Quasilocal energy for three-dimensional massive gravity solutions with chiral deformations of AdS_3 boundary conditions

Alan Garbarz,[1,2,*] Gaston Giribet,[1,†] Andrés Goya,[1,‡] and Mauricio Leston[3,§]

[1] *Departamento de Física, Universidad de Buenos Aires FCEN-UBA, IFIBA-CONICET, Ciudad Universitaria, Pabellón I, 1428, Buenos Aires, Argentina.*

[2] *Instituto de Física de La Plata, Universidad Nacional de La Plata IFLP-UNLP, C.C. 67, 1900, La Plata, Argentina.*

[3] *Instituto de Astronomía y Física del Espacio IAFE-CONICET, Ciudad Universitaria, C.C. 67 Suc. 28, 1428, Buenos Aires, Argentina.*

We consider critical gravity in three dimensions; that is, the New Massive Gravity theory formulated about Anti-de Sitter (AdS) space with the specific value of the graviton mass for which it results dual to a two-dimensional conformal field theory with vanishing central charge. As it happens with Kerr black holes in four-dimensional critical gravity, in three-dimensional critical gravity the Bañados-Teitelboim-Zanelli black holes have vanishing mass and vanishing angular momentum. However, provided suitable asymptotic conditions are chosen, the theory may also admit solutions carrying non-vanishing charges. Here, we give simple examples of exact solutions that exhibit falling-off conditions that are even weaker than those of the so-called Log-gravity. For such solutions, we define the quasilocal stress-tensor and use it to compute conserved charges. Despite the drastic deformation of AdS_3 asymptotic, these solutions have non-zero mass and angular momentum, which we compute.

PACS numbers: 11.25.Tq, 11.10.Kk

arXiv:1307.4791v3 [hep-th] 6 May 2014

I. INTRODUCTION

In three dimensions a fully covariant parity-even theory of gravity that reduces to massive spin-two Fierz-Pauli theory at linearized level does exist. This is the so-called New Massive Gravity theory (NMG), which was proposed in Ref. [1, 2]. When formulated about asymptotically Anti-de Sitter (AdS) spacetime, NMG exhibits features that are reminiscent of those of cosmological Topologically Massive Gravity (TMG) [3, 4]. In particular, a special point of the parameter space that resembles the chiral point of TMG exists in NMG as well. At this point, the central charge of the dual two-dimensional conformal field theory (CFT) vanishes [5] and, as it happens in TMG [6], the asymptotic boundary conditions may be relaxed with respect to the standard Brown-Henneaux boundary conditions [7]. The aim of the present paper is to study how the asymptotic conditions may be relaxed.

Asymptotic boundary conditions are of central importance in three-dimensional gravity, particularly in the case of negative cosmological constant. For the theory on AdS_3, boundary conditions are crucial to realize the action of the conformal group at the boundary of the spacetime [7], what is ultimately interpreted in a natural way within the context of AdS/CFT correspondence [8]. A good example to illustrate in what sense boundary conditions are essential to define the theory is Chiral Gravity: A few years ago, when Chiral Gravity was proposed in Ref. [9], the discussion about the appropriate boundary conditions to be imposed was the key point to determine the consistency of the model [10–12]. It was shown in Ref. [13] how much the properties of TMG at the chiral point depend on the asymptotic conditions considered. In fact, the theory presents quite distinct features depending on whether one considers the orthodox boundary conditions originally proposed in [7] or, on the contrary, one opts for the weakened version proposed in [6, 14]. In the former case, the bulk theory results to be dual to an holomorphic CFT, whose symmetry is generated by a single copy of Virasoro algebra. In contrast, in the latter case, TMG at the chiral point results to be dual to a non-unitary CFT whose symmetry is generated by the product of a Virasoro algebra and a Witt algebra [15]. Other choices of boundary conditions leading to different symmetry algebras at the boundary were also considered in the literature, and the discussions about this point are interesting and extend to different three-dimensional models, including TMG, NMG and Einstein gravity coupled to matter; see for instance [16–22].

In this paper, we will consider solutions obeying a set of boundary conditions different from those of [7] and [6]. The asymptotic conditions studied here turn out to be a deformation of Brown-Henneaux boundary conditions, but of a different type since they not only relax the next-to-leading components of the metric but also some leading pieces of it. Remarkably, although the conditions we will consider change the asymptotic behavior drastically, quasilocal stress tensor can still be consistently defined at the boundary of the spacetime [23] and be used to compute conserved charges associated to exact solutions obeying the new boundary conditions. The quasilocal energy for deformed AdS_3 boundary conditions was computed, for instance, in Refs. [24–28]; however, as said,

*Electronic address: alan-at-df.uba.ar
†Electronic address: gaston-at-df.uba.ar
‡Electronic address: af.goya-at-df.uba.ar
§Electronic address: mauricio-at-iafe.uba.ar

the asymptotic conditions considered here exhibit a substantially weaker damping-off.

The paper is organized as follows: In section 2, we will review three-dimensional massive gravity. We will be involved with the theory proposed in [1] formulated on AdS_3 space at the point of the parameter space at which the central charge of the dual CFT vanishes. This amounts to tune the value of the graviton mass in terms of the effective cosmological constant. In section 3, with the aim of presenting the set of boundary conditions we are interested in, we will first summarize different choices of asymptotic behaviors considered in the literature and compare their falling-off conditions in the near boundary limit. In section 4, we will present a simple example of an exact solution that satisfies the new boundary conditions but can not be accommodated neither in the Brown-Henneaux nor in Log-gravity boundary conditions. The particular solution we will discuss is a chiral deformation of the extremal Bañados-Teitelboim-Zanelli (BTZ) black hole [29, 30]. It generalizes explicit solutions to TMG and NMG previously found in Refs. [31–33]. For these solutions, we will show in section 5 that quasilocal stress-tensor can be consistently defined in such a way that both the mass and angular momentum of the deformed BTZ hole can be computed. Remarkably, despite the drastic deformation of AdS_3 asymptotic behavior we consider, the solutions have finite mass and angular momentum, which we compute.

II. MASSIVE GRAVITY

A. The Fierz-Pauli action

Let us start by reviewing massive gravity in D dimensions. This is given by the Fierz-Pauli action

$$S = \int d^Dx \left(-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{m^2}{2}(h_{\mu\nu}h^{\mu\nu} - h^2) \right) \quad (1)$$

for the symmetric field $h_{\mu\nu}$ that represents the gravitational perturbation about Minkowski space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, with $\kappa^2 = 16\pi l_P^{D-2}$. As usual, we denote $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. Here we will set the Planck length l_P to one.

The kinetic term in (1) coincides with the Einstein-Hilbert action at second order in $h_{\mu\nu}$. The relative factor -1 in the massive term is chosen for the component h^{00} of the metric to appear as a Lagrange multiplier; then the theory has only five local degrees of freedom in $D = 4$ dimensions and two degrees of freedom in $D = 3$ dimensions.

The theory may also be formulated about other backgrounds. Specially about (A)dS spaces, for which one has to introduce a cosmological term to the action above. Nevertheless, let us first discuss the theory about Minkowski space and introduce the cosmological constant later, after discussing the covariant extension of the theory.

The equations of motion derived from (1) are

$$\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda_\nu - \partial_\lambda\partial_\nu h^\lambda_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = 0. \quad (2)$$

Here, the kinetic term coincides with Einstein tensor for metric $g_{\mu\nu}$ at first order in $h_{\mu\nu}$. For this reason, we prefer to denote it $G^{(1)}_{\mu\nu}[h]$; namely

$$G^{(1)}_{\mu\nu}[h] \equiv \Box h_{\mu\nu} - \eta_{\mu\nu}\Box h - \partial_\lambda\partial_\mu h^\lambda_\nu - \partial_\lambda\partial_\nu h^\lambda_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h, \quad (3)$$

where the superindex (1) refers to the linear nature.

Now, let us go back to equation (2). Acting on it with the differential operator ∂^μ, if $m \neq 0$, one gets $\partial^\mu h_{\mu\nu} = \partial_\nu h$. Plugging this back into (3) one finds

$$\Box h_{\mu\nu} - \partial_\mu\partial_\nu h - m^2(h_{\mu\nu} - \eta_{\mu\nu}h) = 0. \quad (4)$$

Taking the trace of (4) one finds that the trace vanishes, $\eta^{\mu\nu}h_{\mu\nu} \equiv h = 0$. This implies $\partial^\mu h_{\mu\nu} = 0$ and consequently, from (3), one gets the Klein-Gordon equation

$$\Box h_{\mu\nu} = m^2 h_{\mu\nu}. \quad (5)$$

This, supplemented by $\partial^\mu h_{\mu\nu} = 0$ and $h = 0$, completes the set of equations of a massive spin-two field about Minkowski spacetime.

B. Massive gravity in three dimensions

Now, let us discuss a curious feature that occurs in three dimensions. Consider the particular case $D = 3$ of theory (1); namely, consider the system of equations

$$(\Box - m^2)h_{\mu\nu} = 0, \quad \partial^\mu h_{\mu\nu} = 0, \quad \eta^{\mu\nu}h_{\mu\nu} = 0. \quad (6)$$

Next, replace the metric $h_{\mu\nu}$ by the linearized Einstein tensor constructed out of it; namely, change

$$h_{\mu\nu} \to G^{(1)}_{\mu\nu}[h] \quad (7)$$

in (6). As we will see, this simple trick will lead to an interesting model that is only possible in three dimensions. After performing (7), the equations of motion turn out to be

$$(\Box - m^2)G^{(1)}_{\mu\nu}[h], \quad R^{(1)}[h] = 0, \quad (8)$$

where $R^{(1)}[h]$ represents the Ricci scalar associated to metric $g_{\mu\nu}$ at first order in the perturbation $h_{\mu\nu}$. Then, taking into account (3), the first of these equations can be written as

$$G^{(1)}_{\mu\nu}[G^{(1)}[h]] = G^{(1)}_{\mu\nu}[m^2h], \quad (9)$$

since, provided $\partial^\mu h_{\mu\nu} = 0$ and $h = 0$, it holds $G^{(1)}_{\mu\nu}[h] = \Box h_{\mu\nu}$. In this way, one obtains a equations that are invariant under linearized diffeomorphisms.

Now, an important observation: The key point to understand the peculiarity of the three-dimensional case is to be reminded that in three dimensions there exists a direct connection (local identification) between Einstein tensor associated to a given metric and the metric itself. In fact, all solutions to three-dimensional Einstein equations are locally equivalent, so that, in what local information regards to, $g_{\mu\nu}$ and $G_{\mu\nu}$ carry exactly the same information. In particular, this implies that (9) expresses the local identify between $G^{(1)}_{\mu\nu}$ and $m^2 h_{\mu\nu}$, which actually justifies having done (7).

Summarizing, something remarkable has been achieved: A theory has been obtained that is invariant under linear diffeomorphisms and, at the same time, is equivalent to massive Fierz-Pauli theory [1]. This is due to the magic of $D = 3$ dimensions, c.f. [34]. Below we will see how to extend this theory in a fully covariant way.

An important observation is that system (8) can be derived from the following action

$$S^{(1)}_{\text{NMG}} = \frac{1}{16\pi}\int d^3x \left(\frac{1}{2} h^{\mu\nu} G^{(1)}_{\mu\nu}[h] - \frac{1}{4m^2} G^{(1)}_{\mu\nu}[h] \left(R^{(1)\mu\nu}[h] - \frac{1}{4}\eta^{\mu\nu} R^{(1)}[h] \right) \right), \quad (10)$$

varying this action with respect to the field $h_{\mu\nu}$.

Then, it is easy to propose a generally covariant extension of (10). Noticing that contracting the Einstein tensor $G_{\mu\nu} \equiv G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + ... \equiv R_{\mu\nu} - (1/2)Rg_{\mu\nu}$ and the Schouten tensor $S_{\mu\nu} \equiv R_{\mu\nu} - (1/4)Rg_{\mu\nu}$ in three dimensions yields $G_{\mu\nu}S^{\mu\nu} = R_{\mu\nu}R^{\mu\nu} - (3/8)R^2$, one can write the NMG action [1]

$$S_{\text{NMG}} = \frac{1}{16\pi}\int_\Sigma d^3x\sqrt{-g}\left(R - \frac{1}{m^2}(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2) \right), \quad (11)$$

which represents a fully covariant parity-even theory of massive gravity.

The equations of motion derived from action (11) read

$$G_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0, \quad (12)$$

which, apart from the Einstein tensor $G_{\mu\nu}$, involve the tensor

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{1}{2}\Box R g_{\mu\nu} - \frac{3}{2}RR_{\mu\nu} - R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} + \frac{3}{8}R^2 g_{\mu\nu} + 4R_{\mu\alpha\nu\beta}R^{\alpha\beta}. \quad (13)$$

These are fourth-order differential equations for the metric $g_{\mu\nu}$. The precise combination of the square-curvature terms in (11) satisfies the notable property of being the trace of the equations it leads to. That is, $g^{\mu\nu}K_{\mu\nu} = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2$. This combination also happens to be the one that makes the dependence $\Box R$ to disappear from the trace of the equations of motion. This results in the decoupling of a ghostly mode of higher-curvature gravity. In turn, the theory contains (no more than) two local degrees of freedom, which can be associated to a massive spin-2 field in $D = 3$.

C. The theory on Anti-de Sitter

NMG action (11) may be supplemented with a cosmological constant term

$$S_\Lambda = -\frac{\Lambda}{8\pi}\int d^3x\sqrt{-g}. \quad (14)$$

The theory admits AdS_3 vacua provided the effective cosmological constant $\Lambda_{\text{eff}} \equiv -l^{-2}$ is negative: Asking for constant curvature solutions $R_{\mu\nu} = -(2/l^2)g_{\mu\nu}$, one finds $-\Lambda_{\text{eff}} \equiv l_\pm^{-2} = -(\Lambda/2)(1 \pm \sqrt{1-\Lambda/m^2})$. Generically, this gives two values for the effective cosmological constant, which we denote l_-^2 and l_+^2. Provided one of these values is negative, AdS_3 space appears as a solution, together with all the other geometries that are locally equivalent to it, like the notable case of the BTZ black hole [29].

The metric of AdS_3 spacetime can be written as

$$ds^2_{\text{AdS}} = -\left(\frac{r^2}{l^2}+1\right)dt^2 + \left(\frac{r^2}{l^2}+1\right)^{-1}dr^2 + r^2 d\phi^2, \quad (15)$$

with $r \in \mathbb{R}_{\geq 0}$, $t \in \mathbb{R}$, $\phi \in [0, 2\pi)$. The boundary of the space is located at $r = \infty$.

According to AdS/CFT conjecture, if NMG on AdS_3 resulted to be a consistent model, then it should be dual to a two-dimensional CFT formulated at $r = \infty$. Say this is actually the case. Then, it is expected that the central charge of the dual CFT_2 will coincide with the central extension of the algebra that generates the asymptotic isometry group. Its value can easily be computed to be

$$c = \frac{3l}{2}\left(1 - \frac{1}{2l^2m^2}\right) \quad (16)$$

which, indeed, agrees with the trace anomaly of the dual CFT_2 [26].

In addition to $S_{NMG} + S_\Lambda$, the theory may be augmented by adding to the action a gravitational Chern-Simons term [3]

$$S_{\text{CS}} = \frac{1}{32\pi\mu}\int_\Sigma d^3x\varepsilon^{\mu\nu\rho}\Gamma^\beta_{\mu\alpha}\left(\partial_\nu\Gamma^\alpha_{\rho\beta} - \frac{2}{3}\Gamma^\alpha_{\nu\delta}\Gamma^\delta_{\rho\beta})\right). \quad (17)$$

In such case, equations of motion (12) acquires an additional term proportional to the Cotton tensor $C_{\mu\nu} = \frac{1}{2}\varepsilon_\mu{}^{\alpha\beta}\nabla_\alpha R_{\beta\nu} + \frac{1}{2}\varepsilon_\nu{}^{\alpha\beta}\nabla_\alpha R_{\mu\beta}$ with a coupling constant $1/\mu$. Equation (16) is also modified by the inclusion of (17): For finite μ the dual CFT_2 exhibits diffeomorphism anomaly and thus the right-moving c_+ and left-moving c_- central charges are different; more precisely, $c_\pm = c \pm 3/(2\mu)$, with c given by (16). Chiral Gravity theory of [9] corresponds to $\mu l = 1$ and $1/m^2 = 0$, and can be naturally generalized to finite m^2 by demanding $c_- = 0$. Here, we will be mainly interested in the case $1/\mu = 0$ at the point of the parameter space where $c = 0$, where the graviton mass is

$$m^2 = \frac{1}{2l^2}. \tag{18}$$

At this point, NMG exhibits peculiar features. For instance, when (18) holds all BTZ black holes have vanishing conserved charges. It is worth mentioning that, despite of this fact, the theory at $c = 0$ also presents solutions with non-vanishing conserved charges [25]. We will discuss solutions of this sort.

III. BOUNDARY CONDITIONS

A. Asymptotic boundary conditions

As we mentioned in the introduction, an important ingredient to define the theory are the boundary conditions. Standard Brown-Henneaux boundary conditions are specified by considering deformations of AdS_3 metric (15) of the form

$$g_{\mu\nu} = g^{(\mathrm{AdS})}_{\mu\nu} + h_{\mu\nu} \tag{19}$$

and demanding the components of $h_{\mu\nu}$ to damp off in the following manner [7]

$$h_{tt} \simeq \mathcal{O}(1), \quad h_{rr} \simeq \mathcal{O}(r^{-4}), \quad h_{\phi t} \simeq \mathcal{O}(1), \tag{20}$$

$$h_{\phi r} \simeq \mathcal{O}(r^{-3}), \quad h_{\phi\phi} \simeq \mathcal{O}(1), \quad h_{rt} \simeq \mathcal{O}(r^{-3}) \tag{21}$$

where $\mathcal{O}(r^{-n})$ stands for arbitrary functions of coordinates ϕ and t that fall off equally or faster than a power r^{-n} at large r. In particular, this implies

$$g_{tt} = -\frac{r^2}{l^2} + \mathcal{O}(1), \quad g_{\phi\phi} = r^2 + \mathcal{O}(1). \tag{22}$$

This set of boundary conditions incorporates, in particular, the BTZ black hole solutions.

Log-gravity boundary conditions proposed in [6, 14] permit relaxation of (22) including terms like $h_{ij} \simeq \mathcal{O}(\log(r))$ with $i, j = t, \phi$. Also, other sets of boundary conditions can be defined [17, 18, 21, 35]. In order to clearly distinguish between different boundary conditions, let us summarize some proposals below. To do this, consider the following form of the metric

$$ds^2 = d\rho^2 + e^{2\rho}\gamma_{ab}dz^a dz^b, \tag{23}$$

which resembles the Fefferman-Graham expansion of General Relativity [36], with $z^\pm$ being two null directions $(a, b = \pm)$, with the asymptotic expansion

$$\gamma_{ab}(\rho) = \gamma^{(0)}_{ab} + e^{-2\rho}\gamma^{(2)}_{ab} + e^{-4\rho}\gamma^{(4)}_{ab} + \ldots \tag{24}$$

where $\gamma^{(n)}_{ab}$ are functions of z^+ and/or z^- that do not depend on ρ; see for instance Ref. [37].

In terms of (23)-(24), Brown-Henneaux boundary conditions (20)-(21) read

$$\gamma^{(0)}_{--} = \gamma^{(0)}_{++} = 0, \quad \gamma^{(0)}_{-+} = \gamma^{(0)}_{+-} = -\frac{1}{2}. \tag{25}$$

To compare with (20)-(21) consider the change of coordinates $\tau \equiv t/l^2$, $\varphi \equiv \phi/l$, with $z^\pm \equiv \tau \pm \varphi$ and $\rho \equiv \log(r)$.

The so-called Log-gravity relaxed boundary conditions [6] correspond to considering NMG at the point $m^2 l^2 = 1/2$ and supplementing expansion (24) with an additional term $\rho e^{-2\rho}\gamma^{(\mathrm{Log})}_{++}$ keeping (25). Provided NMG is parity-even, a $(--)$ version of these conditions also exists [20]. Clearly, the asymptotic behavior with $\gamma^{(\mathrm{Log})}_{++} \neq 0$ is weaker than Brown-Henneaux conditions due to the existence of a term in (23) that grows linearly as $\sim \mathcal{O}(\rho)$. In terms of radial coordinate r this corresponds to a logarithmic term $\sim \mathcal{O}(\log(r))$.

In Ref. [21], the possibility of having contributions to (23) that grow quadratically as $\sim \mathcal{O}(\rho^2) \sim \mathcal{O}(\log^2(r))$ was considered. It was shown that such a behavior is possible if NMG is coupled to TMG at the fine tuned point $ml^2 = -2\mu l = -3/2$. In such case, it is possible to supplement (24) with a term $\rho^2 e^{-2\rho}\gamma^{(\mathrm{Log}^2)}_{++}$ with (25). Explicit solutions obeying these conditions were found in Ref. [35].

A totally different set of boundary conditions for NMG coupled to TMG was proposed in Ref. [17], where it was shown that the theory at the point $m^2 l^2 = -1/2$ with arbitrary μ admits to add to (24) a term like $e^{-\rho}\gamma^{(1)}_{+-}$. This is a term that grows quite rapidly at large ρ, as it gives a next-to-leading contribution $\sim \mathcal{O}(e^\rho) \sim \mathcal{O}(r)$ to (23). The computation of quasilocal stress-tensor of solutions satisfying this asymptotic and/or the Log-gravity boundary conditions was done, for instance, in Refs. [24, 25].

It is interesting to compare the behaviors listed above with the new type of boundary conditions proposed by Compère, Song, and Strominger in [22] in the context of three-dimensional Einstein gravity coupled to matter. These correspond to considering expansion (24) and relaxing (25) by allowing for $\gamma^{(0)}_{++} = \partial_+ f(z^+) \neq 0$ with $\gamma^{(0)}_{-+} = \gamma^{(0)}_{+-} = -1/2$, and $\gamma^{(2)}_{--}$ fixed. These boundary conditions were shown to yield an asymptotic isometry algebra generated by the product of a single Virasoro algebra and an affine Kac-Moody $\widehat{u}(1)$ factor. Reduced conformal symmetry for TMG on AdS_3 with boundary conditions of mixed chirality was also studied in Ref. [18].

Here, we will consider a different set of boundary conditions. We will consider deformations of Brown-Henneaux asymptotic (25) that correspond to supplementing expansion (24) with terms

$$\rho e^{-2\rho}\gamma^{(\mathrm{Log})}_{++} + \rho\gamma^{(\mathrm{New})}_{++}. \tag{26}$$

These asymptotic conditions reduce to Log-gravity conditions only in the case $\gamma^{(\mathrm{New})}_{++} = 0$, while in the case such a term is turned on the metric (23) acquires a dependence like $\sim \mathcal{O}(\rho e^{2\rho}) \sim \mathcal{O}(r^2 \log(r))$. Notice that, in contrast to Brown-Henneaux and Log-gravity boundary conditions, (26) changes the leading behavior (22) and not only the next-to-leading behavior. In fact, (26) permits to change (22) by

$$g_{tt} \simeq -\frac{r^2}{l^2}\log(r) - \frac{r^2}{l^2} + \mathcal{O}(\log(r)). \quad (27)$$

and something similar for $g_{\phi\phi}$. This type of boundary conditions was studied in [18] for the case of TMG, where it was shown that these are consistent with asymptotic conformal symmetry at the boundary. Explicit solutions obeying these asymptotic conditions were analyzed in Refs. [32, 33]. In the next section we will review this type of solutions in the case of NMG.

IV. NON-LINEAR SOLUTION

A. Deformation of BTZ solution

Consider first the extremally rotating BTZ solution

$$ds^2_{e\mathrm{BTZ}} = -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2\left(N_\phi(r)dt - d\phi\right)^2 \quad (28)$$

with $r \in \mathbb{R}_{\geq 0}$, $t \in \mathbb{R}$, $\phi \in [0, 2\pi)$, where

$$N^2(r) = \frac{r^2}{l^2} - 4M + \frac{4M^2l^2}{r^2}, \quad N_\phi(r) = \frac{2Ml}{r^2}. \quad (29)$$

For $M > 0$ this metric exhibits an event horizon at $r = \sqrt{2Ml}$. Being an Einstein space with negative cosmological constant in three dimensions, BTZ geometry is locally equivalent to AdS_3, and it is asymptotically AdS_3 in the sense of [7]. The parameter M in (28)-(29) in the case of General Relativity corresponds to the mass and angular momentum of the extremally rotating black hole. In the case of TMG at the chiral point $\mu = 1/l$, in contrast, the mass of such a solution is zero, and the same happens in NMG at $m^2l^2 = 1/2$.

Now, consider a deformation of (28)-(29) of the form

$$ds^2 = ds^2_{e\mathrm{BTZ}} + H_{ab}dz^a dz^b \quad (30)$$

where $z^\pm$ are the coordinates introduced before, with $a, b = +, -$. Here, H_{ab} are three functions of the coordinates ϕ, t, and r. The large r expansion of H_{ab} determines the asymptotic boundary conditions.

It turns out that an exact solution to NMG at the point $m^2l^2 = 1/2$ ($c = 0$) is given by the particular deformation

$$H_{ab}(r) = l^4 \delta^+_a \delta^+_b \left(k_0 + k_2 r^2\right) \log\left(\frac{r^2 - 2Ml^2}{2Ml^2}\right) \quad (31)$$

with k_0 and k_2 being two arbitrary constant.

Geometry (28)-(31) is not conformally flat, so it is not an Einstein manifold. Still, it presents constant curvature invariants that only depend on l. This geometry presents a curious geodesic structure at the radius $r = \sqrt{2Ml}$, where the extremal BTZ geometry presents its horizon. In general, function (31) diverges at $r = \sqrt{2Ml}$, except in the special case $k_0 = -2Ml^2k_2$ we will refer to later.

Solutions similar to (28)-(31) exist for NMG coupled to TMG at the point $c_- = 0$. For the theory with $1/\mu = 0$, provided it is parity-even, (28)-(31) remains a solution if one changes $N_\phi \to -N_\phi$ and $z^\pm \to z^\mp$. Metrics (28)-(31) were also studied in Ref. [32, 33], and solutions locally equivalent to them appeared already in [31, 35].

V. CONSERVED CHARGES

A. Auxiliary field formulation

In this section, we will compute the conserved charges associated to solution (28)-(31). To do so, we will first write the quasilocal stress-tensor, which first requires the introduction of suitable boundary terms in the action. In order to write down the boundary terms for NMG, it is convenient to rewrite the theory in a different way. In fact, there exists another action, other than (11), from which equations (12)-(13) may de derived. This amounts to introduce a symmetric rank-two auxiliary field $f_{\mu\nu}$ and consider the alternative action

$$S_{\mathrm{A}} = \frac{1}{16\pi}\int_\Sigma d^3x\sqrt{-g}\left(R + f^{\mu\nu}G_{\mu\nu} - \frac{1}{4}m^2(f_{\mu\nu}f^{\mu\nu} - f^2)\right), \quad (32)$$

where, again, $G_{\mu\nu}$ is the Einstein tensor made out of metric $g_{\mu\nu}$, while $f_{\mu\nu}$ is a non-dynamical field. After varying with respect to field $f_{\mu\nu}$ we find the equation $f_{\mu\nu} = (2/m^2)S_{\mu\nu}$, and plugging this back into action (32) we recover (11).

In general, higher derivatives actions like (11) require additional information to be provided about the variation of the fields at the boundary. In such cases, it is not enough to fix the variations of the metric at the boundary, but it is also necessary to specify the variation of the normal derivative of it. In the case of General Relativity this problem is solved by the addition of the Gibbons-Hawking term to the action. In the case of NMG, on the contrary, such boundary term is not enough. The authors of [24] gave a prescription for a variational principle in NMG based on the second order derivative action (32). Following the criterion of [24] it is sufficient to fix the variations of $g_{\mu\nu}$ and $f_{\mu\nu}$ at the boundary and add to the action a suitable boundary term that supplements the Gibbons-Hawking contribution.

B. Boundary terms

Then, as said, boundary terms S_{B} are added to action (11) for the variational principle to be defined in such a way that both the metric $g_{\mu\nu}$ and the auxiliary field $f_{\mu\nu}$

are fixed on the boundary $\partial\Sigma$. With this prescription, the boundary action $S_{\rm B}$ reads

$$S_{\rm B} = -\frac{1}{8\pi}\int_{\partial\Sigma} d^2x\sqrt{-\gamma}\left(K + \frac{1}{2}\hat{f}^{ij}(K_{ij} - \gamma_{ij}K)\right). \tag{33}$$

Here, we use the convention that Latin indices $i, j = 0, 1$ refer to the coordinates on the constant-r surfaces, while the Greek indices are $\mu, \nu = 0, 1, 2$ and do include the radial direction r as well. Now, γ_{ij} is the two-dimensional metric induced on $\partial\Sigma$ and K_{ij} is the extrinsic curvature, with $K = \gamma^{ij}K_{ij}$. Matrix components $\hat{f}^{ij}$ are defined as $\hat{f}^{ij} \equiv f^{ij} + 2f^{r(i}N^{j)} + f^{rr}N^iN^j$, where N^i are the shift functions of the metric $g_{\mu\nu}$ to be written in Arnowitt-Deser-Misner (ADM) form

$$ds^2 = N^2dr^2 + \gamma_{ij}(dx^i + N^idr)(dx^j + N^jdr), \tag{34}$$

with the radial lapse function N^2.

The first term in (33) is of course the Gibbons-Hawking term. The other two terms come from the higher-curvature terms of NMG. Notice that in (33) the field $\hat{f}^{ij}$ couples to the Israel tensor $K_{ij} - \gamma_{ij}K$ in the same manner as the field $f^{\mu\nu}$ couples to the Einstein tensor in the bulk action (32). Besides, pushing this analogy forward, one may suggest to supplement (33) with pure boundary terms of the form

$$S_{\rm C} = \frac{1}{8\pi}\int_{\partial\Sigma} d^2x\sqrt{-\gamma}(m_0 + m_1\,\hat{f} + m_2(\,\hat{f}^2 - \hat{f}_{ij}\hat{f}^{ij}) + ...), \tag{35}$$

with adequate constant mass coefficients m_i, and $\hat{f} \equiv \hat{f}^{ij}\gamma_{ij}$. In fact, these terms are in general needed to regularize the action and define finite conserved charges, for instance in the context of holographic renormalization [24–26, 28]. Here, because of the special properties of the theory at the point $c = 0$, we will not need to introduce such terms in the action to regularize the charges. This is because for the theory on AdS_3 it is required to introduce a counterterm like

$$S_{\rm C} \propto \; c\int d^2x\sqrt{-\gamma} \tag{36}$$

which in this case vanishes. See [24–26, 28] for a discussion on counterterms and the different choices of asymptotic conditions.

C. Quasilocal stress-tensor and charges

Now, having introduced the boundary terms, let us analyze the definition of the Brown-York tensor in NMG as done in Ref. [24]. This tensor is defined by varying the full action $S = S_A + S_B + S_C$ with respect to the metric γ^{ij}, namely

$$T_{ij} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma^{ij}}_{|r=\mathrm{const}}, \tag{37}$$

and then taking the limit $r \to \infty$. The explicit form of T_{ij} is cumbersome and can be found in [24] . It involves the fields γ_{ij}, f_{ij}, and its derivatives

Conserved charges are then defined in terms of integrals [23]

$$Q[\xi] = \int ds\; u^iT_{ij}\xi^j, \tag{38}$$

where ds is the line element of the constant-t surfaces at the boundary, u is a unit vector orthogonal to the constant-t surfaces, and ξ is the Killing vector that generates the isometry on $\partial\Sigma$ to which the charge is associated. In the case of the mass, the components of this vector could be $\xi^i = N_tu^i$, where the lapse function N^t is the lapse function of the two-dimensional metric induced at the boundary expressed in the ADM decomposition.

Let us compute the charges associated to solutions (28)-(31). The action of the theory evaluated at these solutions at large r diverges like $\sim c\; r^2 + \mathcal{O}(1)$, which in this case is finite in virtue of (18). The conserved charge associated to vector $\xi = N_tu$ can be shown to behave like

$$Q[N_tu] = \lim_{r\to\infty}\frac{2(k_0 + 2Ml^2k_2)}{1 + l^2k_2\log(r^2/(2Ml^2))};$$

which tends to zero if $k_2 \neq 0$ while it tends to $2k_0$ if $k_2 = 0$. However, this is not the definition of quasilocal energy we want for $k_2 \neq 0$ configurations. Instead, we prefer to define the mass with respect to the boundary Killing vector $\xi = \partial_t$. The mass associated to it reads

$$Q[\partial_t] = 2(k_0 + 2Ml^2k_2). \tag{39}$$

For $k_2 = 0$ this result coincides with the result $Q[\partial_t] = 2k_0$ found in [25] for a particular case of this geometry, while for $k_2 \neq 0$ this gives a new finite contribution to the mass that happens to be proportional to the parameter M of the extremal BTZ solution. Notice that if $k_2 = 0$ then N_tu tends to ∂_t when r tends to infinity.

The angular momentum, on the other hand, being the charge associated to Killing vector ∂_ϕ, is given by

$$Q[\partial_\phi] = 2(k_0 + 2Ml^2k_2)l. \tag{40}$$

Again, for $k_2 = 0$ one reobtains the result of previous computations [25], while it receives a correction when the $\mathcal{O}(r^2\log(r))$ terms is included. This means that solutions have mass equal to angular momentum for all values of k_0, k_2, and M.

It is a remarkable property of NMG at the critical point $m^2l^2 = 1/2$ (*i.e.* $c = 0$) that non-linear solutions exhibiting asymptotic behavior of this kind happen to have finite conserved charges.

It is worth mentioning that the two terms contributing to (39) and (40) come from different terms in the large r expansion in (31); while the piece that depends on k_0 comes from the $\mathcal{O}(\log(r))$, it being consistent with Log-gravity boundary conditions, the second piece comes from the new $\mathcal{O}(r^2 \log(r))$ dependence. It is notable that the latter depends both on k_2 and M.

There is a special case for which the deformation (31) vanishes at $r^2 = 2Ml^2$; namely, when $k_0 = -2Ml^2k_2$. With this choice of parameters the solution exhibits special features; for instance, the effective potential of geodesics does not diverge. Remarkably, in this case conserved charges (39) and (40) vanish.

This work has been supported by UBA, CONICET, and ANPCyT. The authors are grateful to Stéphane Detournay and Guillem Pérez-Nadal for discussions. G.G. thanks Pontificia Universidad Católica de Valparaíso for the hospitality during his stays, where part of this work was done.

[1] E. Bergshoeff, O. Hohm, P. Townsend, *Phys. Rev. Lett.* **102** (2009) 201301.
[2] E. Bergshoeff, O. Hohm, P. Townsend, *Phys. Rev.* **D79** (2009) 124042.
[3] S. Deser, R. Jackiw, S. Tempelton, *Annals Phys.* **140** (1982) 372.
[4] S. Deser, R. Jackiw, S. Tempelton *Phys. Rev. Lett.* **48** (1982) 975.
[5] D. Grumiller and O. Hohm, *Phys. Lett.* **B686** (2010) 2640905.
[6] D. Grumiller, N. Johansson, *JHEP* **0807** (2008) 134.
[7] J. D. Brown, M. Henneaux, *Commun. Math. Phys.* **104** (1986) 207.
[8] J.M. Maldacena, *Adv. Theor. Math. Phys.* **2** (1998) 231.
[9] Li, W. Song, A. Strominger, *JHEP* **0804** (2008) 082.
[10] S. Carlip, S. Deser, A. Waldron, D. Wise, *Phys. Lett.* **B666** (2008) 272.
[11] S. Carlip, S. Deser, A. Waldron, D. Wise, *Class. Quant. Grav.* **26** (2009) 075008.
[12] G. Giribet, M. Kleban, M. Porrati, *JHEP* **0810** (2008) 045.
[13] A. Maloney, W. Song, A. Strominger, *Phys. Rev.* **D81** (2010) 064007.
[14] D. Grumiller, N. Johansson, *Int. J. Mod. Phys.* **D17** (2009) 2367.
[15] M. Henneaux, C. Martínez, R. Troncoso, *Phys. Rev.* **D82** (2010) 064038.
[16] M. Henneaux, C. Martínez, R. Troncoso, J. Zanelli, *Phys. Rev.* **D65** (2002) 104007.
[17] J. Oliva, D. Tempo, R. Troncoso, *JHEP* **0907** (2009) 011.
[18] M. Henneaux, C. Martínez, R. Troncoso, *Phys. Rev.* **D79** (2009) 081502R.
[19] Y. Liu, Y-W. Sun, *JHEP* **0904** (2009) 106.
[20] Y. Liu, Y-W. Sun, *JHEP* **0905** (2009) 039.
[21] Y. Liu, Y-W. Sun, *Phys. Rev.* **D79** (2009) 126001.
[22] G. Compère, W. Song, A. Strominger, arXiv:1303.2662.
[23] J. Brown, J. York, *Phys. Rev.* **D47** (1993) 1407.
[24] O. Hohm, E. Tonni, *JHEP* **1004** (2010) 093.
[25] G. Giribet, M. Leston, *JHEP* **1009** (2010) 070.
[26] G. Giribet, A. Goya, M. Leston, *Phys. Rev.* **D84** (2011) 066003.
[27] F. Correa, C. Martínez, R. Troncoso, *JHEP* **1202** (2012) 136.
[28] G. Giribet, A. Goya, *JHEP* **1303** (2013)130.
[29] M. Bañados, C. Teitelboim, J. Zanelli, *Phys. Rev. Lett.* **69** (1992) 1849.
[30] M. Bañados, M. Henneaux, C. Teitelboim, J. Zanelli, *Phys. Rev.* **D48** (1993) 1506.
[31] G. Gibbons, C. Pope, E. Sezgin, *Class. Quant. Grav.* **25** (2008) 205005.
[32] A. Garbarz, G. Giribet, Y. Vásquez, *Phys. Rev.* **D79** (2009) 044036.
[33] G. Clément, *Class. Quant. Grav.* **26** (2009) 165002.
[34] E. Bergshoeff, J.. Fernández-Melgarejo, J. Rosseel, P. Townsend, *JHEP* **1204** (2012) 070.
[35] E. Ayón-Beato, G. Giribet, M. Hassaïne, *JHEP* **0905** (2009) 029.
[36] C. Fefferman, C. Graham, *Conformal Invariants,* 1985 in *Elie Cartan et les Mathématiques d'aujourd'hui* (Asterisque) 95.
[37] C. Cunliff, *JHEP* **1304** (2013) 141.