $\mathcal{M}_h$.

By Lemma 3.2 we have that Q_0 maps positive elements of $L^p(\mathcal{M})$ on positive elements of $L^p(\mathcal{N})$. In fact, note that for a positive element $z \in \mathcal{M}$, we have $Q(z) \in L^p(\mathcal{N})$. Since $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{M})$, the lemma implies that

$$\|z - Q_0(z)^+\|_p \leq \|z - Q_0(z)\|_p \,.$$

Hence by the uniqueness of $Q_0(z)$ it follows that $Q_0(z) = Q_0(z)^+$.

Now we prove that Q_0 preserve bounded operators of $\mathcal{M}_h$. Let $z \in \mathcal{M}$ be a positive element. Note that $\|z\| - Q_0(z) = Q_0(\|z\| - z) \geq 0$, then $0 \leq Q_0(z) \leq \|z\|$. Hence $Q_0(z)$ is bounded. Let $z \in \mathcal{M}_h$, then $z + \|z\|$ is a positive operator and $Q_0(z) + c = Q_0(z + \|z\|)$ is bounded, so it follows that $Q_0(x)$ is bounded.

In order to prove our claim, let $z \in \mathcal{M}_h$, then

$$\begin{aligned}\|Q_0(z)\| &= \|Q_0(z + \|z\| - \|z\|)\| = \|Q_0(z + \|z\|) - \|z\|\| \\ &\leq \|Q_0(z + \|z\|)\| + \|z\| \\ &\leq \|z + \|z\|\,\| + \|z\| \leq 3\|z\|.\end{aligned}$$

Finally notice that $Q_0(z^*) = Q_0(z)^*$. In particular, we have that the restriction of Q_0 to $\mathcal{M}_{sh}$ coincides with Q. Since Q_0 is uniformly bounded on $\mathcal{M}_h$, and $i\mathcal{M}_h = \mathcal{M}_{sh}$, we obtain that Q is uniformly bounded with the constant 3. □

Remark 3.6. An example of homogeneous space of $\mathcal{U}_\mathcal{M}$ whose Lie algebra equals to the skew-Hermitian part of the center of $\mathcal{M}$ is the unitary orbit of an inner automorphism. We denote an inner automorphism by $Ad(u) : \mathcal{M} \longrightarrow \mathcal{M}$, $Ad(u)(x) = uxu^*$, for $x \in \mathcal{M}$ and $u \in \mathcal{U}_\mathcal{M}$. Fix $v \in \mathcal{U}_\mathcal{M}$, we consider the unitary orbit given by

$$\mathcal{O} = \{\, Ad(u) \circ Ad(v) \;:\; u \in \mathcal{U}_\mathcal{M} \,\}.$$

The isotropy group at $Ad(v)$ is

$$G = \{\, u \in \mathcal{U}_\mathcal{M} \;:\; Ad(u) \circ Ad(v) = Ad(v) \,\}.$$

The Lie algebra of G can be computed

$$\mathcal{G} = \{ z \in \mathcal{M}_{sh} \;:\; zvxv^* = vxv^* z,\, \forall\, x \in \mathcal{M} \,\} = \mathcal{Z}(\mathcal{M})_{sh}.$$

In order to show that $\mathcal{O}$ is a homogeneous space we need to check that $\mathcal{G}$ is complemented in $\mathcal{M}_{sh}$. To this end, note that the modular group of the trace is equal to $\mathcal{U}_\mathcal{M}$, then there exists a conditional expectation onto any ultraweakly closed subalgebra of $\mathcal{M}$ by a theorem of M. Takesaki (see [10]). The restriction to $\mathcal{M}_{sh}$ of a conditional expectation onto $\mathcal{Z}(\mathcal{M})$ is a projection onto $\mathcal{Z}(\mathcal{M})_{sh}$.

On the other hand, note that by Proposition 3.5 the metric projection of $L^p(\mathcal{M})_{sh}$ onto $\overline{\mathcal{G}}^p$ is uniformly bounded. Hence Theorem 2.1 can be applied, and the curves $\delta(t) = Ad(e^{t(z-Q(z))}) \circ Ad(u)$ are locally minimal with the quotient Finsler metric for all p even.

3.2. **Diagonal algebra in $\mathcal{M} \otimes M_2$.** Let M_2 denote the 2×2 matrix algebra. There is a natural finite trace $\hat{\tau}$ on $\mathcal{M} \otimes M_2$ defined by

$$\hat{\tau}\Big(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \Big) = \frac{1}{2}\tau(x_{11} + x_{22}), \qquad \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in M_2 \otimes \mathcal{M}.$$

It is straightforward to show that $L^p(\mathcal{M} \otimes M_2, \hat{\tau}) = L^p(\mathcal{M}) \otimes M_2$. We will need the following lemma. Its proof can be found in [1, Lemma 2.5].

Lemma 3.7. *Let $p > 1$, $z \in L^p(\mathcal{M})_{sh}$ and Q be metric projection of $L^p(\mathcal{M})_{sh}$ onto a closed subspace $\mathcal{S}$. Then $y = Q(z)$ if and only if $\tau(y^{p-1}x) = 0$ for all $x \in L^p(\mathcal{M})_{sh}$.*

We take the subalgebra $\mathcal{N}$ consisting of diagonal operator matrices, i.e.

$$\mathcal{N} = \Big\{ \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix} \;:\; x_{11}, x_{22} \in \mathcal{M} \,\Big\}.$$

In this example we can explicitly compute the projection Q. Actually, this is a consequence of the following inequality.

Lemma 3.8. *Let $p \geq 2$ a positive even number and $b \in \mathcal{M}$. Then,*

$$\left\| \begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix} \right\|_p \leq \left\| \begin{pmatrix} a & b \\ -b^* & d \end{pmatrix} \right\|_p ,$$

for all $a, d \in \mathcal{M}_{sh}$.

Proof. Let $p = 2k$, $k \geq 1$. By Lemma 3.7, to prove the stated inequality is equivalent to show the orthogonality condition

$$\hat{\tau}(\begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix}^{2k-1} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}) = 0,$$

for all $a, d \in \mathcal{M}_{sh}$. Note that it is easy to compute any power of a co-diagonal matrix. Indeed, for $k \geq 1$ we have

$$\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}^{2k} = (-1)^k \begin{pmatrix} (bb^*)^k & 0 \\ 0 & (b^*b)^k \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}^{2k+1} = (-1)^k \begin{pmatrix} 0 & (bb^*)^k b \\ -(b^*b)^k b^* & 0 \end{pmatrix}.$$

Then, we obtain

$$\hat{\tau}(\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}^{2k-1} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}) = \hat{\tau}(\begin{pmatrix} 0 & (bb^*)^{2(k-1)} b \\ (b^*b)^{2(k-1)} b^* & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}) = 0.$$

Hence our lemma is proved. □

Now it is plain that $Q : (L^p(\mathcal{M}) \otimes M_2)_{sh} \longrightarrow L^p(\mathcal{N})_{sh}$ is the linear map given by

$$Q(\begin{pmatrix} x_{11} & x_{12} \\ -x_{12}^* & x_{22} \end{pmatrix}) = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}.$$

Then Q is the extension to the non-commutative L^p space of the unique trace-invariant conditional expectation from $\mathcal{M} \otimes M_2$ onto $\mathcal{N}$. In particular, Q is uniformly bounded with constant 1.

Example 3.9. Consider the projection in $\mathcal{M} \otimes M_2$ given by $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\mathcal{O}$ denote the unitary orbit, i.e.

$$\mathcal{O} = \{ ueu^* \; : u \in \mathcal{U}_{\mathcal{M} \otimes M_2} \}.$$

The isotropy group at e of the natural action of $\mathcal{U}_{\mathcal{M} \otimes M_2}$ is given by

$$G = \{ u \in \mathcal{U}_{\mathcal{M} \otimes M_2} \; : \; ue = eu \}$$

The Lie algebra of this group is

$$\mathcal{G} = \{ x \in (\mathcal{M} \otimes M_2)_{sh} \; : \; xe = ex \} = \{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \; : \; a, d \in \mathcal{M}_{sh} \}.$$

Therefore by our preceding discussion the projection Q onto the Lie algebra is uniformly bounded, so our result about minimality of curves holds.

3.3. **Special diagonal algebra in $\mathcal{M}\otimes M_2$.** Consider the following subalgebra of $\mathcal{M}\otimes M_2$ given by

$$\mathcal{N}=\{\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix} : x\in\mathcal{M}\}.$$

Let E denote the unique trace-invariant (with respect to the trace $\hat{\tau}$) conditional expectation onto $\mathcal{N}$, i.e.

$$E:\mathcal{M}\otimes M_2\longrightarrow\mathcal{N},\qquad E(\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix})=\frac{1}{2}\begin{pmatrix} x_{11}+x_{22} & 0\\ 0 & x_{11}+x_{22}\end{pmatrix}.$$

We denote by E_p the extension of the above expectation to the corresponding non-commutative L^p spaces.

Lemma 3.10. *Consider*

$$\mathcal{X}=\{\begin{pmatrix} a & b\\ b & -a\end{pmatrix} : a,b\in\mathcal{M}\}.$$

If $A,B\in\mathcal{X}$, then $AB^2\in\mathcal{X}$.

Proof. Let

$$A=\begin{pmatrix} a & b\\ b & -a\end{pmatrix},\qquad B=\begin{pmatrix} d & e\\ e & -d\end{pmatrix}$$

Then

$$B^2=\begin{pmatrix} d & e\\ e & -d\end{pmatrix}\begin{pmatrix} d & e\\ e & -d\end{pmatrix}=\begin{pmatrix} d^2+e^2 & de-ed\\ ed-de & d^2+e^2\end{pmatrix}$$

Since

$$\begin{aligned} AB^2 &= \begin{pmatrix} a & b\\ b & -a\end{pmatrix}\begin{pmatrix} d^2+e^2 & de-ed\\ ed-de & d^2+e^2\end{pmatrix}\\ &= \begin{pmatrix} ad^2+ae^2+bed-bde & ade-aed+be^2+bd^2\\ bd^2+be^2-aed+ade & bde-bed-ad^2-ae^2\end{pmatrix},\end{aligned}$$

then $AB^2\in\mathcal{X}$. □

Lemma 3.11. *Let $2\le p<\infty$, p even. Then:*

$$\left\|\begin{pmatrix} (a-c)/2 & b\\ b & (c-a)/2\end{pmatrix}\right\|_p\le\left\|\begin{pmatrix} a & b\\ b & c\end{pmatrix}+\begin{pmatrix} d & 0\\ 0 & d\end{pmatrix}\right\|_p$$

for any $d\in\mathcal{M}$.

Proof. Note that

$$\begin{pmatrix} (a-c)/2 & b\\ b & (c-a)/2\end{pmatrix}=\begin{pmatrix} a & b\\ b & c\end{pmatrix}+\begin{pmatrix} (-a-c)/2 & 0\\ 0 & (-a-c)/2\end{pmatrix}.$$

Then, the inequality of the lemma is equivalent to

$$\hat{\tau}(\begin{pmatrix} (a-c)/2 & b\\ b & (c-a)/2\end{pmatrix}^{p-1}\begin{pmatrix} d & 0\\ 0 & d\end{pmatrix})=0,$$

for any $d \in \mathcal{M}$. We show that it holds for $d \geq 0$ then it extends by a straightforward argument. The inequality holds for $p = 2$, since

$$\begin{aligned}
&\hat{\tau}(\begin{pmatrix} (a-c)/2 & b \\ b & (c-a)/2 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}) \\
&= \hat{\tau}(\begin{pmatrix} d^{1/2} & 0 \\ 0 & d^{1/2} \end{pmatrix} \begin{pmatrix} (a-c)/2 & b \\ b & (c-a)/2 \end{pmatrix} \begin{pmatrix} d^{1/2} & 0 \\ 0 & d^{1/2} \end{pmatrix}) \\
&= \frac{1}{4}(\tau(d^{1/2}(a-c)d^{1/2}) + \tau(d^{1/2}(c-a)d^{1/2})) = 0.
\end{aligned}$$

On the other hand,

$$X = \begin{pmatrix} (a-c)/2 & b \\ b & (c-a)/2 \end{pmatrix} \in \mathcal{X}.$$

Hence, by Lemma 3.10, $X^3 = XX^2 \in \mathcal{X}$, and the p-orthogonality condition holds for $p = 2$ since

$$\hat{\tau}(X^3 \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}) = \frac{1}{2} \big(\tau(d^{1/2}(X^3)_{11} d^{1/2}) + \tau(d^{1/2}(X^3)_{22} d^{1/2}) \big) = 0$$

and $X^3 \in \mathcal{X}$ implies $(X^3)_{11} = -(X^3)_{22}$. All the other powers can be handled in a similar fashion, for instance $X^5 = X^3X^2 \in \mathcal{X}$. □

Let $\mathcal{L}$ stand for the real subspace of $\mathcal{M}_{sh} \otimes M_2$ given by

$$\mathcal{L} = \{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathcal{M}_{sh} \}$$

and $\mathcal{L}^p$ be the completion with the p-norm. Then, it is easy to check, using the previous lemma, that $E : \mathcal{L} \longrightarrow \mathcal{N}$ and $E_p : \mathcal{L}^p \longrightarrow L^p(\mathcal{N})$ for p even, are contractive maps.

Analogous statements hold for the subspace

$$\mathcal{D} = \{ \begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix} : b \in \mathcal{M} \},$$

invoking Lemma 3.8. If p is even or $p = \infty$, then $Q_{\mathcal{N},p} = E_p$, namely the best approximant can be obtained via the conditional expectation in $\mathcal{L}$. In particular Q is uniformly bounded $\mathcal{L}$. A similar argument shows that Q is uniformly bounded in $\mathcal{D}$. We do not know if Q is uniformly bounded in $\mathcal{M} \otimes M_2$.

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