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t-structures for relative D-modules and t-exactness of the de Rham functor

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t-STRUCTURES FOR RELATIVE $\mathcal{D}$-MODULES AND t-EXACTNESS OF THE DE RHAM FUNCTOR

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Abstract. This paper is a contribution to the study of relative holonomic $\mathcal{D}$-modules. Contrary to the absolute case, the standard t-structure on holonomic $\mathcal{D}$-modules is not preserved by duality and hence the solution functor is no longer t-exact with respect to the canonical, resp. middle-perverse, t-structure. We provide an explicit description of these dual t-structures. We use this description to prove that the solution functor as well as the relative Riemann-Hilbert functor are t-exact with respect to the dual t-structure and to the middle-perverse one while the de Rham functor is t-exact for the canonical, resp. middle-perverse, t-structure and their duals.

Contents

Introduction.

Let X and S be complex manifolds and let p_X denote the projection of $X \times S \to S$. We shall denote by d_X and d_S their respective complex dimensions and will often write p instead of p_X whenever there is no ambiguity.

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An extensive study of holonomic and regular holonomic $\mathcal{D}_{X\times S/S}$-modules as well as of their derived categories was performed in [22] and [23]. Such modules are called for convenience respectively relative holonomic and regular relative holonomic modules. Relative holonomic modules are coherent modules whose characteristic variety, in the product $(T^*X)\times S$, is contained in $\Lambda\times S$ for some Lagrangian conic closed analytic subset Λ of T^*X. Regular relative holonomic modules are holonomic modules whose restriction to the fibers of p_X have regular holonomic $\mathcal{D}_X$-modules as cohomologies.

Another notion introduced in [22] was that of $\mathbb{C}$-constructibility over $p_X^{-1}\mathcal{O}_S$, leading to the (bounded) derived category of sheaves of $p_X^{-1}\mathcal{O}_S$-modules with $\mathbb{C}$-constructible cohomology, the S-$\mathbb{C}$-constructible complexes (this category is denoted by $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$), where a natural notion of perversity was also introduced. In loc.cit. it was proved that the essential image of the de Rham functor DR as well as of the solution functor Sol, when restricted to the bounded derived category of $\mathcal{D}_{X\times S/S}$-modules with holonomic cohomology, is $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. Recall that, denoting by ${}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M})$ (resp. ${}^{\mathrm{p}}\mathrm{DR}(\mathcal{M})$) the complex $\mathrm{Sol}(\mathcal{M})[d_X]$ (resp. $\mathrm{DR}(\mathcal{M})[d_X]$), these two functors satisfy a natural isomorphism of commutation with duality: $\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{Sol}(\cdot)\simeq {}^{\mathrm{p}}\mathrm{DR}(\cdot)$.

Under the assumption $d_S=1$, a right quasi-inverse functor to ${}^{\mathrm{p}}\mathrm{Sol}$, the functor RH^{S}, was introduced in [23], so naturally RH^{S} is a functor from $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ to the bounded derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ of $\mathcal{D}_{X\times S/S}$-modules with regular holonomic modules. RH^{S} is the relative version of Kashiwara's Riemann Hilbert functor RH (cf.[13]) as explained in Section 4.a where we briefly recall its construction. Recall that the importance of this apparently restrictive assumption on S is two-sided: for $d_S=1$, $\mathcal{O}_S$-flatness and absence of $\mathcal{O}_S$ torsion are equivalent, so we can split proofs in the torsion case and in the torsion free case; on the other hand, although we will not enter into details here, the construction of RH^{S} requires, locally on S, the existence of bases of the coverings of the subanalytic site S_{sa} formed by $\mathcal{O}_S$-acyclic open subanaytic sets which is possible in the case $d_S=1$.

The main goal of this paper is to prove the t-exactness of ${}^{\mathrm{p}}\mathrm{Sol}$, ${}^{\mathrm{p}}\mathrm{DR}$ and RH^{S} with respect to the t-structures involved (for any S in the first two cases and for $d_S=1$ in the case of RH^{S}). Recall that when one replaces $\mathcal{O}_S$ by the constant sheaf $\mathbb{C}_X[[\hbar]]$ of formal power series in one parameter $\hbar$, so no longer in the relative case, these questions were studied and solved by A. D'Agnolo, S. Guillermou and P. Schapira in [1].

Here, to be more precise, in the holonomic side we have the standard t-structure P as well as its dual Π, which, contrary to the absolute case proved by Kashiwara in [13], do not coincide if $d_X\geqslant 1$, $d_S\geqslant 1$ which is not surprising due to the possible absence of $\mathcal{O}_S$-flatness. Similarly, on the $\mathbb{C}$-constructible side, we have the perverse t-structure p introduced in [22] and its dual π, which do not coincide if d_X, $d_S\geqslant 1$ as well. Kashiwara's paper [14] provides a wide setting for this kind of problems covering the case $d_X=0$ (the $\mathcal{O}_S$-coherent case) as well as the standard t-structure on the $\mathbb{C}$-constructible case and the correspondent t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{rhol}}(\mathcal{D}_X)$ via RH. We took there our inspiration, adapting the ideas of several proofs.

In Theorems 2.11 and 3.10 we completely describe Π and π for any d_X and d_S. In particular, when $d_S = 1$, we prove in Proposition 2.6 that Π is obtained by left tilting P with respect to a natural torsion pair (respectively P is obtained by right tilting Π with respect to a natural torsion pair) and we conclude in Corollary 2.7 that the category of strict relative holonomic modules is quasi-abelian ([28]). Similar results are deduced for π and p in Proposition 3.8 leading to the conclusion that perverse S-$\mathbb{C}$-constructible complexes with a perverse dual are the objects of a quasi-abelian category. Recall that the procedure of tilting a t-structure $(\mathsf{D}^{\leqslant 0}, \mathsf{D}^{\geqslant 0})$ on a triangulated category $\mathcal{C}$ with respect to a given torsion pair $(\mathcal{T}, \mathcal{F})$ on its heart has been introduced by Happel, Reiten and Smalø in their work [9]. Following the notation of Bridgeland ([5] and [6]) Polishcuk proved in [27] that performing the left tilting procedure one gets all the t-structures $(\overline{\mathsf{D}}^{\leqslant 0}, \overline{\mathsf{D}}^{\geqslant 0})$ satisfying the condition $\mathsf{D}^{\leqslant 0} \subseteq \overline{\mathsf{D}}^{\leqslant 0} \subseteq \mathsf{D}^{\leqslant 1}$. The relations between torsion pairs, tilted t-structures and quasi-abelian categories have been clarified in [3] and [7].

With these informations in hand we have the tools to prove, in Theorem 4.1 that ${}^{\mathrm{p}}\mathrm{DR}$ is exact with respect to P and p (so, by duality, with respect to Π and π) which gives a precision to the behaviour of ${}^{\mathrm{p}}\mathrm{DR}$ already studied in [23]. However, since it is not known if RH^{S} provides an equivalence of categories for general d_X, we do not dispose of a morphism of functors $\boldsymbol{D}\,\mathrm{RH}^{\mathrm{S}}(\cdot) \to \mathrm{RH}^{\mathrm{S}}(\boldsymbol{D}(\cdot))$ allowing us to argue by duality as in the $\mathbb{C}$-constructible framework. Nevertheless, by a direct proof, in Theorem 4.2 we prove that RH^{S} is exact with respect to p and Π as well as to the dual structures π and P.

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1. Torsion pairs, quasi-abelian categories and t-structures

Let $\mathcal{C}$ be an additive category. In what follows any full subcategory $\mathcal{C}'$ of $\mathcal{C}$ will be strictly full (i.e., closed under isomorphisms) and additive and we will use the notation $\mathcal{C}' \subseteq \mathcal{C}$ to indicate such a subcategory. Any functor between additive categories will be an additive functor. In these terms given $\mathcal{C}_i \subseteq \mathcal{C}$ for $i \in \{1, 2\}$ following [2, Definition 1.3.1] we will denote by $\mathcal{C}_1 \cap \mathcal{C}_2$ the strictly full subcategory of $\mathcal{C}$ whose objects belong to both $\mathcal{C}_1$ and $\mathcal{C}_2$.

A *torsion pair* in an abelian category $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ satisfying the following conditions: $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for every $T \in \mathcal{T}$ and every $F \in \mathcal{F}$; for any object $A \in \mathcal{A}$ there exists a short exact sequence: $0 \to t(A) \to A \to f(A) \to 0$ in $\mathcal{A}$ such that $t(A) \in \mathcal{T}$ and $f(A) \in \mathcal{F}$. The class $\mathcal{T}$ is called the *torsion class* and it is closed under extensions, direct sums and quotients, while $\mathcal{F}$ is the *torsion-free class* and it is closed under extensions, subobjects and direct products. In particular, $\mathcal{T}$ is a full subcategory of $\mathcal{A}$ such that the inclusion functor

$i_{\mathcal{T}} : \mathcal{T} \to \mathcal{A}$ admits a right adjoint $t : \mathcal{A} \to \mathcal{T}$ such that $ti_{\mathcal{T}} = \mathrm{id}_{\mathcal{T}}$, and dually, the inclusion functor $i_{\mathcal{F}} : \mathcal{F} \to \mathcal{A}$ admits a left adjoint $f : \mathcal{A} \to \mathcal{F}$ such that $fi_{\mathcal{F}} = \mathrm{id}_{\mathcal{F}}$.

In general, the categories $\mathcal{T}$ and $\mathcal{F}$ are not abelian categories but, as observed in [3, 5.4], they are *quasi-abelian* categories. Let us recall that an additive category $\mathcal{E}$ is called *quasi-abelian* if it admits kernels and cokernels, and the class of short exact sequences $0 \to E_1 \overset{\alpha}{\to} E_2 \overset{\beta}{\to} E_3 \to 0$ with $E_1 \cong \mathrm{Ker}\beta$ and $E_3 \cong \mathrm{Coker}\,\alpha$ is stable by pushouts and pullbacks. Both $\mathcal{T}$ and $\mathcal{F}$ admit kernels and cokernels such that: $\mathrm{Ker}_{\mathcal{T}} = t \circ \mathrm{Ker}_{\mathcal{A}}$, $\mathrm{Coker}_{\mathcal{T}} = \mathrm{Coker}_{\mathcal{A}}$, $\mathrm{Ker}_{\mathcal{F}} = \mathrm{Ker}_{\mathcal{A}}$ and $\mathrm{Coker}_{\mathcal{F}} = f \circ \mathrm{Coker}_{\mathcal{A}}$. Exact sequences in $\mathcal{T}$ (respectively in $\mathcal{F}$) coincide with short exact sequences in $\mathcal{A}$ whose terms belong to $\mathcal{T}$ (respectively $\mathcal{F}$) and hence they are stable by pullbacks and push-outs thus proving that $\mathcal{T}$ and $\mathcal{F}$ are quasi-abelian categories. For more details on quasi-abelian categories we refer to Schneiders work [28].

Definition 1.1. ([9, Ch. I, Proposition 2.1], [5, Proposition 2.5]). Let $\mathcal{H}_{\mathsf{D}}$ be the heart of a t-structure $\mathsf{D} = (\mathsf{D}^{\leqslant 0}, \mathsf{D}^{\geqslant 0})$ on a triangulated category $\mathcal{C}$ and let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\mathcal{H}_{\mathsf{D}}$. Then the pair $\mathsf{D}_{(\mathcal{T},\mathcal{F})} := (\mathsf{D}^{\leqslant 0}_{(\mathcal{T},\mathcal{F})}, \mathsf{D}^{\geqslant 0}_{(\mathcal{T},\mathcal{F})})$ of full subcategories of $\mathcal{C}$

$$\mathsf{D}^{\leqslant 0}_{(\mathcal{T},\mathcal{F})} = \{C \in \mathcal{C} \mid H^1_{\mathsf{D}}(C) \in \mathcal{T},\ H^i_{\mathsf{D}}(C) = 0\ \forall\, i > 1\}$$
$$\mathsf{D}^{\geqslant 0}_{(\mathcal{T},\mathcal{F})} = \{C \in \mathcal{C} \mid H^0_{\mathsf{D}}(C) \in \mathcal{F},\ H^i_{\mathsf{D}}(C) = 0\ \forall\, i < 0\}$$

is a t-structure on $\mathcal{C}$ whose heart is

$$\mathcal{H}_{\mathsf{D}_{(\mathcal{T},\mathcal{F})}} = \{C \in \mathcal{C} \mid H^1_{\mathsf{D}}(C) \in \mathcal{T},\ H^0_{\mathsf{D}}(C) \in \mathcal{F},\ H^i_{\mathsf{D}}(C) = 0\ \forall\, i \notin \{0,1\}\}.$$

Following [5] we say that $\mathsf{D}_{(\mathcal{T},\mathcal{F})}$ is obtained *by left tilting* D *with respect to the torsion pair* $(\mathcal{T}, \mathcal{F})$ while the t-structure $\widetilde{\mathsf{D}}_{(\mathcal{T},\mathcal{F})} := \mathsf{D}_{(\mathcal{T},\mathcal{F})}[1]$ is called the t-structure obtained *by right tilting* D *with respect to the torsion pair* $(\mathcal{T}, \mathcal{F})$ and in this case the right tilted heart is:

$$\mathcal{H}_{\widetilde{\mathsf{D}}_{(\mathcal{T},\mathcal{F})}} = \{C \in \mathcal{C} \mid H^0_{\mathsf{D}}(C) \in \mathcal{T},\ H^{-1}_{\mathsf{D}}(C) \in \mathcal{F},\ H^i_{\mathsf{D}}(C) = 0\ \forall\, i \notin \{0,-1\}\}.$$

Remark 1.2. ([9]). Following the previous notations, whenever one performs a left tilting of D with respect to a given torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{H}_{\mathsf{D}}$ one obtains the new heart $\mathcal{H}_{\mathsf{D}_{(\mathcal{T},\mathcal{F})}}$ and the starting torsion pair is "tilted" in the torsion pair $(\mathcal{F}, \mathcal{T}[-1])$ which is a torsion pair in $\mathcal{H}_{\mathsf{D}_{(\mathcal{T},\mathcal{F})}}$: the class $\mathcal{F}$ placed in degree zero is the torsion class for this torsion pair while the old torsion class $\mathcal{T}$ shifted by $[-1]$ becomes the new torsion-free class and, for any $M \in \mathcal{H}_{\mathsf{D}_{(\mathcal{T},\mathcal{F})}}$, the sequence $0 \to H^0(M) \to M \to H^1(M)[-1] \to 0$ is the short exact sequence associated to the torsion pair $(\mathcal{F}, \mathcal{T}[-1])$.

Performing a right tilting of $\mathsf{D}_{(\mathcal{T},\mathcal{F})}$ with respect to the torsion pair $(\mathcal{F}, \mathcal{T}[-1])$ on $\mathcal{H}_{\mathsf{D}_{(\mathcal{T},\mathcal{F})}}$ one re-obtains the starting t-structure D endowed with its torsion pair $(\mathcal{T}, \mathcal{F})$. In such a way the right tilting by $(\mathcal{F}, \mathcal{T}[-1])$ in $\mathcal{H}_{\mathsf{D}_{(\mathcal{T},\mathcal{F})}}$ is the inverse of the left tilting of D with respect to $(\mathcal{T}, \mathcal{F})$ on $\mathcal{H}_{\mathsf{D}}$.

Any t-structure $\mathsf{D}_{(\mathcal{T},\mathcal{F})}$ obtained by left tilting D with respect to a torsion pair $(\mathcal{T},\mathcal{F})$ in the heart $\mathcal{H}_{\mathsf{D}}$ of a t-structure D in $\mathcal{C}$ satisfies

$$\mathsf{D}^{\leqslant 0} \subseteq \mathsf{D}^{\leqslant 0}_{(\mathcal{T},\mathcal{F})} \subseteq \mathsf{D}^{\leqslant 1} \quad \text{or equivalently} \quad \mathsf{D}^{\geqslant 1} \subseteq \mathsf{D}^{\geqslant 0}_{(\mathcal{T},\mathcal{F})} \subseteq \mathsf{D}^{\geqslant 0}$$

and hence the heart $\mathcal{H}_{\mathsf{D}_{(\mathcal{T},\mathcal{F})}}$ of the t-structure $\mathsf{D}_{(\mathcal{T},\mathcal{F})}$ satisfies $\mathcal{H}_{\mathsf{D}_{(\mathcal{T},\mathcal{F})}} \subseteq \mathsf{D}^{[0,1]} := \mathsf{D}^{\leqslant 1} \cap \mathsf{D}^{\geqslant 0}$. Dually any t-structure $\widetilde{\mathsf{D}}_{(\mathcal{T},\mathcal{F})} := \mathsf{D}_{(\mathcal{T},\mathcal{F})}[1]$ obtained by right tilting D with respect to a torsion pair $(\mathcal{T},\mathcal{F})$ in the heart $\mathcal{H}_{\mathsf{D}}$ of a t-structure D in $\mathcal{C}$ satisfies

$$\mathsf{D}^{\leqslant -1} \subseteq \widetilde{\mathsf{D}}^{\leqslant 0}_{(\mathcal{T},\mathcal{F})} \subseteq \mathsf{D}^{\leqslant 0} \quad \text{or equivalently} \quad \mathsf{D}^{\geqslant 0} \subseteq \widetilde{\mathsf{D}}^{\geqslant 0}_{(\mathcal{T},\mathcal{F})} \subseteq \mathsf{D}^{\geqslant -1}$$

and hence $\mathcal{H}_{\widetilde{\mathsf{D}}_{(\mathcal{T},\mathcal{F})}} \subseteq \mathsf{D}^{[-1,0]} := \mathsf{D}^{\leqslant 0} \cap \mathsf{D}^{\geqslant -1}$.

Polishchuk in [27, Lemma 1.2.2] proved the following:

Lemma 1.3. *In any pair of t-structures $\mathsf{D}, \overline{\mathsf{D}}$ on a triangulated category $\mathcal{C}$ verifying the condition $\mathsf{D}^{\leqslant 0} \subseteq \overline{\mathsf{D}}^{\leqslant 0} \subseteq \mathsf{D}^{\leqslant 1}$ (resp. $\mathsf{D}^{\leqslant -1} \subseteq \overline{\mathsf{D}}^{\leqslant 0} \subseteq \mathsf{D}^{\leqslant 0}$), the t-structure $\overline{\mathsf{D}}$ is obtained by left tilting (resp. right tilting) D with respect to the torsion pair*

$$(\mathcal{T},\mathcal{F}) := (\overline{\mathsf{D}}^{\leqslant -1} \cap \mathcal{H}_{\mathsf{D}}, \overline{\mathsf{D}}^{\geqslant 0} \cap \mathcal{H}_{\mathsf{D}}) \qquad (\textit{resp. } (\mathcal{T},\mathcal{F}) := (\overline{\mathsf{D}}^{\leqslant 0} \cap \mathcal{H}_{\mathsf{D}}, \overline{\mathsf{D}}^{\geqslant 1} \cap \mathcal{H}_{\mathsf{D}})$$

and in particular, for any $A \in \mathcal{H}_{\mathsf{D}}$, the approximating triangle for the t-structure $\overline{\mathsf{D}}$ is the short exact sequence for this torsion pair.

Remark 1.4. In the work [8] and [29] the authors propose a generalization of the previous result. In [8, Theorem 2.14 and 4.3] the authors proved that, under some technical hypotheses, given any pair of t-structures D, $\overline{\mathsf{D}}$ satisfying the condition:

$$\mathsf{D}^{\leqslant 0} \subseteq \overline{\mathsf{D}}^{\leqslant 0} \subseteq \mathsf{D}^{\leqslant \ell}$$

one can recover the t-structure $\overline{\mathsf{D}}$ by an iterated procedure of left tilting of length ℓ starting with D. Equivalently the t-structure D can be obtained by an iterated procedure of right tilting of length ℓ starting with $\overline{\mathsf{D}}$.

In particular by [8, Lemma 2.10 (ii)] these hypotheses are fulfilled whenever, following the definition of Keller and Vossieck [20] (cf. also [8, Definition 6.8]), the t-structure $\overline{\mathsf{D}}$ is *left* D*-compatible* i.e. the class $\overline{\mathsf{D}}^{\leqslant 0}$ is stable under the left truncations $\tau^{\leqslant k}_{\mathsf{D}}$ of D for any $k \in \mathbb{Z}$.

2. t-STRUCTURES ON $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$

Following the notation of the introduction, we denote by $\mathcal{D}_{X\times S/S}$ the subsheaf of $\mathcal{D}_{X\times S}$ of relative differential operators with respect to p_X and by $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X\times S/S})$ the bounded derived category of left $\mathcal{D}_{X\times S/S}$-modules with coherent cohomologies. As in the absolute case (in which S is a point) the category $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X\times S/S})$ is endowed with a duality functor: given $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X\times S/S})$ we set

$$\boldsymbol{D}(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{M}, \mathcal{D}_{X\times S/S} \otimes_{\mathcal{O}_{X\times S}} \Omega^{\otimes^{-1}}_{X\times S/S})[n]$$

(with $n = d_X$) where $\Omega_{X\times S/S}$ denotes the sheaf of relative differential forms of maximal degree, hence $\mathcal{M} \overset{\cong}{\to} \boldsymbol{D}\boldsymbol{D}\mathcal{M}$ (since, as explained in [22, Proposition 3.2], any coherent $\mathcal{D}_{X\times S/S}$-module locally admits a free resolution of length at most $2n+\ell$ with $\ell = d_S$). In [22, 3.4] the authors proved that the dual of a holonomic $\mathcal{D}_{X\times S/S}$-module is an object in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ (the bounded derived category of left $\mathcal{D}_{X\times S/S}$-modules with holonomic cohomologies; [22, Corollary 3.6]). Hence the previous duality restricts into a duality in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$, but despite the absolute case it is no longer true that the dual of a holonomic $\mathcal{D}_{X\times S/S}$-module is a holonomic $\mathcal{D}_{X\times S/S}$-module.

Due to the previous considerations, we can endow the triangulated category $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ with two t-structures P and Π: we denote by P the natural t-structure and by Π its dual t-structure with respect to the functor $\boldsymbol{D}$. Thus, by definition, complexes in ${}^{P}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ (respectively ${}^{P}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$) are isomorphic in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ to complexes of $\mathcal{D}_{X\times S/S}$-modules which have zero entries in positive (respectively negative) degrees and holonomic cohomologies. The dual t-structure Π is by definition:

$$\begin{aligned}
{}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) &= \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid \boldsymbol{D}\mathcal{M} \in {}^{P}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\} \\
{}^{\Pi}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) &= \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid \boldsymbol{D}\mathcal{M} \in {}^{P}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\}.
\end{aligned}$$

Remark 2.1. We have the following statements:

(1) If $S = \{pt\}$ then $\Pi = P$ (cf.[12, 4.11]).

(2) If $X = \{pt\}$ then P is nothing more than the natural t-structure in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$ and Π is its dual t-structure with respect to the functor $\boldsymbol{D}(\cdot) := R\mathcal{H}om_{\mathcal{O}_S}(\cdot, \mathcal{O}_S)$ described by Kashiwara in [14, §4, Proposition 4.3] which we shall denote by π:

$$\begin{aligned}
{}^{\pi}\mathsf{D}^{\leqslant 0}_{\mathrm{coh}}(\mathcal{O}_S) &= \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S) \mid \operatorname{codim}\operatorname{Supp}(\mathcal{H}^k(\mathcal{M})) \geqslant k\} \\
{}^{\pi}\mathsf{D}^{\geqslant 0}_{\mathrm{coh}}(\mathcal{O}_S) &= \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S) \mid \mathcal{H}^k_{[Z]}(\mathcal{M}_{|U}) = 0 \text{ for any analytic closed subset } Z \\
&\qquad \text{of any open subset } U \subseteq S \text{ and } k < \operatorname{codim}_U Z\}.
\end{aligned}$$

Recall that, following [22], for $s \in S$ on denotes by Li^*_s the derived functor $p_X^{-1}(\mathcal{O}_S/m) \overset{L}{\otimes}_{p_X^{-1}\mathcal{O}_S} (\cdot)$ where m is the maximal ideal of functions vanishing at s.

Lemma 2.2. *Let consider the functors* $Li^*_s : \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ *with* s *varying in* S. *The following holds true:*

(1) *the complex* $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ *is isomorphic to* 0 *if and only if* $Li^*_s\mathcal{M} = 0$ *for any* s *in* S;

(2) $Li^*_s\mathcal{M} \in \mathsf{D}^{\leqslant k}_{\mathrm{hol}}(\mathcal{D}_X)$ *for each* $s \in S$ *if and only if* $\mathcal{M} \in {}^{P}\mathsf{D}^{\leqslant k}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$;

(3) *if* $Li^*_s\mathcal{M} \in \mathsf{D}^{\geqslant k}_{\mathrm{hol}}(\mathcal{D}_X)$ *for each* $s \in S$ *then* $\mathcal{M} \in {}^{P}\mathsf{D}^{\geqslant k}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$;

(4) $Li^*_s\mathcal{M} \in \mathsf{D}^{\geqslant k}_{\mathrm{hol}}(\mathcal{D}_X)$ *for each* $s \in S$ *if and only if* $\mathcal{M} \in {}^{\Pi}\mathsf{D}^{\geqslant k}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$.

Proof. These statements are a slight generalization of [23, Corollary 1.11], with exactly the same idea of proof. In particular (4) can be deduced by duality from (2) since we can characterize the objets in ${}^{\Pi}\,\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ as follows:

$$\begin{aligned} &\mathcal{M} \in {}^{\Pi}\,\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \iff \boldsymbol{D}\mathcal{M} \in {}^{P}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \\ \overset{\text{by (2)}}{\iff} &\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \text{ and } \forall\, s \in S,\ Li_s^*\boldsymbol{D}\mathcal{M} \overset{*}{\cong} \boldsymbol{D}Li_s^*\mathcal{M} \in \mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_X) \\ \iff &\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \text{ and } \forall\, s \in S, Li_s^*\mathcal{M} \in \mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_X) \end{aligned}$$

where the last equivalence holds true since in the absolute case the functor $\boldsymbol{D}$ on $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ is exact with respect to the natural t-structure. As a morphism, $* : Li_s^*\boldsymbol{D}\mathcal{M} \to \boldsymbol{D}Li_s^*\mathcal{M}$ is an application of [12, (A.10)] and it is an isomorphism because Li_s^* is the derived tensor product of a coherent module $(p^{-1}\mathcal{O}_S/m)$ over a coherent sheaf $(p^{-1}\mathcal{O}_S)$. q.e.d.

Lemma 2.3. *We have the double inclusion*

$${}^{\Pi}\,\mathsf{D}^{\leqslant -\ell}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \subseteq {}^{P}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \subseteq {}^{\Pi}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$$

hence, given $\mathcal{M}$ *a holonomic* $\mathcal{D}_{X\times S/S}$*-module, its dual satisfies*

$$\boldsymbol{D}\mathcal{M} \in {}^{P}\,\mathsf{D}^{[0,\ell]}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}).$$

Proof. In general, if $\mathcal{M} \in {}^{P}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$, by the right exactness of i_s^* we deduce that for any $s \in S$ the complex $Li_s^*\mathcal{M}$ belongs to $\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_X)$ and hence $Li_s^*\boldsymbol{D}\mathcal{M} \cong \boldsymbol{D}Li_s^*\mathcal{M} \in \mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_X)$ thus, according to (3) of Lemma 2.2, $\boldsymbol{D}\mathcal{M} \in {}^{P}\,\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and so $\mathcal{M} \in {}^{\Pi}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$.

According to the definitions, Lemma 2.2 and by the t-exactness of the functor $\boldsymbol{D}$ in the absolute case, we have the following chain:

$$\begin{aligned} &\mathcal{M} \in {}^{\Pi}\,\mathsf{D}^{\leqslant -\ell}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \iff \boldsymbol{D}\mathcal{M} \in {}^{P}\,\mathsf{D}^{\geqslant \ell}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \\ &\Rightarrow \forall\, s \in S, Li_s^*\boldsymbol{D}\mathcal{M} \cong \boldsymbol{D}Li_s^*\mathcal{M} \in \mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_X) \Leftrightarrow \forall\, s \in S, Li_s^*\mathcal{M} \in \mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_X) \\ &\iff \mathcal{M} \in {}^{P}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}). \end{aligned}$$

q.e.d.

Following [26], $p_X^{-1}\mathcal{O}_S$-flat holonomic $\mathcal{D}_{X\times S/S}$-modules are called *strict*. The following result will be useful in the sequel:

Lemma 2.4. *Let* $\mathcal{N} \in {}^{P}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$*. Then* $\boldsymbol{D}\mathcal{N}$ *is quasi-isomorphic to a bounded complex* $\mathcal{F}^\bullet$ *of coherent* $\mathcal{D}_{X\times S/S}$*-modules whose terms in negative degrees are zero while the terms in positive degrees are strict coherent* $\mathcal{D}_{X\times S/S}$*-modules. In particular* $\mathcal{H}^0\boldsymbol{D}\mathcal{N}$ *is torsion free.*

Proof. Since any coherent $\mathcal{D}_{X\times S/S}$-module locally admits a resolution of finite length by free $\mathcal{D}_{X\times S/S}$-modules of finite rank, any complex $\mathcal{N}\in {}^{P}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ locally admits a resolution $\mathcal{L}^\bullet$ by free $\mathcal{D}_{X\times S/S}$-modules of finite rank such that $\mathcal{L}^i=0$ for any $i>0$ and for $i\ll 0$. Thus $\boldsymbol{D}\mathcal{N}$ can be represented locally by the complex $\mathcal{L}^{*\bullet}:=\mathcal{H}om_{\mathcal{D}_{X\times S/S}}(\mathcal{L}^\bullet,\mathcal{D}_{X\times S/S}\otimes_{\mathcal{O}_{X\times S}}\Omega^{\otimes^{-1}}_{X\times S/S})[n]$ whose terms are free $\mathcal{D}_{X\times S/S}$-modules of finite rank and whose cohomology in negative degrees is zero. By the assumption

$$\boldsymbol{D}\mathcal{N}\simeq {}^{P}\tau^{\geqslant 0}(\boldsymbol{D}\mathcal{N})\simeq \mathcal{F}^\bullet\in {}^{\Pi}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\ \text{(since } \mathcal{N}\in {}^{P}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\text{)}$$

with

$$\mathcal{F}^\bullet:=\cdots 0\longrightarrow 0\longrightarrow \mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}})\longrightarrow \mathcal{L}^{*1}\xrightarrow{d^1_{\mathcal{L}^{*\bullet}}}\mathcal{L}^{*2}\longrightarrow\cdots$$

where $\mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}})$ is placed in degree 0. It remains to prove that $\mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}})$ is a strict coherent $\mathcal{D}_{X\times S/S}$-module.

Let us consider the distinguished triangle induced by the following short exact sequence of complexes of coherent $\mathcal{D}_{X\times S/S}$-modules:

$$\begin{array}{ccccccccccccc}
\mathcal{L}^{*\geqslant 1} & & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{L}^{*1} & \longrightarrow & \mathcal{L}^{*2} & \longrightarrow & \cdots\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathcal{F}^\bullet & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}}) & \longrightarrow & \mathcal{L}^{*1} & \longrightarrow & \mathcal{L}^{*2} & \longrightarrow & \cdots\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}})[0] & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The triangle $Li^*_s\mathcal{L}^{*\geqslant 1}\to Li^*_s\mathcal{F}^\bullet\to Li^*_s(\mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}}))\xrightarrow{+}$ is distinguished, since each $\mathcal{L}^{i*}$ is strict, $Li^*_s\mathcal{L}^{*\geqslant 1}\in\mathsf{D}^{\geqslant 1}_{\mathrm{coh}}(\mathcal{D}_X)$ while $Li^*_s\mathcal{F}^\bullet\in\mathsf{D}^{\geqslant 0}_{\mathrm{coh}}(\mathcal{D}_X)$ in view of Lemma 2.2 (4). Hence, for any $s\in S$, $Li^*_s(\mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}}))\in\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_X)$, so $\mathcal{H}^jLi^*_s(\mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}}))=0,\forall\, j\neq 0$ since $Li^*_s(\mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}}))\in\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_X)$. According to [23, Lemma 1.13] we conclude that $\mathrm{Coker}(d^{-1}_{\mathcal{L}^{*\bullet}})$ is strict and so $\mathcal{H}^0\boldsymbol{D}\mathcal{N}$ is torsion free. q.e.d.

Remark 2.5. In accordance with Lemma 2.4, if $\mathcal{M}$ is a torsion module, $\mathcal{H}^0\boldsymbol{D}(\mathcal{M})$, being torsion free and a torsion module, is zero.

When $d_S=1$, it is well known that $p_X^{-1}\mathcal{O}_S$-flatness is equivalent to absence of $p_X^{-1}\mathcal{O}_S$-torsion, hence a holonomic $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}$ is strict if and only if for any $f\in\mathcal{O}_S$ the morphism $\mathcal{M}\xrightarrow{f}\mathcal{M}$ (multiplication by f) is a monomorphism.

In this case, for a given coherent $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}$, we denote by $\mathcal{M}_t$ the coherent sub-module of sections locally annihilated by some $f\in\mathcal{O}_S$ and we denote by $\mathcal{M}_{tf}$ the quotient $\mathcal{M}/\mathcal{M}_t$. We denote by $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S})_t$ the full subcategory of holonomic $\mathcal{D}_{X\times S/S}$-modules satisfying $\mathcal{M}_t\simeq\mathcal{M}$ and by $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_{tf}$ the full subcategory of holonomic $\mathcal{D}_{X\times S/S}$-modules satisfying $\mathcal{M}\simeq\mathcal{M}_{tf}$. The properties of torsion pair in $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ are clearly satisfied by $(\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t,\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_{tf})$.

Moreover this torsion pair is *hereditary* i.e. the class of torsion modules (which coincides with the class of holonomic $\mathcal{D}_{X\times S/S}$-modules $\mathcal{M}$ satisfying

$\dim p_X(\mathrm{Supp}(\mathcal{M})) = 0$ plus the zero module) is closed under sub-objects and so it forms an abelian category.

Proposition 2.6. *If $d_S = 1$, Π is the t-structure obtained by left tilting P with respect to the torsion pair $(\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t, \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_{tf})$ in $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ while P is the t-structure obtained by right tilting Π with respect to the torsion pair $(\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_{tf}, \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t[-1])$ in $\mathcal{H}_\Pi$.*

Proof. By Lemma 2.3 we have

$$^{\Pi}\,\mathsf{D}^{\leqslant -1}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \subset {}^{P}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \subset {}^{\Pi}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \subset {}^{P}\,\mathsf{D}^{\leqslant 1}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$$

(the last inclusion on the right is obtained by shifting by $[-1]$ the first one) and hence, by Polishchuk's result (Lemma 1.3), the t-structure Π is obtained by left tilting P with respect to the torsion pair

$$({}^{\Pi}\,\mathsf{D}^{\leqslant -1}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \cap \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}), {}^{\Pi}\,\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \cap \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})).$$

Also by Lemma 2.3, if $\mathcal{M}$ is holonomic, then $\boldsymbol{D}\mathcal{M} \in {}^{P}\,\mathsf{D}^{[0,1]}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$, that is, $\boldsymbol{D}\mathcal{M}$ is concentrated in degrees 0 and 1. The result will then be a consequence of the following statements:

- (i) $\mathcal{M}$ is a strict holonomic module if and only if $\boldsymbol{D}(\mathcal{M})$ is concentrated in degree zero and strict.
- (ii) If $\mathcal{M}$ belongs to $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t$ then $\boldsymbol{D}(\mathcal{M})$ is concentrated in degree 1 and ${}^{P}\mathcal{H}^1(\boldsymbol{D}\mathcal{M})$ belongs to $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t$.

Item (i) is contained in Proposition 2 of [23]. Therefore it remains to check item (ii). Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t$. First we remark that, by the functoriality of the action of $p_X^{-1}\mathcal{O}_S$, all cohomology groups ${}^{P}\mathcal{H}^j(\boldsymbol{D}\mathcal{M})$ belong to $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t$. In accordance with Remark 2.5, ${}^{P}\mathcal{H}^0(\boldsymbol{D}\mathcal{M}) = 0$. This ends the proof of (ii) and proves that $\mathcal{M}$ belongs to $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t$ if and only if $\boldsymbol{D}(\mathcal{M})$ is concentrated in degree 1 and ${}^{P}\mathcal{H}^1(\boldsymbol{D}\mathcal{M})$ belongs to $\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t$. This proves the first statement.

As a consequence, the heart of Π can be described as

$$\mathcal{H}_\Pi = \{\mathcal{M} \in {}^{P}\,\mathsf{D}^{[0,1]}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid {}^{P}\mathcal{H}^0(\mathcal{M}) \text{ strict and } {}^{P}\mathcal{H}^1(\mathcal{M}) \text{ torsion}\}$$

and thus the t-structure P is obtained by right tilting Π with respect to $(\mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_{tf}, \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})_t[-1])$ in $\mathcal{H}_\Pi$ (cf. [9] and Remark 1.2). q.e.d.

Corollary 2.7. *If $d_S = 1$ then the full subcategory of strict holonomic $\mathcal{D}_{X\times S/S}$-modules (thus holonomic $\mathcal{D}_{X\times S/S}$-modules with a strict holonomic dual) is quasi-abelian.*

Therefore the problem of expliciting Π only matters for $d_S \geqslant 2$ and $d_X \geqslant 1$. The following Lemmas permit to describe the t-structure Π in terms of support conditions as done by Kashiwara in the case of $X = \{pt\}$ (cf. [14]).

Lemma 2.8. *Let us consider $F : \mathcal{C} \longrightarrow \overline{\mathcal{C}}$ a triangulated functor between two triangulated categories $\mathcal{C}$ and $\overline{\mathcal{C}}$. Let $P := ({}^P\mathsf{D}^{\leqslant 0}, {}^P\mathsf{D}^{\geqslant 0})$ be a bounded t-structure on $\mathcal{C}$ and ${}^{\overline{P}}\mathsf{D}^{\leqslant 0}$ (resp. ${}^{\overline{P}}\mathsf{D}^{\geqslant 0}$)) a class on $\overline{\mathcal{C}}$ closed under extensions and shift by* [1] *(resp. closed under extensions and shift by $[-1]$). The following statements hold true:*

1. *the functor $F({}^P\mathsf{D}^{\leqslant 0}) \subseteq {}^{\overline{P}}\mathsf{D}^{\leqslant 0}$ if and only if $F(\mathcal{H}_P) \subseteq {}^{\overline{P}}\mathsf{D}^{\leqslant 0}$;*
2. *the functor $F({}^P\mathsf{D}^{\geqslant 0}) \subseteq {}^{\overline{P}}\mathsf{D}^{\geqslant 0}$ if and only if $F(\mathcal{H}_P) \subseteq {}^{\overline{P}}\mathsf{D}^{\geqslant 0}$;*
3. *the previous conditions are simultaneously satisfied if and only if $F(\mathcal{H}_P) \subseteq \mathcal{H}_{\overline{P}}$.*

Proof. Let us recall that by definition a t-structure $P := ({}^P\mathsf{D}^{\leqslant 0}, {}^P\mathsf{D}^{\geqslant 0})$ on $\mathcal{C}$ is bounded if for any $X \in \mathcal{C}$ there exist $m \leqslant n \in \mathbb{Z}$ such that $X \in {}^P\mathsf{D}^{\leqslant n} \cap {}^P\mathsf{D}^{\geqslant m}$ and as remarked by Bridgeland in [5, Lemma 2.3] these t-structures are completely determined by their hearts (via its Postnikov tower).

The left to right implication is clear since $\mathcal{H}_P \subseteq {}^P\mathsf{D}^{\leqslant 0}$ so let us suppose that $F(\mathcal{H}_P) \subseteq {}^{\overline{P}}\mathsf{D}^{\leqslant 0}$ and let us prove that $F({}^P\mathsf{D}^{\leqslant 0}) \subseteq {}^{\overline{P}}\mathsf{D}^{\leqslant 0}$. Recall that for any $X \in {}^P\mathsf{D}^{\leqslant 0}$ there exists a suitable $k \in \mathbb{N}$ such that $X \in {}^P\mathsf{D}^{\leqslant 0} \cap {}^P\mathsf{D}^{\geqslant -k}$. Let us proceed by induction on $k \in \mathbb{N}$. For $k = 0$ we get $X \in \mathcal{H}_P$ and thus $F(X) \in {}^{\overline{P}}\mathsf{D}^{\leqslant 0}$ by hypothesis. Let us suppose by inductive hypothesis that the first statement holds true for k and let $X \in {}^P\mathsf{D}^{\leqslant 0} \cap {}^P\mathsf{D}^{\geqslant -k-1}$. By applying the functor F to the distinguished triangle ${}^PH^{-k-1}(X)[k+1] \to X \to {}^P\tau^{\geqslant -k}(X) \overset{+1}{\to}$ we obtain $F({}^PH^{-k-1}(X))[k+1] \to F(X) \to F({}^P\tau^{\geqslant -k}(X)) \overset{+1}{\to}$. By hypothesis $F({}^PH^{-k-1}(X))[k+1] \in {}^{\overline{P}}\mathsf{D}^{\leqslant 0}[k+1] \subseteq {}^{\overline{P}}\mathsf{D}^{\leqslant 0}$ (thanks to the fact that ${}^{\overline{P}}\mathsf{D}^{\leqslant 0}$ is closed under [1]) and by inductive hypothesis $F({}^P\tau^{\geqslant -k}(X)) \in {}^{\overline{P}}\mathsf{D}^{\leqslant 0}$. Thus $F(X) \in {}^{\overline{P}}\mathsf{D}^{\leqslant 0}$ since ${}^{\overline{P}}\mathsf{D}^{\leqslant 0}$ is closed under extensions. The second statement follows similarly and the third is the consequence of the first and second ones. q.e.d.

Lemma 2.9. *Let $\mathcal{N}$ be a coherent $\mathcal{D}_{X\times S/S}$-module. Then, for each k,*

$$\operatorname{codim}\operatorname{Char}(\mathcal{E}xt^k_{\mathcal{D}_{X\times S/S}}(\mathcal{N}, \mathcal{D}_{X\times S/S})) \geqslant k,$$

in particular

$$\operatorname{codim}\operatorname{Char}(\mathcal{H}^k_{\mathcal{D}_{X\times S/S}}\boldsymbol{D}\mathcal{N}) \geqslant k + d_X.$$

Proof. According to the faithfull flatness of $\mathcal{D}_{X\times S}$ over $\mathcal{D}_{X\times S/S}$ and to [12, Theorem 2.19 (2)], we have, for each k,

$$\operatorname{codim}\operatorname{Char}(\mathcal{E}xt^k_{\mathcal{D}_{X\times S}}(\mathcal{D}_{X\times S} \otimes_{\mathcal{D}_{X\times S/S}} \mathcal{N}, \mathcal{D}_{X\times S}))$$

$$= \operatorname{codim}\operatorname{Char}(\mathcal{E}xt^k_{\mathcal{D}_{X\times S/S}}(\mathcal{N}, \mathcal{D}_{X\times S/S}) \otimes_{\mathcal{D}_{X\times S/S}} \mathcal{D}_{X\times S}) \geqslant k$$

Since

$$\operatorname{Char}(\mathcal{E}xt^k_{\mathcal{D}_{X\times S/S}}(\mathcal{N}, \mathcal{D}_{X\times S/S}) \otimes_{\mathcal{D}_{X\times S/S}} \mathcal{D}_{X\times S}) = \pi^{-1}\operatorname{Char}(\mathcal{E}xt^k_{\mathcal{D}_{X\times S/S}}(\mathcal{N}, \mathcal{D}_{X\times S/S}))$$

where $\pi : T^*X \times T^*S \to T^*X \times S$ is the projection, we conclude that

$$\operatorname{codim} \operatorname{Char}(\mathcal{E}xt^k_{\mathcal{D}_{X\times S/S}}(\mathcal{N}, \mathcal{D}_{X\times S/S})) \geqslant k$$

as desired. q.e.d.

Lemma 2.10. *For any holonomic $\mathcal{D}_{X\times S/S}$-module $\mathcal{M}$ we have*

$$\operatorname{Char}(\mathcal{M}) = \bigcup_{i\in I} \Lambda_i \times T_i$$

for some closed $\mathbb{C}^*$-conic irreducible Lagrangian subsets Λ_i of T^*X and some closed analytic subsets T_i of S, and, locally on X, the set I is finite. Moreover $p_X(\operatorname{Supp}(\mathcal{M})) = \bigcup_{i\in I} T_i$, hence it is an analytic subset of S, and

$$\dim \operatorname{Char}(\mathcal{M}) = \dim X + t \qquad \textit{where} \qquad t = \dim p_X(\operatorname{Supp}(\mathcal{M})) = \sup_{i\in I} \dim T_i.$$

Proof. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_{X\times S/S}$-module, and let $\Lambda \subseteq T^*X$ be a Lagrangian analytic $\mathbb{C}^*$-conic (or conic, for short) closed subset such that $\operatorname{Char}(\mathcal{M}) \subseteq \Lambda \times S$.

Let $\Lambda = \bigcup_{i\in I} \Lambda_i$, with Λ_i closed conic irreducible Lagrangian in T^*X, be the (locally finite) decomposition of Λ in irreducible components.

Let us consider the family of the components Λ_j such that $\operatorname{Char}\mathcal{M}\cap(\Lambda_j\times S) \neq \varnothing$. For simplicity, let us denote this family by $\{\Lambda_1, \cdots, \Lambda_K\}$. By the assumption of irreducibility, for each irreducible component W of $\operatorname{Char}\mathcal{M}$ there must exist a Λ_j such that $W \subset \Lambda_j \times S$.

Let W be any irreducible component of $\operatorname{Char}\mathcal{M}$ which is contained in $\Lambda_1 \times S$. Then W is conic involutive in the Poisson manifold $T^*X \times S$ and, for each $s \in S$, $W \cap p_X^{-1}(s)$ is contained in $\Lambda_1 \times \{s\}$. According to [15, Cor.1.1.14], $W \cap p_X^{-1}(s)$ is still involutive in $T^*X \times \{s\}$. Since it is contained in $\Lambda_1 \times \{s\}$, it must be a conic Lagrangian closed analytic set. Since Λ_1 is conic Lagrangian closed analytic irreducible, we must have either $W \cap p_X^{-1}(s) = \varnothing$ or $W \cap p_X^{-1}(s) = \Lambda_1 \times \{s\}$. In particular $W = \Lambda_1 \times \widetilde{T}_1$, for some closed subset $\widetilde{T}_1$ of S. To see that $\widetilde{T}_1$ is analytic (hence irreducible analytic) it suffices to fix a point $p \in \Lambda_1$, then $\{p\} \times \widetilde{T}_1 = q_X^{-1}(p) \cap W$ where q denotes the projection $T^*X \times S \to T^*X$. Hence $\{p\} \times \widetilde{T}_1$ is analytic and so is $\widetilde{T}_1$.

By the preceding argument, the union of the family of irreducible components of $\operatorname{Char}\mathcal{M}$ contained in $\Lambda_1 \times S$ is equal to $\cup_{l\in L_1}(\Lambda_1 \times \widetilde{T}_{1,l}) = \Lambda_1 \times T_1$ for some finite family $(\widetilde{T}_{i,l})_{l\in L}$ of closed irreducible subsets in S. We can now apply this argument to each Λ_i and the first part of the result follows.

Since

$$\operatorname{Supp}(\mathcal{M}) = \operatorname{Char}\mathcal{M} \cap (T^*_X X \times S)$$

we deduce that $p_X(\operatorname{Supp}(\mathcal{M})) = \bigcup_i T_i$ and hence $t := \dim p_X(\operatorname{Supp}(\mathcal{M})) = \sup_{i\in I} \dim T_i$ hence $\dim \operatorname{Char}(\mathcal{M}) = \dim X + t$ which ends the proof.

q.e.d.

We have now the tools to obtain the description of Π for arbitrary d_S:

Theorem 2.11. *The t-structure Π on $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ can be described in the following way:*

$$(*)\ {}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) = \{\mathcal{M}\in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid \forall\, k,\ \operatorname{codim} p_X(\operatorname{Supp}({}^{P}\mathcal{H}^k(\mathcal{M}))) \geqslant k\}$$
$$(**)\ {}^{\Pi}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) = \{\mathcal{M}\in {}^{P}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid {}^{P}\mathcal{H}^k_{[X\times W]}(\mathcal{M}_{|X\times U}) = 0 \text{ for any}$$
closed analytic subset W of any open subset $U\subseteq S$ and $k<\operatorname{codim}_U W\}$.

Proof. Note that the statement is true in the absolute case since we get ${}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_X) = {}^{P}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_X)$ (an holonomic $\mathcal{D}_X$-module whose characteristic variety has codimension grater than d_X is necessarily zero).

Step 1. Let us prove the equality $(**)$. We start by proving the inclusion of the left hand side into the right one.

Let W be a closed analytic subset of an open subset $U\subseteq S$ such that $\operatorname{codim}_U W \geqslant k$. Let us prove that $R\Gamma_{[X\times W]}(\boldsymbol{D}\mathcal{N}_{|X\times U}) \in {}^{P}\mathsf{D}^{\geqslant k}(\mathcal{D}_{X\times U/U})$ for any complex $\mathcal{N}\in {}^{P}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$. This will be a consequence of Lemma 2.4. Indeed, keeping the notation of the proof of this Lemma, we have

$$R\Gamma_{[X\times W]}(\boldsymbol{D}\mathcal{N}_{|X\times U}) \cong R\Gamma_{[W]}(p_X^{-1}\mathcal{O}_U)\otimes_{p_X^{-1}\mathcal{O}_U}\mathcal{F}^{\bullet}_{|X\times U} \in {}^{P}\mathsf{D}^{\geqslant k}(\mathcal{D}_{X\times U/U})$$

since $R\Gamma_{[W]}(p_X^{-1}\mathcal{O}_U)\in \mathsf{D}^{\geqslant k}(p_X^{-1}\mathcal{O}_U)$ and the terms of $\mathcal{F}^{\bullet}_{|X\times U}$ are strict coherent $\mathcal{D}_{X\times U/U}$-modules.

Let us now prove the inclusion of the right hand side in the left one, that is, let us prove that given $\mathcal{M}\in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ such that ${}^{P}\mathcal{H}^k_{[X\times W]}(\mathcal{M}_{|X\times U})) = 0$ for any closed analytic subset W of an open subset $U\subseteq S$ and $k<\operatorname{codim}_S W$ we get $\mathcal{M}\in {}^{\Pi}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$.

We note that the statement is local. In view of Lemma 2.2 (4) it suffices to check that, for each $s\in S$, $Li^*_s\mathcal{M}\in \mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_X)$. We shall argue by induction on d_S. First suppose $d_S=1$. Let $s_0\in S$ and s be a local coordinate on S vanishing in s_0. By the same arguments of Lemma 2.8 we may assume that $\mathcal{M}$ is concentrated in degree 0. Hence $\mathcal{M}$ is strict, since, if $s^P\mathcal{M}=0$ for some natural P, $\Gamma_{[X\times\{s_0\}]}(\mathcal{M})\neq 0$ contradicting the assumption on $\mathcal{M}$; so, according to Proposition 2.6 $\mathcal{M}\in {}^{\Pi}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$.

Let us now treat the general case. It will be a consequence of the following Lemma which is a variation of a formula proved in [24], page 153:

Lemma 2.12. *Let X be an open subset in $\mathbb{C}^n$, let S be an open set of $\mathbb{C}^d$ containing 0, with coordinates $(s_1,\cdots,s_d)$. Let us denote by S_j the submanifold of S of equations $s_1=0,\cdots,s_j=0$, for $j=1,\cdots,d$ and, for any f holomorphic on S_j, denote by $L^*_f := p_X^{-1}(\mathcal{O}_{S_j}/\mathcal{O}_{S_j}f)\otimes^L_{p_X^{-1}(\mathcal{O}_{S_j})}(\cdot)$ the corresponding derived functor.*

Then we have an isomorphism of functors on $\mathsf{D}^{\mathrm{b}}(\mathcal{O}_{X\times S_j})$

$$(A)\ Li^*_{s_{j+1}}R\Gamma_{[X\times S_j]}(\cdot) = R\Gamma_{[X\times S_{j+1}]}Li^*_{s_{j+1}}(\cdot)$$

We consider the local situation where $s_0 = 0 \in \mathbb{C}^d$. Following the notations of the preceding Lemma, let $W = S_1$. Let $S^* := S \smallsetminus S_1$ and we denote by $R\Gamma_{[X\times S^*]}(\cdot)$ the functor of localization relatively to the hypersurface $X \times S_1$. As usual we may assume that $\mathcal{M}$ is concentrated in degree 0. Since S_1 has equation $s_1 = 0$ we deduce, as in the case $d_S = 1$, that $\mathcal{M}$ has no s_1 torsion, since, if $s_1^P\mathcal{M} = 0$ for some natural P, $\Gamma_{[X\times S_1]}(\mathcal{M}) \neq 0$ contradicting the assumption that $R\Gamma_{[X\times S_1]}(\mathcal{M}) \in {}^P\mathsf{D}^{\geqslant 1}(\mathcal{D}_{X\times S/S})$.

Let consider the distinguished triangle

$$(B) \quad Li^*_{s_1}R\Gamma_{[X\times S_1]}\mathcal{M} \longrightarrow Li^*_{s_1}\mathcal{M} \longrightarrow Li^*_{s_1}R\Gamma_{[X\times S^*]}\mathcal{M} \xrightarrow{+}$$

we have $Li^*_{s_1}R\Gamma_{[X\times S_1]}\mathcal{M} \in {}^P\mathsf{D}^{\geqslant 0}(\mathcal{D}_{X\times S_1/S_1})$ and so $Li^*_{s_1}\mathcal{M} \in {}^P\mathsf{D}^{\geqslant 0}(\mathcal{D}_{X\times S_1/S_1})$ since in this case s_1 is invertible on $R\Gamma_{[X\times S^*]}\mathcal{M}$ which is an object in ${}^P\mathsf{D}^{\geqslant 0}(\mathcal{D}_{X\times S/S})$.

Moreover, given a closed analytic subset W_1 of S_1, we have, according to (A)

$$R\Gamma_{[X\times W_1]}(Li^*_{s_1}\mathcal{M}) = Li^*_{s_1}R\Gamma_{[X\times S_1]}(\mathcal{M}) \in \mathsf{D}^{\geqslant codim_{S_1}W_1}(X\times S_1)$$

Hence $Li^*_{s_1}\mathcal{M}$ belongs to ${}^\Pi\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S_1/S_1})$ and we can proceed recursively to conclude the statement.

Step 2. By Lemma 2.10 we know that for any $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ we have

$$\dim \mathrm{Char}({}^P\mathcal{H}^k(\mathcal{M})) = d_X + \dim p_X(\mathrm{Supp}({}^P\mathcal{H}^k(\mathcal{M})))$$

hence

$$\mathrm{codim}\,\mathrm{Char}({}^P\mathcal{H}^k(\mathcal{M})) \geqslant k + d_X \iff \mathrm{codim}\, p_X(\mathrm{Supp}({}^P\mathcal{H}^k(\mathcal{M}))) \geqslant k$$

so we are reduced to prove that

$$\{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid \mathrm{codim}\,\mathrm{Char}({}^P\mathcal{H}^k(\mathcal{M})) \geqslant k + d_X\} = {}^\Pi\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}).$$

First we prove the inclusion:

$$\{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid \mathrm{codim}\,\mathrm{Char}({}^P\mathcal{H}^k(\mathcal{M})) \geqslant k + d_X\} \subseteq {}^\Pi\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}).$$

Let us argue by induction on m such that $\mathcal{M} \in {}^P\mathsf{D}^{\leqslant m}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and that $\mathrm{codim}\,\mathrm{Char}({}^P\mathcal{H}^k(\mathcal{M})) \geqslant k + d_X$. For $m = 0$ we have by Lemma 2.3 that ${}^P\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \subset {}^\Pi\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$. Let us suppose that any complex in ${}^P\mathsf{D}^{\leqslant m}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ satisfying $\mathrm{codim}\,\mathrm{Char}({}^P\mathcal{H}^k(\mathcal{M})) \geqslant k+d_X$ belongs to ${}^\Pi\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and let $\mathcal{M} \in {}^P\mathsf{D}^{\leqslant m+1}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ satisfying $\mathrm{codim}\,\mathrm{Char}({}^P\mathcal{H}^k(\mathcal{M})) \geqslant k + d_X$. By inductive hypothesis we have that ${}^P\tau^{\leqslant m}\mathcal{M} \in {}^\Pi\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and the distinguished triangle

$${}^P\tau^{\leqslant m}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow {}^P\mathcal{H}^{m+1}(\mathcal{M})[-m-1] \xrightarrow{+}$$

proves that $\mathcal{M} \in {}^\Pi\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ if and only if ${}^P\mathcal{H}^{m+1}(\mathcal{M}) \in {}^\Pi\mathsf{D}^{\leqslant -m-1}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$. This last condition is satisfied in view of the assumption on $\mathcal{M}$ according to [12, Theorem 2.19 (1)] together with the faithfull flatness of $\mathcal{D}_{X\times S}$ over $\mathcal{D}_{X\times S/S}$, which shows that $\boldsymbol{D}({}^{P}\mathcal{H}^{m+1}(\mathcal{M}))\in {}^{P}\,\mathsf{D}^{\geqslant m+1}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$.

Let us now prove the inclusion

$$^{\Pi}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\subseteq\{\mathcal{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\mid \operatorname{codim}\operatorname{Char}({}^{P}\mathcal{H}^{k}(\mathcal{M}))\geqslant k+d_X\}.$$

Recalling that ${}^{\Pi}\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}:=\boldsymbol{D}({}^{P}\,\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}))$ we can apply Lemma 2.8 with $F=\boldsymbol{D}$ and so we need only to prove that given $\mathcal{N}$ a holonomic $\mathcal{D}_{X\times S/S}$-module, $\boldsymbol{D}(\mathcal{N})$ satisfies

$$\operatorname{codim}\operatorname{Char}({}^{P}\mathcal{H}^{k}(\boldsymbol{D}(\mathcal{N})))\geqslant k+d_X$$

and this holds true by Lemma 2.9. q.e.d.

Remark 2.13. We conclude by the previous Theorem 2.11 that the t-structure Π is left P-compatible (cf. Remark 1.4) and so, according to Lemma 2.3 and to [8, Theorem 4.3], it can be recovered from P via an iterated right tilting procedure of length ℓ.

3. t-STRUCTURES ON $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$

In $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ the natural dualizing complex is $p_X^{!}\mathcal{O}_S=p_X^{-1}\mathcal{O}_S[2d_X]$ and one defines the duality functor (cf. [22] for details) by setting

$$\boldsymbol{D}(F)=R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,p_X^{-1}\mathcal{O}_S)[2d_X].$$

Hence the canonical morphism $F\to\boldsymbol{D}\boldsymbol{D}(F)$ is an isormorphism for any $F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.

We are now concerned by the corresponding of Lemma 2.10 in the framework of $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. It can be deduced thanks to the functor RH^{S} which will be recalled later (cf. 4.a). Let us be more precise: for a complex F of sheaves on $X\times S$, let $SS\,F$ denote its microsupport (cf. [16] for a detailed introduction to this notion). Given $F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{C}-c}(p^{-1}\mathcal{O}_S)$, as proved in [23, Th. 3], we have a functorial isomorphism $F\simeq{}^{\mathrm{p}}\mathrm{Sol}\,\mathrm{RH}^{\mathrm{S}}(F)$ where $\mathrm{RH}^{\mathrm{S}}(F)$ is a complex with (regular) holonomic cohomology, hence $\operatorname{Char}(\mathrm{RH}^{\mathrm{S}}(F))=SS\,F$. Therefore we conclude:

Corollary 3.1. *Let $F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{C}-c}(p^{-1}\mathcal{O}_S)$. Let Λ be a Lagrangian closed analytic $\mathbb{C}^*$-conic subset of T^*X such that $SS\,F$ is contained in $\Lambda\times S$. Then each closed irreducible component of $SS\,F$ is of the form $\Lambda_j\times T$ where T is an analytic closed irreducible subset of S and Λ_j is a closed irreducible component of Λ. In particular, $p_X(\operatorname{Supp}F)$ is an analytic subset of S.*

Definition 3.2. [22, 2.7] The *perverse* t-structure p on the triangulated category $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ is given by

$$^{p}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)=\{F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)\mid\forall\,\alpha,i_\alpha^{-1}F\in\mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathbb{C}\text{-c}}(p_{X_\alpha}^{-1}\mathcal{O}_S),\ \text{for some adapted }\mu\text{-stratification }(X_\alpha)\}$$

$$^{p}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)=\{F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)\mid\forall\,\alpha,i_\alpha^{!}F\in\mathsf{D}^{\geqslant -d_{X_\alpha}}_{\mathbb{C}\text{-c}}(p_{X_\alpha}^{-1}\mathcal{O}_S),\ \text{for some adapted }\mu\text{-stratification }(X_\alpha)\}$$

or equivalently

$$\begin{aligned}{}^p\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) = \{F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \mid \forall\,\alpha, i_x^{-1}F \in \mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S),\ \text{for any } x \in X_\alpha \\ \text{and for some adapted } \mu\text{-stratification } (X_\alpha)\}\\ {}^p\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) = \{F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \mid \forall\,\alpha, i_x^{!}F \in \mathsf{D}^{\geqslant d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S),\ \text{for any } x \in X_\alpha \\ \text{and for some adapted } \mu\text{-stratification } (X_\alpha)\}.\end{aligned}$$

(See [16, Definition 8.3.19] for the definition of adapted μ-stratification.) Hence its dual π with respect to the functor $\boldsymbol{D}$ is

$$\begin{aligned}{}^\pi\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) = \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \mid \boldsymbol{D}\mathcal{M} \in {}^p\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)\}\\ {}^\pi\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) = \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \mid \boldsymbol{D}\mathcal{M} \in {}^p\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)\}.\end{aligned}$$

Notation 3.3. We shall denote by $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)$ the heart of the t-structure p.

We have the following statements:

(1) If $S = \{pt\}$ then p equals the middle-perversity t-structure (cf.[12, 4.11]).
(2) If $X = \{pt\}$ then p is, as above, the standard t-structure in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$ and π is the dual t-structure in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$ described by Kashiwara in [14] (cf. Remark 2.1.)

Therefore the problem of expliciting π only matters for $d_S \geqslant 1$ and $d_X \geqslant 1$.

Lemma 3.4. *Let us consider the functors $Li_s^* : \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \to \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ with s varying in S. The following holds true:*

(1) *the complex $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ is isomorphic to 0 if and only if $Li_s^*F = 0$ for any s in S;*
(2) *$Li_s^*F \in \mathsf{D}^{\leqslant k}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ for each $s \in S$ if and only if $F \in \mathsf{D}^{\leqslant k}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$;*
(3) *if $Li_s^*F \in \mathsf{D}^{\geqslant k}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ for each $s \in S$ then $F \in \mathsf{D}^{\geqslant k}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.*

Proof. (1) is proved in [22, Proposition 2.2]. The other implications can also be deduced by the proof of Proposition 2.2 in [22]. q.e.d.

Statement (2) of the previous Lemma affirms that a complex F belongs to the aisle of the natural t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ if and only if any Li_s^*F belongs to the aisle of the natural t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. This result admits the following counterpart for the perverse t-structure thus obtaining an analog of Lemma 2.2.

Lemma 3.5. *The following statements hold true:*

(1) *$Li_s^*F \in {}^p\,\mathsf{D}^{\leqslant k}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ for each $s \in S$ if and only if $F \in {}^p\,\mathsf{D}^{\leqslant k}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$;*
(2) *if $Li_s^*F \in {}^p\,\mathsf{D}^{\geqslant k}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ for each $s \in S$ then $F \in {}^p\,\mathsf{D}^{\geqslant k}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$;*
(3) *$Li_s^*F \in {}^p\,\mathsf{D}^{\geqslant k}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ for each $s \in S$ if and only if $F \in {}^\pi\,\mathsf{D}^{\geqslant k}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$;*
(4) *if $F \in {}^p\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ then $Li_s^*F \in {}^p\,\mathsf{D}^{\geqslant -\ell}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ for each $s \in S$.*

Proof. (1) Recall that $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ belongs to ${}^p\,\mathsf{D}^{\leqslant k}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ if for some adapted μ-stratification $(X_\alpha)_{\alpha\in A}$ we have

$$\forall\,\alpha,\ i_\alpha^{-1}F \in \mathsf{D}^{\leqslant k-d_{X_\alpha}}_{\mathbb{C}\text{-c}}(p_{X_\alpha}^{-1}\mathcal{O}_S)$$

or equivalently, by Lemma 3.4 (2),

$$\forall\, \alpha, Li_s^* i_\alpha^{-1} F \cong i_\alpha^{-1} Li_s^* F \in \mathsf{D}^{\leqslant k-d_{X_\alpha}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) \quad \forall\, s \in S$$

which is equivalent to

$$Li_s^* F \in {}^p\,\mathsf{D}^{\leqslant k}_{\mathbb{C}\text{-c}}(\mathbb{C}_X).$$

(2) If $Li_s^* F \in {}^p\,\mathsf{D}^{\geqslant k}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ for each $s \in S$ we get:

$$\forall\, \alpha, Li_s^* i_\alpha^! F \cong i_\alpha^! Li_s^* F \in \mathsf{D}^{\geqslant k-d_{X_\alpha}}_{\mathbb{C}\text{-c}}(\mathbb{C}_{X_\alpha}) \text{ for some adapted } \mu\text{-stratification } (X_\alpha)$$

and so by (3) of Lemma 3.4 we obtain $F \in {}^p\,\mathsf{D}^{\geqslant k}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.

(3) can be deduced by duality from (1) since $Li_s^* \boldsymbol{D} F \cong \boldsymbol{D} Li_s^* F$ for any $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ we have:

$$\begin{aligned} & F \in {}^\pi\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \Longleftrightarrow \boldsymbol{D} F \in {}^p\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \\ \overset{\text{by (1)}}{\Longleftrightarrow}\ & F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \text{ and } \forall\, s \in S,\ Li_s^* \boldsymbol{D} F \cong \boldsymbol{D} Li_s^* F \in \mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) \\ \Longleftrightarrow\ & F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \text{ and } \forall\, s \in S, Li_s^* F \in \mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) \end{aligned}$$

where the last equivalence holds true since in the absolute case the functor $\boldsymbol{D}$ on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ is t-exact with respect to the perverse t-structure.

Let us prove (4): we have

$$\begin{gathered} F \in {}^p\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \Longleftrightarrow \\ R\Gamma_{X_\alpha \times S}(F) \in \mathsf{D}^{\geqslant -d_{X_\alpha}}(p_X^{-1}\mathcal{O}_S)\ \forall\, \alpha, \text{ for some adapted } \mu\text{-stratification } (X_\alpha) \Rightarrow \\ R\Gamma_{X_\alpha}(Li_s^* F) \cong Li_s^* R\Gamma_{X_\alpha \times S}(F) \in \mathsf{D}^{\geqslant -d_{X_\alpha} - \ell}(X)\ \forall\, s \in S, \forall\, \alpha, \\ \text{for some adapted } \mu\text{-stratification } (X_\alpha) \\ \Longleftrightarrow Li_s^* F \in {}^p\,\mathsf{D}^{\geqslant -\ell}_{\mathbb{C}\text{-c}}(X). \end{gathered}$$

q.e.d.

Lemma 3.6. *We have the double inclusion*

$${}^\pi\,\mathsf{D}^{\leqslant -\ell}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \subseteq {}^p\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \subseteq {}^\pi\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$$

hence, given a perverse $p_X^{-1}\mathcal{O}_S$-module F, its dual satisfies

$$\boldsymbol{D} F \in {}^p\,\mathsf{D}^{[0,\ell]}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S).$$

Proof. If $F \in {}^p\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}((p_X^{-1}\mathcal{O}_S)$ by (1) of Lemma 3.5 we get for any $s \in S$, $Li_s^* F \in {}^p\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ and hence $Li_s^* \boldsymbol{D} F \cong \boldsymbol{D} Li_s^* F \in {}^p\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$. Thus, according to (2) of Lemma 3.5, $\boldsymbol{D} F \in {}^p\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}((p_X^{-1}\mathcal{O}_S)$ and so $F \in {}^\pi\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}((p_X^{-1}\mathcal{O}_S)$.

According to the definitions, Lemma 3.5 and by the t-exactness of the functor $\boldsymbol{D}$ for the perverse t-structure in the absolute case, we have:

$$\begin{aligned} F \in {}^{\pi}\,\mathsf{D}^{\leqslant -\ell}_{\mathbb{C}\text{-c}}((p_X^{-1}\mathcal{O}_S) &\iff \boldsymbol{D}F \in {}^{p}\,\mathsf{D}^{\geqslant \ell}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \\ \Rightarrow \forall\, s\in S, Li_s^*\boldsymbol{D}F \cong \boldsymbol{D}Li_s^*F \in \mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) &\Leftrightarrow \forall\, s \in S, Li_s^*F \in \mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) \\ &\iff F \in {}^{p}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}((p_X^{-1}\mathcal{O}_S). \end{aligned}$$

q.e.d.

Definition 3.7. Let $d_S = 1$. A perverse sheaf $F \in \mathrm{perv}(p_X^{-1}\mathcal{O}_S)$ (following the notation 3.3) is called *torsion-free* if for any $s \in S$ we have $Li_s^*F \in \mathrm{perv}(\mathbb{C}_X)$. We will denote by $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{tf}$ the full subcategory of perverse sheaves which are torsion-free.

In other words, for each $s_0 \in S$, given a local coordinate on S vanishing on s_0, the morphism $F \xrightarrow{s} F$ is injective in the abelian category $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)$.

Proposition 3.8. *If $d_S = 1$, π is the t-structure obtained by left tilting p with respect to the torsion pair*

$$({}^{\pi}\mathsf{D}^{\leqslant -1}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \cap \mathrm{perv}(p_X^{-1}\mathcal{O}_S), {}^{\pi}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \cap \mathrm{perv}(p_X^{-1}\mathcal{O}_S))$$

and ${}^{\pi}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \cap \mathrm{perv}(p_X^{-1}\mathcal{O}_S) = \mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{tf}$.

Proof. By Lemma 3.6 ${}^{\pi}\,\mathsf{D}^{\leqslant -1}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \subset {}^{p}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \subset {}^{\pi}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ hence, by Polishchuk result (Lemma 1.3), the t-structure π is obtained by left tilting p with respect to the torsion pair

$$({}^{\pi}\,\mathsf{D}^{\leqslant -1}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \cap \mathrm{perv}(p_X^{-1}\mathcal{O}_S), {}^{\pi}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \cap \mathrm{perv}(p_X^{-1}\mathcal{O}_S)).$$

By [23, Lemma 1.9] ${}^{\pi}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \cap \mathrm{perv}(p_X^{-1}\mathcal{O}_S) = \mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{tf}$. q.e.d.

Corollary 3.9. *If $d_S = 1$ the full subcategory of perverse S-$\mathbb{C}$-constructible sheaves with a perverse dual is quasi-abelian.*

We have the following description of π for arbitrary d_S:

Theorem 3.10. *The t-structure π on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ can be described in the following way:*

$$\begin{aligned} {}^{\pi}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) = \{F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \mid i_x^{-1}F \in {}^{\pi}\mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S) \textit{ for any } x \in X_\alpha \\ \textit{and for some adapted } \mu\textit{-stratification } (X_\alpha)\} \\ {}^{\pi}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) = \{F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \mid i_x^{!}F \in {}^{\pi}\mathsf{D}^{\geqslant d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S) \textit{ for any } x \in X_\alpha \\ \textit{and for some adapted } \mu\textit{-stratification } (X_\alpha)\} \end{aligned}$$

where the t-structure π on $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$ is the dual of the canonical t-structure described in Remark 2.1.

Proof. Following the definition of the perverse t-structure and [22, Remark 2.24]

$$\begin{gathered}F \in {}^{\pi}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \Leftrightarrow \\ \boldsymbol{D}F \in {}^{p}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \Leftrightarrow \\ \forall\,\alpha,\,\forall\,x \in X_\alpha,\; i_x^{!}\boldsymbol{D}F \cong \boldsymbol{D}i_x^{-1}F \in \mathsf{D}^{\geqslant d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S) \Leftrightarrow \\ \forall\,\alpha,\,\forall\,x \in X_\alpha,\; i_x^{-1}F \in {}^{\pi}\mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S).\end{gathered}$$

Dually

$$\begin{gathered}F \in {}^{\pi}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \Leftrightarrow \\ \boldsymbol{D}F \in {}^{p}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \Leftrightarrow \\ \forall\,\alpha,\,\forall\,x \in X_\alpha,\; i_x^{-1}\boldsymbol{D}F \cong \boldsymbol{D}i_x^{!}F \in \mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S) \Leftrightarrow \\ \forall\,\alpha,\,\forall\,x \in X_\alpha,\; i_x^{!}F \in {}^{\pi}\mathsf{D}^{\geqslant d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S)\end{gathered}$$

q.e.d.

Remark 3.11. Let us denote by ${}^{p}\tau^{\leqslant k}$ the truncation functor with respect to the t-structure p on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$. We observe that given $F \in {}^{\pi}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ we get by the previous Theorem 3.10 that ${}^{p}\tau^{\leqslant k}F \in {}^{\pi}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ for any $k \in \mathbb{Z}$ since the functors i_x^{-1} are exact and the t-structure π on $\mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S)$ is stable by truncation with respect to the standard t-structure. So, in analogy with Remark 2.13, the t-structure π is left p-compatible and, according to Lemma 2.3 and to [8, Theorem 4.3], it can be recovered from p via an iterated right tilting procedure of length ℓ.

We can now explicitly describe the torsion class in the abelian category $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)$ as follows:

Proposition 3.12. *Assume that $d_S = 1$. We have:*

$$\begin{aligned}\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_t &:= {}^{\pi}\,\mathsf{D}^{\leqslant -1}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \cap \mathrm{perv}(p_X^{-1}\mathcal{O}_S) \\ &= \{F \in \mathrm{perv}(p_X^{-1}\mathcal{O}_S)|\ \operatorname{codim} p_X(\operatorname{Supp} F) \geqslant 1\}\end{aligned}$$

Proof. We observe that $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_t = {}^{\pi}\,\mathsf{D}^{\leqslant -1}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \cap {}^{p}\,\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ (since ${}^{\pi}\,\mathsf{D}^{\leqslant -1}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S) \subseteq {}^{p}\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$). Let us recall that in the case $d_S = 1$ the dual t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$ described in Remark 2.1 reduces to:

$$\begin{aligned}{}^{\pi}\,\mathsf{D}^{\leqslant 0}_{\mathrm{coh}}(\mathcal{O}_S) &= \{\mathcal{M} \in \mathsf{D}^{\leqslant 1}_{\mathrm{coh}}(\mathcal{O}_S) \mid \operatorname{codim}\operatorname{Supp}(\mathcal{H}^1(\mathcal{M})) \geqslant 1\} \\ {}^{\pi}\,\mathsf{D}^{\geqslant 0}_{\mathrm{coh}}(\mathcal{O}_S) &= \{\mathcal{M} \in \mathsf{D}^{\geqslant 0}_{\mathrm{coh}}(\mathcal{O}_S) \mid \mathcal{H}^0(\mathcal{M}) \text{ is strict}\}\end{aligned}$$

where we recall that since $d_S = 1$ the condition $\operatorname{codim}\operatorname{Supp}(\mathcal{H}^1(\mathcal{M})) \geqslant 1$ is equivalent to $d_{\operatorname{Supp}(\mathcal{H}^1(\mathcal{M}))} = 0$ or $\mathcal{M} = 0$.

Accordingly to Theorem 3.10 an object F belongs to $\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_t$ if and only if it verifies the following two conditions where (X_α) is a μ-stratification of

X adapted to F:

(i) $\forall\,\alpha, i_x^{-1}F \in \mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathrm{coh}}\mathcal{O}_S$ and $\operatorname{codim}\operatorname{Supp}(i_x^{-1}(\mathcal{H}^{-d_{X_\alpha}}(F))) \geqslant 1, \forall\, x\in X_\alpha$.
(ii) $\forall\,\alpha, i_\alpha^{!}F \in \mathsf{D}^{\geqslant -d_{X_\alpha}}_{\mathbb{C}\text{-c}}(p_{X_\alpha}^{-1}\mathcal{O}_S)$.

Recall that, locally on X_α, $i_\alpha^{-1}F \simeq p_{X_\alpha}^{-1}G$, for some $G\in\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_S)$ and so (i) is equivalent to the following

(i') $i_\alpha^{-1}F \in \mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathbb{C}\text{-c}}(p_{X_\alpha}^{-1}\mathcal{O}_S)$ and $\operatorname{codim} p_{X_\alpha}(\operatorname{Supp}(i_\alpha^{-1}\mathcal{H}^{-d_{X_\alpha}}(F)))\geqslant 1$.

Step 1. Let us prove that, for any $F\in\operatorname{perv}(p_X^{-1}\mathcal{O}_S)_t$, $\mathcal{H}om_{\operatorname{perv}(p_X^{-1}\mathcal{O}_S)}(F,F)\cong \mathcal{H}om_{\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)}(F,F) := \mathcal{H}^0 R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)$ satisfies:

$$\operatorname{codim} p_X(\operatorname{Supp}(\mathcal{H}^0 R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)))\geqslant 1.$$

We recall that $R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F,F)\in\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ since $F\in\operatorname{perv}(p_X^{-1}\mathcal{O}_S)$ (see [22, Proposition 2.26]). For each X_α, $i_\alpha^{-1}\mathcal{H}^0R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)$ is coherent S-locally constant as a $p_{X_\alpha}^{-1}\mathcal{O}_S$-module. Hence, according to Corollary 3.1, $p_{X_\alpha}(\operatorname{Supp}(i_\alpha^{-1}\mathcal{H}^0R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)))$ is an analytic subset of S.

If $\operatorname{codim} p_X(\operatorname{Supp}(\mathcal{H}^0R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)))=0$, let X_α be a stratum of maximal dimension such that

$$\operatorname{codim} p_{X_\alpha}(\operatorname{Supp}(i_\alpha^{-1}\mathcal{H}^0R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)))=0.$$

Such a stratum X_α can not satisfy $d_{X_\alpha}=d_X$ since locally on $X_\alpha\times S$, $i_\alpha^{-1}\mathcal{H}^{-d_{X_\alpha}}F\simeq p_{X_\alpha}^{-1}G$ for some $G\in\operatorname{Mod}_{\mathrm{coh}}(\mathcal{O}_S)$ and condition (i') gives $\operatorname{codim}\operatorname{Supp}G\geqslant 1$ hence $\operatorname{codim} p_{X_\alpha}(\operatorname{Supp}(i_\alpha^{-1}\mathcal{H}^0R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)))\geqslant 1$. In particular X_α can not be open in X.

Let V be an open neighbourhood of X_α in X such that $V\smallsetminus X_\alpha$ intersects only strata of dimension $>d_{X_\alpha}$, and let $j_\alpha:(V\smallsetminus X_\alpha)\times S\hookrightarrow V\times S$ be the inclusion.

Then the complex $i_\alpha^{-1}Rj_{\alpha*}j_\alpha^{-1}R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)$ belongs to $\mathsf{D}^{\geqslant 0}_{\mathrm{coh}}(p_{X_\alpha}^{-1}\mathcal{O}_S)$ and $\mathcal{H}^0 i_\alpha^{-1}Rj_{\alpha,*}j_\alpha^{-1}R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)\cong i_\alpha^{-1}j_{\alpha,*}j_\alpha^{-1}\mathcal{H}^0R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)$ and so

$$\operatorname{codim} p_{X_\alpha}(\operatorname{Supp}(i_\alpha^{-1}\mathcal{H}^0Rj_{\alpha,*}j_\alpha^{-1}R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)))\geqslant 1.$$

By the conditions (i') and (ii) we deduce that

$$\begin{aligned}\mathcal{H}^0 i_\alpha^{!}R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F) &\simeq \mathcal{H}^0R\mathcal{H}om_{p_{X_\alpha}^{-1}\mathcal{O}_S}(i_\alpha^{-1}F, i_\alpha^{!}F)\\ &\simeq \mathcal{H}om_{p_{X_\alpha}^{-1}\mathcal{O}_S}(\mathcal{H}^{-d_{X_\alpha}}(i_\alpha^{-1}F),\mathcal{H}^{-d_{X_\alpha}}(i_\alpha^{!}F))\end{aligned}$$

and since $\operatorname{codim} p_{X_\alpha}(\operatorname{Supp}(i_\alpha^{-1}\mathcal{H}^{-d_{X_\alpha}}(F)))\geqslant 1$ we obtain

$$\operatorname{codim} p_{X_\alpha}(\operatorname{Supp}(\mathcal{H}^0 i_\alpha^{!}R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F,F)))\geqslant 1.$$

From the distinguished triangle

$$\begin{aligned}i_\alpha^{!}R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F) &\longrightarrow i_\alpha^{-1}R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)\\ &\longrightarrow i_\alpha^{-1}Rj_{\alpha,*}j_\alpha^{-1}R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)\xrightarrow{+1}\end{aligned}$$

we obtain the short left exact sequence

$$0 \longrightarrow \mathcal{H}^0 i_\alpha^! R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F) \longrightarrow \mathcal{H}^0 i_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F) \longrightarrow \mathcal{H}^0 i_\alpha^{-1} Rj_{\alpha,*} j_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)$$

which proves that $\operatorname{codim} p_{X_\alpha}(\operatorname{Supp}(i_\alpha^{-1} R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F))) \geqslant 1$ since both the first and the third term of the sequence satisfy this condition.

Step 2. Let us now deduce from step 1 that, for any $F \in \operatorname{perv}(p_X^{-1}\mathcal{O}_S)_t$, $\operatorname{codim} p_X(\operatorname{Supp} F) \geqslant 1$.

The previous condition implies $\dim(p_X(\operatorname{Supp} \mathcal{H}om_{\operatorname{perv}(p_X^{-1}\mathcal{O}_S)}(F,F))) = 0$ for any $F \neq 0$ and hence $\forall (x_0,s_0) \in X \times S$, choosing a local coordinate s in S vanishing in s_0, by the S-$\mathbb{C}$-constructibility of $\mathcal{H}om_{\operatorname{perv}(p_X^{-1}\mathcal{O}_S)}(F,F) \simeq \mathcal{H}^0 R\mathcal{H}om_{p_X^{-1}\mathcal{O}_S}(F,F)$ there exists a positive integer N such that in a neighbourhood of (x_0,s_0), $(s-s_0)^N \operatorname{Hom}_{\operatorname{perv}(p_X^{-1}\mathcal{O}_S)}(F,F) = 0$. Therefore $(s-s_0)^N \operatorname{id}_F = 0$ and so $\operatorname{id}_{(s-s_0)^N F} = 0$ which entails the result. q.e.d.

Remark 3.13. Assume that $d_S = 1$. By Proposition 3.8) π is the t-structure obtained by left tilting p with respect to the torsion pair $(\operatorname{perv}(p_X^{-1}\mathcal{O}_S)_t, \operatorname{perv}(p_X^{-1}\mathcal{O}_S)_{tf})$ in $\operatorname{perv}(p_X^{-1}\mathcal{O}_S)$ while p is the t-structure obtained by right tilting π with respect to the tilted torsion pair $(\operatorname{perv}(p_X^{-1}\mathcal{O}_S)_{tf}, \operatorname{perv}(p_X^{-1}\mathcal{O}_S)_t[-1])$ in $\mathcal{H}_\pi$. In particular we obtain that

$$\mathcal{H}_\pi = \{F \in {}^p\mathsf{D}^{[0,1]}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)) \mid {}^p\mathcal{H}^0(F) \text{ torsion free and } {}^p\mathcal{H}^1(\mathcal{M}) \text{ torsion}\}.$$

4. t-exactness of the ${}^{\mathrm{p}}\mathrm{DR}$ and the RH^{S} functors

4.a. **Reminder on the construction of** RH^{S}**.** For details on the relative subanalytic site and construction of relative subanalytic sheaves we refer to [21]. For details on the construction of RH^{S} we refer to [23].

We shall denote by $\operatorname{Op}(Z)$ the family of open subsets of a subanalytic site Z. One denotes by ρ, without reference to $X \times S$ unless otherwise specified, the natural functor of sites $\rho : X \times S \to (X \times S)_{sa}$ associated to the inclusion $\operatorname{Op}((X\times S)_{sa}) \subset \operatorname{Op}(X \times S)$. Accordingly, we shall consider the associated functors $\rho_*, \rho^{-1}, \rho!$ introduced in [18] and studied in [25].

One also denotes by $\rho' : X \times S \to X_{sa} \times S_{sa}$ the natural functor of sites. We have well defined functors ρ'_* and $\rho'_!$ from $\operatorname{Mod}(\mathbb{C}_{X\times S})$ to $\operatorname{Mod}(\mathbb{C}_{X_{sa}\times S_{sa}})$.

Note that $W \in \operatorname{Op}(X_{sa} \times S_{sa})$ if and only if W is a locally finite union of relatively compact subanalytic open subsets W of the form $U \times V$, $U \in \operatorname{Op}(X_{sa})$, $V \in \operatorname{Op}(S_{sa})$. Note that there is a natural morphism of sites $\eta : (X \times S)_{sa} \to X_{sa} \times S_{sa}$ associated to the inclusion $\operatorname{Op}(X_{sa} \times S_{sa}) \hookrightarrow \operatorname{Op}((X \times S)_{sa})$.

In the absolute case, the Riemann-Hilbert reconstruction functor RH introduced by Kashiwara in [13] from $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ to $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_X)$ was later denoted by $T\mathcal{H}om(\cdot,\mathcal{O}_X)$ in [18] where it was extensively studied. In [19] the authors showed that it can be recovered using the language of subanalytic sheaves as

$\rho^{-1} R\mathcal{H}om(\cdot, \mathcal{O}^t_X)$ where $\mathcal{O}^t_X$ is the subanalytic complex of tempered holomorphic functions on X_{sa}.

Let F be a subanalytic sheaf on $(X \times S)_{sa}$. Following [21], one denotes by $F^{S,\sharp}$ the sheaf on $X_{sa} \times S_{sa}$ associated to the presheaf

$$\begin{aligned} Op(X_{sa} \times S_{sa}) &\longrightarrow \mathrm{Mod}(\mathbb{C}) \\ U \times V &\longmapsto \Gamma(X \times V; \rho^{-1}\Gamma_{U\times S}F) \simeq \mathrm{Hom}(\mathbb{C}_U \boxtimes \rho_! \mathbb{C}_V, F) \\ &\qquad \simeq \varprojlim_{\substack{W \Subset V \\ W \in Op^c(S_{sa})}} \Gamma(U \times W; F). \end{aligned}$$

One also denotes by $(\bullet)^{RS,\sharp}$ the associated right derived functor.

Then $\mathcal{O}^{t,S,\sharp}_{X\times S} := (\mathcal{O}^t_{X\times S})^{RS,\sharp}$ is an object of $\mathsf{D}^{\mathrm{b}}(\rho'_* p^{-1}\mathcal{O}_S)$ and we also have $\mathcal{O}_{X\times S} \simeq \rho'^{-1}(\mathcal{O}^{t,S,\sharp}_{X\times S})$ (cf. [21] for details).

The functor $\mathrm{RH}^{\mathrm{S}} : \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(p_X^{-1}\mathcal{O}_S) \to \mathsf{D}^{\mathrm{b}}(\mathcal{D}_{X\times S/S})$ was then defined in [23] by the expression

$$\mathrm{RH}^{\mathrm{S}}(F) := \rho'^{-1} R\mathcal{H}om_{\rho'_* p_X^{-1}\mathcal{O}_S}(\rho'_* F, \mathcal{O}^{t,S,\sharp}_{X\times S})[d_X].$$

When F is $S-\mathbb{C}$ constructible, then $\mathrm{RH}^{\mathrm{S}}(F)$ has regular holonomic $\mathcal{D}_{X\times S/S}$-cohomologies ([23, Th. 3]).

4.b. **Main results and proofs.** The main results of this section are Theorem 4.1 and Theorem 4.2 below.

Theorem 4.1. *The functor* ${}^{\mathrm{p}}\mathrm{DR}$ *is t-exact with respect to the t-structures P and p above and consequently,* ${}^{\mathrm{p}}\mathrm{DR}$ *is also t-exact with respect to the dual t-structures Π and π.*

Theorem 4.2. *If $d_S = 1$ the functor* RH^{S} *is t-exact with respect to the t-structures p and Π as well as with respect to the their dual t-structures π and P.*

Proof of Theorem 4.1 The second statement follows obviously from the first thanks to the t-exactness of the duality functors (by definition of the dual t-structures) and the commutation of ${}^{\mathrm{p}}\mathrm{DR}$ with duality (cf. [22, Th. 3.11]). Let us now prove the first part of the statement. According to Lemma 2.8, it is sufficient to prove that if $\mathcal{M}$ is a holonomic relative module then ${}^{\mathrm{p}}\mathrm{DR}(\mathcal{M})$ is perverse.

In [23, Proposition 1.15 (1)] the authors proved that ${}^{\mathrm{p}}\mathrm{DR}({}^{P}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})) \subseteq {}^{p}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.

It remains to prove that ${}^{\mathrm{p}}\mathrm{DR}({}^{P}\mathsf{D}^{\geqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})) \subseteq {}^{p}\mathsf{D}^{\geqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ which, by duality (and by the commutativity of $\boldsymbol{D}$ and ${}^{\mathrm{p}}\mathrm{DR}$), is equivalent to prove that ${}^{\mathrm{p}}\mathrm{DR}({}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})) \subseteq {}^{\pi}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.

Recall that we proved in Theorem 2.11 that

$${}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) = \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid \operatorname{codim} p_X(\mathrm{Supp}({}^{P}\mathcal{H}^k(\mathcal{M}))) \geqslant k\}.$$

We denote by ${}^{P}\tau^{\leqslant k}$ the truncation functor with respect to the t-structure P on $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$. Given $\mathcal{M} \in {}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$, for any $k \in \mathbb{Z}$ both ${}^{P}\tau^{\leqslant k}\mathcal{M}$ and

${}^P\tau^{\geqslant k+1}\mathcal{M}$ belong to ${}^\Pi\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ (since ${}^P\mathcal{H}^i({}^P\tau^{\leqslant k}\mathcal{M}))) = {}^P\mathcal{H}^i(\mathcal{M})))$ for $i\leqslant k$ or zero otherwise).

Let us prove that:

$$(I_k)\qquad \mathcal{M}\in {}^\Pi\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\cap {}^P\,\mathsf{D}^{\leqslant k}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \Rightarrow {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M})\in {}^\pi\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$$

by induction on $k\geqslant 0$.

Let $k=0$. By Lemma 2.3 and Lemma 3.6 we get ${}^P\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\subseteq {}^\Pi\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and ${}^p\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)\subset {}^\pi\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ and so (I_0) holds true by Lemma 3.6. Let us suppose that (I_k) holds true and let us prove (I_{k+1}). Let consider $\mathcal{M}\in {}^\Pi\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\cap {}^P\,\mathsf{D}^{\leqslant k+1}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$. The distinguished triangle

$${}^P\tau^{\leqslant k}\mathcal{M}\longrightarrow\mathcal{M}\longrightarrow {}^P\mathcal{H}^{k+1}(\mathcal{M})[-k-1]\underset{+}{\longrightarrow}$$

induces the distinguished triangle

$${}^{\mathrm{p}}\mathrm{DR}({}^P\tau^{\leqslant k}\mathcal{M})\longrightarrow {}^{\mathrm{p}}\mathrm{DR}(\mathcal{M})\longrightarrow {}^{\mathrm{p}}\mathrm{DR}({}^P\mathcal{H}^{k+1}(\mathcal{M}))[-k-1]\underset{+}{\longrightarrow}$$

By inductive hypothesis ${}^{\mathrm{p}}\mathrm{DR}({}^P\tau^{\leqslant k}\mathcal{M})\in {}^\pi\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ since ${}^P\tau^{\leqslant k}\mathcal{M}\in {}^\Pi\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})\cap {}^P\,\mathsf{D}^{\leqslant k}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$. In order to conclude it is enough to prove that ${}^{\mathrm{p}}\mathrm{DR}({}^P\mathcal{H}^{k+1}(\mathcal{M}))[-k-1]\in {}^\pi\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$.

By Proposition 3.10 we have to prove that

$$i_x^{-1}\left({}^{\mathrm{p}}\mathrm{DR}({}^P\mathcal{H}^{k+1}(\mathcal{M}))[-k-1]\right)\in {}^\Pi\mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S)$$

for any α and any $x\in X_\alpha$, for some adapted μ-stratification (X_α). By the first item we have ${}^{\mathrm{p}}\mathrm{DR}({}^P\mathcal{H}^{k+1}(\mathcal{M}))\in {}^p\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)\subseteq {}^\pi\,\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ and thus (see Definition 3.2)

$$i_x^{-1}\left({}^{\mathrm{p}}\mathrm{DR}({}^P\mathcal{H}^{k+1}(\mathcal{M}))[-k-1]\right)\in \mathsf{D}^{\leqslant -d_{X_\alpha}+k+1}_{\mathrm{coh}}(\mathcal{O}_S)$$

for any α and any $x\in X_\alpha$, for some adapted μ-stratification (X_α). Moreover $\operatorname{codim} p_X(\operatorname{Supp}({}^P\mathcal{H}^{k+1}(\mathcal{M})))\geqslant k+1$ since $\mathcal{M}\in {}^\Pi\,\mathsf{D}^{\leqslant 0}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ and thus

$$\operatorname{codim} p_X(\operatorname{Supp} i_x^{-1}({}^{\mathrm{p}}\mathrm{DR}({}^P\mathcal{H}^{k+1}(\mathcal{M}))[-k-1]))\geqslant k+1$$

which proves (see Remark 2.1) that $i_x^{-1}\left({}^{\mathrm{p}}\mathrm{DR}({}^P\mathcal{H}^{k+1}(\mathcal{M}))[-k-1]\right)\in {}^\Pi\mathsf{D}^{\leqslant -d_{X_\alpha}}_{\mathrm{coh}}(\mathcal{O}_S)$.
q.e.d.

Corollary 4.3. *The functor* ${}^{\mathrm{p}}\mathrm{Sol}$ *is t-exact with respect to the t-structures respectively P on* $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})^{Op}$ *and π on* $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$.

Proof. The statement follows immediately from the relation $\boldsymbol{D}\,{}^{\mathrm{p}}\mathrm{DR} = {}^{\mathrm{p}}\mathrm{Sol}$ (cf. [22, Corollary 3.9]). q.e.d.

Remark 4.4. However the functor ${}^{\mathrm{p}}\mathrm{Sol}:\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})^{Op}\to\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ is not t-exact with respect to the t-structures respectively P on $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})^{Op}$ and p on $\mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(p^{-1}\mathcal{O}_S)$ as shown by the following example:

Example 4.5. Let $X = \mathbb{C}^*$ and $S = \mathbb{C}$ with respective coordinates x and s. Let $\mathcal{M}$ be the quotient of $\mathcal{D}_{X\times S/S}$ by the left ideal generated by ∂_x and s. Then $\mathcal{M}$ can be identified with $\mathcal{O}_{X\times\{0\}}$ with the s-action being zero and the standard ∂_x-action. We notice that $\mathcal{M}$ is holonomic, but not strict. As a $\mathcal{D}_{X\times S/S}$-module, it has the following resolution:

$$0 \to \mathcal{D}_{X\times S/S} \xrightarrow{P\mapsto(P\partial_x, Ps)} \mathcal{D}^2_{X\times S/S} \xrightarrow{(Q,R)\mapsto R\partial_x - Qs} \mathcal{D}_{X\times S/S} \to \mathcal{M} \to 0.$$

Then ${}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M})$ is represented by the complex

$$0 \longrightarrow \underset{-1}{\mathcal{O}_{X\times S}} \xrightarrow{\phi} \underset{0}{\mathcal{O}^2_{X\times S}} \xrightarrow{\psi} \underset{1}{\mathcal{O}_{X\times S}} \longrightarrow 0,$$

where $\phi(f) = (\partial_x f, sf)$ and $\psi(g,h) = sg - \partial_x h$. We know that ${}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M})$ is constructible, and since we work on $\mathbb{C}^*$, we see that its cohomology is S-locally constant. We note that $\mathcal{H}^0({}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M}))|_{X\times\{0\}} \neq 0$, since $(g,h) = (0,1)$ is a nonzero section of it. Therefore, ${}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M})$ does not belong to ${}^{\mathrm{p}}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ and so ${}^{\mathrm{p}}\mathrm{Sol}(\mathcal{M})$ is not perverse.

However ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is a perverse object: it is realized by the complex

$$0 \longrightarrow \underset{-1}{\mathcal{M}} \xrightarrow{\partial_x} \underset{0}{\mathcal{M}} \longrightarrow 0$$

and the surjectivity of ∂_x on $\mathcal{O}_{X\times\{0\}}$ entails that $\mathcal{H}^0\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) = 0$. Moreover $\mathcal{H}^j R\Gamma_{X\times S}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = \mathcal{H}^j\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = 0$, for $j < -1$.

Proof of Theorem 4.2

i) Let us prove the first t-exactness. By Lemma 2.8 we have to prove that $\mathrm{RH}^{\mathrm{S}}(\mathrm{perv}(p_X^{-1}\mathcal{O}_S)) \subseteq {}^{\Pi}\mathsf{D}^{\geqslant 0}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S}) \cap {}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$. Recall that $\mathrm{RH}Li_s^*(F) \cong Li_s^*\,\mathrm{RH}^{\mathrm{S}}(F)$ by [23, Proposition 3.25]. According to Lemma 3.5 (4) given $F \in \mathrm{perv}(p_X^{-1}\mathcal{O}_S)$ we have $Li_s^*F \in {}^{p}\mathsf{D}^{\leqslant 0}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$ for each $s \in S$ and hence $\mathrm{RH}\,Li_s^*(F) \cong Li_s^*\,\mathrm{RH}^{\mathrm{S}}(F) \in {}^{P}\mathsf{D}^{\geqslant 0}_{\mathrm{rhol}}(\mathcal{D}_X)$ for each $s \in S$ (since the functor RH is t-exact in the absolute case) and so by Lemma 2.2 we obtain $\mathrm{RH}^{\mathrm{S}}(F) \in {}^{\Pi}\mathsf{D}^{\geqslant 0}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$.

It remains to prove that $\mathrm{RH}^{\mathrm{S}}(\mathrm{perv}(p_X^{-1}\mathcal{O}_S)) \subseteq {}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$. Let $F \in \mathrm{perv}(p_X^{-1}\mathcal{O}_S)$. According to Lemma 3.5 (4), for any $s \in S$, $Li_s^*F \in {}^{p}\mathsf{D}^{\geqslant -1}_{\mathbb{C}\text{-c}}(X)$. Hence $Li_s^*(\mathrm{RH}^{\mathrm{S}}\,F) \cong \mathrm{RH}(Li_s^*F) \in \mathsf{D}^{\leqslant 1}_{\mathrm{rhol}}(\mathcal{D}_X)$ and thus by (2) of Lemma 2.2 we obtain $(*)$ $\mathrm{RH}^{\mathrm{S}}(F) \in {}^{P}\mathsf{D}^{\leqslant 1}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$. By Proposition 2.6 and Definition 1.1 it is sufficient to prove that $(**)$ ${}^{P}\mathcal{H}^1(\mathrm{RH}^{\mathrm{S}}(F))$ is a torsion module.

We divide the question in two cases, the torsion case and the torsion free case. Let us first suppose that $F \in \mathrm{perv}(p_X^{-1}\mathcal{O}_S)_t$. According to Proposition 3.12 we have $\operatorname{codim} p_X(\operatorname{Supp} F) \geqslant 1$ and so also $\operatorname{codim} p_X(\operatorname{Supp} {}^{P}\mathcal{H}^1(\mathrm{RH}^{\mathrm{S}}(F))) \geqslant 1$.

Let us now suppose that $F \in \mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{tf}$. According to [23, Cor.4], $\mathrm{RH}^{\mathrm{S}}(F)$ is a regular strict holonomic $\mathcal{D}_{X\times S/S}$-module so it belongs to ${}^{\Pi}\mathsf{D}^{\leqslant 0}_{\mathrm{rhol}}(\mathcal{D}_{X\times S/S})$ which achieves the proof of i).

ii) Let us now prove the second t-exactness. By Lemma 2.8 we have to prove that $\mathrm{RH}^{\mathrm{S}}(\mathcal{H}_\pi) \subseteq \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$. Given $F \in \mathcal{H}_\pi$ we know, according to Remark 3.13, that $F \in {}^p\mathsf{D}^{[0,1]}_{\mathbb{C}\text{-c}}(p_X^{-1}\mathcal{O}_S)$ with ${}^p\mathcal{H}^0(F)$ strict while ${}^p\mathcal{H}^1(F)$ is a torsion module. So, by Proposition 3.12, we have $\operatorname{codim} p_X(\operatorname{Supp} {}^p\mathcal{H}^1(F)) \geqslant 1$. Let us consider the distinguished triangle ${}^p\mathcal{H}^0(F) \to F \to {}^p\mathcal{H}^1(F)[-1] \overset{+1}{\to}$ (which provides the short exact sequence of F with respect to the torsion pair $(\mathrm{perv}(p_X^{-1}\mathcal{O}_S)_{tf}, \mathrm{perv}(p_X^{-1}\mathcal{O}_S)_t[-1])$ in $\mathcal{H}_\pi$). According to [23, Cor.4] we conclude that $\mathrm{RH}^{\mathrm{S}}({}^p\mathcal{H}^0(F))$ is a strict relative holonomic $\mathcal{D}_{X\times S/S}$-module while, by the previous t-exactness, $\mathrm{RH}^{\mathrm{S}}({}^p\mathcal{H}^1(F)[-1]) = \mathrm{RH}^{\mathrm{S}}({}^p\mathcal{H}^1(F))[1] \in \mathcal{H}_\Pi[1]$. According to Proposition 2.6 we have

$$\mathcal{H}_\Pi[1] = \{\mathcal{M} \in {}^P\mathsf{D}^{[-1,0]}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S}) \mid {}^P\mathcal{H}^{-1}(\mathcal{M}) \text{ strict and } {}^P\mathcal{H}^0(\mathcal{M}) \text{ torsion}\}.$$

On the other hand, since $\operatorname{codim} p_X(\operatorname{Supp} {}^p\mathcal{H}^1(F)) \geqslant 1$, the cohomology sheaves of $\mathrm{RH}^{\mathrm{S}}({}^p\mathcal{H}^1(F)[-1])$ are torsion $\mathcal{D}_{X\times S/S}$-modules. Therefore ${}^P\mathcal{H}^{-1}(\mathrm{RH}^{\mathrm{S}}({}^p\mathcal{H}^1(F))[1])$, being strict, must be equal to 0, in other words $\mathrm{RH}^{\mathrm{S}}({}^p\mathcal{H}^1(F)[-1]) \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_{X\times S/S})$ which ends the proof. q.e.d.

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