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DOCTORAL THESIS

Global existence for a hyperbolic model of multiphase flows with few interfaces

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Abstract

In this thesis we consider a hyperbolic system of three conservation laws modeling the one-dimensional flow of a fluid that undergoes phase transitions. We address the issue of the global in time existence of weak entropic solutions to the initial-value problem for large $\mathbf{BV}$ data: such is a challenging problem in the field of hyperbolic conservation laws, that can be tackled only for very special systems. In particular, we focus on initial data consisting of either two or three different phases separated by interfaces. This translates into the modeling of a tube divided into either two or three regions where the fluid lies in a specific phase. In the case of two interfaces this possibly gives rise to a drop of liquid in a gaseous environment or a bubble of gas in a liquid one.

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"We are all stories, in the end. Just make it a good one, eh?"
Eleventh Doctor

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Introduction

This thesis deals with the global in time existence of solutions to the initial-value problem for a system of hyperbolic conservation laws and gives an overall review of the results contained in [8, 9, 10]. Conservation laws are nonlinear first order partial differential equations that frequently arise in physics, for example in fluid dynamics and models of car traffic. These laws state that a measurable property of an isolated physical system does not change evolving over time. Many situations in nature are modeled according to the general principle that physical quantities are neither created nor destroyed and their variation in a domain is due to the flux across the boundary. This is expressed by systems of homogeneous equations of the form

$$\partial_t U + \mathrm{div}_x f(U) = 0\,, \qquad t \geq 0, \quad x \in \mathbb{R}^d, \quad d \geq 1\,,$$

where $U(x,t) = (U_1(x,t), \ldots, U_n(x,t))$ represents the n-tuple of the conserved quantities and $f(U) = (f_1(U), \ldots, f_n(U))$ is the *flux*.

The mathematical treatment of conservation laws is demanding even in the one-dimensional case, where the above system is rewritten as

$$\partial_t U + \partial_x f(U) = 0\,, \qquad t \geq 0, \quad x \in \mathbb{R}\,.$$

One of the major issues on the study of initial-value problems is the existence of smooth solutions: even starting from smooth initial data, solutions may develop discontinuities in finite time. For this reason, it is more convenient to consider $\mathbf{BV}$ functions and distributional solutions sorted according to entropy-admissibility conditions. As a consequence, the standard tools of functional analysis do not apply and it is only by developing ad hoc techniques (Glimm scheme, vanishing viscosity, front tracking algorithm) that in the last decades the theory of one-dimensional hyperbolic conservation laws has grown and several deep results on the well-posedeness of initial-value problems have been proved; see the reference books [17, 21, 26, 38].

The global in time existence of weak entropic solutions to the initial-value problem for any strictly hyperbolic system of conservation laws with genuinely nonlinear or linearly degenerate fields and for sufficiently small $\mathbf{BV}$ data is a well-established fact. Instead, no analogous result can be proved under the assumption of initial data with merely bounded total variation (not necessarily small), as the examples of [29] show. Nevertheless, investigating the possibility for large data is a challenging problem that can be tackled for some special systems, as it is for the Temple ones [40].

This thesis focuses on the issue of large data for a system of the gas dynamics that was first considered in [2]. In particular, the system arises in a model of phase transitions for the one-dimensional inviscid flow of a fluid and is the conservative part of a more complex system theorized in [24]. Denoting by $v > 0$ the specific volume of the fluid, u the velocity, p the pressure and λ the mass-density fraction of the vapor in the fluid, the system is written in Lagrangian coordinates as

$$\begin{cases} v_t - u_x = 0\,, \\ u_t + p(v,\lambda)_x = 0\,, \\ \lambda_t = 0\,. \end{cases}$$

The phase states of the fluid are characterized by the variable λ ranging over the real interval of values between 0 (pure liquid phase) and 1 (pure vapor phase). The variable λ is also incorporated in the pressure

$$p(v,\lambda) = \frac{a^2(\lambda)}{v}\,,$$

where $a(\lambda) > 0$ is a C^1 function defined on $[0,1]$. The system is strictly hyperbolic with two genuinely nonlinear characteristic fields supporting sonic waves and a linearly degenerate one supporting contact discontinuities, which are interpreted as *interfaces* keeping the fluid separated into two different phases.

The initial-value problem includes $\mathbf{BV}$ data

$$(v_o(x), u_o(x), \lambda_o(x))\,, \qquad v_o(x) \geq \overline{v} > 0\,, \qquad x \in \mathbb{R}\,.$$

If the function λ_o is constant, we immediately get the global existence of solutions for any initial data. Indeed, in this case $\lambda(x,t)$ remains constant also w.r.t x and our system becomes an (isentropic) isothermal p-system

$$\begin{cases} v_t - u_x = 0\,, \\ u_t + p(v)_x = 0\,, \end{cases}$$

for which Nishida in [33] proves the existence for arbitrarily large $\mathbf{BV}$ data. When λ_o is non-constant, instead, Nishida's theorem does not apply, but in [2] Amadori and Corli find that $\mathrm{TV}(v_o)$, $\mathrm{TV}(u_o)$ can be taken large provided that $\mathrm{TV}(\lambda_o)$ is small, and vice versa. Motivated by [2], one could wonder if the Nishida's theorem is recovered at least in the case where λ_o is piecewise constant with few jumps, say one or two. Indeed, in this case the model reduces to either two or three p-systems coupled through either one or two interfaces. Furthermore, the problem could be understood as the perturbation of a Riemann solution. This subject was thoroughly studied in [31, 37] for generic hyperbolic conservation laws and large data. Nevertheless, not all the hypotheses required there are satisfied in this case (see [6]) and, hence, the global existence of solutions can not be inferred from the theorems contained therein, but has to be dealt with differently. A satisfactory answer is given in [8, 9, 10].

In particular, [10] considers the case of an initial datum λ_o with a single discontinuity at $x = 0$ that gives rise to a contact discontinuity referred to as the *phase wave*. This assumption on λ_o allows to weaken the hypotheses of [2] (larger bound on the $\mathbf{BV}$ norm of the data) and to improve the final existence theorem. The most relevant novelties of the proof are the introduction of a peculiar Glimm functional that controls the total variation of the approximate solutions and a careful tailoring of the front tracking algorithm used to construct them. Front tracking approximations [17, 26] are piecewise constant functions whose jumps are located along finitely many straight lines in the (x, t)-plane, that are called *fronts* and can be of two main types, shocks and rarefactions. The standard scheme prescribes also the use of fictitious non-physical fronts that are not needed for systems of only two conservation laws, see [7, 13]. Since our initial data reduce the system to two 2×2 systems, it would be reasonable to avoid this technicality. However, this is not the case: non-physical waves must be taken into consideration, but the simple structure of the data allows for an unusual treatment that simplifies the estimates.

On the other hand, [8, 9] study the initial-value problem for data with two phase interfaces, i.e. corresponding to λ_o piecewise constant with two jumps, say at $x = 0$ and $x = 1$. Clearly, this case is more complicated than the previous one, because of the possible bouncing back and forth of the waves in the middle region $[0, 1]$. Here, the model describes a fluid consisting of three homogeneous mixtures of liquid and vapor and three main configurations (that may have a physical interpretation) can be considered:

- **(d) the drop case**;
- **(b) the bubble case**;
- **(p) the increasing (decreasing) pressure case**.

The first case is analyzed in [9], while the other two are treated in [8]. For the proof of the main result, two novel ideas are employed: the first one is a further simplification of the Glimm functional, in which some nonlinear terms are dropped; the second one is another original variant of the front tracking algorithm that is essential for the adoption of this new functional.

The thesis is organized as follows. In Chapter 1 we recall some preliminary concepts on hyperbolic conservation laws and we discuss the model studied throughout the sequel. In Chapter 2 we describe the peculiar front tracking algorithm introduced in [9] and that we need to construct the approximate solutions in the proofs of Chapter 3 and 4. In particular, Chapter 3 contains the existence result for the single interface case. Remark that the analysis of this chapter differs from that of the related paper [10] for two reasons: the use of a more sophisticated front tracking algorithm and the change of approach in the proof of the decreasing of the Glimm functional. The latter is also an ingredient in the proof of the main theorem of Chapter 4, where we treat the case of the two phase waves and we conclude with some open questions.

Chapter 1

Preliminaries

In this chapter we briefly recall some preliminary notions on the theory of conservation laws with particular attention to p-systems, that are at the core of the model studied throughout the sequel. As mentioned in the Introduction, in this thesis we address the issue of the existence of solutions to the initial-value problem for a system of conservation laws. Hence, we devote the first two sections to review some basics on the Riemann problem and the front tracking algorithm, which is a classical technique used to construct approximate solutions. In Section 1.3 we focus on some well-known existence results for the p-system and, finally, in Section 1.4 we introduce the model we are going to analyze in the remaining chapters. As general references we cite [17, 21, 26, 38].

1.1 The Riemann problem

Let $\Omega \subseteq \mathbb{R}^n$, $n > 1$, be an open set and let $f : \Omega \to \mathbb{R}^n$ be a smooth vector field. We focus on systems of one-dimensional conservation laws of the form

$$U_t + f(U)_x = 0\,, \qquad t \geq 0, \quad x \in \mathbb{R}\,, \tag{1.1.1}$$

where $U = U(x,t) = (U_1(x,t), \ldots, U_n(x,t))$.

Definition 1.1.1. *Given $U_-, U_+ \in \Omega$ and*

$$U(x,0) = \begin{cases} U_- & \textit{if } x < 0, \\ U_+ & \textit{if } x > 0\,, \end{cases} \tag{1.1.2}$$

the Riemann problem (1.1.1), (1.1.2) *is the initial-value problem consisting of system* (1.1.1) *and initial data* (1.1.2).

In this section, we show how to construct a weak solution to (1.1.1), (1.1.2). In general, given an initial condition

$$U(x,0) = U_o(x)\,, \qquad x \in \mathbb{R}\,, \tag{1.1.3}$$

by *weak* solution to the problem (1.1.1), (1.1.3) we mean a function $u : \mathbb{R} \times [0,T] \to \mathbb{R}^n$ that satisfies the following requests:

- the map $t \to U(\cdot,t)$ is continuous with values in L^1_{loc};
- (1.1.3) is satisfied;
- for every C^1 function φ with compact support contained in the open strip $\mathbb{R}\times]0,T[$ it holds

$$\int_0^T \int_{-\infty}^{+\infty} [\varphi_t(x,t)U(x,t) + \varphi_x(x,t)f(U(x,t))]\,dx\,dt = 0\,.$$

Moreover, given a convex entropy η with entropy flux q, we say that the weak solution is *η-admissible* if it satisfies the entropy inequality

$$\eta(U)_t + q(U)_x \leq 0$$

in the distributional sense, i.e. if for every non-negative C^1 function φ with compact support contained in $\mathbb{R}\times]0,T[$ it holds

$$\int_0^T \int_{-\infty}^{+\infty} [\varphi_t(x,t)\eta(U(x,t)) + \varphi_x(x,t)q(U(x,t))]\,dx\,dt \geq 0\,.$$

We now make some assumptions on the flux f. For example, in order to have finite speed of propagation (which characterizes hyperbolic equations), we have to require that the Jacobian matrix of f, denoted by $Df(U)$, has n real eigenvalues $\lambda_1(U) \leq \cdots \leq \lambda_n(U)$ for every $U \in \Omega$. When these eigenvalues are all distinct we say that the system is *strictly hyperbolic*.

Under the assumption of strict hyperbolicity, the system admits a basis of right eigenvectors $\{r_1(U), \cdots, r_n(U)\}$ depending smoothly on U. For each characteristic family $i = 1, \ldots, n$ we say that the i-th characteristic field is:

- *genuinely nonlinear* if for every $U \in \Omega$ it holds $\nabla\lambda_i(U) \cdot r_i(U) \neq 0\,;$
- *linearly degenerate* if for every $U \in \Omega$ it holds $\nabla\lambda_i(U) \cdot r_i(U) = 0$.

To be able to construct weak entropic solutions to (1.1.1), (1.1.2), we work under the hypotheses that the system is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. The next example is a classical system of the gas dynamics and will be used both as a motivation for the subsequent analysis and as an excuse for computing the basic quantities just defined.

Example 1.1.2. *A* p-system *is a one-dimensional model for the dynamics of an isentropic gas (with costant entropy), for which one has conservation of mass and momentum but not energy. In Lagrangian coordinates it is given by*

$$\begin{cases} v_t - u_x = 0\,, \\ u_t + p(v)_x = 0\,, \end{cases} \qquad t \geq 0, \quad x \in \mathbb{R}\,, \tag{1.1.4}$$

where $v > 0$ *is the specific volume,* $u \in \mathbb{R}$ *is the velocity of the gas and* $p = p(v)$ *is a prescribed pressure law satisfying* $p'(v) < 0$ *and* $p''(v) > 0$*. The class of systems like* (1.1.4) *includes the interesting case of the so-called* γ*-laws*

$$p(v) = \frac{a^2}{v^\gamma}\,, \tag{1.1.5}$$

where $a > 0$ *and* $\gamma \geq 1$ *are constants. In particular,* γ *is called* the adiabatic constant *and it holds* $1 < \gamma < 3$ *for the majority of the gases. System* (1.1.4) *is strictly hyperbolic with eigenvalues*

$$\lambda_1 := -\sqrt{-p'(v)} < 0 < \sqrt{-p'(v)} =: \lambda_2 \,.$$

The right eigenvectors are $r_1 = (1, \sqrt{-p'(v)})$, $r_2 = (-1, \sqrt{-p'(v)})$ *and both the characteristc fields are genuinely nonlinear. Indeed,*

$$\begin{aligned}
\nabla\lambda_1 \cdot r_1 &= \left(\frac{p''(v)}{2\sqrt{-p'(v)}}, 0\right) \cdot \left(1, \sqrt{-p'(v)}\right) > 0\,, \\
\nabla\lambda_2 \cdot r_2 &= \left(-\frac{p''(v)}{2\sqrt{-p'(v)}}, 0\right) \cdot \left(-1, \sqrt{-p'(v)}\right) > 0\,.
\end{aligned}$$

Since both the system (1.1.1) and the Riemann data (1.1.2) are self-similar, i.e. invariant under the transformation $(x,t) \to (cx, ct)$, for $c \in \mathbb{R}$ constant, we observe that the solutions should have the same property. Thus, we search for solutions

$$U(x,t) = \widetilde{U}(\xi)\,, \qquad \xi = \frac{x}{t}\,.$$

Inserting this into (1.1.1), we end up with

$$Df(\widetilde{U})\,\dot{\widetilde{U}} = \xi\,\dot{\widetilde{U}}\,,$$

where $\dot{\widetilde{U}}$ is the derivative of $\widetilde{U}$ w.r.t. ξ. Hence, $\dot{\widetilde{U}}$ is an eigenvector of the Jacobian matrix $Df(\widetilde{U})$ with eigenvalue ξ, i.e. for some $i \in \{1, \ldots, n\}$ we have

$$\xi = \lambda_i(\widetilde{U}(\xi))\,, \qquad \dot{\widetilde{U}}(\xi) = r_i(\widetilde{U}(\xi))\,,$$

by a suitable normalization of r_i. Assume that the i-th characteristic field is genuinely nonlinear and, in particular, that $\nabla\lambda_i \cdot r_i > 0$. Let $U_- \in \Omega$ be fixed. Then, there exists a curve $R_i(U_-)$ in Ω emanating from U_- and such that, for any $U_+ \in R_i(U_-)$, the Riemann problem (1.1.1), (1.1.2) has a weak solution given by the continuos function

$$U(x,t) = \begin{cases} U_- & \text{if } x < \lambda_i(U_-)t\,, \\ \widetilde{U}(x/t) & \text{if } \lambda_i(U_-)t \le x \le \lambda_i(U_+)t\,, \\ U_+ & \text{if } x > \lambda_i(U_+)t\,, \end{cases} \tag{1.1.6}$$

where $\widetilde{U}(\lambda_i(U_-)) = U_-$, $\widetilde{U}(\lambda_i(U_+)) = U_+$ and $\lambda_i(\widetilde{U}(x/t))$ is strictly increasing (for t fixed). We call (1.1.6) a *centred rarefaction wave* and we denote by $R_i(\sigma)(U_-)$ its orbit parameterized by σ, i.e. the solution to the Cauchy problem

$$\frac{dU}{d\sigma} = r_i(U)\,, \qquad U\restriction_{\sigma=0} = U_-\,. \tag{1.1.7}$$

Thus, we see that for any left state U_- in Ω there are n such integral curves emanating from U_-, on which U_+ can lie allowing for a solution like (1.1.6).

Now, assume that the i-th characteristic field is linearly degenerate. In this case, we denote by $C_i(\sigma)(U_-)$ the solution to (1.1.7) and we notice that along this curve λ_i is constant. If there exists $\sigma_o \in \mathbb{R}$ such that $U_+ = C_i(\sigma_o)(U_-)$, then

$$U(x,t) = \begin{cases} U_- & \text{if } x \le \lambda_i(U_-)t\,, \\ U_+ & \text{if } x > \lambda_i(U_-)t\,. \end{cases} \tag{1.1.8}$$

is a solution to the Riemann problem (1.1.1), (1.1.2) called *contact discontinuity*. Moreover, (1.1.8) is weak since the *Rankine-Hugoniot* condition, namely

$$f(U) - f(U_-) = s_i(U - U_-)\,, \tag{1.1.9}$$

is satisfied along $U = C_i(\sigma)(U_-)$ with $s_i = \lambda_i(U_-)$.

This condition characterizes the i-th *Hugoniot locus* $S_i(U_-)$ through a point U_- as the set of points U for which there is $s_i \in \mathbb{R}$ such that (1.1.9) is satisfied. More precisely, we can prove the *local* existence around U_- of n smooth curves $S_1(\sigma)(U_-), \ldots, S_n(\sigma)(U_-)$, parameterized by a suitable σ that allows for a second order tangency with the corresponding rarefaction curves of the same family $R_1(\sigma)(U_-), \ldots, R_n(\sigma)(U_-)$, when the field is genuinely nonlinear. Moreover, we have that the i-th rarefaction curve and the i-th Hugoniot locus through U_- coincide when the i-th characteristic field is linearly degenerate.

Once we have derived the Hugoniot loci for system (1.1.1) and U_-, we select the parts of these curves that give admissible shocks, i.e. that satisfy some entropy conditions. If the i-th characteristic field is genuinely nonlinear (for example, $\nabla\lambda_i \cdot r_i > 0$), given two points U_- and $U_+ = S_i(\sigma_o)(U_-)$ for a suitable $\sigma_o \in \mathbb{R}$, a *shock wave* of family i connecting U_+ to U_- is a solution to the Riemann problem (1.1.1), (1.1.2) of the form

$$U(x,t) = \begin{cases} U_- & \text{if } x \le s_i t\,, \\ U_+ & \text{if } x > s_i t\,. \end{cases} \tag{1.1.10}$$

In particular, (1.1.10) is called a *compressive shock* when $\sigma_o < 0$ and a *rarefaction shock* when $\sigma_o > 0$. Moreover, we say that (1.1.10) is a *Lax* shock if the shock speed s_i verifies the Lax inequalities:

$$\lambda_{i-1}(U_-) < s_i < \lambda_i(U_-)\,, \qquad \lambda_i(U_+) < s_i < \lambda_{i+1}(U_+)\,, \tag{1.1.11}$$

(where $\lambda_0 = -\infty$ and $\lambda_{n+1} = +\infty$). We have that compressive shocks are Lax shocks, while rarefaction shocks violate (1.1.11).

Example 1.1.3. *Here we obtain the expressions for the rarefaction curves and the shock curves of system* (1.1.4)*, that are depicted in Figure 1.1. The rarefaction curves of family $i = 1, 2$ through a point $U_- = (v_-, u_-) \in \Omega$ are*

$$U = R_1(v)(U_-)\ :\quad v > v_-\,,\quad u = u_- + \int_{v_-}^{v} \sqrt{-p'(z)}\,dz\,,$$

$$U = R_2(v)(U_-)\ :\quad v < v_-\,,\quad u = u_- - \int_{v_-}^{v} \sqrt{-p'(z)}\,dz\,,$$

where in both cases the curves are parameterized by v and we select only the branch of the integral curves along which λ_i increases.

As for the shock curves of family $1, 2$ *connecting* U *to* U_-*, we recall that the speeds are*

$$s_1 = -\sqrt{\frac{p(v_-) - p(v)}{v - v_-}} < 0\,, \qquad s_2 = \sqrt{\frac{p(v_-) - p(v)}{v - v_-}} > 0\,,$$

and, according to (1.1.11)*,* 1*-shocks must satisfy the conditions*

$$s_1 < \lambda_1(U_-)\,, \qquad \lambda_1(U) < s_1 < \lambda_2(U)\,,$$

while 2*-shocks must satisfy*

$$\lambda_1(U_-) < s_2 < \lambda_2(U_-)\,, \qquad \lambda_2(U) < s_2\,.$$

The expressions for the shock curves are

$$\begin{aligned} U = S_1(v)(U_-) &: \quad v < v_-\,, \quad u = u_- - \sqrt{(v - v_-)(p(v_-) - p(v))}\,, \\ U = S_2(v)(U_-) &: \quad v > v_-\,, \quad u = u_- - \sqrt{(v - v_-)(p(v_-) - p(v))}\,. \end{aligned}$$

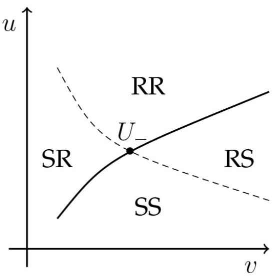

FIGURE 1.1: Rarefaction and shock curves through U_- in the (v, u)-plane. The thick curves are of the first characteristic family, while the dashed ones are the curves of the second family.

Now, we combine the properties of the rarefaction waves and the shock waves to obtain the unique solution to the Riemann problem for small initial data. We start by defining the *wave curves*. Let $U_- \in \Omega$ and $i \in \{1, \dots, n\}$.

- If the i-th characteristic family is genuinely nonlinear, we set

$$W_i(\sigma)(U_-) := \begin{cases} R_i(\sigma)(U_-) & \text{if } \sigma \geq 0\,, \\ S_i(\sigma)(U_-) & \text{if } \sigma < 0\,. \end{cases}$$

- If the i-th characteristic family is linearly degenerate, we set

$$W_i(\sigma)(U_-) := C_i(\sigma)(U_-)\,.$$

Remark that, in the genuinely nonlinear case we parameterize these wave curves by σ allowing for a second order tangency at U_- between the shock and the rarefaction curves. The importance of the wave curves is that they almost form a local coordinate system around U_- and this gives the possibility to prove the existence of solutions to the Riemann problem for U_+ close to U_-.

For $(\sigma_1, \dots, \sigma_n)$ in a neighborhood of $0 \in \mathbb{R}^n$, we inductively define the states $\omega_0, \dots, \omega_n$ by

$$\omega_0 := U_-\,, \qquad \omega_i := W_i(\sigma_i)(\omega_{i-1})\,. \tag{1.1.12}$$

By the previous discussion, each Riemann problem with data

$$U(x,0) = \begin{cases} \omega_{i-1} & \text{if } x < 0\,, \\ \omega_i & \text{if } x > 0\,, \end{cases} \tag{1.1.13}$$

has a weak entropy-admissible solution consisting of a simple wave of the i-th family. More precisely,

- if the i-th characteristic field is genuinely nonlinear and $\sigma_i > 0$, the solution is a rarefaction wave whose characteristic speeds range over the interval $[\lambda_i^-, \lambda_i^+]$, where $\lambda_i^- := \lambda_i(\omega_{i-1})$ and $\lambda_i^+ = \lambda_i(\omega_i)$;

- if either the i-th field is genuinely nonlinear and $\sigma_i < 0$ or the i-th field is linearly degenerate, the solution is either a shock travelling with speed $\lambda_i^- := \lambda_i^+ := s_i$ or a contact discontinuity travelling with speed $\lambda_i^- := \lambda_i^+ := \lambda_i(\omega_{i-1})$.

Assume that $\omega_n = U_+$. The solution to the Riemann problem (1.1.1), (1.1.2) can be constructed by piecing together the solutions to the n Riemann problems (1.1.1), (1.1.13) on different sectors of the (x,t)-plane. Indeed, for sufficiently small $\sigma_1, \dots \sigma_n$, the speeds $\lambda_i^\pm$ remain close to the corresponding eigenvalues $\lambda_i(U_-)$. By the strict hyperbolicity and the closeness of the initial states, we can assume that the n intervals $[\lambda_i^-, \lambda_i^+]$ are disjoint. Therefore, the following piecewise smooth function $U : \mathbb{R} \times [0, +\infty[\to \Omega$ is well-defined:

$$U(x,t) = \begin{cases} U_- & \text{if } x < \lambda_1^- t\,, \\ \omega_i & \text{if } x \in]\lambda_i^+ t, \lambda_{i+1}^- t[\,, i = 1, \dots, n\,, \\ R_i(\sigma)(\omega_{i-1}) & \text{if } x \in [\lambda_i^- t, \lambda_i^+ t], x = \lambda_i(R_i(\sigma_i)(\omega_{i-1}))t\,, \\ U_+ & \text{if } x > \lambda_n^+ t\,. \end{cases} \tag{1.1.14}$$

The following theorem states the existence of solutions to a Riemann problem.

Theorem 1.1.4 (Lax)**.** *Assume that f is a smooth vector field defined on Ω, the system* (1.1.1) *is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. Then, for every compact set $K \subset \Omega$, there exists $\delta > 0$ such that the Riemann problem* (1.1.1), (1.1.2) *has a unique weak solution of the form* (1.1.14) *whenever $U_- \in K$ and $|U_+ - U_-| \leq \delta$.*

For the proof and other fundamentals on Riemann problems we cite the seminal paper [30].

Example 1.1.5. *We return to system* (1.1.4) *and consider the Riemann problem for initial data* (1.1.2)*. For any point $U_- \in \Omega$ and $i = 1, 2$, the wave curves $W_i(v)(U_-)$ separate the plane into four distinct regions SS, SR, RS and RR; see Figure 1.1. If U_+ lies in any of the two above curves, then the solution is a simple wave of rarefaction or shock type depending on which branch of the* *curve it is located. On the other hand, if U_+ lies in the interior of one of the four regions, consider the family of curves*

$$\mathcal{F} := \{W_2(v)(U_m) : \, U_m \in W_1(v)(U_-)\} \; .$$

Remark that the three regions SS, SR and RS are covered univalently by $\mathcal{F}$, i.e. through each point U_+ in one of these regions there passes exactly one curve $W_2(v)(U_m)$. This is no longer true for the region RR, as we will see below. The solution to the Riemann problem is constructed as follows: we connect U_m to U_- on the right by a wave of family 1 and we connect U_+ to U_m on the right by a wave of family 2. Clearly, whether these waves are of rarefaction or shock type depends on the position of U_+. The four possible outcomes are depicted in Figure 1.3.

In the peculiar case where U_+ belongs to RR and fails to be in a suitably small neighborhood of U_-, we observe that it cannot be connected to U_-. Indeed, not every point in this region can always be reached by a curve $R_2(v)(U_m)$.

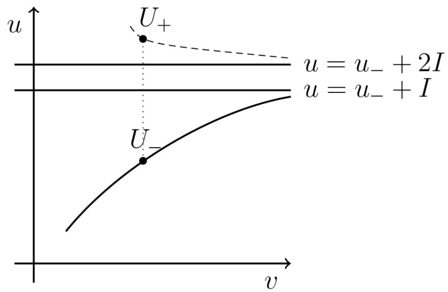

FIGURE 1.2: The appearance of vacuum corresponds to the situation where the integral (1.1.15) converges and the 1-rarefaction curve through U_- has the horizontal asymptote $u = u_- + I$.

For example, if

$$I = \int_{v_-}^{+\infty} \sqrt{-p'(y)} \, dy < \infty \, , \tag{1.1.15}$$

the curve $R_1(v)(U_-)$ has the horizontal asymptote $u = u_- + I$. If we take $U_+ = (v_-, u_+)$ with $u_+ > u_- + 2I$, one can see that there is no way that U_+ lies on any $R_2(v)(U_m)$, as in Figure 1.2. In particular, (1.1.15) *is verified for pressure laws of the kind* (1.1.5) *if and only if $\gamma > 1$; the convergence of the integral physically corresponds to the possible appearance of vacuum. However, by Theorem 1.1.4 it is still possible to solve the Riemann problem if $|U_+ - U_-|$ is small enough, see Chapter 17 §A of [38].*

1.2 The Cauchy problem

In this section we study the initial-value problem for a general $n \times n$ system of conservation laws (1.1.1) and data (1.1.3). We are in the setting of Theorem 1.1.4, i.e. we have a strictly hyperbolic system where each characteristic field is either genuinely nonlinear or linearly degenerate. Given an initial condition U_o with sufficiently small total variation, it is possible to construct a weak, entropy-admissible solution U defined for all times. The smallness hypothesis on the data is important since the Riemann problems may fail to have a solution if the initial states are far apart.

The following is a classical result on the global existence of solutions for conservation laws; see [17, 21, 26] and references therein for a detailed analysis.

Theorem 1.2.1 (Global existence of entropy weak solutions)**.** *Under the basic assuptions of strict hyperbolicity of the system, genuine nonlinearity or linar degeneracy of the characteristic fields, there exists a constant $\delta_o > 0$ with the following property. For every initial datum $U_o \in L^1$ with*

$$\mathrm{TV}\, U_o \le \delta_o \,, \tag{1.2.1}$$

the initial-value problem (1.1.1), (1.1.3) *has a weak solution $U = U(x,t)$ defined for all $t \ge 0$. In addition, if the system admits a convex entropy η, then one can find a solution which is also η-admissible.*

The proof of Theorem 1.2.1 consists of four main steps:

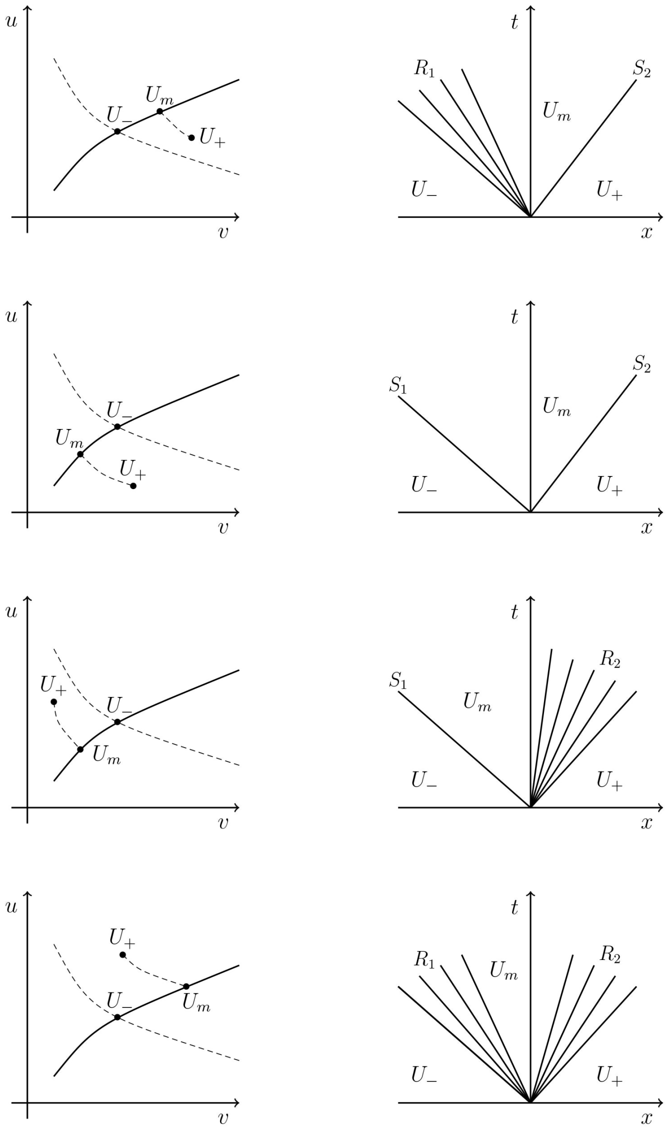

FIGURE 1.3: The Riemann problem for the p-system and initial states U_-, U_+.

- the construction of approximate solutions;
- the interaction estimates;
- the compactness of a subsequence of approximate solutions;
- showing that the limit is indeed a solution.

Theorem 1.2.1 was first proved in the paper of Glimm [25], where the fundamental approach was formulated and all the basic estimates can be found. The Glimm's result for small initial data uses approximate solutions constructed via a difference scheme which involves a random choice and is called the *Glimm scheme*. An alternative line of proof that is exploited in the next chapters is based on a front tracking algorithm. A front tracking for scalar equations was first proposed by Dafermos in [22] and, subsequently, was extended to 2×2 systems by DiPerna in [23]. To cover the general $n \geq 2$ case, one needs to overcome the problem of possible blow-up in the number of wave fronts. This was accomplished by Risebro in [36] through a back-stepping procedure and by Bressan in [16] thanks to the introduction of non-physical fronts. Early applications of this method can be found in [1] and [42]. Here, we essentially follow Chapter 7 of [17], where the version of the algorithm refined in [14] is presented.

1.2.1 The front tracking algorithm

The algorithm can be roughly described as follows. For a fixed $\varepsilon > 0$, an ε-approximate front tracking solution is a piecewise constant function $U = U_\varepsilon(x,t)$, whose discontinuities are located along finitely many straight line $x = x_\alpha(t)$ in the (x,t) plane (*fronts*) and approximately satisfy the Rankine-Hugoniot condition (1.1.9). At each time $t > 0$ the following estimate is expected:

$$\sum_\alpha \Big| [f(U_+) - f(U_-)] - \dot{x}_\alpha [U_+ - U_-] \Big| = \mathcal{O}(1)\varepsilon \,,$$

where $U_+ = U(x_\alpha+, t)$ and $U_- = U(x_\alpha-, t)$. For $i = 1, \ldots, n$, recall the i-th rarefaction curve $R_i(\sigma)(U_-)$ and the i-th shock curve $S_i(\sigma)(U_-)$, through the state U_- and parameterized by some σ that measures the size of the wave. Jumps can be of three types: shocks (or contact discontinuities in the linearly degenerate case), rarefactions and non-physical waves.

- Along each shock front (or contact discontinuity) x_α, the points U_+ and U_- are related by $U_+ = S_{k_\alpha}(\sigma_\alpha)(U_-)$, for a genuinely nonlinear characteristic family $k_\alpha \in \{1, \ldots, n\}$ and $\sigma_\alpha < 0$ (or $U_+ = C_{k_\alpha}(\sigma_\alpha)(U_-)$, for a linearly degenerate k_α). We call $|\sigma_\alpha|$ the *strength* of the wave and the speed s_α satisfies $|s_\alpha - \dot{x}_\alpha| \leq \varepsilon$ (or $\lambda_{k_\alpha}(U_-)$ satisfies $|\lambda_{k_\alpha}(U_-) - \dot{x}_\alpha| \leq \varepsilon$).

- Along each rarefaction front x_α, one has $U_+ = R_{k_\alpha}(\sigma_\alpha)(U_-)$ for some genuinely nonlinear characteristic family $k_\alpha \in \{1, \ldots, n\}$ and $\sigma_\alpha \in]0, \varepsilon]$. Again, we call $|\sigma_\alpha|$ the strength of the rarefaction and the speed satisfies $|\lambda_{k_\alpha}(U_+) - \dot{x}_\alpha| \leq \varepsilon$.

- Along a non-physical front x_α, we set $|\sigma_\alpha| = |U_+ - U_-|$ for the strength of the wave and $\dot{x}_\alpha = \hat{\lambda}$ for the speed, where $\hat{\lambda}$ is fixed and strictly bigger than all characteristic speeds. For convenience, we say that non-physical fronts belong to a fictitious $n+1$ characteristic family.

Definition 1.2.2 (ε-approximate solution). *For $\varepsilon > 0$, we say that a continuous map $t \mapsto U_\varepsilon(\cdot, t) \in L^1_{loc}(\mathbb{R}, \mathbb{R}^n)$ is an ε-approximate solution of* (1.1.1), (1.1.3) *if the following conditions hold: 1) as a function of (x,t), it is piecewise constant with jumps as above; 2) the total strength of all the non-physical waves is $\leq \varepsilon$ for all $t \geq 0$; 3) it holds*

$$\|U_\varepsilon(\cdot, 0) - U_o(\cdot)\|_{L^1} \leq \varepsilon\,.$$

For a proof of Theorem 1.2.1, we first need to establish the existence of front tracking approximations defined for all times, i.e. we have to prove that there exists a constant $\delta_o > 0$ such that, for every initial condition $U_o \in L^1$ that satisfies (1.2.1) and for every $\varepsilon > 0$, the Cauchy problem (1.1.1), (1.1.3) admits an ε-approximate solution globally defined in time. Secondly, one has to show that a suitable sequence of front tracking approximations converges to a limit $U = U(x,t)$ that provides an entropy-admissible weak solution.

Let $\varepsilon > 0$ be fixed. The construction of U_ε by means of the front tracking algorithm starts at time $t = 0$ by taking a piecewise constant approximation $U_{o,\varepsilon}$ of the initial data U_o satisfying

$$\mathrm{TV}\, U_{o,\varepsilon} \leq \mathrm{TV}\, U_o \leq \delta_o\,, \qquad \|U_{o,\varepsilon} - U_o\|_{L^1} < \varepsilon\,.$$

Let $x_1 < \cdots < x_N$ be the points where $U_{o,\varepsilon}$ is discontinuous. For each $\alpha = 1, \ldots, N$, the Riemann problem of initial states $U_{o,\varepsilon}(x_\alpha-)$ and $U_{o,\varepsilon}(x_\alpha+)$ is approximately solved on a forward neighborhood of $(x_\alpha, 0)$ in the (x,t)-plane by a function of the form $U_\varepsilon(x,t) = \widetilde{U}_\varepsilon((x - x_\alpha)/t)$, with $\widetilde{U}_\varepsilon$ piecewise constant. If the exact solution to the Riemann problem contains only shocks and contact discontinuities, then we let U_ε coincide with the exact solution; if, instead, centred rarefaction waves are present, they are approximated by a centred rarefaction fan containing several small jumps travelling at a speed close to the characteristic one.

Piecing together the solutions of all the Riemann problems, we obtain an approximate solution defined on a small time interval. Then, the fronts are prolonged until the first time two of them interact and a new Riemann problem appears. Since at this time U_ε is still a piecewise constant function, the newly generated Riemann problems can be approximately solved within the class of piecewise constant functions and so on. By slightly perturbing the speed of just one incoming front at a point of interaction, we can assume that at any time only one interaction between two fronts can occur. Moreover, for interaction times $t > 0$ we choose not to partition an outgoing rarefaction when there is an incoming rarefaction of the same characteristic family.

For $n \times n$ systems the main source of technical difficulty is that the number of fronts may approach infinity in finite time: this is due to the fact that at each interaction point there are two incoming fronts, while the number of outgoing ones is n or even larger if rarefaction waves are involved. Thus, one should provide a uniform (in time) bound on the number of fronts and interactions. To overcome this issue, the algorithm in [17] adopts two different procedures to solve Riemann problems: an *accurate Riemann solver*, which introduces several new fronts, and a *simplified Riemann solver*, which generates at most two physical outgoing fronts and collect the remaining new waves into a single *non-physical* front that is not directly related to elementary waves of the system. Below we describe these two solvers represented in Figure 1.4.

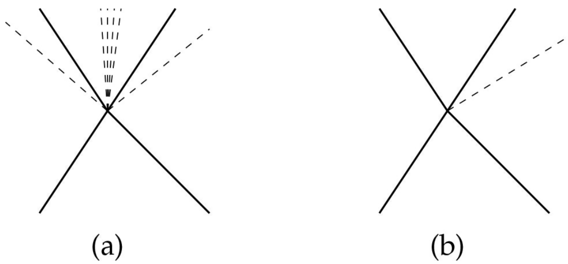

FIGURE 1.4: The accurate Riemann solver (a) and the simplified Riemann solver (b). The dashed lines represent the newly generated fronts, i.e. of a different characteristic family from those of the incoming fronts.

Consider two wave fronts of size σ, σ' interacting at a point $(\bar{x}, \bar{t})$ of the (x, t)-plane. For a fixed $\rho_\varepsilon > 0$, the emerging Riemann problem of initial states U_-, U_+ is treated with the accurate solver if $|\sigma\sigma'| \geq \rho_\varepsilon$, otherwise is treated with the simplified solver.

1. **Accurate Riemann solver**. We determine $\sigma_1, \ldots, \sigma_n$ and $\omega_0, \ldots, \omega_n$ as in (1.1.12). If all the jumps (ω_{i-1}, ω_i) were shocks or contact discontinuities, then the Riemann problem would have a piecewise constant solution with $\leq n$ fronts. Generally, though, the exact solution is not piecewise constant because of the presence of rarefaction waves. Indeed, if one of the jump (ω_{i-1}, ω_i) is of rarefaction type and has strength $|\sigma_i|$ greater than a small parameter $\eta_\varepsilon > 0$ (fixed at the beginning), then it is partitioned in a number $e_i = \lfloor |\sigma_i|/\eta_\varepsilon \rfloor + 1$ of discontinuities by inserting additional states

$\omega_{i,j}$ between ω_{i-1} and ω_i. For $j = 1, \ldots, e_i$, we set

$$\omega_{i,j} := R_i(j\sigma_i)(\omega_{i-1}), \qquad x_{i,j}(t) := \bar{x} + \lambda_i(\omega_{i,j})(t - \bar{t}).$$

As soon as $\omega_{i,j}$ and their locations have been computed, we can define an approximate solution as it is shown in Figure 1.5.

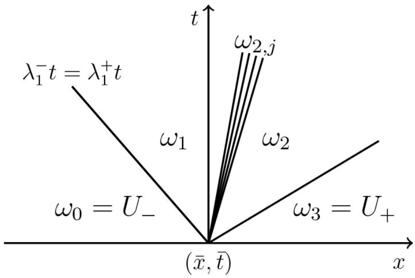

FIGURE 1.5: An example of approximate solution constructed with the accurate solver. The picture represents the outgoing fronts from the interaction at $(\bar{x}, \bar{t})$ for a system of $n = 3$ equations.

2. **Simplified Riemann solver**. Let $j \geq j'$ be the families of the two incoming waves of sizes σ, σ'. We assume that the left, middle and right states U_-, U_m, U_+ before the interaction are related by

$$U_m = W_j(\sigma)(U_-), \qquad U_+ = W_{j'}(\sigma')(U_m),$$

and we define the auxiliary right state

$$V_+ := \begin{cases} W_j(\sigma) \circ W_{j'}(\sigma')(U_-) & \text{if } j > j', \\ W_j(\sigma + \sigma')(U_-) & \text{if } j = j'. \end{cases}$$

Let $V = V(x,t)$ be the piecewise constant solution of the Riemann problem with data U_-, V_+ constructed as in the accurate case. By our definition of auxiliary state, V contains exactly two wave fronts of sizes σ and σ' if $j > j'$, or a single wave front of size

$\sigma+\sigma'$ if $j = j'$. Of course, one may have $V_+ \neq U_+$: if this is the case, we let the jump (V_+, U_+) travel with a fixed speed $\hat{\lambda}$ strictly bigger than all characteristic speeds. Thus, in a forward neighborhood of $(\bar{x}, \bar{t})$ we define an approximate solution as

$$U(x,t) = \begin{cases} V(x,t) & \text{if } x - \bar{x} \leq \hat{\lambda}(t-\bar{t})\,, \\ U_+ & \text{if } x - \bar{x} > \hat{\lambda}(t-\bar{t})\,, \end{cases}$$

see Figure 1.6 (a) and (b).

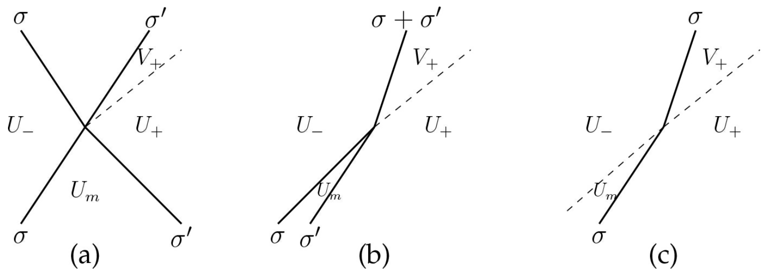

FIGURE 1.6: Examples of approximate solutions constructed with the simplified solver. The picture represents the interacting fronts in the (x,t)-plane. Non-physical fronts are denoted by dashed lines.

This procedure introduces a new *non-physical* wave front that may collide with other (physical) fronts. When this happens, let U_-, U_m and U_+ be the left, middle and right states as before and σ be the size of a physical wave of the i-th family. If $U_+ = W_i(\sigma)(U_m)$, we define the auxiliary right state

$$V_+ := W_i(\sigma)(U_-)\,.$$

Now, call V the solution to the Riemann problem with data U_-, V_+ given by a single front of family i and size σ. Since, in general, it holds $V_+ \neq U_+$, we let the jump (V_+, U_+) travel with fixed speed $\hat{\lambda}$ and we define the approximate solution as above (Figure 1.6 (c)).

It is not a-priori obvious whether the scheme is well-defined, since we do not know if the states remain sufficiently close in order that all the Riemann problems are solvable. In other words, we must prove that such approximate solutions can be defined for all times. To do this, we need to obtain a quantitative estimate on the strengths of the interacting waves, so that it is possible to show that the approximations have uniformly bounded total variation. To keep track of the total variation of an approximate solution U_ε, we introduce two functionals defined in terms of the strengths of the waves. At time $t > 0$, let x_α be the locations of the fronts carrying the jumps of $U_\varepsilon(\cdot, t)$ and let $|\sigma_\alpha|$ be the strength of the wave at x_α. We define a linear functional measuring the total strength of waves in $U_\varepsilon(\cdot, t)$

$$L(t) = L(U_\varepsilon(\cdot, t)) := \sum_\alpha |\sigma_\alpha| \tag{1.2.2}$$

and a quadratic functional measuring the *wave interaction potential*

$$Q(t) = Q(U_\varepsilon(\cdot, t)) := \sum_{\mathcal{A}} |\sigma_\alpha \sigma_\beta| \,, \tag{1.2.3}$$

where $\mathcal{A}$ is the set of the approaching waves at time t. More precisely, we say that two fronts located at $x_\alpha < x_\beta$ and belonging to the characteristic families $k_\alpha, k_\beta \in \{1, \dots, n+1\}$ are *approaching* if and only if $k_\alpha > k_\beta$ or else if $k_\alpha = k_\beta$ and at least one of them is a shock. We observe that L and Q are defined and constant outside interaction times. In order to see how they vary across interaction times, we need some estimates on the difference between the strengths of the incoming and the outgoing waves.

Lemma 1.2.3 (Glimm interaction estimates)**.** *Let $n \geq 2$ and consider an interaction between two wave fronts.*

i) Let σ'_i, σ'_j be the sizes of the two incoming fronts belonging to the distinct characteristic families i, j and assume $|\sigma'_i \sigma'_j| \geq \rho_\varepsilon$. Their interaction determines a Riemann problem whose exact solution consists of outgoing waves of sizes $\sigma_1, \dots, \sigma_n$. These are related to the incoming waves by the *estimate*

$$|\sigma_i - \sigma_i'| + |\sigma_j - \sigma_j'| + \sum_{k \neq i,j} |\sigma_k| = \mathcal{O}(1)|\sigma_i'\sigma_j'|\,.$$

ii) *Let σ_i', σ_i'' be the sizes of two incoming fronts, both belonging to the same i-th characteristic family, and assume $|\sigma_i'\sigma_i''| \geq \rho_\varepsilon$. The sizes of the outgoing waves $\sigma_1, \ldots, \sigma_n$ are related to the incoming ones by*

$$|\sigma_i - \sigma_i' - \sigma_i''| + \sum_{k \neq i} |\sigma_k| = \mathcal{O}(1)|\sigma_i'\sigma_i''|(|\sigma_i'| + |\sigma_i''|)\,.$$

iii) *Let σ_i', σ_j' be the sizes of two incoming fronts of families i, j and such that $|\sigma_i'\sigma_j'| < \rho_\varepsilon$. Then, if σ_{n+1} denotes the strength of the outgoing non-physical wave, we have*

$$|\sigma_{n+1}| = \mathcal{O}(1)|\sigma_i'\sigma_j'|\,.$$

iv) *Let a non-physical front σ_{n+1}' interact with a wave of size σ_i'. Then, if σ_{n+1} denotes the strength of the outgoing non-physical wave, we have*

$$|\sigma_{n+1}| - |\sigma_{n+1}'| = \mathcal{O}(1)|\sigma_i'\sigma_{n+1}'|\,.$$

For a proof see Lemma 7.2 of [17]. We consider a time τ where two fronts of sizes σ', σ'' interact and we estimate the change in the functionals L, Q across τ. Concerning L, the estimates of Lemma 1.2.3 yield

$$\Delta L(\tau) = L(\tau+) - L(\tau-) = \mathcal{O}(1)|\sigma'\sigma''|\,.$$

Notice that after τ the two fronts σ', σ'' are no longer approaching, while the newly generated fronts can approach all the other waves. Hence, by Lemma 1.2.3 we can derive

$$\Delta Q(\tau) = Q(\tau+) - Q(\tau-) = -|\sigma'\sigma''| + \mathcal{O}(1)|\sigma'\sigma''|L(\tau-)\,.$$

In particular, if L remains sufficiently small, the previous estimate implies

$$\Delta Q(\tau) \leq -\frac{|\sigma'\sigma''|}{2}\,.$$

Thus, we can choose a suitable constant C_o such that the *Glimm functional*, defined as

$$F(t) := L(t) + C_o Q(t)\,, \tag{1.2.4}$$

decreases at every interaction time τ, provided that $L(\tau-)$ is small enough. Moreover, let $C_1 \in \mathbb{R}$ satisfy

$$\frac{1}{C_1}\,\mathrm{TV}\,U_\varepsilon(\cdot,t) \leq L(t) \leq C_1\,\mathrm{TV}\,U_\varepsilon(\cdot,t)\,.$$

Since we have assumed $\mathrm{TV}\,U_{o,\varepsilon} \leq \delta_o$ and it holds $Q(t) \leq L^2(t)$ for all $t \geq 0$, we get

$$\begin{aligned}\mathrm{TV}\,U_\varepsilon(\cdot,t) &\leq C_1[L(t) + C_o Q(t)] \leq C_1[L(0) + C_o Q(0)]\\ &\leq C_1[C_1\delta_o + C_o(C_1\delta_o)^2]\,.\end{aligned}$$

This means that the total variation of U is uniformly bounded independently from the approximation parameter ε and the time t.

To prove that the total number of fronts remains finite, we recall that the accurate Riemann solver is used when the strengths of two interacting waves σ', σ'' satisfy $|\sigma'\sigma''| \geq \rho_\varepsilon$. This can happen only finitely many times, since Q is decreasing and across these interactions it holds $Q(\tau+) - Q(\tau-) \leq -\rho_\varepsilon/2$ by Lemma 1.2.3. Hence, new physical fronts are introduced only at a number $\leq 2Q(0)/\rho_\varepsilon$ of interaction points and their total number is finite. On the other hand, a new non-physical front is generated only when two physical fronts interact and any two physical fronts can cross only once. By this reason, we deduce that also the total number of non-physical fronts is finite.

As a consequence, for any values of the parameters $\eta_\varepsilon, \rho_\varepsilon$ and for initial data with sufficiently small total variation, a piecewise constant approximate solution can be constructed for all times $t \geq 0$. To verify that it is indeed an ε-front tracking approximation satisfying the requests of Definition 1.2.2, it remains to prove that the maximum strength of any rarefaction wave is uniformly bounded and that the total strengths of the non-physical waves remains small. We refer to the end of Chapter 7 of [17] for the details.

In conclusion, we take a sequence of parameters $(\varepsilon_\nu)_{\nu \geq 1}$ decreasing to zero. For each $\nu \geq 1$, the previous analysis yields the existence of an ε_ν-approximate solution U_ν of (1.1.1), (1.1.3) with uniformly bounded total variation. Moreover, the maps $t \mapsto U_\nu(\cdot, t)$ are uniformly Lipschitz continuous with values in $L^1(\mathbb{R}, \mathbb{R}^n)$. We can thus apply Helly's Theorem [17] and extract a subsequence which converges to a limit function $U \in L^1_{loc}$ which is proven to be the weak entropic solution of (1.1.1), (1.1.3).

1.2.2 Front tracking for systems of two conservation laws

The main technical difficulty in the proof of the well-definiteness of the algorithm is to control the overall error generated by non-physical fronts. This requires to introduce the concept of *generation order* and perform a rigorous analysis of interactions (see Section 7.3 of [17]). Nevertheless, for a 2×2 system it is possible to avoid the introduction of non-physical waves and always use an accurate solver to construct approximate solutions. In this section, we give a sketch of the proof that is contained in [13] and will turn useful in the next chapter.

In a few words, in the $n = 2$ case the only problem comes from the fact that rarefaction fronts can be partitioned generating several other fronts. Then, in order to ensure that the total number of wave fronts and interactions remains finite, it suffices to verify that the number of interactions creating possibly large rarefactions is finite. This is what is required in the following lemma; for a proof see Lemma 2.3 of [7].

Lemma 1.2.4. *Let a wave front tracking pattern made of segments of the two families be given in $[0, T[\times\mathbb{R}$. Assume that the speeds of the fronts of the first family lay between two constants $a_1 < a_2$ and those of the fronts of the second family lay between $b_1 < b_2$, with $a_2 < b_1$. Assume that the wave front-tracking pattern has also the following properties:*

i) at $t = 0$ there is a finite number of waves;

ii) the interactions occur only between two wave fronts at any single time;

iii) except a finite number of interactions, there is at most one outgoing wave of each family for each interaction.

Then, the number of interactions in the region $\mathbb{R} \times [0, T[$ is finite.

The only non trivial property to verify is the third one. The interactions giving rise to rarefactions that have to be split are among those that involve two incoming waves of the same family and an outgoing rarefaction wave of the other family and with strength $> \eta_\varepsilon$. This can be inferred from Lemma 1.2.3. We recall that for any couple of interacting waves σ_i', σ_i'' belonging to the same i-th family, the strength $|\sigma_j|$ of the outgoing wave of family $j \neq i$ satisfies

$$|\sigma_j| \leq \mathcal{O}(1)|\sigma_i'\sigma_i''|\big(|\sigma_i'| + |\sigma_i''|\big).$$

Recall also that, if the initial data have sufficiently small total variation, then it holds $\Delta Q \leq -|\sigma_i'\sigma_i''|/2$. Thus, for a suitable constant C_2 we get

$$\eta_\varepsilon < |\sigma_j| \leq \mathcal{O}(1)|\sigma_i'\sigma_i''|(|\sigma_i'| + |\sigma_i''|) \leq C_2|\sigma_i'\sigma_i''| \leq -2C_2\Delta Q\,,$$

as long as the total variation remains bounded (and small). This means that $\Delta Q < -\eta_\varepsilon/(2C_2)$ and, whenever such interactions occur, the potential Q decreases by a fixed positive amount: this can happen only finitely many times since Q is decreasing and $Q(0)$ is bounded.

Therefore, by Lemma 1.2.4 we obtain that the total number of wave fronts and interactions remains finite in time and the algorithm is well-defined with no need of non-physical fronts.

1.3 Some existence results for large data

It is clear that Theorem 1.2.1 applies to the p-systems introduced in (1.1.4). However, by the simple form of these equations one can prove the existence of global solutions under more general assumptions on the size of the data. In this section, we recall (without proofs) some well-known results regarding this issue.

Consider the system of (isentropic) isothermal gas dynamics (1.1.4) with pressure law (1.1.5) and $\gamma = 1$, namely

$$\begin{cases} v_t - u_x = 0\,, \\ u_t + \left(\dfrac{a^2}{v}\right)_x = 0\,, \end{cases} \qquad t \ge 0, \quad x \in \mathbb{R}\,. \tag{1.3.1}$$

From Example 1.1.5 we recall that in the case $\gamma = 1$ vacuum cannot appear and Riemann problems are solvable for any couple of initial states. Moreover, in [33] Nishida proves that it is sufficient that the total variation of the initial data

$$(v(x,0), u(x,0)) = (v_o(x), u_o(x))\,, \qquad v_o(x) \ge \underline{v} > 0\,, \qquad x \in \mathbb{R}\,, \tag{1.3.2}$$

is finite in order to have globally defined solutions. The uniform boundedness of the total variation of the approximate solutions (constructed via a Glimm scheme) is accomplished by the decreasing of a linear functional that measures the wave strengths in Riemann invariants

$$w(v,u) = u + a\log v\,, \qquad z(v,u) = u - a\log v\,, \tag{1.3.3}$$

(see Definition 17.1 of [38]). And no interaction potential is needed.

Theorem 1.3.1 ([33]). *The Cauchy problem for* (1.3.1) *and data* (1.3.2) *with bounded total variation admits a weak entropic solution defined for all times* $t \ge 0$.

This result was subsequently extended by Nishida and Smoller in [34] to any pressure law (1.1.5) with $\gamma > 1$. Here, the total variation of the initial data can be taken large but bounded by a constant depending on γ.

Theorem 1.3.2 ([34]). *For the p-system* (1.1.4) *with pressure law* (1.1.5) *and* $\gamma \ge 1$, *let* (v_o, u_o) *be the initial data with bounded total variation and assume that*

$$0 < \underline{v} \le v_o(x) \le \overline{v} < +\infty$$

for some constants $\underline{v}, \overline{v} \in \mathbb{R}$. There exists a constant C depending only on $\underline{v}, \overline{v}$ such that if

$$(\gamma - 1)\,\mathrm{TV}(v_o, u_o) < C\,, \tag{1.3.4}$$

then the initial-value problem has a weak solution defined for all times $t \geq 0$.

This means that the data can have large total variation when $(\gamma - 1)$ is small: indeed, the smaller is $(\gamma - 1)$, the larger can be taken $\mathrm{TV}(v_o, u_o)$. In particular, if $\gamma = 1$, there is no restriction on the size of the data and we recover Theorem 1.3.1.

A further generalization of Theorem 1.3.1 that is worth mentioning is due to Bakhvalov. The author in [15] extends the global existence result for the isothermal gas dynamics to any pressure law satisfying the condition

$$3(p'')^2 \leq 2p'p''' \quad \text{for all } v > 0\,. \tag{1.3.5}$$

Roughly speaking, (1.3.5) determines whether the strength of a shock decreases by crossing a shock of the opposite family. In particular, for (1.1.5) the Bakhvalov's condition holds if and only if $\gamma \in \,]0, 1]$. Hence, it is not satisfied in the case of interest $1 < \gamma < 3$.

A condition that has the same flavor of (1.3.4) is required in the existence theorem proved by Liu [32]. Consider the non-isentropic 3×3 system of the gas dynamics in Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0\,, \\ u_t + p_x = 0\,, \\ E_t + (pu)_x = 0\,, \end{cases} \tag{1.3.6}$$

where $p = p(v, s) = p(v, e)$ is the pressure, e is the internal energy, s is the entropy and $E = u^2/2 + e$ is the total energy. The third equation in (1.3.6) represents the conservation of energy. In [32] the author works under the assumption that the gas is polytropic, i.e. the equation of state is

$$p(v, s) = \frac{a^2}{v^\gamma} \exp\left(\frac{(\gamma - 1)s}{R}\right),$$

where R is a positive constant and $1 < \gamma \leq 5/3$. In brief, given initial data

$$(v(x,0), u(x,0), E(x,0)) = (v_o(x), u_o(x), E_o(x)), \qquad x \in \mathbb{R},$$

the global existence of solutions to the Cauchy problem holds true provided that v_o, u_o and $s_o = s(e_o, p_o)$ have bounded total variation and

$$(\gamma - 1)\,\mathrm{TV}(u_o, v_o, s_o)$$

is sufficiently small. Similar results were obtained also by Peng in [35] and Temple in [40] and all these papers use the Glimm scheme. A front tracking technique, instead, was exploited in [11] and will be used in the sequel. We point out that a front tracking scheme in case of initial data with large total variation is far from being simple and a careful study of interactions is required.

1.4 A model of phase transitions

At last, we introduce the model studied in the thesis. We focus on a hyperbolic system of conservation laws for the flow of a fluid capable of undergoing phase transitions from pure liquid to pure gas, including mixtures of both. The system was first considered by Amadori and Corli in [2] and is a simplified version of the model discussed in [20], that in turn derives from [24]. The original model of [24] analyzes the wave patterns observed in experiments performed on retrograde fluids (i.e. with large heat capacity) with the use of *shock tubes*.

A shock tube is an instrument that is used in numerous experiments and computational studies on the phase-changes for fluids, see [24] and the references therein. The apparatus consists of a cylindrical metal tube with a piston on one side and the other side either open or closed. By compressing or withdrawing the piston, shock or rarefaction waves pass through the fluid and phase changes are induced.

In Lagrangian variables the system of [20] is written as

$$\begin{cases} v_t - u_x = 0\,, \\ u_t + p(v,\lambda)_x = \varepsilon u_{xx}\,, \\ \lambda_t = \dfrac{1}{\tau}(p - p_e)\lambda(\lambda - 1) + b\varepsilon\lambda_{xx}\,, \end{cases} \tag{1.4.1}$$

where v, u are the usual state variables for the specific volume and the velocity of the fluid, $\varepsilon > 0$ is a viscosity coefficient and b is a positive real parameter, the ratio $1/\tau$ is the typical reaction time and p_e is a fixed equilibrium pressure. The quantity $\lambda \in [0,1]$ is the mass density fraction of the vapor in the fluid: the value $\lambda = 0$ refers to the liquid phase, while the value $\lambda = 1$ refers to the vapor phase; the intermediate values of λ describe mixtures of the pure phases. Remark that here the pressure p depends not only on v but also on λ. More precisely, p is of the form

$$p = p(v,\lambda) = \frac{a^2(\lambda)}{v}\,, \tag{1.4.2}$$

where $v \in]0,+\infty[$ and a is a smooth positive function for $\lambda \in [0,1]$. In particular, (1.4.2) satisfies

$$p > 0\,, \qquad p_v < 0\,, \qquad p_{vv} > 0\,. \tag{1.4.3}$$

For $a^2(\lambda)$ a possible choice is a linear interpolation between the two pure phases, i.e. $a^2(\lambda) = k_0 + \lambda(k_1 - k_0)$, for some constants $0 < k_0 < k_1$. A motivation for the pressure to depend on λ can be given by considering a pressure law for real gases (the van der Waals pressure law, for instance) when the temperature is above the critical point. In such cases what is observable in laboratory are the two hyperbolic branches of the pressure associated to the liquid and to the vapor phase, i.e. $a^2(0)/v$ and $a^2(1)/v$, respectively. The pressure curves $a^2(\lambda)/v$, for $0 < \lambda < 1$, interpolate the curves of the pressure in the liquid and vapor phases for the case of mixtures. See Figure 1.7.

Notice that system (1.4.1) is isothermal, an assumption that can be seen as a consequence of the large heat capacity of retrograde fluids [39].

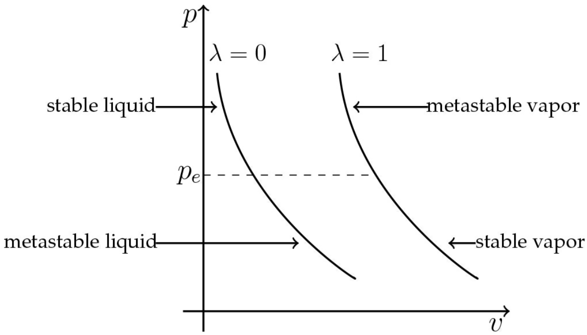

FIGURE 1.7: Pressure curves.

The reaction term $(1/\tau)(p - p_e)\lambda(\lambda - 1)$ on the right-hand side of the third equation allows for metastable states because of the presence of the equilibrium pressure p_e: namely, metastable states are the vapor states lying above the line $p = p_e$ or the liquid states lying below it. The line $p = p_e$ in the (v, p) plane plays the role of the equal-area *Maxwell line* in the standard theory of phase transitions for fluids [41].

1.4.1 The model of Amadori and Corli

Clearly, the mathematical analysis of (1.4.1) is non-trivial and some simplifications are in order. If we let $\varepsilon = 0$, for example, (1.4.1) becomes a system of balance laws with a reaction term that is studied in [4]. In turn, this requires the study of the homogenous part of the system, namely

$$\begin{cases} v_t - u_x = 0\,, \\ u_t + p(v,\lambda)_x = 0\,, \\ \lambda_t = 0\,. \end{cases} \tag{1.4.4}$$

System (1.4.4) is the protagonist of this thesis and was first considered by Amadori and Corli in [2], where they prove the global in time existence of solutions to the initial-value problem for $\mathbf{BV}$ data

$$(v(x,0), u(x,0), \lambda(x,0)) = (v_o(x), u_o(x), \lambda_o(x)), \qquad x \in \mathbb{R}. \tag{1.4.5}$$

Motivated by Theorem 1.3.1 above, the authors assume that the initial data have suitably bounded total variation but not necessarily small. Recall that for small enough data the global existence to the Cauchy problem (1.4.4), (1.4.5) is guaranteed by Theorem 1.2.1.

The proof of [2] exploits a front tracking technique analogous to that described in Section 1.2.1; for a variant including a decomposition by path method we refer to [12]. Instead, for proofs based on the Glimm scheme we mention [35], that contains an interesting analysis of the wave-pattern interactions for (1.4.4) and, in the case $p = p(v,\gamma) = v^{-\gamma}$ with $\gamma > 1$ playing the role of the variable λ, we refer to [27, 28], where the authors use both the Glimm scheme and the front tracking algorithm.

Under the assumption (1.4.2), by $(1.4.3)_2$ we have that system (1.4.4) is strictly hyperbolic in the whole domain $\Omega =]0,\infty[\times\mathbb{R}\times[0,1]$ with eigenvalues

$$e_1 = -\sqrt{-p_v(v,\lambda)}, \qquad e_2 = 0, \qquad e_3 = \sqrt{-p_v(v,\lambda)}. \tag{1.4.6}$$

We write $c = \sqrt{-p_v(v,\lambda)} = a(\lambda)/v$. The corresponding right eigenvectors associated to e_1, e_2, e_3 are

$$r_1 = \begin{pmatrix} 1 \\ c \\ 0 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} -p_\lambda \\ 0 \\ p_v \end{pmatrix}, \qquad r_3 = \begin{pmatrix} -1 \\ c \\ 0 \end{pmatrix}. \tag{1.4.7}$$

By $(1.4.3)_3$ the eigenvalues e_1 and e_3 are genuinely nonlinear with

$$\nabla e_i \cdot r_i = \frac{p_{vv}(v,\lambda)}{2c} > 0, \qquad i = 1, 3,$$

while e_2 is linearly degenerate.

In [2] the hypotheses on the initial data are expressed in terms of a so-called *weighted total variation* WTV of $a(\lambda_o)$. This quantity arises naturally in the problem and reduces to the logarithmic variation of a when the function is continuous. The idea is to prescribe a bound on $\mathrm{WTV}(a(\lambda_o))$ such that the larger is $\mathrm{TV}(v_o, u_o)$, the smaller must be $\mathrm{WTV}(a(\lambda_o))$, and vice versa.

Let $f : \mathbb{R} \to]0, +\infty[$, we define the weighted total variation of f by

$$\mathrm{WTV}(f) := 2 \sup \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|}{f(x_j) + f(x_{j-1})}\,, \tag{1.4.8}$$

where the supremum is taken over all $n \geq 1$ and $(n+1)$-tuples of points x_j with $x_0 < x_1 < \cdots < x_n$. This definition is motivated by the choice of strengths for 2-waves that we will see in (2.1.4). Moreover, $\mathrm{WTV}(f)$ is essentially related to $\mathrm{TV}(\log f)$ as it is clear from the following formula deduced from Proposition 2.1 of [2]:

$$\frac{\inf f}{\sup f}\,\mathrm{TV}(\log f) \leq \mathrm{WTV}(f) \leq \mathrm{TV}(\log f)\,.$$

In particular, if f is continuous we have $\mathrm{WTV}(f) = \mathrm{TV}(\log f)$. Denote $a_o(x) = a(\lambda_o(x))$, $A_o = \mathrm{WTV}(a_o)$, $p_o(x) = p(v_o(x), \lambda_o(x))$ and assume

$$a(\lambda) > 0\,, \qquad a'(\lambda) > 0\,. \tag{1.4.9}$$

The quantity A_o measures the total strength of the 2-waves at time $0+$. The main result of Amadori and Corli states the following.

Theorem 1.4.1 ([2]). *Assume* (1.4.2) *and* (1.4.9). *Consider initial data* (1.4.5) *with* $v_o(x) \geq \underline{v}$ *for some constant* $\underline{v} > 0$. *For every* $m > 0$ *and a suitable function* $k :]0, +\infty[\to]0, 1/2[$ *the following holds. If*

$$A_o < k(m)\,, \tag{1.4.10}$$

$$\mathrm{TV}(\log p_o) + \frac{1}{\inf a_o}\,\mathrm{TV}(u_o) < 2\,(1 - 2A_o)\,m\,, \tag{1.4.11}$$

then, the Cauchy problem (1.4.4), (1.4.5) *has a weak entropic solution* (v, u, λ) *defined for all* $t \geq 0$. *Moreover, the solution is valued in a compact subset of* Ω

and there is a constant $C(m)$ such that for every $t \geq 0$

$$\mathrm{TV}(v(\cdot,t), u(\cdot,t)) \leq C(m)\,.$$

The function k deserves some comments. The interaction of two waves α_i, β_i of the same family $i = 1, 3$ produces a wave ε_i of the same family and a "reflected" wave ε_j of the other family ($j = 1, 3$, $j \neq i$). For a suitable definition of strengths (see (2.1.4)), the authors prove that

$$|\varepsilon_j| \leq d(m) \min\{|\alpha_i|, |\beta_i|\}\,, \tag{1.4.12}$$

where $d < 1$ is a *damping coefficient* depending on α_i, β_i and $|\alpha_i|, |\beta_i| \leq m$, see Lemma 5.6 of [2]. The function k is defined by

$$k(m) := \frac{1 - \sqrt{d(m)}}{2 - \sqrt{d(m)}}\,, \tag{1.4.13}$$

it satisfies $k(0) = 1/2$ and it is decreaseing to 0 as $m \to +\infty$. Then, $\mathrm{WTV}(a_o) < 1/2$. Furthermore, (1.4.10), (1.4.11) can be read as analogous to (1.3.4): the larger is m, the smaller is $k(m)$ and vice versa. The possible occurring of a blow-up in the $\mathbf{BV}$ norm when these bounds do not hold is still an open problem.

The proof of Theorem 1.4.1 exploits the special structure of (1.4.4) and differentiates the treatment of 1- and 3-waves from that of 2-waves. More precisely, the authors consider a linear functional that is analogous to that of [7] and accounts only for waves of family 1 and 3, with a weight $\xi > 1$ to be assigned to shocks' strengths. As we will see in the next chapters, the choice of ξ as a function of m is a crucial passage in the proof. The linear functional referred to an approximate solution $(v, u, \lambda)(\cdot, t)$ is given by

$$L_\xi(t) = \sum_{\substack{i=1,3\\ \gamma_i>0}} |\gamma_i| + \xi \sum_{\substack{i=1,3\\ \gamma_i<0}} |\gamma_i| + K_{np} \sum_{\gamma\in\mathcal{NP}} |\gamma|\,, \qquad K_{np} > 0\,. \tag{1.4.14}$$

where the sum varies over the set of waves γ_i of family $i = 1, 3$ and the

set of non-physical waves $\gamma \in \mathcal{NP}$. Moreover, for 2-waves we introduce $L_{cd} = \sum |\gamma_2|$, that does not depend on t and is assumed to satisfy $L_{cd} \leq A_o = \mathrm{WTV}(a_o)$.
Motivated again by [7], the authors do not use a simplified solver for interactions between 1- and 3-waves, but only for interactions involving a contact discontinuity of family 2. Thus, the interaction potential contains only the products of approaching 2-waves to either 1- or 3-waves, i.e.

$$Q(t) = \sum_{\substack{i=1,3 \\ (\gamma_i,\delta_2)\in\mathcal{A}}} |\gamma_i\,\delta_2|\,. \tag{1.4.15}$$

We remark that Theorem 1.4.1 can be restated in a slightly different form as in Theorem 3.1 of [4]. More precisely, we rewrite conditions (1.4.10) and (1.4.11) as

$$A_o < \frac{1}{2} \tag{1.4.16}$$

and

$$\mathrm{TV}(\log p_o) + \frac{1}{\inf a_o}\,\mathrm{TV}\,u_o < H(A_o)\,, \tag{1.4.17}$$

respectively. The function $H :]0, 1/2] \to [0, +\infty[$ has the explicit expression

$$H(A_o) := 2(1 - 2A_o)k^{-1}(A_o) \tag{1.4.18}$$

and satisfies $H(1/2) = 0$ and $H(A) \to +\infty$ when $A \to 0+$. This reformulation is deduced from Theorem 1.4.1 as explained in Section 3 of [4] and turns out to be very useful whenever we need to make some comparisons.

1.4.2 The model of this thesis

The analysis of [2] is the starting point for the main results of this thesis, that were originally proved in the papers [8, 9, 10], where initial data with a simple structure allow to obtain more refined theorems.

In particular, we will consider initial data (1.4.5) with λ_o piecewise constant with either one or two jumps. From a physical point of view, such models represent the dynamics of a fluid in a one-dimensional tube divided into two or three regions where the fluid lies in a specific phase (which can be pure liquid, pure gas or a mixture of both).

i) **The single phase wave** [10]. Let λ_o be a piecewise constant function with a single discontinuity at $x = 0$, such as

$$\lambda_o(x) = \begin{cases} \lambda_\ell & \text{if } x < 0\,, \\ \lambda_r & \text{if } x > 0\,, \end{cases} \qquad \lambda_\ell, \lambda_r \in [0,1]\,. \tag{1.4.19}$$

This gives rise to a single contact discontinuity traveling along the line $x = 0$ in the (x,t)-plane and referred to as the *phase wave*. This assumption reduces the initial-value problem (1.4.4), (1.4.5) to two initial-value problems for two isothermal p-systems coupled through the interface at $x = 0$. See Figure 1.8, where $a_\ell = a(\lambda_\ell)$ and $a_r = a(\lambda_r)$.

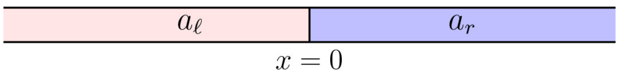

FIGURE 1.8: The single phase wave.

ii) **The two phase waves** [8, 9]. Let λ_o be a piecewise constant function with two discontinuities at $x = 0$ and $x = 1$, such as

$$\lambda_o(x) = \begin{cases} \lambda_\ell & \text{if } x < 0\,, \\ \lambda_m & \text{if } 0 < x < 1\,, \\ \lambda_r & \text{if } x > 1\,, \end{cases} \qquad \lambda_\ell, \lambda_m, \lambda_r \in [0,1]\,. \tag{1.4.20}$$

We define $a_\ell = a(\lambda_\ell)$, $a_m = a(\lambda_m)$, $a_r = a(\lambda_r)$ and three main configurations can be considered:

(d) the drop case, $a_m < \min\{a_\ell, a_r\}$;

(b) the bubble case, $a_m > \max\{a_\ell, a_r\}$;

(p) the increasing (decreasing) pressure case, $a_\ell < a_m < a_r$ ($a_\ell > a_m > a_r$).

If we assume a to be a strictly increasing function (which is a physically meaningful assumption), case **(d)** corresponds to the situation where $\lambda_m < \min\{\lambda_\ell, \lambda_r\}$, i.e. the mixture is more liquid in the middle region of the tube $[0, 1]$ than in the surrounding ones (liquid drop in a gaseous environment). On the other hand, always assuming that a is an increasing function, case **(b)** corresponds to $\lambda_m > \max\{\lambda_\ell, \lambda_r\}$ and models a bubble of gas surrounded by liquid. See Figure 1.9.

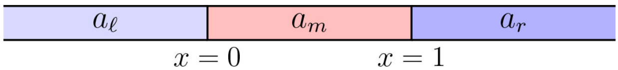

FIGURE 1.9: The two phase waves.

The purpose of the thesis is to prove the existence of solutions to the Cauchy problem for (1.4.4) and initial data (1.4.5) satisfying either (1.4.19) or (1.4.20). For brevity, we will use the following notation

$$\mathrm{TV}\left(\log(p_o), \frac{u_o}{\min a_o}\right) := \mathrm{TV}(\log(p_o)) + \frac{1}{\min a_o}\mathrm{TV}(u_o)\,, \tag{1.4.21}$$

and we will call δ the phase wave in case i) and η, ζ the two phase waves in case ii). Now, let A_o indicate a measure of the strengths of the 2-waves present in the model, i.e. $A_o := |\delta|$ in the single interface case and $A_o := \mathcal{H}_{d,b,p}(|\eta|, |\zeta|)$ (for some functions $\mathcal{H}_{d,b,p}$ defined on suitable sets to be specified later) in the two interfaces case. Remark that in the framework of [2] A_o would be given by the sum $|\eta| + |\zeta| \neq \mathcal{H}_{d,b,p}(|\eta|, |\zeta|)$. Then, the main theorem will roughly state the following.

> There exists a strictly decreasing function $\mathcal{K} :]0, +\infty[\, \to\,]0, +\infty[$ such that if
>
> $$\mathrm{TV}\left(\log(p_o), \frac{u_o}{\min a_o}\right) < \mathcal{K}(A_o)\,, \tag{1.4.22}$$
>
> then the Cauchy problem has a weak entropic solution globally defined in time.

Our analysis begins in Chapter 2 with the description of the front tracking algorithm used to construct the approximate solutions. Then, we will carry out the estimates on the functionals and complete the proof for the single phase wave in Chapter 3 and for the two phase waves case in Chapter 4, respectively. To conclude, we will also make some comparisons between the results obtained here and the corresponding ones of [2].

Chapter 2

A front tracking algorithm

In this chapter we recall some other basic facts about system (1.4.4) from [2, 3], i.e. wave curves, strengths of waves and related notations. More importantly, we outline the peculiar front tracking algorithm used in the next chapters. The most remarkable novelty of this algorithm is the introduction of the *composite waves*, that sum up the effects of contact discontinuities and non-physical waves. This choice is made accordingly to the adoption of two Riemann solvers that allow to exploit an asymmetric Glimm functional, as described in Section 2.3.3. In Section 2.1 we show how to solve Riemann problems for system (1.4.4) and in Section 2.2 we list the main instructions of the algorithm. The following section is entirely devoted to the analysis of interactions between waves, which is a fundamental step in the proof of the decreasing of the Glimm functional. We conclude with Section 2.4 by showing that the algorithm is well-defined.

2.1 The Riemann problem

Recall system (1.4.4) introduced at the end of the previous chapter and denote by $U = (v, u, \lambda)$ the state variables in the domain

$$\Omega =]0, +\infty[\, \times \mathbb{R} \times [0, 1].$$

We have already seen in Section 1.4.1 that under the assumption (1.4.2) system (1.4.4) is strictly hyperbolic in the whole Ω and the first and third characteristic field are genuinely nonlinear, while the second one is linearly degenerate.

For $v > 0$ and $U_- = (v_-, u_-, \lambda_-) \in \Omega$, we introduce the quantities

$$i_1(v, U_-) := u_- + a(\lambda_-) \log\left(\frac{v}{v_-}\right), \qquad s_1(v, U_-) := u_- + a(\lambda_-)\frac{v - v_-}{\sqrt{vv_-}},$$
$$i_3(v, U_-) := u_- - a(\lambda_-) \log\left(\frac{v}{v_-}\right), \qquad s_3(v, U_-) := u_- - a(\lambda_-)\frac{v - v_-}{\sqrt{vv_-}}.$$

Recall the p-system for the (isentropic) isothermal gas dynamics (1.3.1) and the wave curves of Example 1.1.3. We find that the integral curves of system (1.4.4) of family $1, 3$, with initial point $U_- \in \Omega$ and parameterized by v, are

$$\begin{aligned} U = I_1(v)(U_-) \ : &\qquad v > 0, \qquad u = i_1(v, U_-), \qquad \lambda = \lambda_-, \\ U = I_3(v)(U_-) \ : &\qquad v > 0, \qquad u = i_3(v, U_-), \qquad \lambda = \lambda_-. \end{aligned}$$

The rarefaction curves of family $1, 3$ coincide with the branch of I_1 and I_3 along which the corresponding eigenvalue increases, i.e.

$$R_1(v)(U_-) = I_1(v, U_-)\restriction_{\{v > v_-\}}, \qquad R_3(v)(U_-) = I_3(v, U_-)\restriction_{\{v < v_-\}} .$$

On the other hand, the shock curves of family $1, 3$, through a point U_- and parameterized by v, are

$$\begin{aligned} U = S_1(v)(U_-) : &\qquad v < v_-, \qquad u = s_1(v, U_-), \qquad \lambda = \lambda_-, \\ U = S_3(v)(U_-) : &\qquad v > v_-, \qquad u = s_3(v, U_-), \qquad \lambda = \lambda_-. \end{aligned}$$

Remark that all these curves lie on the plane of equation $\lambda = \lambda_-$ contained in Ω. We set

$$w_1(v, U_-) := \begin{cases} i_1(v, U_-) & \text{if } v > v_-, \\ s_1(v, U_-) & \text{if } v < v_-, \end{cases}$$

$$w_3(v, U_-) := \begin{cases} i_3(v, U_-) & \text{if } v < v_-, \\ s_3(v, U_-) & \text{if } v > v_-. \end{cases}$$

From the above expressions we can deduce the form of the Lax wave-curves of family $1, 3$ (see Figure 1.1, where the role of 2-waves is now played by 3-waves):

$$U = W_1(v)(U_-): \qquad v > 0, \qquad u = w_1(v, U_-), \qquad \lambda = \lambda_-, \tag{2.1.1}$$

$$U = W_3(v)(U_-): \qquad v > 0, \qquad u = w_3(v, U_-), \qquad \lambda = \lambda_-. \tag{2.1.2}$$

Concerning the second characteristic family, the Lax wave curves through a point U_- are both of rarefaction and shock type by the linear degeneracy of the field and are parameterized by λ:

$$U = W_2(\lambda)(U_-): \qquad v = v_- \frac{a^2(\lambda)}{a^2(\lambda_-)}, \qquad u = u_-, \qquad \lambda \in [0, 1]. \tag{2.1.3}$$

Notice that this wave curve lies on the plane of equation $u = u_-$. Moreover, the pressure is constant across contact discontinuities: indeed, for $U, U_- \in \Omega$ it holds

$$U = W_2(\lambda)(U_-) \quad \Rightarrow \quad p(v, \lambda) = \frac{a^2(\lambda)}{v} = \frac{a^2(\lambda)}{v_-} \frac{a^2(\lambda_-)}{a^2(\lambda)} = p(v_-, \lambda_-).$$

As in [2], we introduce the following important quantities.

Definition 2.1.1 (Size of a wave)**.** *The size γ_i of a wave of family $i = 1, 2, 3$ connecting two states $U, U_- \in \Omega$ is given by*

$$\gamma_1 = \frac{1}{2} \log\left(\frac{v}{v_-}\right), \qquad \gamma_2 = 2\,\frac{a(\lambda) - a(\lambda_-)}{a(\lambda) + a(\lambda_-)}, \qquad \gamma_3 = \frac{1}{2} \log\left(\frac{v_-}{v}\right). \tag{2.1.4}$$

According to this definition, rarefaction waves have positive size and shock waves have negative size. For convenience, we introduce the function h as

$$h(\gamma) := \begin{cases} \gamma & \text{if } \gamma \geq 0, \\ \sinh \gamma & \text{if } \gamma < 0, \end{cases} \tag{2.1.5}$$

and notice that

$$w_i(v, U_-) = u_- + 2a(\lambda_-)h(\gamma_i), \qquad i = 1, 3.$$

In this way, we can view the previous curves as parameterized by γ_i, i.e. as $W_i(\gamma_i)(U_-)$, $I_i(\gamma_i)(U_-)$ and so on, for $i = 1, 3$.

Let us consider the Riemann problem for (1.4.4) with data

$$U(x, 0) = \begin{cases} U_- = (v_-, u_-, \lambda_-) & \text{if } x < 0, \\ U_+ = (v_+, u_+, \lambda_+) & \text{if } x > 0. \end{cases} \tag{2.1.6}$$

To simplify the notations, we also write $a_\pm = a(\lambda_\pm)$, $p_\pm = p(v_\pm, \lambda_\pm)$. If $\lambda_- = \lambda_+$, then the solution to the Riemann problem is as in Example 1.1.5 of Chapter 1; otherwise, the following result proved in Theorem 1 of [3] holds true.

Proposition 2.1.2. *Fix any pair of states U_-, U_+ in Ω. Then the Riemann problem* (1.4.4), (2.1.6) *has a unique Ω-valued solution made of the juxtaposition of three simple Lax waves for each characteristic family. If γ_i is the size of the waves of family $i = 1, 2, 3$, then*

$$\begin{gathered} \gamma_3 - \gamma_1 = \frac{1}{2}\log\left(\frac{p_+}{p_-}\right), \qquad a_- h(\gamma_1) + a_+ h(\gamma_3) = \frac{u_+ - u_-}{2}, \\ \gamma_2 = 2\,\frac{a_+ - a_-}{a_+ + a_-}. \end{gathered} \tag{2.1.7}$$

Moreover,

$$|\gamma_1| + |\gamma_3| \leq \frac{1}{2}\,|\log(p_+) - \log(p_-)| + \frac{1}{2\min\{a_-, a_+\}}|u_+ - u_-|. \tag{2.1.8}$$

See Figure 2.1 for a picture of the two cases $\lambda_- = \lambda_+$ and $\lambda_- \neq \lambda_+$. Proposition 2.1.2 displays the relations that must be satisfied by the sizes of the Lax waves in the exact solution of a Riemann problem. In particular, these formulas are needed when we solve a Riemann problem by the accurate solver. Instead, for our simplified Riemann solver we rely on the following proposition.

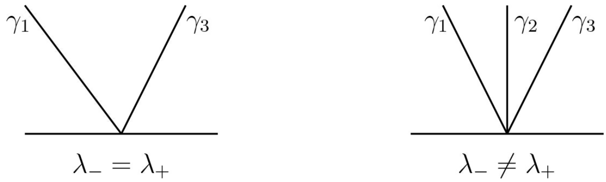

FIGURE 2.1: The Riemann problem.

Proposition 2.1.3. *Fix two functions Θ_1, Θ_3 that can be either the identity Id or the function h defined in* (2.1.5)*. For any pair of states $U_-, U_+ \in \Omega$ there exist unique $\gamma_1, \gamma_3 \in \mathbb{R}$ such that:*

$$\gamma_3 - \gamma_1 = \frac{1}{2}\log\left(\frac{p_+}{p_-}\right), \qquad a_-\Theta_1(\gamma_1) + a_+\Theta_3(\gamma_3) = \frac{u_+ - u_-}{2}. \quad (2.1.9)$$

Proof. Let us call $\log(p_+/p_-)/2 =: A$ and $(u_+ - u_-)/2 =: B$, since they are two constant quantities once we fixed U_- and U_+. Thus, we have four possible cases to examine:

$$\begin{cases} \gamma_3 - \gamma_1 = A, \\ a_-h(\gamma_1) + a_+h(\gamma_3) = B, \end{cases} \qquad \begin{cases} \gamma_3 - \gamma_1 = A, \\ a_-\gamma_1 + a_+\gamma_3 = B, \end{cases} \quad (2.1.10)$$

$$\begin{cases} \gamma_3 - \gamma_1 = A, \\ a_-h(\gamma_1) + a_+\gamma_3 = B, \end{cases} \qquad \begin{cases} \gamma_3 - \gamma_1 = A, \\ a_-\gamma_1 + a_+h(\gamma_3) = B. \end{cases} \quad (2.1.11)$$

System $(2.1.10)_1$ ($\Theta_1 = \Theta_3 = h$) is solved in Proposition 2.1.2 above, while system $(2.1.10)_2$ is linear. To conclude, it suffices to study $(2.1.11)_1$, since the two systems in (2.1.11) are analogous. In this case, by setting $k = a_+/a_-$, it holds $h(\gamma_1) + k\gamma_1 = B/(a_-) - kA$. If $G(x) := kx + h(x)$, then we have $G(\gamma_1) = B/(a_-) - kA$. Since G is invertible and onto, we get $\gamma_1 = G^{-1}(B/(a_-) - kA)$. □

We stress that only for $(2.1.10)_1$ the quantities γ_1, γ_3 are the sizes of the waves in the exact Lax solution to (1.4.4), (2.1.6). This is no longer true for the other three cases.

In addition to the above proposition, our definition of the accurate and simplified procedures requires the concepts of Pre-Riemann solver and composite wave. A composite wave is a stationary wave that is the result of the composition of a 2-wave with waves taken along integral curves of family 1 and 3. In what follows, we use the subscripts 'L' to refer to the Lax curves W_i and 'I' to refer to the integral curves I_i, $i = 1, 3$. For $i = 1, 3$ and $\theta_i \in \{L, I\}$ we let

$$\Theta_i = \begin{cases} h & \text{if } \theta_i = L, \\ Id & \text{if } \theta_i = I. \end{cases} \tag{2.1.12}$$

Definition 2.1.4 (Pre-Riemann solver)**.** *Fix $\theta_i \in \{L, I\}$, for $i = 1, 3$. A* Pre-Riemann solver $\mathcal{R}_{\theta_1\theta_3} : \Omega \times \Omega \to \mathbb{R} \times]-2, 2[\times \mathbb{R}$ *is the map defined by*

$$\mathcal{R}_{\theta_1\theta_3}(U_-, U_+) = (\gamma_1, \gamma, \gamma_3), \tag{2.1.13}$$

where by (2.1.12) γ_1, γ_3 *are as in* (2.1.9) *and* $\gamma = 2(a_+ - a_-)/(a_+ + a_-)$.

By Proposition 2.1.3 there are four possible Pre-Riemann solvers that we denote by R_{LL}, R_{II}, R_{LI} and R_{IL}. However, only R_{LL} provides the sizes of the waves of a Lax solution, while the outcome for R_{IL}, R_{LI} and R_{II} may be non-entropic waves with assigned zero speed. In particular, R_{II} is used to define the composite waves as follows.

Definition 2.1.5 (Composite wave)**.** *A composite wave $\gamma_0 = (\gamma_0^1, \gamma, \gamma_0^3)$ associated to a 2-wave γ and connecting two states $U_-, U_+ \in \Omega$ with $\lambda_- \neq \lambda_+$, is the wave with* zero speed *defined by $\gamma_0 = R_{II}(U_-, U_+)$.*

In this way, we are left to deal with waves of family $1, 3$ and composite waves belonging to a fictitious family 0. Remark that a wave γ_0 reduces to an elementary 2-wave as long as $\gamma_0^1 = \gamma_0^3 = 0$. The γ_0^1, γ_0^3 components are given zero speed and may be non-entropic. Figure 2.2 (b) is an auxiliary picture where these components are treated as waves traveling with positive or negative speed, only because it turns out to be useful when handling interactions in the front tracking algorithm; the real picture is Figure 2.2 (a) where γ_0^1, γ_0^3 are depicted as vertical fronts sticked to the 2-wave. Note that sometimes the term "i-waves" ($i = 1, 3$)

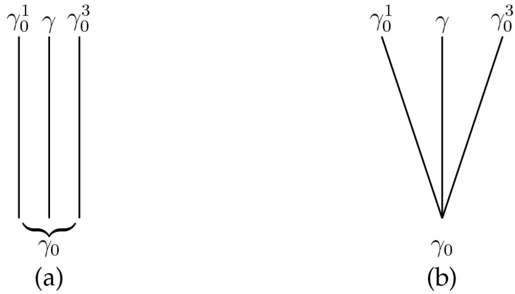

FIGURE 2.2: A composite wave in the (x,t)-plane: in (a) γ_0 is drawn as three parallel close lines, while (b) is the auxiliary picture that is used to determine the states in the interactions.

will be used improperly to denote both real physical waves (i.e. connecting states that lie on a Lax curve) and not (i.e. when referring to states that lie on a general integral curve or on a combination of Lax curves and integral curves). To take into account both these possibilities, we introduce the notion of generic "wave" curves of family 1 and 3 as follows:

$$U = \Phi_1(\gamma_1)(U_-) : \quad \begin{aligned} & v = v_- \exp(\gamma_1), \quad u = u_- + 2a_-\,\Theta_1(\gamma_1), \\ & \lambda = \lambda_-, \end{aligned} \tag{2.1.14}$$

$$U = \Phi_3(\gamma_3)(U_-) : \quad \begin{aligned} & v = v_- \exp(-\gamma_3), \quad u = u_- + 2a_-\,\Theta_3(\gamma_3), \\ & \lambda = \lambda_-, \end{aligned} \tag{2.1.15}$$

for all the choices of Θ_1, Θ_3, of the initial state $U_- \in \Omega$ and of the sizes γ_1, γ_3.

2.2 Approximate front tracking solutions

In this section we build the front tracking approximate solutions to the Cauchy problem (1.4.4), (1.4.5). For $\nu \in \mathbb{N}$ approximation parameter, a $1/\nu$-front tracking approximate solution $U_\nu = (v_\nu, u_\nu, \lambda_\nu)$ is a piecewise constant function, whose discontinuities are located along finitely many fronts in the (x,t)-plane. For system (1.4.4) jumps can be of three types: shocks, rarefactions and composite waves. The latter travel along vertical straight lines carrying each a contact discontinuity of the second characteristic family and some other contributions playing the role of non-physical waves.

Step 1 We approximate the initial data (1.4.5) by taking a sequence of piecewise constant functions with a finite number of jumps $(v_{o,\nu}, u_{o,\nu})$ such that, denoting $p_{o,\nu} = a^2(\lambda_o)/v_{o,\nu}$, we have:

- $\mathrm{TV}\,(\log(p_{o,\nu})) \leq \mathrm{TV}\,(\log(p_o))$ and $\mathrm{TV}\,(u_{o,\nu}) \leq \mathrm{TV}\,(u_o)$;
- $\lim_{x\to-\infty}(v_{o,\nu}, u_{o,\nu})(x) = \lim_{x\to-\infty}(v_o, u_o)(x)$;
- $\|(v_{o,\nu}, u_{o,\nu}) - (v_o, u_o)\|_{\mathbf{L}^1} \leq 1/\nu$.

Notice that we do not approximate λ_o since it is already a piecewise constant function in the cases considered in this thesis. Then, we introduce two strictly positive parameters: $\sigma = \sigma_\nu$, that controls the size of rarefactions, and a threshold $\rho = \rho_\nu$, that depends on the initial data and determines whether the accurate or the simplified Riemann solver is used.

Step 2 At time $t = 0$ we solve the Riemann problems at the points of discontinuity of $(v_{o,\nu}, u_{o,\nu}, \lambda_o)$ as follows. By applying R_{LL} to the side states of each discontinuity, we determine the outgoing waves: shocks are not modified while rarefactions are approximated by fans of waves, each of them with size less than σ (as we saw in Section 1.2.1). At this stage, composite waves are simply contact discontinuities and they are not modified. Then, $(v_\nu, u_\nu, \lambda_o)(\cdot, t)$ is defined until the first time two fronts interact.

Step 3 When two wave fronts of family 1 or 3 interact, we get to solve a new Riemann problem of initial states U_+, U_- with $\lambda_- = \lambda_+$. If one of the incoming waves is a rarefaction, then after the interaction it is prolonged (if it still exists) as a single discontinuity with speed equal to the characteristic speed of the state at the right. If a new rarefaction is generated, we proceed as in Step 2 and split the rarefaction into a fan of waves having size less than σ.

$\boxed{\textbf{Step 4}}$ When a wave front of family $i = 1, 3$ and size γ_i interacts with one of the composite waves, we proceed as follows:

1. if $|\gamma_i| \geq \rho$, we use the *accurate Riemann solver* of Proposition 2.2.2 below, possibly partitioning the newly generated rarefactions according to Step 2;

2. if $|\gamma_i| < \rho$, we use the *simplified Riemann solver* of Proposition 2.2.2 below.

Before stating Proposition 2.2.2, we prove a useful property of commutation which is essential in the construction.

Lemma 2.2.1 (Commutation of "waves"). *Let $i, j = 1, 3$ and $\alpha_i, \beta_j \in \mathbb{R}$. Fix a choice $\Theta_i(\alpha_i)$, $\Theta_j(\beta_j)$ and recall* (2.1.14), (2.1.15). *Assume that two states $U_-, U_+ \in \Omega$ lie in the same phase ($\lambda_- = \lambda_+$). If U_-, U_+ are connected by an "i-wave" of size α_i followed by a "j-wave" of size β_j*

$$U_+ = \Phi_j(\beta_j) \circ \Phi_i(\alpha_i)(U_-),$$

then, they can be connected also by a "j-wave" of size β_j followed by an "i-wave" α_i

$$U_+ = \Phi_i(\alpha_i) \circ \Phi_j(\beta_j)(U_-).$$

Proof. We define $U^* = (v^*, u^*, \lambda^*) := \Phi_i(\alpha_i) \circ \Phi_j(\beta_j)(U_-)$. Clearly, it holds $\lambda^* = \lambda_- = \lambda_+$.

- If $i \neq j$, for example $i = 1$ and $j = 3$, we have

$$\begin{aligned} v^* &= v_- \exp(2\alpha_1 - 2\beta_3) = v_+, \\ u^* &= u_- + 2a_- \left(\Theta_1(\alpha_1) + \Theta_3(\beta_3)\right) = u_+. \end{aligned}$$

- If $i = j = 3$ as in Figure 2.3 (the case $i = j = 1$ is analogous), we have

$$\begin{aligned} v^* &= v_- \exp(-2\alpha_3 - 2\beta_3) = v_+, \\ u^* &= u_- + 2a_- \left(\Theta_3(\alpha_3) + \Theta_3(\beta_3)\right) = u_+. \end{aligned}$$

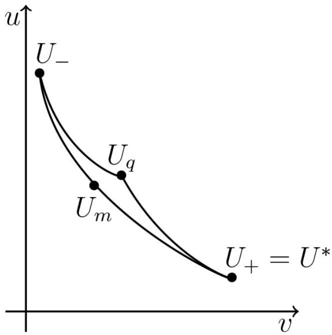

FIGURE 2.3: The commutation of two "3-waves": $\alpha_3, \beta_3 < 0$, $\Theta_3(\alpha_3) = h(\alpha_3)$ and $\Theta_3(\beta_3) = \beta_3$. Here U_m and U_q are the states connected to U_- along the 3-Lax curve by α_3 and, respectively, along the 3-integral curve by β_3.

It follows that $U^* = U_+$ and the lemma is completely proved. □

Notice that, when $\Theta_i(\alpha_i) = h(\alpha_i)$ and $\Theta_j(\beta_j) = h(\beta_j)$, Lemma 2.2.1 is a consequence of the invariance by translation of Lax curves for the system (1.3.1); see [33].

Finally, we have all the tools to describe the two Riemann solvers mentioned above. When dealing with interactions with the composite waves, we will use the notation δ to denote the incoming waves and ε to denote the outgoing ones, respectively. Moreover, if δ_i is a wave of family $i = 1, 3$, we refer to ε_i as the "transmitted wave" and to ε_j, $j = 1, 3$, $j, \neq i$, as the "reflected" wave.

Proposition 2.2.2. *Let $i = 1, 3$. Consider the interaction of a wave δ_i with a composite wave $\delta_0 = (\delta_0^1, \delta, \delta_0^3)$ and let U_-, U_+ be the initial states for the newly generated Riemann problem. Refer to Figure 2.4 and 2.5 and define*

$$\widetilde{U}_- := I_1(\delta_0^1)(U_-), \qquad \widetilde{U}_+ := I_3(-\delta_0^3)(U_+).$$

1. Accurate Riemann solver, $|\delta_i| \geq \rho$. *The solution is given by the three waves* $\varepsilon_1, \varepsilon_0, \varepsilon_3$ *satisfying:*

$$(\varepsilon_1, \delta, \varepsilon_3) = R_{LL}(\widetilde{U}_-, \widetilde{U}_+), \qquad \varepsilon_0 = \delta_0.$$

2. Simplified Riemann solver, $|\delta_i| < \rho$. *The solution is given by the two waves* $\varepsilon_i, \varepsilon_0$ *satisfying:*

$$\begin{aligned} i = 1: \quad & (\varepsilon_1, \delta, \varepsilon_3) = R_{LI}(\widetilde{U}_-, \widetilde{U}_+), \qquad \varepsilon_0 = (\delta_0^1, \delta, \delta_0^3 + \varepsilon_3), \\ i = 3: \quad & (\varepsilon_1, \delta, \varepsilon_3) = R_{IL}(\widetilde{U}_-, \widetilde{U}_+), \qquad \varepsilon_0 = (\delta_0^1 + \varepsilon_1, \delta, \delta_0^3). \end{aligned}$$

Proof. Referring to Figure 2.4 and 2.5 for the case $i = 3$, the interaction of a wave δ_i with δ_0 is handled letting δ_i interact first with the component δ_0^j of the opposite family $j \neq i$. In some sense, we have δ_i and δ_0^j crossing each other without changing size as a consequence of Lemma 2.2.1. Hence, we look at the interaction of δ_i with the 2-wave δ: this gives rise to the Riemann problem of initial states $\widetilde{U}_-, \widetilde{U}_+$ defined in the statement. Depending on whether $|\delta_i| \geq \rho$ or $< \rho$ and, in this last case, on whether $i = 1$ or $i = 3$, we choose suitable $\theta_1, \theta_3 \in \{L, I\}$ and compute $\mathcal{R}_{\theta_1\theta_3}(\widetilde{U}_-, \widetilde{U}_+)$.

1. *Accurate Riemann solver*. We choose $\theta_1 = \theta_3 = L$, i.e. the solution to the Riemann problem is constituted by Lax waves. Once we have computed
$$R_{LL}(\widetilde{U}_-, \widetilde{U}_+) = (\varepsilon_1, \delta, \varepsilon_3),$$
we let ε_1 and ε_3 commute in the sense of Lemma 2.2.1 with δ_0^1 and δ_0^3, respectively. In this way, they are prolonged as outgoing waves of family $1, 3$; see Figure 2.4 (a), (b) for a picture of case $i = 3$. Then, the resulting composite wave connects U_p to U_q, where
$$U_p = W_1(\varepsilon_1)(U_-), \qquad U_q = W_3(-\varepsilon_3)(U_+).$$
Hence, $\varepsilon_0 = R_{II}(U_p, U_q) = (\delta_0^1, \delta, \delta_0^3) = \delta_0$.

2. *Simplified Riemann solver*. We have to distinguish between the case $i = 1$ and $i = 3$. Once the triple $(\varepsilon_1, \delta, \varepsilon_3)$ has been determined by

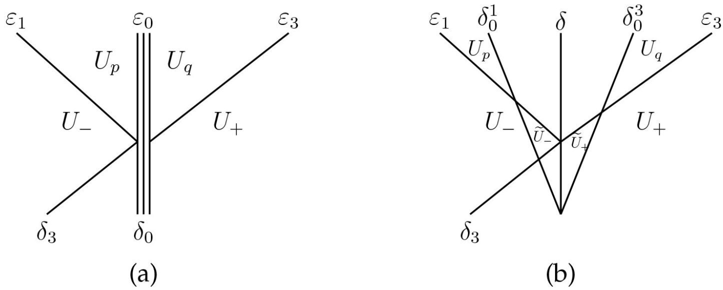

FIGURE 2.4: Interaction of a 3-wave δ_3 with a composite wave δ_0 solved by the accurate Riemann solver. The actual Riemann solver is represented in (a), while (b) is an auxiliary picture.

$\mathcal{R}_{\theta_1\theta_3}$, the idea is to 'project' the reflected wave along the integral curve of the same family; see Figure 2.5 (a), (b) for a picture of case $i = 3$.

If $i = 1$, we choose $\theta_1 = L$ and $\theta_3 = I$ and compute

$$R_{LI}(\widetilde{U}_-, \widetilde{U}_+) = (\varepsilon_1, \delta, \varepsilon_3),$$

i.e. the solution is formally given by a physical wave ε_1 and a non-entropic one ε_3. Then, we let ε_1 commute with δ_0^1 by Lemma 2.2.1. The outgoing composite wave connects U_p to U_+, where

$$U_p = W_1(\varepsilon_1)(U_-), \qquad U_+ = I_3(\delta_0^3 + \varepsilon_3) \circ W_2(\delta) \circ I_1(\delta_0^1)(U_p).$$

Hence, $\varepsilon_0 = R_{II}(U_p, U_+) = (\delta_0^1, \delta, \delta_0^3 + \varepsilon_3)$.

If $i = 3$, we choose $\theta_1 = I$ and $\theta_3 = L$ and compute

$$R_{IL}(\widetilde{U}_-, \widetilde{U}_+) = (\varepsilon_1, \delta, \varepsilon_3),$$

i.e. the solution is formally given by a non-entropic wave ε_1 and a physical one ε_3. Then, we let ε_3 commute with δ_0^3 by Lemma 2.2.1.

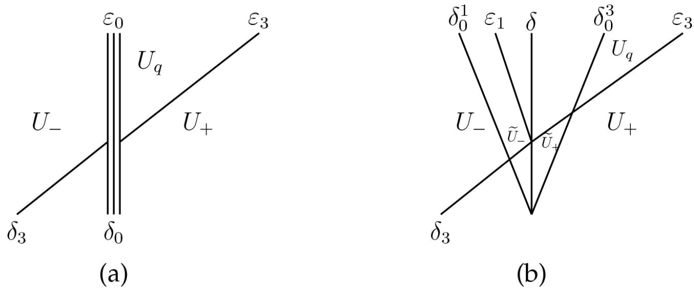

FIGURE 2.5: Interaction of a 3-wave δ_3 with a composite wave δ_0 solved by the simplified Riemann solver. The actual solver is represented in (a), while (b) is an auxiliary picture.

The outgoing composite wave connects U_- to U_q, where

$$U_q = I_3(\delta_0^3) \circ W_2(\delta) \circ I_1(\delta_0^1 + \varepsilon_1)(U_-).$$

Hence, $\varepsilon_0 = R_{II}(U_-, U_q) = (\delta_0^1 + \varepsilon_1, \delta, \delta_0^3)$.

□

By Proposition 2.2.2, a composite wave $\gamma_0(t) = (\gamma_0^1(t), \gamma(t), \gamma_0^3(t))$ at a time t can be understood as follows. We have that $\gamma(t) = \gamma$, where γ is the size of a 2-wave that remains constant for all times t. Moreover, for the components of family $i = 1, 3$ we set

$$\gamma_0^i(0+) = 0\,, \qquad \gamma_0^i(t) = \sum_{\tau<t} \Delta\gamma_0^i(\tau) \quad \text{for } t > 0\,,$$

where $\Delta\gamma_0^i(\tau) := \varepsilon_i$, if τ is an interaction time where the simplified solver is used with an incoming wave of family $j = 1, 3$, $j \neq i$, and ε_i is the size of the reflected wave attached to γ_0; otherwise, $\Delta\gamma_0^i(\tau) := 0$. Remark that sometimes we simply write γ_0 instead of $\gamma_0(t)$ when it is clear from the context that we are referring to time t. With this notation we can now introduce the strength of a composite wave.

Definition 2.2.3 (Strength of a composite wave)**.** *Let a composite wave at time $t > 0$ be denoted by $\gamma_0(t) = (\gamma_0^1(t), \gamma(t), \gamma_0^3(t))$. For the strength of $\gamma_0(t)$ we define*

$$|\gamma_0(t)| = \|\gamma_0^1(t)\| + \|\gamma_0^3(t)\|,$$

where

$$\|\gamma_0^i(t)\| := \sum_{\tau<t} |\Delta\gamma_0^i(\tau)|, \qquad i = 1, 3. \tag{2.2.1}$$

Remark 2.2.4. *According to this definition, we have that the strengths of ε_0 and δ_0 in the proof of Proposition 2.2.2 (Simplified solver) are related by*

$$|\varepsilon_0| = |\delta_0| + \begin{cases} |\varepsilon_3| & \text{if } i = 1\,, \\ |\varepsilon_1| & \text{if } i = 3\,. \end{cases}$$

Remark 2.2.5. *The simplified Riemann solver described in this section was first introduced in [9] and subsequently used in [8]. The preceding papers [2, 10] contain two different versions of the simplified procedure. More precisely, in [2] the authors use the standard solver discussed in Section 1.2.1, where the errors are collected into non-physical waves traveling with a fixed positive speed. In [10], instead, non-physical waves are attached to the fronts carrying the contact discontinuities and travel with zero speed, thus forming so-called* composite $(2, 0)$-waves*. This idea reminds of an analogous trick used in the important paper [36].*

The main advantage of the solver of Proposition 2.2.2 in comparison with those of [2, 10] is that the transmitted i-wave ε_i is taken along an integral curve and does not maintain the same size δ_i of the incoming one, as explained in Lemma 2.3.2 below. This is the key feature that guarantees the decrease in time of an asymmetric Glimm functional as the one chosen in (2.3.28)*. Indeed, such a functional would not be decreasing with the solvers of [2, 10].*

2.3 Interactions

Fix an approximation index ν. In order to prove that the algorithm is defined for all $t > 0$ and provides a piecewise constant solution U_ν for any initial data $U_{o,\nu}$, first we have to see how waves change their strengths across interactions and prove that the total variation of U^ν remains bounded for any t, independently from ν. This is accomplished by introducing a suitable Glimm functional F, that does not increase across interaction times and remains constant otherwise. This section contains the analysis of the different types of interaction: as in [2], we distinguish between interactions that involve a composite wave and those occurring between two waves of family 1 or 3.

2.3.1 Interactions with the composite waves

Proposition 2.3.1. *Consider the interaction of a wave δ_i with a composite wave $\delta_0 = (\delta_0^1, \delta, \delta_0^3)$ and let $\varepsilon_i, \varepsilon_j, \varepsilon_0$ be as in Proposition 2.2.2, for $i, j = 1, 3$, $j \neq i$. We have $\Theta_i = h$ in every case and we choose Θ_j between Id and h depending on the solver used. Then,*

$$\begin{aligned} \varepsilon_3 - \varepsilon_1 &= \begin{cases} -\delta_1 & \textit{if } i = 1, \\ \delta_3 & \textit{if } i = 3, \end{cases} \\ a_-\Theta_1(\varepsilon_1) + a_+\Theta_3(\varepsilon_3) &= \begin{cases} a_+\Theta_1(\delta_1) & \textit{if } i = 1, \\ a_-\Theta_3(\delta_3) & \textit{if } i = 3. \end{cases} \end{aligned} \tag{2.3.1}$$

Moreover, the signs of $\varepsilon_1, \varepsilon_3$ satisfy:

$$\operatorname{sgn} \varepsilon_i = \operatorname{sgn} \delta_i, \qquad \operatorname{sgn} \varepsilon_j = \begin{cases} \operatorname{sgn} \delta \cdot \operatorname{sgn} \delta_i & \textit{if } i = 1, \\ -\operatorname{sgn} \delta \cdot \operatorname{sgn} \delta_i & \textit{if } i = 3. \end{cases} \tag{2.3.2}$$

Proof. To prove (2.3.1), notice that for $i = 1$ we use R_{LL} or R_{LI} (i.e. $\Theta_1 = h$), while for $i = 3$ we use R_{LL} or R_{IL} (i.e. $\Theta_3 = h$). Hence, $(2.3.1)_2$ is equivalent to

$$a_-\Theta_1(\varepsilon_1) + a_+\Theta_3(\varepsilon_3) = \begin{cases} a_+h(\delta_1) & \text{if } i = 1, \\ a_-h(\delta_3) & \text{if } i = 3. \end{cases}$$

Recall the states $\widetilde{U}_- = I_1(\delta_0^1)(U_-)$ and $\widetilde{U}_+ = I_3(-\delta_0^3)(U_+)$ from Proposition 2.2.2. By (1.4.2) and (2.1.4) we have that

$$\frac{1}{2}\log\left(\frac{\widetilde{p}_+}{\widetilde{p}_-}\right) = \begin{cases} -\delta_1 & \text{if } i = 1, \\ \delta_3 & \text{if } i = 3, \end{cases} \qquad \frac{\widetilde{u}_+ - \widetilde{u}_-}{2} = \begin{cases} a_+ h(\delta_1) & \text{if } i = 1, \\ a_- h(\delta_3) & \text{if } i = 3, \end{cases}$$

and by Proposition 2.1.3 we notice that

$$\varepsilon_3 - \varepsilon_1 = \frac{1}{2}\log\left(\frac{\widetilde{p}_+}{\widetilde{p}_-}\right), \qquad a_-\Theta_1(\varepsilon_1) + a_+\Theta_3(\varepsilon_3) = \frac{\widetilde{u}_+ - \widetilde{u}_-}{2}.$$

Hence, (2.3.1) is completely proved.

Now, we verify (2.3.2) for $\delta > 0$ (the other case is similar). Since for interactions solved in the accurate way the relations contained in (2.3.2) have already been proved in Proposition 1 of [3], we focus on the case where the simplified solver is used. Again we have to distinguish case $i = 1$ and $i = 3$. Denote $k = a_+/a_- > 1$.

i) When $i = 1$, by (2.3.1) $\varepsilon_1, \varepsilon_3$ solve

$$\begin{cases} \varepsilon_3 - \varepsilon_1 = -\delta_1, \\ a_- h(\varepsilon_1) + a_+\varepsilon_3 = a_+ h(\delta_1). \end{cases} \tag{2.3.3}$$

By substituting the expression for ε_3 from the first equation of (2.3.3) into the second one, we obtain $h(\varepsilon_1) + k\varepsilon_1 = k\delta_1 + kh(\delta_1)$. Hence, it follows $\operatorname{sgn}\varepsilon_1 = \operatorname{sgn}\delta_1$. If $\delta_1 > 0$, then $R_{LI}(\widetilde{U}_-, \widetilde{U}_+) = R_{II}(\widetilde{U}_-, \widetilde{U}_+)$ and system (2.3.3) is linear. Thus, we get $\varepsilon_3 = \delta_1\delta/2$ and $\operatorname{sgn}\varepsilon_3 = \operatorname{sgn}\delta_1 = \operatorname{sgn}\delta \cdot \operatorname{sgn}\delta_1$, as wished. If, instead, $\delta_1 < 0$, then the second formula in (2.3.3) becomes

$$\sinh\varepsilon_1 + k\varepsilon_3 = k\sinh\delta_1. \tag{2.3.4}$$

By substituting the expression for ε_1 from the first equation of (2.3.3) in (2.3.4), we get $k(\varepsilon_3 + \delta_1) + \sinh(\varepsilon_3 + \delta_1) = k\,(\sinh\delta_1 + \delta_1)$. If we call $\Gamma(x) := kx + \sinh x$, then $\Gamma(\varepsilon_3 + \delta_1) = k\,(\sinh\delta_1 + \delta_1)$ and

$$\Gamma(\varepsilon_3 + \delta_1) - \Gamma(\delta_1) = (k-1)\sinh\delta_1. \tag{2.3.5}$$

Since Γ is a strictly increasing function and $\delta_1 < 0$, we get $\varepsilon_3 < 0$, that is $\operatorname{sgn} \varepsilon_3 = \operatorname{sgn} \delta_1 = \operatorname{sgn} \delta \cdot \operatorname{sgn} \delta_1$.

ii) When $i = 3$, by (2.3.1) $\varepsilon_1, \varepsilon_3$ solve

$$\begin{cases} \varepsilon_3 - \varepsilon_1 = \delta_3, \\ a_- \varepsilon_1 + a_+ h(\varepsilon_3) = a_- h(\delta_3). \end{cases} \tag{2.3.6}$$

By substituting the expression for ε_1 coming from the first equation of (2.3.6) into the second one, we obtain $\varepsilon_3 + kh(\varepsilon_3) = \delta_3 + h(\delta_3)$. Hence, we have that $\operatorname{sgn} \varepsilon_3 = \operatorname{sgn} \delta_3$. Now, take $\delta_3 < 0$ and suppose to use R_{LL} to solve the Riemann problem at the same interaction point. The corresponding outgoing waves $\varepsilon_1^*, \varepsilon_3^*$ solve

$$\begin{cases} \varepsilon_3^* - \varepsilon_1^* = \delta_3, \\ a_- h(\varepsilon_1^*) + a_+ h(\varepsilon_3^*) = a_- h(\delta_3). \end{cases} \tag{2.3.7}$$

Since $\varepsilon_1^* > 0$, then system (2.3.7) reduces to (2.3.6) and by uniqueness its solutions must coincide precisely with $\varepsilon_1, \varepsilon_3$. Hence, (2.3.2) is valid. If $\delta_3 > 0$, instead, we have that $h(\delta_3) = \delta_3$ and $h(\varepsilon_3) = \varepsilon_3$, i.e. in this case it holds $R_{IL}(\widetilde{U}_-, \widetilde{U}_+) = R_{II}(\widetilde{U}_-, \widetilde{U}_+)$. This amounts to solve a linear system in $\varepsilon_1, \varepsilon_3$ and we find $\varepsilon_1 = -\delta_3 \delta / 2$. Hence, $\operatorname{sgn} \varepsilon_1 = -\operatorname{sgn} \delta_3 = -\operatorname{sgn} \delta \cdot \operatorname{sgn} \delta_3$, as wished.

$\square$

Summarizing, we find the following patterns of solutions at interaction points. When the accurate Riemann solver is used, the possible configurations of waves are shown in Table 2.1, where by 0 we denote composite waves and by iR, iS we denote rarefactions and shocks of family $i = 1, 3$. When the simplified procedure is used, instead, the outcome is an outgoing wave of the same family and type of the incoming one, plus a composite wave that carries an additional error formally computed as a reflected wave.

The next lemma is concerned with the strengths of the waves involved in an interaction.

TABLE 2.1: Patterns of solutions for the accurate Riemann solver

	$\delta > 0$	$\delta < 0$
INCOMING	OUTGOING	OUTGOING
$0 \times 1R$	$1R + 0 + 3R$	$1R + 0 + 3S$
$0 \times 1S$	$1S + 0 + 3S$	$1S + 0 + 3R$
$3R \times 0$	$1S + 0 + 3R$	$1R + 0 + 3R$
$3S \times 0$	$1R + 0 + 3S$	$1S + 0 + 3S$

Lemma 2.3.2 (Interaction estimates)**.** *For the interaction of a wave δ_i with a composite wave δ_0, let $\varepsilon_i, \varepsilon_j, \varepsilon_0$ be as in the previous propositions, for $i, j = 1, 3$, $j \neq i$. Then, $|\varepsilon_i - \delta_i| = |\varepsilon_j|$ and the following estimates are verified.*

1. Accurate Riemann solver, $|\delta_i| \geq \rho$:

$$|\varepsilon_0| = |\delta_0|, \qquad\qquad |\varepsilon_j| \leq \frac{1}{2}|\delta_i \delta|. \tag{2.3.8}$$

2. Simplified Riemann solver, $|\delta_i| < \rho$:

$$|\varepsilon_0| = |\delta_0| + |\varepsilon_j|\,, \quad |\varepsilon_j| \leq \begin{cases} \dfrac{C_o}{2}|\delta_i\delta| & \text{if } \delta_i < 0 \text{ and } \operatorname{sgn}((2-i)\delta) > 0, \\ \dfrac{1}{2}|\delta_i\delta| & \text{otherwise}, \end{cases} \tag{2.3.9}$$

where

$$C_o = C_o(\rho) := \frac{\sinh(\rho)}{\rho} > 1 \tag{2.3.10}$$

is such that $C_o(\rho) \to 1+$ for $\rho \to 0^+$.

Proof. The equality $|\varepsilon_i - \delta_i| = |\varepsilon_j|$ is a consequence of $(2.3.1)_1$. Moreover, notice that the estimate for $|\varepsilon_j|$ in (2.3.9) concides with that in (2.3.8) except for the interactions of a 1-shock with $\delta > 0$ and of a 3-shock with $\delta < 0$.

1. *Accurate Riemann solver*. The first equality in (2.3.8) is immediate by the definition of the solver, while $(2.3.8)_2$ is proved following the same steps of Theorem 2 of [3].

2. *Simplified Riemann solver*. We study only the case $\delta > 0$ and refer to Figure 2.5 (a), (b). The equality $|\varepsilon_0| = |\delta_0| + |\varepsilon_j|$ in (2.3.9) is a consequence of the choice to attach the reflected wave to the composite wave, see Proposition 2.2.2. To prove $(2.3.9)_2$, we distinguish cases according to the characteristic family and the sign of size of the interacting wave. Recall $\widetilde{U}_-, \widetilde{U}_+$ from Proposition 2.2.2.

   If δ_i is a rarefaction, i.e. $\delta_i > 0$, then $R_{LI}(\widetilde{U}_-, \widetilde{U}_+) = R_{II}(\widetilde{U}_-, \widetilde{U}_+)$ for $i = 1$ and $R_{IL}(\widetilde{U}_-, \widetilde{U}_+) = R_{II}(\widetilde{U}_-, \widetilde{U}_+)$ for $i = 3$. This means that in both cases the two expressions in (2.3.1) are linear and, by a straightforward calculation, we obtain $|\varepsilon_j| = |\delta_i \delta|/2$.

   If δ_i is a shock and $i = 3$, i.e. $\delta_3 < 0$, we have $R_{IL}(\widetilde{U}_-, \widetilde{U}_+) = R_{LL}(\widetilde{U}_-, \widetilde{U}_+)$ and the estimate (2.3.9) follows exactly as in the accurate case.

   Finally, let δ_i be a shock of family $i = 1$, i.e. consider $\delta_1 < 0$. Recall from Proposition 2.2.2 that we have $\varepsilon_1 = \varepsilon_3 + \delta_1$ and $\varepsilon_3 < 0$, $\delta_1 < 0$; moreover, (2.3.5) holds. By the Mean Value Theorem there exists an intermediate s such that $\Gamma(\varepsilon_3 + \delta_1) - \Gamma(\delta_1) = \Gamma'(s)\varepsilon_3$. Hence, we have
$$(k+1)|\varepsilon_3| \le \Gamma'(s)|\varepsilon_3| = (k-1)\sinh|\delta_1|,$$
   and we deduce
$$|\varepsilon_3| \le \frac{k-1}{k+1}\sinh|\delta_1| = \frac{\delta}{2}\sinh|\delta_1| \le \frac{C_o}{2}|\delta_1\delta|.$$

□

Remark 2.3.3. *Denote* $[\delta]_+ = \max\{\delta, 0\}$ *and* $[\delta]_- = \max\{-\delta, 0\}$ *the positive and negative part of* $\delta \in \mathbb{R}$*, respectively. An important consequence of* (2.3.1), (2.3.2) *and* (2.3.8) *is that*
$$|\varepsilon_1| + |\varepsilon_3| \le \begin{cases} |\delta_i| + |\delta_i|[\delta]_+ & \textit{if } i = 1 \textit{ and } |\delta_i| \ge \rho, \\ |\delta_i| + |\delta_i|[\delta]_- & \textit{if } i = 3 \textit{ and } |\delta_i| \ge \rho. \end{cases} \tag{2.3.11}$$

Indeed, if $\delta > 0$ and $i = 1$ then by (2.3.1), (2.3.2) *we get* $|\varepsilon_1| + |\varepsilon_3| = |\delta_1| + 2|\varepsilon_3|$, *which is estimated* $\leq |\delta_1| + |\delta_1\delta|$ *by* (2.3.8). *If $\delta < 0$, instead, we get* $|\varepsilon_1|+|\varepsilon_3| = |\delta_1|$ *simply by* (2.3.1), (2.3.2). *The case $i = 3$ is entirely analogous.*

2.3.2 Interaction between waves of family 1 and 3

Here we analyze interactions between 1- and 3-waves. Two situations may occur: either the waves belong to different characteristic families, Figure 2.6 (a), or they belong to the same family, Figure 2.6 (b). In this last case at least one of them must be a shock. Moreover, remark that all these interactions take place in a fixed phase, i.e. all the states involved have the same value for λ. In the following lemma we collect (without proof) some useful results contained in Section 5.2 of [2].

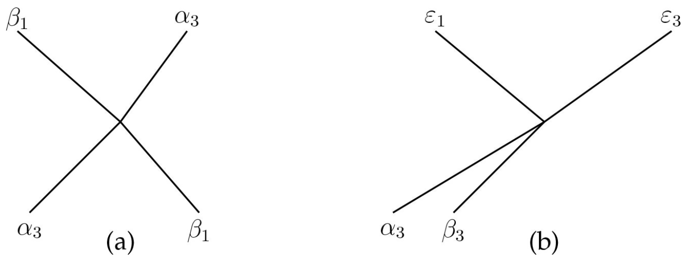

FIGURE 2.6: Interaction between two waves of different family (a) and between two waves of family 3 (b).

Lemma 2.3.4. *For $i, j, k = 1, 3$, $i \neq j$, let α_i, β_k refer to the sizes of some interacting waves of family i and k and $\varepsilon_i, \varepsilon_j$ refer to the two outgoing waves as in Figure 2.6.*

1. *If $i \neq k$, then $\varepsilon_i = \alpha_i$ and $\varepsilon_j = \beta_k$.*
2. *If $i = k$, then the following relations on the sign of the waves hold:*
   - *i) when $\alpha_i < 0$ and $\beta_k < 0$, we get $\varepsilon_i < 0$ and $\varepsilon_j > 0$;*
   - *ii) when $\alpha_i \cdot \beta_k < 0$, we get $\varepsilon_j < 0$.*

TABLE 2.2: Patterns of solutions for interactions of two i-waves

INCOMING	OUTGOING
$iS \times iS$	$iS + jR$
$iS \times iR$	$iS + jS$
$iS \times iR$	$iR + jS$

Moreover, for $i = k$ we can derive the following useful identities as in [35]:

$$\varepsilon_3 - \varepsilon_1 = \operatorname{sgn}(i-2)\alpha_i + \operatorname{sgn}(k-2)\beta_k, \tag{2.3.12}$$

$$h(\varepsilon_1) + h(\varepsilon_3) = h(\alpha_i) + h(\beta_k). \tag{2.3.13}$$

In Table 2.2 we summarize the patterns of solutions deduced from this lemma for an interaction between two waves of the same family $i = 1, 3$. Now, we give some sharper estimates for the change in the strengths of the waves across an interaction of this kind. In particular, we find that the bound on the strength of the reflected wave depends on a damping factor smaller than 1. This coefficient is not constant, but it depends on the strengths of the incoming waves. Therefore, for later need we assume that the size of any interacting wave γ_i, $i = 1, 3$, is bounded by a fixed constant $m_o > 0$:

$$|\gamma_i| \le m_o\,. \tag{2.3.14}$$

We stress that this bound has to be imposed particularly to shock waves, since rarefaction waves remain small.

Lemma 2.3.5 (Interaction estimates)**.** *For $i, j = 1, 3$, $j \neq i$, consider the interaction of two i-waves α_i, β_i and denote by $\varepsilon_i, \varepsilon_j$ the transmitted and reflected wave, respectively.*

i) If both $\alpha_i < 0$ and $\beta_i < 0$, then $|\varepsilon_i| > \max\{|\alpha_i|, |\beta_i|\}$.

ii) If $\alpha_i \cdot \beta_i < 0$, then both the amounts of shock and rarefaction of family i decrease across the interaction. Moreover, when $\alpha_i < 0 < \beta_i$ it holds

$$|\varepsilon_j| \leq c(\alpha_i) \cdot \min\{|\alpha_i|, |\beta_i|\}\,, \qquad c(z) := \frac{\cosh z - 1}{\cosh z + 1}\,. \tag{2.3.15}$$

Proof. We prove only (2.3.15), the rest being already seen in Lemmas 5.4–5.6 of [2]. For simplicity, we assume $i = 3$ and distinguish between two cases according to the outgoing wave ε_3. Indeed, by Lemma B.1 of [2] there exists a function $x_o(\cdot)$ such that ε_3 is a rarefaction if and only if $\beta_3 \geq x_o(|\alpha_3|)$. In the limiting case $\beta_3 = x_o(|\alpha_3|)$, the shock and the rarefaction wave cancel each other and $\varepsilon_3 = 0$. Thus, the interaction gives rise only to the reflected wave ε_1. By setting $x = |\beta_3| = \beta_3$ and $z = |\alpha_3|$, from (2.3.12), (2.3.13) and $\varepsilon_3 = 0$ it follows

$$\sinh(x - z) - \sinh z + x = 0,$$

which implicitly defines the function $x = x_o(z)$.

$\boxed{3S \times 3R \to 1S + 3R}$ First we specialize (2.3.12) and (2.3.13) to this case:

$$|\varepsilon_1| + |\varepsilon_3| = -|\alpha_3| + |\beta_3|, \tag{2.3.16}$$

$$\sinh(|\varepsilon_1|) - |\varepsilon_3| = \sinh(|\alpha_3|) - |\beta_3|. \tag{2.3.17}$$

By summing (2.3.16) and (2.3.17), we obtain

$$\sinh(|\varepsilon_1|) + |\varepsilon_1| = \sinh(|\alpha_3|) - |\alpha_3|. \tag{2.3.18}$$

Remark that in this case (2.3.15) reduces to

$$|\varepsilon_1| \leq c(\alpha_3)|\alpha_3|, \tag{2.3.19}$$

since $|\alpha_3| < |\beta_3|$ by (2.3.16). We write $y = |\varepsilon_1|$ and call $G(y, z) = \sinh y + y - \sinh z + z$, so that (2.3.18) becomes $G(y, z) = 0$. By a simple application of the Implicit Function Theorem, there exists a function $y = y(z) \geq 0$, defined for all $z \geq 0$ and satisfying $G(y(z), z) = 0$. Since $G_y(y, z) = \cosh y + 1 > 0$, in order to prove $y(z) \leq c(z)z$ it suffices to show that $g(z) = G(c(z)z, z) > 0$, i.e. $g(z) = (c(z) + 1)z + \sinh(c(z)z) - \sinh z > 0$. Using the Mean Value Theorem together with the fact that $c(z)z < z$ and $c(z) + 1 = (1 - c(z)) \cosh z$, we find that

$$g(z) = (c(z) + 1)z + (c(z)z - z) \cosh \zeta > z[c(z) + 1 + (c(z) - 1) \cosh z] = 0,$$

for $c(z)z < \zeta < z$. Hence, (2.3.15) is proved.

$\boxed{3S \times 3R \to 1S + 3S}$ In this case, the identities (2.3.12) and (2.3.13) can be rewritten as

$$\begin{aligned} |\varepsilon_1| - |\varepsilon_3| &= -|\alpha_3| + |\beta_3|, \\ \sinh(|\varepsilon_1|) + \sinh(|\varepsilon_3|) &= \sinh(|\alpha_3|) - |\beta_3|. \end{aligned}$$

We set $y = |\varepsilon_1|$ and we define

$$F(x, y; z) = \sinh y + \sinh(y - x + z) - \sinh z + x,$$

which is subjected to the constraints

$$z \geq 0, \quad 0 \leq x < x_o(z), \quad \max\{0, x - z\} < y < \min\{x, z\}.$$

By the Implicit Function Theorem, there exists a function $y = y(x; z)$ such that $F(x, y(x; z); z) = 0$. Moreover, denoting by y' the derivative of y with respect to x and so on, we have

$$y' = -\frac{F_x}{F_y}, \qquad y'' = -\frac{F_{xx} + 2F_{xy}y' + F_{yy}(y')^2}{F_y},$$

where

$$\begin{aligned} F_x = 1 - \cosh(y - x + z) < 0, &\quad F_y = \cosh(y - x + z) + \cosh y > 0, \\ F_{xx} = -F_{xy} = \sinh(y - x + z) > 0, &\quad F_{yy} = \sinh(y - x + z) + \sinh y > 0. \end{aligned}$$

Therefore, $y' > 0$ and

$$y''(x;z) = -\frac{\sinh(y-x+z)(1-y')^2 + \sinh(y)(y')^2}{F_y} < 0.$$

Hence, $x \mapsto y(x;z)$ is concave down and $y(x;z) \le y'(0;z)x = c(z)x$. To conclude, it remains to prove $y(x;z) \le c(z)z$. Since $y' > 0$, we get

$$y(x;z) \le y(x_o(z);z) \le c(z)z,$$

where the last inequality coincides with (2.3.19) in the limiting case $\beta_3 = x_o(z)$, $z = |\alpha_3|$. □

Corollary 2.3.6. *If α_i, β_i denote two interacting waves of family $i = 1,3$ and verify $\alpha_i \cdot \beta_i < 0$, then by* (2.3.14) *and* (2.3.15) *it holds*

$$|\varepsilon_j| \le c(m_o)\min\{|\alpha_i|, |\beta_i|\}, \qquad j = 1,3,\ j \ne i. \tag{2.3.20}$$

Remark 2.3.7. *We anticipated in Section 1.4.1 that a different damping coefficient denoted by d is used in [2] in place of c in the estimate* (2.3.15). *As pointed out in Remark 5.7 of [2], d is defined as*

$$d(m_o) := \max_{|\alpha_i|,|\beta_i| \le m_o} \frac{|\varepsilon_j(\alpha_i,\beta_i)|}{\min\{|\alpha_i|,|\beta_i|\}}, \quad j \ne i,$$

where the function $\varepsilon_j(|\alpha_i|,|\beta_i|)$ implicitly solves $h(\varepsilon_j) + h(\varepsilon_j + \alpha_i + \beta_i) - h(\alpha_i) - h(\beta_i) = 0$, see (5.4) in [2]. Hence, d is an increasing function of m_o and vanishes as $m_o \to 0$ because quadratic interaction estimates hold for m_o small. Moreover, $d(m_o) \to 1$ for m_o large and $d(m_o) \ge c(m_o)$.

Remark 2.3.8. *Under the notation of the proof of Lemma 2.3.5, i.e. $x = \beta_i$, $z = |\alpha_i|$, we see that the size of the reflected shock is*

$$|\varepsilon_j| = \begin{cases} y(x;z) & \text{if } x \le x_o(z), \\ y(z) & \text{if } x > x_o(z). \end{cases} \tag{2.3.21}$$

The strength ε_j is a continuous function of x, since $y(x_o(z);z) = y(z)$ for every z. In particular, assume that $\beta_i > x_o(|\alpha_i|)$, so that ε_i is a rarefaction.

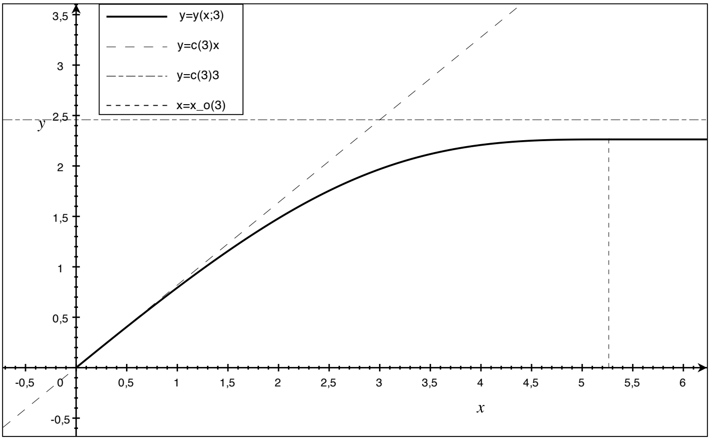

FIGURE 2.7: The reflected shock for interactions between two i-waves $\alpha_i < 0 < \beta_i$.

For β_i in this range, the size of ε_j does not change by (2.3.21) *and the part of β_i exceeding $x_o(|\alpha_i|)$ is entirely propagated along ε_i. This holds since the interaction affects only that part of β_i whose amplitude is exactly $x_o(|\alpha_i|)$. We refer to Figure 2.7 for a graph of $|\varepsilon_j| = y$ (solid curve) as a function of $\beta_i = x$, for $|\alpha_i| = z = 3$. There, the vertical line marks the transition of ε_i from shock to rarefaction. The other two dashed lines refer to the bounds in* (2.3.15)*; in particular, since $\lim_{z\to+\infty}(c(z)z - y(x_o(z); z)) = 0$, the horizontal bound becomes asymptotically accurate.*

2.3.3 An asymmetric Glimm functional

In this section we outline the main features of our Glimm functional. We postpone its precise definition to the next chapters since it is different in each of the cases considered (one and two phase interfaces). As seen in Section 1.2, the Glimm functional is used to control the total variation of the approximate solutions and is usually denoted by F. Moreover, it is given by the sum of a linear functional L and a quadratic interaction potential Q.

Fix an approximate solution U_ν. For $t > 0$ not an interaction time, we define

$$L(t) = L^1(t) + L^3(t), \tag{2.3.22}$$

and

$$L^i(t) = L^i(U_\nu(\cdot, t)) := \sum_{\gamma_i > 0} |\gamma_i| + \xi \sum_{\gamma_i < 0} |\gamma_i| + \sum_{\gamma_0} \|\gamma_0^i(t)\|, \qquad i = 1, 3,$$

where the last sum varies over the set of all the composite waves γ_0. More precisely, $|\gamma_i|$ denotes the strength of a wave of family i and $\|\gamma_0^i(t)\|$ is defined at (2.2.1). Notice that the strengths of the shock waves ($\gamma_i < 0$) are weighted by a parameter $\xi \geq 1$ to be determined. We also introduce

$$\overline{L}(t) = \overline{L}^1(t) + \overline{L}^3(t) := \sum_{\gamma_1 \in \mathbb{R}} |\gamma_1| + \sum_{\gamma_3 \in \mathbb{R}} |\gamma_3|. \tag{2.3.23}$$

We have that at an interaction between a composite wave $\gamma_0 = (\gamma_0^1, \gamma, \gamma_0^3)$ and a wave γ_i of family $i = 1, 3$, by (2.3.11) it holds

$$\Delta\overline{L}(t) = \overline{L}(t^+) - \overline{L}(t^-) \leq \begin{cases} |\gamma_i|[\gamma]_+ & \text{if } i = 1, \\ |\gamma_i|[\gamma]_- & \text{if } i = 3. \end{cases} \tag{2.3.24}$$

The four outcomes for $\Delta\overline{L}$ are depicted in Figure 2.8.

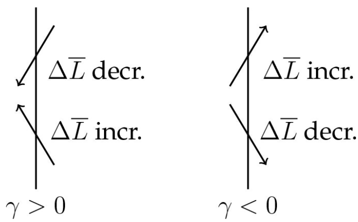

FIGURE 2.8: The total variation across an interaction with the composite wave $\gamma_0 = (\gamma_0^1, \gamma, \gamma_0^3)$.

This suggests a possible way to define an asymmetric interaction potential. Recall that an interaction potential like that of [2] would contain all the various products $|\gamma_i\gamma|$, where γ_i is a wave of family $i = 1, 3$ approaching a wave γ_0 of second component γ. Across a time of interaction between γ_i and γ_0, by (2.3.24) we have that $\overline{L}$ decreases spontaneously when either $i = 1$ and $\gamma < 0$ or $i = 3$ and $\gamma > 0$. Therefore, it seems reasonable to simplify the potential by getting rid of such products $|\gamma_i\gamma|$, at least when γ_i is a shock. This trimming procedure leads to the functional Q defined as

$$Q(t) = Q^1(t) + Q^3(t)\,, \tag{2.3.25}$$

where

$$Q^1(t) = Q^1(U_\nu(x,t)) := \sum_{\gamma_0} K^1_\gamma \Big(\sum_{\mathcal{A}\,,\gamma_1>0} |\gamma_1\gamma| + \xi \sum_{\mathcal{A}\,,\gamma_1<0} |\gamma_1|[\gamma]_+ \Big)\,, \tag{2.3.26}$$

$$Q^3(t) = Q^3(U_\nu(x,t)) := \sum_{\gamma_0} K^3_\gamma \Big(\sum_{\mathcal{A}\,,\gamma_3>0} |\gamma_3\gamma| + \xi \sum_{\mathcal{A}\,,\gamma_3<0} |\gamma_3|[\gamma]_- \Big)\,, \tag{2.3.27}$$

and the first sum varies over the set of all the composite waves $\gamma_0 = (\gamma_0^1, \gamma, \gamma_0^3)$ present in the model. The set $\mathcal{A}$ denotes the waves γ_i approaching γ_0 at time t and $K^{1,3}_\gamma$ are suitable positive coefficients to be determined. By "approaching" we mean waves with negative speed located at the right of γ_0 or waves with positive speed located at its left. By this choice of Q, our Glimm functional

$$F = F^1 + F^3\,, \qquad F^i = L^i + Q^i\,, \quad i = 1, 3\,, \tag{2.3.28}$$

has an asymmetric character that eventually improves the conditions to impose on the initial data. Remark that, since the parameters $K^i_{\gamma_0}$ depend also on the composite waves γ_0, i.e. on the phase waves of the model, we obtain a different potential Q for each of the cases studied in Chapter 3 and 4.

The main purpose is to prove that F does not increase across interaction times and, thus, remains bounded by $F(0)$. In general, at any interaction time $t > 0$ the variation $\Delta F(t) = F(t+) - F(t-)$ can be decomposed into the sum of two terms

$$\Delta F(t) = \Delta^R F(t) + \Delta^T F(t)\,,$$

where $\Delta^R F$ refers to the quantity of wave that is reflected and is always positive, while $\Delta^T F$ is the variation of the transmitted waves and generally has no definite sign; see Figure 2.9 for a representation of both $\Delta^R F$ and $\Delta^T F$ in the two main cases of interaction, where the thick lines denote the "transmitted quantity" and the dashed ones denote the "reflected quantity". More precisely, consider an interaction occurring at a time $t > 0$.

- If the interaction is either between a composite wave and a wave of family i or between two waves of the same family i, for $i = 1, 3$, then we set

$$\Delta^R F = \Delta F^j\,, \quad j = 1, 3\,, j \neq i\,, \qquad \Delta^T F = \Delta F^i\,. \tag{2.3.29}$$

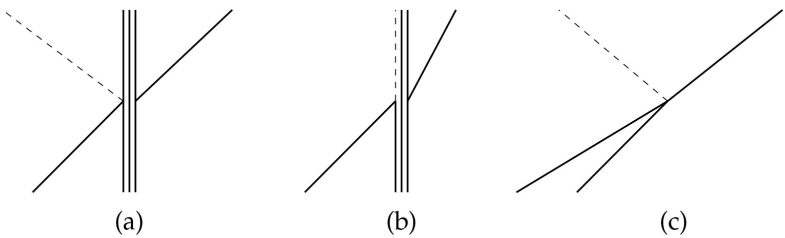

FIGURE 2.9: The variations $\Delta^R F$ and $\Delta^T F$: (a), (b) represents an interaction with a composite wave solved by the accurate and the simplified Riemann solver, respectively; (c) represents the interaction of two waves of family 3.

- If the interaction is between two waves of different family, we set

$$\Delta^R F = 0\,, \qquad \Delta^T F = \Delta F^1 + \Delta F^3\,. \tag{2.3.30}$$

Notice that in the second case it holds $\Delta^T F = \Delta F = 0$ by Lemma 2.3.4, which means that the functional remains constant across these interactions. In the first case, instead, the decrease of the functional F will follow from the following claim.

Claim 2.3.9. *For any $t > 0$, we can show that $\Delta^R F(t) \geq 0$ and under suitable assumptions on the parameters $\xi, K_\gamma^{1,3}, \rho$ it holds $\Delta^T F(t) \leq 0$. Moreover, there exists a constant $0 < \mu \leq 1$ such that*

$$[\Delta^R F(t)]_+ \leq \mu[\Delta^T F(t)]_-\,, \tag{2.3.31}$$

whence

$$\Delta F(t) = [\Delta^R F(t)]_+ - [\Delta^T F(t)]_- \leq (\mu - 1)[\Delta^T F(t)]_- \leq 0\,.$$

In the next chapters we will determine this μ and, by requiring that it is ≤ 1, we will prove a local decreasing property for F. Afterwords, we will combine together all the conditions found on the parameters and we will get the global decreasing of F. See Section 3.2.1, 4.3, 4.4 and 4.5.

Remark 2.3.10. *The functional F obtained above has been proved to provide larger bounds on the initial data in comparison with that used in [2, 10], which on the contrary includes all the possible terms in the potential. We add also that the approach of this thesis using $\Delta^R F$ and $\Delta^T F$ is not the same followed in [2, 8, 9, 10].*

2.4 The consistence of the algorithm

Finally, we discuss the consistence of the front tracking algorithm described in the previous sections. Namely, we show that for a fixed ν the algorithm gives an approximate solution U_ν defined for all $t \geq 0$ and then a subsequence of $(U_\nu)_{\nu\geq 1}$ converges to a weak entropic solution of the problem.

As mentioned in Section 1.2.1, the algorithm must satisfy three main requirements to be well-defined:

- the size of the rarefaction waves must remain small;
- the total number of wave fronts and interactions must be finite;
- the total size of non-physical waves must vanish as the approximation parameter ν tends to $+\infty$.

The last of the three requirements above is adapted to the current situation by replacing "non-physical waves" by "composite waves". We briefly sketch the proofs of the first two requirements, following Lemma 6.1, Lemma 6.2 and Proposition 6.3 of [2].

Lemma 2.4.1. *Let $i = 1, 3$ and consider a rarefaction with size $\gamma_i(t)$ at time t. Then, as long as the rarefaction exists, it holds*

$$|\gamma_i(t)| < \sigma \cdot \exp\Big(\sum_{\gamma_0} |\gamma|/2\Big), \tag{2.4.1}$$

where the sum varies over all the composite waves $\gamma_0 = (\gamma_0^1, \gamma, \gamma_0^3)$ of the model.

Proof. When the rarefaction is generated at some time $t_0 \geq 0$, one has that $0 < |\gamma_i(t_0)| < \sigma$. Then, if it interacts with a 1- or a 3-wave, the size does not increase. Indeed, the size does not change for interactions with waves of the other family (Lemma 2.3.4) and it does decrease for interactions with waves of the same family (Lemma 2.3.5). On the other hand, when γ_i interacts with a composite wave $\gamma_0 = (\gamma_0^1, \gamma, \gamma_0^3)$ at a time $t > t_0$, the strength may increase. Without loss of generality, assume $i = 1$ and recall (2.3.8), (2.3.9). If $\gamma < 0$, then the size of the outgoing rarefaction decreases; instead, if $\gamma > 0$,

$$|\gamma_1(t+)| = |\gamma_1(t-)| + |\varepsilon_3| \leq |\gamma_1(t-)| \left(1 + \frac{1}{2}|\gamma|\right) < |\gamma_1(t-)| \exp\big(|\gamma|/2\big),$$

where ε_3 denotes the reflected outgoing wave of the other family. Notice that, for rarefaction waves the above estimate holds not only when the interaction is handled with the accurate solver, but also when the simplified solver is used. Hence, we can derive (2.4.1). □

Since the model we are considering is provided with either one or two composite waves only, the quantity at the right hand-side of (2.4.1) is bounded and can be made small by choosing a suitable σ.

As for the second requirement of the list, we refer to Lemma 6.2 and Proposition 6.3 of [2]. More precisely, it suffices to prove that the number of interactions remains bounded in finite time.

Lemma 2.4.2. *Consider the front tracking algorithm described in Section 2.2. Then,*

i) the number of interactions involving a composite wave and solved by the accurate solver is finite;

ii) the number of interactions where a new rarefaction of strength $>\sigma$ arises is finite.

Sketch of the proof. Under suitable conditions on the parameters $\xi, K_\gamma^{1,3}, \rho$, one is able to show that the functional F decreases by a fixed positive amount any time an interaction of the kind described in i) or ii) occurs. Since F is decreasing and $F(0)$ is finite, this can happen only finitely many times. □

Notice that the statement of the previous lemma can be rephrased by "except for finitely many interactions, the number of outgoing fronts is always at most two". Indeed, excluding the interactions of i) and ii), all the other ones either involve only waves of family $1, 3$ or involve composite waves and require the simplified solver. Then, from an application of Lemma 1.2.4 it follows that the total number of interactions is finite.

Remark 2.4.3. *One could wonder why non-physical waves must be taken into consideration or, equivalently, why such composite waves of Definition 2.1.5 must be introduced. In particular, the question arises in the case of the single interface, where the assumption on λ_o reduces system* (1.4.4) *to two 2×2 systems of conservation laws. Nevertheless, a formal example of [5] represented in*

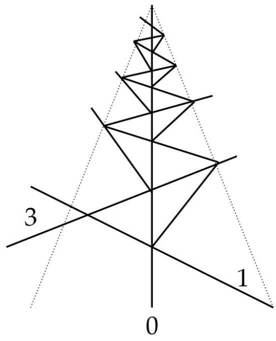

FIGURE 2.10: A case of possible interactions with a composite wave. The numbers denote the three waves giving rise to the whole interaction pattern. The dotted lines pass through the interactions points of the waves in each of the two different phases.

Figure 2.10 shows that the number of interactions might explode near a 2*-wave if we were to use exclusively the accurate Riemann solver. Therefore, even in the simplest case, composite waves (non-physical waves) seem to be unavoidable.*

To conclude the part regarding the consistence of the algorithm, we devote the next section to prove that the total strength of the composite waves vanishes as $\nu \to +\infty$. Finally, the convergence follows from a standard application of Helly's Theorem (see Theorem 2.3 of [17]).

2.4.1 The total size of the composite waves vanishes

In this section, we exploit the notion of generation order of a wave to prove that the composite waves have total strength that goes to zero as $\nu \to +\infty$, i.e. by (2.2.1) they become entropic 2-waves in the limit. For a wave γ_i of family $i = 1, 3$ we define its generation order k_{γ_i} as in Section 6.2 of [2], while for a composite wave γ_0 we proceed as follows. We assign order 1 to the middle component (which never changes) and, when dealing with the interaction of a wave γ_i of strength $< \rho$, we assign order k_{γ_i} to the γ_0^i component and order $k_{\gamma_i}+1$ to the other one. A complete and more precise definition is given below.

Definition 2.4.4 (Generation order of a wave)**.** *Let $i, j = 1, 3$, $i \neq j$. We assign a* generation order *to each wave according to the following inductive procedure. At $t = 0$, any wave takes order 1. At $t > 0$, we distinguish three cases. See Figure 2.11 and 2.12.*

- *If $\varepsilon_i, \varepsilon_j, \varepsilon_0$ are the waves produced by the interaction of a wave δ_i, of strength $|\delta_i| \geq \rho$ and order k_{δ_i}, with a composite wave $\delta_0 = (\delta_0^1, \delta, \delta_0^3)$ of order $(k_{\delta_0^1}, 1, k_{\delta_0^3})$, we set*

$$k_{\varepsilon_i} = k_{\delta_i}, \qquad k_{\varepsilon_j} = k_{\delta_i}+1, \qquad (k_{\varepsilon_0^1}, 1, k_{\varepsilon_0^3}) = (k_{\delta_0^1}, 1, k_{\delta_0^3}).$$

See Figure 2.11 (a).

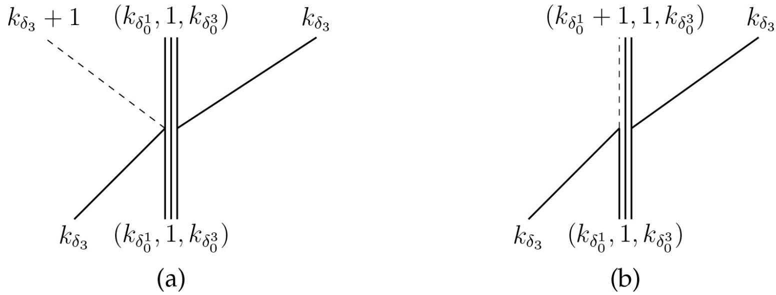

FIGURE 2.11: Generation order for interactions with a composite wave. In (a) the interaction is solved with the accurate Riemann solver, while in (b) the simplified solver is used.

- *If $\varepsilon_i, \varepsilon_0$ are the waves produced by the interaction of a wave δ_i, of strength $|\delta_i| < \rho$ and order k_{δ_i}, with a composite wave $\delta_0 = (\delta_0^1, \delta, \delta_0^3)$ of order $(k_{\delta_0^1}, 1, k_{\delta_0^3})$, we set*

$$k_{\varepsilon_i} = k_{\delta_i}, \qquad (k_{\varepsilon_0^1}, 1, k_{\varepsilon_0^3}) = \begin{cases} (k_{\delta_0^1}, 1, k_{\delta_i}+1) & \text{if } i = 1, \\ (k_{\delta_i}+1, 1, k_{\delta_0^3}) & \text{if } i = 3. \end{cases}$$

See Figure 2.11 (b).

- *If $\varepsilon_i, \varepsilon_j$ are the waves produced by the interaction of two waves of the same family α_i, β_i, we set*

$$k_{\varepsilon_i} = \min\{k_{\alpha_i}, k_{\beta_i}\}, \qquad k_{\varepsilon_j} = \max\{k_{\alpha_i}, k_{\beta_i}\} + 1.$$

See Figure 2.12 (b).

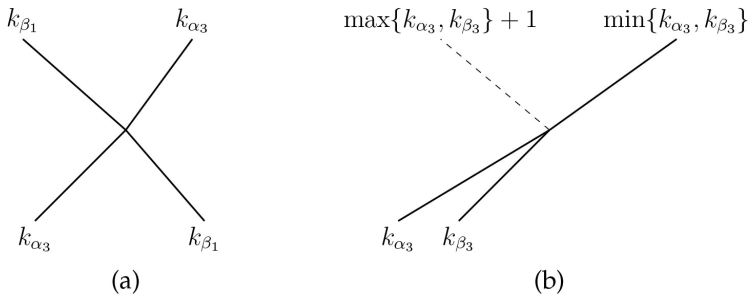

FIGURE 2.12: Generation order for interactions between waves of different family (a) and between two waves of family 3 (b).

Trivially, the generation order for two interacting waves of different family does not change across the interaction, as one can see in Figure 2.12 (a).

Now, we introduce the functionals L_k, Q_k, F_k by restricting L, Q, F to include only waves with generation order k. Let $t \geq 0$ be a time where no interaction occurs. For $\xi \geq 1$ and for any $k = 1, 2, \dots$ we define

$$L_k = L_k^1 + L_k^3,$$

where

$$L_k^i(t) := \sum_{\substack{\gamma_i > 0 \\ k_{\gamma_i} = k}} |\gamma_i| + \xi \sum_{\substack{\gamma_i < 0 \\ k_{\gamma_i} = k}} |\gamma_i| + \sum_{\gamma_0} \sum_{k_{\gamma_0^i} = k} \|\gamma_0^i(t)\|, \qquad i = 1, 3.$$

The interaction potential Q_k is given by

$$Q_k = Q_k^1 + Q_k^3,$$

where

$$Q_k^1(t) := \sum_{\gamma_0} K_\gamma^1 \Big(\sum_{\substack{\mathcal{A},\gamma_1>0\\ k_{\gamma_1}=k}} |\gamma_1\gamma| + \xi \sum_{\substack{\mathcal{A},\gamma_1<0\\ k_{\gamma_1}=k}} |\gamma_1|[\gamma]_+ \Big),$$

$$Q_k^3(t) := \sum_{\gamma_0} K_\gamma^3 \Big(\sum_{\substack{\mathcal{A},\gamma_3>0\\ k_{\gamma_3}=k}} |\gamma_3\gamma| + \xi \sum_{\substack{\mathcal{A},\gamma_3<0\\ k_{\gamma_3}=k}} |\gamma_3|[\gamma]_- \Big),$$

where $K_\gamma^{1,3}$ are the same coefficients of (2.3.26) and (2.3.27). Finally, we have

$$F_k = L_k + Q_k.$$

For $k = 1, 2, \ldots$ we consider:

- a set $\mathcal{I}_k$ of times when two waves α_i and β_i, belonging to the same family $i = 1, 3$ and satisfying $\max\{k_{\alpha_i}, k_{\beta_i}\} = k$, interact;
- a set $\mathcal{J}_k$ of times when a 1- or a 3-wave of order k interacts with a composite wave;
- a set $\mathcal{T}_k$ given by $\mathcal{I}_k \cup \mathcal{J}_k$.

Let $t \in \mathcal{T}_k$, for $k \geq 1$, and look at Figure 2.11 and 2.12. We notice that

$$\Delta^R F(t) = \Delta F_{k+1}(t), \qquad \Delta^T F(t) = \Delta F_k(t) + \sum_{h=1}^{k-1} \Delta F_h(t),$$

where $\Delta^R F$, $\Delta^T F$ are the variations introduced in Section 2.3.3. It is easy to see that:

- if $t \in \mathcal{J}_k$, we have $\Delta F_h = 0$ for $h \leq k-1$ and, hence, $\Delta^T F = \Delta F_k$;
- if $t \in \mathcal{I}_k$, we have $\Delta F_k \leq 0$, while ΔF_h has no definite sign for $h \leq k-1$.

Moreover, by Claim 2.3.9 it follows that $\Delta F_{k+1} \geq 0$ and, under suitable assumptions on $\xi, K_\gamma^{1,3}, \rho$, it follows that

$$\Delta F_k + \sum_{h=1}^{k-1} \Delta F_h \leq 0.$$

Then, by (2.3.31) we get

$$[\Delta F_{k+1}]_+ \leq \mu \left[\Delta F_k + \sum_{h=1}^{k-1} \Delta F_h \right]_- = \mu \left([\Delta F_k]_- - \sum_{h=1}^{k-1} \Delta F_h \right), \quad (2.4.2)$$

for $0 < \mu \leq 1$. Remark that the right-hand side of (2.4.2) reduces to $[\Delta F_k]_-$ when $t \in \mathcal{J}_k$ and that $\Delta F_h = 0$ for $h \geq k+2$.

Remark 2.4.5. *This line of proof leading to Claim 2.3.9 is not the same followed in [2, 8, 9, 10], where formula* (2.4.2) *was not directly inferable by the analysis on the decrease of F.*

For later convenience, we suitably restrict the conditions required on the parameters to get $\mu < 1$ (see Proposition 2.4.6) and we proceed as in Proposition 6.7 of [2] to obtain a recursive estimate for F_k. Remark that the functional F_k increases at times $\tau \in \mathcal{T}_{k-1}$, it decreases at $\tau \in \mathcal{T}_k$, while it has no definite sign for times $\tau \in \mathcal{T}_h$ with $h \geq k+1$. For F_1 we have

$$F_1(t) = F_1(0) - \sum_{\mathcal{T}_1} [\Delta F_1]_- + \sum_{h>1} \sum_{\mathcal{T}_h} \Delta F_1, \quad (2.4.3)$$

while for F_k, $k \geq 2$, we use the fact that $F_k(0) = 0$ to obtain

$$F_k(t) = \sum_{\mathcal{T}_{k-1}} [\Delta F_k]_+ - \sum_{\mathcal{T}_k} [\Delta F_k]_- + \sum_{h>k} \sum_{\mathcal{T}_h} \Delta F_k. \quad (2.4.4)$$

Here we assume that the summation index varies over interaction times $\tau < t$. Now, we consider the last term in (2.4.3), (2.4.4):

$$\sum_{h>k} \sum_{\mathcal{T}_h} \Delta F_k, \qquad k \geq 1.$$

This term is not zero (possibly positive) only if the interaction involves two waves of the same family, one of order k and the other one of order h, with $h > k$. We denote by $\mathcal{T}_{h,k}$ the set of times at which an interaction of this kind occurs. Clearly, $\mathcal{T}_{h,k} \subset \mathcal{T}_h$. Moreover, we define the quantity

$$\alpha_k(t) = \sum_{\tau \in \mathcal{T}_{k-1}} [\Delta F_k(\tau)]_+, \qquad k \geq 2, \tag{2.4.5}$$

that appears in (2.4.4), too. Hence, we rewrite (2.4.3), (2.4.4) respectively as

$$0 \leq F_1(t) = F_1(0) - \sum_{\mathcal{T}_1} [\Delta F_1]_- + \sum_{h>1} \sum_{\mathcal{T}_{h,1}} \Delta F_1, \tag{2.4.6}$$

$$0 \leq F_k(t) = \alpha_k - \sum_{\mathcal{T}_k} [\Delta F_k]_- + \sum_{h>k} \sum_{\mathcal{T}_{h,k}} \Delta F_k, \qquad k \geq 2. \tag{2.4.7}$$

Proposition 2.4.6. *For $k \geq 2$ it holds*

$$\alpha_k \leq \mu^{k-1} F_1(0) + \sum_{h \geq k} \sum_{\ell=1}^{k-1} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell. \tag{2.4.8}$$

Proof. For $k = 2$, we use (2.4.2) and the positivity of F_1 to get

$$\begin{aligned} \alpha_2 = \sum_{\mathcal{T}_1} [\Delta F_2]_+ \ \leq \ \mu \sum_{\mathcal{T}_1} [\Delta F_1]_- \ \leq \ \mu \left(F_1(0) + \sum_{h>1} \sum_{\mathcal{T}_{h,1}} \Delta F_1 \right) \\ \leq \mu F_1(0) + \sum_{h \geq 2} \sum_{\mathcal{T}_{h,1}} \Delta F_1, \end{aligned}$$

which is (2.4.8) for $k = 2$.

By induction, assume that (2.4.8) holds for some $k \geq 2$. Since $F_k \geq 0$, from (2.4.7) we get

$$\sum_{\mathcal{T}_k} [\Delta F_k]_- \leq \alpha_k \ + \sum_{h>k} \sum_{\mathcal{T}_{h,k}} \Delta F_k.$$

Now, by (2.4.5), (2.4.2) and by the previous inequality we find that

$$\begin{aligned}\alpha_{k+1} = \sum_{\mathcal{T}_k}[\Delta F_{k+1}]_+ &\le \mu \sum_{\mathcal{T}_k}[\Delta F_k]_- \,-\, \mu \sum_{\ell<k}\sum_{\mathcal{T}_{k,\ell}} \Delta F_\ell \\ &\le \mu\alpha_k \,+\, \mu \sum_{h>k}\sum_{\mathcal{T}_{h,k}} \Delta F_k \,-\, \mu \sum_{\ell<k}\sum_{\mathcal{T}_{k,\ell}} \Delta F_\ell .\end{aligned}$$

Using the induction hypothesis (2.4.8), we get

$$\alpha_{k+1} \le \mu^k F_1(0) + \mu \underbrace{\sum_{\substack{h,\ell \\ h\ge k>\ell}} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell}_{(I)} + \mu \sum_{h>k}\sum_{\mathcal{T}_{h,k}} \Delta F_k \,-\, \mu \underbrace{\sum_{\ell<k}\sum_{\mathcal{T}_{k,\ell}} \Delta F_\ell}_{(II)} .$$

Notice that

$$(I) = (II) + \sum_{\substack{h,\ell \\ h>k>\ell}} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell ,$$

so that

$$\begin{aligned}\alpha_{k+1} &\le \mu^k F_1(0) + \mu \sum_{\substack{h,\ell \\ h>k>\ell}} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell \,+\, \mu \sum_{h>k}\sum_{\mathcal{T}_{h,k}} \Delta F_k \\ &= \mu^k F_1(0) + \mu \sum_{\substack{h,\ell \\ h>k\ge\ell}} \sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell ,\end{aligned}$$

whence we deduce (2.4.8) for $k+1$, since $\mu < 1$. □

Proposition 2.4.7. *For $k \ge 2$ it holds*

$$\widetilde{F}_k(t) := \sum_{j\ge k} F_j(t) \le \mu^{k-1} F_1(0). \tag{2.4.9}$$

Proof. For $k \ge 2$ we have $\widetilde{F}_k(0) = 0$. Moreover, we deduce also:

- $\Delta\widetilde{F}_k(\tau) = 0$ for $\tau \in \mathcal{T}_h$, $h \le k-2$;
- $\Delta\widetilde{F}_k(\tau) = \Delta F_k(\tau) > 0$ for $\tau \in \mathcal{T}_{k-1}$;

- for all $\tau \in \mathcal{T}_h$, $h \geq k$,

$$\Delta\widetilde{F}_k(\tau) \leq -\sum_{\ell=1}^{k-1} \Delta F_\ell(\tau),$$

  since $\Delta F(\tau) \leq 0$ (under suitable assuptions).

As a consequence of the above properties, by (2.4.5) and (2.4.8) we get

$$\begin{aligned}\widetilde{F}_k(t) &= \alpha_k + \sum_{h\geq k}\sum_{\mathcal{T}_h} \Delta\widetilde{F}_k \\ &\leq \mu^{k-1}F_1(0) + \sum_{h\geq k}\sum_{\ell=1}^{k-1}\sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell - \sum_{h\geq k}\sum_{\ell=1}^{k-1}\sum_{\mathcal{T}_{h,\ell}} \Delta F_\ell \;=\; \mu^{k-1}F_1(0)\,.\end{aligned}$$

□

Finally, we are able to prove the vanishing of the the total strength of the composite waves as follows. Since in the model considered there is a finite number of composite waves, it suffices to focus on just one of them, say $\delta_0 = (\delta_0^1, \delta, \delta_0^3)$. Once all the parameters $\xi, K_\gamma^{1,3}, \sigma$ (with $\sigma = \sigma_\nu \to 0+$ as $\nu \to \infty$) have been chosen, we fix $k > 1$ and estimate the total number of waves of order $< k$. Then, the strength of δ_0 at a time t can be bounded by

$$|\delta_0(t)| = |\delta_0(t)|\restriction_{\{\geq k\}} + |\delta_0(t)|\restriction_{\{<k\}} \leq \widetilde{F}_k(t) + |\delta_0(t)|\restriction_{\{<k\}},$$

where $|\delta_0(t)|\restriction_{\{\gtreqless k\}}$ is the sum of all the terms $|\Delta\delta_0^i(\tau)|$ for $\tau < t$ and $i = 1, 3$, that have generation order $\gtreqless k$ (see (2.2.1)). Hence, by Proposition 2.4.7 we have

$$|\delta_0(t)| \leq \mu^{k-1} \cdot F(0) + C_o\rho\,|\gamma| \cdot [\text{number of fronts of order} < k],$$

which is $< 1/\nu$ by choosing k sufficiently large to have the first term $\leq 1/(2\nu)$ and, then, $\rho = \rho_\nu$ small enough to have also the second term $\leq 1/(2\nu)$. Now, the consistence of the algorithm is completely proved.

Chapter 3

The single phase wave

In this chapter we study the initial-value problem for (1.4.4), (1.4.5) in the single phase wave case, i.e. when λ_o is piecewise constant with a single jump

$$\lambda_o(x) = \begin{cases} \lambda_\ell & \text{if } x < 0, \\ \lambda_r & \text{if } x > 0, \end{cases}$$

as in (1.4.19). In the first section we state the main theorem and make some comments. In Section 3.2 we specify the Glimm functional and show that it is decreasing along the approximate solutions provided that some conditions are verified. Finally, in Section 3.3 we prove the existence theorem by translating the conditions found on the parameters into hypotheses on the initial data.

The content of this chapter comes from [10], although several changes were made. Two of the main novelties consist in the adoption of an asymmetric functional F of the kind defined in (2.3.28) and in the proof of its decrease relying on the study of the variations $\Delta^R F$ and $\Delta^T F$.

3.1 Main result

First, we set $a_r = a(\lambda_r)$, $a_\ell = a(\lambda_\ell)$ and define

$$\delta := 2\,\frac{a_r - a_\ell}{a_r + a_\ell}. \tag{3.1.1}$$

Notice that δ ranges over $]-2,2[$ for a_r, a_ℓ positive. As in [2], the quantity δ measures the size of the contact discontinuity located at $x=0$ at the outset of the front tracking algorithm, which is referred to as the *phase wave*. We denote by

$$\delta_0 = (\delta_0^1, \delta, \delta_0^3)$$

the composite wave originating from δ, see Definition 2.1.5. Clearly, at the beginning we have $\delta_0 = (0, \delta, 0)$.

Denote $p_o(x) = p\left(v_o(x), \lambda_o(x)\right)$ and recall the notation introduced in (1.4.21). Below we state the main result of the chapter.

Theorem 3.1.1. *Assume* (1.4.2) *and consider initial data* (1.4.5) *satisfying* (1.4.19) *and* $v_o(x) \geq \underline{v} > 0$*, for some constant* $\underline{v}$*. Let* δ *be as in* (3.1.1).

There exists a strictly decreasing function $\mathcal{K}$ *defined by*

$$\mathcal{K}(r) := \frac{2}{1+r}\log\left(\frac{2}{r}+1+\frac{2}{r}\sqrt{1+r}\right), \qquad r \in]0,2[\,, \tag{3.1.2}$$

such that, if $\delta \neq 0$ *and the initial data satisfy*

$$\frac{1}{1+[\delta]_+}\mathop{\mathrm{TV}}_{x<0}\left(\log(p_o), \frac{u_o}{a_\ell}\right) + \frac{1}{1+[\delta]_-}\mathop{\mathrm{TV}}_{x>0}\left(\log(p_o), \frac{u_o}{a_r}\right) < \mathcal{K}(|\delta|), \tag{3.1.3}$$

then the Cauchy problem (1.4.4), (1.4.5), (1.4.19) *has a weak entropic solution* (v,u,λ) *defined for* $t \in [0,+\infty[$*. If* $\delta = 0$*, the same conclusion holds with* $\mathcal{K}(|\delta|)$ *replaced by* $+\infty$ *in* (3.1.3).

Moreover, the solution is valued in a compact set of Ω *and there is a constant* $C = C(\delta)$ *such that for every* $t \in [0,+\infty[$ *we have*

$$\mathrm{TV}\left(v(\cdot,t), u(\cdot,t)\right) \leq C. \tag{3.1.4}$$

Remark that condition (3.1.3) is *explicit*, differently from other analogous results of global existence for large data such as [27, 28, 34]. We also observe that (3.1.3) is trivially satisfied if

$$\frac{1}{1+[\delta]_+}\mathop{\mathrm{TV}}_{x<0}\left(\log(p_o), \frac{u_o}{a_\ell}\right) + \frac{1}{1+[\delta]_-}\mathop{\mathrm{TV}}_{x>0}\left(\log(p_o), \frac{u_o}{a_r}\right) \leq \frac{2}{3}\log\left(2+\sqrt{3}\right), \tag{3.1.5}$$

since

$$\lim_{r\to 0+} \mathcal{K}(r) = +\infty, \qquad \lim_{r\to 2-} \mathcal{K}(r) = \frac{2}{3}\log(2+\sqrt{3}).$$

Then, the Cauchy problem (1.4.4), (1.4.5), (1.4.19) has a global solution if (3.1.5) is verified and $v_o(x) \geq \underline{v} > 0$ holds. This is a striking difference in comparison with the results of [2, 4], where the corresponding bound on the right-hand side vanishes at a critical threshold. Moreover, Theorem 3.1.1 improves the existence theorem of [2] when restricted to the case of a single contact discontinuity; we refer to Section 3.3.1 for related comments.

Remark 3.1.2. *If* $\delta = 0$, *i.e.* $a_r = a_\ell$, *then the initial data* (1.4.19) *reduce* (1.4.4) *to a* p*-system where the pressure* p *only depends on* v*. In this case, the results of [2, 7] hold and we recover Theorem 1.3.1.*

3.2 Interactions

In this section, we use the tools and the estimates discussed in Chapter 2 to analyze interactions between waves. First, we introduce some notation and we specify the Glimm functional used to prove the boundedness of the total variation of the approximate solutions.

Without loss of generality, we will assume $\delta > 0$ for the rest of the chapter. Recall the content of Section 2.3.3 and, in particular, the functionals (2.3.22), (2.3.26) and (2.3.27). For t not an interaction time and $\xi \geq 1$, K^ℓ, K^r suitable positive parameters, we define

$$L(t) = L^1(t) + L^3(t), \qquad L^i(t) = \sum_{\gamma_i>0} |\gamma_i| + \xi \sum_{\gamma_i<0} |\gamma_i| + \|\delta_0^i(t)\|, \quad i = 1, 3,$$

and

$$Q(t) = Q^1(t) + Q^3(t),$$

$$Q^1(t) = K^r \Big(\sum_{\substack{x>0 \\ \gamma_1>0}} |\gamma_1 \delta| + \xi \sum_{\substack{x>0 \\ \gamma_1<0}} |\gamma_1 \delta| \Big), \qquad Q^3(t) = K^\ell \sum_{\substack{x<0 \\ \gamma_3>0}} |\gamma_3 \delta|.$$

Notice that the summation in Q^1 is performed over the set of 1-waves approaching the composite wave from the right, while Q^3 refers to the 3-waves approaching δ_0 from the left and does not include 3-shocks. Moreover, we use the indexes ℓ, r instead of $i = 1, 3$ for the coefficients

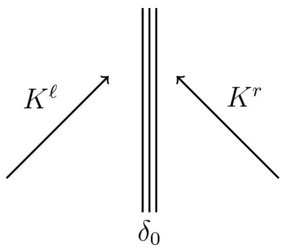

FIGURE 3.1: The parameters K^ℓ and K^r refer to the side from which waves approach δ_0.

of Q to keep track of the side from which a wave approach δ_0, see Figure 3.1: this choice will make sense at the end in the proof of Proposition 3.3.1. The Glimm functional F is, then, given by $F = F^1 + F^3$ as in (2.3.28).

Finally, in order to prove Claim 2.3.9 and verify that F is decreasing, we also introduce

$$M_1 = \left\{ \frac{1}{\xi}, \frac{1}{2K^r - 1}, \frac{\xi}{1 + 2K^\ell}, \frac{C_o}{\xi(2K^r - C_o)} \right\},$$

$$M_2 = \left\{ \frac{K^\ell|\delta| + 1}{\xi}, \frac{K^r|\delta| + 1}{\xi} \right\},$$

and in the sequel we separately derive (2.3.31) for μ_1 and μ_2, where

$$\mu_1 = \max M_1 , \qquad \mu_2 = \max M_2. \tag{3.2.1}$$

3.2.1 Interactions with the composite wave

Here we collect all the estimates concerning the composite wave. Recall the variations $\Delta^R F$ and $\Delta^T F$ defined at (2.3.29). In particular, consider the interaction between δ_0 and a wave of family $i = 1, 3$ occurring at a time $\bar{t}$, as in Figure 3.2 and 3.3. Then, we have

$$\Delta^R F(\bar{t}) = \Delta L^j(\bar{t}) + \Delta Q^j(\bar{t}), \quad j = 1, 3,\ j \neq i, \quad \Delta^T F(\bar{t}) = \Delta L^i(\bar{t}) + \Delta Q^i(\bar{t}).$$

Moreover, by Table 2.1 of Chapter 2 the four possible interaction patterns obtained with the accurate solver (in the case $\delta > 0$) are the following:

$$\begin{array}{lcllcl} 0 \times 1R & \to & 1R + 0 + 3R, & \quad 0 \times 1S & \to & 1S + 0 + 3S, \\ 3R \times 0 & \to & 1S + 0 + 3R, & \quad 3S \times 0 & \to & 1R + 0 + 3S. \end{array}$$

Proposition 3.2.1. *Let a wave δ_i of family $i = 1, 3$ interact with the composite wave δ_0 at some time $\bar{t} > 0$. Then, we have*

$$[\Delta^R F]_+ \leq \mu_1 [\Delta^T F]_-,$$

provided that

$$K^r > \frac{C_o}{2}, \tag{3.2.2}$$

where μ_1 is defined in (3.2.1) *and $C_o > 1$ is the coefficient introduced in* (2.3.10).

Proof. As usual we denote by $\varepsilon_1, \varepsilon_3$ the outgoing waves of family $1, 3$ and recall Table 2.1. By $(2.3.1)_1$ and (2.3.2), we have

$$\begin{cases} \varepsilon_1 - \delta_1 = \varepsilon_3, & |\varepsilon_1| - |\delta_1| = |\varepsilon_3|, & \text{if } i = 1, \\ \varepsilon_3 - \delta_3 = \varepsilon_1, & |\varepsilon_3| - |\delta_3| = -|\varepsilon_1|, & \text{if } i = 3. \end{cases}$$

$\boxed{\text{Case } i = 1}$ See Figure 3.2. If the interacting wave is a rarefaction, then by (2.3.8) and (2.3.9) we get

$$[\Delta^R F]_+ = |\varepsilon_3| \leq \frac{1}{2} |\delta_1 \delta|,$$

while

$$\Delta^T F = |\varepsilon_1| - |\delta_1| - K^r |\delta_1 \delta| = |\varepsilon_3| - K^r |\delta_1 \delta| \leq \frac{1}{2} |\delta_1 \delta| \left(1 - 2K^r\right).$$

Since $C_o > 1$, (3.2.2) implies $\Delta^T F < 0$. Hence, $[\Delta^T F]_- \geq (2K^r - 1)|\delta_1 \delta|/2$ and

$$[\Delta^R F]_+ \leq \frac{1}{2}|\delta_1 \delta| \leq \frac{1}{2K^r - 1}[\Delta^T F]_- \leq \mu_1 [\Delta^T F]_-.$$

Instead, if the interacting wave is a shock, then by (2.3.8) and (2.3.9) it holds

$$[\Delta^R F]_+ = \begin{cases} \xi|\varepsilon_3| \leq \dfrac{\xi}{2}|\delta_1 \delta| & \text{if } |\delta_1| \geq \rho, \\ |\varepsilon_3| \leq \dfrac{C_o}{2}|\delta_1 \delta| & \text{if } |\delta_1| < \rho, \end{cases}$$

and

$$\begin{aligned} \Delta^T F =& \xi|\varepsilon_1| - \xi|\delta_1| - K^r \xi|\delta_1 \delta| \\ =& \xi|\varepsilon_3| - K^r \xi|\delta_1 \delta| \leq \begin{cases} \dfrac{\xi}{2}|\delta_1 \delta|(1 - 2K^r) & \text{if } |\delta_1| \geq \rho, \\ \dfrac{\xi}{2}|\delta_1 \delta|(C_o - 2K^r) & \text{if } |\delta_1| < \rho. \end{cases} \end{aligned}$$

Hence, by (3.2.2) we get $\Delta^T F \leq 0$ and

$$[\Delta^R F]_+ \leq \begin{cases} \dfrac{\xi}{2}|\delta_1 \delta| \leq \dfrac{1}{2K^r - 1}[\Delta^T F]_- & \text{if } |\delta_1| \geq \rho, \\ \dfrac{C_o}{2}|\delta_1 \delta| \leq \dfrac{C_o}{\xi(2K^r - C_o)}[\Delta^T F]_- & \text{if } |\delta_1| > \rho, \end{cases}$$

i.e. $[\Delta^R F]_+ \leq \mu_1 [\Delta^T F]_-$.

$\boxed{\text{Case } i = 3}$ See Figure 3.3. If the interacting wave is a rarefaction, then

$$\Delta^R F = \begin{cases} \xi|\varepsilon_1| & \text{if } |\delta_3| \geq \rho, \\ |\varepsilon_1| & \text{if } |\delta_3| < \rho, \end{cases}$$

and by (2.3.8), (2.3.9) we have

$$\Delta^T F = |\varepsilon_3| - |\delta_3| - K^\ell |\delta_3 \delta| = -|\varepsilon_1| - K^\ell |\delta_3 \delta| \leq -(1 + 2K^\ell)|\varepsilon_1|.$$

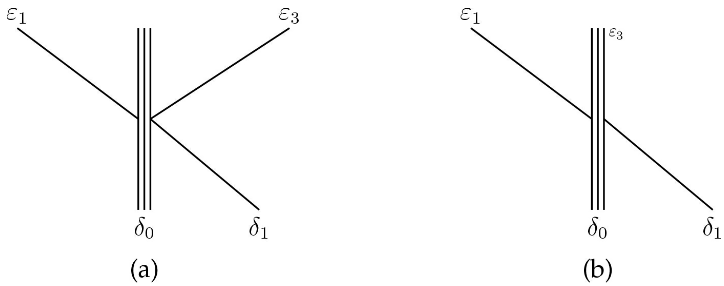

FIGURE 3.2: An interaction of a wave of family 1 with δ_0 solved with the accurate solver (a) and with the simplified solver (b).

Then, $[\Delta^T F]_- \geq (1+2K^\ell)|\varepsilon_1|$ and

$$\Delta^R F \leq \xi|\varepsilon_1| \leq \frac{\xi}{1+2K^\ell}[\Delta^T F]_- \leq \mu_1[\Delta^T F]_-.$$

Instead, if the interacting wave is a shock, we get $[\Delta^R F]_+ = |\varepsilon_1|$ and $\Delta^T F = \xi|\varepsilon_3| - \xi|\delta_3| = -\xi|\varepsilon_1|$. Thus,

$$[\Delta^R F]_+ = |\varepsilon_1| = \frac{1}{\xi}[\Delta^T F]_- \leq \mu_1[\Delta^T F]_-.$$

□

Corollary 3.2.2. *The Glimm functional F is non-increasing across time $\bar{t}$ if the following conditions hold:*

$$\xi \geq 1, \qquad K^r \geq 1, \qquad K^\ell \geq \frac{\xi-1}{2}, \qquad C_o \leq \frac{2K^r\xi}{\xi+1}. \tag{3.2.3}$$

Proof. Remark that $(3.2.3)_4$ implies also (3.2.2). Moreover, by (3.2.3) it holds $\mu_1 \leq 1$ and, thus, we can infer

$$\Delta F = [\Delta^R F]_+ - [\Delta^T F]_- \leq (\mu_1 - 1)[\Delta^T F]_- \leq 0.$$

□

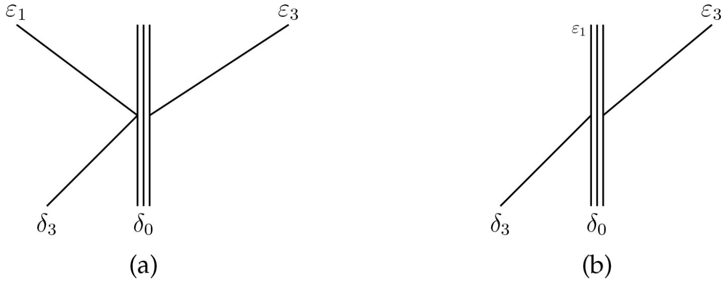

FIGURE 3.3: An interaction of a wave of family 3 with δ_0 solved with the accurate solver (a) and with the simplified solver (b).

3.2.2 Interactions between waves of family 1 and 3

In this section we analyze interactions between waves of the same family $i = 1, 3$ taking place entirely in either $\{x < 0\}$ or $\{x > 0\}$. Thus, by (2.3.29) we have

$$\Delta^R F = \Delta L^j + \Delta Q^j, \qquad j = 1, 3, \, j \neq i, \qquad \Delta^T F = \Delta L^i + \Delta Q^i,$$

and by Table 2.2 and (2.3.12) we have

$$iS \times iS \to jR + iS \qquad : \qquad |\varepsilon_i| - |\alpha_i| - |\beta_i| = -|\varepsilon_j|, \tag{3.2.4}$$

$$iS \times iR \to jS + iR \quad (\alpha_i < 0 < \beta_i) \; : \qquad |\varepsilon_i| - |\beta_i| = -|\varepsilon_j| - |\alpha_i|, \tag{3.2.5}$$

$$iS \times iR \to jS + iS \quad (\alpha_i < 0 < \beta_i) \; : \qquad |\varepsilon_i| - |\alpha_i| = |\varepsilon_j| - |\beta_i|. \tag{3.2.6}$$

We also recall (2.3.15) for the definition of the coefficient c and the important assumption (2.3.14), by which we require that any shock wave must have strength bounded by a parameter $m_o > 0$.

Proposition 3.2.3. *Consider the interaction of two waves α_i, β_i of the same family $i = 1, 3$ at a time $\bar{t}$ giving rise to ε_i and ε_j, $j = 1, 3$, $j \neq i$. If we assume*

(2.3.14) *and*

$$1 \leq \xi \leq \frac{1}{c(m_o)}, \tag{3.2.7}$$

then we have

$$-\min\{|\alpha_i|, |\beta_i|\} \leq -\xi|\varepsilon_j|. \tag{3.2.8}$$

Moreover, it holds

$$[\Delta^R F]_+ \leq \mu_2[\Delta^T F]_-,$$

where μ_2 *is defined in* (3.2.1).

Proof. Together with (3.2.4)–(3.2.6), the estimate (3.2.8) is essential to prove the second statement. Notice that by (2.3.20) and (3.2.7) we have

$$|\varepsilon_j| \leq c(m_o)\min\{|\alpha_i|, |\beta_i|\} \leq \frac{1}{\xi}\min\{|\alpha_i|, |\beta_i|\},$$

of which (3.2.8) is an easy consequence.

To get (2.3.31) for μ_2, we distinguish between the case where the two interacting waves are both shocks ($\alpha_i, \beta_i < 0$) and the case where they are of different type, say α_i shock and β_i rarefaction ($\alpha_i < 0 < \beta_i$).

$\boxed{\text{Case } \alpha_i, \beta_i < 0}$ We have

$$\Delta^R F = |\varepsilon_j| P^R, \qquad \Delta^T F = \xi(|\varepsilon_i| - |\alpha_i| - |\beta_i|)P^T,$$

where $P^R = P^R(|\delta|, K^{\ell,r})$ and $P^T = P^T(|\delta|, K^{\ell,r})$ are suitable positive polynomials. Notice that by (3.2.4) the variation $\Delta^T F$ is rewritten as $\Delta^T F = -\xi|\varepsilon_j|\, P^T$. In particular, let $x_{\alpha_i}, x_{\beta_i}$ denote the locations of α_i and β_i. Then,

$$P^R = \begin{cases} 1 + K^\ell|\delta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K^r|\delta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} > 0, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$P^T = \begin{cases} 1 + K^r|\delta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since it holds $P^T \geq 1$, in all the cases we have $[\Delta^T F]_- \geq \xi|\varepsilon_j|$. Hence,

$$[\Delta^R F]_+ = |\varepsilon_j| P^R \leq \frac{P^R}{\xi}[\Delta^T F]_- \leq \mu_2 [\Delta^T F]_-.$$

$\boxed{\text{Case } \alpha_i < 0 < \beta_i}$ We have

$$\Delta^R F = \xi|\varepsilon_j| P^R, \qquad \Delta^T F = \begin{cases} (|\varepsilon_i| - |\beta_i|)\, P_1^T - \xi|\alpha_i| P_2^T & \text{if } \varepsilon_i > 0, \\ \xi(|\varepsilon_i| - |\alpha_i|) P_2^T - |\beta_i|\, P_1^T & \text{if } \varepsilon_i < 0, \end{cases}$$

for suitable polynomials $P^R = P^R(|\delta|, K^{\ell,r})$ and $P_{1,2}^T = P_{1,2}^T(|\delta|, K^{\ell,r})$. Now, notice that by (3.2.5) and (3.2.6) $\Delta^T F$ can be rewritten as

$$\Delta^T F = \begin{cases} (-|\varepsilon_j| - |\alpha_i|)\, P_1^T - \xi|\alpha_i| P_2^T & \text{if } \varepsilon_i > 0, \\ \xi(|\varepsilon_j| - |\beta_i|) P_2^T - |\beta_i|\, P_1^T & \text{if } \varepsilon_i < 0. \end{cases}$$

In particular, we have

$$P^R = \begin{cases} 1 + K^r|\delta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} > 0, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$P_1^T = \begin{cases} 1 + K^\ell|\delta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K^r|\delta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} > 0, \\ 1 & \text{otherwise,} \end{cases}$$

$$P_2^T = \begin{cases} 1 + K^r|\delta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Since it holds $P_{1,2}^T \geq 1$, by (3.2.8) we have

$$\Delta^T F \leq \begin{cases} (-|\varepsilon_j| - |\alpha_i|) - \xi|\alpha_i| \leq -\xi|\alpha_i| & \text{if } \varepsilon_i > 0, \\ \xi(|\varepsilon_j| - |\beta_i|) - |\beta_i| \leq |\beta_i| - \xi|\beta_i| - |\beta_i| \leq -\xi|\beta_i| & \text{if } \varepsilon_i < 0. \end{cases}$$

Hence, again by (3.2.8) we get $\Delta^T F \leq -\xi^2|\varepsilon_3|$ in both cases and, consequently,

$$[\Delta^R F]_+ = \xi|\varepsilon_j|P^R \leq \frac{P^R}{\xi}[\Delta^T F]_- \leq \mu_2[\Delta^T F]_-.$$

□

Corollary 3.2.4. *Under the assumptions of the previous proposition, the Glimm functional F is non-increasing across time $\bar{t}$ if the following conditions hold for $\xi > 1$:*

$$K^\ell \leq \frac{\xi - 1}{|\delta|}, \qquad K^r \leq \frac{\xi - 1}{|\delta|}. \tag{3.2.9}$$

Proof. By (3.2.7) and (3.2.9) we have that $\mu_2 \leq 1$, whence it follows

$$\Delta F = [\Delta^R F]_+ - [\Delta^T F]_- \leq (\mu_2 - 1)\,[\Delta^T F]_- \leq 0.$$

□

3.3 Proof of Theorem 3.1.1 and final comments

Assume that (3.2.3) and (3.2.9) hold with strict inequalities. This is required in order that

$$\max\{\mu_1, \mu_2\} < 1,$$

which is fundamental in the analysis on the vanishing of the strength of δ_0, as explained in Section 2.4.

Now, we focus on the choice of the parameters. By $(3.2.3)_{2,3}$ and (3.2.9) the coefficients K^r, K^ℓ must satisfy

$$1 < K^r < \frac{\xi - 1}{|\delta|}, \tag{3.3.1}$$

$$\frac{\xi - 1}{2} < K^\ell < \frac{\xi - 1}{|\delta|}. \tag{3.3.2}$$

The parameter K^ℓ can always be chosen in the interval given by (3.3.2) since $|\delta| < 2$; while K^r can be chosen in (3.3.1) only if $1 + |\delta| < \xi$. Hence, by (3.2.7) we require that ξ satisfies

$$1 + |\delta| < \xi \leq \frac{1}{c(m_o)}. \tag{3.3.3}$$

In turn, this is possible if

$$c(m_o) < \frac{1}{1 + |\delta|}. \tag{3.3.4}$$

Notice that (3.3.4) is certainly satisfied if $c(m_o) < 1/3$ because $|\delta| < 2$.

Therefore, the parameters m_o, ξ, K^r, K^ℓ and ρ are taken as follows.

i) We determine the maximum size m_o of the waves in the approximate solution by (3.3.4) (c is invertible since it is strictly increasing).

ii) We choose ξ in the non-empty interval defined by (3.3.3) and then K^r in that defined by (3.3.1).

iii) Finally, we choose K^ℓ as in (3.3.2) and ρ satisfying

$$C_o(\rho) < \frac{2K^r\xi}{\xi + 1}. \tag{3.3.5}$$

Proposition 3.3.1. *Let m_o, ξ, K^r, K^ℓ and ρ satisfy* (3.3.1)–(3.3.5). *Then, the following two statements are verified.*

i) Local Decreasing. *For any interaction at time $t > 0$ between two waves satisfying* (2.3.14), *it holds*

$$\Delta F(t) < 0.$$

ii) Global Decreasing. *Recall the functional defined in* (2.3.23). *If*

$$c(m_o)\,\overline{L}\restriction_{\{x<0\}}(0+) + \overline{L}\restriction_{\{x>0\}}(0+) \leq m_o c(m_o), \tag{3.3.6}$$

and the approximate solution is defined in $[0, T]$, then $F(0+) < m_o$, $\Delta F(t) < 0$ for every $t \in]0, T]$ and (2.3.14) *is satisfied.*

Proof. The local decreasing property of F has already been proved above. As for the second statement, we proceed as follows. For convenience, we use the indexes R, S to denote rarefaction and shock waves, respectively. If we restrict ourselves to consider only waves located in $\{x < 0\}$, then by (3.2.9) we have

$$\begin{aligned} F(0+) &\le L^{1S}(0+) + L^{1R}(0+) + L^{3S}(0+) + L^{3R}(0+)\left(1 + K^{\ell}|\delta|\right) \\ &< L^{1S}(0+) + L^{1R}(0+) + L^{3S}(0+) + \xi L^{3R}(0+) < \xi \overline{L}\restriction_{\{x<0\}}(0+). \end{aligned}$$

Instead, if we restrict to $\{x > 0\}$, by (3.2.9) we get

$$F(0+) \le L^{1}(0+)\left(1 + K^{r}|\delta|\right) + L^{3}(0+) < \xi^2 \overline{L}\restriction_{\{x>0\}}(0+).$$

Then, we can infer

$$F(0+) < \xi \overline{L}\restriction_{\{x<0\}}(0+) + \xi^2 \overline{L}\restriction_{\{x>0\}}(0+).$$

Now, for a fixed $t \le T$, suppose by induction that $F(\tau) \le m_o$ and $\Delta F(\tau) < 0$ for every $0 < \tau < t$ interaction time. Then, by the local decreasing property we have $\Delta F(t) < 0$. This implies

$$F(t) \le F(0+) < \xi \overline{L}\restriction_{\{x<0\}}(0+) + \xi^2 \overline{L}\restriction_{\{x>0\}}(0+).$$

Hence, by (3.2.7) and (3.3.6) the strength of a shock of family $i = 1, 3$ at time t is bounded by

$$\begin{aligned} |\delta_i| &\le \frac{1}{\xi} F(t) < \overline{L}\restriction_{\{x<0\}}(0+) + \xi \overline{L}\restriction_{\{x>0\}}(0+) \\ &< \overline{L}\restriction_{\{x<0\}}(0+) + \frac{1}{c(m_o)} \overline{L}\restriction_{\{x>0\}}(0+) \le m_o \end{aligned}$$

and we recover (2.3.14). □

In this last part we conclude the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. It only remains to reinterpret the choice of the parameter m_o in terms of the assumption (3.1.3) on the initial data. Observe that we can approximate the initial data (already satisfying the three requirements of Step 1 in Section 2.2) in such a way that the jump $\big((p_\ell, u_\ell, \lambda_\ell), (p_r, u_r, \lambda_r)\big)$ at the interface $x = 0$ is substituted by the jump related to the 2-wave connecting $(p_\ell, u_\ell, \lambda_\ell)$ to $(p_\ell, u_\ell, \lambda_r)$ and the solution to the newly generated Riemann problem at $x = 0+$ of initial states $(p_\ell, u_\ell, \lambda_r)$ and (p_r, u_r, λ_r). This is possible because p and u remain constant across the 2-wave δ. Thus, we can relate hypothesis (3.1.3) to (3.3.6) by including in $\overline{L}\restriction_{\{x>0\}}(0+)$ the total variation of p_o and u_o at $x = 0$ and by (2.1.8) we can prove that

$$\begin{aligned} \overline{L}\restriction_{\{x<0\}}(0+) &\le \frac{1}{2} \operatorname*{TV}_{x<0}\left(\log(p_o), \frac{u_o}{a_\ell}\right), \\ \overline{L}\restriction_{\{x>0\}}(0+) &\le \frac{1}{2} \operatorname*{TV}_{x>0}\left(\log(p_o), \frac{u_o}{a_r}\right). \end{aligned} \tag{3.3.7}$$

Now, for $m_o > 0$ consider the functions

$$w(m_o) := \frac{1}{c(m_o)} - 1 = \frac{2}{\cosh m_o - 1}, \qquad z(m_o) := 2\, m_o c(m_o),$$

and notice that w is strictly decreasing from $\mathbb{R}_+$ to $\mathbb{R}_+$, while z is increasing on the same sets. By (3.3.4) and (3.3.7) we have to look for a value of m_o such that the following relations hold:

$$|\delta| < w(m_o), \tag{3.3.8}$$

$$c(m_o) \operatorname*{TV}_{x<0}\left(\log(p_o), \frac{u_o}{a_\ell}\right) + \operatorname*{TV}_{x>0}\left(\log(p_o), \frac{u_o}{a_r}\right) < z(m_o). \tag{3.3.9}$$

Since $|\delta| < 2$, we restrict the choice of the parameter m_o to have $w(m_o) \in \,]0, 2[$, that is $\cosh m_o > 2$: this corresponds to consider

$$m_o > \cosh^{-1}(2) = \log\left(2 + \sqrt{3}\right).$$

Since it holds $c(w^{-1}(r)) = (1 + r)^{-1}$, we notice that

$$z\left(w^{-1}(r)\right) = \frac{2}{1+r}\, c^{-1}\left(\frac{1}{1+r}\right) = \mathcal{K}(r), \qquad r \in \,]0, 2[\,, \tag{3.3.10}$$

which can be written explicitly as in (3.1.2). Hence, if the assumption (3.1.3) is verified, we can easily choose $m_o > \cosh^{-1}(2)$ such that (3.3.8), (3.3.9) hold. Thus, Theorem 3.1.1 is completely proved. □

3.3.1 Some comparisons

It is interesting to make a comparison between Theorem 3.1.1 and the analogous existence results of [2, 10]. First, we address the main result of [2], which was proved to be equivalent to Theorem 3.1 of [4], as remarked at the end of Section 1.4.1. There, when applied to the case of a single contact discontinuity, condition (1.4.17) can be written as

$$\mathrm{TV}\left(\log(p_o), \frac{u_o}{\min\{a_r, a_\ell\}}\right) < H(|\delta|), \tag{3.3.11}$$

where $H = H(r)$ is defined as in (1.4.18) only for $r < 1/2$, i.e.

$$H(r) = 2(1-2r)k^{-1}(r)\,, \qquad k(m) = \frac{1-\sqrt{d(m)}}{2-\sqrt{d(m)}}, \tag{3.3.12}$$

and d is the damping coefficient introduced in (1.4.13).

We can immediately see that the result of Theorem 3.1.1 is new for the range $1/2 \le |\delta| < 2$ and includes the case where the phase wave may be arbitrarily large, i.e. $|\delta|$ close to 2. In order to compare (3.3.11) with (3.1.3) in the common range $|\delta| < 1/2$, we set $r = |\delta| \in]0, 1/2[$ and rewrite H as

$$H(r) = 2(1-2r)\,d^{-1}\left[\left(\frac{1-2r}{1-r}\right)^2\right].$$

Comparing this expression with (3.3.10), we see that $1/(1+r) > (1-2r)$ and

$$\frac{1}{1+r} > \left(\frac{1-2r}{1-r}\right)^2, \qquad c^{-1}\left(\frac{1}{1+r}\right) > d^{-1}\left[\left(\frac{1-2r}{1-r}\right)^2\right],$$

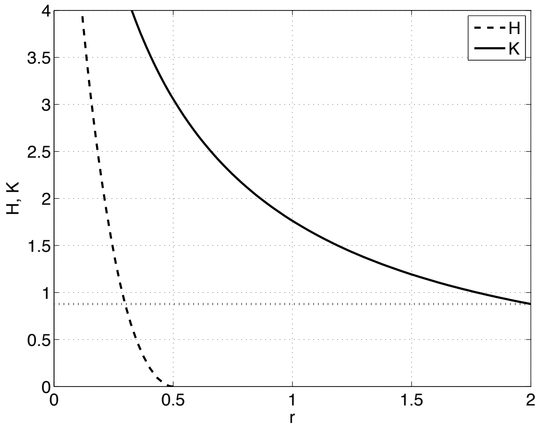

FIGURE 3.4: The function H is represented as a dashed curve and $\mathcal{K}$ as a thick curve. The horizontal dotted line gives the asymptotic value $2\log(2+\sqrt{3})/3$ of $\mathcal{K}$ as $r \to 2-$.

since $c < d$ and c is strictly increasing. We deduce that $\mathcal{K}(r) > H(r)$ for $0 \le r < 1/2$; see Figure 3.4. Then, the conditions on the initial data obtained here improve the ones required in the previous works [2, 4], albeit the latter were given for a more general case.

Finally, we briefly discuss the result of [10]. The main difference with the analysis of this chapter is the adoption in [10] of a Glimm functional that lacks the asymmetric property. More precisely, in [10] the linear functional is the same of (1.4.14), while the interaction potential differs from (1.4.15) only for the presence of the weight ξ attached to the shock waves. Moreover, in [10] no distinction between the region $\{x < 0\}$ and $\{x > 0\}$ is made. By following the same reasoning as above (except for the introduction of $\Delta^R F$ and $\Delta^T F$) in [10] one ends up with the following hypothesis analogous to (3.1.3):

$$\operatorname*{TV}_{x<0}\left(\log(p_o), \frac{u_o}{a_\ell}\right) + \operatorname*{TV}_{x>0}\left(\log(p_o), \frac{u_o}{a_r}\right) < \mathcal{K}(|\delta|). \tag{3.3.13}$$

Notice that (3.1.3) allows to differentiate the amount of total variation of the data that can be taken in the two regions of the tube separated by the interface. This is a remarkable feature that is missing in (3.3.13) and, consequently, also in Theorem 2.1 of [10].

3.3.2 Conclusions

To conclude this chapter, we observe that it is still unknown whether the global existence of solutions for the system (1.4.4) and data (1.4.5) satisfying (1.4.19) holds for any $\mathbf{BV}$ initial data v_o, u_o. We refer to the final section of Chapter 4 for a few more words on open problems and future works.

Chapter 4

The two phase waves

In this chapter we continue the analysis of system (1.4.4) by considering the case of the two phase waves, i.e. we study the initial-value problem (1.4.4), (1.4.5) with λ_o as in (1.4.20)

$$\lambda_o(x) = \begin{cases} \lambda_\ell & \text{if } x < 0, \\ \lambda_m & \text{if } 0 < x < 1, \\ \lambda_r & \text{if } x > 1. \end{cases}$$

Similarly to Chapter 3, in the following sections we specify the Glimm functional and we prove an existence result in each of the three cases: **(d) drop**, **(b) bubble** and **(p) increasing (decreasing) pressure**. The content is taken from [8, 9] and has been slightly modified to adapt to the current setting.

4.1 Main result

Under the notation $a_\ell = a(\lambda_\ell)$, $a_m = a(\lambda_m)$ and $a_r = a(\lambda_r)$, we introduce

$$\eta := 2\,\frac{a_m - a_\ell}{a_m + a_\ell}, \qquad \zeta := 2\,\frac{a_r - a_m}{a_r + a_m}.$$

The quantities η and ζ range over $]-2, 2[$ and represent the strengths of the contact discontinuities located at $x = 0$ and $x = 1$, respectively. We refer to η and ζ as the *phase waves*. Moreover, we call η_0 and ζ_0 the two composite waves originating from η and ζ,

$$\eta_0 = (\eta_0^1, \eta, \eta_0^3), \qquad \zeta_0 = (\zeta_0^1, \zeta, \zeta_0^3).$$

We consider three main configurations depending on the signs of η, ζ:

- **(d) the drop case**, $\eta < 0$ and $\zeta > 0$;
- **(b) the bubble case**, $\eta > 0$ and $\zeta < 0$;
- **(p) the increasing (decreasing) pressure case**, $\eta > 0$ and $\zeta > 0$ ($\eta < 0$ and $\zeta < 0$).

For brevity, we let $\iota = d, b, p$ refer to these three cases: $\iota = d$ for the drop case, $\iota = b$ for the bubble case and $\iota = p$ for the increasing/decreasing pressure case. In particular, by $\iota = p$ we refer only to the increasing pressure case, since the decreasing one is analogous.

In order to state the existence theorem, we have to introduce some threshold functions. First, we recall the strictly decreasing function defined in (3.1.2), i.e.

$$\mathcal{K}(r) := \frac{2}{1+r} \log\left(\frac{2}{r} + 1 + \frac{2}{r}\sqrt{1+r}\right), \qquad r \in \mathbb{R}_+, \tag{4.1.1}$$

that satisfies

$$\lim_{r\to 0^+} \mathcal{K}(r) = +\infty, \qquad \lim_{r\to+\infty} \mathcal{K}(r) = 0.$$

Then, we introduce the following continuous functions related to the *stability* of the two phase waves

$$\mathcal{H}_\iota : S_\iota \to [0, +\infty[\,, \qquad \iota = d, b, p,$$

where S_ι are suitable subsets of $[0,2[\times[0,2[$ in which $(|\eta|, |\zeta|)$ must be chosen. In particular,

$$S_d := \left\{ (|\eta|, |\zeta|) \in [0,2[\times[0,2[:\ \max\left\{ \left(1 + \frac{|\zeta|}{2}\right)\frac{|\eta|}{2}, \left(1 + \frac{|\eta|}{2}\right)\frac{|\zeta|}{2} \right\} < 1 \right\},$$

$$S_b := [0,2[\times[0,2[\,,$$

while S_p is contained in $]0,2[\times]0,2[$ and for its definition we refer to (4.5.26) since it is more complicated. See Figure 4.1 for a picture of the domains S_d and S_p. Notice that in the bubble case the pair $(|\eta|,|\zeta|)$ can vary inside the whole square $[0,2[\times[0,2[$, while in the drop and in the increasing pressure case they can cover only a portion of it. Moreover, when one of the two waves η or ζ vanishes, say $\zeta \to 0$, then these stability conditions reduce to $|\eta| < 2$, which is always true.

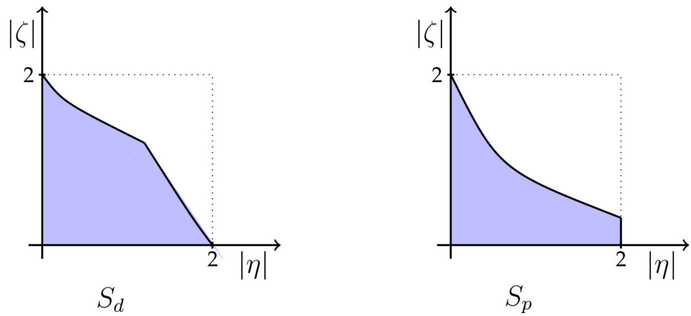

FIGURE 4.1: The domains S_d and S_p.

We define

$$\mathcal{H}_d(|\eta|,|\zeta|) := 4\max\left\{\frac{|\zeta|}{4-2|\eta|-|\eta\zeta|},\frac{|\eta|}{4-2|\zeta|-|\eta\zeta|}\right\}, \tag{4.1.2}$$

$$\mathcal{H}_b(|\eta|,|\zeta|) := \frac{4}{4-|\eta\zeta|}\max\left\{|\eta|\,\frac{2+|\zeta|}{2-|\zeta|},|\zeta|\,\frac{2+|\eta|}{2-|\eta|}\right\}, \tag{4.1.3}$$

while for the definition of $\mathcal{H}_p$ we refer to (4.5.27). For $\iota = d,b$ it holds $\mathcal{H}_\iota = 0$ only when $\eta = \zeta = 0$ and, more importantly, $\mathcal{H}_\iota(|\eta|,0) = |\eta|$ and $\mathcal{H}_\iota(0,|\zeta|) = |\zeta|$: this allows to recover the single phase wave result. Instead, for $\iota = p$ we have that $\mathcal{H}_p(|\eta|,|\zeta|) \to +\infty$ when either $|\eta| \to 0$ or $|\zeta| \to 0$. Moreover, for $\iota = d,p$ it holds $\mathcal{H}_\iota(|\eta|,|\zeta|) \to +\infty$ when $(|\eta|,|\zeta|)$ tends to the curved edges of S_ι.

We denote $p_o(x) = p(v_o(x), \lambda_o(x))$ and recall the useful notation (1.4.21). The following theorem states the global in time existence of solutions in all the three cases.

Theorem 4.1.1. *Assume* (1.4.2) *and consider initial data satisfying* (1.4.20) *and* $v_o(x) \geq \underline{v} > 0$, *for some constant* $\underline{v}$. *Let* $\iota = d, b, p$ *and* $(|\eta|, |\zeta|) \in S_\iota$. *If it holds*

$$\mathrm{TV}\left(\log(p_o), \frac{u_o}{\min\{a_\ell, a_m, a_r\}}\right) < \mathcal{K}\left(\mathcal{H}_\iota(|\eta|, |\zeta|)\right), \tag{4.1.4}$$

then the initial-value problem (1.4.4), (1.4.5), (1.4.20) *has a weak entropic solution* (v, u, λ) *defined for* $t \in [0, +\infty[$. *If* $\eta = \zeta = 0$, *the same conclusion holds with* $\mathcal{K}\left(\mathcal{H}_\iota(|\eta|, |\zeta|)\right)$ *replaced by* $+\infty$ *in* (4.1.4).

Moreover, $(v(\cdot, t), u(\cdot, t)) \in L^\infty([0, \infty[\,, \mathbf{BV}(\mathbb{R}))$ *and the solution is valued in a compact set.*

Hypothesis (4.1.4) can be read as follows: the larger are $|\eta|, |\zeta|$, the smaller must be the total variation of p_o, u_o; vice versa, the smaller are $|\eta|, |\zeta|$, the larger can be the total variation of p_o, u_o. In addition, only for the drop case condition (4.1.4) can be further improved by a localization of the total variation in each of the three intervals $\{x < 0\}$, $\{0 < x < 1\}$ and $\{x > 1\}$. Indeed, when $\iota = d$ the left-hand side of (4.1.4) can be replaced by

$$\begin{aligned}\mathop{\mathrm{TV}}_{x<0}\left(\log(p_o), \frac{u_o}{a_\ell}\right) + \frac{1}{1 + \mathcal{H}_d(|\eta|, |\zeta|)} \mathop{\mathrm{TV}}_{0<x<1}\left(\log(p_o), \frac{u_o}{a_m}\right)& \\ + \mathop{\mathrm{TV}}_{x>1}\left(\log(p_o), \frac{u_o}{a_r}\right)&,\end{aligned} \tag{4.1.5}$$

which shows that we can take a total variation of the data in the middle region multiplied by a factor < 1. In this case, Theorem 4.1.1 improves the main result of [2] not only because $\mathcal{K}$ is sharper than H of hypothesis (1.4.17) as seen at the end of Chapter 3, but also because the total variation of the initial data can be taken larger in the middle region of the tube than in the external ones. See Remark 4.6.2 and 4.6.3 for the details. The asymmetrical character of (4.1.5) is due to the particular choice of the Glimm functional which is introduced and carefully analyzed in the sequel.

We conclude this section by extracting from (4.1.4) some more information that reminds of (3.1.5). For $\iota = d, b$, we introduce the sub-level sets of $\mathcal{H}_\iota$

$$S_\iota^\kappa = \{(|\eta|, |\zeta|) \in S_\iota : \quad \mathcal{H}_\iota (|\eta|, |\zeta|) < \kappa\}, \qquad \kappa > 0.$$

See, for example, Figure 4.2 for the drop case. Since $\mathcal{K}$ is decreasing,

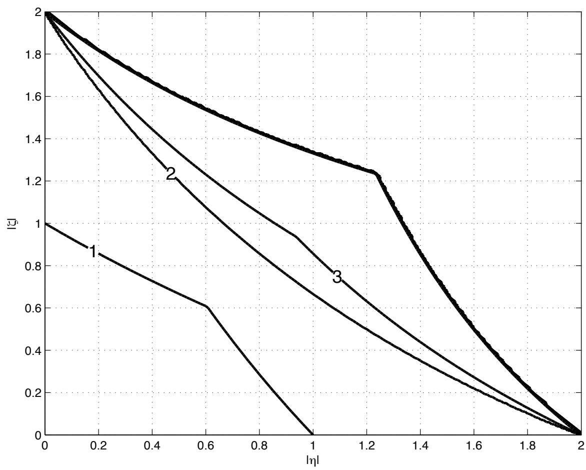

FIGURE 4.2: The sub-level sets S_d^κ for $\kappa = 1, 2, 3$.

then for every $(|\eta|, |\zeta|) \in S_\iota^\kappa$ condition (4.1.4) holds if

$$\mathrm{TV}\left(\log(p_o), \frac{u_o}{\min\{a_\ell, a_m, a_r\}}\right) < \mathcal{K}(\kappa).$$

In particular, we have $\mathcal{K}(2) = 2\log(2+\sqrt{3})/3$ and the domain S_ι^2 includes the segments $[0, 2[$ on each axis. Therefore, for $\eta = 0$ or $\zeta = 0$ we recover the same exact condition of (3.1.5).

4.2 Functionals

Here we analyze interactions between waves. For $\iota = d, b, p$, we separately study interactions that involve one of the two composite waves and interactions between 3- and 1-waves entirely occurring in one of the regions separated by the interfaces.

As in Section 2.3.3, for $t > 0$ (not an interaction time) and $\xi > 1$ parameter to be determined, we introduce the linear functional $L(t) = L^1(t) + L^3(t)$, where

$$L^i(t) = \sum_{\gamma_i > 0} |\gamma_i| + \xi \sum_{\gamma_i < 0} |\gamma_i| + \|\eta_0^i(t)\| + \|\zeta_0^i(t)\|, \quad i = 1, 3.$$

In the next sections we will also specify the interaction potential

$$Q_\iota(t) = Q_\iota^1(t) + Q_\iota^3(t), \qquad \iota = d, b, p, \tag{4.2.1}$$

where the coefficients denoted by $K_{\eta,\zeta}^{\ell,m,r}$ will keep track also of the interval $\{x < 0\}$, $\{0 < x < 1\}$ or $\{x > 1\}$, from which 1- and 3-waves approach η_0 or ζ_0; see Figure 4.3. The resulting Glimm functional $F_\iota =$

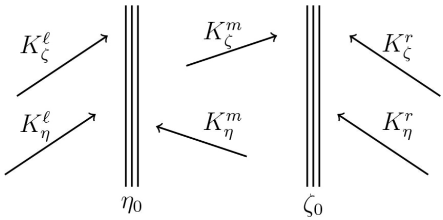

FIGURE 4.3: The parameters $K_{\eta,\zeta}^{\ell,m,r}$ refer to composite wave approached (η_0 or ζ_0) and to the region of provenience of the approaching waves ($\{x < 0\}$, $\{0 < x < 1\}$ and $\{x > 1\}$).

$L + Q_\iota$ has an asymmetrical character and decreases under certain conditions on the parameters ξ, $K_{\eta,\zeta}^{\ell,m,r}$ and ρ (which are different in the three cases considered). We recall Tables 2.1 and 2.2 for the interaction patterns and the variations $\Delta^R F$ and $\Delta^T F$ defined in (2.3.29). In particular, if an interaction occurs at a time $\bar{t} > 0$ and at least one of interacting wave is of family $i = 1, 3$, then by (2.3.29) we have

$$\Delta^R F_\iota = \Delta L^j + \Delta Q^j_\iota, \quad j = 1, 3\,, j \neq i\,, \quad \Delta^T F_\iota = \Delta L^i + \Delta Q^i_\iota.$$

Recall also that one of the parameters of the proof is $m_o > 0$, coming from the assumption that the strength of any shock wave satisfies (2.3.14).

4.3 The drop case

In this section, we look into the estimates for F_d. Recall that in this case it holds $\eta < 0$ and $\zeta > 0$. For $t > 0$ (not an interaction time) and $K^{\ell,m,r}_{\eta,\zeta}$ positive parameters to be determined, we define the phase-dependent interaction potentials of (4.2.1) as

$$Q^1_d(t) = \Big(K^m_\eta \sum_{\substack{0<x<1\\ \delta_1>0}} |\delta_1| + K^r_\eta \sum_{\substack{x>1\\ \delta_1>0}} |\delta_1|\Big)|\eta| + K^r_\zeta\Big(\sum_{\substack{x>1\\ \delta_1>0}} |\delta_1| + \xi \sum_{\substack{x>1\\ \delta_1<0}} |\delta_1|\Big)|\zeta|,$$

$$Q^3_d(t) = K^\ell_\eta\Big(\sum_{\substack{x<0\\ \delta_3>0}} |\delta_3| + \xi \sum_{\substack{x<0\\ \delta_3<0}} |\delta_3|\Big)|\eta| + \Big(K^\ell_\zeta \sum_{\substack{x<0\\ \delta_3>0}} |\delta_3| + K^m_\zeta \sum_{\substack{0<x<1\\ \delta_3>0}} |\delta_3|\Big)|\zeta|.$$

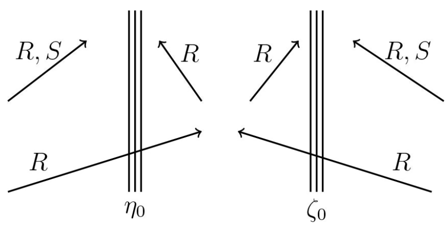

FIGURE 4.4: The terms of Q_d.

Notice that the shocks missing among the terms of Q_d are those of family 1 interacting with η_0 and those of family 3 interacting with ζ_0; see Figure 4.4.

In order to prove Claim 2.3.9, we introduce the following sets:

$$M_1^d = \left\{ \frac{1}{2K_\zeta^r - 1}, \frac{1}{2K_\eta^\ell - 1}, \frac{\xi}{1+2K_\zeta^m}, \frac{\xi}{1+2K_\eta^m}, \frac{1+K_\eta^m|\eta|}{\xi}, \frac{1+K_\zeta^m|\zeta|}{\xi}, \right.$$
$$\frac{1}{2|\eta|[K_\eta^r - K_\eta^m(1+|\zeta|/2)]/|\zeta| + (2K_\zeta^r - 1)}, \frac{C_o}{\xi(2K_\eta^\ell - C_o)},$$
$$\left. \frac{1}{2|\zeta|[K_\zeta^\ell - K_\zeta^m(1+|\eta|/2)]/|\eta| + (2K_\eta^\ell - 1)}, \frac{C_o}{\xi(2K_\zeta^r - C_o)} \right\},$$

$$M_2^d = \left\{ \frac{1+K_\eta^\ell|\eta| + K_\zeta^\ell|\zeta|}{\xi}, \frac{1+K_\eta^r|\eta| + K_\zeta^r|\zeta|}{\xi}, \frac{1+K_\eta^m|\eta|}{\xi}, \frac{1+K_\zeta^m|\zeta|}{\xi} \right\}.$$

The next step consists in finding the conditions to impose on the parameters so that (2.3.31) holds for

$$\mu_1^d = \max M_1^d, \qquad \mu_2^d = \max M_2^d. \tag{4.3.1}$$

Proposition 4.3.1 (Interactions with the composite waves)**.** *Assume that at a time $\bar t > 0$ a wave δ_i, $i = 1, 3$, interacts with one of the composite waves η_0 or ζ_0. Then, we have*

$$[\Delta^R F_d]_+ \le \mu_1^d \, [\Delta^T F_d]_-,$$

provided that the following conditions hold:

$$\min\left\{K_\zeta^r, K_\eta^\ell\right\} > \frac{C_o}{2}, \tag{4.3.2}$$

$$2\left(K_\eta^m(1+\frac{|\zeta|}{2}) - K_\eta^r\right)|\eta| + (1-2K_\zeta^r)|\zeta| < 0\,, \tag{4.3.3}$$

$$2\left(K_\zeta^m(1+\frac{|\eta|}{2}) - K_\zeta^\ell\right)|\zeta| + (1-2K_\eta^\ell)|\eta| < 0. \tag{4.3.4}$$

Here, μ_1^d is defined in (4.3.1) *and C_o in* (2.3.10).

Proof. Since the two cases give symmetric conditions, we only analyze interactions involving ζ_0. As usual, let $\varepsilon_1, \varepsilon_3$ denote the outgoing waves; see Figure 4.5 and 4.6. By $(2.3.1)_1$ and (2.3.2) it holds

$$\begin{cases} \varepsilon_1 - \delta_1 = \varepsilon_3, & |\varepsilon_1| - |\delta_1| = |\varepsilon_3|, \qquad \text{if } i = 1, \\ \varepsilon_3 - \delta_3 = \varepsilon_1, & |\varepsilon_3| - |\delta_3| = -|\varepsilon_1|, \qquad \text{if } i = 3. \end{cases} \tag{4.3.5}$$

$\boxed{\text{Case } i = 1}$ If the interacting wave is a rarefaction, then by (2.3.8), (2.3.9) we have

$$\Delta^R F_d = |\varepsilon_3| \leq \frac{1}{2}|\delta_1\zeta|,$$

and

$$\begin{aligned} \Delta^T F_d &= |\varepsilon_1| - |\delta_1| + K_\eta^m|\varepsilon_1\eta| - K_\eta^r|\delta_1\eta| - K_\zeta^r|\delta_1\zeta| \\ &= |\varepsilon_3| + K_\eta^m|\varepsilon_1\eta| - K_\eta^r|\delta_1\eta| - K_\zeta^r|\delta_1\zeta| \\ &\leq \frac{1}{2}|\delta_1\zeta| + K_\eta^m|\varepsilon_1\eta| - K_\eta^r|\delta_1\eta| - K_\zeta^r|\delta_1\zeta| \\ &\leq \frac{1}{2}|\delta_1\zeta| \left[2\left(K_\eta^m(1 + \frac{|\zeta|}{2}) - K_\eta^r\right)\frac{|\eta|}{|\zeta|} + (1 - 2K_\zeta^r)\right]. \end{aligned}$$

Then, by (4.3.3) it holds $\Delta^T F_d < 0$ and

$$[\Delta^R F_d]_+ \leq \frac{1}{2}|\delta_1\zeta| \leq \frac{1}{2|\eta|[K_\eta^r - K_\eta^m(1 + |\zeta|/2)]/|\zeta| + (2K_\zeta^r - 1)}\,[\Delta^T F_d]_-,$$

i.e. $[\Delta^R F_d]_+ \leq \mu_1^d[\Delta^T F_d]_-$.

Instead, if the interacting wave is a shock, then by the interaction estimates (2.3.8), (2.3.9) we obtain

$$\Delta^R F_d = \begin{cases} \xi|\varepsilon_3| \leq \dfrac{\xi}{2}|\delta_1\zeta| & \text{if } |\delta_1| \geq \rho, \\ |\varepsilon_3| \leq \dfrac{C_o}{2}|\delta_1\zeta| & \text{if } |\delta_1| < \rho. \end{cases}$$

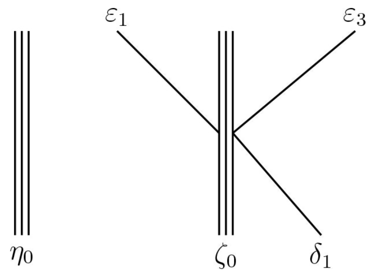

FIGURE 4.5: Interaction of a 1-wave with ζ_0 solved with the accurate Riemann solver.

On the other hand,

$$\begin{aligned}\Delta^T F_d &= \xi|\varepsilon_1| - \xi|\delta_1| - K_\zeta^r \xi|\delta_1\zeta| \\ &= \xi|\varepsilon_3| - K_\zeta^r \xi|\delta_1\zeta| \leq \begin{cases} \frac{\xi}{2}|\delta_1\zeta|\left(1-2K_\zeta^r\right) & \text{if } |\delta_1| \geq \rho, \\ \frac{\xi}{2}|\delta_1\zeta|\left(C_o-2K_\zeta^r\right) & \text{if } |\delta_1| < \rho, \end{cases}\end{aligned}$$

and it holds $\Delta^T F_d < 0$ by (4.3.2) (since $C_o > 1$). Hence,

$$[\Delta^R F_d]_+ \leq \begin{cases} \frac{\xi}{2}|\delta_1\zeta| \leq \frac{1}{2K_\zeta^r - 1}[\Delta^T F_d]_- & \text{if } |\delta_1| \geq \rho, \\ \frac{C_o}{2}|\delta_1\zeta| \leq \frac{C_o}{\xi(2K_\zeta^r - C_o)}[\Delta^T F_d]_- & \text{if } |\delta_1| < \rho, \end{cases}$$

i.e. $[\Delta^R F_d]_+ \leq \mu_1^d[\Delta^T F_d]_-$.

$\boxed{\text{Case } i = 3}$ If the interacting wave is a rarefaction, then

$$\Delta^R F_d = \begin{cases} \xi|\varepsilon_1| & \text{if } |\delta_3| \geq \rho, \\ |\varepsilon_1| & \text{if } |\delta_3| < \rho, \end{cases}$$

and by the interaction estimates (2.3.8), (2.3.9) we have

$$\Delta^T F_d = |\varepsilon_3| - |\delta_3| - K_\zeta^m |\delta_3 \zeta| = -|\varepsilon_1| - K_\zeta^m |\delta_3 \zeta| \le -(1 + 2K_\zeta^m)|\varepsilon_1|.$$

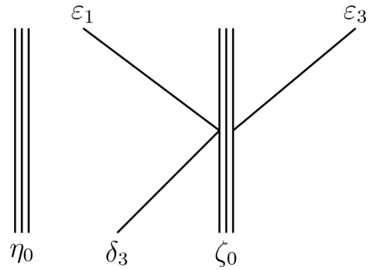

FIGURE 4.6: Interaction of a 3-wave with ζ_0 solved with the accurate Riemann solver.

Hence, $\Delta^T F_d \le 0$ and

$$[\Delta^R F_d]_+ \le \frac{\xi}{2}|\delta_3 \zeta| \le \frac{\xi}{1 + 2K_\zeta^m}[\Delta^T F_d]_- \le \mu_1^d [\Delta^T F_d]_-.$$

On the other hand, if the interacting wave is a shock, we have

$$\Delta^R F_d = \begin{cases} |\varepsilon_1| + K_\eta^m |\varepsilon_1 \eta| & \text{if } |\delta_3| \ge \rho, \\ |\varepsilon_1| & \text{if } |\delta_3| < \rho, \end{cases}$$

and $\Delta^T F_d = \xi|\varepsilon_3| - \xi|\delta_3| = -\xi|\varepsilon_1| \le 0$. Hence,

$$[\Delta^R F_d]_+ = \begin{cases} (1 + K_\eta^m |\eta|)|\varepsilon_1| = \dfrac{1 + K_\eta^m |\eta|}{\xi}[\Delta^T F_d]_- & \text{if } |\delta_3| \ge \rho, \\ |\varepsilon_1| = \dfrac{1}{\xi}[\Delta^T F_d]_- & \text{if } |\delta_3| < \rho, \end{cases}$$

i.e. $[\Delta^R F_d]_+ \le \mu_1^d [\Delta^T F_d]_-$. $\square$

Corollary 4.3.2. *The Glimm functional F_d is non-increasing across time $\bar{t}$ if the following conditions hold for $\xi > 1$:*

$$\min\left\{K_\zeta^r, K_\eta^\ell\right\} \geq 1, \quad \frac{\xi-1}{2} \leq K_\zeta^m \leq \frac{\xi-1}{|\zeta|}, \quad \frac{\xi-1}{2} \leq K_\eta^m \leq \frac{\xi-1}{|\eta|}, \tag{4.3.6}$$

$$\left(K_\eta^m \big(1+\frac{|\zeta|}{2}\big) - K_\eta^r\right)|\eta| + (1-K_\zeta^r)|\zeta| \leq 0, \tag{4.3.7}$$

$$\left(K_\zeta^m \big(1+\frac{|\eta|}{2}\big) - K_\zeta^\ell\right)|\zeta| + (1-K_\eta^\ell)|\eta| \leq 0, \tag{4.3.8}$$

$$C_o \leq \frac{2\xi}{\xi+1}\min\{K_\zeta^r, K_\eta^\ell\}. \tag{4.3.9}$$

Proof. Notice that (4.3.7) implies (4.3.3), (4.3.8) implies (4.3.4) and (4.3.9) implies (4.3.2). Moreover, from (4.3.6)–(4.3.9) it follows $\mu_1^d \leq 1$. Thus,

$$\Delta F_d = [\Delta^R F_d]_+ - [\Delta^T F_d]_- \leq (\mu_1^d - 1)[\Delta^T F_d]_- \leq 0.$$

□

In the next proposition we analyze interactions between waves of the same family.

Proposition 4.3.3 (Interactions between i-waves)**.** *Consider the interaction at time $\bar{t} > 0$ of two waves α_i, β_i of the same family $i = 1, 3$ and let $\varepsilon_i, \varepsilon_j$ denote the outgoing waves, for $j = 1, 3$, $j \neq i$. If* (2.3.14) *is satisfied and it holds*

$$1 \leq \xi \leq \frac{1}{c(m_o)}, \tag{4.3.10}$$

Then, we have

$$[\Delta^R F_d]_+ \leq \mu_2^d [\Delta^T F_d]_-,$$

where μ_2^d is defined in (4.3.1)*.*

Proof. We proceed as in the proof of Proposition 3.2.3. First, recall (3.2.4)–(3.2.6) and the important estimate (3.2.8), which holds true by (4.3.10). We distinguish between the case where α_i and β_i are both shock waves

($\alpha_i, \beta_i < 0$) and the case where α_i and β_i are of different type (for example, $\alpha_i < 0 < \beta_i$).

$\boxed{\text{Case } \alpha_i, \beta_i < 0}$ By (3.2.4) we have

$$\Delta^R F_d = |\varepsilon_j| P^R, \qquad \Delta^T F_d = -\xi |\varepsilon_j| P^T,$$

for $P^R = P^R(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r})$ and $P^T = P^T(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r})$ positive polynomials. In particular,

$$P^R(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r}) = \begin{cases} 1 + K_\eta^\ell |\eta| + K_\zeta^\ell |\zeta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K_\zeta^m |\zeta| & \text{if } i = 1 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\eta^m |\eta| & \text{if } i = 3 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\eta^r |\eta| + K_\zeta^r |\zeta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} > 1, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$P^T(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r}) = \begin{cases} 1 + K_\eta^\ell |\eta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K_\zeta^r |\zeta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} > 1, \\ 1 & \text{otherwise.} \end{cases}$$

Since it holds $P^T \geq 1$, we have $[\Delta^T F_d]_- \geq \xi |\varepsilon_j|$. Hence,

$$[\Delta^R F_d]_+ = |\varepsilon_j| P^R \leq \frac{P^R}{\xi} [\Delta^T F_d]_- \leq \mu_2^d [\Delta^T F_d]_-.$$

$\boxed{\text{Case } \alpha_i < 0 < \beta_i}$ By (3.2.5), (3.2.6) we get

$$\Delta^R F_d = \xi |\varepsilon_j| P^R, \qquad \Delta^T F_d = \begin{cases} (-|\varepsilon_j| - |\alpha_i|) P_1 - \xi |\alpha_i| P_2^T & \text{if } \varepsilon_i > 0, \\ \xi(|\varepsilon_j| - |\beta_i|) P_2^T - |\beta_i| P_1^T & \text{if } \varepsilon_i < 0, \end{cases}$$

for $P^R = P^R(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r})$ and $P_{1,2}^T = P_{1,2}^T(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r})$.

In particular,

$$P^R(|\eta|,|\zeta|,K^{\ell,m,r}_{\eta,\zeta}) = \begin{cases} 1+K^\ell_\eta|\eta| & \text{if } i=1 \text{ and } x_{\alpha_i},x_{\beta_i}<0,\\ 1+K^r_\zeta|\zeta| & \text{if } i=3 \text{ and } x_{\alpha_i},x_{\beta_i}>1,\\ 1 & \text{otherwise,}\end{cases}$$

and

$$P^T_1(|\eta|,|\zeta|,K^{\ell,m,r}_{\eta,\zeta}) = \begin{cases} 1+K^\ell_\eta|\eta|+K^\ell_\zeta|\zeta| & \text{if } i=3 \text{ and } x_{\alpha_i},x_{\beta_i}<0,\\ 1+K^m_\eta|\eta| & \text{if } i=1 \text{ and } 0<x_{\alpha_i},x_{\beta_i}<1,\\ 1+K^m_\zeta|\zeta| & \text{if } i=3 \text{ and } 0<x_{\alpha_i},x_{\beta_i}<1,\\ 1+K^r_\eta|\eta|+K^r_\zeta|\zeta| & \text{if } i=1 \text{ and } x_{\alpha_i},x_{\beta_i}>1,\\ 1 & \text{otherwise,}\end{cases}$$

$$P^T_2(|\eta|,|\zeta|,K^{\ell,m,r}_{\eta,\zeta}) = \begin{cases} 1+K^\ell_\eta|\eta| & \text{if } i=3 \text{ and } x_{\alpha_i},x_{\beta_i}<0,\\ 1+K^r_\zeta|\zeta| & \text{if } i=1 \text{ and } x_{\alpha_i},x_{\beta_i}>1,\\ 1 & \text{otherwise,}\end{cases}$$

Since $P^T_{1,2}\geq 1$, by (3.2.8) we have $\Delta^T F_d \leq -\xi^2|\varepsilon_j|$. Hence,

$$[\Delta^R F_d]_+ \leq \frac{P^R}{\xi}[\Delta^T F_d]_- \leq \mu^d_2[\Delta^T F_d]_-.$$

□

Corollary 4.3.4. *Under the assumptions of the previous propositions, F_d is non-increasing across time $\bar t$ if the following conditions hold for $\xi>1$:*

$$K^m_\eta \leq \frac{\xi-1}{|\eta|}, \qquad K^m_\zeta \leq \frac{\xi-1}{|\zeta|}, \tag{4.3.11}$$

$$K^\ell_\eta|\eta|+K^\ell_\zeta|\zeta| \leq \xi-1, \qquad K^r_\eta|\eta|+K^r_\zeta|\zeta| \leq \xi-1. \tag{4.3.12}$$

Proof. It suffices to notice that (4.3.11), (4.3.12) imply $\mu^d_2\leq 1$. □

4.3.1 The choice of the parameters

In (4.3.6)–(4.3.9), (4.3.11), (4.3.12) we keep strict inequalities in order to have

$$\max\{\mu_1^d, \mu_2^d\} < 1,$$

that is a fundamental requirement for the control of the strength of the composite waves; see Section 2.4. Then, the various parameters are chosen in the following order: m_o, ξ, $K^m_{\eta,\zeta}$, $K^{\ell,r}_{\eta,\zeta}$ and finally ρ.

We notice that, for the choice of $K^m_{\zeta,\eta}$, by (4.3.6)$_{2,3}$ and (4.3.11) it must hold

$$\frac{\xi-1}{2} < \min\left\{\frac{\xi-1}{|\eta|}, \frac{\xi-1}{|\zeta|}\right\}, \tag{4.3.13}$$

which is always satisfied since $|\eta|, |\zeta| < 2$. Moreover, by combining together the conditions obtained in (4.3.6)$_3$ with (4.3.7) and (4.3.12)$_2$ we get necessarily

$$(\xi-1)\big(1+\frac{|\zeta|}{2}\big)\frac{|\eta|}{2} < K^r_\eta|\eta| + (K^r_\zeta - 1)|\zeta| < (\xi-1) - |\zeta|. \tag{4.3.14}$$

Hence, it follows

$$(\xi-1)\big(1+\frac{|\zeta|}{2}\big)\frac{|\eta|}{2} < (\xi-1) - |\zeta|,$$

which is equivalent to

$$1 + \frac{4|\zeta|}{4-2|\eta|-|\eta\zeta|} < \xi, \tag{4.3.15}$$

provided that $4-2|\eta|-|\eta\zeta| > 0$. Analogously, from (4.3.6)$_2$, (4.3.8) and (4.3.12)$_1$ we get

$$1 + \frac{4|\eta|}{4-2|\zeta|-|\eta\zeta|} < \xi,$$

provided that $4-2|\zeta|-|\eta\zeta| > 0$. Therefore, it must hold

$$1 + 4\max\left\{\frac{|\zeta|}{4-2|\eta|-|\eta\zeta|}, \frac{|\eta|}{4-2|\eta|-|\eta\zeta|}\right\} < \xi \tag{4.3.16}$$

under the *stability* condition (otherwise rewritten)

$$\max\left\{\left(1+\frac{|\zeta|}{2}\right)\frac{|\eta|}{2},\left(1+\frac{|\eta|}{2}\right)\frac{|\zeta|}{2}\right\}<1,$$

that is $(|\eta|,|\zeta|)\in S_d$. Recalling the function $\mathcal{H}_d$ defined in (4.1.2), by (4.3.10) and (4.3.16) we obtain the condition

$$1+\mathcal{H}_d(|\eta|,|\zeta|)<\xi\leq\frac{1}{c(m_o)}.$$

Below, we summarize the choice of the parameters. Let $(|\eta|,|\zeta|)\in S_d$.

i) Recalling (4.3.10), we fix m_o such that

$$c(m_o)<\frac{1}{1+\mathcal{H}_d(|\eta|,|\zeta|)}. \tag{4.3.17}$$

ii) Then, we choose ξ in the non-empty interval given by

$$1+\mathcal{H}_d(|\eta|,|\zeta|)<\xi\leq\frac{1}{c(m_o)}. \tag{4.3.18}$$

iii) We choose K_η^m, K_ζ^m such that

$$\frac{\xi-1}{2}<K_\eta^m<\min\left\{\frac{\xi-1}{|\eta|},\frac{(\xi-1)-|\zeta|}{(1+|\zeta|/2)|\eta|}\right\}=\frac{(\xi-1)-|\zeta|}{(1+|\zeta|/2)|\eta|}, \tag{4.3.19}$$

$$\frac{\xi-1}{2}<K_\zeta^m<\min\left\{\frac{\xi-1}{|\zeta|},\frac{(\xi-1)-|\eta|}{(1+|\eta|/2)|\zeta|}\right\}=\frac{(\xi-1)-|\eta|}{(1+|\eta|/2)|\zeta|}. \tag{4.3.20}$$

This is possible since these two intervals are non-empty by (4.3.16). Thus, $(4.3.6)_{2,3}$ and (4.3.11) follow and it holds

$$K_\eta^m\big(1+\frac{|\zeta|}{2}\big)|\eta|<(\xi-1)-|\zeta|,\qquad K_\zeta^m\big(1+\frac{|\eta|}{2}\big)|\zeta|<(\xi-1)-|\eta|. \tag{4.3.21}$$

iv) By (4.3.21), we choose K^r_η, K^ℓ_ζ that satisfy

$$K^m_\eta\big(1+\frac{|\zeta|}{2}\big)|\eta| < K^r_\eta|\eta| < (\xi-1)-|\zeta|, \tag{4.3.22}$$

$$K^m_\zeta\big(1+\frac{|\eta|}{2}\big)|\zeta| < K^\ell_\zeta|\zeta| < (\xi-1)-|\eta|, \tag{4.3.23}$$

and, then, we can take K^ℓ_η and K^r_ζ such that

$$1 < K^\ell_\eta < 1+\frac{(\xi-1)-|\eta|-K^\ell_\zeta|\zeta|}{|\eta|}, \tag{4.3.24}$$

$$1 < K^r_\zeta < 1+\frac{(\xi-1)-|\zeta|-K^r_\eta|\eta|}{|\zeta|}. \tag{4.3.25}$$

Notice that, by (4.3.22), (4.3.23) and the fact that $K^r_\zeta, K^\ell_\eta > 1$, the conditions (4.3.7) and (4.3.8) are verified; while (4.3.25) and (4.3.24) imply (4.3.12).

v) Finally, we choose ρ such that $C_o = C_o(\rho)$ satisfies (4.3.9).

4.4 The bubble case

In this section we carry out the estimates for F_b. Recall that here $\eta > 0$ and $\zeta < 0$. For $t > 0$ and $K^{\ell,m,r}_{\eta,\zeta} > 0$, the potentials of (4.2.1) are

$$\begin{aligned}
Q^1_b(t) = \Bigg[&K^m_\eta\Big(\sum_{\substack{0<x<1\\ \delta_1>0}}|\delta_1| + \xi\sum_{\substack{0<x<1\\ \delta_1<0}}|\delta_1|\Big) + K^r_\eta\Big(\sum_{\substack{x>1\\ \delta_1>0}}|\delta_1| + \xi\sum_{\substack{x>1\\ \delta_1<0}}|\delta_1|\Big)\Bigg]|\eta| \\
&+ K^r_\zeta\sum_{\substack{x>1\\ \delta_1>0}}|\delta_1\zeta|, \\
Q^3_b(t) = \Bigg[&K^\ell_\zeta\Big(\sum_{\substack{x<0\\ \delta_3>0}}|\delta_3| + \xi\sum_{\substack{x<0\\ \delta_3<0}}|\delta_3|\Big) + K^m_\zeta\Big(\sum_{\substack{0<x<1\\ \delta_3>0}}|\delta_3| + \xi\sum_{\substack{0<x<1\\ \delta_3<0}}|\delta_3|\Big)\Bigg]|\zeta| \\
&+ K^\ell_\eta\sum_{\substack{x<0\\ \delta_3>0}}|\delta_3\eta|.
\end{aligned}$$

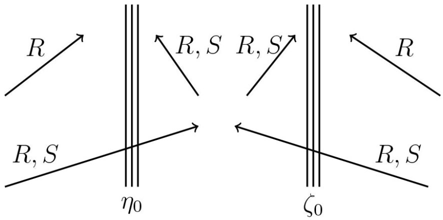

FIGURE 4.7: The terms of Q_b.

In this case, the missing terms in Q_b are the 3-shocks approaching η_0 and the 1-shocks approaching ζ_0; see Figure 4.7. As before, in the next two propositions we list the conditions on the parameters ξ, $K^{\ell,m,r}_{\eta,\zeta}$ and ρ for the decrease of F_b.

We define

$$M_1^b = \left\{ \frac{1}{\xi}, \frac{\xi}{1+2K^r_\zeta}, \frac{\xi}{1+2K^\ell_\eta}, \frac{1+K^m_\eta|\eta|}{2K^m_\zeta - 1}, \frac{1+K^m_\zeta|\zeta|}{2K^m_\eta - 1}, \right. \\ \left. \frac{C_o}{\xi(2K^m_\zeta - C_o)}, \frac{C_o}{\xi(2K^m_\eta - C_o)} \right\},$$

$$M_2^b = \left\{ \frac{1+K^\ell_\eta|\eta| + K^\ell_\zeta|\zeta|}{\xi}, \frac{1+K^r_\eta|\eta| + K^r_\zeta|\zeta|}{\xi}, \frac{1+K^m_\eta|\eta|}{\xi}, \frac{1+K^m_\zeta|\zeta|}{\xi} \right\},$$

and, following the exact same reasoning as in the drop case, we recover (2.3.31) for

$$\mu_1^b = \max M_1^b, \qquad \mu_2^b = \max M_2^b. \tag{4.4.1}$$

Proposition 4.4.1 (Interactions with the composite waves)**.** *Assume that at a time $\bar{t} > 0$ a wave δ_i, $i = 1, 3$, interacts with one of the composite waves η_0 or ζ_0. Then, we have*

$$[\Delta^R F_b]_+ \le \mu_1^b [\Delta^T F_b]_-,$$

provided that the following conditions hold:

$$\min\left\{K_\eta^m, K_\zeta^m\right\} > \frac{C_o}{2}, \qquad K_\eta^m \le K_\eta^r, \qquad K_\zeta^m \le K_\zeta^\ell. \tag{4.4.2}$$

Here, μ_1^b *is defined in* (4.4.1) *and* C_o *in* (2.3.10).

Proof. Since the two cases give symmetric conditions, we only analyze interactions involving ζ_0; see Figure 4.5 and 4.6. Let $\varepsilon_1, \varepsilon_3$ denote the outgoing waves of family $1, 3$ and recall that by (2.3.1) and (2.3.2) we have

$$\begin{cases} \varepsilon_1 - \delta_1 = \varepsilon_3, & |\varepsilon_1| - |\delta_1| = -|\varepsilon_3|, & \text{if } i = 1, \\ \varepsilon_3 - \delta_3 = \varepsilon_1, & |\varepsilon_3| - |\delta_3| = |\varepsilon_1|, & \text{if } i = 3. \end{cases}$$

$\boxed{\text{Case } i = 1}$ If the interacting wave is a rarefaction, then we have

$$\Delta^R F_b = \begin{cases} \xi|\varepsilon_3| & \text{if } |\delta_1| \ge \rho, \\ |\varepsilon_3| & \text{if } |\delta_1| < \rho, \end{cases}$$

and

$$\begin{aligned} \Delta^T F_b &= |\varepsilon_1| - |\delta_1| + K_\eta^m|\varepsilon_1\eta| - K_\eta^r|\delta_1\eta| - K_\zeta^r|\delta_1\zeta| \\ &= -|\varepsilon_3| + K_\eta^m|\varepsilon_1\eta| - K_\eta^r|\delta_1\eta| - K_\zeta^r|\delta_1\zeta| \\ &\le -|\varepsilon_3| + (K_\eta^m - K_\eta^r)|\delta_1\eta| - K_\zeta^r|\delta_1\zeta|, \end{aligned}$$

where the last inequality is verified since $|\varepsilon_1| \le |\delta_1|$. Then, by $(4.4.2)_2$ and by (2.3.8), (2.3.9) we get $\Delta^T F_b \le -|\varepsilon_3|(1 + 2K_\zeta^r) \le 0$. Hence,

$$[\Delta^R F_b]_+ \le \xi|\varepsilon_3| \le \frac{\xi}{1 + 2K_\zeta^r}[\Delta^T F_b]_- \le \mu_1^b[\Delta^T F_b]_-.$$

Instead, if the interacting wave is a shock, then in both the accurate and simplified cases we have $\Delta^R F_b = |\varepsilon_3|$ and

$$\Delta^T F_b = \xi|\varepsilon_1| - \xi|\delta_1| + K_\eta^m\xi|\varepsilon_1\eta| - \xi K_\eta^r|\delta_1\eta|$$

$$= -\xi|\varepsilon_3| + K_\eta^m \xi|\varepsilon_1\eta| - K_\eta^r \xi|\delta_1\eta| \leq -\xi|\varepsilon_3| + (K_\eta^m - K_\zeta^r)\xi|\delta_1\eta|.$$

Thus, by $(4.4.2)_2$ it holds $[\Delta^T F_b]_- \geq \xi|\varepsilon_3|$ and

$$[\Delta^R F_b]_+ = |\varepsilon_3| \leq \frac{1}{\xi}[\Delta^T F_b]_- \leq \mu_1^b [\Delta^T F_b]_-.$$

$\boxed{\text{Case } i = 3}$ If the interacting wave is a rarefaction, then by (2.3.8), (2.3.9) we have

$$\Delta^R F_b = \begin{cases} |\varepsilon_1| + K_\eta^m|\varepsilon_1\eta| \leq \frac{1}{2}|\delta_3\zeta|(1 + K_\eta^m|\eta|) & \text{if } |\delta_3| \geq \rho, \\ |\varepsilon_1| \leq \frac{1}{2}|\delta_3\zeta| & \text{if } |\delta_3| < \rho, \end{cases}$$

while for $\Delta^T F_b$ we have

$$\Delta^T F_b = |\varepsilon_3| - |\delta_3| - K_\zeta^m|\delta_3\zeta| = |\varepsilon_1| - K_\zeta^m|\delta_3\zeta| \leq \frac{1}{2}|\delta_3\zeta|(1 - 2K_\zeta^m).$$

Notice that, since $C_o > 1$, by $(4.4.2)_1$ it holds $\Delta^T F_b \leq 0$. Hence,

$$[\Delta^R F_b]_+ \leq \begin{cases} \frac{1}{2}|\delta_3\zeta|(1 + K_\eta^m|\eta|) \leq \frac{1 + K_\eta^m|\eta|}{2K_\zeta^m - 1}[\Delta^T F_b]_- & \text{if } |\delta_3| \geq \rho, \\ \frac{1}{2}|\delta_3\zeta| \leq \frac{1}{2K_\zeta^m - 1}[\Delta^T F_b]_- & \text{if } |\delta_3| < \rho, \end{cases}$$

i.e. $[\Delta^R F_b]_+ \leq \mu_1^b [\Delta^T F_b]_-$.

On the other hand, if the interacting wave is a shock, then by (2.3.8) and (2.3.9) we have

$$[\Delta^R F_b]_+ = \begin{cases} \xi|\varepsilon_1| + K_\eta^m \xi|\varepsilon_1\eta| \leq \frac{\xi}{2}|\delta_3\zeta|(1 + K_\eta^m|\eta|) & \text{if } |\delta_3| \geq \rho, \\ |\varepsilon_1| \leq \frac{C_o}{2}|\delta_3\zeta| & \text{if } |\delta_3| < \rho, \end{cases}$$

and

$$\Delta^T F_b = \xi|\varepsilon_3| - \xi|\delta_3| - K_\zeta^m \xi|\delta_3\zeta|$$
$$= \xi|\varepsilon_1| - K_\zeta^m \xi|\delta_3\zeta| \leq \begin{cases} \frac{\xi}{2}|\delta_3\zeta|(1 - 2K_\zeta^m) & \text{if } |\delta_3| \geq \rho, \\ \frac{\xi}{2}|\delta_3\zeta|(C_o - 2K_\zeta^m) & \text{if } |\delta_3| < \rho. \end{cases}$$

By $(4.4.2)_1$ we have that $\Delta^T F_b \leq 0$. Hence,

$$[\Delta^R F_b]_+ \leq \begin{cases} \frac{\xi}{2}|\delta_3\zeta|(1 + K_\eta^m|\eta|) \leq \frac{1 + K_\eta^m|\eta|}{2K_\zeta^m - 1}[\Delta^T F_b]_- & \text{if } |\delta_3| \geq \rho, \\ \frac{C_o}{2}|\delta_3\zeta| \leq \frac{C_o}{\xi(2K_\zeta^m - C_o)}[\Delta^T F_b]_- & \text{if } |\delta_3| < \rho. \end{cases}$$

i.e. $[\Delta^R F_b]_+ \leq \mu_1^b [\Delta^T F_b]_-$. □

Corollary 4.4.2. *The Glimm functional F_b is non-increasing across time $\bar{t}$ if the following conditions hold for $\xi > 1$:*

$$\min\{K_\eta^m, K_\zeta^m\} \geq 1, \qquad \frac{\xi - 1}{2} \leq \min\{K_\zeta^r, K_\eta^\ell\}, \tag{4.4.3}$$
$$K_\eta^m \leq K_\eta^r, \qquad K_\zeta^m \leq K_\zeta^\ell, \tag{4.4.4}$$
$$1 + K_\eta^m \frac{|\eta|}{2} - K_\zeta^m \leq 0, \qquad 1 + K_\zeta^m \frac{|\zeta|}{2} - K_\eta^m \leq 0, \tag{4.4.5}$$
$$C_o \leq \frac{2\xi}{\xi + 1} \min\{K_\eta^m, K_\zeta^m\}. \tag{4.4.6}$$

Proof. Notice that $(4.4.2)_1$ is implied by (4.4.6). Moreover, by (4.4.3)–(4.4.6) it holds $\mu_1^b \leq 1$ and, thus, we can infer $\Delta F_b \leq 0$. □

Proposition 4.4.3 (Interactions between i-waves). *Consider the interaction at time $\bar{t} > 0$ of two waves α_i and β_i of the same family $i = 1, 3$ and let ε_i, ε_j be the outgoing waves, for $j = 1, 3$, $j \neq i$. If we assume* (2.3.14) *and it holds*

$$1 \leq \xi \leq \frac{1}{c(m_o)}, \tag{4.4.7}$$

then, we have

$$[\Delta^R F_b]_+ \le \mu_2^b [\Delta^T F_b]_-,$$

where μ_2^b *is defined in* (4.4.1).

Proof. Again we proceed as in the proof of Proposition 3.2.3. Recall (3.2.4)–(3.2.6) and (3.2.8), which follows from (4.4.7).

$\boxed{\text{Case } \alpha_i, \beta_i < 0}$ By (3.2.4) we have

$$\Delta^R F_b = |\varepsilon_j| P^R, \qquad \Delta^T F_b = -\xi |\varepsilon_j| P^T,$$

where $P^R = P^R(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r})$ and $P^T = P^T(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r})$. In particular,

$$P^R(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r}) = \begin{cases} 1 + K_\eta^\ell |\eta| + K_\zeta^\ell |\zeta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K_\zeta^m |\zeta| & \text{if } i = 1 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\eta^m |\eta| & \text{if } i = 3 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\eta^r |\eta| + K_\zeta^r |\zeta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} > 1, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$P^T(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r}) = \begin{cases} 1 + K_\zeta^\ell |\zeta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K_\eta^m |\eta| & \text{if } i = 1 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\zeta^m |\zeta| & \text{if } i = 3 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\eta^r |\eta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} > 1, \\ 1 & \text{otherwise.} \end{cases}$$

Since it holds $P^T \ge 1$, we have $[\Delta^T F_b]_- \ge \xi |\varepsilon_j|$. Hence,

$$[\Delta^R F_b]_+ \le \frac{P^R}{\xi} [\Delta^T F_b]_- \le \mu_2^b [\Delta^T F_b]_-.$$

$\boxed{\text{Case } \alpha_i < 0 < \beta_i}$ By (3.2.5), (3.2.6) we get

$$\Delta^R F_b = \xi|\varepsilon_j|P^R, \qquad \Delta^T F_b = \begin{cases} (-|\varepsilon_j| - |\alpha_i|)P_1^T - \xi|\alpha_i|P_2^T & \text{if } \varepsilon_i > 0, \\ \xi(|\varepsilon_j| - |\beta_i|)P_2^T - |\beta_i|P_1^T & \text{if } \varepsilon_i < 0, \end{cases}$$

where

$$P^R(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r}) = \begin{cases} 1 + K_\zeta^\ell|\zeta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K_\eta^m|\eta| & \text{if } i = 3 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\zeta^m|\zeta| & \text{if } i = 1 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\eta^r|\eta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} > 1, \\ 1 & \text{otherwise}, \end{cases}$$

and

$$P_1^T(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r}) = \begin{cases} 1 + K_\eta^\ell|\eta| + K_\zeta^\ell|\zeta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K_\eta^m|\eta| & \text{if } i = 1 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\zeta^m|\zeta| & \text{if } i = 3 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\eta^r|\eta| + K_\zeta^r|\zeta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} > 1, \\ 1 & \text{otherwise}, \end{cases}$$

$$P_2^T(|\eta|, |\zeta|, K_{\eta,\zeta}^{\ell,m,r}) = \begin{cases} 1 + K_\zeta^\ell|\zeta| & \text{if } i = 3 \text{ and } x_{\alpha_i}, x_{\beta_i} < 0, \\ 1 + K_\eta^m|\eta| & \text{if } i = 1 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\zeta^m|\zeta| & \text{if } i = 3 \text{ and } 0 < x_{\alpha_i}, x_{\beta_i} < 1, \\ 1 + K_\eta^r|\eta| & \text{if } i = 1 \text{ and } x_{\alpha_i}, x_{\beta_i} > 1, \\ 1 & \text{otherwise}. \end{cases}$$

Since $P_{1,2}^T \geq 1$, by (3.2.8) we have $[\Delta^T F_b]_- \geq \xi^2|\varepsilon_j|$. Hence,

$$[\Delta^R F_b]_+ \leq \frac{P^R}{\xi}[\Delta^T F_b]_- \leq \mu_2^b[\Delta^T F_b]_-.$$

$\square$

Corollary 4.4.4. *Under the assumptions of the previous proposition, F_b is non-increasing across time $\bar{t}$ provided that the following conditions hold for $\xi > 1$:*

$$K_\eta^m \le \frac{\xi - 1}{|\eta|}, \qquad K_\zeta^m \le \frac{\xi - 1}{|\zeta|}, \tag{4.4.8}$$

$$K_\eta^\ell|\eta| + K_\zeta^\ell|\zeta| \le \xi - 1, \qquad K_\eta^r|\eta| + K_\zeta^r|\zeta| \le \xi - 1. \tag{4.4.9}$$

Proof. From (4.4.8), (4.4.9) it follows $\mu_2^b \le 1$. □

4.4.1 The choice of the parameters

Now, we determine the order in which we choose the parameters. To simplify the analysis, we let $K_\eta^m = K_\eta^r$ and $K_\zeta^m = K_\zeta^\ell$ and the final result remains unchanged. Moreover, we keep (4.4.3)–(4.4.6) and (4.4.8), (4.4.9) with strict inequalities, in order to get

$$\max\{\mu_1^b, \mu_2^b\} < 1.$$

Once η, ζ have been fixed, we choose in order: m_o, ξ, K_η^m and K_ζ^m, K_ζ^r and K_η^ℓ; at last, we choose ρ. First, notice that the conditions in (4.4.5) identify a cone in the (K_η^m, K_ζ^m)-plane, represented in Figure 4.8. Hence, by (4.4.5) we deduce

$$K_\eta^m > 1 + K_\zeta^m \frac{|\zeta|}{2} > 1 + \frac{|\zeta|}{2}\big(1 + K_\eta^m \frac{|\eta|}{2}\big),$$
$$K_\zeta^m > 1 + K_\eta^m \frac{|\eta|}{2} > 1 + \frac{|\eta|}{2}\big(1 + K_\zeta^m \frac{|\zeta|}{2}\big),$$

that imply

$$K_\eta^m > \frac{1 + |\zeta|/2}{1 - |\eta\zeta|/4}, \qquad K_\zeta^m > \frac{1 + |\eta|/2}{1 - |\eta\zeta|/4}. \tag{4.4.10}$$

In particular, the right-hand sides in (4.4.10) are the coordinates of the intersection point V between the two lines of Figure 4.8. Notice also that (4.4.10) implies $(4.4.3)_1$. Since we have chosen $K_\eta^m = K_\eta^r$ and $K_\zeta^m = K_\zeta^\ell$, conditions (4.4.9) imply (4.4.8). By $(4.4.3)_2$ and $(4.4.9)_2$, we get $K_\eta^m|\eta| +$

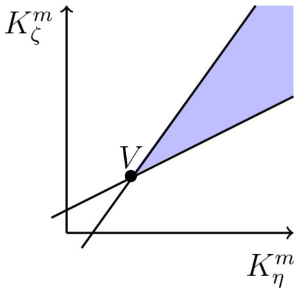

FIGURE 4.8: Graphical representation of conditions (4.4.5) for $|\eta| = 1/2$ and $|\zeta| = 3/2$.

$(\xi - 1)|\zeta|/2 < \xi - 1$, which is equivalent to

$$K_\eta^m \frac{|\eta|}{1 - |\zeta|/2} < \xi - 1. \tag{4.4.11}$$

Similarly, by $(4.4.3)_2$ and $(4.4.9)_1$ we get

$$K_\zeta^m \frac{|\zeta|}{1 - |\eta|/2} < \xi - 1. \tag{4.4.12}$$

By (4.4.10), (4.4.11) and (4.4.12) it follows that ξ must satisfy the inequality

$$1 + \frac{4}{4 - |\eta\zeta|} \max\left\{ |\eta|\frac{2 + |\zeta|}{2 - |\zeta|}, |\zeta|\frac{2 + |\eta|}{2 - |\eta|} \right\} < \xi.$$

Since this condition has to match with (4.4.7), we must require

$$1 + \mathcal{H}_b(|\eta|, |\zeta|) < \xi \leq \frac{1}{c(m_o)}, \tag{4.4.13}$$

where $\mathcal{H}_b$ is the same of (4.1.3).

Summarizing, we choose the parameters as follows. We let $(|\eta|, |\zeta|) \in S_b$ be given.

i) First, we fix m_o such that

$$1 + \mathcal{H}_b(|\eta|, |\zeta|) < \frac{1}{c(m_o)} \tag{4.4.14}$$

and take ξ in the interior of the interval given by (4.4.13).

ii) In the (K_η^m, K_ζ^m)-plane we choose a point in the affine cone defined by (4.4.5) and sufficiently close to V. Moreover, we require (4.4.11) and (4.4.12).

iii) We choose $K_\eta^r = K_\eta^m$, $K_\zeta^\ell = K_\zeta^m$ and, then, by (4.4.11) and (4.4.12) we choose K_ζ^r and K_η^ℓ such that

$$\frac{\xi - 1}{2} < K_\zeta^r < \frac{\xi - 1}{|\zeta|} - K_\eta^m \frac{|\eta|}{|\zeta|}, \qquad \frac{\xi - 1}{2} < K_\eta^\ell < \frac{\xi - 1}{|\eta|} - K_\zeta^m \frac{|\zeta|}{|\eta|}, \tag{4.4.15}$$

and (4.4.9) holds.

iv) Finally, we choose ρ such that $C_o = C_o(\rho)$ satisfies (4.4.6).

4.5 The increasing pressure case

Finally, we prove the decreasing of the functional F_p in the increasing-pressure case. We recall that both η and ζ are positive. For $t > 0$ not an interaction time, we define the terms of (4.2.1) as follows:

$$\begin{aligned} Q_p^1(t) =& \left[K_\eta^m \Big(\sum_{\substack{0<x<1 \\ \delta_1>0}} |\delta_1| + \xi \sum_{\substack{0<x<1 \\ \delta_1<0}} |\delta_1| \Big) + K_\eta^r \Big(\sum_{\substack{x>1 \\ \delta_1>0}} |\delta_1| + \xi \sum_{\substack{x>1 \\ \delta_1<0}} |\delta_1| \Big) \right] |\eta| \\ & + K_\zeta^r \Big(\sum_{\substack{x>1 \\ \delta_1>0}} |\delta_1| + \xi \sum_{\substack{x>1 \\ \delta_1<0}} |\delta_1| \Big) |\zeta|, \\ Q_p^3(t) =& K_\eta^\ell \sum_{\substack{x<0 \\ \delta_3>0}} |\delta_3 \eta| + \Big(K_\zeta^\ell \sum_{\substack{x<0 \\ \delta_3>0}} |\delta_3| + K_\zeta^m \sum_{\substack{0<x<1 \\ \delta_3>0}} |\delta_3| \Big) |\zeta|. \end{aligned}$$

Notice that the missing shocks in Q_p are those of family 3 approaching both η_0 and ζ_0, see Figure 4.9.

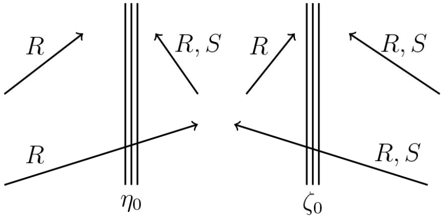

FIGURE 4.9: The terms of Q_p.

In order to prove Claim 2.3.9, we set

$$M_1^p = \left\{ \frac{1+K_\zeta^m|\zeta|}{2K_\eta^m - 1}, \frac{\xi(1+K_\eta^m|\eta|)}{1+2K_\zeta^m}, \frac{1+K_\eta^m|\eta|}{\xi}, \frac{C_o}{\xi(2K_\eta^m - C_o)}, \frac{\xi}{1+2K_\eta^\ell + 2|\zeta|(K_\zeta^\ell - K_\zeta^m(1-|\eta|/2))/|\eta|}, \frac{1}{2|\eta|(K_\eta^r - K_\eta^m(1+|\zeta|/2))/|\zeta| + (2K_\zeta^r - 1)}, \frac{C_o}{2\xi|\eta|(K_\eta^r - K_\eta^m(1+C_o|\zeta|/2))/|\zeta| + \xi(2K_\zeta^r - C_o)} \right\},$$

$$M_2^p = \left\{ \frac{1+K_\eta^\ell|\eta| + K_\zeta^\ell|\zeta|}{\xi}, \frac{1+K_\eta^r|\eta| + K_\zeta^r|\zeta|}{\xi}, \frac{1+K_\eta^m|\eta|}{\xi}, \frac{1+K_\zeta^m|\zeta|}{\xi} \right\}.$$

In the sequel, we recover (2.3.31) for

$$\mu_1^p = \max M_1^p, \qquad \mu_2^p = \max M_2^p. \tag{4.5.1}$$

Proposition 4.5.1 (Interactions with the composite waves). *Assume that at time $\bar{t} > 0$ a wave δ_i, $i = 1, 3$, interacts with one of the composite waves η_0 or ζ_0. Then, we have*

$$[\Delta^R F_p]_+ \le \mu_1^p [\Delta^T F_p]_-,$$

provided that the following conditions hold:

$$K_\eta^m > \frac{C_o}{2}, \qquad K_\eta^\ell|\eta| + (K_\zeta^\ell - K_\zeta^m)|\zeta| \geq 0, \tag{4.5.2}$$

$$2\left(K_\eta^m\big(1 + \frac{C_o}{2}|\zeta|\big) - K_\eta^r\right)|\eta| + (C_o - 2K_\zeta^r)|\zeta| < 0. \tag{4.5.3}$$

Here, μ_1^p *is defined in* (4.5.1) *and* C_o *in* (2.3.10).

Proof. Let $\varepsilon_1, \varepsilon_3$ denote the outgoing waves and recall that for interactions with both η_0 and ζ_0 it holds

$$\begin{cases} \varepsilon_1 - \delta_1 = \varepsilon_3, & |\varepsilon_1| - |\delta_1| = |\varepsilon_3|, & \text{if } i = 1, \\ \varepsilon_3 - \delta_3 = \varepsilon_1, & |\varepsilon_3| - |\delta_3| = -|\varepsilon_1|, & \text{if } i = 3, \end{cases}$$

by (2.3.1), (2.3.2). In this case we separately treat the interactions with the two phase waves since the interaction potential is not symmetric with respect to η_0 and ζ_0.

$\boxed{\text{Interactions with } \eta_0}$ Assume $i = 1$. If δ_1 is a rarefaction, then by (2.3.8) and (2.3.9) we have

$$\Delta^R F_p = \begin{cases} |\varepsilon_3| + K_\zeta^m|\varepsilon_3\zeta| \leq \frac{1}{2}|\delta_1\eta|(1 + K_\zeta^m|\zeta|) & \text{if } |\delta_1| \geq \rho, \\ |\varepsilon_3| \leq \frac{1}{2}|\delta_1\eta| & \text{if } |\delta_1| < \rho, \end{cases}$$

and in both the accurate and the simplified cases it holds

$$\Delta^T F_p = |\varepsilon_1| - |\delta_1| - K_\eta^m|\delta_1\eta| = |\varepsilon_3| - K_\eta^m|\delta_1\eta| \leq \frac{1}{2}|\delta_1\eta|(1 - 2K_\eta^m).$$

Thus, by $(4.5.2)_1$ (since $C_o > 1$) we have $\Delta^T F_p < 0$ and

$$[\Delta^R F_p]_+ \leq \frac{1}{2}|\delta_1\eta|(1 + K_\zeta^m|\zeta|) \leq \frac{1 + K_\zeta^m|\zeta|}{2K_\eta^m - 1}[\Delta^T F_p]_- \leq \mu_1^p[\Delta^T F_p]_-.$$

Instead, if δ_1 is a shock, then by (2.3.8) and (2.3.9) we have

$$\Delta^R F_p = \begin{cases} \xi|\varepsilon_3| \le \dfrac{\xi}{2}|\delta_1\eta| & \text{if } |\delta_1| \ge \rho, \\ |\varepsilon_3| \le \dfrac{C_o}{2}|\delta_1\eta| & \text{if } |\delta_1| < \rho, \end{cases}$$

and

$$\begin{aligned} \Delta^T F_p &= \xi|\varepsilon_1| - \xi|\delta_1| - K_\eta^m \xi|\delta_1\eta| \\ &= \xi|\varepsilon_3| - K_\eta^m \xi|\delta_1\eta| \le \begin{cases} \dfrac{\xi}{2}|\delta_1\eta|(1-2K_\eta^m) & \text{if } |\delta_1| \ge \rho, \\ \dfrac{\xi}{2}|\delta_1\eta|(C_o-2K_\eta^m) & \text{if } |\delta_1| < \rho. \end{cases} \end{aligned}$$

By $(4.5.2)_1$ we get $\Delta^T F_p < 0$. Hence,

$$[\Delta^R F_p]_+ \le \begin{cases} \dfrac{\xi}{2}|\delta_1\eta| \le \dfrac{1}{2K_\eta^m - 1}[\Delta^T F_p]_- & \text{if } |\delta_1| \ge \rho, \\ \dfrac{C_o}{2}|\delta_1\eta| \le \dfrac{C_o}{\xi(2K_\eta^m - C_o)}[\Delta^T F_p]_- & \text{if } |\delta_1| < \rho, \end{cases}$$

i.e. $[\Delta^R F_p]_+ \le \mu_1^p[\Delta^T F_p]$.

Now, let $i = 3$. If δ_3 is a rarefaction, we have

$$\Delta^R F_p = \begin{cases} \xi|\varepsilon_1| & \text{if } |\delta_1| \ge \rho, \\ |\varepsilon_1| & \text{if } |\delta_1| < \rho, \end{cases}$$

and in both the accurate and the simplified case we get

$$\begin{aligned} \Delta^T F_p &= |\varepsilon_3| - |\delta_3| + K_\zeta^m|\varepsilon_3\zeta| - K_\eta^\ell|\delta_3\eta| - K_\zeta^\ell|\delta_3\zeta| \\ &= -|\varepsilon_1| - K_\zeta^m|\varepsilon_1\zeta| - (K_\zeta^\ell - K_\zeta^m)|\delta_3\zeta| - K_\eta^\ell|\delta_3\eta| \\ &= -|\varepsilon_1| - K_\zeta^m|\varepsilon_1\zeta| - \left(K_\eta^\ell + (K_\zeta^\ell - K_\zeta^m)\frac{|\zeta|}{|\eta|}\right)|\delta_3\eta|. \end{aligned}$$

Then, by (2.3.8), (2.3.9) and $(4.5.2)_2$ it holds

$$\Delta^T F_p \le -|\varepsilon_1| - K^m_\zeta |\varepsilon_1 \zeta| - 2|\varepsilon_1| \left(K^\ell_\eta + (K^\ell_\zeta - K^m_\zeta) \frac{|\zeta|}{|\eta|} \right)$$
$$= -|\varepsilon_1| \left[1 + 2K^\ell_\eta + 2 \left(K^\ell_\zeta - K^m_\zeta (1 - \frac{|\eta|}{2}) \right) \frac{|\zeta|}{|\eta|} \right] < 0.$$

Thus,

$$[\Delta^R F_p]_+ \le \xi |\varepsilon_1| \le \frac{\xi}{1 + 2K^\ell_\eta + 2|\zeta|(K^\ell_\zeta - K^m_\zeta(1 - |\eta|/2))/|\eta|} [\Delta^T F_p]_-,$$

i.e. $[\Delta^R F_p]_+ \le \mu^p_1 [\Delta^T F_p]_-$.

Instead, if δ_3 is a shock, $\Delta^R F_p = |\varepsilon_1|$ and $\Delta^T F_p = \xi|\varepsilon_3| - \xi|\delta_3| = -\xi|\varepsilon_1|$. Hence,

$$[\Delta^R F_p]_+ = |\varepsilon_1| = \frac{1}{\xi} [\Delta^T F_p]_- \le \mu^p_1 [\Delta^T F_p]_-.$$

$\boxed{\text{Interactions with } \zeta_0}$ Assume $i = 1$ and refer to Figure 4.5 and 4.6. If δ_1 is a rarefaction, by (2.3.8) and (2.3.9) in both the accurate and the simplified case we have

$$\Delta^R F_p = |\varepsilon_3| \le \frac{1}{2} |\delta_1 \zeta|$$

and

$$\Delta^T F_p = |\varepsilon_1| - |\delta_1| + K^m_\eta |\varepsilon_1 \eta| - K^r_\zeta |\delta_1 \zeta| - K^r_\eta |\delta_1 \eta|$$
$$= |\varepsilon_3| + K^m_\eta |\varepsilon_1 \eta| - K^r_\zeta |\delta_1 \zeta| - K^r_\eta |\delta_1 \eta|$$
$$\le |\varepsilon_3| + \left(K^m_\eta (1 + \frac{|\zeta|}{2}) - K^r_\eta \right) |\delta_1 \eta| - K^r_\zeta |\delta_1 \zeta|$$
$$\le \frac{1}{2} |\delta_1 \zeta| \left[2 \left(K^m_\eta (1 + \frac{|\zeta|}{2}) - K^r_\eta \right) \frac{|\eta|}{|\zeta|} + (1 - 2K^r_\zeta) \right].$$

Hence, by (4.5.3) (since $C_o > 1$) it holds $\Delta^T F_p < 0$ and

$$[\Delta^R F_p]_+ \le \frac{1}{2} |\delta_1 \zeta| \le \frac{1}{2|\eta|(K^r_\eta - K^m_\eta(1 + |\zeta|/2))/|\zeta| + (2K^r_\zeta - 1)} [\Delta^T F_p]_-,$$

i.e. $[\Delta^R F_p]_+ \le \mu_1^p [\Delta^T F_p]_-$.

On the other hand, if δ_1 is a shock we get the same estimates as before in the accurate case, while in the simplified case we have

$$\Delta^R F_p = |\varepsilon_3| \le \frac{C_o}{2}|\delta_1 \zeta|$$

and by (4.5.3) it holds

$$\Delta^T F_p \le \frac{\xi}{2}|\delta_1\zeta| \left[2\left(K_\eta^m (1 + \frac{C_o}{2}|\zeta|) - K_\eta^r \right) \frac{|\eta|}{|\zeta|} + (C_o - 2K_\zeta^r) \right] < 0.$$

Hence,

$$[\Delta^R F_p]_+ \le \frac{C_o}{2\xi|\eta|(K_\eta^r - K_\eta^m(1 + C_o|\zeta|/2))/|\zeta| + \xi(2K_\zeta^r - C_o)} [\Delta^T F_p]_-,$$

i.e. $[\Delta^R F_p]_+ \le \mu_1^p [\Delta^T F_p]_-$.

Now, assume $i = 3$. If δ_3 is a rarefaction, then by (2.3.8), (2.3.9) we have

$$\Delta^R F_p = \begin{cases} \xi|\varepsilon_1| + K_\eta^m \xi|\varepsilon_1 \eta| = \xi|\varepsilon_1|(1 + K_\eta^m|\eta|) & \text{if } |\delta_3| \ge \rho, \\ \\ |\varepsilon_1| & \text{if } |\delta_3| < \rho, \end{cases}$$

and

$$\Delta^T F_p = |\varepsilon_3| - |\delta_3| - K_\zeta^m |\delta_3\zeta| = -|\varepsilon_1| - K_\zeta^m|\delta_3\zeta| \le -|\varepsilon_1|(1 + 2K_\zeta^m).$$

Hence,

$$[\Delta^R F_p]_+ \le \xi|\varepsilon_1|(1 + K_\eta^m|\eta|) \le \frac{\xi(1 + K_\eta^m|\eta|)}{1 + 2K_\zeta^m}[\Delta^T F_p]_- \le \mu_1^p [\Delta^T F_p]_-.$$

If, instead, δ_3 is a shock we have

$$\Delta^R F_p = \begin{cases} |\varepsilon_1| + K_\eta^m|\varepsilon_1\eta| = |\varepsilon_1|(1 + K_\eta^m|\eta|) & \text{if } |\delta_3| \ge \rho, \\ |\varepsilon_1| & \text{if } |\delta_3| < \rho, \end{cases}$$

and $\Delta^T F_p = \xi|\varepsilon_3| - \xi|\delta_3| = -\xi|\varepsilon_1|$. Hence,

$$[\Delta^R F_p]_+ \leq |\varepsilon_1|(1 + K_\eta^m|\eta|) \leq \frac{1 + K_\eta^m|\eta|}{\xi}[\Delta^T F_p]_- \leq \mu_1^p[\Delta^T F_p]_-.$$

□

Corollary 4.5.2. *The Glimm functional F_p is non-increasing across time $\bar{t}$ if the following conditions hold for $\xi > 1$:*

$$K_\eta^m \leq \frac{\xi - 1}{|\eta|}, \qquad \frac{\xi - 1}{2} + K_\eta^m \xi \frac{|\eta|}{2} - K_\zeta^m \leq 0, \qquad C_o \leq \frac{2\xi K_\eta^m}{\xi + 1}, \tag{4.5.4}$$

$$K_\eta^\ell|\eta| + (K_\zeta^\ell - K_\zeta^m)|\zeta| \geq 0, \qquad 1 + K_\zeta^m \frac{|\zeta|}{2} - K_\eta^m \leq 0, \tag{4.5.5}$$

$$\left(\frac{\xi - 1}{2} - K_\eta^\ell\right)|\eta| + \left(K_\zeta^m\big(1 - \frac{|\eta|}{2}\big) - K_\zeta^\ell\right)|\zeta| \leq 0 \tag{4.5.6}$$

$$(1 - K_\zeta^r)|\zeta| + \left(K_\eta^m\big(1 + \frac{|\zeta|}{2}\big) - K_\eta^r\right)|\eta| \leq 0, \tag{4.5.7}$$

$$\left((\xi + 1)\frac{C_o}{2} - \xi K_\zeta^r\right)|\zeta| + \xi\left(K_\eta^m\big(1 + \frac{C_o}{2}|\zeta|\big) - K_\eta^r\right)|\eta| \leq 0. \tag{4.5.8}$$

Proof. Notice that $(4.5.2)_1$ is implied by $(4.5.4)_3$ and (4.5.3) is implied by (4.5.8). Moreover, by (4.5.4)–(4.5.8) it holds $\mu_1^p \leq 1$. □

Remark 4.5.3. *Thanks to a more careful analysis of the interactions between 3-rarefaction waves and η_0, here we obtain that* $(4.5.5)_1$ *and* (4.5.6) *improve the corresponding condition (5.2)$_1$ found in [8]. Then, the existence result contained in this thesis is slightly better than that of [8].*

Proposition 4.5.4 (Interactions between i-waves)**.** *Consider the interaction at time $\bar{t} > 0$ of two waves of the same family $i = 1, 3$ and assume* (2.3.14)*. Then, we have*

$$[\Delta^R F_p]_+ \leq \mu_2^p[\Delta^T F_p]_-$$

under the condition

$$1 \leq \xi \leq \frac{1}{c(m_o)}. \tag{4.5.9}$$

Here, μ_2^p is defined in (4.5.1).

Proof. The proof is omitted since it is completely analogous to that of Proposition 4.3.3 and 4.4.3. □

Corollary 4.5.5. *Under the assumption of the previous proposition, F_p is non-increasing across time $\bar{t}$ provided that the following conditions hold for $\xi > 1$:*

$$K_\eta^m \le \frac{\xi - 1}{|\eta|}, \qquad K_\zeta^m \le \frac{\xi - 1}{|\zeta|}, \tag{4.5.10}$$

$$K_\eta^r|\eta| + K_\zeta^r|\zeta| \le \xi - 1, \qquad K_\eta^\ell|\eta| + K_\zeta^\ell|\zeta| \le \xi - 1. \tag{4.5.11}$$

Proof. By (4.5.10), (4.5.11) we get $\mu_2^p \le 1$. □

4.5.1 The choice of the parameters

Here, we make some comments and we establish the order in which we can choose the parameters. Remark that we keep strict inequalities on the conditions (4.5.4)–(4.5.8) and (4.5.10), (4.5.11).

First, we can rewrite $(4.5.5)_1$ and (4.5.6) as

$$K_\zeta^m|\zeta| < K_\eta^\ell|\eta| + K_\zeta^\ell|\zeta|,$$
$$\frac{\xi - 1}{2}|\eta| + K_\zeta^m|\zeta|\big(1 - \frac{|\eta|}{2}\big) < K_\eta^\ell|\eta| + K_\zeta^\ell|\zeta|,$$

and (4.5.7) as

$$K_\eta^m\big(1 + \frac{|\zeta|}{2}\big)|\eta| + |\zeta| < K_\eta^r|\eta| + K_\zeta^r|\zeta|.$$

By (4.5.11) we have the following necessary conditions

$$K_\zeta^m|\zeta| < \xi - 1, \qquad \frac{\xi - 1}{2}|\eta| + K_\zeta^m|\zeta|\big(1 - \frac{|\eta|}{2}\big) < \xi - 1. \tag{4.5.12}$$

$$K_\eta^m\big(1 + \frac{|\zeta|}{2}\big)|\eta| + |\zeta| < \xi - 1. \tag{4.5.13}$$

Notice that $(4.5.12)_1$ and $(4.5.12)_2$ are equivalent since the latter inequality can be rewritten as

$$\left(1-\frac{|\eta|}{2}\right)\big(K_\zeta^m|\zeta|-(\xi-1)\big)<0$$

and $|\eta|<2$. Moreover, by $(4.5.5)_2$ and $(4.5.4)_2$ we have to require

$$K_\eta^m > 1+K_\zeta^m\frac{|\zeta|}{2}, \qquad K_\zeta^m > \frac{\xi-1}{2}+\xi K_\eta^m\frac{|\eta|}{2}, \tag{4.5.14}$$

that give the following lower bounds on K_η^m and K_ζ^m:

$$K_\eta^m > \frac{1+(\xi-1)|\zeta|/4}{1-\xi|\eta\zeta|/4}, \qquad K_\zeta^m > \frac{(\xi-1)+\xi|\eta|}{2(1-\xi|\eta\zeta|/4)}. \tag{4.5.15}$$

Remark that (4.5.14) represents an affine cone in the (K_η^m, K_ζ^m)-plane under the condition

$$\xi < \frac{4}{|\eta\zeta|}. \tag{4.5.16}$$

The vertex is the point whose coordinates are given by the right-hand sides of (4.5.15). Hence, K_ζ^m must be chosen in the non-empty intervals identified by (4.5.12), $(4.5.15)_2$, while K_η^m must be chosen in that identified by (4.5.13), $(4.5.15)_1$. By $(4.5.12)_1$, for K_ζ^m we get the necessary conditions

$$\frac{(\xi-1)+\xi|\eta|}{2(1-\xi|\eta\zeta|/4)}|\zeta| < K_\zeta^m|\zeta| < \xi-1, \tag{4.5.17}$$

while for K_η^m we get

$$\frac{1+(\xi-1)|\zeta|/4}{1-\xi|\eta\zeta|/4}\big(1+\frac{|\zeta|}{2}\big)|\eta|+|\zeta| < K_\eta^m\big(1+\frac{|\zeta|}{2}\big)|\eta|+|\zeta| < \xi-1. \tag{4.5.18}$$

To simplify the expressions, we introduce the notation $|\eta|=x$, $|\zeta|=y$

and $\xi - 1 = z$. Then, the previous inequalities can be rewritten as

$$\frac{xy}{2}z^2 + \left(y - 2 + \frac{3}{2}xy\right)z + xy < 0, \tag{4.5.19}$$

$$\frac{xy}{4}z^2 + \left(\frac{xy}{8}(4-y) - 1\right)z + \left(1 + \frac{y}{2}\right)x + y\left(1 - \frac{xy}{4}\right) < 0, \tag{4.5.20}$$

respectively. We also denote

$$a(x,y) = \frac{xy}{2}, \quad b(x,y) = y - 2 + \frac{3}{2}xy, \quad c(x,y) = xy, \quad d(x,y) = \frac{xy}{4},$$
$$e(x,y) = \frac{xy}{8}(4-y) - 1, \quad f(x,y) = \left(1 + \frac{y}{2}\right)x + y\left(1 - \frac{xy}{4}\right),$$

so that (4.5.19), (4.5.20) become

$$a(x,y)z^2 + b(x,y)z + c(x,y) < 0, \tag{4.5.21}$$
$$d(x,y)z^2 + e(x,y)z + f(x,y) < 0, \tag{4.5.22}$$

respectively. Notice that the coefficients a, c, d, f are positive, e is negative and b may change sign. In order that each of these equations have distinct solutions, the discriminants $b^2 - 4ac$ and $e^2 - 4df$ must be strictly positive. If $b < 0$ and $e < 0$, all the solutions are positive. Thus, for (4.5.19) we impose

$$y - 2 + \frac{3}{2}xy < 0, \qquad \left(y - 2 + \frac{3}{2}xy\right)^2 - 2x^2y^2 > 0, \tag{4.5.23}$$

while for (4.5.20) we require

$$\left(\frac{xy}{8}(4-y) - 1\right)^2 - xy\left[\left(1 + \frac{y}{2}\right)x + y\left(1 - \frac{xy}{4}\right)\right] > 0. \tag{4.5.24}$$

Assuming (4.5.23) and (4.5.24), we denote by $z_{1,2}(x,y)$ the solutions to the equation in (4.5.21) and by $z_{3,4}(x,y)$ the solutions to the equation in (4.5.22).

Hence, by (4.5.16)–(4.5.20) we get

$$1+\max\Big\{z_1(x,y)\,,z_3(x,y)\Big\}<\xi<1+\min\Big\{z_2(x,y)\,,z_4(x,y)\,,\frac{4}{xy}-1\Big\}. \tag{4.5.25}$$

Therefore, we can introduce the domain S_p represented in Figure 4.1 as the set given by

$$S_p:=\Bigg\{(x,y):\ \max\Big\{z_1(x,y),z_3(x,y)\Big\}<\min\Big\{z_2(x,y),z_4(x,y),\frac{4}{xy}-1\Big\}\Bigg\}, \tag{4.5.26}$$

and the function $\mathcal{H}_p$ as defined by

$$\mathcal{H}_p(|\eta|,|\zeta|):=\max\Big\{z_1(|\eta|,|\zeta|),z_3(|\eta|,|\zeta|)\Big\}. \tag{4.5.27}$$

By (4.5.9) we find that the condition that relates m_o to $|\eta|,|\zeta|$ is

$$1+\mathcal{H}_p(|\eta|,|\zeta|)<\frac{1}{c(m_o)}. \tag{4.5.28}$$

Once fixed $(|\eta|,|\zeta|)\in S_p$, we proceed with the choice of the parameters.

i) We choose m_o such that (4.5.28) holds and, in turn, we choose ξ satisfying both (4.5.25) and

$$1+\mathcal{H}_p(|\eta|,|\zeta|)<\xi\leq\frac{1}{c(m_o)}\,, \tag{4.5.29}$$

so that (4.5.9) holds.

ii) We choose K_η^m, K_ζ^m in the interval identified by (4.5.17) and (4.5.18), i.e. satisfying

$$\frac{(\xi-1)+\xi|\eta|}{2(1-\xi|\eta\zeta|/4)}<K_\zeta^m<\frac{\xi-1}{|\zeta|}, \tag{4.5.30}$$

$$\frac{1+(\xi-1)|\zeta|/4}{1-\xi|\eta\zeta|/4}<K_\eta^m<\frac{\xi-1-|\zeta|}{(1+|\zeta|/2)|\eta|}. \tag{4.5.31}$$

In this way, $(4.5.4)_2$, $(4.5.5)_2$ and (4.5.10) hold. Then, we choose $K_\eta^\ell = K_\zeta^\ell$, $K_\eta^r = K_\zeta^r$ such that

$$\frac{K_\zeta^m|\zeta|}{|\eta|+|\zeta|} < K_\eta^\ell = K_\zeta^\ell < \frac{\xi-1}{|\eta|+|\zeta|}, \tag{4.5.32}$$

$$\frac{K_m^\eta\big(1+|\zeta|/2\big)|\eta|+|\zeta|}{|\eta|+|\zeta|} < K_\eta^r = K_\zeta^r < \frac{\xi-1}{|\eta|+|\zeta|}. \tag{4.5.33}$$

Thus, $(4.5.5)_1$ and (4.5.7) hold.

iii) Finally, we notice that (4.5.8) is equivalent to

$$\left((\xi+1)\frac{C_o}{2\xi} - K_\zeta^r\right)|\zeta| + \left(K_\eta^m\big(1+\frac{C_o}{2}|\zeta|\big) - K_\eta^r\right)|\eta| < 0\,. \tag{4.5.34}$$

Then, by taking ρ sufficiently small (since $C_o(\rho) \to 1$ if $\rho \to 0+$) and $\xi > 1$, (4.5.34) is implied by $(4.5.4)_3$.

4.6 The Proof of Theorem 4.1.1

Now, we collect into a single proposition all the results obtained so far.

Proposition 4.6.1. *For $\iota = d, b, p$ and $(|\eta|, |\zeta|) \in S_\iota$, let $m_o > 0$ satisfy*

$$1 + \mathcal{H}_\iota\left(|\eta|, |\zeta|\right) \le \frac{1}{c(m_o)} \tag{4.6.1}$$

and assume the following:

(d) *in the* **drop case** *the parameters ξ, $K_{\eta,\zeta}^{\ell,m,r}$ and ρ satisfy* (4.3.18)–(4.3.25) *and* (4.3.9)*;*

(b) *in the* **bubble case** *the parameters ξ, $K_{\eta,\zeta}^{\ell,m,r}$ and ρ satisfy* (4.4.11), (4.4.12), (4.4.14), (4.4.15) *and* (4.4.6)*;*

(p) *in the* **increasing pressure case** *the parameters ξ, $K_{\eta,\zeta}^{\ell,m,r}$ and ρ satisfy* (4.5.29)–(4.5.33) *and* $(4.5.4)_3$ *and* (4.5.8).

Then, the following two statements are verified.

i) Local Decreasing. *For any interaction at time $t > 0$ between two waves satisfying* (2.3.14), *it holds*

$$\Delta F_\iota(t) \leq 0.$$

ii) Global Decreasing. *Recall the functional defined in* (2.3.23). *If*

$$\overline{L}(0+) \leq m_o c(m_o) \tag{4.6.2}$$

and the approximate solution is defined in $[0, T]$, then $F_\iota(0+) \leq m_o$, $\Delta F_\iota(t) \leq 0$ for every $t \in]0, T]$ and (2.3.14) *is satisfied.*

Proof. Fix $\iota = d, b, p$. The first statement concerning the local decreasing property was proved in the previous sections. As for the global decreasing property, observe that, if we restrict to consider only waves located in $\{0 < x < 1\}$, by (4.3.11), (4.4.8) (4.5.10) we have

$$\begin{aligned} F_\iota(0+) &= L(0+) + Q_\iota(0+) \leq L(0+) \left(1 + \max\{K_\eta^m |\eta|, K_\zeta^m |\zeta|\}\right) \\ &\leq \xi^2 \overline{L}\restriction_{\{0<x<1\}}(0+), \end{aligned}$$

while if we restrict to either $\{x < 0\}$ or $\{x > 1\}$, by (4.3.12), (4.4.9) (4.5.11) we have

$$\begin{aligned} F_\iota(0+) &= L(0+) + Q_\iota(0+) \leq L(0+) \left(1 + K_\eta^{\ell,r} |\eta| + K_\zeta^{\ell,r} |\zeta|\right) \\ &\leq \xi^2 \overline{L}\restriction_{\{x<0\}\cup\{x>1\}}(0+). \end{aligned}$$

Then,

$$F_\iota(0+) \leq \xi^2 \overline{L}(0+)\,.$$

Fix $t \leq T$ and suppose by induction that $F_\iota(\tau) \leq m_o$ and $\Delta F_\iota(\tau) \leq 0$ for every $0 < \tau < t$, interaction time. Then, the inequality $\Delta F_\iota(t) \leq 0$ implies that

$$F_\iota(t) \leq F_\iota(0+) \leq \xi^2 \overline{L}(0+).$$

Hence, by (4.6.2) the size of a shock δ_i ($i = 1, 3$) at time t satisfies

$$|\delta_i| \leq \frac{1}{\xi} F_\iota(t) \leq \xi \overline{L}(0+) \leq \frac{1}{c(m_o)} \overline{L}(0+) \leq m_o$$

and (2.3.14) is verified. □

Remark 4.6.2. *In the drop case, the estimate on the global decreasing of the Glimm functional can be improved by localizing the variations in the three regions separated by the interfaces. For convenience we use R, S to indicate rarefaction waves and shock waves, respectively. By* (4.3.11) *in the middle region* $\{0 < x < 1\}$ *we have*

$$\begin{aligned} F_d(0+) =& L(0+) + Q_d(0+) \\ \leq& L^{1S}(0+) + L^{1R}(0+)\left(1 + K_\eta^m|\eta|\right) + L^{3S}(0+) \\ &+ L^{3R}(0+)\left(1 + K_\zeta^m|\zeta|\right) \\ \leq& L^{1S}(0+) + \xi L^{1R}(0+) + L^{3S}(0+) + \xi L^{3R}(0+) \\ \leq& \xi \overline{L}\restriction_{\{0<x<1\}}(0+). \end{aligned}$$

Then,

$$F_d(0+) \leq \xi^2 \overline{L}\restriction_{\{x<0\}}(0+) + \xi \overline{L}\restriction_{\{0<x<1\}}(0+) + \xi^2 \overline{L}\restriction_{\{x>1\}}(0+),$$

and (4.6.2) *becomes*

$$\overline{L}\restriction_{\{x<0\}}(0+) + c(m_o)\overline{L}\restriction_{\{0<x<1\}}(0+) + \overline{L}\restriction_{\{x>1\}}(0+) \leq m_o c(m_o). \quad (4.6.3)$$

Remark that an analogous localization property of the total variation is not true in the remaining cases $\iota = b, p$.

The proof of Theorem 4.1.1 is similar to that of Theorem 3.1.1 and fits into the general framework outlined in Chapter 2. In this last section, we finish it and add some final comments.

End of the Proof of Theorem 4.1.1. It only remains to reinterpret the choice of the parameter m_o in terms of the assumption (4.1.4) on the initial data.

Fix $\iota = d, b, p$. By (2.1.8) we can prove that

$$\overline{L}(0+) \leq \frac{1}{2}\,\mathrm{TV}\left(\log(p_o), \frac{u_o}{\min\{a_\ell, a_m, a_r\}}\right). \tag{4.6.4}$$

Now, by (4.3.17), (4.4.14), (4.5.28), (4.6.2), (4.6.4) and $(2.3.15)_2$ we look for an m_o satisfying the following inequalities:

$$\mathcal{H}_\iota(|\eta|, |\zeta|) < w(m_o), \tag{4.6.5}$$

$$\mathrm{TV}\left(\log(p_o), \frac{u_o}{\min\{a_\ell, a_m, a_r\}}\right) < z(m_o), \tag{4.6.6}$$

where

$$w(m_o) := \frac{1}{c(m_o)} - 1 = \frac{2}{\cosh m_o - 1}, \qquad z(m_o) := 2m_o\, c(m_o).$$

Notice that $w(m_o)$ is strictly decreasing from $\mathbb{R}_+$ to $\mathbb{R}_+$, while $z(m_o)$ is strictly increasing on the same sets. Recalling (4.1.1), we have

$$z\left(w^{-1}(r)\right) = \mathcal{K}(r), \qquad r \in]0, +\infty[\,.$$

Hence, if the assumption (4.1.4) is verified, it is easy to prove that one can choose m_o such that (4.6.5), (4.6.6) hold. Thus, Theorem 4.1.1 is completely proved. $\square$

Remark 4.6.3. *Consider the drop case. As in the proof of Theorem 3.1.1, we can approximate the initial datum in such a way that we can relate hypothesis* (4.1.4) *to* (4.6.3) *by including in* $\overline{L}\restriction_{\{0<x<1\}}(0+)$ *(*$\overline{L}\restriction_{\{x>1\}}(0+)$*, respectively) the total variation of* p_o *and* u_o *at the interface* $x = 0$ *(*$x = 1$*, respectively) and by* (2.1.8) *we can prove that*

$$\overline{L}\restriction_{\{x<0\}}(0+) \leq \frac{1}{2}\,\underset{x<0}{\mathrm{TV}}\left(\log(p_o), \frac{u_o}{a_\ell}\right), \quad \overline{L}\restriction_{\{x>1\}}(0+) \leq \frac{1}{2}\,\underset{x>1}{\mathrm{TV}}\left(\log(p_o), \frac{u_o}{a_r}\right),$$
$$\overline{L}\restriction_{\{0<x<1\}}(0+) \leq \frac{1}{2}\,\underset{0<x<1}{\mathrm{TV}}\left(\log(p_o), \frac{u_o}{a_m}\right). \tag{4.6.7}$$

Now, by (4.3.17), (4.6.3), (4.6.7) *and* (2.3.15)$_2$, *we have that* m_o *must satisfy*

$$\operatorname*{TV}_{x<0}\left(\log(p_o), \frac{u_o}{a_\ell}\right) + c(m_o) \operatorname*{TV}_{0<x<1}\left(\log(p_o), \frac{u_o}{a_m}\right) + \operatorname*{TV}_{x>1}\left(\log(p_o), \frac{u_o}{a_r}\right) < z(m_o), \quad (4.6.8)$$

where the left-hand side corresponds to (4.1.5).

We conclude the chapter with a comparison between Theorem 4.1.1 and Theorem 3.1 of [4]. First, notice that condition (1.4.16) can be written as

$$|\eta| + |\zeta| < 1/2,$$

when applied to the current problem. Then, we have that the set considered by Amadori and Corli in [2, 4], namely

$$S_{AC} := \left\{(|\eta|, |\zeta|) \in [0, 2[\times[0, 2[: \quad |\eta| + |\zeta| < \frac{1}{2}\right\},$$

is contained in S_ι, for $\iota = d, b$: indeed, for $\iota = d$ it follows from the fact that

$$\max\left\{\big(1 + \frac{|\zeta|}{2}\big)\frac{|\eta|}{2}, \big(1 + \frac{|\eta|}{2}\big)\frac{|\zeta|}{2}\right\} < \frac{|\eta| + |\zeta|}{2} < \frac{1}{4},$$

while for $\iota = b$ it is trivial. As for $\iota = p$, only by a numerical computation we can verify that $S_{AC}\backslash(\{(|\eta|, 0) : |\eta| < 2\} \cup \{(0, |\zeta|) : |\zeta| < 2\})$ is contained in S_ι. Moreover, we claim that

$$\mathcal{H}_\iota(|\eta|, |\zeta|) \le |\eta| + |\zeta|. \quad (4.6.9)$$

when $(|\eta|, |\zeta|) \in S_{AC}$. We show (4.6.9) just for $\iota = d, b$.

(d) Consider the drop case. By (4.1.2) we have that

$$\frac{4|\zeta|}{4 - 2|\eta| - |\eta\zeta|} \le |\eta| + |\zeta| \quad \Longleftrightarrow \quad (|\eta| + |\zeta|)\big(1 + \frac{|\zeta|}{2}\big) \le 2,$$

which holds true in S_{AC}. Since $\mathcal{H}_d$ is symmetric w.r.t. $|\eta|$ and $|\zeta|$, the claim is completely verified.

(b) Now, consider the bubble case. Since $\mathcal{H}_b$ is symmetric w.r.t. $|\eta|$ and $|\zeta|$, to prove the claim it suffices to verify that it holds

$$\frac{(2+|\eta|)4|\zeta|}{(2-|\eta|)(4-|\eta\zeta|)} \leq |\eta| + |\zeta|. \tag{4.6.10}$$

Simplifying the expression (4.6.10), we find that it is equivalent to

$$|\eta|^2|\zeta| + |\eta||\zeta|^2 - (2|\eta\zeta| + 2|\zeta|^2 + 4|\eta| + 8|\zeta|) + 8 \geq 0,$$

which is satisfied if $|\eta\zeta| + |\zeta|^2 + 2|\eta| + 4|\zeta| \leq 4$. Since $|\eta| < 1/2 - |\zeta|$, this last inequality is verified if

$$\left(\frac{1}{2} - |\zeta|\right)|\zeta| + |\zeta|^2 + 2\left(\frac{1}{2} - |\zeta|\right) + 4|\zeta| \leq 4,$$

that is when $|\zeta| \leq 6/5$. Therefore, (4.6.10) holds.

Now, condition (1.4.17) here becomes

$$\mathrm{TV}\left(\log(p_o), \frac{1}{\min\{a_\ell, a_m, a_r\}} u_o\right) < H(|\eta| + |\zeta|), \tag{4.6.11}$$

where recall that the function $H(r)$ is only defined for $r < 1/2$ and is given explicitly in (1.4.18) and (3.3.12). Since $\mathcal{K}(r) > H(r)$ in the common range $r < 1/2$ (refer to the end of Chapter 3) and it holds

$$\mathcal{K}\left(\mathcal{H}_\iota(|\eta|, |\zeta|)\right) > \mathcal{K}(|\eta| + |\zeta|) > H(|\eta| + |\zeta|),$$

we have that (4.1.4) improves (4.6.11). Consequently, we obtain enhanced conditions on the initial data in comparison with [2, 4], even though the latter results apply to a wider class of λ_o.

4.6.1 Conclusions and open problems

This chapter concludes the analysis on the global existence of solutions to the initial-value problem (1.4.4), (1.4.5) with λ_o piecewise constant with either one or two jumps. As already mentioned, the results obtained so far are among the few existing theorems for large $\mathbf{BV}$ data.

Concerning system (1.4.4), some important questions remain open that are worth looking into. For example, at present we are not sure whether or not the global existence of $\mathbf{BV}$ solutions for the inital value problem (1.4.4), (1.4.5) with one or two phase interfaces fails if we do not assume the aforementioned threshold bounds on the initial data. Thus, it would be interesting to investigate the possibility for patterns of waves in the front tracking scheme that lead to the blow-up in finite time of the $\mathbf{BV}$ norm of the approximate solutions when the threshold is violated. However, this would not confirm the failure of the global existence at all, since the blow-up might be due to instabilities of the front tracking approximations. Such is a problem that comes up also in the case of a p-system with pressure p that does not satisfy the Bakhvalov condition (1.3.5): in [18] the authors stress that, even if they are able to construct a pattern of waves for which there are no uniform bounds on the total variation, this does not mean that the global existence may not hold.

Moreover, in order to complete the well-posedness picture for (1.4.4), (1.4.5) in the case of large data, it remains to study the uniqueness and continuous dependence of solutions from the initial data. It is not clear yet if, thanks to the simple form of the equations in (1.4.4) and the consequent explicit definition of wave curves and strengths of waves, the approach with the functional of [19] would turn out to be as easily manageable in the case of large data as are the estimates in the proof of the existence part.

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