The intrinsic fundamental group of a linear category

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Abstract

We provide an intrinsic definition of the fundamental group of a linear category over a ring as the automorphism group of the fibre functor on Galois coverings. If the universal covering exists, we prove that this group is isomorphic to the Galois group of the universal covering. The grading deduced from a Galois covering enables us to describe the canonical monomorphism from its automorphism group to the first Hochschild-Mitchell cohomology vector space.

1   Introduction

The purpose of this work is to provide a positive answer to the question of the existence of an intrinsic and canonical fundamental group π_1 associated to a k-category $\mathcal{B}$, where k is a commutative ring. The fundamental group we introduce takes into account the linear structure of the category $\mathcal{B}$, it differs from the fundamental group of the underlying category obtained as the classifying space of its nerve ([24, 23, 25]).

The fundamental group that we define is intrinsic in the sense that it does not depend on the presentation of the k-category by generators and relations. In case a universal covering exists, we obtain that the fundamental groups constructed by R. Martínez-Villa and J.A. de la Peña (see [20], and [4, 13, 1]) depending on a presentation of the category by a quiver and relations are in fact quotients of the intrinsic π_1 that we introduce. Note that those groups can vary according to different presentations of the same k-category (see for instance [1, 5, 18]) while the group that we introduce is intrinsic, since we define it as the automorphisms of the fibre functor of the Galois coverings over a fixed object.

In fact if a universal covering $U : \mathcal{U} \to \mathcal{B}$ exists, the fundamental group that we define is isomorphic to the automorphism group $\text{Aut}\,U$, and in this case changing the base object provides isomorphic intrinsic fundamental groups.

The methods we use are inspired in the topological case considered for instance in R. Douady and A. Douady's book [9]. They are closely related to the way in which the fundamental group is considered in algebraic geometry after A. Grothendieck and C. Chevalley.

In algebraic topology a space has a universal cover if it is connected, locally path-connected and semi-locally simply connected. In other words, usually a space has a universal cover. By contrast, linear categories do not have in general universal coverings.

Our work is very much indebted to the pioneer work of P. Le Meur in his thesis [19], see also [17]. He has shown that under some hypotheses on the category, there exists an optimal fundamental group in the sense that all other "fundamental groups" deduced from different presentations are quotients of the optimal one. His method consists mainly in tracing all the possible presentations of a given category, and relating the diverse resulting "fundamental groups". As already quoted, we adopt a different point of view in this paper.

In Section 2 we recall the definition of a covering of a k-category and we prove properties about morphisms between coverings as initiated in [17, 19]. In Section 3 we define Galois coverings and next we study some properties of this kind of coverings. The main results are Theorem 3.7, which describes the structure of Galois coverings and Theorem 3.10, which concerns morphisms between Galois coverings and the relation between the associated groups of automorphisms. We provide the definition of the universal covering in the category of Galois coverings of a fixed k-category $\mathcal{B}$. In a forthcoming paper we will study the behaviour of Galois coverings through fibre products, as well as a criterion for a covering to be Galois or universal. Differences with the usual algebraic topology setting will also appear, since the fibre product of coverings of k-categories does not provide in general a covering through the projection functor.

In Section 4 we define the intrinsic fundamental group $\pi_1(\mathcal{B}, b_0)$ and we prove some properties of this new object. If the universal covering exists, we prove that this group is isomorphic to the Galois group of the universal covering. In [8] we provide explicit computations of the intrinsic fundamental group of some algebras. In particular we compute the fundamental group of $M_p(k)$, where p is prime and k an algebraically closed field of characteristic zero, which is the direct product of the free group on $p - 1$ generators with the cyclic group of order p. The fundamental group of triangular matrix algebras is the free group on $n - 1$ generators. The fundamental group of the truncated polynomial algebra $k[x]/(x^p)$ in characteristic p is the product of the infinite cyclic group and the cyclic group of order p. In case k is a field containing all roots of unity of order 2 and 3, we prove that $\pi_1(k^3) = C_2 \times C_3$, while if k contains all roots of unity of order 3 and 4, we obtain that $\pi_1(k^4) = (C_2 * C_2) \times C_6 \times C_4 \times C_2$.

In section 4 we also show that if the universal covering exists the fibre functor induces an equivalence between the category of Galois coverings of $\mathcal{B}$ and the subcategory of $\pi_1(\mathcal{B}, b_0)$-Sets whose objects are sets with a transitive action of the group $\pi_1(\mathcal{B}, b_0)$ such that the isotropy group of an element is invariant.

In the last section we suppose that k is a field and that the endomorphism algebra of each object of the k-category is reduced to k. We recover in a simple

way the canonical k-linear embedding (see [1, 22, 7]) of the abelian characters of the automorphism group of a Galois covering to the first Hochschild-Mitchell cohomology vector space of the category. As an immediate consequence, if there exists a Galois covering whose group is isomorphic to the fundamental group, the abelian characters of the fundamental group embed into the cohomology of degree one. For this, we use our description of Galois coverings as well as the canonical grading of the k-category deduced from a Galois covering, as obtained in [6]. In this way Euler derivations are considered, see also [10, 11].

Note that it will be interesting to explore the behaviour of the intrinsic fundamental group with respect at least to Morita equivalences of k-categories. As expected the fundamental group is an invariant of the equivalence class of a k-category but not of its Morita class. Of course in case the category admits a unique basic representative in its Morita class, the fundamental group attached to this category can be considered as the canonical fundamental group of the Morita equivalence class.

2 k-categories, stars and coverings

Let k be a commutative ring. A k-*category* is a small category $\mathcal{B}$ such that each morphism set $_y\mathcal{B}_x$ from an object $x \in \mathcal{B}_0$ to an object $y \in \mathcal{B}_0$ is a k-module, the composition of morphisms is k-bilinear and k-multiples of the identity at each object are central in the endomorphism ring of the object. Note that such k-categories are also called *linear categories over* k. In particular each endomorphism set of an object is a k-algebra, and $_y\mathcal{B}_x$ is a $_y\mathcal{B}_y - {}_x\mathcal{B}_x$-bimodule.

Each k-algebra A provides a single object k-category $\mathcal{B}_A$ with endomorphism ring A. The structure of A can be described more precisely by choosing a finite set E of orthogonal idempotents of A, such that $\sum_{e \in A} e = 1$ in the following way: the k-category $\mathcal{B}_{A,E}$ has set of objects E and morphisms from e to f the k-module fAe. Note that $\mathcal{B}_{A,\{1\}} = \mathcal{B}_A$. This approach is meaningful since clearly the category of left A-modules is isomorphic to the category of k-functors from $\mathcal{B}_{A,E}$ to the category of k-modules, where a k-functor is a functor which is k-linear when restricted to morphisms.

Definition 2.1. *The star* $\mathsf{St}_{b_0}\mathcal{B}$ *of a k-category $\mathcal{B}$ at an object b_0 is the direct sum of all the morphisms with source or target b_0 :*

$$\mathsf{St}_{b_0}\mathcal{B} = \left(\bigoplus_{y \in \mathcal{B}_0} {}_y\mathcal{B}_{b_0} \right) \oplus \left(\bigoplus_{y \in \mathcal{B}_0} {}_{b_0}\mathcal{B}_y \right).$$

Note that this k-module counts twice the endomorphism algebra at b_0.

Definition 2.2. *Let $\mathcal{C}$ and $\mathcal{B}$ be k-categories. A k-functor $F : \mathcal{C} \to \mathcal{B}$ is a covering of $\mathcal{B}$ if it is surjective on objects and if F induces k-isomorphisms between the corresponding*

stars. More precisely, for each $b_o \in \mathcal{B}_0$ and each x in the non-empty fibre $F^{-1}(b_0)$, the map

$$F^x_{b_0} : \mathsf{St}_x \mathcal{C} \longrightarrow \mathsf{St}_{b_0} \mathcal{B}.$$
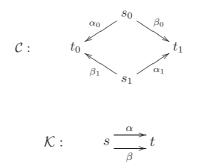
induced by F is a k-isomorphism.

Remark 2.3. *Each star is the direct sum of the source star $\mathsf{St}^-_{b_0}\mathcal{B} = \bigoplus_{y \in \mathcal{B}_0} {}_y\mathcal{B}_{b_0}$ and the target star $\mathsf{St}^+_{b_0}\mathcal{B} = \bigoplus_{y \in \mathcal{B}_0} {}_{b_0}\mathcal{B}_y$. Since St^- and St^+ are preserved under any k-functor, the condition of the definition is equivalent to the requirement that the corresponding target and source stars are isomorphic through F.*
Moreover this splitting goes further : for $b_1 \in \mathcal{B}_0$, the restriction of F to $\bigoplus_{y \in F^{-1}(b_1)} {}_y\mathcal{C}_x$, for all $x \in F^{-1}(b_0)$ is k-isomorphic to the corresponding k-module ${}_{b_1}\mathcal{B}_{b_0}$. The same holds with respect to the target star and morphisms starting at all objects in a single fibre.

Remark 2.4. *The previous facts show that Definition 2.2 coincides with the one given by K. Bongartz and P. Gabriel in [4].*

To each small category $\mathcal{A}$ one may associate its linearization $k\mathcal{A}$ in the following way: objects of $k\mathcal{A}$ are the objects of $\mathcal{A}$, while morphisms are free k-modules on the sets of morphisms of $\mathcal{A}$. Such linearized k-categories admit by construction a multiplicative basis of morphisms, which is not usually the case, see for instance [3]. Hence a k-category is not in general the linearization of a small category. Note that any usual covering $F : \mathcal{C} \to \mathcal{B}$ of categories provides by linearization a covering $kF : k\mathcal{C} \to k\mathcal{B}$.

Example 2.5. *[17] Consider the following k-categories $\mathcal{C}$ and $\mathcal{K}$ obtained by linearization of the categories given by the corresponding diagrams ($\mathcal{K}$ is called the Kronecker category):*

$$\mathcal{C}: \quad t_0 \overset{\alpha_0}{\underset{\beta_1}{\swarrow\, s_0 \,\searrow}} \quad \overset{\beta_0}{\underset{s_1}{\searrow\,\,\swarrow}} \quad \overset{\alpha_1}{\nearrow} t_1$$

$$\mathcal{K}: \quad s \underset{\beta}{\overset{\alpha}{\rightrightarrows}} t$$

and the three following coverings F_0, F_1 and $F_2 : \mathcal{C} \longrightarrow \mathcal{K}$ given by

$$F_i(s_0) = F_i(s_1) = s, \ \ F_i(t_0) = F_i(t_1) = t, \ \ F_i(\beta_0) = F_i(\beta_1) = \beta, \text{ for } i = 0, 1, 2$$

while

- $F_0(\alpha_0) = F_0(\alpha_1) = \alpha$
- $F_1(\alpha_0) = F_1(\alpha_1) = \alpha + \beta$
- $F_2(\alpha_0) = \alpha + \beta, \ F_2(\alpha_1) = \alpha$

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Note that F_0 is the linearization of a covering of small categories, and F_1 is F_0 followed by an automorphism of $\mathcal{K}$. We will observe that F_0 and F_1 are in fact Galois coverings (see Definition 3.1), while F_2 is not.

Definition 2.6. *A morphism from a covering $F : \mathcal{C} \to \mathcal{B}$ to a covering $G : \mathcal{D} \to \mathcal{B}$ is a pair of k-linear functors (H, J) where $H : \mathcal{C} \to \mathcal{D}$, $J : \mathcal{B} \to \mathcal{B}$ are such that J is an isomorphism, J is the identity on objects and $GH = JF$. The category of coverings of $\mathcal{B}$ is denoted $\mathsf{Cov}(\mathcal{B})$.*

Our next purpose is to show that the automorphism group of a connected covering acts freely on each fibre.

Definition 2.7. *A k-category $\mathcal{B}$ is connected if any two objects b and c of $\mathcal{B}$ can be linked by a finite walk made of non zero morphisms, more precisely there exist a finite sequence of objects $x_1, \ldots, x_n$ and non zero morphisms $\varphi_1, \ldots, \varphi_n$ such that $x_1 = b$, $x_n = c$, where φ_i belongs either to $_{x_{i+1}}\mathcal{B}_{x_i}$ or to $_{x_i}\mathcal{B}_{x_{i+1}}$.*

Proposition 2.8. *Let $F : \mathcal{C} \longrightarrow \mathcal{B}$ be a covering of k-categories. If $\mathcal{C}$ is connected, then $\mathcal{B}$ is connected.*

Proof. Let b and c be objects in $\mathcal{B}_0$, and let x_0 and y_0 be two objects respectively chosen in their fibres. Consider a walk of non zero morphisms connecting x_0 and y_0 in $\mathcal{C}$. Since F induces k-isomorphisms at each star, the image by F of a non zero morphism is a non zero morphism in $\mathcal{B}$. □

Proposition 2.9. *[17] Let $F : \mathcal{C} \longrightarrow \mathcal{B}$ and $G : \mathcal{D} \longrightarrow \mathcal{B}$ be coverings of k-linear categories. Assume $\mathcal{C}$ is connected. Two morphisms (H_1, J), (H_2, J) from F to G such that H_1 and H_2 coincide on some object are equal.*

Proof. Let (H, J) be a morphism of coverings, let x_0 be an object of $\mathcal{C}$ and consider the map between stars induced by H:

$$H_{H(x_0)}^{x_0} : \mathsf{St}_{x_0}\mathcal{C} \longrightarrow \mathsf{St}_{H(x_0)}\mathcal{D}.$$

Observe that $GH(x_0) = JF(x_0)$. There is a commutative diagram

$$
\begin{array}{ccc}
\mathsf{St}_{x_0}\mathcal{C} & \xrightarrow{\ H_{H(x_0)}^{x_0}\ } & \mathsf{St}_{H(x_0)}\mathcal{D} \\
{\scriptstyle F_{F(x_0)}^{x_0}}\downarrow & & \downarrow{\scriptstyle G_{G(H(x_0))}^{H(x_0)}} \\
\mathsf{St}_{F(x_0)}\mathcal{B} & \xrightarrow[\ J_{J(F(x_0))}^{F(x_0)}\ ]{} & \mathsf{St}_{JF(x_0)}\mathcal{B}
\end{array}
$$

where the morphisms $F_{F(x_0)}^{x_0}$, $J_{J(F(x_0))}^{F(x_0)}$ and $G_{G(H(x_0))}^{H(x_0)}$ are k-isomorphisms. Consequently H is an isomorphism at each star level, determined by F, J and G. In case H_1 and H_2 are morphisms such that $H_1(x_0) = H_2(x_0)$, the k-linear maps $\mathsf{St}_{x_0}\mathcal{C} \longrightarrow \mathsf{St}_{H_1(x_0)}\mathcal{D} = \mathsf{St}_{H_2(x_0)}\mathcal{D}$ induced by H_1 and H_2 are equal, hence H_1 and H_2 coincide on morphisms starting or ending at x_0, in particular H_1 and H_2 coincide on objects related to x_0 by a non-zero morphism. Since $\mathcal{C}$ is connected, it follows that H_1 and H_2 coincide on every object and on every morphism of $\mathcal{C}$. □

Corollary 2.10. *Let $F : \mathcal{C} \to \mathcal{B}$ be a connected covering of a k-linear category $\mathcal{B}$. The group $\mathrm{Aut}_1(F) = \{(H, 1) : F \to F \mid H$ an isomorphism of $\mathcal{C}\}$ acts freely on each fibre.*

3 Galois and universal coverings

We start this section with the definition of a Galois covering in order to study properties of this kind of coverings. The main results are the description of the structure of Galois coverings, and the relation between Galois coverings and the group of automorphisms. Finally we consider universal objects in the category of Galois coverings of a fixed k-category $\mathcal{B}$.

Definition 3.1. *A covering $F : \mathcal{C} \longrightarrow \mathcal{B}$ of k-categories is a* Galois covering *if $\mathcal{C}$ is connected and if $\mathrm{Aut}_1 F$ acts transitively on some fibre. We denote* $\mathrm{Gal}(\mathcal{B})$ *the full subcategory of* $\mathrm{Cov}(\mathcal{B})$ *whose objects are the Galois coverings of $\mathcal{B}$.*

It is natural to expect that $\mathrm{Aut}_1 F$ should act transitively on each fibre whenever it acts transitively on a particular one. In order to prove this fact, we shall use a construction introduced in [13, 4], see also [7].

Definition 3.2. *Let G be a group acting by k-isomorphisms on a k-category $\mathcal{C}$, such that the action on the objects is free, meaning that if $sx = x$ for some object, then $s = 1$. The set of objects of the categorical quotient $\mathcal{C}/G$ is the set of G-orbits of $\mathcal{C}_0$. The k-module of morphisms from an orbit α to an orbit β is*

$$_\beta (\mathcal{C}/G)_\alpha = \left(\bigoplus_{x \in \alpha, \, y \in \beta} {}_y\mathcal{C}_x \right) /G$$

where for a kG-module X we denote X/G the k-module of coinvariants $X/(\ker \epsilon)X$, which is the quotient of X by the augmentation ideal, where $\epsilon : kG \to k$ is given by $\epsilon(s) = 1$ for all $s \in G$.

Remark 3.3. *The previous definition provides a k-category: the composition is well defined precisely because the action of G on the objects is free.*

Proposition 3.4. *Let G be a group acting by k-isomorphisms on a connected k-category $\mathcal{C}$, and assume that the action on the objects is free. Then the projection functor $P : \mathcal{C} \longrightarrow \mathcal{C}/G$ is a Galois covering with $\mathrm{Aut}_1(P) = G$.*

Proof. The projection functor is a covering since it is surjective on objects. For each choice of an object $x_0 \in \alpha$ and $y_0 \in \beta$ we clearly have k-isomorphisms

$$\bigoplus_{y \in \beta} {}_y\mathcal{C}_{x_0} \to {}_\beta (\mathcal{C}/G)_\alpha \quad \text{and} \quad \bigoplus_{x \in \alpha} {}_{y_0}\mathcal{C}_x \to {}_\beta (\mathcal{C}/G)_\alpha$$

which can be assembled in order to provide the required isomorphism of stars. Observe that the fibres of P are the orbits sets by construction, therefore the action of $\mathrm{Aut}_1 P = G$ is transitive on each fibre.

Consider now $(H, 1) \in \mathrm{Aut}_1 P$ and let $x_0 \in \mathcal{C}_0$. Since the action of G on $\mathcal{C}_0$ is free, there exists a unique $s \in G$ such that $sx_0 = H(x_0)$. By definition, the element s provides an isomorphism of $\mathcal{C}$ such that $Ps = P$. The isomorphisms $(s, 1), (H, 1)$ of the connected covering P coincide on an object, consequently they are equal as isomorphisms of P by Proposition 2.9. $\square$

Lemma 3.5. *Let $F : \mathcal{C} \longrightarrow \mathcal{B}$ be a connected covering of k-categories and suppose there exists a singleton fibre. Then every fibre is a singleton and F is an isomorphism of k-categories.*

Proof. Let $b \in \mathcal{B}_0$ be an object such that $F^{-1}(b) = \{x\}$. Since $\mathcal{C}$ is connected it is enough to show that for a non zero morphism in $\mathcal{C}$ with target or source x, the other extreme object y is such that $F^{-1}(F(y)) = \{y\}$. We denote $c = F(y)$. Assume $\varphi \in {}_y\mathcal{C}_x$ is non-zero and let $y' \in F^{-1}(c)$, then $\varphi \in \mathrm{St}_x\mathcal{C}$ and $F(\varphi) \in \mathrm{St}_b\mathcal{B}$. Moreover, $F(\varphi)$ belongs to $\mathrm{St}_c\mathcal{B}$. Since F induces an isomorphism $F_c^{y'} : \mathrm{St}_{y'}\mathcal{C} \longrightarrow \mathrm{St}_c\mathcal{B}$, there is a unique k-linear combination $\sum_z {}_{y'}h_z$ of morphisms from the fibre of b to y' such that $F_c^{y'}(\sum_z {}_{y'}h_z) = F(\varphi)$. Now the fibre of b is reduced to x, which means that there is a non-zero morphism $\psi \in {}_{y'}\mathcal{C}_x$ such that $F\psi = F\varphi$. Note that ψ also belongs to $\mathrm{St}_x\mathcal{C}$, and recall that F_b^x is an isomorphism between the corresponding stars. Hence $\varphi = \psi$ and in particular their ending objects are the same, namely $y = y'$. Finally since all the fibres are singletons, the star property of a covering implies immediately that F is an isomorphism. $\qquad\square$

We are now able to prove the following result.

Proposition 3.6. *Let $F : \mathcal{C} \longrightarrow \mathcal{B}$ be a Galois covering. Then $\mathrm{Aut}_1 F$ acts transitively on each fibre.*

Proof. First consider the categorical quotient $P : \mathcal{C} \longrightarrow \mathcal{C}/\mathrm{Aut}_1 F$. There is a unique functor $F' : \mathcal{C}/\mathrm{Aut}_1 F \longrightarrow \mathcal{B}$ such that $F'P = F$, defined as follows: let α be an object of $\mathcal{C}/\mathrm{Aut}_1 F$, that is, an orbit of $\mathcal{C}_0$ under the action of $\mathrm{Aut}_1 F$. Choose an object $x \in \alpha$ and define $F'\alpha = Fx$. Clearly F' is well defined on objects. In order to define F' on morphisms, let α and β be objects in $\mathcal{C}/\mathrm{Aut}_1 F$, and recall that

$$_\beta \left(\mathcal{C}/\mathrm{Aut}_1 F\right)_\alpha = \left(\bigoplus_{x \in \alpha, \, y \in \beta} {}_y\mathcal{C}_x \right) /\mathrm{Aut}_1 F.$$

Next observe that the morphism

$$F : \bigoplus_{x \in \alpha, \, y \in \beta} {}_y\mathcal{C}_x \to {}_{F'(\beta)}\mathcal{B}_{F'(\alpha)}$$

is sucht that $F(s\varphi) = F(\varphi)$ for any $\varphi \in \bigoplus_{x \in \alpha, \, y \in \beta} {}_y\mathcal{C}_x$ and any $s \in \mathrm{Aut}_1 F$. Finally the commutative triangle of morphisms between corresponding stars shows that F' is indeed a covering.

Since F is a Galois covering, there exists a fibre where the action of $\mathrm{Aut}_1 F$ is transitive, which means that the corresponding fibre of F' is a singleton. Since F is a Galois covering, $\mathcal{C}$ is connected as well as $\mathcal{C}/\mathrm{Aut}_1 F$ by Proposition 2.8. The preceding Lemma asserts that all the fibres of F' are singletons, which exactly means that the action of $\mathrm{Aut}_1 F$ is transitive on each fibre of F. $\qquad\square$

As a consequence we obtain the following description of Galois coverings.

Theorem 3.7. *Let $F : \mathcal{C} \longrightarrow \mathcal{B}$ be a Galois covering. Then there exists a unique isomorphism of categories $F' : \mathcal{C}/\mathrm{Aut}_1 F \longrightarrow \mathcal{B}$ such that $F'P = F$, where $P : \mathcal{C} \longrightarrow \mathcal{C}/\mathrm{Aut}_1 F$ is the Galois covering given by the categorical quotient.*

Proof. The proof of the preceding results provides the covering F', which has a singleton fibre. Then all the fibres of F' are singletons and F' is an isomorphism. $\square$

Example 3.8. *[17, 19] An easy computation shows that* $\mathrm{Aut}_1(F_2)$ *is trivial for the covering* F_2 *in Example 2.5. However each fibre has two objects, hence the action of the trivial group is not transitive on the fibres, consequently* F_2 *is not Galois. Observe that* F_0 *and* F_1 *are Galois coverings.*

Next we recall a result of Patrick Le Meur concerning factorizations of Galois coverings.

Lemma 3.9. *Let* $F : \mathcal{C} \longrightarrow \mathcal{B}$ *and* $G : \mathcal{D} \longrightarrow \mathcal{B}$ *be Galois coverings, and let* (H, J) *be a morphism from* F *to* G. *Then* H *is surjective on objects.*

Proof. Let $H(c) = d \in \mathcal{D}_0$ be an object which is in the image of H. First we prove that any object d' linked to d by a non-zero morphism is also in the image of H. Let for instance $0 \neq f \in {}_{d'}\mathcal{D}_d$. Note that GH is a covering since $JF = GH$. Considering $G(f)$, there exists a finite set of morphisms (f_i) starting at c and ending at objects x_i such that $GH(\sum f_i) = G(f)$, hence $G(\sum H(f_i)) = G(f)$. Note that $H(f_i)$ is a morphism from d to $H(x_i)$. Since G is a covering and f is a morphism starting at d, we infer $f = \sum H(f_i)$. This implies that all the $H(x_i)$ coincide with d', hence d' is in the image of H.

Finally, using that $\mathcal{D}$ is connected we conclude that any object of $\mathcal{D}$ is in the image of H. $\square$

Theorem 3.10. *[17, 19] Let* $F : \mathcal{C} \longrightarrow \mathcal{B}$ *and* $G : \mathcal{D} \longrightarrow \mathcal{B}$ *be Galois coverings, and let* (H, J) *be a morphism from* F *to* G. *Then there is a unique surjective group morphism* $\Lambda : \mathrm{Aut}_1 F \to \mathrm{Aut}_1 G$ *with* $\Lambda(f, 1) = (\lambda(f), 1)$ *such that* $\lambda(f)H = Hf$ *for each* $(f, 1) \in \mathrm{Aut}_1 F$. *Moreover* $\ker\Lambda = \mathrm{Aut}_1 H$ *and* H *is a Galois covering.*

Proof. Given J, we assert that the set of morphisms (H', J) from F to G is in one-to-one correspondence with $\mathrm{Aut}_1 G$ through the map which assigns (gH, J) to each $g \in \mathrm{Aut}_1 G$. Firstly each (gH, J) is a morphism from F to G. Secondly if (H', J) is such a morphism, given an object c_0 of $\mathcal{C}$, both $H'(c_0)$ and $H(c_0)$ are in the same G-fibre. Since the action of $\mathrm{Aut}_1 G$ is free and transitive on the fibres, there exists a unique $g \in \mathrm{Aut}_1 G$ such that $gH(c_0) = H'(c_0)$. Consequently (H', J) and (gH, J) are equal by Proposition 2.9.

Then for each $(f, 1) \in \mathrm{Aut}_1 F$ there exists a unique element $\Lambda(f, 1) \in \mathrm{Aut}_1 G$ such that $\lambda(f)H = Hf$. The uniqueness of $\Lambda(f)$ and the equalities

$$\lambda(f_1 f_2)H = Hf_1 f_2 = \lambda(f_1)Hf_2 = \lambda(f_1)\lambda(f_2)H$$

imply that Λ is a group morphism.

Moreover Λ is surjective. Let $(g, 1) \in \mathrm{Aut}_1 G$. Using the previous Lemma consider an object c in the H-fibre of gHc_0. A simple computation shows that c and c_0 are in the same F-fibre. Since F is a Galois covering, there exists $(f, 1) \in \mathrm{Aut}_1 F$ such that $fc_0 = c$, then $Hfc_0 = Hc = gHc_0$. Then $Hf = gH$ and $\lambda(f) = g$.

Note that $(f, 1) \in \ker\Lambda$ if and only if $Hf = H$ which means precisely that $(f, 1) \in \mathrm{Aut}_1 H$.
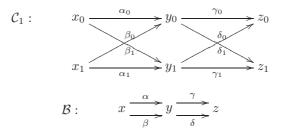
In order to prove that H is a Galois covering, we already know that H is surjective on objects. The functor H induces isomorphisms between stars since $GH = JF$, hence the same equality is valid at the stars level where F, J and G induce k-isomorphisms. This proves that H is a covering. In order to show that H is Galois, let x and x' be in the same H-fibre. They are also in the same F-fibre, hence there exists $(f, 1) \in \mathrm{Aut}_1 F$ such that $fx = x'$. We assert that in fact $(f, 1) \in \mathrm{Aut}_1 H$: indeed, (Hf, J) and (H, J) are both morphisms from F to G with the same value on x, hence they are equal by Proposition 2.9. $\qquad\square$

Remark 3.11. *Two isomorphic k-linear categories have isomorphic categories of Galois coverings.*

Definition 3.12. *A universal covering $U : \mathcal{U} \to \mathcal{B}$ is an object in $\mathrm{Gal}(\mathcal{B})$ such that for any Galois covering $F : \mathcal{C} \to \mathcal{B}$, and for any $u_0 \in \mathcal{U}_0$, $c_0 \in \mathcal{C}_0$ with $U(u_0) = F(c_0)$, there exists a unique morphism $(H, 1)$ from U to F such that $H(u_0) = c_0$.*

In case of existence, a universal covering is unique up to isomorphisms of Galois coverings. In general universal coverings do not exist, as the following Example shows. It has been obtained by Geiss and de la Peña in [14]:

Example 3.13. *Let k be a field and $char(k) = 2$. Consider the k-linear categories*

with $\mathcal{C}_1$ satisfying all commutativity relations and $\mathcal{B}$ satisfying the relations

$$\gamma\alpha = \delta\beta, \qquad \gamma\beta = \delta\alpha.$$

It is clear that $\mathcal{C}_1$ is a Galois covering of $\mathcal{B}$. Since $\mathrm{char}(k) = 2$, if we set $a = \alpha + \beta, b = \beta, c = \gamma + \delta, d = \delta$, we get that $\mathcal{B}$ satisfies the relations

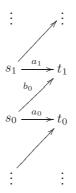$$ca = 0, \qquad cb = da.$$

In this case,

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with all commutativity relations and $c_i a_{i-1} = 0$, is also a Galois covering of $\mathcal{B}$. Now $\mathcal{C}_1$ and $\mathcal{C}_2$ admit no proper Galois covering since they are simply connected, see [20, 2], and there is no morphism between them.

Now we will study the Kronecker category $\mathcal{K}$. Recall that this category is given by two objects s, t, one-dimensional morphism spaces $_s\mathcal{K}_s$ and $_t\mathcal{K}_t$ while $\dim_k {}_t\mathcal{K}_s = 2$ and $_s\mathcal{K}_t = 0$. Observe that for each choice of a vector basis of $_t\mathcal{K}_s$, the category $\mathcal{K}$ is presented by the quiver

$$ s \underset{\beta}{\overset{\alpha}{\rightrightarrows}} t $$

We start with a description of all Galois coverings of the category $\mathcal{K}$ up to isomorphisms.

Let $\{a, b\}$ be a basis of $_t\mathcal{K}_s$. Let $\mathcal{C}_{\{a,b\}}$ be the free k-category presented by the quiver

and let $F_{\{a,b\}} : \mathcal{C}_{\{a,b\}} \to \mathcal{K}$ be given by

- $F_{\{a,b\}}(s_i) = s$,

- $F_{\{a,b\}}(t_i) = t$,

- $F_{\{a,b\}}(a_i) = a$,

- $F_{\{a,b\}}(b_i) = b$.

Proposition 3.14. *Let $\{a, b\}$ be a fix chosen basis of $_t\mathcal{K}_s$. Then any Galois covering of $\mathcal{K}$ is isomorphic to $F_{\{a,b\}}$ if it has infinite Galois group, and to a quotient of it otherwise.*

Proof. It can be seen that any Galois covering of $\mathcal{K}$ is isomorphic to $F_{\{c,d\}}$ or a quotient of it, where $\{c, d\}$ is a basis of $_t\mathcal{K}_s$. Now $F_{\{c,d\}} \simeq F_{\{a,b\}}$ with an isomorphism of type $(1, J)$. $\qquad\square$

Note that the Kronecker category has no universal covering since the Definition of a universal covering only takes into account morphisms of type $(H, 1)$.

4   Fundamental group

As quoted in the Introduction, our main purpose is to provide an intrinsic definition of the fundamental group π_1 of a k-category, where k is a commutative ring. Previous definitions, provided for instance by J. A. de la Peña and R. Martínez-Villa [20], see also K. Bongartz and P. Gabriel [4], depend on the presentation of the category as a quotient of a free k-category by an ideal generated by some set of minimal relations. Different presentations of the same k-category may provide different groups through this construction, see for instance [1, 5, 18].

We will prove the following fact concerning the group π_1 that we will define: if the universal covering U exists, then the group $\mathrm{Aut}_1 U$ is isomorphic to π_1; in this case any group obtained through the presentation construction is a quotient of π_1.

Definition 4.1. *Let $\mathcal{B}$ be a k-category, and let b_0 be a fixed object in $\mathcal{B}_0$. Consider $\mathsf{Gal}(\mathcal{B}, b_0)$ the subcategory of $\mathsf{Gal}\mathcal{B}$ with the same objects and morphisms (H, J) with $J(b_0) = b_0$. Let $\Phi : \mathsf{Gal}(\mathcal{B}, b_0) \to \mathsf{Sets}$ be the fibre functor which associates to each Galois covering F the F-fibre $F^{-1}(b_0)$. We define*

$$\pi_1(\mathcal{B}, b_0) = \mathbf{Aut}\Phi.$$

Remark 4.2. *This fundamental group $\pi_1(\mathcal{B}, b_0)$ is the group of natural isomorphisms $\sigma : \Phi \to \Phi$. In other words an element of the fundamental group is a family of invertible set maps $\sigma_F : F^{-1}(b_0) \to F^{-1}(b_0)$ for each Galois covering F, which are compatible with morphisms of Galois coverings; namely for each morphism $(H, J) : F \to G$ in $\mathsf{Gal}(\mathcal{B}, b_0)$ the corresponding square*

$$
\begin{array}{ccc}
F^{-1}(b_0) & \xrightarrow{\ \sigma_F\ } & F^{-1}(b_0) \\
\Big\downarrow{\scriptstyle H} & & \Big\downarrow{\scriptstyle H} \\
G^{-1}(b_0) & \xrightarrow{\ \sigma_G\ } & G^{-1}(b_0)
\end{array}
$$

is commutative.

In case the universal covering exists, our purpose is to prove that the fundamental group is isomorphic to its automorphism group.

Proposition 4.3. *Let $\mathcal{B}$ be a connected k-category and let b_0 be an object. Assume there exists a universal covering U. Then $\pi_1(\mathcal{B}, b_0)$ acts freely and transitively on $U^{-1}(b_0)$.*

Proof. Let $\sigma \in \pi_1(\mathcal{B}, b_0)$, we define the action by

$$\sigma u = \sigma_U(u).$$

This is an action by the definition of composition of automorphisms of the fibre functor.

Let us first prove that the action is free. Assume $\sigma u = u$. Let F be a Galois covering and let c be an object in $F^{-1}(b_0)$. Consider the unique morphism $(H, 1)$ from U to F such that $Hu = c$. Using Remark 4.2 we obtain $\sigma_F(c) = c$.

In order to prove transitivity, let u and u' be objects in $U^{-1}(b_0)$. We are going to define an automorphism σ of Φ such that $\sigma_U(u) = u'$. Let F be a Galois covering and c some element in $F^{-1}(b_0)$, and let $(H, 1)$ be the unique morphism such that $Hu = c$. We define

$$\sigma_F(c) = H(u').$$

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Using the uniqueness of the morphisms starting at the universal covering, one can prove that the family (σ_F) is indeed an automorphism of Φ. $\square$

Proposition 4.4. *The actions of $\pi_1(\mathcal{B}, b_0)$ and $\mathrm{Aut}_1(U)$ on $U^{-1}(b_0)$ commute.*

Proof. This is an immediate consequence of Remark 4.2. $\square$

We recall that an anti-morphism $\varphi : G \to G'$ is a map such that $\varphi(g_1 g_2) = \varphi(g_2)\varphi(g_1)$ for any elements g_1 and g_2 in G. Of, course an anti-morphism φ provides a unique usual group morphism ψ given by $\psi(g) = \varphi(g^{-1})$.

Lemma 4.5. *Let G and G' be groups acting freely and transitively on a non empty set X. Assume the actions commute. Then each choice of an element in X determines an anti-isomorphism from G to G'.*

Proof. Choose an element $x \in X$. Define $\varphi : G \to G'$ by $gx = \varphi(g)x$. This map is well defined and bijective. Moreover it is a group anti-morphism precisely because the actions commute. $\square$

Theorem 4.6. *Suppose that a connected k-category $\mathcal{B}$ admits a universal covering U. Then*
$$\pi_1(\mathcal{B}, b_0) \simeq \mathrm{Aut}_1 U.$$

Corollary 4.7. *Let $\mathcal{B}$ be a connected k-category admitting a universal covering U and let b_0 and b_1 be two objects. Then $\pi_1(\mathcal{B}, b_0)$ and $\pi_1(\mathcal{B}, b_1)$ are isomorphic.*

Corollary 4.8. *Let $\mathcal{B}$ be a k-category admitting a universal covering, and consider a presentation of $\mathcal{B}$ given by a quiver Q and an admissible two-sided ideal I provided with a minimal set of generators R given by parallel paths. Let $\pi_1(Q, R, b_0)$ be the group of the presentation as defined in [20], with respect to a vertex b_0. Then there is a group surjection $\pi_1(\mathcal{B}, b_0) \longrightarrow \pi_1(Q, R, b_0)$.*

Proof. In [20] it is proven that the group $\pi_1(Q, R, b_0)$ can be realized as $\mathrm{Aut}_1 F$ for a Galois covering F. Then the universal covering U of $\mathcal{B}$ provides an epimorphism from $\mathrm{Aut}_1 U$ to $\pi_1(Q, R, b_0)$. $\square$

For any group Γ, let $\Sigma_l(\Gamma)$ be the full subcategory of the category of left Γ-sets whose objects are sets with a transitive left action of the group Γ such that the isotropy group of an element is invariant. Note that in this case the isotropy group of any element is invariant, since the action is transitive. $\Sigma_r(\Gamma)$ denotes the analogous category, where objects are right Γ-sets.

We will prove that the fibre functor Φ is an equivalence when considered as a functor from $\mathsf{Gal}_1(\mathcal{B}, b_0)$ to the category $\Sigma_l(\pi_1(\mathcal{B}, b_0))$, where $\mathsf{Gal}_1(\mathcal{B}, b_0)$ is the subcategory of $\mathsf{Gal}(\mathcal{B}, b_0)$ with same objects and morphisms of type $(H, 1)$.

Proposition 4.9. *Let $\mathcal{B}$ be a k-category admitting a universal covering $U : \mathcal{U} \to \mathcal{B}$. Let*
$$S : \mathsf{Gal}_1(\mathcal{B}, b_0) \to \Sigma_r(\mathrm{Aut}_1 U)$$
be the functor given by $S(F) = \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B}, b_0)}(U, F)$ and defined by composition on morphisms. Then S is an equivalence.

Proof. First we assert that the action of $\mathrm{Aut}_1(U)$ on $\mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$ is transitive. Let $X,X' \in \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$, let $t_0 \in \mathcal{U}_0$ and let $t_1 \in \mathcal{U}_0$ such that $X(t_0) = X'(t_1)$. Let g be the unique endomorphism of U such that $g(t_1) = t_0$, which exists since U is universal. Moreover, g belongs to $\mathrm{Aut}_1 U$ with inverse given by the endomorphism g' sending t_1 to t_0, and $Xg = X'$ since they coincide on t_1. Observe that in general the action is not free since t_1 is not uniquely determined.

Moreover this action has invariant isotropy group: let $X \in \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$ and $g \in \mathrm{Aut}_1 U$ such that $Xg = X$. For any $h \in \mathrm{Aut}_1 U$, using Theorem 3.10, we have
$$Xhgh^{-1} = \lambda_X(h)Xgh^{-1} = \lambda_X(h)Xh^{-1} = Xhh^{-1} = X.$$

Next we prove that S is faithful. Let F and G be Galois coverings, and let $H, H' \in \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(F,G)$ such that $S(H) = S(H')$, that is, $HX = H'X$ for any $X \in \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$, a non-empty set. Then H and H' coincide on some object, and hence they are equal.

In order to prove that S is full, let $f : \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F) \to \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,G)$ be an $\mathrm{Aut}_1 U$-morphism. We are looking for a morphism $H \in \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(F,G)$ such that $S(H) = f$, that is, $f(X) = HX$ for any $X \in \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$. Given X let H be the unique morphism from F to G such that $H(X(t_0)) = f(X)(t_0)$. It is clear that $f(X) = HX$ since they coincide on some object. For any other X', we know that there exists $g \in \mathrm{Aut}_1 U$ such that $Xg = X'$. Then

$$f(X')(t_0) = f(Xg)(t_0) = f(X)g(t_0) = f(X)(gt_0)$$

$$= HX(gt_0) = HXg(t_0) = HX'(t_0)$$

and hence $f(X') = HX'$.

Finally we prove that S is dense. Let $E \in \Sigma_r(\mathrm{Aut}_1 U)$, I the isotropy group of any element in E. Let $P : \mathcal{U} \to \mathcal{U}/I$ be the projection functor, and let $F : \mathcal{U}/I \to \mathcal{B}$ as constructed in the proof of Proposition 3.6. We define

$$\mathrm{Aut}_1 U \ /I \to \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$$

given by $\overline{g} \mapsto Pg$. This map is well-defined and injective. In order to prove that it is surjective, let $H \in \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$. Since $\mathrm{Aut}_1 U$ acts transitively on $\mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$, there exists $g \in \mathrm{Aut}_1 U$ such that $Pg = H$. On the other hand, the map $\mathrm{Aut}_1 U/I \to E$ given by $\overline{g} \mapsto ge$ is an isomorphism in $\Sigma_r(\mathrm{Aut}_1 U)$. $\qquad\square$

Theorem 4.10. *Let $\mathcal{B}$ be a k-category admitting a universal covering $U : \mathcal{U} \to \mathcal{B}$. Then the fibre functor Φ is an equivalence when considered as a functor from $\mathsf{Gal}_1(\mathcal{B},b_0)$ to the category $\Sigma_l(\pi_1(\mathcal{B},b_0))$.*

Proof. From the previous Proposition, it is enough to see that Φ and S are naturally isomorphic.

Let t_0 be a fixed element in $U^{-1}(b_0)$ and let $\gamma : \pi_1(\mathcal{B},b_0) \to \mathrm{Aut}_1(U)$ be the anti-isomorphism defined by the equality $\sigma_U(t_0) = \gamma(\sigma)(t_0)$ (see Lemma 4.5). The functor $\Sigma(\gamma)$ induced by γ is a natural isomorphism from $\Sigma_r(\mathbf{Aut}_1 U)$ to $\Sigma_l(\pi_1(\mathcal{B},b_0))$.

Given $F \in \mathsf{Gal}_1(\mathcal{B},b_0)$ one must prove that $\Sigma(\gamma)(\mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F))$ is naturally isomorphic to $F^{-1}(b_0)$ as left π_1-sets. Let e be the map defined by $e(X) = X(t_0)$ for $X \in \mathrm{Hom}_{\mathsf{Gal}_1(\mathcal{B},b_0)}(U,F)$. Since U is universal, e is a bijection which is

clearly natural. We assert that e commutes with the left action of $\pi_1(\mathcal{B}, b_0)$. Let $\sigma \in \pi_1(\mathcal{B}, b_0)$, then

$$e(\sigma X) = e(X\gamma(\sigma)) = (X\gamma(\sigma))(t_0) = X(\gamma(\sigma)(t_0)) = X\sigma_U(t_0).$$

On the other hand

$$\sigma e(X) = \sigma X(t_0) = \sigma_F(X(t_0)).$$

Both elements are equal by means of Remark 4.5. $\qquad\square$

5   First Hochschild cohomology and Galois groups

In this section our main purpose is to provide a canonical embedding from the additive characters of the intrinsic π_1 that we have defined to the first Hochschild-Mitchell cohomology vector space of $\mathcal{B}$. This will be achieved in case k is a field and assuming that the endomorphism ring of each object of the category is reduced to k, and that there exists a Galois covering whose group is isomorphic to the fundamental group (for instance if there exists a universal covering).

First we will provide an intrinsic and direct way of describing the injective morphism from the additive characters of the group of automorphisms of a Galois covering to the first Hochschild-Mitchell cohomology vector space of $\mathcal{B}$.

Assem and de la Peña have described this map in [1] when G is the fundamental group of a triangular finite dimensional algebra presented by a quiver with relations. In [22] de la Peña and Saorín noticed that the triangular hypothesis is superfluous. This map has also been obtained in a spectral sequence context in [7].

In order to provide the canonical morphism, we first recall (see [16, 6]) that a Galois covering of $\mathcal{B}$ provides a grading for each choice of objects in the fibres. As expected another choice of objects provides a conjugated grading. We will translate in this setting the connectivity hypothesis of the Galois covering. Finally the definition of Hochschild-Mitchell derivations as well as of the inner ones will provide the context for a natural definition of the required map.

Definition 5.1. *Let $\mathcal{B}$ be a k-category, where k is a ring. A G- grading Z of $\mathcal{B}$ by a group G is a decomposition of each k-module of morphisms as a direct sum of k-modules Z_s indexed by G. For each couple of objects b and c we have ${}_c\mathcal{B}_b = \oplus_{s \in G} Z_s ({}_c\mathcal{B}_b)$ and*

$$Z_t ({}_d\mathcal{B}_c) Z_s ({}_c\mathcal{B}_b) \subset Z_{ts} ({}_d\mathcal{B}_b),$$

where elements of $Z_s ({}_c\mathcal{B}_b)$ are called homogeneous morphisms of degree s from b to c.

The following result is clear:

Proposition 5.2. *Let $\mathcal{B}$ be a k-category as above, equipped with a G-grading Z. Let $(t_b)_{b \in \mathcal{B}_0}$ be a family of elements of G associated to the objects of $\mathcal{B}$. Define $Y_s ({}_c\mathcal{B}_b) = Z_{t_c s t_b^{-1}} ({}_c\mathcal{B}_b)$. Then Y is also a G-grading of $\mathcal{B}$.*

Theorem 5.3. *[6] Let $\mathcal{C} \to \mathcal{B}$ be a Galois covering of categories. Let $(x_b)_{b \in \mathcal{B}_0}$ be a choice of objects of $\mathcal{C}$, where x_b belongs to the F-fibre of b for each $b \in \mathcal{B}_0$. Then there is an $\mathrm{Aut}_1 F$-grading of $\mathcal{B}$. Another choice of fibre objects provides a grading with the same homogeneous components as described in the preceding Proposition.*

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Proof. Since F is a covering, each morphism space of $\mathcal{B}$ is equipped with a direct sum decomposition $\oplus_{y \in F^{-1}c} F\left({}_y\mathcal{C}_{x_b}\right)$. Moreover since F is Galois, for each y in the F-fibre of c there exists a unique automorphism s such that $y = sx_c$. This element s will provide the degree of the direct summand, more precisely

$$Z_s\left({}_c\mathcal{B}_b\right) = F\left({}_{sx_c}\mathcal{C}_{x_b}\right).$$

It is straightforward to check that this is indeed an $\mathrm{Aut}_1 F$-grading. Moreover, a different choice of objects in the fibres $(t_b x_b)_{b \in \mathcal{B}_0}$ where $(t_b)_{b \in \mathcal{B}_0}$ is a family of elements of $\mathrm{Aut}_1 F$ provides another grading, which is precisely the grading described in the proposition above. $\qquad\square$

Remark 5.4. *In [6] the converse is obtained: in case $\mathcal{B}$ is G-graded where G is an arbitrary group, the smash product category construction provides a Galois covering with automorphism group G. As expected, one recovers the original grading as the one induced by the smash Galois covering as defined in [6].*

Definition 5.5. *A homogeneous walk w in a G-graded k-category from an object b to an object c is a sequence of non zero homogeneous paths. It consists of a sequence of objects $x_1 = b, \dots, x_i \dots, x_n = c$, a sequence of signs $\epsilon_1, \dots, \epsilon_n$ where $\epsilon_i \in \{-1, +1\}$, and non-zero homogeneous morphisms $\varphi_1, \dots, \varphi_n$ such that if $\epsilon_i = 1$ then $\varphi_i \in {}_{x_{i+1}}\mathcal{B}_{x_i}$ while if $\epsilon_i = -1$ then $\varphi_i \in {}_{x_i}\mathcal{B}_{x_{i+1}}$. The* degree *of w is the following ordered product of elements of G:*

$$\deg w = (\deg \varphi_n)^{\epsilon_n} \cdots (\deg \varphi_i)^{\epsilon_i} \cdots (\deg \varphi_1)^{\epsilon_1}$$

Remark 5.6. *Note that a homogeneous non zero endomorphism involved in a homogeneous walk at position i appears with its degree, or the inverse of its degree, according to the value of ϵ_i.*

Definition 5.7. *Let $\mathcal{B}$ be a G-graded k-category. The grading is called* connected *if for any couple of objects b and c of $\mathcal{B}$ and for any element $s \in G$, there exists a homogeneous walk from b to c of degree s.*

Theorem 5.8. *Let $F : \mathcal{C} \to \mathcal{B}$ be a Galois covering. Then the induced grading on $\mathcal{B}$ is connected.*

Proof. Let (x_b) be a choice of an object in each F-fibre, providing a grading of $\mathcal{B}$. Note that a morphism φ in $\mathcal{C}$ from sx_b to tx_c has homogeneous image of degree $s^{-1}t$, since $F\varphi = Fs^{-1}\varphi$. This observation shows that a walk in $\mathcal{C}$ from x_b to some sx_d projects to a homogeneous walk from b to d of degree s. Since $\mathcal{C}$ is connected, the theorem is proved. $\qquad\square$

Let us briefly recall the definition of Hochschild-Mitchell cohomology in degree one (see for instance [21]). This cohomology coincides with usual Hochschild cohomology of algebras in case the k-category has a finite number of objects.

Definition 5.9. *Let $\mathcal{B}$ be a k-category. A derivation of $\mathcal{B}$ is a collection $D = \left({}_c D_b\right)_{c,b \in \mathcal{D}_0}$ of k-linear endomorphisms of each k-module of morphisms ${}_c\mathcal{B}_b$, such that $D(gf) = gD(f) + D(g)f$. More precisely if ${}_d g_c$ and ${}_c f_b$ are morphisms of $\mathcal{B}$, then*

$${}_d D_b(gf) = g_c D_b(f) + {}_d D_b(g) f.$$

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An inner derivation D_α associated to a collection α of endomorphisms at each object $(\alpha_b)_{a \in \mathcal{B}_0}$ is obtained in the usual way, namely

$$D_\alpha \left({}_c f_b \right) = \alpha_c f - f \alpha_b.$$

The first Hochschild cohomology k-module $H^1(\mathcal{B}, \mathcal{B})$ is the quotient of the k-module of derivations by the inner ones.

Theorem 5.10. *Let k be a field and let $\mathcal{B}$ be a k-category such that the endomorphism ring of each object is reduced to k. Let $F : \mathcal{C} \to \mathcal{B}$ be a Galois covering. Then there exists a canonical injective morphism*

$$\Delta : \mathrm{Hom}(\mathrm{Aut}_1 F, k^+) \longrightarrow H^1(\mathcal{B}, \mathcal{B}).$$

Proof. Let $\chi : \mathrm{Aut}_1 F \to k^+$ be an abelian character of $\mathrm{Aut}_1 F$. In order to define $\Delta \chi$ as a k-endomorphism of each k-module of morphisms of $\mathcal{B}$, we define Δ_χ on the homogeneous components of a grading induced by the covering F. Let f be a morphism of degree s. By definition $\Delta \chi(f) = \chi(s) f$.

A standard computation shows that $\Delta \chi$ is indeed a derivation, which corresponds to the well known construction of Euler derivations, see also [10, 11]. Moreover if the grading is changed through a different choice of objects in the fibres according to Theorem 5.3, the derivation $\Delta \chi$ is modified by an inner derivation, hence the morphism Δ is canonic.

Assume $\Delta \chi$ is an inner derivation. Let f be a non zero homogeneous morphism of degree s, then $\chi(s) f = \alpha_c f - f \alpha_b = (\alpha_c - \alpha_b) f$ since $\alpha_b \in k$ for each object b. Since k is a field and f is a non zero element of a vector space, $\chi(s) = \alpha_c - \alpha_b$. Moreover if w is a homogeneous walk of degree s from b to c, we also have $\chi(s) = \alpha_c - \alpha_b$ by an easy computation. Finally since F is Galois, we know by Theorem 5.8 that for each automorphism s there exists a homogeneous walk from an object b to itself of degree s. Consequently $\chi(s) = 0$ for every s and Δ is injective. $\square$

We have already observed that the Kronecker category $\mathcal{K}$ does not admit a universal covering, nevertheless there exist Galois coverings of $\mathcal{K}$ such that their group of automorphisms are isomorphic to the fundamental group. This observation provides interest to the following result.

Corollary 5.11. *Let k be a field and let $\mathcal{B}$ be a k-category such that the endomorphism ring of each object is reduced to k. Assume that $\mathcal{B}$ admits a Galois covering whose group is isomorphic to π_1 (for instance if $\mathcal{B}$ admits a universal covering). Then there exists a canonical injective morphism*

$$\Delta : \mathrm{Hom}(\pi_1(\mathcal{B}, b_0), k^+) \longrightarrow H^1(\mathcal{B}, \mathcal{B}).$$

Observe that for the Kronecker category $\dim H^1(\mathcal{K}, \mathcal{K}) = 3$, then in general the above morphism is not an isomorphism.

References

[1] Assem, I.; de la Peña, J.A. The fundamental groups of a triangular algebra. Comm. Algebra 24 (1996), 187–208.

[2] Assem, I.; Skowroński, A. On some classes of simply connected algebras. Proc. London Math. Soc. 56 (1988), 417–450.

[3] Bautista, R.; Gabriel, P.; Roĭter, A. V.; Salmerón, L. Representation-finite algebras and multiplicative bases. Invent. Math. 81 (1985), 217–285.

[4] Bongartz, K.; Gabriel, P. Covering spaces in representation-theory, Invent. Math. 65 (1981/82), 331–378.

[5] Bustamante, J.C.; Castonguay, D. Fundamental groups and presentations of algebras. J. Algebra Appl. 5 (2006), 549–562.

[6] Cibils, C.; Marcos, E.N. Skew category, Galois covering and smash product of a k-category. Proc. Amer. Math. Soc. 134 (2006), 39–50.

[7] Cibils, C.; Redondo M.J. Cartan-Leray spectral sequence for Galois coverings of categories. J. Algebra 284 (2005), 310–325.

[8] Cibils, C.; Redondo M.J.; Solotar A. Connected gradings and fundamental group. Algebra Number Theory **4** (2010), no. 5, 625–648.

[9] Douady, R.; Douady, A. Algèbre et théories galoisiennes. Paris: Cassini. 448 p. (2005).

[10] Farkas, D.R.; Geiss, C.; Green, E.L.; Marcos, E.N. Diagonalizable derivations of finite-dimensional algebras. I. Israel J. Math. 117 (2000), 157–181.

[11] Farkas, D.R.; Green, E.L.; Marcos, E.N. Diagonalizable derivations of finite-dimensional algebras. II. Pacific J. Math. 196 (2000), 341–351.

[12] Freyd, P. *Abelian categories*. Harper and Row, New York, 1964.
http://www.tac.mta.ca/tac/reprints/articles/3/tr3.pdf

[13] Gabriel, P. The universal cover of a representation-finite algebra. Representations of algebras (Puebla, 1980), 68–105, Lecture Notes in Math., 903, Springer, Berlin-New York, 1981.

[14] Geiss, C.; de la Peña, J.A. An interesting family of algebras. Arch. Math. 60 (1993), 25–35.

[15] Gelfand, S.I.; Manin, Y.I. Methods of homological algebra. Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[16] Green, E. L. Graphs with relations, coverings, and group-graded algebras. Trans. Amer. Math. Soc. 279 (1983), 297-310.

[17] Le Meur, P. The universal cover of an algebra without double bypass. J. Algebra 312 (2007), 330–353.

[18] Le Meur, P. The fundamental group of a triangular algebra without double bypasses. C. R. Math. Acad. Sci. Paris 341 (2005), 211–216.

[19] Le Meur, P. Revêtements galoisiens et groupe fondamental d'algébres de dimension finie. Ph.D. thesis, Université Montpellier 2 (2006).
http://tel.archives-ouvertes.fr/tel-00011753

[20] Martínez-Villa, R.; de la Peña, J. A. The universal cover of a quiver with relations. J. Pure Appl. Algebra 30 (1983), 277–292.

[21] Mitchell, B. Rings with several objects. Advances in Math. 8, 1–161 (1972).

[22] de la Peña, J.A.; Saorín, M. On the first Hochschild cohomology group of an algebra. Manuscripta Math. 104 (2001), 431–442.

[23] Quillen, D. Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[24] Segal, G. Classifying spaces and spectral sequences. Inst. Hautes Études Sci. Publ. Math. 34 (1968), 105–112.

[25] Schwede, S. An exact sequence interpretation of the Lie bracket in Hochschild cohomology. J. Reine Angew. Math. 498 (1998), 153–172.

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