A method for calculation of program package elements for singular clusters

Nikita Strelkovskii, Sergey Orlov

International Institute for Applied Systems Analysis
Lomonosov Moscow State University

I I A S A

"The problem of constructing optimal closed-loop control strategies under uncertainty is one of the key problems of the mathematical control theory. Its solution would give a new impetus to the theory's development and create the foundation for its new applications"   Arkady Kryazhimskiy (2013)

1. Problem statement

LET US CONSIDER A LINEAR DYNAMIC CONTROLLED SYSTEM, described by an ordinary differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + c(t), t_0 \leq t \leq \vartheta \qquad (1)$$

Open-loop control (program) $u(\cdot)$ is a measurable function on $[t_0, \vartheta], u(t) \in P \subset \mathbb{R}^r$, P is a convex compact set. The initial state of the system maybe not known a priori: $x(t_0) = x_0 \in X_0 \subset \mathbb{R}^n$, where X_0 is a finite known set. The terminal condition $x(\vartheta) \in M \subset \mathbb{R}^n$, where M is a closed and convex set, should hold.
A linear signal $y(t) = Q(t)x(t)$, where $Q(\cdot)$ is left piecewise continuous matrix-function, $Q(t) \in \mathbb{R}^{q \times n}, t \in [t_0, \vartheta]$, is observed by the controlling side.

X_0

x_0''

x_0'

M

$[M]^\varepsilon$
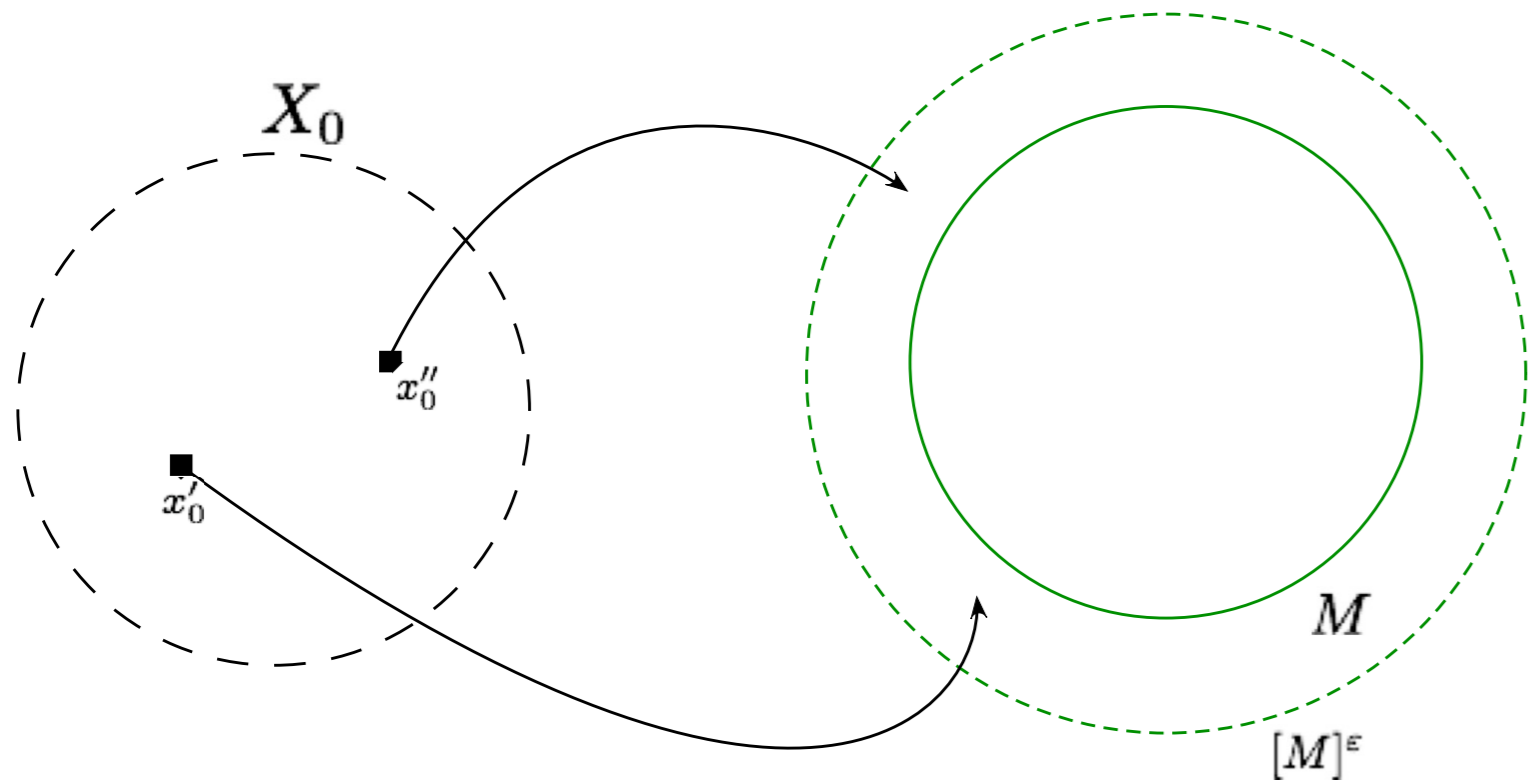
The problem of positional guidance is formulated as follows - based on the given arbitrary $\varepsilon > 0$ choose a closed-loop control strategy with memory, **whatever the system's initial state** x_0 **from the set** X_0, the system's motion $x(\cdot)$ corresponding to the chosen closed-loop strategy and starting at the time t_0 from the state x_0 reaches the state $x(\vartheta)$ belonging to the ε-neighbourhood of the target set M at the time ϑ.

2. Program packages method

Homogeneous system, corresponding to (1)

$$\dot{x}(t) = A(t)x(t)$$

For each $x_0 \in X_0$ its solution is given by the Cauchy formula:

$$x(t) = F(t, t_0)x_0; \; F(t, s) \; (t, s \in [t_0, \vartheta]) \text{ is the fundamental matrix.}$$
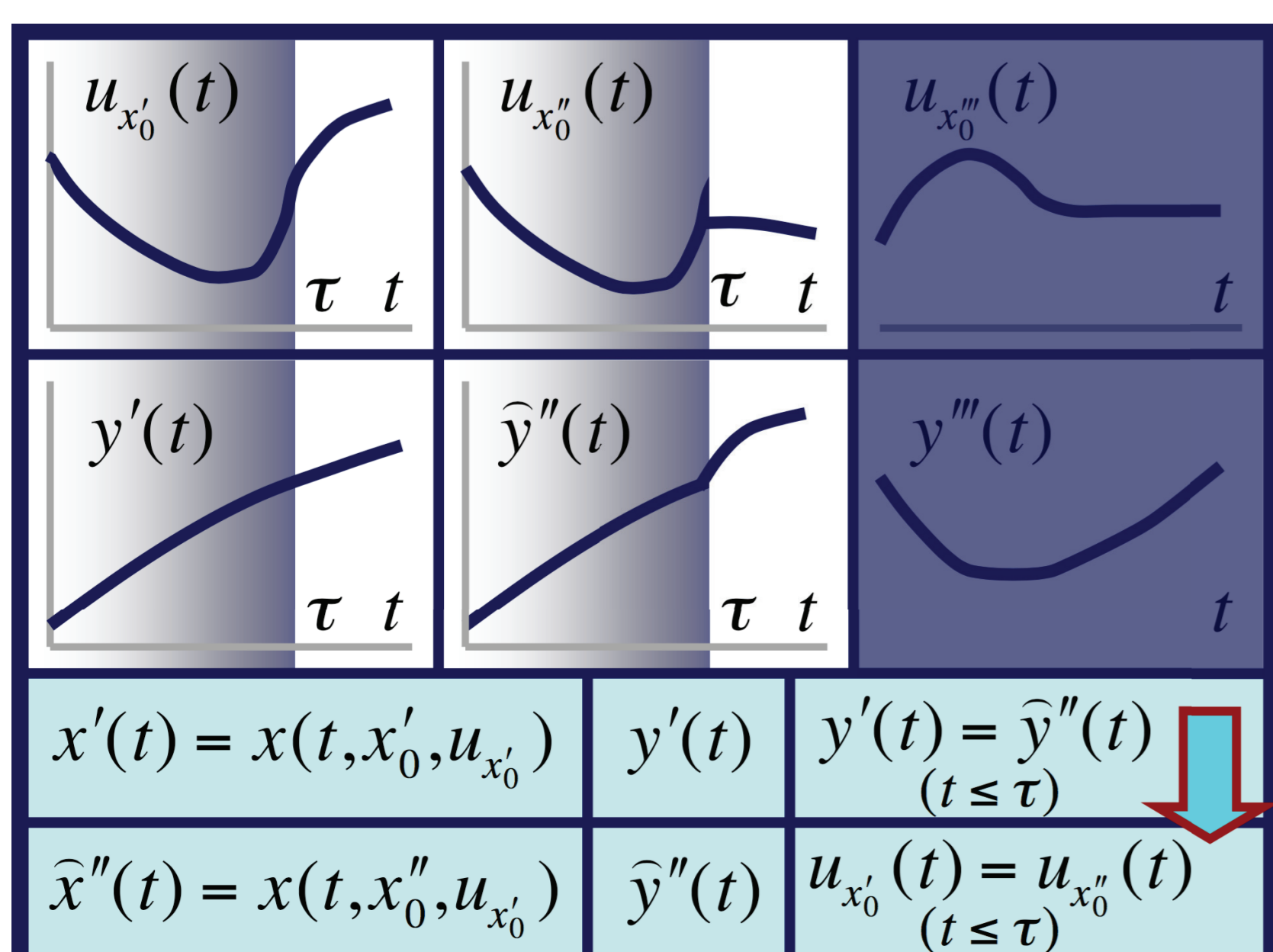
Homogeneous signal, corresponding to an admissible initial state $x_0 \in X_0$:

$$g_{x_0}(t) = Q(t)F(t, t_0)x_0 \; (t \in [t_0, \vartheta], \; x_0 \in X_0).$$

Let $G = \{g_{x_0}(\cdot) | x_0 \in X_0\}$ be the set of all homogeneous signals and let $X_0(\tau | g(\cdot))$ be the set of all admissible initial states $x_0 \in X_0$, corresponding to the homogeneous signal $g(\cdot) \in G$ till time point $\tau \in [t_0, \vartheta]$:

$$X_0(\tau | g(\cdot)) = \{x_0 \in X_0 : g(\cdot)|_{[t_0, \tau]} = g_{x_0}(\cdot)|_{[t_0, \tau]}\}.$$

Program package is an open-loop controls family $(u_{x_0}(\cdot))_{x_0 \in X_0}$, satisfying **non-anticipatory condition**: for any homogeneous signal $g(\cdot)$, any time $\tau \in (t_0, \vartheta]$ and any admissible initial states $x_0', x_0'' \in X_0(\tau | g(\cdot))$ the equality $u_{x_0'}(t) = u_{x_0''}(t)$ holds for almost all $t \in [t_0, \tau]$.

$u_{x_0'}(t)$	$u_{x_0''}(t)$	$u_{x_0'''}(t)$
$\tau \quad t$	$\tau \quad t$	t
$y'(t)$	$\widehat{y}''(t)$	$y'''(t)$
$\tau \quad t$	$\tau \quad t$	t
$x'(t) = x(t, x_0', u_{x_0'})$	$y'(t)$	$y'(t) = \widehat{y}''(t)$ $(t \leq \tau)$
$\widehat{x}''(t) = x(t, x_0'', u_{x_0''})$	$\widehat{y}''(t)$	$u_{x_0'}(t) = u_{x_0''}(t)$ $(t \leq \tau)$

Program package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding**, if for all $x_0 \in X_0$ holds $x(\vartheta | x_0, u_{x_0}(\cdot)) \in M$. **Package guidance problem** is solvable, if a guiding program package exists.
Theorem 1 ([1]). *The problem of positional guidance is solvable if and only if the problem of package guidance is solvable.*

3. Algorithm for singular clusters

An algorithm for the package guidance problem solution is proposed in paper [4], however, in some cases (so called *singular clusters* of the initial states set) it may not be applied, because the minimum condition degenerates. In this work an additional algorithm for tackling such cases is presented.
At a step $i = 0, 1, \ldots$ of the algorithm for arbitrary cluster $X_{0r}(\tau_k) \in \mathcal{X}_0(\tau_k), r = 1, \ldots, R(\tau_k), k = 1, \ldots, K$ a non-empty set w_0 of indices is estimated $j, 1 \leq j \leq m$ such that to $j \in w_0$ corresponds non-empty set $\sigma_j \subset [\tau_{k-1}, \tau_k]$, such that

$$\sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l_{x_{0q}}^{*(i)} \equiv 0 \quad (t \in \sigma_j). \qquad (2)$$

The problem of estimation of the components $(u_{X_{0r}(\tau_k)}^*(t))_j$ of the guiding program package elements on the sets $\sigma_j, j \in w_0$ reduces to the following system of equations

$$\begin{cases} \sum_{j \in w_0} \int_{\sigma_j} e_j(t)u_j(t)dt = v_{x_0}, \quad |u_j(t)| \leq a_* p, \\ x_0 \in X_{0r}(\tau_k) \end{cases} \qquad (3)$$

where

$$v_{x_0} = z_{x_0} - F(\vartheta, t_0)x_0 + a_* p \sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{j=1}^{r} e_j(t) \operatorname{sign} \left(\sum_{\substack{\bar{x}_0 \in X_{0r}(\tau_k) \\ r: x_0 \in X_{0r}(\tau_k)}} \sum_{q=1}^{n} D_{jq}(t) l_{x_{0q}}^* \right) d\tau,$$

$$z_{x_0} \in \operatorname{Arg} \max_{z \in M} \langle z, l_{x_0}^* \rangle_{\mathbb{R}^n},$$

$$e(t) = \begin{pmatrix} D(t)_{j1} \\ \ldots \\ D(t)_{jn} \end{pmatrix}, j = 1, \ldots, m.$$

Let us introduce functions $\lambda(\cdot, \cdot, \cdot, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \times [\tau_{k-1}, \tau_k] \times \mathbb{N} \mapsto \mathbb{R}$ $\Lambda(\cdot, \cdot, \cdot, \cdot, \cdot) : \mathbb{R}^n \times [0, 1] \times [\tau_{k-1}, \tau_k] \times \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$:

$$\lambda(z_{x_0}, l_{x_0}, \mu, \sigma, w) = \sum_{x_0 \in X_{0r}(\tau_k)} \langle l_{x_0}, z_{x_0} \rangle_{\mathbb{R}^n} - \mu \sum_{j \in w} \int_{\sigma_j} \left| \sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l_{x_{0q}} \right| dt,$$

$$\Lambda(z_{x_0}, \mu, \sigma, w, h) = \max_{\substack{\sum_{x_0 \in X_{0r}(\tau_k)} \|l_{x_0}\|_{\mathbb{R}^n} = 1 \\ l_{x_0} \in \mathbb{R}^h}} \lambda(z_{x_0}, l_{x_0}, \mu, \sigma, w).$$

The input of the algorithm is

$$\mu^{(0)} = 1, \sigma_j^{(0)} = \sigma_j, w^{(0)} = w_0, z_{x_0}^{(0)} = v_{x_0} \quad (j \in w_0).$$

After the step $h - 1, h = 1, 2, \ldots)$ of the algorithm the following is known

$$\mu^{(h-1)}, w^{(h-1)}, \sigma_j^{(h-1)}, z_{x_0}^{(h-1)}, l_{x_0}^{*(1)}, \ldots, l_{x_0}^{*(h-1)}, (u_{X_{0r}(\tau_k)}^*(t))_j \; (t \in \sigma_j / \sigma_j^{(h-1)}, j \in w_0).$$

Let us describe the step $h, h = 1, 2, \ldots$ of the algorithm:
1. $\mu^{(h)} \; (0 \leq \mu^{(h)} \leq \mu^{(h-1)})$ is calculated as the smallest root of the equation

$$\Lambda\left(z_{x_0}^{(h-1)}, \mu, \sigma^{(h-1)}, w^{(h-1)}, n - h + 1\right) = 0.$$

2. If $\mu^{(h)} = 0$, then $(u_{X_{0r}(\tau_k)}^*(t))_j = 0, t \in \sigma_j^{(h-1)}, j \in w^{(h-1)}$, and the required element of the guiding program package is found. Otherwise the algorithm proceeds to the next step immediately.
3. The vector family $(l_{x_0}^{*(h)})_{x_0 \in X_0}$ is estimated such, that

$$\sum_{x_0 \in X_{0r}(\tau_k)} \left\langle l_{x_0}^{*(h)}, l_{x_0}^{*(d)} \right\rangle = 0$$

for all $d = 1, \ldots, h - 1$ and

$$\Lambda\left(z_{x_0}^{(h-1)}, \mu^{(h)}, \sigma^{(h-1)}, w^{(h-1)}, n - h + 1\right) =$$
$$= \max_{\substack{\sum_{x_0 \in X_{0r}(\tau_k)} \|l_{x_0}\| = 1 \\ l_{x_0} \in \mathbb{R}^{n-h+1}}} \lambda\left(z^{(h-1)}, l_{x_0}^{(h)}, \mu^{(h)}, \sigma_j^{(h-1)}, w^{(h-1)}\right) =$$
$$= \sum_{x_0 \in X_{0r}(\tau_k)} \left\langle l_{x_0}^{*(h)}, z_{x_0}^{(h-1)} \right\rangle - \mu^{(h)} \sum_{j \in w^{(h-1)}} \int_{\sigma_j^{(h-1)}} \left| \sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l_{x_{0q}}^{*(h)} \right| =$$
$$= 0. \qquad (4)$$

4. The set of indices $w^{(h)} \subset w^{(h-1)}$ is estimated and for each $j \in w^{(h)}$ a set $\sigma_j^{(h)} \subset \sigma_k^{(h-1)}, k \in w^{(h-1)}$ is found such that

$$\sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l_{x_{0q}1}^{*(h)} \equiv 0 \quad (j \in w^{(h)}, t \in \sigma_j^{(h)}).$$

5. On the sets $\sigma_k^{(h-1)} / \sigma_j^{(h)}, k \in w^{(h-1)}, j \in w^{(h)}$ the components $(u_{X_{0r}(\tau_k)}^*(t))_j$ of the guiding program package elements are

$$(u_{X_{0r}(\tau_k)}^*(t))_j = a_* p \operatorname{sign} \left(\sum_{x_0 \in X_{0r}(\tau_k)} \sum_{q=1}^{n} D_{jq}(t) l_{x_{0q}}^{*(h)} \right).$$

6. If $\operatorname{mes} \sigma^{(h)} = \operatorname{mes} \bigcup_{j \in w^{(h)}} \sigma_j^{(h)} = 0$, then $(u_{X_{0r}(\tau_k)}^*(t))_j$ is known on the whole set $\sigma = \bigcup \sigma_j$ and the problem is solved. Otherwise the algorithm proceeds to the next step
7. The following approximation is calculated

$$z^{(h)} = z^{(h-1)} - a_* p \mu^{(h)} \sum_{j \in w^{(h)}} \int_{\sigma_j^{(h-1)}} e_j(t) \operatorname{sign} \left(\sum_{\substack{\bar{x}_0 \in X_{0r}(\tau_k) \\ r: x_0 \in X_{0r}(\tau_k)}} \sum_{q=1}^{n} D_{jq}(t) l_{x_{0q}}^{*(h)} \right) dt;$$

and the algorithm proceeds to the step $h + 1$.

After $n \times m$ steps, in a generic case, m-dimensional control in n-dimensional problem is calculated on the whole half-interval $[\tau_{k-1}, \tau_k)$. Convergence of the algorithm follows from convergence of the corresponding algorithm in Euclidean space [3].

4. Example

Let us consider a linear control system

$$\begin{cases} \dot{x}_1 = x_2, x_1(0) = x_{01} \\ \dot{x}_2 = u, x_2(0) = x_{02} \end{cases} \qquad (5)$$

on the time segment $[0, 2]$.

$$M = \left\{ \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2 : |x_1| \leq 1, x_2 \in \mathbb{R} \right\}; X_0 = \left\{ \begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix} \right\}$$

$$Q(t) = \begin{cases} (0, 0), t \in [0, 1] \\ (1, 0), t \in (1, 2] \end{cases} \quad u(t) \in P = \{u : |u| \leq 1\}, \; t \in [0, 2].$$

Since the solvability criterion holds, the package guidance problem is solvable. Using the algorithm from [4] the guiding program package elements for x_0' and x_0'' can be calculated. However, for the cluster $X_0 = \{x_0', x_0''\}$ on the time interval $[0, 1]$ there is an ambiguity in the minimum condition and the standard algorithm is not applicable. The presented method allows to reduce the problem to the system of two integral equations:

$$\begin{cases} x_1(2|x_0') = \frac{3}{2} + \frac{1}{2}u_{X_0}(t) + \int_1^2 \left(u_{X_0}(t) - t + 1\right)dt = 1 \\ x_1(2|x_0'') = -\frac{3}{2} + \frac{1}{2}u_{X_0}(t) + \int_1^2 \left(u_{X_0}(t) + t - 1\right)dt = -1 \end{cases}$$

Its solution gives that $u_{X_0}(t) \equiv 0, \; t \in [0, 1]$.

$u(t)$

$u_{x_0'}^*(t)$

$u_{x_0'x_0''}^*(t)$

$u_{x_0''}^*(t)$

$x_1(t)$

x_0'

M

x_0''
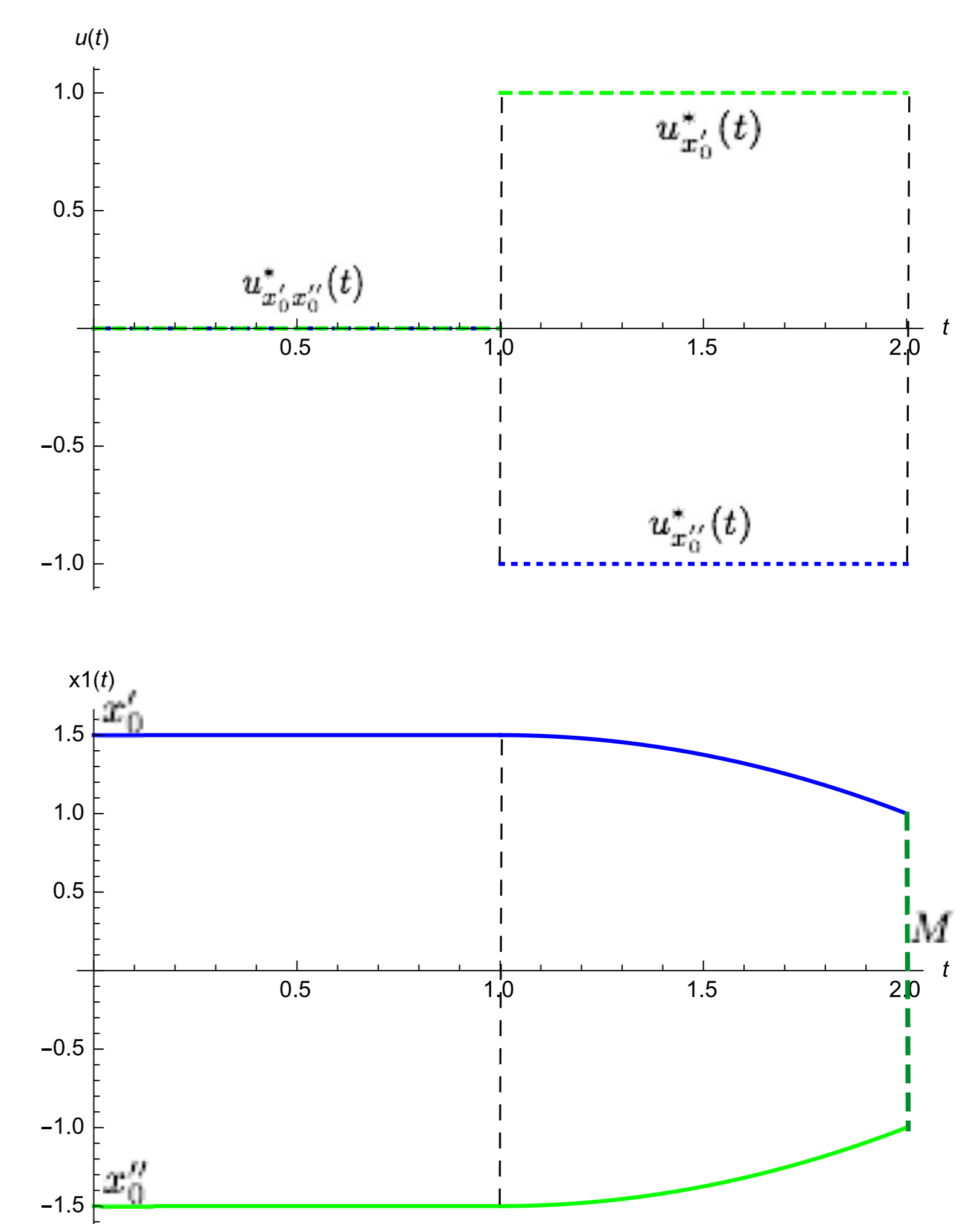
References

[1] *Kryazhimskiy A.V., Osipov Yu.S.* On the solvability of problems of guaranteeing control for partially observable linear dynamical systems // Proc. Steklov Inst. Math. 2012. No. 277. Pp. 144-159.

[2] *Kryazhimskiy A.V., Strelkovskii N.V.* An open-loop criterion for the solvability of a closed-loop guidance problem with incomplete information. Linear control systems // Trudy IMM UrO RAN. 2014. Vol. 20. No. 3. Pp. 132-147.

[3] *Gindes V.B.* Singular control in optimal systems // Izv. Vyssh. Uchebn. Zaved. Mat. 1967. No. 7. Pp. 34-42.

[4] *Strelkovskii, N.V., Orlov, S.M.* Moscow Univ. Comput. Math. Cybern. 2018. Vol. 42. No 2. Pp. 69-79.