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An iterative method for a second order problem with nonlinear two-point boundary conditions

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Abstract

A semi-linear second order ODE under a nonlinear two-point boundary condition is considered. Under appropriate conditions on the nonlinear term of the equation, we define a two-dimensional shooting argument which allows to obtain solutions for some specific situations by the use of Poincaré-Miranda's theorem. Finally, we apply this result combined with the method of upper and lower solutions and develop an iterative sequence that converges to a solution of the problem.

Keywords: Nonlinear two-point boundary conditions; upper and lower solutions; iterative methods.

MSC(2000): 34B15

1 Introduction

We study the semi-linear second order ODE

$$u''(t) + g(t, u(t), u'(t)) = 0, \qquad 0 < t < T \tag{1}$$

under a nonlinear two-point boundary condition.

Problem (1) under various boundary conditions has been studied by many authors. In the pioneering work of Picard [18], the existence of a solution for the Dirichlet problem was proved by the well-known method of successive approximations, assuming that g is Lipschitz and T is small. These conditions have been improved by Hamel [9], for the special case of a forced pendulum equation (see also [13], [14]). For general $g = g(\cdot, u)$, the variational approach has been employed already in 1915 by Lichtenstein [12]. However, when g depends on u' the problem has non-variational structure, and different techniques are required. As a historical antecedent of the topological methods, we may mention the shooting method introduced in 1905 by Severini [20]; later on, more abstract topological tools have been applied, such as the Leray-Schauder degree theory. For an overview of the use of topological methods to this kind of problems, we refer the reader to [15].

The above-mentioned two-point boundary conditions, as well as some other standard ones, such as the Neumann or the Sturm-Liouville conditions, are *linear*;

it is worthy to mention, however, that the general nonlinear case

$$\phi(u(0), u(T), u'(0), u'(T)) = 0, \tag{2}$$

where $\phi : \mathbb{R}^4 \to \mathbb{R}$ is continuous is very important in applications and, in recent years, a considerable number of works have been developed in this direction.

We shall study the existence of solutions of (1) under a particular case of condition (2): namely, nonlinear boundary conditions of the type

$$u'(0) = f_0(u(0)), \quad u'(T) = f_T(u(T)) \tag{3}$$

where $f_0, f_T : \mathbb{R} \to \mathbb{R}$ are given continuous functions. The special case $f_i(x) = a_i x + b_i$ for $i = 0, T$ corresponds to the Sturm-Liouville conditions, and Neumann conditions when $a_0 = a_T = 0$. Our interest in (3) relies on some models in nonlinear beam theory, usually leading to fourth order problems [7], but that admit second order analogues (see e.g. [19]). The results in the present paper complement and extend those in [1].

The paper is organized as follows. In the second section, we impose a growth condition on g, which allows to prove the unique solvability of the associated Dirichlet problem. Furthermore, we prove that the trace mapping $Tr : \mathcal{S} \to \mathbb{R}^2$ given by $Tr(u) = (u(0), u(T))$, where

$$\mathcal{S} := \{u \in H^2(0, T) : u''(t) + g(t, u(t), u'(t)) = 0\} \tag{4}$$

is a homeomorphism for the H^2-norm.

Then, we define a two-dimensional shooting argument, which proves to be successful with the aid of the Poincaré-Miranda theorem (see e.g. [11]) in some particular situations, which include the Sturm-Liouville boundary conditions. This generalizes some of the results in [2], and constitutes the main tool for our iterative method for problem (1)-(3), developed in the third section.

Our method, based on the existence of an ordered couple (α, β) of a lower and an upper solution, has been successfully applied to different boundary value problems when g does not depend on u'. For general g, existence results can still be obtained if one assumes a Nagumo-Bernstein type condition (see [3], [16])). However, these results are usually proved by fixed point or degree arguments and, in consequence, they are non-constructive.

We shall assume instead a Lipschitz condition on u', which is more restrictive, but allows the construction of a non-increasing (resp. non-decreasing) sequence of upper (lower) solutions that converges to a solution of the problem. Our method is slightly different from the monotone techniques known in the literature for linear boundary conditions, see e.g. [4], [17] among others (for upper and lower solutions in the reversed order, see [10]).

2 A continuum of solutions of (1)

For simplicity, let us assume that g is continuous, and write it as

$$g(t,u(t),u'(t)) = r(t)u'(t) + h(t,u(t),u'(t)),$$

with $r \in W^{1,\infty}(0,T)$. We shall assume that h satisfies a global Lipschitz condition on u', namely

$$\Big|\frac{h(t,u,A)-h(t,u,B)}{A-B}\Big| \le k < \frac{\pi}{T} \qquad \text{for} \quad A \neq B. \tag{5}$$

Furthermore, in this section we shall assume the following one-side growth condition on u:

$$\frac{h(t,u,A)-h(t,v,A)}{u-v} \le c \tag{6}$$

for $u \neq v$, where the constant $c \in \mathbb{R}$ satisfies

$$c + \frac{k\pi}{T} < \Big(\frac{\pi}{T}\Big)^2 + \frac{1}{2}\inf_{0\le t\le T} r'(t). \tag{7}$$

Under these assumptions, the set $\mathcal{S}$ of solutions of (1) defined by (4) is homeomorphic to $\mathbb{R}^2$. More precisely,

Theorem 1. *Assume that (5) and (6) hold and let $x, y \in \mathbb{R}$. Then there exists a unique solution $u_{x,y}$ of (1) satisfying the non-homogeneous Dirichlet condition*

$$u_{x,y}(0) = x, \qquad u_{x,y}(T) = y$$

Furthermore, the mapping $Tr : \big(\mathcal{S}, \|\cdot\|_{H^2}\big) \to \mathbb{R}^2$ given by $Tr(u) = (u(0), u(T))$ is a homeomorphism.

Proof. For fixed $v \in H^1(0,T)$, let $u := \mathcal{T}v$ be defined as the unique solution of the linear problem

$$u'' = -[rv' + h(\cdot, v, v')]$$
$$u(0) = x, \quad u(T) = y.$$

It is immediate that $\mathcal{T} : H^1(0,T) \to H^1(0,T)$ is compact. Moreover, if $S_\sigma : H^2(0,T) \to L^2(0,T)$ is the semilinear operator defined by $S_\sigma u := u'' + \sigma[ru' + h(\cdot,u,u')]$, with $\sigma \in [0,1]$, then using (6) and (7) it is seen that the following a *priori* bound holds for any $u, v \in H^2(0,T)$ with $u - v \in H_0^1(0,T)$:

$$\|u' - v'\|_{L^2} \le \mu \|S_\sigma u - S_\sigma v\|_{L^2} \tag{8}$$

for some constant μ independent of σ.

Hence, if $u = \sigma\mathcal{T}u$ for some $\sigma \in [0,1]$, then setting $l_{x,y}(t) = \frac{y-x}{T}t + x$ we obtain:

$$\|u' - \sigma l'_{x,y}\|_{L^2} \le \mu \|S_\sigma(\sigma l_{x,y})\|_{L^2} \le C$$

for some constant C depending only on x and y. Existence of solutions follows from the Leray-Schauder Theorem. Uniqueness is an immediate consequence of (8) for $\sigma = 1$.

Thus, Tr is bijective, and its continuity is clear. On the other hand, if $(x, y) \to (x_0, y_0)$, then applying (8) to $u = u_{x,y} - l_{x,y}$ and $v = u_{x_0,y_0} - l_{x_0,y_0}$ it is easy to see that $u_{x,y} \to u_{x_0,y_0}$ for the H^1-norm. As $u_{x,y}$ and u_{x_0,y_0} satisfy (1), we conclude from (5) that also $u''_{x,y} \to u''_{x_0,y_0}$ for the L^2-norm and so completes the proof. □

It is worth noticing that the previous result allows to define a two-dimensional shooting argument as follows: let $\Theta : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\Theta(x, y) = (u'_{x,y}(0) - f_0(x), u'_{x,y}(T) - f_T(y)).$$

From the previous theorem, we deduce that Θ is continuous, and it is clear that, if $\Theta(x, y) = (0, 0)$, then $u_{x,y}$ is a solution of the problem.

Example 1. *Assume that (5) and (6) hold, and that*

$$h(t, u, 0)sgn(u) < 0 \quad \text{for } |u| \geq M, \tag{9}$$

$$f_0(M^+) \geq 0 \geq f_0(M^-), \qquad f_T(M^+) \leq 0 \leq f_T(M^-) \tag{10}$$

for some constants $M^- \leq -M < M \leq M^+$. Then (1)-(3) admits at least one solution.

In particular, the result holds for the Sturm-Liouville conditions

$$u'(0) = a_0 u(0) + b_0, \quad u'(T) = a_T u(T) + b_T, \qquad a_0 > 0 > a_T. \tag{11}$$

Furthermore, in this case the solution is unique, provided that $c < 0$ in (6).

Indeed, it follows from (9) that $u_{x,y}$ cannot attain in $(0, T)$ neither a maximum value larger than M , nor a minimum value smaller than $-M$. Moreover, for $M^- \leq y \leq M^+$ we obtain:

$$u_{M^+,y}(0) = M^+ \geq y = u_{M^+,y}(T), \qquad u_{M^-,y}(0) = M^- \leq y = u_{M^-,y}(T).$$

Thus, $u'_{M^+,y}(0) \leq 0 \leq u'_{M^-,y}(0)$, and hence $\Theta_1(M^+, y) \leq 0 \leq \Theta_1(M^-, y)$. In the same way, it follows that $\Theta_2(x, M^+) \geq 0 \geq \Theta_2(x, M^-)$ for $M^- \leq x \leq M^+$. By the Poincaré-Miranda's generalized intermediate value theorem, we conclude that Θ has at least one zero $(x, y) \in [M^-, M^+] \times [M^-, M^+]$.

On the other hand, if u and v are solutions of (1)-(11), then

$$(u - v)'' + (r + \psi)(u - v)' + h(\cdot, u, v') - h(\cdot, v, v') = 0$$

where

$$\psi = \frac{h(\cdot, u, u') - h(\cdot, u, v')}{u' - v'} \in L^\infty(0, T).$$

Next, take $p(t) = e^{\int_0^t (r(s)+\psi(s))\,ds}$*, multiply the previous equality by* $(u-v)p$ *and integrate. We obtain:*

$$0 = p(u'-v')(u-v)\Big|_0^T - \int_0^T p(u'-v')^2 + \int_0^T p[h(\cdot,u,v') - h(\cdot,v,v')](u-v)$$

$$\leq p(T)a_T(u-v)^2(T) - a_0(u-v)^2(0) - \int_0^T p(u'-v')^2 + c\int_0^T p(u-v)^2.$$

Hence, for $c < 0$ *it is seen that* $u = v$*.*

3 Iterative sequences of upper and lower solutions

In this section we shall construct solutions of (1) under the two-point boundary condition (3) by an iterative method, based upon the existence of upper and lower solutions.

Let us recall that (α,β) is an ordered couple of a lower and an upper solution for (1) if $\alpha \leq \beta$ and

$$\alpha'' + g(\cdot,\alpha,\alpha') \geq 0 \geq \beta'' + g(\cdot,\beta,\beta').$$

Existence results under various boundary conditions in presence of an ordered couple of a lower and an upper solution are known (see e. g. [6]). In our particular case, we shall assume the boundary constraints:

$$\alpha'(0) - f_0(\alpha(0)) \geq 0 \geq \beta'(0) - f_0(\beta(0)),$$

$$\alpha'(T) - f_T(\alpha(T)) \leq 0 \leq \beta'(T) - f_T(\beta(T))$$

and a Nagumo type condition adapted from [5]:

$$|g(t,u,v)| \leq \psi(|v|), \qquad \text{for } \alpha(t) \leq u \leq \beta(t), m \leq |v| \leq M \tag{12}$$

where $\psi : [0,+\infty) \to (0,+\infty)$ is continuous and satisfies:

$$\int_m^M \frac{1}{\psi(t)}\,dt > T,$$

and

$$m = \min\left\{\frac{|\alpha(0)-\beta(T)|}{T}, \frac{|\alpha(T)-\beta(0)|}{T}, \max_{\alpha(0)\leq s\leq\beta(0)} |f_0(s)|, \max_{\alpha(T)\leq s\leq\beta(T)} |f_T(s)|\right\}$$

$$M > \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty, m\}.$$

Then, the following existence result can be obtained as in [1]:

Theorem 2. *Assume there exists an ordered couple* (α,β) *of a lower and an upper solution as before, and that (12) holds. Then the boundary value problem (1)-(3) admits at least one solution* u*, with* $\alpha \leq u \leq \beta$*.*

Sketch of the proof. The proof follows the outline of the standard results on the subject. Let $P(t,u) = \max\{\alpha(t), \min\{u, \beta(t)\}\}$ and $Q(v) = sgn(v)\min\{|v|, M\}$, and apply Schauder's Theorem in order to obtain a solution of the problem

$$u''(t) - \lambda u(t) = -g(t, P(t,u(t)), Q(u'(t))) - \lambda P(t,u(t)),$$
$$u'(0) = f_0(P(0,u(0))), \qquad u'(T) = f_T(P(T,u(T))$$

for some fixed $\lambda > 0$. It is easy to see that $\alpha \le u \le \beta$, and hence $P(t,u(t)) = u(t)$ for every t. Furthermore, if we suppose that for example $u'(t_1) = M$, then there exists t_0 such that $u'(t_0) = m$ and $m < u'(t) < M$ for t between t_0 and t_1. Hence

$$T < \int_m^M \frac{1}{\psi(s)} ds = \int_{t_0}^{t_1} \frac{u''(t)}{\psi(u'(t))} dt \le |t_1 - t_0|,$$

a contradiction. The same conclusion holds if we suppose $u'(t_1) = -M$; thus, $|u'(t)| < M$ and the proof is complete.

□

Example 2. *The previous result applies when (9) and (10) hold: indeed, in this case it is clear (M^-, M^+) is an ordered couple of a lower and an upper solution. Thus, conditions (5) and (6) in example 1 can be dropped.*

Also, we may consider the forced pendulum equation with friction

$$u'' + ru' + \sin u = \theta,$$

and assume that the forcing term θ is a measurable function satisfying:

$$-1 \le \theta(t) \le 1 \qquad \forall\, t \in [0,T].$$

Then $\alpha \equiv \frac{\pi}{2}$ and $\beta \equiv \frac{3}{2}\pi$ are respectively a lower and an upper solution. Hence, (1)-(3) has a solution for any continuous f_0 and f_T such that

$$f_0\left(\frac{\pi}{2}\right) \le 0 \le f_0\left(\frac{3\pi}{2}\right)$$

and

$$f_T\left(\frac{\pi}{2}\right) \ge 0 \ge f_T\left(\frac{3\pi}{2}\right).$$

Our last result is concerned with the construction of solutions by iteration, provided that h and f satisfy some stronger assumptions.

Let us firstly establish the following auxiliary lemmas:

Lemma 1. *Assume that (5) holds and let λ be a positive constant satisfying $\lambda \ge k\frac{\pi}{T} - \left(\frac{\pi}{T}\right)^2 - \frac{1}{2}\inf r'$. Then for any $z, \theta \in C([0,T])$ the equation*

$$u'' + ru' + h(\cdot, z, u') - \lambda u = \theta$$

is uniquely solvable under the Sturm-Liouville conditions (11). Furthermore, the mapping $\mathcal{K} : C([0,T])^2 \to C([0,T])$ given by $\mathcal{K}(z,\theta) = u$ is compact.

Proof. Existence and uniqueness follow as a particular case of example 1, with $\overline{g}(\cdot, u, u') = ru' + \overline{h}(\cdot, u, u')$, where

$$\overline{h}(\cdot, u, u') = h(\cdot, z, u') - \lambda u - \theta.$$

Indeed, it is clear that $\overline{h}$ satisfies (5) and (6) with $c = -\lambda$. Moreover,

$$\overline{h}(t, u, 0)sgn(u) = (h(t, z(t), 0) - \theta(t))sgn(u) - \lambda|u| < 0$$

when $|u| > \|h(\cdot, z, 0) - \theta\|_\infty$. Thus, (9) is also satisfied.

Let (z, θ) tend to (z_0, θ_0), and set $u = \mathcal{K}(z, \theta)$, $u_0 = \mathcal{K}(z_0, \theta_0)$. Then

$$(u - u_0)'' + (r + \psi)(u - u_0)' - \lambda(u - u_0) = h(\cdot, z, u_0') - h(\cdot, z_0, u_0') + \theta - \theta_0$$

where $\psi = \frac{h(\cdot,z,u')-h(\cdot,z,u_0')}{u'-u_0'}$. Hence, continuity of $\mathcal{K}$ is a consequence of the following estimate, which is valid for any w satisfying (11) with $b_0 = b_T = 0$ and some constant c depending only on k:

$$\|w\|_{H^1} \leq c\|w'' + (r + \psi)w' - \lambda w\|_{L^2}.$$

Indeed, this is easily deduced by applying the Cauchy-Schwartz inequality to the integral $\int_0^T pLw.w$, where $Lw = w'' + (r + \psi)w' - \lambda w$ and $p(t) = e^{\int_0^t (r(s)+\psi(s))\, ds}$, and the fact that $0 < m \leq p \leq M$ for some m and M depending only on k. Finally, compactness of $\mathcal{K}$ follows from the imbedding $H^1(0, T) \hookrightarrow C([0, T])$.

□

Remark 1. *In the previous proof, an analogous estimate can be also obtained for the H^2-norm of w. This implies the compactness of $\mathcal{K}$, but now regarded as an operator from $C([0, T])^2$ to $C^1([0, T])$. More generally, one might consider also a_i and b_i as variables for $i = 0, T$: in this case, $\mathcal{K}$ could be seen as a compact operator from $\mathbb{R}^4 \times C([0, T])^2$ to $C^1([0, T])$.*

Lemma 2. *Let $\phi \in L^\infty(0, T)$ and assume that $w'' + \phi w' - \lambda w \geq 0$ (in the weak sense) for some $\lambda \geq 0$, and*

$$w'(0) - a_0 w(0) \geq 0 \geq w'(T) - a_T w(T)$$

with $a_0 > 0 > a_T$. Then $w \leq 0$.

Proof. If $w(0), w(T) \leq 0$, the result is the well-known maximum principle for Dirichlet conditions.

If for example $w(0) > 0$, then restricting w up to its first zero if necessary, it suffices to consider only the case $w \geq 0$. Taking $p(t) = e^{\int_0^t \phi(s)\, ds}$, it is observed that $(pw')' \geq \lambda pw \geq 0$. Thus, pw' is nondecreasing on $[0, T]$, and hence

$$0 \geq p(T)a_T w(T) \geq p(T)w'(T) \geq p(0)w'(0) \geq p(0)a_0 w(0) > 0,$$

a contradiction. The proof is similar when $w(T) > 0$. □

In order to define our iterative scheme, we shall assume that f_0 and f_T satisfy a one-side Lipschitz condition:

(F) There exists a positive constant R such that

$$f_0(y) - f_0(x) \le R(y-x)$$

if $\alpha(0) \le x < y \le \beta(0)$, and

$$f_T(y) - f_T(x) \ge -R(y-x)$$

if $\alpha(T) \le x < y \le \beta(T)$.

In virtue of Lemma 1, if (5) holds then for $\lambda = \min\{R, k\frac{\pi}{T} - \left(\frac{\pi}{T}\right)^2 - \frac{1}{2}\inf r'\}$, we may define the compact operator $\mathcal{T} : C([0,T]) \to C([0,T])$ given by $\mathcal{T}v = u$, where u is the unique solution of the problem

$$u'' + ru' + h(\cdot, v, u') - \lambda u = -\lambda v$$

satisfying the following Sturm-Liouville condition:

$$u'(0) - Ru(0) = f_0(v(0)) - Rv(0), \quad u'(T) + Ru(T) = f_T(v(T)) + Rv(T).$$

From Remark 1, we observe, moreover, that the set $\mathcal{T}(\{v : \alpha \le v \le \beta\})$ is bounded for the C^1-norm. In particular, this implies the existence of a constant $M = M(R)$ such that if $u = \mathcal{T}v$ for some v lying between α and β, then $\|u'\|_\infty \le M$. This suggests to consider the following Lipschitz condition on h:

(H)

$$|h(t,u,A) - h(t,v,A)| \le R|u-v|$$

for u, v such that $\alpha(t) \le u < v \le \beta(t)$ and $|A| \le M(R)$.

Remark 2. *Conditions (F) and (H) are trivially satisfied if f_0, f_T and h are C^1 functions, and $\frac{\partial h}{\partial u}$ is bounded with respect to u'.*

Theorem 3. *Assume there exists an ordered couple (α, β) of a lower and an upper solution as before. Further, assume that (5), (H) and (F) hold. Set λ as before, and define the sequences $\{\underline{u}_n\}$ and $\{\overline{u}_n\}$ recursively by*

$$\underline{u}_0 = \alpha, \qquad \overline{u}_0 = \beta$$

and

$$\overline{u}_{n+1} = \mathcal{T}\overline{u}_n, \qquad \underline{u}_{n+1} = \mathcal{T}\underline{u}_n$$

Then $(\underline{u}_n\, \overline{u}_n)$ is an ordered couple of a lower and an upper solution. Furthermore, $\{\overline{u}_n\}$ (resp. $\{\underline{u}_n\}$) is non-increasing (non-decreasing) and converges to a solution of the problem.

Proof. Let us firstly prove that $\alpha \leq \overline{u}_1 \leq \beta$. From the definition,

$$\overline{u}_1'' + r\overline{u}_1' + h(\cdot, \beta, \overline{u}_1') - \lambda \overline{u}_1 = -\lambda\beta \geq -\lambda\beta + \beta'' + r\beta' + h(\cdot, \beta, \beta').$$

Hence, setting

$$\psi = \frac{h(\cdot, \beta, \overline{u}_1') - h(\cdot, \beta, \beta')}{\overline{u}_1' - \beta'} \in L^\infty(0, T)$$

we deduce that

$$(\overline{u}_1 - \beta)'' + (r + \psi)(\overline{u}_1 - \beta)' - \lambda(\overline{u}_1 - \beta) \geq 0.$$

On the other hand,

$$\overline{u}_1'(0) - R\overline{u}_1(0) = f_0(\beta(0)) - R\beta(0)$$

and

$$\overline{u}_1'(T) + R\overline{u}_1(T) = f_T(\beta(T)) + R\beta(T).$$

Thus,

$$(\overline{u}_1 - \beta)'(0) - R(\overline{u}_1 - \beta)(0) = 0 = (\overline{u}_1 - \beta)'(T) - R(\overline{u}_1 - \beta)(T),$$

and from Lemma 2 we obtain that $\overline{u}_1 \leq \beta$.

In the same way,

$$\overline{u}_1'' + r\overline{u}_1' + h(\cdot, \beta, \overline{u}_1') - \lambda \overline{u}_1 \leq -\lambda\beta + \alpha'' + r\alpha' + h(\cdot, \alpha, \alpha')$$

and hence

$$(\overline{u}_1 - \alpha)'' + (r + \psi)(\overline{u}_1 - \alpha)' - \lambda(\overline{u}_1 - \alpha) \geq 0$$

where

$$\psi = \frac{h(\cdot, \alpha, \overline{u}_1') - h(\cdot, \alpha, \alpha')}{\overline{u}_1' - \alpha'} \in L^\infty(0, T).$$

Also

$$\overline{u}_1'(0) - R\overline{u}_1(0) = f_0(\beta(0)) - R\beta(0) \leq f_0(\alpha(0)) - R\alpha(0)$$

and

$$\overline{u}_1'(T) + R\overline{u}_1(T) = f_T(\beta(T)) + R\beta(T) \geq f_T(\alpha(T)) + R\alpha(T),$$

and we conclude that $\overline{u}_1 \geq \alpha$.

On the other hand,

$$\overline{u}_1'' + r\overline{u}_1' + h(\cdot, \overline{u}_1, \overline{u}_1') = (\lambda - R)(\overline{u}_1 - \beta) + [h(\cdot, \overline{u}_1, \overline{u}_1') + R\overline{u}_1] - [h(\cdot, \beta, \overline{u}_1') + R\beta] \leq 0,$$

and we deduce that $\overline{u}_1$ is an upper solution of the problem. Inductively, it follows that $\overline{u}_n$ is an upper solution for every n, with $\alpha \leq \overline{u}_{n+1} \leq \overline{u}_n$, which by Dini's theorem implies that $\overline{u}_n$ converges uniformly to a function $\overline{u}$. From the definition of $\{\overline{u}_n\}$,

$$\overline{u}''_{n+1} + r\overline{u}'_{n+1} + h(\cdot, \overline{u}_n, \overline{u}'_{n+1}) \to 0$$

uniformly. Moreover, from Lemma 1 and Remark 1 we know that $\{\overline{u}_n\}$ is bounded in $H^2(0,T)$, and it follows easily that

$$\overline{u}'' + r\overline{u}' + h(\cdot, \overline{u}, \overline{u}') = 0.$$

Thus, $\overline{u}$ is a solution of the problem. The proof for $\underline{u}_n$ is analogous. Moreover, if we assume as inductive hypothesis that $\underline{u}_n \le \overline{u}_n$, then

$$\overline{u}''_{n+1} + r\overline{u}'_{n+1} + h(\cdot, \overline{u}_n, \overline{u}'_{n+1}) - \lambda\overline{u}_{n+1} = -\lambda\overline{u}_n$$
$$\le -\lambda\underline{u}_n = \underline{u}''_{n+1} + r\underline{u}'_{n+1} + h(\cdot, \underline{u}_n, \underline{u}'_{n+1}) - \lambda\underline{u}_{n+1}.$$

In the same way as before, we may define

$$\psi = \frac{h(\cdot, \underline{u}_n, \overline{u}'_{n+1}) - h(\cdot, \underline{u}_n, \underline{u}'_{n+1})}{\overline{u}'_{n+1} - \underline{u}'_{n+1}} \in L^\infty(0,T),$$

and hence for $w = \overline{u}_{n+1} - \underline{u}_{n+1}$ we deduce:

$$w'' + (r+\psi)w' - \lambda w \le h(\cdot, \underline{u}_n, \overline{u}'_{n+1}) - h(\cdot, \overline{u}_n, \overline{u}'_{n+1}) \le -R(\underline{u}_n - \overline{u}_n) \le -Rw.$$

From Lemma 2, we conclude that $w \ge 0$, i.e. $\underline{u}_{n+1} \le \overline{u}_{n+1}$. □

Remark 3. *It is interesting to observe that, even if (5) is somewhat too restrictive, some condition regarding the growth of h with respect to u' is required. We may recall, for instance, the following example by Habets and Pouso [8] for the mean curvature operator:*

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = u + a,$$

where the function $a \in L^\infty(0,T)$ is defined by

$$a(t) = 2[\chi_{[0,\frac{T}{2}]}(t) - \chi_{(\frac{T}{2},T]}(t)] = \begin{cases} 2 & 0 \le t \le \frac{T}{2} \\ -2 & \frac{T}{2} < t \le T \end{cases}$$

Under conditions (11) with $b_0 = b_T = 0$, it is seen that $\alpha = -3$ and $\beta = 3$ is an ordered couple of a lower and an upper solution, but the equation has no solutions when $T > 2\sqrt{2}$. However, here

$$h(\cdot, u, u') = (u+a)\left(\sqrt{1+u'^2}\right)^{3/2},$$

and (9) is satisfied. This explains the need of the Nagumo condition, or at least a similar one, in Theorem 2.

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