Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 899 (2015) 149–164

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Partition function of $N = 2$ gauge theories on a squashed S^4 with $SU(2) \times U(1)$ isometry

Alejandro Cabo-Bizet [a], Edi Gava [b], Victor I. Giraldo-Rivera [a,b,c,*], M. Nouman Muteeb [c,*], K.S. Narain [a]

[a] *ICTP, Strada Costiera 11, 34014 Trieste, Italy*
[b] *INFN, sezione di Trieste, Italy*
[c] *SISSA, Via Bonomea 265, 34128 Trieste, Italy*

Received 22 June 2015; accepted 24 July 2015

Available online 29 July 2015

Editor: Herman Verlinde

Abstract

We study $N = 2$ supersymmetric gauge theories on a large family of squashed 4-spheres preserving $SU(2) \times U(1) \subset SO(4)$ isometry and determine the conditions under which this background is supersymmetric. We then compute the partition function of the theories by using localization technique. The results indicate that for $N = 2$ SUSY, including both vector-multiplets and hypermultiplets, the partition function is independent of the arbitrary squashing functions as well as of the other supergravity background fields.

1. Introduction

Supersymmetric localization techniques furnish a rich ground for exact computation of various quantities in Supersymmetric Quantum Field Theories. This program started with the work of [1], later pursued by [2,3] and more recently brought back by [4] which gave rise to an intense activity of exact calculations in various dimensions and/or manifolds [5–12]. A systematic way to put rigid SUSY on curved spaces in the case of $N = 1$ theories was worked out by [13,14],

* Corresponding authors.
E-mail addresses: acabo@sissa.it (A. Cabo-Bizet), gava@sissa.it (E. Gava), vgiraldo@ictp.it (V.I. Giraldo-Rivera), mmuteeb@sissa.it (M.N. Muteeb), narain@ictp.it (K.S. Narain).

http://dx.doi.org/10.1016/j.nuclphysb.2015.07.029
0550-3213/

and, for $N = 2$ theories in [15]. The partition function on squashed spheres depends in general on the squashing parameters [8,11]. However for some squashing, preserving a particular isometry of the manifold, the partition function comes out to be independent of squashing parameters. Detailed studies of 3-dimensional cases had appeared in [8,16,17] and [18–20]. For the four dimensional case, the analysis of which geometrical background data the partition function depends on, has been performed for $N = 1$ SUSY. The four dimensional squashed sphere has also been considered, first in [11], and later in [9,21,22]. The SUSY partition function on the branched S^4 in [9,21] computes the SUSY Rènyi entropy of a circular region in a 4-dimensional space [23,24].

In this paper we calculate $N = 2$ supersymmetric partition function on a very general squashed S^4 with $SU(2) \times U(1)$ isometry, and show that it is independent of the squashing metric parameters and of the other supergravity backgrounds. In the case of $N = 2$ theories on the ellipsoid considered in [11] the isometry is generically $U(1) \times U(1)$. In the limit $l = r$ in [11] this symmetry is enhanced to the $SO(3) \times SO(2)$ subgroup of $SO(5)$. On the other hand, the $SU(2) \times U(1)$ isometry in our case is a subgroup of $SO(4) \equiv SU(2)_L \times SU(2)_R$. The paper is organized as follows. In Section 2, the Killing spinor equations for $N = 2$ rigid SUSY on squashed S^4 are given, in Section 3 the squashed S^4 metric and spin connection components are given and we solve the Killing spinor equations, calculating various background fields and then giving the conditions for their regularity. Section 4 contains the calculation of the Q^2 action on the fields of vector-multiplets and hypermultiplets. In Sections 5 and 6, we find the saddle point configurations and one-loop determinants for vector multiplet and hypermultiplet respectively. In Section 7 we comment on the contribution of point-like instantons and anti-instantons to the supersymmetric partition function. A brief summary of the main result is given in Section 8.

2. Rigid supersymmetric theories on curved spaces

By now a systematic way to put rigid SUSY on a curved spaces has been developed: the procedure is to start from the supergravity transformations [13,14,25,15] and obtain a rigid SUSY theory on a given curved manifold by freezing the quantum fluctuations of the gravitational background by taking the Planck mass limit $M_P \to \infty$, setting to zero and the fermionic fields in the supergravity multiplet. By following this procedure for $N = 2$ one obtains a set of Killing spinor equations which have to be satisfied in order to obtain rigid 4D $N = 2$ SUSY and at the same time constrain the background geometry. They are:

$$\begin{aligned} &D_m \xi_A + T^{kl}\sigma_{kl}\sigma_m \bar{\xi}_A = -\iota \sigma_m \bar{\xi}'_A, \\ &D_m \bar{\xi}_A + \bar{T}^{kl}\bar{\sigma}_{kl}\bar{\sigma}_m \xi_A = -\iota \bar{\sigma}_m \xi'_A \quad \text{for a given pair} \quad \xi'_A, \bar{\xi}'_A, \end{aligned} \tag{1}$$

(where $\iota \equiv \sqrt{-1}$) coming from the gravitino variation, and:

$$\begin{aligned} &\sigma^m \bar{\sigma}^n D_m D_n \xi_A + 4 D_l T_{mn} \sigma^{mn} \sigma^l \bar{\xi}_A = M \xi_A, \\ &\bar{\sigma}^m \sigma^n D_m D_n \bar{\xi}_A + 4 D_l \bar{T}_{mn} \bar{\sigma}^{mn} \bar{\sigma}^l \xi_A = M \bar{\xi}_A, \end{aligned} \tag{2}$$

with M a real scalar background field, which is a consequence of the variation of a spin $1/2$ field in the supergravity multiplet.

Here ξ_A and $\bar{\xi}_A$ (spinor indices are omitted) are chiral and anti-chiral Killing spinors satisfying reality conditions to be specified later and are the parameters of $N = 2$ SUSY. The index A is a $SU(2)_R$ R-symmetry index of the $N = 2$ theory. The fields T^{kl}, $\bar{T}^{kl}$ are self-dual and anti-self-dual real tensor background fields respectively. The covariant derivatives in (1) and (2) are covariantized also with respect to a background $SU(2)_R$ gauge field $(V_m)^A_B$, in addition to the local Lorenz and gauge transformations. We work in four component notation, where (1) is written compactly as

$$D_m\xi + T.\Gamma\Gamma_m\xi = -\iota\Gamma_m\xi', \tag{3}$$

where $T.\Gamma \equiv T_{kl}\Gamma^{kl}$.[1] Now multiplying from left by Γ^m and using the identity $\Gamma_m\Gamma_{kl}\Gamma^m = 0$ we get

$$\xi_p \equiv \Gamma^m D_m\xi = -4\iota\xi'. \tag{4}$$

Here a new spinor ξ_p is defined which will be useful later on, when we will calculate the square of supersymmetry transformation Q^2 acting on different fields of $N = 2$ theory.

3. Supersymmetry on the squashed S^4

The family of squashed 4-spheres which we will consider is defined by the following metric or vielbein one-forms:

$$\begin{aligned}
&ds^2 = \mathrm{d}r^2 + \frac{f(r)^2}{4}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right) + \frac{h(r)^2}{4}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi)^2,\\
&e^4 = \mathrm{d}r, \quad e^3 = -\frac{h(r)}{2}\left(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi\right), \quad e^2 = \frac{f(r)}{2}\left(\sin\psi\mathrm{d}\theta - \sin\theta\cos\psi\mathrm{d}\phi\right),\\
&e^1 = -\frac{f(r)}{2}\left(\cos\psi\mathrm{d}\theta + \sin\theta\sin\psi\mathrm{d}\phi\right),
\end{aligned} \tag{5}$$

where $f(r)$ and $h(r)$ are smooth arbitrary functions of r. The above metric has $SU(2) \times U(1)$ isometry. The spin connection is given by the following non-zero components Ω_m^{ab},

$$\begin{aligned}
&\Omega_1^{21} = 1 - \frac{h(r)^2}{2f(r)^2}, \quad \Omega_1^{43} = \frac{h'(r)}{2}, \quad \Omega_2^{31} = \frac{h(r)\sin(\psi)}{2f(r)}, \quad \Omega_2^{32} = \frac{h(r)\cos(\psi)}{2f(r)},\\
&\Omega_2^{41} = \frac{1}{2}\cos(\psi)f'(r), \quad \Omega_2^{42} = -\frac{1}{2}\sin(\psi)f'(r), \quad \Omega_3^{21} = \cos(\theta) - \frac{h(r)^2\cos(\theta)}{2f(r)^2},\\
&\Omega_3^{31} = -\frac{h(r)\sin(\theta)\cos(\psi)}{2f(r)}, \quad \Omega_3^{32} = \frac{h(r)\sin(\theta)\sin(\psi)}{2f(r)}, \quad \Omega_3^{41} = \frac{1}{2}\sin(\theta)\sin(\psi)f'(r),\\
&\Omega_3^{42} = \frac{1}{2}\sin(\theta)\cos(\psi)f'(r), \quad \Omega_3^{43} = \frac{1}{2}\cos(\theta)h'(r),
\end{aligned} \tag{6}$$

where $a, b = 1, \dots, 4$ are flat indices and $m = 1, \dots 4$ is curved space index.

3.1. Solution of killing spinor equation on the squashed S^4

The purpose of this section is to show that if the background fields $(V_m)^A_B, T_{mn}, \bar{T}_{mn}, M$ are chosen appropriately, the squashed S^4 admits a Killing spinor which is solution of the two stets of Killing spinor equations (1) and (2). We write the backgrounds T and V in a complexified version:

[1] Our conventions of Γ matrices can be simply read off the Killing spinor equations of [11] and their σ matrices.

$$V_m = \begin{pmatrix} \iota v_{3m} & \iota(v_{1m} + \iota v_{2m}) \\ \iota(v_{1m} - \iota v_{2m}) & -\iota v_{3m} \end{pmatrix},$$

$$T = \begin{pmatrix} \iota t_3 & \iota(t_1 - \iota t_2) & 0 & 0 \\ \iota(t_1 + \iota t_2) & -\iota t_3 & 0 & 0 \\ 0 & 0 & \iota t_3 & \iota(\bar{t}_1 - \iota \bar{t}_2) \\ 0 & 0 & \iota(\bar{t}_1 + \iota \bar{t}_2) & -\iota \bar{t}_3 \end{pmatrix}. \tag{7}$$

We will consider the following ansatz for the Killing spinor and we will calculate the background fields T, V and M such that this ansatz satisfies the set of Killing spinor equations

$$\xi = \begin{pmatrix} s_1(r) & 0 \\ 0 & t_2(r) \\ s_3(r) & 0 \\ 0 & t_4(r) \end{pmatrix}. \tag{8}$$

The Killing spinor satisfies the reality condition given in [11]:

$$(\xi_{\alpha A})^\dagger = \epsilon^{AB}\epsilon^{\alpha\beta}\xi_{\beta B}, \qquad (\bar{\xi}_{\dot{\alpha} A})^\dagger = \epsilon^{AB}\epsilon^{\dot{\alpha}\dot{\beta}}\xi_{\dot{\beta} B}. \tag{9}$$

The parameters in the Killing spinor are arbitrary smooth functions of r. After solving the Killing spinor equations, it turns out that some of these parameters are constrained.

The general solution to the main and auxiliary equations using the ansatz (8) takes the following form:

$$\begin{aligned}
&s_1(r) = s(r), \quad s_3(r) = \frac{\iota\, c\, h(r)}{s(r)}, \quad t_2(r) = s(r), \\
&t_4(r) = -\frac{\iota\, c\, h(r)}{s(r)}, \\
&t_3 = \frac{s(r)\left(f(r)\left(2f(r)s'(r) - s(r)f'(r)\right) + h(r)s(r)\right)}{4cf(r)^2h(r)}, \\
&\bar{t}_3 = \frac{c\left(f(r)h(r)\left(s(r)f'(r) + 2f(r)s'(r)\right) - 2f(r)^2s(r)h'(r) + h(r)^2s(r)\right)}{4f(r)^2s(r)^3}, \\
&v_{33} = \frac{1}{2}\left(\frac{h(r)}{f(r)^2} + \frac{h'(r)-2}{h(r)} - \frac{2s'(r)}{s(r)}\right), \\
&M = \frac{2f''(r)}{f(r)} + \frac{f'(r)^2 - 2h'(r) + \frac{4h(r)s'(r)}{s(r)}}{f(r)^2} + \frac{h(r)^2}{f(r)^4} + \frac{4s'(r)\left(s(r)h'(r) - h(r)s'(r)\right)}{h(r)s(r)^2}.
\end{aligned} \tag{10}$$

Here only the non-zero part of the background fields and Killing spinor components are given, c is a real arbitrary constant which sets the normalization of the killing vector we will use to localize, $s(r)$ is a smooth function of r and the background fields T and V_m are indexed by flat tangent space indices. For these background fields to be well defined on the squashed S^4, it is necessary that $s(r)$ has no zero between the two poles. We thus determined the form of all the additional background fields in order for $N = 2$ SUSY to be preserved on the squashed four-sphere. We have set $v_{12} = 0$, this choice of background preserves $SU(2) \times U(1) \times U(1)_R$ symmetry. Should we take $v_{12} \neq 0$ it can be shown that the symmetry is reduced to $SU(2) \times U(1)'$ where $U(1)' \equiv (U(1) \times U(1)_R)_{diagonal}$.

3.2. Regularity of the background fields

Our metric should look like the round S^4 at the North and South poles, this implies that $f(r) = h(r) = 0$ at $r = 0$ and $r = \pi$. Moreover for our metric to be non-singular in the interval $\pi > r > 0$, the functions $f(r)$ and $h(r)$ are strictly non-zero and do not change sign inside the interval.

North pole ($r = 0$): Near the North pole the regularity of invariant quantities R, $R_{\mu\nu}R^{\mu\nu}$ and of the background fields both in flat tangent space indices and curved space indices, fixes $f(r)$, $h(r)$ and $s(r)$ in the following form:

$$\begin{aligned}
h(r) &= r + \mathrm{h}_{n_3} r^3 + O(r^4),\\
f(r) &= r + \mathrm{f}_{n_3} r^3 + O(r^4),\\
s(r) &= s_{n_0} + s_{n_2} r^2 + s_{n_3} r^3 + O(r^4).
\end{aligned} \tag{11}$$

There are higher order terms, but those are irrelevant to the present analysis.

South pole ($r = \pi$): Similarly near the South pole the regularity requirements fix $f(r)$, $h(r)$ and $s(r)$ in the following way

$$\begin{aligned}
h(r) &= \pi - r + \mathrm{h}_{s_3}(\pi - r)^3 + O\left((\pi - r)^4\right),\\
f(r) &= \pi - r + \mathrm{f}_{s_3}(\pi - r)^3 + O\left((\pi - r)^4\right),\\
s(r) &= (\pi - r) s_{s_1} + (\pi - r)^3 s_{s_3} + O\left((\pi - r)^4\right).
\end{aligned} \tag{12}$$

Here $h_{n_3}, f_{n_3}, s_{n_0}, s_{n_2}, s_{n_3}, h_{s_3}, f_{s_3}, s_{s_1}, s_{s_3}$ are arbitrary real constants.

For reasons that will become clear later, a quantity of interest which we want to calculate is $(s(r)^2 - \frac{c^2 h(r)^2}{s(r)^2})$. At the North pole it evaluates to $s_{n_0}^2$, whereas at the South pole it evaluates to $-\frac{c^2}{s_{s_1}^2}$. So it has the interesting property that it changes sign between North and South poles and hence passes through zero. This result will have important consequences later on, in Section 6 when we will calculate the one-loop determinant, where we show that the relevant differential operators are transversally elliptic. Before proceeding, we want to comment that there is an ambiguity in the choice of the functions $f(r)$, $h(r)$ and $s(r)$ at the North and South poles, that is, if we take the following choice for these functions at the North pole

$$\begin{aligned}
h(r) &= -r + \mathrm{h}_{n_3} r^3 + O(r^4),\\
f(r) &= r + \mathrm{f}_{n_3} r^3 + O(r^4),\\
s(r) &= s_{n_1} + s_{n_3} r^3 + O(r^4),
\end{aligned} \tag{13}$$

and the following choice at the South pole

$$\begin{aligned}
h(r) &= r - \pi + \mathrm{h}_{s_3}(\pi - r)^3 + O\left((\pi - r)^4\right),\\
f(r) &= \pi - r + \mathrm{f}_{s_3}(\pi - r)^3 + O\left((\pi - r)^4\right),\\
s(r) &= s_{s_0} + s_{s_2}(\pi - r)^2 + s_{s_3}(\pi - r)^3 + O\left((\pi - r)^4\right),
\end{aligned} \tag{14}$$

all the background fields are still regular there. The only difference is that the quantity $(s(r)^2 - \frac{c^2 h(r)^2}{s(r)^2})$ evaluates to $-\frac{c^2}{s_{n_1}^2}$ at the South pole and to $s_{s_0}^2$ at the South pole. Every other result remains the same.

4. Multiplets

4.1. Vector multiplet

In 4D $N = 2$ SUSY with Euclidean signature, vector multiplets are made of a gauge field A_m, two independent gauginos $\lambda_{\alpha A}, \bar{\lambda}_{\dot{\alpha}A}$, two scalar fields $\phi, \bar{\phi}$ and an auxiliary field $D_{AB} = D_{BA}$, all Lie algebra valued. The supersymmetric Yang–Mills Lagrangian with the additional couplings to the backgrounds was written in [11], we write it again for completeness:

$$
\begin{aligned}
L_{YM} = \mathrm{Tr}[&\frac{1}{2}F^{mn}F_{mn} + 16F_{mn}(\bar{\phi}T^{mn} + \phi\bar{T}^{mn}) + 64\bar{\phi}^2 T_{mn}T^{mn} + 64\phi^2\bar{T}^{mn}\bar{T}_{mn} \\
&- 4D_m\bar{\phi}D^m\phi + 2M\phi\bar{\phi} - 2\iota\lambda^A\sigma^m D_m\bar{\lambda}_A - 2\lambda^A[\phi, \bar{\lambda}_A] \\
&+ 2\bar{\lambda}^A[\phi, \bar{\lambda}] + 4[\phi, \bar{\phi}]^2 - \frac{1}{2}D^{AB}D_{AB}],
\end{aligned} \tag{15}
$$

with the inclusion of the θ-term:

$$
S_{YM} = \frac{1}{g_{YM}^2}\int d^4x\sqrt{g}L_{YM} + \iota\frac{\theta}{8\pi^2}\int \mathrm{Tr}(F \wedge F). \tag{16}
$$

4.2. Hypermultiplet

The hypermultiplet consists of scalars q_{AI} and fermions $\psi_{\alpha A}, \bar{\psi}_I^{\dot{\alpha}}$ satisfying reality conditions [11]. The index I runs from 1 to $2r$. There is also an auxiliary scalar F_{IA} transforming as a doublet under a local $SU(2)_{\check{R}}$ symmetry. This symmetry and the auxiliary field are introduced in the theory by the requirement that the SUSY algebra of matter multiplet is closed off shell respect to the supercharge that is used to localize [4]. From [11] the gauge covariant kinetic Lagrangian for the hypermultiplet is

$$
\begin{aligned}
L_{mat} = &\frac{1}{2}D_m q^A D^m q_A - q^A\{\phi, \bar{\phi}\}q_A + \frac{\iota}{2}q^A D_{AB}q^B + \frac{1}{8}(R + M)q^A q_A - \frac{\iota}{2}\bar{\psi}\bar{\sigma}^m D_m\psi \\
&- \frac{1}{2}\psi\phi\psi + \frac{1}{2}\bar{\psi}\bar{\phi}\bar{\psi} + \frac{\iota}{2}\psi\sigma^{kl}T_{kl}\psi - \frac{\iota}{2}\bar{\psi}\bar{\sigma}^{kl}\bar{T}_{kl}\bar{\psi} \\
&- q^A\lambda_A\psi + \bar{\psi}\bar{\lambda}q^A - \frac{1}{2}F^A F_A.
\end{aligned} \tag{17}
$$

4.3. Closure of the supercharge algebra

For localization computation we need to identify a continuous fermionic symmetry $\mathbf{Q}$ and the corresponding Killing spinor is taken to be commuting. The supersymmetry transformation Q acting on the fields of $N = 2$ SUSY theory squares into a combination of bosonic symmetries:

$$
\begin{aligned}
\mathbf{Q}^2 \equiv\ &L_v + Gauge(\hat{\Phi}) + Lorentz(L_{ab}) + Scale(\omega) \\
&+ R_{U(1)}(\Theta) + R_{SU(2)}(\hat{\Theta}_{AB}) + \check{R}_{SU(2)}(\hat{\check{\Theta}}),
\end{aligned} \tag{18}
$$

with various parameters defined as in [11]. For the vector multiplet the SUSY algebra is closed off shell, the only requirement being that the Killing spinor equations be satisfied. For the hypermultiplet the closure of full $N = 2$ SUSY algebra requires the existence of infinite number of auxiliary spinors and auxiliary fields. But for localization computation we need only one supercharge corresponding to a particular Killing spinor and in this case only finite number of auxiliary spinors are required. These auxiliary spinors are required to satisfy certain constraint equations (see [4]).

Next we compute these transformation parameters for our background. First of all, we observe that $\xi^A \xi_{pA} = \bar{\xi}^A \bar{\xi}_{pA} = 0$. This condition implies that $\omega = \Theta = 0$. In other words the square of the supersymmetry transformation does not give rise to dilation or $U(1)_R$ transformation. This condition is necessary because the non-zero values of the background fields T_{ab} and $\bar{T}_{ab}$ break the $U(1)_R$ symmetry anyway.

The explicit expression for other transformation parameters are given below

$$
\begin{aligned}
L_{ab} &= \begin{pmatrix} 0 & -8c & 0 & 0 \\ 8c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Theta_{AB} &= \begin{pmatrix} 0 & 2c\left(\frac{h(r)^2}{f(r)^2} - \frac{2s'(r)h(r)}{s(r)} + h'(r)\right) \\ 2c\left(\frac{h(r)^2}{f(r)^2} - \frac{2s'(r)h(r)}{s(r)} + h'(r)\right) & 0 \end{pmatrix}, \\
\hat{\Theta}^A_B &= \begin{pmatrix} 4c & 0 \\ 0 & -4c \end{pmatrix}, \\
Lie_v \xi &= \begin{pmatrix} -\frac{2cs(r)\left(\left(h'(r)-2\right)f(r)^2+h(r)^2\right)}{f(r)^2} & 0 \\ 0 & \frac{2cs(r)\left(\left(h'(r)-2\right)f(r)^2+h(r)^2\right)}{f(r)^2} \\ \frac{2\iota c^2 h(r)\left(f(r)^2\left(h'(r)+2\right)-h(r)^2\right)}{f(r)^2 s(r)} & 0 \\ 0 & \frac{2\iota c^2 h(r)\left(f(r)^2\left(h'(r)+2\right)-h(r)^2\right)}{f(r)^2 s(r)} \end{pmatrix},
\end{aligned}
\tag{19}
$$

where the Lie derivative Lie_v is defined as $L_v \xi \equiv v^m D_m \xi + \frac{1}{4} D_{[a} v_{b]} \Gamma^{ab} \xi$. The non-zero L_{ab} implies the fact that the $U(1)$ group which is used to find the fixed points of the manifold, belongs to the Cartan of $SU(2)$ part of the isometry group $SU(2) \times U(1)$. Therefore it follows that our Killing spinor is invariant under Q^2. In 4-component notation:

$$
Q^2 \xi = \iota Lie_v \xi - \xi \hat{\Theta} = 0. \tag{20}
$$

The auxiliary spinor, which helps to close off-shell the supersymmetry, is given by:

$$
\check{\xi} = \begin{pmatrix} \frac{ch(r)}{s(r)} & 0 \\ 0 & \frac{ch(r)}{s(r)} \\ -\iota s(r) & 0 \\ 0 & \iota s(r) \end{pmatrix}. \tag{21}
$$

To fix the background $SU(2)_{\check{R}}$, we have to fix the corresponding gauge field $\check{V}_m$:

$$\check{V}_m = \begin{pmatrix} \iota\check{v}_{3m} & \iota(\check{v}_{1m} + \iota\check{v}_{2m}) \\ \iota(\check{v}_{1m} - \iota\check{v}_{2m}) & -\iota\check{v}_{3m} \end{pmatrix}. \tag{22}$$

The requirement that all the background fields be invariant under the action of $\mathbf{Q}^2$ fixes all the components of $\check{V}_m$ to zero except $\check{v}_{33}$, $\check{v}_{34}$, which remain arbitrary.

After the gauge fixing, $\hat{\check{\Theta}}{}^A_B$ becomes

$$\hat{\check{\Theta}}{}^A_B = \begin{pmatrix} -4(h(r)\check{v}_{33}(r)c + c) & 0 \\ 0 & 4(h(r)\check{v}_{33}(r)c + c) \end{pmatrix}. \tag{23}$$

And also the auxiliary spinor $\check{\xi}$ is proven to be invariant under Q^2

$$\mathbf{Q}^2\check{\xi} = 0. \tag{24}$$

5. Localization

5.1. S_{YM} saddle points

The path integral computation of the expectation value of an observable of a supersymmetric YM theory which is invariant under a supercharge $\mathbf{Q}$ localizes to a subset S_Q of the entire field space. The zero locus of the supercharge $\mathbf{Q}$ coincides with the set of bosonic configurations for which the supersymmetry variations of the fermions vanish:

$$\mathbf{Q}\Psi = 0 \quad \text{for all fermions} \quad \Psi. \tag{25}$$

This is easily seen if we take as regulator the $\mathbf{Q}$-exact deformation: $\mathbf{Q}V = \mathrm{Q}((\mathbf{Q}\Psi)^\dagger\Psi)$.

To take into account the gauge fixing, the supercharge $\mathbf{Q}$ is generalized to $\hat{\mathbf{Q}} \equiv \mathbf{Q} + \mathbf{Q}_B$, where $\mathbf{Q}_B$ is the BRST-supercharge. However as pointed out in [4], this does not affect the zero locus. To effectively calculate the zero locus of the supercharge, we add to the Lagrangian a $\mathbf{Q}$-exact quantity $\mathbf{Q}V$, whose critical point set is S_Q and whose bosonic part is semi-positive definite. Now either solve the localization equation

$$\hat{\mathbf{Q}}\lambda = 0 \tag{26}$$

directly or analyze the $\hat{\mathbf{Q}}$-transform of the following quantity,

$$V = \mathrm{Tr}[(\hat{\mathbf{Q}}\lambda_{\alpha A})^\dagger\lambda_{\alpha A} + (\hat{\mathbf{Q}}\bar{\lambda}^{\dot{\alpha}}_A)^\dagger\bar{\lambda}^{\dot{\alpha}}_A], \tag{27}$$

which has semi-positive definite bosonic part. In writing explicitly (27) we use the proper reality conditions which make the action well defined. We get the analogous expression to the equation (4.2) in [11]. Analyzing that expression we get two partial differential equations for $\phi - \bar{\phi} \equiv \phi_2(\psi, \theta, \varphi, r)$, where we make use of Bianchi identities to get the second one:

$$\partial_\psi\phi_2(\psi, \theta, \varphi, r) = 0, \tag{28}$$

and

$$\tilde{\nabla}^2\phi_2(\theta, \varphi, r) + \frac{f(r)^2}{2h(r)}\xi\Gamma^m\xi_p\partial_m\phi_2(\theta, \varphi, r) + G(r)\phi_2(\theta, \varphi, r) = 0, \tag{29}$$

in the second equation we used the fact that $\phi_2(\psi, \theta, \varphi, r)$ is independent of ψ-coordinate, $\tilde{\nabla}^2$ is the following Laplacian like operator:

$$\tilde{\nabla}^2 * = \frac{f(r)^2}{2h(r)} \frac{h(r)}{\sqrt{g} f(r) \xi_n} \nabla_\mu \Big(\sqrt{g} \xi_n^2 g^{\mu\nu} \nabla_\nu \big(\frac{f(r)}{h(r)} * \big) \Big) \tag{30}$$

$\xi_n = \xi.\xi$ is the proper norm of the four component spinor and $G(r)$,

$$\begin{aligned} G(r) = \frac{1}{h(r)^3 s(r)^3} \Bigg(& -c^2 h(r)^4 \left(s(r) \left(f'(r)^2 + 2h'(r) \right) - 2 f(r) f'(r) s'(r) \right) \\ & - h(r)^2 \left(-3c^2 f(r)^2 s(r) h'(r)^2 + 2 f(r) s(r)^4 f'(r) s'(r) + s(r)^5 f'(r)^2 \right) \\ & + h(r)^3 \left(c^2 f(r)^2 s(r) h''(r) + 2s'(r) \left(s(r)^4 - 2c^2 f(r)^2 h'(r) \right) \right) + 2c^2 h(r)^5 s'(r) \\ & + f(r) h(r) s(r)^4 \left(2h'(r) \left(s(r) f'(r) + 2 f(r) s'(r) \right) + f(r) s(r) h''(r) \right) \\ & - f(r)^2 s(r)^5 h'(r)^2 \Bigg), \end{aligned} \tag{31}$$

For the round sphere

$$f(r) = \sin r, \quad h(r) = \sin r, \quad s(r) = \frac{1}{\sqrt{2}} \cos(\frac{r}{2}), \quad c = \frac{1}{4}, \tag{32}$$

the field $\phi_2 = 0$ at the localization locus, which will also ensure that $A_m = 0$ at the locus. This result is true in an open neighborhood of the round S^4, as appears also in [11], and so we will assume, it is the solution to the locus equations.

The saddle points are thus labeled by a Lie Algebra valued constant a_0, and are given by the equations [4,11]:

$$A_m = 0, \qquad \phi = \bar{\phi} = a_0, \qquad D_{AB} = -\imath a_0 \omega_{AB}. \tag{33}$$

The value of the Super-Yang–Mills action on this saddle point is then:

$$\frac{1}{g_{YM}^2} \int d^4x \sqrt{g} L_{YM}|_{saddle\ point} = \frac{2\pi^3 \operatorname{Tr}\left[\mathrm{a}_0^2\right]}{c^2 \mathrm{g}_{YM}^2}. \tag{34}$$

5.2. Saddle points for matter multiplet

To find the saddle points of the matter multiplet we will use the following fermionic functional

$$V_{mat} = \operatorname{Tr}[(\hat{\mathbf{Q}}\psi_{\alpha I})^\dagger \psi_{\alpha I} + (\hat{\mathbf{Q}}\bar{\psi}_I^{\dot{\alpha}})^\dagger \bar{\psi}_I^{\dot{\alpha}}]. \tag{35}$$

The bosonic part of $\hat{\mathbf{Q}} V_{mat}$ is

$$\hat{\mathbf{Q}} V_{mat}|_{bos} = \operatorname{Tr}[(\hat{\mathbf{Q}}\psi_{\alpha I})^\dagger \hat{\mathbf{Q}}\psi_{\alpha I} + (\hat{\mathbf{Q}}\bar{\psi}_I^{\dot{\alpha}})^\dagger \hat{\mathbf{Q}}\bar{\psi}_I^{\dot{\alpha}}]. \tag{36}$$

It is easy to check that:

$$\hat{\mathbf{Q}} V_{mat}|_{bos} = 4\|\xi\|^2 (\frac{1}{2}(D_m q^{AI} - P_m q^{AI})^2 + M_q(r) q^{AI} q_{IA} - \frac{1}{2} F^{AI} F_{IA}), \tag{37}$$

where

$$P_{mA}^B = \frac{1}{\|\xi\|^2} (2(\epsilon\xi\gamma_m\xi_p + \epsilon\xi T \gamma_m \xi)^B{}_A + D^n \operatorname{Log}(\|\xi\|^2)(\epsilon\xi\gamma_{nm}\xi)^B{}_A), \tag{38}$$

and

$$M_q = -\frac{1}{4}R + \frac{1}{\|\xi\|^2}(8\xi_p^A\xi_{pA} + \xi^A\gamma^m T^2\gamma_m\xi_A - D^n \operatorname{Log}(\|\xi\|^2)\xi^A(3\gamma_m\xi_{pA} + T\gamma_m\xi_A) + \frac{1}{2}(P^{mA}{}_B P^B_{mA})) - \frac{1}{2\|\xi\|^2}P^{mA}{}_A P^B_{mB}, \tag{39}$$

where $\xi_A = (\xi_{\alpha A}, \bar{\xi}_{\dot{\alpha}A})$,$\epsilon^{AB}$ is the $SU(2)_R$ tensor and R is the Ricci scalar. As a result of the condition $F^{\dagger}_{IA} = -F^{AI}$ which is imposed along the contour of path integration, all the bosonic terms are manifestly positive definite, except the term containing $M_q(r)$. For the round S^4

$$M_q(r) = \frac{7}{8} + \frac{\cos(2r)}{8}, \tag{40}$$

and it is bounded from below by $\frac{3}{4}$. Therefore there is a large open neighborhood of the round sphere for which $M_q(r)$ is positive definite. So we get the result for the saddle points of the hypermultiplet as

$$q_{IA} = 0, \qquad F_{IA} = 0. \tag{41}$$

Hence there will be no classical contribution from the hypermultiplet sector.

6. One-loop determinant

To calculate the one-loop determinant we have to first fix the gauge. We choose the following gauge function [11].

$$G = \iota\partial_m A^m + \iota L_v((\xi^A\xi_A - \bar{\xi}_A\bar{\xi}^A)\phi_2 - v^m A_m). \tag{42}$$

The saddle point conditions do not change under the new supercharge $\hat{Q}^2 \equiv (Q_B + Q)^2$, with the zero mode of $\phi_1 = a_0$ at the saddle point.

6.1. Vector multiplet contribution

The basic idea of localization is that the actual value of the path integral or any other Q-closed observable remains unchanged under any $\hat{\mathbf{Q}}$-exact deformation $L \to L + s\hat{\mathbf{Q}}(V + V_{GF})$. By choosing the bosonic part of $L \to L + s\hat{\mathbf{Q}}(V + V_{GF})$ positive definite and sending $s \to \infty$, Gaussian approximation becomes exact for the path integral over the fluctuations around the locus. The Gaussian integral evaluates to the square root of the ratio between the determinant of a fermionic kinetic operator $K_{fermion}$ and that of a bosonic kinetic operator K_{boson}. These kinetic operators coming from the quadratic part of the $\hat{\mathbf{Q}}$-exact regulator.

To compute the 1-loop contribution it is convenient to change variables in the path integral to a set, X, Ξ, which makes manifest the cohomology of $\hat{\mathbf{Q}}$ [4,11]. After doing that, the quadratic part of $V + V_{GF}$ can be written as:

$$(V + V_{GF})|_{quadratic} = (\hat{\mathbf{Q}}\mathbf{X}, \Xi)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}\begin{pmatrix} \mathbf{X} \\ \hat{\mathbf{Q}}\Xi \end{pmatrix}, \tag{43}$$

where D_{ij} are differential operators and X, Ξ are cohomologically paired bosonic and fermionic fields respectively,

$$\Xi \equiv (\Xi_{AB}, \bar{C}, C), \qquad \mathbf{X} = (\phi_2, A_m; \bar{a}_0, B_0), \tag{44}$$

and

$$\Xi_{AB} \equiv 2\bar{\xi}_{(A}\bar{\lambda}_{B)} - 2\xi_{(A}\lambda_{B)}, \tag{45}$$

where $\bar{C}, C, \bar{a}_0, B_0$ belong to the ghost multiplets The fields X and Ξ can be regarded as sections of bundles E_0, E_1 over the squashed sphere and hence D_{10} acts on the complex as $D_{10} : \Gamma(E_0) \to \Gamma(E_1)$. The invariance of the deformation term $\hat{Q}(V + V_{GF})$ under the action of $\hat{Q}$ and the pairing of the fields under $\hat{\mathbf{Q}}^2 = \mathbf{H}$ leads to the cancellations between bosonic and fermionic fluctuations, which gives the following ratio [4,11]:

$$\frac{\mathbf{det}_{CokerD_{10}}\mathbf{H}}{\mathbf{det}_{KerD_{10}}\mathbf{H}}. \tag{46}$$

The fact that $\hat{\mathbf{Q}}^2$ commutes with the differential operators D_{ij} is used in the derivation of the last expression and is a result of the invariance of $(V + V_{GF})$ under $\hat{Q}^2$. This can readily be seen by considering $\hat{Q}^2(V + V_{GF})_{Quad}$.

$$\begin{aligned}\hat{Q}(V + V_{GF})_{Quad} = &\begin{pmatrix} X & \hat{Q}\Sigma \end{pmatrix} \mathbb{D} \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ \hat{Q}\Sigma \end{pmatrix} \\ &- \begin{pmatrix} \hat{Q}X & \Sigma \end{pmatrix} \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}X \\ \Sigma \end{pmatrix},\end{aligned} \tag{47}$$

where $\mathbb{D} \equiv \begin{pmatrix} \mathrm{D}_{00} & \mathrm{D}_{01} \\ \mathrm{D}_{10} & \mathrm{D}_{11} \end{pmatrix}$.

Then

$$\begin{aligned}&\hat{Q}^2(V + V_{GF})_{Quad} \\ &= \begin{pmatrix} \hat{Q}X & \hat{Q}^2\Sigma \end{pmatrix} \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} X \\ \hat{Q}\Sigma \end{pmatrix} + \begin{pmatrix} X & \hat{Q}\Sigma \end{pmatrix} \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} \hat{Q}X \\ \hat{Q}^2\Sigma \end{pmatrix} \\ &\quad - \begin{pmatrix} \hat{Q}^2X & \hat{Q}\Sigma \end{pmatrix} \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & -\hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}X \\ \Sigma \end{pmatrix} + \begin{pmatrix} \hat{Q}X & \Sigma \end{pmatrix} \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}^2X \\ \hat{Q}\Sigma \end{pmatrix} \\ &= \begin{pmatrix} \hat{Q}X & \Sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\hat{Q}^2 \end{pmatrix} \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} X \\ \hat{Q}\Sigma \end{pmatrix} + \begin{pmatrix} X & \hat{Q}\Sigma \end{pmatrix} \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}X \\ \Sigma \end{pmatrix} - \begin{pmatrix} X & \hat{Q}\Sigma \end{pmatrix} \begin{pmatrix} -\hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}X \\ \Sigma \end{pmatrix} \\ &\quad + \begin{pmatrix} \hat{Q}X & \Sigma \end{pmatrix} \mathbb{D} \begin{pmatrix} 1 & 0 \\ 0 & \hat{Q}^2 \end{pmatrix} \begin{pmatrix} \hat{Q}^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ \hat{Q}\Sigma \end{pmatrix}.\end{aligned} \tag{48}$$

Now with the requirement that $[\hat{Q}^2, D_{ij}] = 0$, different terms cancel among each other and we get

$$\hat{Q}^2(V + V_{GF})_{Quad} = 0. \tag{49}$$

6.2. Index of D_{10}

To evaluate the ratio (46) through the index computation, we first note that the constant fields $B_0, \bar{a}_0$ have each weight 0 under the action of $U(1)$ at the poles and are thus regarded as sitting in the kernel of D_{10} and making a contribution of 2. For the contribution of other fields we need an explicit expression for D_{10},[2] which is read from equation (43) To compute the index of D_{10} it is better to use its, symbol $\sigma(D_{10})$, this is computed by taking the Fourier transform of the operator D_{10} and then retaining only the highest order derivative (momentum) terms [4]. To write the symbol explicitly we have to express the Fourier transform of D_{10} in the following orthonormal basis of four unit vector fields μ_a^m $(a = 1, 2, 3, 4)$, which relabels the original vielbein basis

$$-2\imath(\tau^a)^A_B\bar{\xi}^B\bar{\sigma}^m\xi_A = 4ch(r)\mu_a^m, \qquad 2\bar{\xi}^A\bar{\sigma}^m\xi_A = 4ch(r)\mu_4^m \quad (a = 1, 2, 3). \tag{50}$$

Here c is the constant appearing in the definition of the Killing spinor. So the symbol is given by:

$$\sigma(D_{10}) = \begin{pmatrix} p_4W(r) & p_3 & -p_2 & -p_1W(r) & -4cp_1h(r) \\ -p_3 & p_4W(r) & p_1 & -p_2W(r) & -4cp_2h(r) \\ p_2 & -p_1 & p_4W(r) & -p_3W(r) & -4cp_3h(r) \\ pp_1p_4 & p_2p_4 & p_3p_4 & p_4^2 - 8c\left(p_1^2+p_2^2+p_3^2\right)h(r) & 2\left(p_1^2+p_2^2+p_3^2\right)W(r) \end{pmatrix}, \tag{51}$$

where $W(r) \equiv 2s(r)^2 - \frac{2c^2h(r)^2}{s(r)^2}$. This matrix can be block diagonalized in terms of 1×1 and 4×4 factors, the relevant part of the symbol to compute the index is the following 4×4 block,

$$\sigma(D'_{10}) = \begin{pmatrix} p_4W(r) & p_3 & -p_2 & -p_1 \\ -p_3 & p_4W(r) & p_1 & -p_2 \\ p_2 & -p_1 & p_4W(r) & -p_3 \\ p_1 & p_2 & p_3 & p_4W(r) \end{pmatrix}. \tag{52}$$

The determinant of this symbol is:

$$\mathrm{Det}(\sigma(D'_{10})) = \left(\frac{4c^4p_4^2h(r)^4}{s(r)^4} - 8c^2p_4^2h(r)^2 + p_1^2 + p_2^2 + p_3^2 + 4p_4^2s(r)^4\right)^2. \tag{53}$$

For $p_1 = p_2 = p_3 = 0$ and $p_4 \neq 0$, this value of determinant changes sign between North and South poles as discussed in Section 3.2, hence it has at least one zero. Therefore the symbol is not invertible at the location of that zero and by definition D_{10} cannot be elliptic. But restricting the momentum to $p_4 = 0$, σ is always invertible provided (p_1, p_2, p_3) are not all zero simultaneously. Therefore D_{10} is a transversally elliptic operator with respect to the symmetry generated by υ. In general the kernel and cokernel of such transversally elliptic operator are infinite dimensional, but since $[\hat{Q}^2, D_{ij}] = 0$, they both can be split into irredeucible representations of $\mathbf{H}$ with finite multiplicities, these multiplicities can be read off from the index theorem as explained in [4]. The index theorem localizes the contributions to the fixed points of the action of $\mathbf{H}$, that is to the North and South poles of the squashed S^4. According to the Atiyah–Bott [26] formula, the index is given by

[2] Strictly speaking the relevant differential operator for the index computation is a combination of the original D_{10} and D_{11}. But it turns out that this operator commutes with $\mathbf{H}$ and the distinction becomes irrelevant.

$$ind(D'_{10}) = \sum_{x=\mathit{fixed\ points}} \frac{\mathrm{Tr}_{E_0}(\gamma) - \mathrm{Tr}_{E_1}(\gamma)}{\det(1 - \frac{\partial \tilde{x}}{\partial x})}, \tag{54}$$

where γ denotes the eigenvalue of the action of the operator $e^{\iota \mathbf{H} t}$ on the vector and $SU(2)_R$ indices of the fields. So we need the action of $e^{\iota \mathbf{H} t}$ Near the North and South poles, on the local coordinates $z_1 \equiv x_1 + \iota x_2$, $z_2 \equiv x_3 + \iota x_4$, where we are defining near the North pole:

$$\begin{aligned} x_1 + \iota x_2 &= r \cos\left(\frac{\theta}{2}\right) e^{\iota \frac{\psi+\varphi}{2}}, \\ x_3 + \iota x_4 &= r \sin\left(\frac{\theta}{2}\right) e^{\iota \frac{\psi-\varphi}{2}}, \end{aligned} \tag{55}$$

so,

$$z_1 \to e^{4\iota c t} z_1 \equiv q_1 z_1, \qquad z_2 \to e^{4\iota c t} z_2 \equiv q_2 z_2, \tag{56}$$

with $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 4\pi$. As for the action of Q^2 on the fields of vector multiplet, its eigenvalues turn out to be of the same form as in [11], except that in our case $q_1 = q_2 = q = e^{4\iota c t}$. Putting all together, also the similar contribution from the South pole, we get the index D_{10}.

The one loop determinant can be computed by extracting the spectrum of eigenvalues of $\mathbf{H}$ from the index. For a non-abelian group G, with a_0 in its Cartan subalgebra, the one loop contribution of the vector-multiplet can be written as [11]:

$$\begin{aligned} Z^{vec}_{1-loop} &= \left(\frac{\det K_{fermion}}{\det K_{boson}}\right)^{\frac{1}{2}} \\ &= \prod_{\alpha \in \Delta_+} \frac{1}{(\hat{a}_0.\alpha)^2} \prod_{m,n \geq 0} ((m+n) + \iota \hat{a}_0.\alpha)((m+n+2) + \iota \hat{a}_0.\alpha) \\ &\quad \times ((m+n) - \iota \hat{a}_0.\alpha)((m+n+2) - \iota \hat{a}_0.\alpha) \\ &= \prod_{\alpha \in \Delta_+} \frac{\Upsilon_1(\iota \hat{a}_0.\alpha)\Upsilon_1(-\iota \hat{a}_0.\alpha)}{(\hat{a}_0.\alpha)^2}, \end{aligned} \tag{57}$$

where $\hat{a}_0 \equiv \frac{a_0}{4c}$. The function $\Upsilon(x)$ has zeros at $x = -(m+n), (m+n+2)$, this function is implemented to regularized the infinite products. It is defined by:

$$\Upsilon_b(x) = \prod_{n_1, n_2 \geq 0} (b n_1 + \frac{n_2}{b} + x)(b n_1 + \frac{n_2}{b} + b + \frac{1}{b} - x), \tag{58}$$

where b is a constant that in the case of [11] is exactly the squashing parameter, while and in our case $b = 1$.

6.3. Hypermultiplet one-loop contribution

We begin also with cohomological pairing [4,11] for the matter sector, the computation of the one-loop determinant reduces to that of the index of an operator D^{mat}_{10}. This operator corresponds to the terms bilinear in the fields Ξ and q_{IA} in the functional V_{mat}. Its symbol $\sigma(D^{mat}_{10})$ is given by

$$\sigma(D^{mat}_{10}) = \begin{pmatrix} \frac{2\left((p_3-\iota p_4)s(r)^4+c^2h(r)^2(p_3+\iota p_4)\right)}{s(r)^4+c^2h(r)^2} & 2(p_1+\iota p_2) \\ 2(p_1-\iota p_2) & -\frac{2\left((p_3+\iota p_4)s(r)^4+c^2h(r)^2(p_3-\iota p_4)\right)}{s(r)^4+c^2h(r)^2} \end{pmatrix}. \tag{59}$$

The determinant of this symbol is

$$\mathrm{Det}[\sigma(D^{mat}_{10})] = -\frac{4\left(s(r)^4-c^2h(r)^2\right)^2}{\left(c^2h(r)^2+s(r)^4\right)^2}p_4^2 - 4p_1^2 - 4p_2^2 - 4p_3^2. \tag{60}$$

For $p_1 = p_2 = p_3 = 0$, $p_4 \neq 0$, the determinant changes sign somewhere between North and South poles (see Section 3.2) and hence it possesses at least one zero. Therefore the operator D^{mat}_{10} is again transversally elliptic with respect to the isometry generated by L_v in the p_4 direction.

The index for the action of **H** on different fields at the poles can be calculated by using Atiyah–Bott formula. With $q_1 = q_2 = e^{4\iota ct}$ in our case of squashed S^4, the eigenvalues for the action of Q^2 on the matter multiplet case again turn out to have the same form as in [11].

For the hypermultiplets coupled to gauge symmetry, in the representation $R \bigoplus \bar{R}$ the final result for the one-loop determinant for the hypermultiplets becomes:

$$\begin{aligned} Z^{hyp}_{1-loop} &= \prod_{\rho\in R}\prod_{m,n\geq 0}((m+n+1)-\iota\hat{a}_0.\alpha)^{-1}((m+n+1)+\iota\hat{a}_0.\alpha)^{-1} \\ &= \prod_{\rho\in R}\Upsilon_1(\iota\hat{a}_0.\rho+1)^{-1}, \end{aligned} \tag{61}$$

where ρ runs over all the weights in a given representation.

7. Instanton contribution

Near the North pole the Killing spinor evaluates to

$$\xi = \begin{pmatrix} s_{n_0} & 0 \\ 0 & s_{n_0} \\ \frac{\iota cr}{s_{n_0}} & 0 \\ 0 & -\frac{\iota cr}{s_{n_0}} \end{pmatrix}, \tag{62}$$

so that $\xi^A\xi_A = 2s_{n_0}^2$ and $\bar{\xi}_A\bar{\xi}^A = \frac{2c^2r^2}{s_{n_0}^2}$. Since $\bar{\xi}_A\bar{\xi}^A \to 0$ at the North pole, the localization equation has to be evaluated away from the North pole to have smooth gauge field configurations.

Similarly near the South pole

$$\xi = \begin{pmatrix} (\pi-r)s_{s_1} & 0 \\ 0 & (\pi-r)s_{s_1} \\ \frac{\iota c}{s_{s_1}} & 0 \\ 0 & -\frac{\iota c}{s_{s_1}} \end{pmatrix}, \tag{63}$$

and $\xi^A\xi_A = 2(\pi-r)^2 s_{s_1}^2$ and $\bar{\xi}_A\bar{\xi}^A = \frac{2c^2}{s_{s_1}^2}$. In this case $\xi^A\xi_A \to 0$. Therefore the South pole has also to be excluded if smooth gauge field configurations are assumed.

To include the contribution from the poles, we first notice that because $\bar{\xi}_A \bar{\xi}^A \to 0$ at the North pole, in general $F^+_{mn} \neq 0$, $F^-_{mn} = 0$ there and still solve the localization equation. These configurations are the pointlike anti-instantons contribution.

Also at the North pole the following condition is satisfied for our background

$$\frac{1}{4}\Omega^{ab}_m \sigma_{ab}\xi_A + \iota\xi_B V^B_{mA} = 0. \tag{64}$$

Likewise, at the South pole $\xi^A \xi_A \to 0$, and we get the point instanton contribution $F^+_{mn} = 0$, $F^-_{mn} \neq 0$ and the following twisting condition is satisfied

$$\frac{1}{4}\Omega^{ab}_m \bar{\sigma}_{ab}\bar{\xi}_A + \iota\bar{\xi}_B V^B_{mA} = 0. \tag{65}$$

The Killing vector near the North pole can be written as

$$\upsilon^m \frac{\partial}{\partial x_m} = 4c(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}) + 4c(x_3 \frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_3}). \tag{66}$$

Notice that near the South pole our $N = 2$ theory on squashed S^4 approaches topologically twisted theory with Omega deformation parameters $\epsilon_1 = 4c$, $\epsilon_2 = 4c$ [2,3], and the contribution of these point-instantons is given by $Z_{inst}(a_0, \epsilon_1, \epsilon_2, \tau)$, where the parameter τ is defined by $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi\iota}{g^2_{YM}}$.

Whereas near the North pole, the contribution of point anti-instantons is given by $Z_{inst}(a_0, \epsilon_1, \epsilon_2, \bar{\tau})$. Putting all together, the final form of the squashed S^4 partition function is

$$Z = \int d\hat{a}_0 e^{-\frac{2\pi^3 \,\mathrm{Tr}[a_0^2]}{c^2 g^2_{YM}}} |Z_{inst}|^2 \prod_{\alpha\in\Delta_+} \Upsilon_1(\iota\hat{a}_0.\alpha)\Upsilon_1(-\iota\hat{a}_0.\alpha) \prod_{\rho\in R} \Upsilon_1(\iota\hat{a}_0.\rho + 1)^{-1}. \tag{67}$$

8. Conclusions

We have computed the partition function of $N = 2$ SUSY on squashed S^4 which admits $SU(2) \times U(1)$ isometry, using SUSY localization technique. We find that the full partition function is independent of the squashing parameters as well as the other supergravity background fields.

The squashing functions independence of the one-loop part of the partition function, which is obvious from the form of the relevant Killing vector v, can perhaps be attributed to the fact that in our squashed S^4 the theory is topologically twisted at the poles. This is because the $SU(2)_R$ symmetry which is generically broken down to $U(1)_R$ on the squashed S^4 excluding the poles, is again enhanced to $SU(2)_R$ at the poles. So this $SU(2)_R$ can be identified at the poles with the $SU(2)$ Lorentz isometry to topologically twist the theory. The classical part can be written as a total derivative and gives to a contribution which is independent of the squashing parameters.

It will be interesting to explain this independence along the same lines given in [17]. That is to say, if we deform the vector multiplet and hypermultiplet actions around the round S^4 with respect to e.g. $f(r)$, it might be possible to write these deformed actions as Q-exact terms separately. This Q-exactness of the deformed action will explain the independence of partition function of the parameter $f(r)$ in the sense of [17]. However we have to consider perturbations around the round S^4, unlike [17], where it is perturbed around flat R^4.

Acknowledgements

We would like to thank G. Bonelli for discussions, and L.F. Alday and L. Pando-Zayas for comments.

References

[1] E. Witten, Two-dimensional gauge theories revisited, J. Geom. Phys. 9 (1992) 303–368, http://dx.doi.org/10.1016/0393-0440(92)90034-X, arXiv:hep-th/9204083.
[2] N.A. Nekrasov, Seiberg–Witten prepotential from instanton counting, arXiv:hep-th/0306211.
[3] N. Nekrasov, A. Okounkov, Seiberg–Witten theory and random partitions, arXiv:hep-th/0306238.
[4] V. Pestun, Localization of gauge theory on a four-sphere and supersymmetric Wilson loops, Commun. Math. Phys. 313 (2012) 71–129, http://dx.doi.org/10.1007/s00220-012-1485-0, arXiv:0712.2824.
[5] A. Kapustin, B. Willett, I. Yaakov, Exact results for Wilson loops in superconformal Chern–Simons theories with matter, J. High Energy Phys. 1003 (2010) 089, http://dx.doi.org/10.1007/JHEP03(2010)089, arXiv:0909.4559.
[6] I. Yaakov, Localization of gauge theories on the three-sphere, Phd Thesis, Caltech, http://thesis.library.caltech.edu/7116/.
[7] R.K. Gupta, S. Murthy, All solutions of the localization equations for $N = 2$ quantum black hole entropy, J. High Energy Phys. 1302 (2013) 141, http://dx.doi.org/10.1007/JHEP02(2013)141, arXiv:1208.6221.
[8] N. Hama, K. Hosomichi, S. Lee, Notes on SUSY gauge theories on three-sphere, J. High Energy Phys. 1103 (2011) 127, http://dx.doi.org/10.1007/JHEP03(2011)127, arXiv:1012.3512.
[9] T. Nosaka, S. Terashima, Supersymmetric gauge theories on a squashed four-sphere, J. High Energy Phys. 1312 (2013) 001, http://dx.doi.org/10.1007/JHEP12(2013)001, arXiv:1310.5939.
[10] J. Gomis, S. Lee, Exact Kahler potential from gauge theory and mirror symmetry, J. High Energy Phys. 1304 (2013) 019, http://dx.doi.org/10.1007/JHEP04(2013)019, arXiv:1210.6022.
[11] N. Hama, K. Hosomichi, Seiberg–Witten theories on ellipsoids, J. High Energy Phys. 1209 (2012) 033, http://dx.doi.org/10.1007/JHEP10(2012)051, arXiv:1206.6359.
[12] Y. Imamura, Perturbative partition function for squashed S^5, arXiv:1210.6308.
[13] G. Festuccia, N. Seiberg, Rigid supersymmetric theories in curved superspace, J. High Energy Phys. 1106 (2011) 114, http://dx.doi.org/10.1007/JHEP06(2011)114, arXiv:1105.0689.
[14] T.T. Dumitrescu, G. Festuccia, N. Seiberg, Exploring curved superspace, J. High Energy Phys. 1208 (2012) 141, http://dx.doi.org/10.1007/JHEP08(2012)141, arXiv:1205.1115.
[15] C. Klare, A. Zaffaroni, Extended supersymmetry on curved spaces, J. High Energy Phys. 1310 (2013) 218, http://dx.doi.org/10.1007/JHEP10(2013)218, arXiv:1308.1102.
[16] L.F. Alday, D. Martelli, P. Richmond, J. Sparks, Localization on three-manifolds, J. High Energy Phys. 1310 (2013) 095, http://dx.doi.org/10.1007/JHEP10(2013)095, arXiv:1307.6848.
[17] C. Closset, T.T. Dumitrescu, G. Festuccia, Z. Komargodski, The geometry of supersymmetric partition functions, J. High Energy Phys. 1401 (2014) 124, http://dx.doi.org/10.1007/JHEP01(2014)124, arXiv:1309.5876.
[18] T. Nishioka, I. Yaakov, Supersymmetric Rényi entropy, J. High Energy Phys. 1310 (2013) 155, http://dx.doi.org/10.1007/JHEP10(2013)155, arXiv:1306.2958.
[19] X. Huang, S.-J. Rey, Y. Zhou, Three-dimensional SCFT on conic space as hologram of charged topological black hole, J. High Energy Phys. 1403 (2014) 127, http://dx.doi.org/10.1007/JHEP03(2014)127, arXiv:1401.5421.
[20] Y. Imamura, D. Yokoyama, $\mathcal{N} = 2$ supersymmetric theories on squashed three-sphere, Int. J. Mod. Phys. Conf. Ser. 21 (2013) 171–172, http://dx.doi.org/10.1142/S2010194513009665.
[21] X. Huang, Y. Zhou, $N = 4$ super-Yang–Mills on conic space as hologram of STU topological black hole, arXiv:1408.3393.
[22] M. Crossley, E. Dyer, J. Sonner, Super-Rényi entropy & Wilson loops for $N = 4$ SYM and their gravity duals, arXiv:1409.0542.
[23] H. Casini, M. Huerta, R.C. Myers, Towards a derivation of holographic entanglement entropy, J. High Energy Phys. 1105 (2011) 036, http://dx.doi.org/10.1007/JHEP05(2011)036, arXiv:1102.0440.
[24] I.R. Klebanov, S.S. Pufu, S. Sachdev, B.R. Safdi, Rényi entropies for free field theories, J. High Energy Phys. 1204 (2012) 074, http://dx.doi.org/10.1007/JHEP04(2012)074, arXiv:1111.6290.
[25] T.T. Dumitrescu, G. Festuccia, Exploring curved superspace (II), J. High Energy Phys. 1301 (2013) 072, http://dx.doi.org/10.1007/JHEP01(2013)072, arXiv:1209.5408.
[26] M.F. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes: I, Ann. Math. 86 (2) (1967) 374–407, http://www.jstor.org/stable/1970694.