Approaching fundamental limits to free-space communication through atmospheric turbulence

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ABSTRACT

We have determined the optimal beams for free-space optical transmission through atmospheric turbulence. These are stochastic eigenmodes derived analytically from a canonical turbulence model, assuming known turbulence statistics. Under weak or strong turbulence, using these modes as transmit and receive bases minimizes signal degradation by turbulence, and minimizes the complexity of any signal processing method employed to compensate for turbulence. These modes can be mapped to/from single-mode waveguides by fundamentally lossless modal multiplexers and demultiplexers. Adaptive optics can be replaced by adaptive multi-input multi-output signal processing, enabling compensation of fast fluctuations of both phase and amplitude.

Keywords: Free-space communications, space multiplexing, atmospheric eigenmodes, capacity limits.

1. INTRODUCTION

The propagation of beams through random media is of fundamental importance in applications such as optical communications, remote sensing, and imaging. However, coherent fields that propagate through random media such as atmospheric turbulence are subject to distortion and scintillation that can cause considerable degradation in system performance. In a modal communication system [1][2], which exploits spatial degrees of freedom by launching modulated data signals onto different orthogonal spatial optical modes, aberrations induced by turbulence on the transmitted modes result in signal degradation from mode coupling and mode cross-talk, thus reducing the channel capacity of the system.

Here, we consider the problem of finding optimal transmission beams [3][4] for propagation through the random atmosphere based on the knowledge of the second-order moment of the propagated field. The covariance function $C(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ is the crucial ingredient in the coherent mode decomposition analysis, as it encodes the notion of similarity between the fields at points $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$.[1] The statistics of a field that has propagated through atmospheric turbulence can be characterized by the covariance or, similarly, by its associated normal-mode decomposition, a statistical procedure that uses an orthogonal transformation to convert a set of observations of correlated fields into a set of amplitudes of linearly uncorrelated fields called eigenmodes.

Assuming that the covariance function C of the atmospheric fields is integrable over the region $\boldsymbol{R}_1$ occupied by the fields in the transmitter plane, it represents a Hilbert–Schmidt kernel and, by *Mercer's theorem*, it may be expanded in a series of orthogonal functions of the form[2]

$$C(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \sum_n \lambda_n \, \psi_n^*(\boldsymbol{\rho}_1) \, \psi_n(\boldsymbol{\rho}_2) \,, \tag{1}$$

where the eigenvalues λ_n and the eigenmodes $\psi_n(\boldsymbol{\rho})$ satisfy the Fredholm integral equation

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[1] It is a basic assumption that points with inputs $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ that are close in the receiver aperture are likely to have similar field values, and thus, under a Gaussian process view, it is the covariance function that defines nearness or similarity.

[2] Mercer's theorem and Hilbert–Schmidt kernels are introduced in the theory of integral equations. (See, for instance, [5].)

$$\int_{\boldsymbol{R}_1} C(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\, \psi_n(\boldsymbol{\rho}_1)\, d\boldsymbol{\rho}_1 = \lambda_n\, \psi_n(\boldsymbol{\rho}_2)\,. \tag{2}$$

The summation in Eq. (1), in general, may be a finite or infinite sum. The eigenvalues are non-negative and, loosely speaking, describe the amount of power allocated on average to the eigenmode ψ_n. Indeed, for $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = \boldsymbol{\rho}$, we have $C(\boldsymbol{\rho}, \boldsymbol{\rho}) = \sum_n \lambda_n\, |\psi_n(\boldsymbol{\rho})|^2$. The eigenmodes are orthogonal and are typically taken to be orthonormal. The orthonormality implies that $\int_{\boldsymbol{R}_1} \psi_n^*(\boldsymbol{\rho})\, \psi_m(\boldsymbol{\rho})\, d\boldsymbol{\rho} = \delta_{nm}$, with δ_{nm} the Kronecker symbol. Equation (1) is often called the coherent mode representation of the covariance function.

2. STATISTICAL EIGENMODES OF LASER BEAMS IN THE ATMOSPHERE

The eigenvalues λ_n and the eigenmodes $\psi_n(\boldsymbol{\rho})$ of the coherent mode representation are obtained as the solution of the homogeneous Fredholm integral equation Eq. (2). For coherent beams in Kolmogorov atmospheric turbulence, a propagation model using the extended Huygens-Fresnel principle expresses the covariance function C at the receiver aperture as[3]

$$C(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \exp\left(-\frac{|\boldsymbol{\rho}_1|^2 + |\boldsymbol{\rho}_1|^2}{2\omega_0^2}\right) \exp\left(-\frac{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^{5/3}}{2\delta^2}\right), \tag{3}$$

where ω_0 is the beam intensity radius, and $\delta^2 = r_0^2/6.88$ is proportional to the coherence diameter r_0 describing the spatial correlation of field fluctuations in the receiver plane. Unfortunately, as happens frequently in the study of propagation effects through the turbulent atmosphere, the 5/3-power law makes the analytic solution to Eq. (2) intractable.

In order to overcome the difficulties with the 5/3-power law, when the coherence width δ is small ($\omega_0 > \delta$), we can consider a statistical model in which the random fields are expressed as finite sums over statistically independent cells in the receiver aperture. To a good approximation, the field can be considered to consist of $(\omega_0/\delta)^2$ independent speckle cells, each of radius δ. As the spatial variation of the random field in any of these independent cells can be described as a quadratic power law, we can approximate the total field covariance as a linear superposition of a number of Gaussian-shaped basis functions. We have shown that this expansion leads to a squared exponential covariance function:

$$C(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{\sigma_e^2}{\sigma_a^2}\, \exp\left(-\frac{|\boldsymbol{\rho}_1|^2 + |\boldsymbol{\rho}_1|^2}{2\sigma_g^2}\right)\, \exp\left(-\frac{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^2}{2\sigma_s^2}\right), \tag{4}$$

with σ_e^2, σ_g^2, and σ_s^2 described in terms of ω_0^2 and δ^2 as

$$\frac{1}{\sigma_e^2} = \frac{2}{\delta^2} + \frac{2}{\omega_0^2}$$

$$\sigma_s^2 = \delta^2\left(1 + \frac{\delta^2}{\omega_0^2}\right)$$

$$\sigma_g^2 = \omega_0^2\left(1 + \frac{\delta^2}{\omega_0^2}\right).$$

We have verified numerically that this approach is very accurate, even when the number of speckle cells $(\omega_0/\delta)^2$ within the receiver area is very small.

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3 Atmospheric turbulence has been studied extensively, and various theoretical models has been proposed to describe turbulence-induced phase degradation and amplitude fluctuations. The extended Huygens Fresnel principle is often used to formulate the field correlations at the receiver aperture. (See, for instance, [6].)

The advantage of considering the modulated squared exponential covariance function (4) is that its coherent mode decomposition can be developed analytically. We have shown, as can be verified by direct substitution, that its eigenmodes $\psi_n(\boldsymbol{\rho})$ are given by the normalized Laguerre-Gauss modes

$$\psi_n(\rho,\phi) = \frac{1}{\omega}\sqrt{\frac{2\,p!}{\pi\,(\,p+|l|\,)!}}\left(\sqrt{2}\,\frac{\rho}{\omega}\right)^{|l|} L_p^{|l|}\left(2\,\frac{\rho^2}{\omega^2}\right)\exp\left(-\,\frac{\rho^2}{\omega^2}\right)\exp(jl\phi)\,, \tag{5}$$

where $L_p^{|l|}$ are the normalized Laguerre polynomials and $l = 0, \pm 1 \pm 2, \ldots$ and $p = 0,\ 1,\ 2, \ldots$ are arranged in a double-indexed sequence. The ψ_n are sorted in ascending order $n = 1,\ 2, \ldots$ by decreasing eigenvalues λ_n:

$$\lambda_n = \frac{\pi}{2}\omega^2(1-\xi)\,\xi^{\frac{|l|}{2}+p}\,. \tag{6}$$

Here,

$$\xi = \frac{1 + 2\frac{\sigma_g^2}{\sigma_s^2} - \sqrt{1 + 2\frac{\sigma_g^2}{\sigma_s^2}}}{1 + 2\frac{\sigma_g^2}{\sigma_s^2} + \sqrt{1 + 2\frac{\sigma_g^2}{\sigma_s^2}}}$$

and

$$\frac{1}{\omega^2} = \frac{1}{2\sigma_g^2}\sqrt{1 + 2\frac{\sigma_g^2}{\sigma_s^2}}\,,$$

where ω is the effective beam width of the modes, which, as expected, depends on the coherence width δ (see Fig. **Error! Reference source not found.**).

3. SPACE-MULTIPLEXED LASER COMMUNICATION

In this section, we investigate transmission using the statistical eigenmodes ψ_n over a general atmospheric channel, applying mode-division multiplexing to near-field line-of-sight (LOS) free-space optical communication. Here, the term "near-field" refers to link geometries in which, on average, the minimum spot size on the receiver plane is smaller than the receiver aperture, yielding near-perfect power coupling when the beam is focused on the receiver. We consider a modal multiplexing system that transmits a complete set of spatial modes, such as all the LG, with a common central rotation axis. We assume that all modes are efficiently multiplexed at the transmitter, and spatially co-propagate through the atmospheric channel. We consider that at the receiver, the modes are demultiplexed without loss or crosstalk (see, e.g., [7][8]) and are detected coherently [9][10].

Given N transmit and receive modes, and frequency-flat fading, the baseband complex model we consider is

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n}\,,$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the $N{\times}1$ input and output vectors, respectively, while $\boldsymbol{n}\sim\mathcal{N}(\boldsymbol{0},\sigma^2)$ is additive circularly symmetric complex Gaussian noise. The input vector component x_j represents the signal amplitude transmitted in eigenmode ψ_j, while the output vector component y_i represents the signal amplitude received in eigenmode ψ_i. The channel is characterized by the $N{\times}N$ random matrix $\boldsymbol{H}$, with entries H_{ij} representing the scattering gain from transmit mode j to receive mode i. Assuming a constraint on the total transmitted power P for each channel realization, the spectral efficiency, in bit/s/Hz, can be represented as

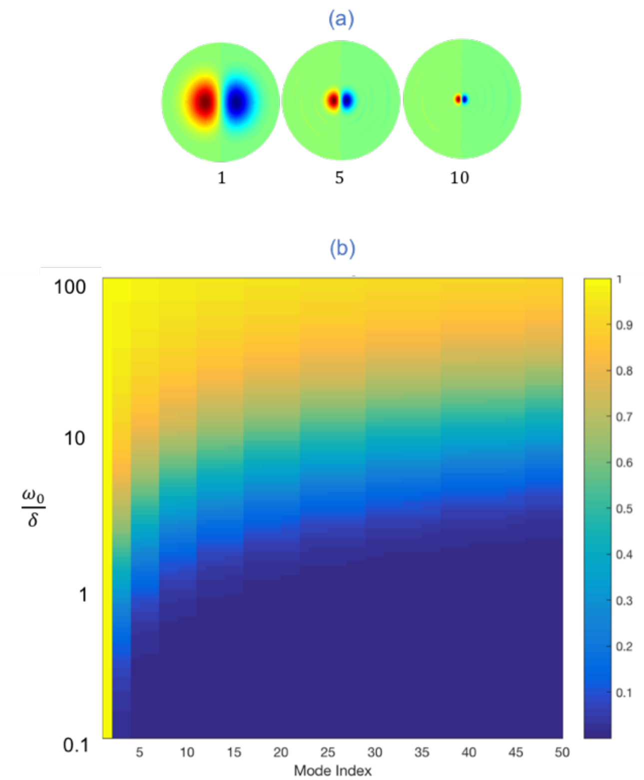

Fig. 1. (a) The effective beam width ω of the modes depends on the coherence width δ. As an example, the plot shows the evolution of the second eigenmode ($n = 2$) with the coherence parameter (ω_0/δ). Modes under weak, medium , and strong turbulence conditions, with (ω_0/δ) equal to 1, 5, and 10, respectively, are considered. (b) Color map plot of the eigenvalues λ_{n} of the atmospheric covariance function. They have been normalized to the largest eigenvalue λ_1. Eigenvalues are sorted in descending order, as given by the mode index n, as a function of coherence parameter (ω_0/δ).

$$S = \max_{\boldsymbol{Q}} E_{\boldsymbol{H}}\{\log_2(\boldsymbol{I} + \gamma\, \boldsymbol{H}^*\boldsymbol{Q}\, \boldsymbol{H})\}\,, \tag{7}$$

where the $N{\times}N$ matrix $\boldsymbol{Q} = E[\boldsymbol{x}\, \boldsymbol{x}^*]$ denotes the covariance of the input.[4] Without loss of generality, the variance σ^2 of the white Gaussian noise $\boldsymbol{n}$, the input power trace$\{\boldsymbol{Q}\}$, and the channel power trace$\{\boldsymbol{H}\}$ are set to one, and the information transfer rates are obtained as a function of the channel SNR $\gamma = P/\sigma^2$. Note that the expectation $E_{\boldsymbol{H}}$ is computed using the statistical distribution of $\boldsymbol{H}$.

If we consider channels that either vary slowly over time and/or are reciprocal, information on the instantaneous channel state, i.e., the realization of $\boldsymbol{H}$ can be assumed known at the transmitter. In this case, the channel can be partitioned by eigenbeam-forming into orthogonal sub-channels that are determined by the instantaneous channel state. Due to the non-uniform gains of these orthogonal sub-channels, channel capacity can be maximized by allocating transmit power

[4] A given transmission strategy is completely characterized by its covariance matrix $\boldsymbol{Q}$. Let us decompose the normalized input covariance as $\mathrm{Q} = \boldsymbol{VPV}^*$ identifying the eigenvectors of $\boldsymbol{Q}$ with the columns of the unitary matrix $\boldsymbol{V}$ and its eigenvalues with the diagonal entries of $\boldsymbol{P}$. Both the eigenvectors and the eigenvalues have immediate engineering significance. The strategy consists of transmitting independent symbols along the eigenvectors of $\boldsymbol{Q}$, with the corresponding eigenvalues specifying the powers allocated to each eigenvector. It is worth considering the capacity-achieving forms of $\boldsymbol{Q}$ for two relevant regimes, i.e., when instantaneous information is available for tracking the states of the atmospheric channel $\boldsymbol{H}$, and when limited statistical information about $\boldsymbol{H}$ is used. (See [11].)

control to each sub-channel based on the water-pouring theorem. In this case, the ergodic-average capacity S given by Eq. (7) simplifies to the sum

$$\mathrm{S} = \sum_{n=1}^{N} \log_2\left(1 + \gamma_n\, \lambda_n\right), \tag{8}$$

where λ_n are the singular values of the channel covariance matrix and γ_n are the average SNRs in the orthogonal sub-channels using the optimized power allocation, given by $\gamma_n = P_n/\sigma^2$, with P_n the water-pouring-based transmitted power in the nth sub-channel and σ^2 the noise variance in each receiver element. The standard water-pouring algorithm allocates power such that $\sigma^2/\lambda_n + P_n$ is the same for all active orthogonal sub-channels and is zero for all idle sub-channels, subject to the total input power constraint P. As can be seen from Eq. (8), the spectral efficiency is determined by the number of independently addressable spatially multiplexed sub-channels and by their corresponding λ_n.[5]

If the transmitter has access to the statistical distribution of $\boldsymbol{H}$, but not to $\boldsymbol{H}$ itself, the transmit sub-channel directions and power allocations cannot be based on knowledge of the instantaneous channel state, but can be optimized based on statistics of the channel states that are fed back from the receiver to the transmitter.[6] Now, we consider that the eigenvectors of the input covariance $\boldsymbol{Q}$ are equal the eigenmodes of the channel covariance C. We address the optimization of the eigenvalues of the transmit covariance matrix, i.e., the power allocation policy, with a water-pouring algorithm. The fading channel cannot be partitioned into orthogonal independent channels and the ergodic-average spectral efficiency S is computed as

$$S = E_{\boldsymbol{H}}\left\{\log_2\left(\boldsymbol{I} + \gamma\, \boldsymbol{H}^{*}\boldsymbol{C}\, \boldsymbol{H}\right)\right\}. \tag{9}$$

In order to understand the benefit offered by the transmission strategies described above, we also consider the extreme case in which the transmitter has access to neither instantaneous nor statistical channel state information. When the transmitter has statistical knowledge of the channel state, it can adjust its transmission strategy to the correlation properties of the channel and adapt the beam width ω of the set of stochastic eigenmodes to the spatial statistics of the received fields, as described by δ. When no channel information is available at the transmitter, however, the transmitter cannot adapt to potential correlation in the channel. In this scenario, the only possible strategy is to use a family of Laguerre-Gauss beams with fixed beam width ω_0 and split the transmit power equally over all transmit beams, even though some of these beams will not propagate effectively through the channel. In this extreme case, the fading channel is completely unknown at the transmitter and the ergodic-average spectral efficiency S is computed as

$$S = E_{\boldsymbol{H}}\left\{\log_2\left(\boldsymbol{I} + \gamma\, \boldsymbol{H}^{*}\boldsymbol{U}\boldsymbol{I}\boldsymbol{U}^{*}\boldsymbol{H}\right)\right\}. \tag{10}$$

Here, the covariance matrix $\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{I}\boldsymbol{U}^{*}$ for the set of Laguerre-Gaussian beams considered in this transmission strategy is represented by the change of basis matrix $\boldsymbol{U}$, of which each column consists of the components of the corresponding beam of the set on the basis of the atmospheric eigenmodes.

Figure **Error! Reference source not found.** considers Eqs. (8)-(10) and show plots of the spectral efficiency as a function of coherence parameter (ω_0/δ) and a signal level described by SNR γ. It shows that better performance is obtained when statistical information about the channel is used. The capacity improvement obtained by exploiting even partial channel knowledge is substantial. It is interesting to observe the anticipated weak-turbulence behavior at small coherence parameter (ω_0/δ), where the entire transmit power is allocated to a small number of individual eigenvectors.

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5 For a single-beam system, when the total system power is increased by a factor 2^{δ}, the spectral efficiency $S = log_2(1+\gamma)$ is increased by δ. If, on the other hand, EDOF beams are transmitting in parallel, the system S should increase by EDOF $\times\delta$. Hence, the EDOF can be defined as EDOF $= (d/d\delta)\, S(2^{\delta}\gamma)|_{\delta=0}$. (See [12].)

6 If the transmitter receives the channel covariance matrix C only instead of the concrete channel realization $\boldsymbol{H}$, it does not have any information about the actual fading of each transmit-receive pair but possesses directional information regarding the signal subspaces that can be used for beamforming. The motivation for this approach stems from the fact that the channel statistics –i.e., the covariance C of the atmospheric channel— vary over much larger time scales than the instantaneous channel $\boldsymbol{H}$. Therefore, the statistical information can be easily obtained by exploiting reciprocity, or by employing feedback channels with significantly lower bandwidth compared with instantaneous $\boldsymbol{H}$ feedback systems.

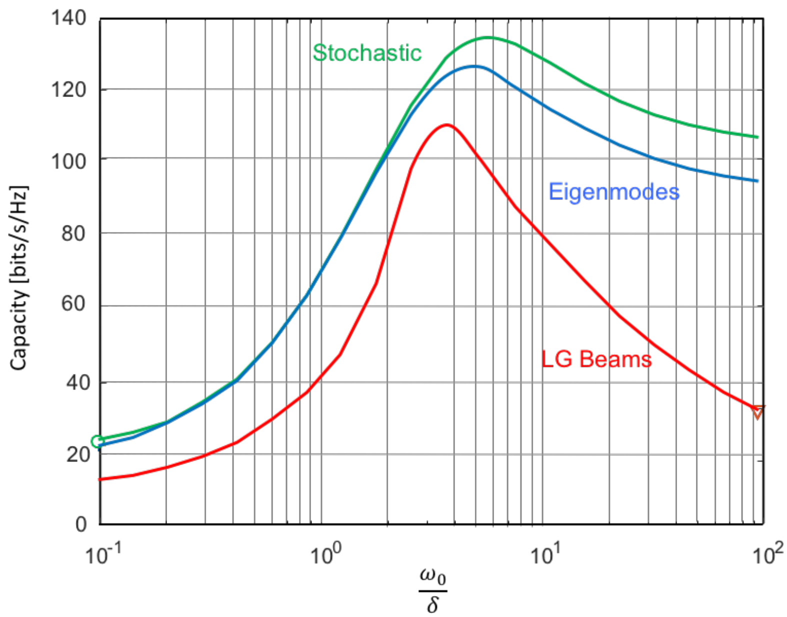

Fig. 2. Mutual information is shown as a function of coherence parameter (ω_0/δ) and considering a 50-dB SNR γ. The analysis assumes several transmission strategies $\mathbf{Q}$.

Only when $\omega \sim \omega_0$, for values of the coherence parameter $(\omega_0/\delta) \sim 1$, do the performance of the Laguerre-Gauss beams and the ideal eigenmodes become comparable.

4. CONCLUSIONS

We have considered the problem of finding optimal transmission beams, or eigenmodes, for propagation through the random atmosphere based on the knowledge of the second-order moment of the propagated field. Our optimal transmission method employs multiple eigenmodes that correspond to the orthogonal eigenvectors of the covariance matrix of the received field. We have found that the eigenvectors of the atmospheric covariance function are well-approximated by a family of Laguerre-Gauss modes whose beam width depends on the spatial statistics of the propagated fields, i.e., the shorter the field coherence length δ, the smaller the eigenmode beam width ω. Our results demonstrate that transmission of this mode family over an atmospheric channel can significantly increase channel capacity. One of the major strengths of the proposed approach is that can characterize any atmospheric propagation channel by only two parameters, namely, the beam intensity radius ω and the coherence length δ describing the spatial correlation of field fluctuations in the receiver plane. It should be straightforward to extract these parameters from measurements.

More results and details of our analysis will be presented at the meeting.

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