Homogenisation of the Stokes equations in a perforated domain by the energy decomposition method

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Abstract

We consider homogenisation of the Stokes equations in a domain perforated with small holes, representing a porous medium. We propose an approach called the energy decomposition method, where the problem is divided into two components in different subspaces. One of these can be understood as representing the perturbation due to the microscale inhomogeneities. By finding an approximate solution to this subproblem, we are able to represent the effect of the inhomogeneities on the total problem. The method is simple to implement and does not presuppose a periodic distribution of the holes, making it a promising starting point for stochastic homogenisation.

Keywords homogenisation, Stokes equations, Darcy's law, Brinkman's law, numerical analysis

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Tiivistelmä

Tutkimus käsittelee Stokesin yhtälöiden homogenisaatiota alueessa, jossa on pieniä reikiä eli esteitä, joiden läpi virtaus ei pääse etenemään. Tilanne vastaa virtausta huokoisen aineen läpi. Energiahajotelmaksi kutsutussa menetelmässä Stokesin yhtälöitä vastaava minimointitehtävä jaetaan kahteen aliavaruuteen, joista toinen edustaa mikroskooppisten esteiden ratkaisuun aiheuttamaa häiriötä. Kun tämä alitehtävä ratkaistaan likimääräisesti, häiriö pystytään esittämään ratkaisematta alkuperäistä ongelmaa suoraan. Menetelmän toteutus on yksinkertainen, eikä se edellytä väliaineelta periodista rakennetta. Energiahajotelma vaikuttaakin lupaavalta lähtökohdalta stokastiselle homogenisaatiolle.

Avainsanat homogenisaatio, Stokesin yhtälö, Darcyn laki, Brinkmanin laki, numeerinen analyysi

Thanks

My advisor Antti Hannukainen, Tuomo Kuusi, and the local Stokes experts Rolf Stenberg and Daniele Boffi.

To Grégoire Allaire, who was not involved in the making of this thesis in any way and strictly not to be held responsible for the sketchy presentation of his work that follows, but who personally told me not to quit mathematics in 2012. *"Vous allez mettre un trait sur quatre années de votre vie! Moi, je ne le ferais pas."* Ultimately, I did not.

Finally, I wish to use this opportunity to express my long overdue gratitude to Hauenkuonon Igni Ferroque the dachshund, who tirelessly cheered me on during the long and arduous process of completing my Master's thesis in Political History during the summer of 2016.

Otaniemi, 28.2.2018

Emma-Karoliina Kurki

Contents

1 Introduction: Down the rabbit hole

The theory of homogenisation addresses the numerical solution of partial differential equations in a medium whose properties vary on a "microscopic" scale, that is, small compared to the global dimension of the problem. The direct solution of such problems, e.g. by the finite element method (FEM), would have to happen on a scale comparable to that of the heterogeneities. This is generally not feasible in practice because of the computational time required. The goal of homogenisation is to find an effective macroscopic model that takes the fine structure of the problem into account, yielding a good approximation of the global solution of the original problem.

One common example of a multiscale environment is a transformer core, where steel laminae alternate with non-conducting layers to reduce eddy-current losses. The electromagnetic properties of the core, and consequently the coefficients of the corresponding equation, vary periodically from one material to another. Homogenisation was first developed for such periodic media, but is not limited to them. Rather than enumerate well-known techniques here, we refer the reader to e.g. Cioranescu and Donato's book [7] for an introduction to the topic.

In this thesis, we take on a slightly different problem where the microscale oscillations are not represented by the coefficients of the equation, but the geometry of the domain. We seek to simulate an incompressible viscous flow through a porous medium, such as groundwater in the sandstone of Figure 1. When the velocity is small, such flows can be approximated by the Stokes equations

$$\begin{cases} \nabla p - \mu \Delta \boldsymbol{u} = \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases} \quad \text{in } \Omega.$$

For the purposes of the mathematician, the viscosity of the fluid μ is usually set to one. The fluid domain Ω is described by a set perforated with small, preferably random "holes" representing obstacles. The wording may appear somewhat contrary to intuition, because these holes represent where the fluid is *not* able to penetrate in real life and not vice versa.

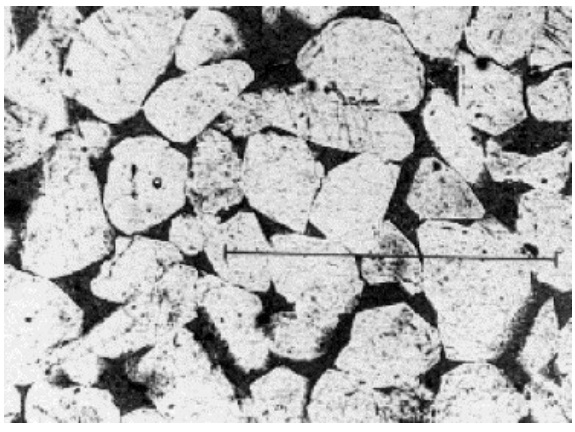


Figure 1: Optical micrograph of thin section of Fontainebleau sandstone. The scale bar represents 500 μm. [13]

The problem of *homogenisation of the Stokes equations in a perforated domain* has been thoroughly investigated by Grégoire Allaire in his 1990 papers [1], [2]; see also [3] and [4] for an abridged version. The classical approach is to model the porous medium as a periodic array of unit cells of size comparable to ε, each containing a small obstacle. The periodicity assumption is neither realistic nor absolutely necessary, but greatly simplifies the analysis: namely, methods that rely on the construction of test functions in a given geometry do not lend themselves easily to the case of truly random media. Letting ε approach zero, one obtains one of the three possible homogeneous problems, depending on the ratio of the obstacle and cell sizes.

The first of them, *Darcy's law*, was first formulated by the eponymous scientist and engineer Henry Darcy (École Polytechnique, class of 1821) in his 1856 report on the construction of the municipal water system of the town of Dijon, France [11]. Measuring the filtration of water through layers of sand, Darcy found that the velocity of the flow was directly proportional to the pressure gradient:

$$\mu \boldsymbol{u} = -k \nabla p$$

(or $\mu \boldsymbol{u} = -k \nabla p + \boldsymbol{f}$ in the presence of an external force), where μ denotes the viscosity of the fluid. The coefficient k is called the *permeability* of the medium, whose order of magnitude is similar to the cross section of a single pore; a typical value for sand, for instance, is between 20 and 200 $(\mu m)^2$ [21, p. 474]. Homogenisation is one way to justify this relation mathematically.

An amendment of Darcy's law for larger permeabilities was suggested by H. C. Brinkman in his 1949 paper [6]. An intermediate model between the Stokes and Darcy equations, Brinkman's equation involves both the Laplacian of the velocity and a first-order velocity term:

$$\mu \Delta \boldsymbol{u} - \nabla p - \frac{\mu}{k} \boldsymbol{u} = \boldsymbol{f}.$$

Finally, when the permeability is large enough and the obstacles asymptotically nil, they cease to have any effect and the process converges to an ordinary Stokes problem

$$\mu \Delta \boldsymbol{u} - \nabla p = \boldsymbol{f}.$$

The ultimate goal of Stokes homogenisation studies could be taken to be the case of a domain with *random* holes. This is likewise where the theory remains lacking, though general nonperiodic homogenisation has been discussed since the 1970s (see e.g. [7, Chapter 13] for further references). Relevant to the Stokes problem, in 1985 Jacob Rubinstein [17] derived the Brinkman law for a flow past a random array of spheres combining probabilistic methods with a homogenisation result based on resolvent expansions. In the following, we will commit a slight abuse of language and employ the words *classical* and *periodic* homogenisation as quasi-synonyms as opposed to the energy decomposition method.

In sum, it is well known that the Stokes problem in a perforated domain can be homogenised, what the solution looks like and how the error behaves – as long as the

holes are distributed periodically and we do not want to compute anything. To the applied mathematician, the existing results are rather disappointing. The techniques are not suited for practical implementation, and the notion of an ε approaching zero does not describe any process that we might observe in the wild.

The energy decomposition arises from the observation that the existing theory of homogenisation is not, in a certain sense, natural. The energy decomposition seems like a promising basis for stochastic homogenisation, since the concept is indifferent to the microscopic geometry (essentially, periodicity or lack thereof) of the material. What is more, the error analysis is conceptually simple, consisting of perturbation arguments. Difficulties arise in quantifying the two-scale structure, which was classically swept under the carpet by assuming a periodic medium.

The solution of any two-scale problem can be seen as the combination of a "slowly" changing baseline and "rapid" oscillations representing the perturbation due to the microscopic inhomogeneities; we would like to avoid direct solution of the latter. The "slow" and "fast" components can be thought of as living in two separate function spaces altogether, say, W_s and W_f. Thanks to this property, we may split the solution $\boldsymbol{u} \in V$ in two by means of a projection operator:

$$\boldsymbol{u} = P_f \boldsymbol{u} + (\boldsymbol{I} - P_f)\boldsymbol{u},$$

where $P_f : V \to W_f$ picks the rapidly oscillating part of the solution, the one we want to rid ourselves of. As a result, we obtain two different energy minimisation problems in the two subspaces. Homogenisation in this context is equivalent to finding an approximate, readily computable solution to the fast problem that represents the effect of the inhomogeneities on the total solution.

A similar approach (called *multiscale splitting* by the original author) has been previously applied in a F context by Axel Målqvist [15]. Målqvist's "fine-scale" (in our terminology, "fast") space V^f is the kernel of a Clément-type interpolation operator with nodal values

$$\left(\mathcal{I}_H \boldsymbol{v}\right)(\boldsymbol{x}) = \frac{\int_\Omega \boldsymbol{v}\boldsymbol{\lambda}_{\boldsymbol{x}}}{\int_\Omega \boldsymbol{\lambda}_{\boldsymbol{x}}},$$

where $\boldsymbol{\lambda}_{\boldsymbol{x}}$ is the corresponding nodal finite element basis function. The fast components of the basis functions are their projections into the fast space $\boldsymbol{\varphi}_{\boldsymbol{x}} = \mathcal{P}_f \boldsymbol{\lambda}_{\boldsymbol{x}} \in V^f$. That is, each satisfies a *corrector problem*

$$a(\boldsymbol{\varphi}_{\boldsymbol{x}}, \boldsymbol{v}) = a(\boldsymbol{\lambda}_{\boldsymbol{x}}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V^f.$$

Then, a basis of the "coarse" (or "slow") finite element space $V_H^s = V_H - V_H^f$ where the solution is computed is given by $\{\boldsymbol{\lambda}_{\boldsymbol{x}} - \boldsymbol{\varphi}_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\mathcal{N}}$. In other words, the "fine-scale" basis functions $\boldsymbol{\varphi}_{\boldsymbol{x}}$ act as correctors to the "coarse" solution. In practice, thanks to the fact that the correctors decay exponentially fast away from $\boldsymbol{x}$ (a localisation result comparable to our Proposition 3.5), it is enough to solve them on a local patch of elements to obtain a good approximation. This is relatively costly, but once the correctors are at hand, one may proceed to solve multiscale problems directly using

the modified basis. It should be stressed that Målqvist's aim is not homogenisation of any kind, but the creation of an efficient multiscale finite element method.

Most of the results concerning homogenisation via energy decomposition are put into writing for the first time in this thesis. Kristian Moring has previously written a Bachelor's thesis [14] on the energy decomposition for the Poisson equation. However, because of the greater complexity of the Stokes problem, we were obliged to rethink many of the developments. The new material mainly stems from me and my advisor Antti Hannukainen's discussions.

In Chapter 2, we briefly introduce the classical derivation of Brinkman's law via homogenisation and the techniques needed in it, in particular the so-called method of oscillating test functions. This is to provide a context for our discussion of the energy decomposition, the topic of the following Chapter 3. In Chapter 4, we present a numerical example for the sake of illustration. The "toolbox" in the appendix collects relevant theorems and definitions, stated either with references to literature or without proof altogether.

2 Classical homogenisation with obstacles

The aim of this chapter is to overview the classical procedure of homogenisation of the Stokes equations in a perforated domain. We begin by stating the problem and the solution, followed by an outline of the proof. The presentation is based on Allaire's papers [1] and [2], with many ideas dating back to Cioranescu and Murat's treatment of the Poisson equation (see [10]; these results were first published in 1982 [8], [9]).

2.1 Problem setting and results

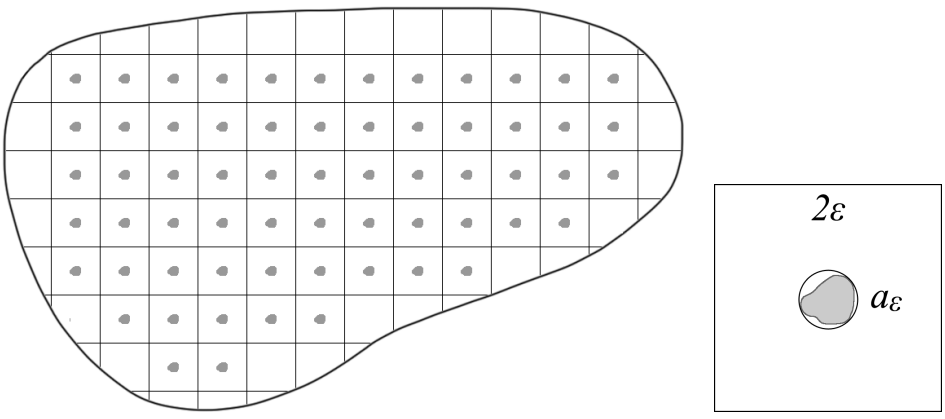
Approximating the medium. Let the bounded Lipschitz domain $\Omega \in \mathbb{R}^N$, $N \geq 2$, represent a porous medium. We cover Ω with a regular mesh of size 2ε, each cell being a cube identical to $(-\varepsilon, \varepsilon)^N$. At the center of each cube, we make a hole Γ_i^ε identical to a reference obstacle Γ scaled to size a_ε. Now the set Ω_ε that stands for the fluid domain can be described as $\Omega_\varepsilon = \Omega \setminus \cup_i \Gamma_i^\varepsilon$.

The hole size (in fact, diameter) a_ε is assumed to be small compared to the distance between holes, i.e. $\lim_{\varepsilon\to 0} a_\varepsilon/\varepsilon = 0$. We define the *critical size* of the holes $a_\varepsilon^{\text{crit}}$ to be such that

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{a_\varepsilon^{\text{crit}}}{\varepsilon^{N/(N-2)}} &= C_0 \text{ for } N \geq 3, \\ \lim_{\varepsilon\to 0} -\varepsilon^2 \log a_\varepsilon &= C_0 \text{ for } N = 2, \end{aligned} \tag{1}$$

where $C_0 > 0$ is a constant. Furthermore, we define the quantity σ_ε describing the ratio of the critical hole size and the current size a_ε:

$$\frac{a_\varepsilon^{\text{crit}}}{a_\varepsilon} \sim \sigma_\varepsilon = \begin{cases} \left(\frac{\varepsilon^N}{a_\varepsilon^{N-2}}\right)^{1/2} & \text{for } N \geq 3, \\ \varepsilon \left|\log\left(\frac{a_\varepsilon}{\varepsilon}\right)\right|^{1/2} & \text{for } N = 2. \end{cases} \tag{2}$$



(a) (b)

Figure 2: The premise of classical homogenisation: the domain Ω (a) is covered with identical copies of the reference cell (b) with side length 2ε. Each "hole" is contained within a sphere of diameter a_ε.

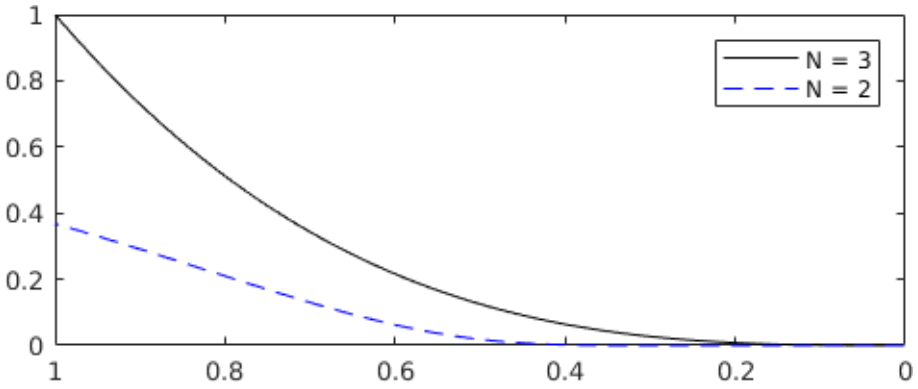


Figure 3: The critical hole size $a_\varepsilon^{\text{crit}}$ (3) decreases rapidly as a function of ε.

By this definition, the hole size is critical whenever $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \sigma > 0$. If the ratio tends to zero, the current hole size is larger than the critical one; if to infinity, the current size is smaller than the critical one. The definitions (1) and (2) are equivalent, with the constants C_0 and σ related by $C_0 = \sigma^{-\frac{2}{N-2}}$ for $N \geq 3$ and $C_0 = \sigma^2$ for $N = 2$.

The definition may seem arbitrary at this stage, but arises as a necessary condition from the construction of the test functions upon which the proof relies. This was likely first observed by Cioranescu and Murat discussing the Poisson equation in [10, Theorem 2.2]. Asymptotically, we have

$$a_\varepsilon^{\text{crit}} \sim \begin{cases} C_0\varepsilon^{\frac{N}{N-2}} & \text{for } N \geq 3, \\ \exp\left(-\frac{C_0}{\varepsilon^2}\right) & \text{for } N = 2. \end{cases} \tag{3}$$

This would mean $\mathcal{O}(\varepsilon^3)$ in three dimensions, $\mathcal{O}(\exp(-\varepsilon^{-2}))$ in two. In the classical homogenisation process, we "zoom out" of the microscale by letting ε approach zero. In order for us to obtain anything but Darcy's law as a result, the holes must vanish from view very rapidly (Figure 3).

Remark 2.1. When the obstacles are of the same order of magnitude as the cell, Darcy's law can likewise be derived by homogenisation. The proof employs the standard technique of *two-scale convergence*. See e.g. [4] for a brief discussion, and the introduction to [1] for further references.

Problem setting. We wish to solve a steady viscous flow in the porous medium represented by the domain Ω_ε. We model this by the Stokes equations with a no-slip (Dirichlet) boundary condition on $\partial\Omega_\varepsilon = \partial\Omega \cup \partial\Gamma$, where $\partial\Gamma = \cup_i \partial\Gamma_i$ represents the surface of the obstacles. For a given body force $\boldsymbol{f} \in [L^2(\Omega)]^N$ and setting the viscosity of the fluid equal to one, the problem becomes

$$\begin{cases} \nabla p_\varepsilon - \Delta \boldsymbol{u}_\varepsilon = \boldsymbol{f} & \text{in } \Omega_\varepsilon \\ \nabla \cdot \boldsymbol{u}_\varepsilon = 0 & \text{in } \Omega_\varepsilon \\ \boldsymbol{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{4}$$

or, in the equivalent variational formulation: Find $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon) \,/\, \mathbb{R}]$ such that

$$\begin{cases} \int_{\Omega_\varepsilon} \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{v} - \int_{\Omega_\varepsilon} p_\varepsilon \, \nabla \cdot \boldsymbol{v} = \int_{\Omega_\varepsilon} \boldsymbol{f} \cdot \boldsymbol{v} & \text{for each } \boldsymbol{v} \in [H_0^1(\Omega_\varepsilon)]^N \\ \int_{\Omega_\varepsilon} q \, \nabla \cdot \boldsymbol{u}_\varepsilon = 0 & \text{for each } q \in L^2(\Omega_\varepsilon) \,/\, \mathbb{R}. \end{cases} \tag{5}$$

It is well known that for a fixed ε, there exists a unique solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in [H_0^1(\Omega_\varepsilon)]^N \times [L^2(\Omega_\varepsilon) \,/\, \mathbb{R}]$ (the pressure is determined in $L^2(\Omega_\varepsilon)$ only up to a constant). However, these spaces are not the same for different values of ε. To overcome this issue, we extend the solutions to the entire domain Ω, including the holes. Let $\widetilde{\boldsymbol{u}}_\varepsilon$ and $P_\varepsilon(p_\varepsilon)$ be the extensions of the velocity $\boldsymbol{u}_\varepsilon$ and the pressure p_ε respectively, defined by

$$\widetilde{\boldsymbol{u}}_\varepsilon = \begin{cases} \boldsymbol{u}_\varepsilon & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \Gamma_i^\varepsilon, \end{cases} \tag{6}$$

$$P_\varepsilon(p_\varepsilon) = \begin{cases} p_\varepsilon & \text{in } \Omega_\varepsilon \\ |C_i^\varepsilon|^{-1} \int_{C_i^\varepsilon} p_\varepsilon & \text{in } \Gamma_i^\varepsilon. \end{cases} \tag{7}$$

Here each C_i^ε is a control volume around the corresponding hole Γ_i^ε, defined as the part of the ball of radius ε outside the hole; see Figure 2 in [1, p. 230]. The main result is summarised in the following theorem:

Theorem 2.1 (Allaire)**.** *According to the scaling of the hole size, there are three different limit flow regimes:*

1. *"Small obstacles": If* $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \infty$*, then* $(\widetilde{\boldsymbol{u}}_\varepsilon, P_\varepsilon)$ *converges strongly to* $(\boldsymbol{u}, p)$ *in* $[H_0^1(\Omega)]^N \times [L^2(\Omega) \,/\, \mathbb{R}]$*, where* $(\boldsymbol{u}, p)$ *is the unique solution of the Stokes equations:*

$$\begin{cases} \nabla p - \mu \, \Delta \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

2. *Critical: If* $\lim_{\varepsilon \to 0} \sigma_\varepsilon = \sigma > 0$*, then* $(\widetilde{\boldsymbol{u}}_\varepsilon, P_\varepsilon)$ *converges weakly to* $(\boldsymbol{u}, p)$ *in* $[H_0^1(\Omega)]^N \times [L^2(\Omega) \,/\, \mathbb{R}]$*, where* $(\boldsymbol{u}, p)$ *is the unique solution of the Brinkman problem*

$$\begin{cases} \nabla p - \mu \, \Delta \boldsymbol{u} + \frac{\mu}{\sigma^2} \boldsymbol{M} \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

3. *"Large obstacles": If* $\lim_{\varepsilon \to 0} \sigma_\varepsilon = 0$*, then* $(\widetilde{\boldsymbol{u}}_\varepsilon \sigma_\varepsilon^{-2}, P_\varepsilon)$ *converges strongly to* $(\boldsymbol{u}, p)$ *in* $[H_0^1(\Omega)]^N \times [L^2(\Omega) \,/\, \mathbb{R}]$*, where* $(\boldsymbol{u}, p)$ *is the unique solution of the Darcy's law*

$$\begin{cases} \boldsymbol{u} = \boldsymbol{M}^{-1}(\boldsymbol{f} - \nabla p) & \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Moreover, the matrix $\boldsymbol{M}$, which appears in both the Brinkman-type and Darcy's laws, is the same in both cases and only depends on the reference obstacle Γ. □

The matrix $\boldsymbol{M}$ is found by solving a local problem in the reference cell:

Proposition 2.2.

- *For $N \geq 3$, the k:th local problem is: Find $(\boldsymbol{w}_k, q_k) \in [H_0^1(\Omega_\varepsilon)]^N \times L^2(\Omega_\varepsilon)$ such that*

$$\begin{cases} \nabla q_k - \Delta \boldsymbol{w}_k = 0 & \text{in } \mathbb{R}^N \setminus \Gamma \\ \nabla \cdot \boldsymbol{w}_k = 0 & \text{in } \mathbb{R}^N \setminus \Gamma \\ \boldsymbol{w}_k = 0 & \text{on } \partial\Gamma \\ \boldsymbol{w}_k = \boldsymbol{e}_k & \text{at infinity,} \end{cases} \tag{11}$$

  where $\boldsymbol{e}_k$ is the k:th unit basis vector of $\mathbb{R}^N$. Then the matrix $\boldsymbol{M}$ is given by

$$[\boldsymbol{M}]_{ik} = \frac{1}{2^N} \left[\int_{\mathbb{R}^N \setminus \Gamma} \nabla \boldsymbol{w}_k : \nabla \boldsymbol{w}_i \right]_{1 \leq i,k \leq N}. \tag{12}$$

- *For $N = 2$, the local problem for $k = 1, 2$ is: s: Find $(\boldsymbol{w}_k, q_k) \in [H_0^1(\Omega_\varepsilon)]^2 \times L^2(\Omega_\varepsilon)$ such that*

$$\begin{cases} \nabla q_k - \Delta \boldsymbol{w}_k = 0 & \text{in } \mathbb{R}^N \setminus \Gamma \\ \nabla \cdot \boldsymbol{w}_k = 0 & \text{in } \mathbb{R}^N \setminus \Gamma \\ \boldsymbol{w}_k = 0 & \text{on } \partial\Gamma \\ \boldsymbol{w}_k = \boldsymbol{e}_k \cdot \log|\boldsymbol{x}| & \text{at infinity.} \end{cases} \tag{13}$$

  In this case, we obtain $\boldsymbol{M} = \pi \boldsymbol{I}$, regardless of the exact size and shape of the model obstacle Γ.

□

Remark 2.2. There are no periodic boundary conditions on the local problem. The hole size a_ε was defined to be asymptotically smaller than the period ε. Therefore, when the hole size is rescaled to 1, the boundary of the cell goes to infinity, and the periodic boundary condition becomes a uniform boundary condition at infinity. [1, p. 231]

Remark 2.3. The strange result in two dimensions is a consequence of the *Stokes paradox,* which asserts that the system (11) does not have a solution in two dimensions. To obtain a nontrivial solution, we must allow logarithmic growth at infinity in (13). The reason behind the "paradox" is the limited validity of the Stokes equations: the assumption that the Reynolds number of the flow be small at an arbitrary distance from the obstacle is too strong to model the situation accurately.

2.2 Proof: Deriving Brinkman's law

Let us proceed to prove Theorem 2.1, with the purpose to illustrate the method of homogenisation via *oscillating test functions* due to Luc Tartar [19] (see [7, p. 138ff] for an accessible introduction). While it should not be impossible to rearrange the material into a combined proof for all three limit flow regimes, we will restrict ourselves to the Brinkman case and assume throughout that the hole size is critical. The choice is not immediately visible in this abridged presentation, but lurks behind the framework of Lemma 2.3; a slightly modified set of hypotheses – some stronger, some weaker – is needed in the Darcy case. Allaire discusses the sub- and supercritical cases in [2].

To begin with, we should verify that the sequences of extensions $(\widetilde{\boldsymbol{u}}_\varepsilon, P_\varepsilon)_{\varepsilon>0}$ are bounded in $[H^1_0(\Omega)]^N \times L^2(\Omega)\,/\,\mathbb{R}$; this is because they must eventually converge to the solution of the homogenised problem. It is not at all hard to see this for the extension of the velocity by zero (6). Setting $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ in the variational formulation (5) and taking into account the fact that $\widetilde{\boldsymbol{u}}_\varepsilon = 0$ in the holes $\Omega \setminus \Omega_\varepsilon$, we obtain in the entire domain

$$\int_\Omega |\nabla \widetilde{\boldsymbol{u}}_\varepsilon|^2 = \int_\Omega \boldsymbol{f} \cdot \widetilde{\boldsymbol{u}}_\varepsilon \le \|\boldsymbol{f}\|_{L^2(\Omega)} \, \|\widetilde{\boldsymbol{u}}_\varepsilon\|_{L^2(\Omega)} \,.$$

The Poincaré inequality $\|\widetilde{\boldsymbol{u}}_\varepsilon\|_{L^2(\Omega)} \le C(\Omega)\, \|\nabla \widetilde{\boldsymbol{u}}_\varepsilon\|_{L^2(\Omega)}$ then yields

$$\|\nabla \widetilde{\boldsymbol{u}}_\varepsilon\|_{L^2(\Omega)} \le C\, \|\boldsymbol{f}\|_{L^2(\Omega)} \,, \tag{14}$$

and we obtain $\|\widetilde{\boldsymbol{u}}_\varepsilon\|_{H^1_0(\Omega)} \le C\, \|\boldsymbol{f}\|_{L^2(\Omega)}$, where the constant C depends only on the domain Ω and not on ε. Hence, the sequence $(\widetilde{\boldsymbol{u}}_\varepsilon)_{\varepsilon>0}$ is bounded and thus contains a subsequence, still denoted $(\widetilde{\boldsymbol{u}}_\varepsilon)_{\varepsilon>0}$, that converges weakly to a limit $\boldsymbol{u} \in [H^1_0(\Omega)]^N$ when $\varepsilon \to 0$.

The construction of an extension of the pressure that is bounded in $L^2(\Omega)\,/\,\mathbb{R}$ is not quite this straightforward. To this end – and to find the homogenised problem satisfied by the limit $(\boldsymbol{u}, p)$ – we introduce some test functions and a linear operator, on whose existence the proof relies.

Lemma 2.3 (Brinkman regime)**.** *There exist functions $(\boldsymbol{w}^\varepsilon_k, q^\varepsilon_k, \boldsymbol{\mu}_k)_{1\le k\le N}$ and a linear map R_ε such that*

(H1) $\boldsymbol{w}^\varepsilon_k \in [H^1(\Omega)]^N$, $q^\varepsilon_k \in L^2(\Omega)$.

(H2) $\nabla \cdot \boldsymbol{w}^\varepsilon_k = 0$ *in* Ω *and* $w^\varepsilon_k = 0$ *in the holes* Γ^ε_i.

(H3) $\boldsymbol{w}^\varepsilon_k \rightharpoonup \boldsymbol{e}_k$ *in* $[H^1(\Omega)]^N$ *weakly,* $q^\varepsilon_k \rightharpoonup 0$ *in* $L^2(\Omega)\,/\,\mathbb{R}$ *weakly.*

(H4) $\boldsymbol{\mu}_k \in [W^{-1,\infty}(\Omega)]^N$.

(H5) For each sequence $\boldsymbol{v}_\varepsilon$, for each $\boldsymbol{v}$ such that $\boldsymbol{v}_\varepsilon \rightharpoonup \boldsymbol{v}$ in $[H^1(\Omega)]^N$ weakly and $\boldsymbol{v}_\varepsilon = 0$ in the holes, and for each $\phi \in \mathcal{D}(\Omega)$ we have

$$\langle \nabla q^\varepsilon_k - \Delta \boldsymbol{w}^\varepsilon_k,\ \phi \boldsymbol{v}_\varepsilon \rangle_{H^{-1}(\Omega),\, H^1_0(\Omega)} \to \langle \boldsymbol{\mu}_k, \phi \boldsymbol{v} \rangle_{H^{-1}(\Omega),\, H^1_0(\Omega)}.$$

(H6)
- $R_\varepsilon \in \mathcal{L}\left([H_0^1(\Omega)]^N; [H_0^1(\Omega_\varepsilon)]^N\right)$;
- $\boldsymbol{u} \in [H_0^1(\Omega_\varepsilon)]^N$ *implies* $R_\varepsilon \widetilde{\boldsymbol{u}} = \boldsymbol{u}$ *in* Ω_ε;
- $\nabla \cdot \boldsymbol{u} = 0$ *in* Ω *implies* $\nabla \cdot (R_\varepsilon \boldsymbol{u}) = 0$ *in* Ω_ε;
- $\|R_\varepsilon \boldsymbol{u}\|_{H_0^1(\Omega_\varepsilon)} \leq C \|\boldsymbol{u}\|_{H_0^1(\Omega)}$ *with* C *independent of* ε.

□

The hypotheses put forth in Lemma 2.3 are verified essentially by constructing the desired test functions and the map in the underlying geometry, which is where the scaling of the hole size comes into play. This the most technical and involved part of the proof and will not be reproduced here. The full story for the Stokes equations can be found in [1], while Cioranescu and Murat [10] give some explicit examples of test functions for the Poisson case. Taking the test functions for granted, we are now ready to describe the matrix $\boldsymbol{M}$ that will appear in the homogenised problem.

Proposition 2.4. *Let* $(\boldsymbol{w}_k^\varepsilon, q_k^\varepsilon, \boldsymbol{\mu}_k)_{1\leq k\leq N}$ *be the functions prescribed by Lemma 2.3. Let* $\boldsymbol{M}$ *be the matrix whose entries are given by* $[\boldsymbol{M}]_{ik} = M_{ik} = \boldsymbol{\mu}_k \cdot \boldsymbol{e}_i$. *Then for each* $\phi \in \mathcal{D}(\Omega)$ *we have*

$$\langle M_{ik}, \phi\rangle_{\mathcal{D}'(\Omega),\,\mathcal{D}(\Omega)} = \langle \boldsymbol{\mu}_k, \phi \boldsymbol{e}_i\rangle_{\mathcal{D}'(\Omega),\,\mathcal{D}(\Omega)} = \lim_{\varepsilon\to 0} \int_\Omega \phi\, \nabla \boldsymbol{w}_k^\varepsilon : \nabla \boldsymbol{w}_i^\varepsilon. \tag{15}$$

Thus $\boldsymbol{M}$ *is a symmetric matrix, and positive in the sense that* $\langle \boldsymbol{M}\boldsymbol{\Phi}, \boldsymbol{\Phi}\rangle_{H^{-1}(\Omega),\,H_0^1(\Omega)} \geq 0$ *for each* $\boldsymbol{\Phi} \in [\mathcal{D}(\Omega)]^N$.

Proof. Let $\boldsymbol{v}_\varepsilon = \boldsymbol{w}_i^\varepsilon$ and $\boldsymbol{v} = \boldsymbol{e}_i$ in (H5). Then

$$\langle \nabla q_k^\varepsilon - \Delta \boldsymbol{w}_k^\varepsilon,\ \phi \boldsymbol{w}_i^\varepsilon\rangle_{H^{-1}(\Omega),\,H_0^1(\Omega)} \to \langle \boldsymbol{\mu}_k, \phi \boldsymbol{e}_i\rangle_{H^{-1}(\Omega),\,H_0^1(\Omega)}. \tag{16}$$

On the other hand, integrating by parts, we obtain

$$\langle \nabla q_k^\varepsilon - \Delta \boldsymbol{w}_k^\varepsilon,\ \phi \boldsymbol{w}_i^\varepsilon\rangle_{H^{-1}(\Omega),\,H_0^1(\Omega)} = -\int_\Omega q_k^\varepsilon \boldsymbol{w}_i^\varepsilon \cdot \nabla\phi + \int_\Omega \nabla \boldsymbol{w}_k^\varepsilon : \boldsymbol{w}_i^\varepsilon \nabla\phi + \int_\Omega \phi\, \nabla \boldsymbol{w}_k^\varepsilon : \nabla \boldsymbol{w}_i^\varepsilon. \tag{17}$$

Rellich's theorem and Proposition A.1 allow us to pass to the limit in (17):

$$\lim_{\varepsilon\to 0} \langle \nabla q_k^\varepsilon - \Delta \boldsymbol{w}_k^\varepsilon,\ \phi \boldsymbol{w}_i^\varepsilon\rangle_{H^{-1}(\Omega),\,H_0^1(\Omega)} = \lim_{\varepsilon\to 0} \int_\Omega \phi\, \nabla \boldsymbol{w}_k^\varepsilon : \nabla \boldsymbol{w}_i^\varepsilon. \tag{18}$$

Combining (16) and (18) yields

$$\lim_{\varepsilon\to 0} \int_\Omega \phi\, \nabla \boldsymbol{w}_k^\varepsilon : \nabla \boldsymbol{w}_i^\varepsilon = \langle \boldsymbol{\mu}_k,\ \phi \boldsymbol{e}_i\rangle_{H^{-1}(\Omega),\,H_0^1(\Omega)}.$$

Moreover, for each $\boldsymbol{\Phi} \in [\mathcal{D}(\Omega)]^N$,

$$\langle \boldsymbol{M}\boldsymbol{\Phi},\ \boldsymbol{\Phi}\rangle_{H^{-1}(\Omega),\,H_0^1(\Omega)} = \lim_{\varepsilon\to 0} \int_\Omega \left| \sum_{k=1}^N \phi_k\, \nabla \boldsymbol{w}_k^\varepsilon \right|^2 \geq 0.$$

□

Remark 2.4. The physical interpretation of the functions $\boldsymbol{w}_k^\varepsilon$ and q_k^ε introduced in Lemma 2.3 is, respectively, the velocity and pressure of a unit boundary layer around the holes. The matrix $\boldsymbol{M}$ can be understood as the energy of these boundary layers. [1, 232]

Moreover, the linear operator hypothesized in (H6) allows a description of the extension of the pressure:

Proposition 2.5. *If there exists a linear operator R_ε satisfying (H6), then the operator P_ε defined by*

$$\langle \nabla \left[P_\varepsilon(q_\varepsilon)\right], \boldsymbol{v}\rangle_{H^{-1},H_0^1(\Omega)} = \langle \nabla q_\varepsilon, R_\varepsilon \boldsymbol{v}\rangle_{H^{-1}(\Omega_\varepsilon),\, H_0^1(\Omega_\varepsilon)} \text{ for each } \boldsymbol{v} \in \left[H_0^1(\Omega)\right]^N \tag{19}$$

is a linear continuous extension operator from $L^2(\Omega_\varepsilon)\,/\,\mathbb{R}$ to $L^2(\Omega)\,/\,\mathbb{R}$ such that the following hold for each $q_\varepsilon \in L^2(\Omega_\varepsilon)\,/\,\mathbb{R}$:

(i) $P_\varepsilon(q_\varepsilon) = q_\varepsilon$ *in* $L^2(\Omega_\varepsilon)\,/\,\mathbb{R}$;

(ii) $\|P_\varepsilon(q_\varepsilon)\|_{L^2(\Omega)/\mathbb{R}} \le C\,\|q_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}}$;

(iii) $\|\nabla\left[P_\varepsilon(q_\varepsilon)\right]\|_{H^{-1}(\Omega)} \le C\,\|\nabla q_\varepsilon\|_{H^{-1}(\Omega_\varepsilon)}$,

where C is a constant independent of ε. [1, pp. 216–218] □

Proposition 2.6. *There exists a map R_ε satisfying (H6), such that the extension operator $P_\varepsilon : L^2(\Omega_\varepsilon)\,/\,\mathbb{R} \to L^2(\Omega)\,/\,\mathbb{R}$ defined by Proposition 2.5 satisfies (7):*

$$P_\varepsilon(q_\varepsilon) = \begin{cases} q_\varepsilon & \text{in } \Omega_\varepsilon \\ |C_i^\varepsilon|^{-1} \int_{C_i^\varepsilon} q_\varepsilon & \text{in } \Gamma_i^\varepsilon \end{cases}$$

for all $q_\varepsilon \in L^2(\Omega_\varepsilon)\,/\,\mathbb{R}$ and each hole $\Gamma_i^\varepsilon \subset \Omega$. [1, pp. 230, 233ff] □

We now have the means to verify the boundedness of the extension of the pressure P_ε in $L^2(\Omega)\,/\,\mathbb{R}$. $P_\varepsilon(p_\varepsilon)$ is defined by (19):

$$\langle \nabla \left[P_\varepsilon(p_\varepsilon)\right], \boldsymbol{v}\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} = \langle \nabla p_\varepsilon, R_\varepsilon \boldsymbol{v}\rangle_{H^{-1}(\Omega_\varepsilon),\, H_0^1(\Omega_\varepsilon)} \text{ for each } \boldsymbol{v} \in \left[H_0^1(\Omega)\right]^N.$$

Introducing the original equation (4) and integrating by parts, we obtain

$$\langle \nabla \left[P_\varepsilon(p_\varepsilon)\right], \boldsymbol{v}\rangle_{H^{-1}(\Omega)\,,H_0^1(\Omega)} = -\int_{\Omega_\varepsilon} \nabla \boldsymbol{u}_\varepsilon : \nabla\left(R_\varepsilon \boldsymbol{v}\right) + \int_{\Omega_\varepsilon} \boldsymbol{f} \cdot R_\varepsilon \boldsymbol{v}.$$

Hence

$$\begin{aligned} \left|\langle \nabla \left[P_\varepsilon(p_\varepsilon)\right], \boldsymbol{v}\rangle\right| &\le \|\nabla \widetilde{\boldsymbol{u}}_\varepsilon\|_{L^2(\Omega)}\, \|\nabla(R_\varepsilon \boldsymbol{v})\|_{L^2(\Omega_\varepsilon)} + \|\boldsymbol{f}\|_{L^2(\Omega)}\, \|R_\varepsilon \boldsymbol{v}\|_{L^2(\Omega_\varepsilon)} \\ &\le C\,\|\boldsymbol{f}\|_{L^2(\Omega)}\, \|\boldsymbol{v}\|_{H_0^1(\Omega)}, \end{aligned}$$

where the last inequality follows from (14) and (H6); this means $\|\nabla\left[P_\varepsilon(p_\varepsilon)\right]\|_{H^{-1}(\Omega)} \le C\,\|\boldsymbol{f}\|_{L^2(\Omega)}$. Lemma A.2 (a corollary of the Peetre–Tartar theorem) allows us to conclude that $\|P_\varepsilon(p_\varepsilon)\|_{L^2(\Omega)/\mathbb{R}} \le C\,\|\boldsymbol{f}\|_{L^2(\Omega)}$, where the constant C depends only on Ω. Hence, the sequence $\left(P_\varepsilon(p_\varepsilon)\right)_{\varepsilon>0}$ is bounded and one can extract a subsequence, still denoted $\left(P_\varepsilon(p_\varepsilon)\right)_{\varepsilon>0}$, that converges weakly to a limit $p \in L^2(\Omega)\,/\,\mathbb{R}$ when $\varepsilon \to 0$.

We are now ready for the main result:

Proof of Theorem 2.1. Let $(\boldsymbol{w}_k^\varepsilon, q_k^\varepsilon, \mu_k)_{1\le k\le N}$ be the functions that satisfy Hypotheses (H1) – (H5) of Lemma 2.3. For a fixed $\phi \in \mathcal{D}(\Omega)$, we apply the test functions $\boldsymbol{v} = \phi \boldsymbol{w}_k^\varepsilon \in [H_0^1(\Omega_\varepsilon)]^N$, $q = \phi q_k^\varepsilon \in L^2(\Omega_\varepsilon)\,/\,\mathbb{R}$ in the variational formulation (5):

$$\begin{cases} \int_{\Omega_\varepsilon} \nabla \boldsymbol{u}_\varepsilon : \nabla\left(\phi \boldsymbol{w}_k^\varepsilon\right) - \int_{\Omega_\epsilon} p_\varepsilon \nabla \cdot \left(\phi \boldsymbol{w}_k^\varepsilon\right) = \int_{\Omega_\varepsilon} \boldsymbol{f} \cdot \left(\phi \boldsymbol{w}_k^\varepsilon\right), \\ \int_{\Omega_\varepsilon} \left(\phi q_k^\varepsilon\right) \nabla \cdot \boldsymbol{u}_\varepsilon = 0. \end{cases}$$

Expanding this and taking into account (H2), which requires that $\boldsymbol{w}_k^\varepsilon$ be divergence-free, we obtain

$$\int_{\Omega_\varepsilon} \phi \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{w}_k^\varepsilon + \int_{\Omega_\varepsilon} \nabla \boldsymbol{u}_\varepsilon : \boldsymbol{w}_k^\varepsilon \nabla \phi - \int_{\Omega_\epsilon} p_\varepsilon \boldsymbol{w}_k^\varepsilon \cdot \nabla \phi = \int_{\Omega_\varepsilon} \phi \boldsymbol{f} \cdot \boldsymbol{w}_k^\varepsilon, \tag{20}$$

$$\int_{\Omega_\varepsilon} \phi q_k^\varepsilon \nabla \cdot \boldsymbol{u}_\varepsilon = 0. \tag{21}$$

Now we may write in the sense of distributions

$$\int_{\Omega_\varepsilon} \phi \nabla \boldsymbol{u}_\varepsilon : \nabla \boldsymbol{w}_k^\varepsilon = -\langle \Delta \boldsymbol{w}_k^\varepsilon,\, \phi \widetilde{\boldsymbol{u}}_\varepsilon \rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} - \int_{\Omega_\varepsilon} \boldsymbol{u}_\varepsilon \nabla \phi : \nabla \boldsymbol{w}_k^\varepsilon,$$

$$\int_{\Omega_\varepsilon} \phi q_k^\varepsilon \nabla \cdot \boldsymbol{u}_\varepsilon = -\langle \nabla q_k^\varepsilon,\, \phi \widetilde{\boldsymbol{u}}_\varepsilon \rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} - \int_{\Omega_\varepsilon} q_k^\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla \phi.$$

Inserting these into (20) and (21) and adding up the two equations,

$$\sigma_\varepsilon^2 \left\langle \nabla q_k^\varepsilon - \Delta \boldsymbol{w}_k^\varepsilon,\, \phi \frac{\widetilde{\boldsymbol{u}}_\varepsilon}{\sigma_\varepsilon^2} \right\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} + \int_\Omega q_k^\varepsilon \widetilde{\boldsymbol{u}}_\varepsilon \cdot \nabla \phi - \int_\Omega \widetilde{\boldsymbol{u}}_\varepsilon \nabla \phi : \nabla \boldsymbol{w}_k^\varepsilon$$
$$+ \int_\Omega \nabla \widetilde{\boldsymbol{u}}_\varepsilon : \boldsymbol{w}_k^\varepsilon \nabla \phi - \int_{\Omega_\varepsilon} p_\varepsilon \boldsymbol{w}_k^\varepsilon \cdot \nabla \phi = \int_\Omega \phi \boldsymbol{f} \cdot \boldsymbol{w}_k^\varepsilon. \tag{22}$$

Here, because $P_\varepsilon(p_\varepsilon) = p_\varepsilon$ in Ω_ε and $\boldsymbol{w}_k^\varepsilon = 0$ in $\Omega \setminus \Omega_\varepsilon$ by (H2), we may write

$$\int_{\Omega_\varepsilon} p_\varepsilon \boldsymbol{w}_k^\varepsilon \cdot \nabla \phi = \int_\Omega P_\varepsilon(p_\varepsilon) \boldsymbol{w}_k^\varepsilon \cdot \nabla \phi.$$

We let $\varepsilon \to 0$ in (22). The sequence $(\widetilde{\boldsymbol{u}}_\varepsilon \sigma_\varepsilon^{-2})_{\varepsilon \ge 0}$ fulfills the conditions of hypothesis (H5), which implies $\sigma_\varepsilon^2 \langle \nabla q_k^\varepsilon - \Delta \boldsymbol{w}_k^\varepsilon,\, \phi \widetilde{\boldsymbol{u}}_\varepsilon \sigma_\varepsilon^{-2} \rangle_{H^{-1},\, H_0^1(\Omega)} \to \int_\Omega \phi \boldsymbol{\mu}_k \cdot \boldsymbol{u}$. Furthermore, we have the following convergences:

$$\begin{aligned} \widetilde{\boldsymbol{u}}_\varepsilon &\rightharpoonup \boldsymbol{u} && \text{in } \left[H_0^1(\Omega)\right]^N \text{ weakly}, \\ \boldsymbol{w}_k^\varepsilon &\rightharpoonup \boldsymbol{e}_k && \text{in } \left[H^1(\Omega)\right]^N \text{ weakly}, \\ q_k^\varepsilon &\rightharpoonup 0 && \text{in } L^2(\Omega)\,/\,\mathbb{R} \text{ weakly}, \\ P_\varepsilon(p_\varepsilon) &\rightharpoonup p && \text{in } L^2(\Omega)\,/\,\mathbb{R} \text{ weakly}. \end{aligned}$$

By Rellich's theorem, $(\widetilde{\boldsymbol{u}}_\varepsilon)_{\varepsilon\geq 0}$ has a subsequence that converges strongly in $L^2(\Omega)$. Hence and by Proposition A.1, (22) converges to

$$\langle \boldsymbol{\mu}_k,\, \phi\boldsymbol{u}\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} + \int_\Omega \nabla\boldsymbol{u} : \boldsymbol{e}_k\, \nabla\phi + \int_\Omega p\boldsymbol{e}_k \cdot \nabla\phi = \int_\Omega \phi\boldsymbol{f}\cdot\boldsymbol{e}_k.$$

Integrating this by parts, we may write

$$\langle \boldsymbol{\mu}_k,\, \phi\boldsymbol{u}\rangle - \langle \Delta\boldsymbol{u},\, \phi\boldsymbol{e}_k\rangle + \langle \nabla p,\, \phi\boldsymbol{e}_k\rangle = \langle \boldsymbol{f},\, \phi\boldsymbol{e}_k\rangle \text{ for } k = 1, 2, \ldots, N. \tag{23}$$

However, because $\boldsymbol{M}$ is a symmetric matrix, we have $\langle \boldsymbol{\mu}_k,\, \phi\boldsymbol{u}\rangle = \sum_{i=1}^N \langle \boldsymbol{\mu}_k,\, \phi u_i\boldsymbol{e}_i\rangle = \sum_i \langle M_{ki},\, \phi u_i\rangle = \langle \boldsymbol{M}\boldsymbol{u},\, \phi\boldsymbol{e}_k\rangle$ and (23) reads

$$\nabla p - \Delta\boldsymbol{u} + \boldsymbol{M}\boldsymbol{u} = \boldsymbol{f}.$$

Furthermore, knowing that $\nabla\cdot\widetilde{\boldsymbol{u}}_\varepsilon = 0$ in Ω and $\widetilde{\boldsymbol{u}}_\varepsilon \rightharpoonup \boldsymbol{u}$ in $[H_0^1(\Omega)]^N$, we have $\nabla\cdot\boldsymbol{u} = 0$ in Ω, and we obtain the homogenised problem: Find $(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^N \times [L^2(\Omega)\,/\,\mathbb{R}]$ such that

$$\begin{cases} \nabla p - \Delta\boldsymbol{u} + \boldsymbol{M}\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega. \end{cases} \tag{24}$$

Thanks to the positivity of $\boldsymbol{M}$, the operator $-\Delta + \boldsymbol{M}$ is immediately seen to be coercive:

$$\begin{aligned} \langle (-\Delta + \boldsymbol{M})\,\boldsymbol{u},\, \boldsymbol{u}\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} &= \int_\Omega |\nabla\boldsymbol{u}|^2 + \langle \boldsymbol{M}\boldsymbol{u},\, \boldsymbol{u}\rangle_{H^{-1}(\Omega),\, H_0^1(\Omega)} \\ &\geq \|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2 \geq C\,\|\boldsymbol{u}\|_{L^2(\Omega)}^2. \end{aligned}$$

Boundedness follows from (H4). Hence, the homogeneous system (24) admits a unique solution. □

The convergence estimate, which we state without proof, requires a slightly stronger version of the convergence of the test functions in (H5). However, this is not a problem; see [1, p. 222-223].

Proposition 2.7. *Let the solution $(\boldsymbol{u}, p)$ satisfying Brinkman's law (9) be sufficiently smooth, say, $(\boldsymbol{u}, p) \in [W^{2,\infty}(\Omega)]^N \times [L^2(\Omega)\,/\,\mathbb{R}]$. Then there exists a constant $C > 0$, $C = C(\Omega, \Gamma)$, such that*

$$\begin{aligned} \|\widetilde{\boldsymbol{u}}_\varepsilon - \boldsymbol{W}_\varepsilon\boldsymbol{u}\|_{H_1^0(\Omega)} &\leq C\varepsilon\,\|\boldsymbol{u}\|_{W^{2,\infty}(\Omega)}, \\ \|p_\varepsilon - p - \boldsymbol{u}\cdot\boldsymbol{Q}_\varepsilon\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} &\leq C\varepsilon\,\|\boldsymbol{u}\|_{W^{2,\infty}(\Omega)}. \end{aligned}$$

Here, W_ε is the matrix defined by $\boldsymbol{W}_\varepsilon\boldsymbol{e}_k = \boldsymbol{w}_k^\varepsilon$, and $\boldsymbol{Q}_\varepsilon$ the vector defined by $\boldsymbol{Q}_\varepsilon\cdot\boldsymbol{e}_k = q_k^\varepsilon$. [1, pp. 232–233] □

3 Homogenisation via energy decomposition

The remainder of this work is a first step toward the method of homogenisation via energy decomposition. We begin by introducing the basic premise of energy decomposition itself, then proceed to state the problem and verify that it has a unique solution. The extension and localisation chapters are technical prerequisites to the actual homogenisation, which is achieved in chapter 3.5. Finally, we estimate the error made along the way.

3.1 The energy decomposition

Let $\Omega \in \mathbb{R}^N$, $N \geq 2$, be a bounded Lipschitz domain containing the holes $\cup_i \Gamma_i$, and let $\Omega_{\mathrm{H}} = \Omega \setminus \cup_i \Gamma_i$ describe the fluid domain. The holes Γ_i are likewise assumed Lipschitz, and their size and distribution such that the direct solution of the nonhomogenised problem has a *two-scale structure*. To be quantified later, the two-scale property simply means that the holes and the perturbation due to them are small (yet not negligible) compared to the overall scale of the problem; in other words, there is something to homogenise. Note that we need not assume a periodic setting to discuss the energy decomposition.

Consider the Stokes problem in Ω_{H}: Find $(\boldsymbol{u}, p) \in [H_0^1(\Omega_{\mathrm{H}})]^N \times L^2(\Omega_{\mathrm{H}})$ such that

$$\begin{cases} \int_{\Omega_{\mathrm{H}}} \nabla \boldsymbol{u} : \nabla \boldsymbol{v} - \int_{\Omega_{\mathrm{H}}} p \nabla \cdot \boldsymbol{v} = \int_{\Omega_{\mathrm{H}}} \boldsymbol{f} \cdot \boldsymbol{v} & \text{for each } \boldsymbol{v} \in [H_0^1(\Omega_{\mathrm{H}})]^N \\ \int_{\Omega_{\mathrm{H}}} q \nabla \cdot \boldsymbol{u} = 0 & \text{for each } q \in L^2(\Omega_{\mathrm{H}}), \end{cases} \tag{25}$$

where $\boldsymbol{f} \in [L^2(\Omega_{\mathrm{H}})]^N$. Let

$$V_0(\omega) = \left\{ \boldsymbol{v} \in \left[H_0^1(\omega)\right]^N : \nabla \cdot \boldsymbol{v} = 0 \right\}.$$

It is well known that the problem (25) can be equivalently written as one of energy minimisation: Find $\boldsymbol{u} \in V_0(\Omega_{\mathrm{H}})$ such that

$$J(\boldsymbol{u}) = \min_{\boldsymbol{v} \in V_0(\Omega_{\mathrm{H}})} J(\boldsymbol{v}) = \min_{\boldsymbol{v} \in V_0(\Omega_{\mathrm{H}})} \left[\frac{1}{2} \|\nabla \boldsymbol{v}\|^2_{L^2(\Omega_{\mathrm{H}})} - (\boldsymbol{f}, \boldsymbol{v})_{L^2(\Omega_{\mathrm{H}})} \right]. \tag{26}$$

Again, we would like to avoid the computationally unfriendly domain Ω_{H} by deriving a homogenised problem in the whole Ω. With the minimisation problem (26) as the starting point, we aim to separate the "slowly" and "rapidly" oscillating parts of the solution, both living in their respective function spaces. This makes sense as long as the original problem has a two-scale structure, in which case the modifiers have meaning.

To begin with, we must define the problem in the entire domain. As we have chosen to impose the divergence-free condition on the space V_0, we can concentrate on solving the velocity for now: the pressure can eventually be obtained by postprocessing. Let $\boldsymbol{u} \in V_0(\Omega_{\mathrm{H}})$ be the minimiser of the energy functional (26). As in the classical formalism, we extend the velocity into the holes by zero:

$$\widetilde{\boldsymbol{u}} = \begin{cases} \boldsymbol{u} & \text{in } \Omega_{\mathrm{H}}, \\ 0 & \text{in } \cup_i \Gamma_i. \end{cases} \tag{27}$$

It clearly holds that $\widetilde{\boldsymbol{u}} \in V_0(\Omega)$; in addition, $J(\widetilde{\boldsymbol{u}}) = J(\boldsymbol{u})$.

Furthermore, let $\{\Omega_i\}_{i\in\mathcal{I}}$ denote a partition of the domain Ω into a finite number of disjoint, convex, Lipschitz subdomains. For the time being, we make no assumptions on the distribution of the holes with regard to the partition. In particular, and in contrast to classical homogenisation, a single partition set Ω_i might contain several holes.

Decomposing the problem. The *fast space*, housing the rapidly oscillating components of the solution, is defined

$$W_f = \left\{ \boldsymbol{v} \in V_0(\Omega) : \int_{\Omega_i} \boldsymbol{v} = 0 \text{ for all } i = 1, \dots, N \right\}. \tag{28}$$

This clearly is a closed subspace of $V_0(\Omega)$. Let the projection operator $P_f : V_0(\Omega) \to W_f$ be such that

$$(\nabla [P_f \widetilde{\boldsymbol{u}}], \nabla \boldsymbol{v}_f)_{L^2(\Omega)} = (\nabla \widetilde{\boldsymbol{u}}, \nabla \boldsymbol{v}_f)_{L^2(\Omega)} \quad \text{for all } \boldsymbol{v}_f \in W_f. \tag{29}$$

We define the *slow space* as the orthogonal complement of the fast space (28) with respect to the gradient inner product:

$$W_s = \left\{ \boldsymbol{w} \in V_0(\Omega) : (\nabla \boldsymbol{w}, \nabla \boldsymbol{v}_f)_{L^2(\Omega)} = 0 \text{ for all } \boldsymbol{v}_f \in W_f \right\}.$$

By virtue of the zero mean condition imposed on W_f, it holds that

$$V_0(\Omega) \,/\, \mathbb{R}^N = W_f \oplus W_s$$

and each $\boldsymbol{u} \in V_0(\Omega_{\mathrm{H}})$ can be uniquely decomposed so that $\widetilde{\boldsymbol{u}} = P_f \widetilde{\boldsymbol{u}} + (\boldsymbol{I} - P_f)\widetilde{\boldsymbol{u}}$, where $P_f \widetilde{\boldsymbol{u}} \in W_f$ and $(\boldsymbol{I} - P_f)\widetilde{\boldsymbol{u}} \in W_s$. Furthermore, we have $P_f \widetilde{\boldsymbol{u}} = -(\boldsymbol{I} - P_f)\widetilde{\boldsymbol{u}}$ in the holes $\cup_i \Gamma_i$. Writing $\boldsymbol{u}_f = P_f \widetilde{\boldsymbol{u}}$ and $\boldsymbol{u}_s = (\boldsymbol{I} - P_f)\widetilde{\boldsymbol{u}}$, we have the following decomposition in Ω for $\boldsymbol{u} \in V_0(\Omega_H)$:

$$\begin{cases} \boldsymbol{u} = \boldsymbol{u}_s + \boldsymbol{u}_f & \text{in } \Omega_H, \\ \boldsymbol{u}_f = -\boldsymbol{u}_s & \text{in } \cup_i \Gamma_i, \\ (\nabla \boldsymbol{u}_f, \nabla \boldsymbol{v})_{L^2(\Omega)} = 0 & \text{for all } \boldsymbol{v} \in W_s. \end{cases} \tag{30}$$

It follows that the minimisation problem (26) breaks down into a "slow" and a "fast" problem in spaces W_s and W_f respectively: Find $\boldsymbol{u} \in V_0(\Omega)$ such that

$$\min_{\boldsymbol{v}\in V_0(\Omega)} J(\boldsymbol{v}) = J(\boldsymbol{u}) = \min_{\boldsymbol{v}_s \in W_s} \left[J(\boldsymbol{v}_s) + \min_{\substack{\boldsymbol{v}_f \in W_f \\ \boldsymbol{v}_f = -\boldsymbol{v}_s \text{ in } \cup_i \Gamma_i}} J\left(\boldsymbol{v}_f(\boldsymbol{v}_s)\right) \right]. \tag{31}$$

The two-scale assumption. As a quantitative manifestation of the two-scale structure, we make a further assumption on the slow space associated with the problem: for each $\boldsymbol{u}_s \in W_s$, it holds that

$$\begin{aligned} \|\boldsymbol{u}_s - \pi_0 \boldsymbol{u}_s\|_{L^2(\Omega^*)} &\leq C\varepsilon^{1+\alpha}, \\ \|\nabla \boldsymbol{u}_s - \pi_0 \nabla \boldsymbol{u}_s\|_{L^2(\Omega^*)} &\leq C\varepsilon^{1+\alpha} \end{aligned} \quad \text{for all } i \in \mathcal{I}. \tag{32}$$

where $\alpha > 0$, and $\pi_m : V \to \mathcal{P}_m(\omega)$ denotes componentwise interpolation by a mth-degree polynomial, in this case a constant. This guarantees that the "slow" component indeed varies slowly compared to some characteristic scale of the heterogeneities, represented by ε. In the periodic case, this would be the mesh size, as seen in the previous chapter.

For now, we take the two-scale assumption for granted, as well as leave unidentified the sets Ω^* and ω over which, respectfully, the requirement is to be met and the constant interpolation performed. The statement will be assessed in more depth after its role in the homogenisation process has been made clear.

Who is in the slow space? The energy decomposition can be tentatively understood from a spectral viewpoint. Recall that the eigenfunctions of the Laplacian form a basis of $H_0^1(\Omega)$. It turns out that we may choose the partition $\{\Omega_i\}_{i\in\mathcal{I}}$ so that the lowest eigenfunctions very nearly span the slow space W_s:

Proposition 3.1. *Let $(\boldsymbol{\psi}_i, \lambda_i)_{i\geq 1} \in V_0(\Omega) \times \mathbb{R}_+$ be the normalized eigenpairs of the Laplacian, i.e. $\|\boldsymbol{\psi}_i\|_{L^2(\Omega)} = 1$ and*

$$(\nabla \boldsymbol{\psi}_i, \nabla \boldsymbol{v})_{L^2(\Omega)} = \lambda_i (\boldsymbol{\psi}_i, \boldsymbol{v}_f)_{L^2(\Omega)} \quad \textit{for all } \boldsymbol{v} \in V_0(\Omega). \tag{33}$$

Then we have for all $\boldsymbol{v}_s \in W_s$

$$\min_{\boldsymbol{v}_s \in W_s} \|\nabla(\boldsymbol{\psi}_i - \boldsymbol{v}_s)\|^2_{L^2(\Omega)} \leq C\lambda_i^{3/2} h^2,$$

where $h = C \max_{j\in\mathcal{I}} (\operatorname{diam} \Omega_j)$.

Proof. Writing $\boldsymbol{\psi}_i = P_f \boldsymbol{\psi}_i + (\boldsymbol{I} - P_f)\boldsymbol{\psi}_i$, (33) immediately yields

$$(\nabla \boldsymbol{\psi}_i, \nabla \boldsymbol{v}_f)_{L^2(\Omega)} = (\nabla [P_f \boldsymbol{\psi}_i], \nabla \boldsymbol{v}_f)_{L^2(\Omega)} \quad \text{for all } \boldsymbol{v} \in W_f.$$

Let $h_j = C \operatorname{diam} \Omega_j$. Take a $\boldsymbol{v}_f \in W_f$ and estimate:

$$\begin{aligned} (\nabla [P_f \boldsymbol{\psi}_i], \nabla \boldsymbol{v}_f)_{L^2(\Omega)} &= (\nabla \boldsymbol{\psi}_i, \nabla \boldsymbol{v}_f)_{L^2(\Omega)} = \lambda_i (\boldsymbol{\psi}_i, \boldsymbol{v}_f)_{L^2(\Omega)} \\ &= \lambda_i \sum_{j\in\mathcal{I}} (\boldsymbol{\psi}_i - \pi_0 \boldsymbol{\psi}_i, \boldsymbol{v}_f)_{L^2(\Omega_j)} \\ &\leq \lambda_i \sum_j \|\boldsymbol{\psi}_i - \pi_0 \boldsymbol{\psi}_i\|_{L^2(\Omega_j)} \|\boldsymbol{v}_f\|_{L^2(\Omega_j)} \\ &\leq \lambda_i \sum_j \pi^{-1} h_j^2 \|\nabla \boldsymbol{\psi}_i\|^2_{L^2(\Omega_j)} \|\nabla \boldsymbol{v}_f\|_{L^2(\Omega_j)} \qquad (34) \\ &\leq \pi^{-1} \lambda_i h_{max}^2 \left(\sum_j \|\nabla \boldsymbol{\psi}_i\|^2_{L^2(\Omega_j)} \right)^{1/2} \left(\sum_j \|\nabla \boldsymbol{v}_f\|^2_{L^2(\Omega_j)} \right)^{1/2} \\ &= C\lambda_i h_{max}^2 \|\nabla \boldsymbol{\psi}_i\|_{L^2(\Omega)} \|\nabla \boldsymbol{v}_f\|_{L^2(\Omega)}, \end{aligned}$$

where (34) follows from the zero-mean Poincaré inequality with an estimate for the constant (A.4). Choosing $\boldsymbol{v}_f = P_f\boldsymbol{\psi}_i$ in the above, we obtain

$$\|\nabla\left[P_f\boldsymbol{\psi}_i\right]\|_{L^2(\Omega)} \leq C\lambda_i h_{max}^2 \|\nabla\boldsymbol{\psi}_i\|_{L^2(\Omega)} = C\lambda_i^{3/2} h_{max}^2 \tag{35}$$

(because $\lambda\|\boldsymbol{\psi}\|^2 = \lambda(\boldsymbol{\psi},\boldsymbol{\psi}) = \lambda(\boldsymbol{\psi},\boldsymbol{\psi}_f) + \lambda(\boldsymbol{\psi},\boldsymbol{\psi}_s) = (\nabla\boldsymbol{\psi},\nabla\boldsymbol{\psi}) + (\nabla\boldsymbol{\psi}_s,\nabla\boldsymbol{\psi}_f) = \|\nabla\boldsymbol{\psi}\|^2$). Hence, for an arbitrary $\boldsymbol{v}_s \in W_s$,

$$\begin{aligned}\min_{\boldsymbol{v}_s\in W_s} \|\nabla(\boldsymbol{\psi}_i - \boldsymbol{v}_s)\|_{L^2(\Omega)}^2 &= \min_{\boldsymbol{v}_s\in W_s}\left[\|\nabla\left[P_f\boldsymbol{\psi}_i\right]\|_{L^2(\Omega)}^2 + \|\nabla\left[(\boldsymbol{I}-P_f)(\boldsymbol{\psi}_i-\boldsymbol{v}_s)\right]\|_{L^2(\Omega)}^2\right]\\ &= \|\nabla\left[P_f\boldsymbol{\psi}_i\right]\|_{L^2(\Omega)}^2 \leq C\lambda_i^{3/2}h^2.\end{aligned}$$

□

In other words, assuming that the eigenvalues are in ascending order, it holds that $W_s \approx \operatorname{span}\{\boldsymbol{\psi}_i\}_{i\leq n}$, where the cut-off index n depends on the partition $\{\Omega_i\}_{i\in\mathcal{I}}$. The parameter h is related to the mesh size. For instance, for a N-dimensional mesh consisting of hypercubes, the natural choice would be $h_j = N^{-1/2}\operatorname{diam}\Omega_j$; in addition, if the medium is taken to be periodic, this choice would correspond to $h = \varepsilon$ in accordance with the classical case.

3.2 Setting the stage: The fast problem

The fast problem in $V_0(\Omega)$, as extracted from (31), is

$$\min_{\substack{\boldsymbol{u}_f\in W_f\\ \boldsymbol{u}_f=-\boldsymbol{u}_s \text{ in } \cup_i\Gamma_i}} J\left(\boldsymbol{u}_f(\boldsymbol{u}_s)\right), \tag{36}$$

with $J(\boldsymbol{u}_f) = \frac{1}{2}(\nabla\boldsymbol{u}_f,\nabla\boldsymbol{u}_f)_{L^2(\Omega)} - (\boldsymbol{f},\boldsymbol{u}_f)_{L^2(\Omega)}$. With the energy decomposition as the starting point, our aim is to avoid the direct solution of this problem and replace $\boldsymbol{u}_f$ with a readily computable approximation. Then, plugging the approximate solution back into (31), we will have homogenised the problem.

As a first approximative step, we notice that the fast component of the load is small. Writing $h_\kappa = C\operatorname{diam}\Omega_\kappa$ and $h = \max_{\kappa\in\mathcal{I}} h_\kappa$ as in the proof of Proposition 3.1, we have

$$\begin{aligned}(\boldsymbol{f},\boldsymbol{u}_f)_{L^2(\Omega)} = \left|\sum_{\kappa\in\mathcal{I}}\int_{\Omega_\kappa}\boldsymbol{f}\cdot\boldsymbol{u}_f\right| &\leq \left|\sum_\kappa \|\boldsymbol{f}\|_{L^2(\Omega_\kappa)}\|\boldsymbol{u}_f\|_{L^2(\Omega_\kappa)}\right|\\ &\leq \left|\sum_\kappa \|\boldsymbol{f}\|_{L^2(\Omega_\kappa)} C_\kappa h_\kappa \|\nabla\boldsymbol{u}_f\|_{L^2(\Omega_i)}\right|\\ &\leq C_{max}h_{max}\|\nabla\boldsymbol{u}_f\|_{L^2(\Omega)}\left|\sum_\kappa\|\boldsymbol{f}\|_{L^2(\Omega_\kappa)}\right|\\ &\leq Ch\|\boldsymbol{f}\|_{L^2(\Omega)}^2,\end{aligned} \tag{37}$$

where the constant C depends on the partition. Ignoring the load term, the constrained minimisation problem (36) is equivalent to the following variational problem on Ω: Find $(\boldsymbol{u}_f, \boldsymbol{\lambda}) \in W_f \times V_0(\cup_i \Gamma_i)$, such that

$$\begin{cases} (\nabla \boldsymbol{u}_f, \nabla \boldsymbol{v}_f)_{L^2(\Omega)} + (\boldsymbol{\lambda}, \boldsymbol{v}_f)_{H^1(\cup_i \Gamma_i)} = 0 & \text{for all } \boldsymbol{v}_f \in W_f, \\ (\boldsymbol{u}_f, \boldsymbol{\mu})_{H^1(\cup_i \Gamma_i)} = (-\boldsymbol{u}_s, \boldsymbol{\mu})_{H^1(\cup_i \Gamma_i)} & \text{for all } \boldsymbol{\mu} \in V_0(\cup_i \Gamma_i). \end{cases} \tag{38}$$

Proposition 3.2. *The problem* (38) *is well-posed.*

Proof. The second equation of problem (38) states that $b(\boldsymbol{u}_f, \boldsymbol{\mu}) = l_{\boldsymbol{u}_s}(\boldsymbol{\mu})$ for all $\boldsymbol{\mu} \in V_0(\cup_i \Gamma_i)$. By the Babuška–Lax–Milgram lemma (A.3), there is a unique solution whenever three conditions are satisfied. First, the bilinear form b is merely the inner product on $H^1(\cup_i \Gamma_i)$, whose continuity is obvious. Second, for the inf–sup condition, let $\boldsymbol{w} = \mathcal{E}\boldsymbol{\lambda}$ be the extension given by Lemma 3.3. Then, we have $\boldsymbol{w} \in W_f(\Omega)$, and

$$\|\boldsymbol{w}\|_{H^1(\Omega)} \leq \max_{i \in \mathcal{I}} C_i \cdot \|\boldsymbol{\lambda}\|_{H^1(\cup_i \Gamma_i)},$$

$$\sup_{\boldsymbol{w} \in W_f,\, \boldsymbol{w} \neq 0} \frac{(\boldsymbol{w}, \boldsymbol{\lambda})_{H^1(\cup_i \Gamma_i)}}{\|\boldsymbol{w}\|_{H^1(\Omega)}} \geq \frac{(\mathcal{E}\boldsymbol{\lambda}, \boldsymbol{\lambda})_{H^1(\cup_i \Gamma_i)}}{\|\boldsymbol{w}\|_{H^1(\Omega)}} \geq \frac{(\boldsymbol{\lambda}, \boldsymbol{\lambda})_{H^1(\cup_i \Gamma_i)}}{C \|\boldsymbol{\lambda}\|_{H^1(\cup_i \Gamma_i)}} = \frac{1}{C} \|\boldsymbol{\lambda}\|_{H^1(\cup_i \Gamma_i)}.$$

The third condition follows for any $\boldsymbol{u}_f$ by letting $\boldsymbol{\mu} = \boldsymbol{u}_f|_{\cup_i \Gamma_i}$, which clearly is in $V_0(\cup_i \Gamma_i)$. □

Remark 3.1. The choice of the H^1 inner product in problem (38) may not be immediately obvious, but necessary lest we end up with an ill-posed problem. Why is the L^2 inner product not good enough for the job? The normalized eigenpairs of the Laplacian, $(\boldsymbol{\lambda}_i, \mu_i)_{i \geq 1} \in V_0(\cup_i \Gamma_i) \times \mathbb{R}^n$, provide a counterexample. The eigenvalues satisfy $\mu_i > 0$ and $\mu_i \to \infty$ as $i \to \infty$ (see [20, pp. 38–39]); in particular, there exists a j such that $\mu_i \geq 1$ for all $i \geq j$. Furthermore, we have

$$|\boldsymbol{\lambda}_i|^2_{H^1(\cup_i \Gamma_i)} = \mu_i \|\boldsymbol{\lambda}_i\|^2_{L^2(\cup_i \Gamma_i)}.$$

Applying the above estimate for $(\boldsymbol{\lambda}_i, \mu_i)_{i \geq j}$, we observe that a finite upper bound does not exist:

$$\sup_{\boldsymbol{w} \in W_f,\, \boldsymbol{w} \neq 0} \frac{(\boldsymbol{w}, \boldsymbol{\lambda}_i)_{L^2(\cup_i \Gamma_i)}}{\|\boldsymbol{w}\|_{H^1(\Omega)}} \geq \frac{(\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_i)_{L^2(\cup_i \Gamma_i)}}{\|\boldsymbol{\lambda}_i\|_{H^1(\Omega)}} \geq \frac{\|\boldsymbol{\lambda}_i\|^2_{L^2(\cup_i \Gamma_i)}}{\sqrt{2\mu_i}\, \|\boldsymbol{\lambda}_i\|_{L^2(\cup_i \Gamma_i)}} = \frac{1}{\sqrt{2\mu_i}} \xrightarrow[i \to \infty]{} \infty.$$

Remark 3.2. In particular, the Babuška–Lax–Milgram lemma guarantees the existence of a unique Lagrange multiplier $\boldsymbol{\lambda} \in V_0(\cup_i \Gamma_i)$. Solving the multiplier is not our main interest, but it is immediately seen to represent the part of the flow that has been suppressed by the constraint $\boldsymbol{u} = -\boldsymbol{u}_s$ in $\cup_i \Gamma_i$. In principle, we are solving the variational problem for $\boldsymbol{u}$: Find $\boldsymbol{u} \in V = \{\boldsymbol{u} \in W_f(\Omega) : \boldsymbol{u} = \boldsymbol{u}_s \text{ in } \cup_i \Gamma_i\}$ such that

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega \setminus \cup_i \Gamma_i} = 0 \text{ for all } \boldsymbol{v} \in V_0(\Omega \setminus \cup_i \Gamma_i).$$

Let $\mathcal{E}\boldsymbol{v}$ be the extension of $\boldsymbol{v}$ into the space $V_0(\Omega)$; we may define $\boldsymbol{v}$ in each hole Γ_i as the solution of a Stokes problem with the appropriate boundary conditions. Thus the problem satisfied by $\boldsymbol{\lambda}$ is

$$(\boldsymbol{\lambda}, \boldsymbol{v})_{H^1(\cup_i \Gamma_i)} = (\nabla \boldsymbol{u}, \nabla \mathcal{E}\boldsymbol{v})_{L^2(\cup_i \Gamma_i)} = l_{\boldsymbol{u}}(\boldsymbol{v}),$$

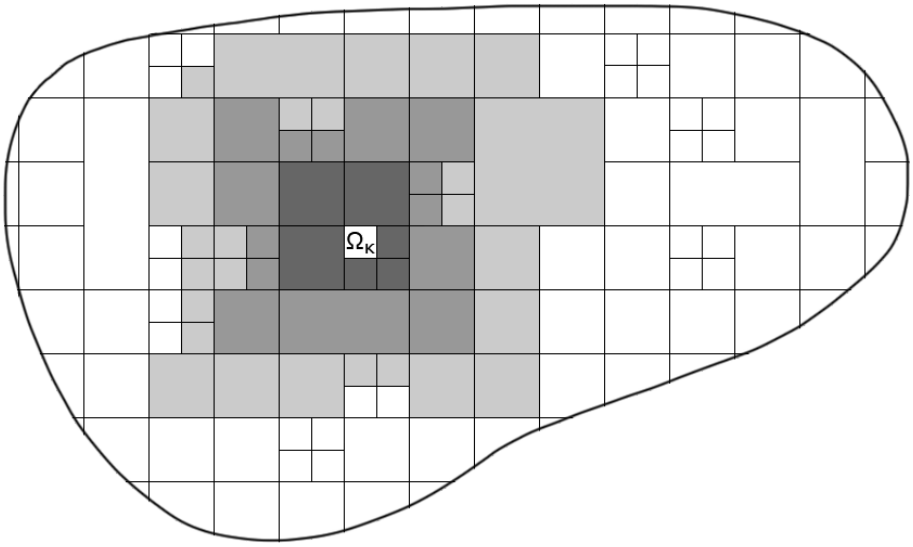


Figure 4: Three recursive patches Ω_κ^n, $n = 1, 2, 3$, with R chosen as the side length of Ω_κ. The tiles, rectangular for convenience only, represent a partition $\{\Omega_i\}_{i\in\mathcal{I}}$ of Ω. For the time being, we may imagine ourselves far from the domain boundary, unlike in this picture.

from where we see that $\boldsymbol{\lambda}$ can be understood as a solution to the Stokes problem in $\cup_i \Gamma_i$:

$$\begin{cases} \Delta\boldsymbol{\lambda} + \boldsymbol{\lambda} - \nabla p = \Delta \boldsymbol{u}, \\ \nabla \cdot \lambda = 0. \end{cases}$$

3.3 A friendly local extension lemma

Before beginning with the actual homogenisation process, we introduce a technical tool for the analysis of the decomposed problem, namely an extension of the hole data into the surrounding domain. The extension technique can be considered the closest counterpart of oscillating test functions in our framework. Namely, it generates functions in the fast space W_f that will be used to extract information from the fast problem.

For a partition set $\Omega_\kappa \in \{\Omega_i\}_{i\in\mathcal{I}}$, we define the *extended patch* $\Omega_\kappa^n \supset \Omega_\kappa$ recursively by $\Omega_\kappa^0 = \Omega_\kappa$ and

$$\Omega_\kappa^n = \bigcup_{i\in\mathcal{I}^n} \Omega_i, \text{ where } \mathcal{I}^n = \left\{ i \in \mathcal{I} : \operatorname{dist}\left(\Omega_i, \Omega_\kappa^{n-1}\right) < R \right\}, \ n = 1, 2, \ldots \tag{39}$$

with $R > 0$ to be chosen later.

Remark 3.3. In the following, we assume that the patch does not meet the domain boundary: $\Omega_\kappa^n \cap \partial\Omega = \varnothing$. The domain boundary layer is a significant source of error and its effect will have to be taken into account, but is sadly omitted in the following. In addition, we assume that the holes are situated at a minimum distance from each other and the patch boundary.

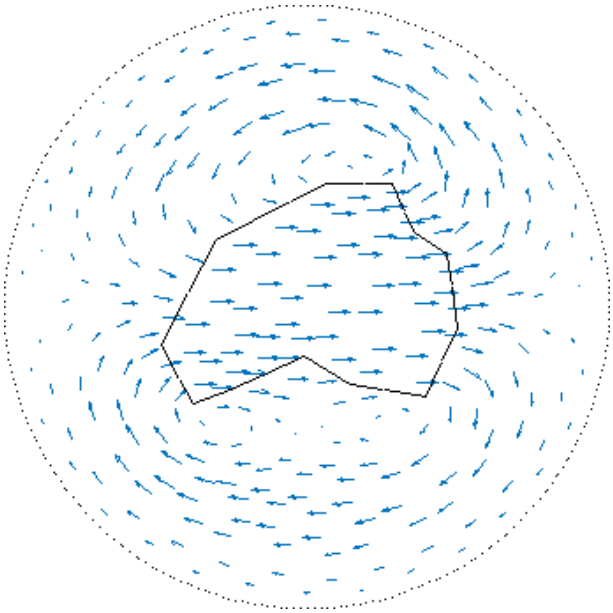


Figure 5: The extension $\widetilde{\boldsymbol{v}}$ in $\widetilde{\Gamma_\iota}$, made up of a hole and its neighborhood.

For each hole $\Gamma_\iota \subset \Omega^n_\kappa$, let $\boldsymbol{x}_\iota = |\Gamma_\iota|^{-1} \int_{\Gamma_\iota} \boldsymbol{x}\, \mathrm{d}V$ be its centroid and $\widetilde{\Gamma}_\iota = \mathcal{B}(\boldsymbol{x}_\iota, R_\iota) \supset \Gamma_\iota$. The simplifying assumptions made in Remark 3.3 guarantee the existence of radii $\{R_\iota\}_\iota$ such that the sets $\{\widetilde{\Gamma}_\iota \setminus \Gamma_\iota\}_\iota$ are nonempty and disjoint.

Lemma 3.3. *Let $\boldsymbol{v} \in V_0(\cup_\iota \Gamma\iota)$. There exists an extension $\mathcal{E}\boldsymbol{v} = \widetilde{\boldsymbol{v}} \in V_0(\Omega^n_\kappa)$ of the hole data*

$$\begin{cases} \widetilde{\boldsymbol{v}} = \boldsymbol{0} & \text{in } \Omega^n_\kappa \setminus \cup_\iota \widetilde{\Gamma}_\iota, \\ \widetilde{\boldsymbol{v}} = \boldsymbol{v} & \text{in } \cup_\iota \Gamma_\iota \end{cases}$$

with the following properties:

(i) $\int_{\Omega^n_\kappa} \widetilde{\boldsymbol{v}} = 0$;

(ii) $\|\widetilde{\boldsymbol{v}}\|_{H^1(\Omega^n_\kappa)} \leq C(\Omega^n_\kappa)\, \|\boldsymbol{v}\|_{H^1(\cup_\iota \Gamma_\iota)}$.

Proof. Having agreed that the hole neighborhoods $\widetilde{\Gamma}_\iota \setminus \Gamma_\iota$ do not coincide, the extension can be sought for one hole at a time. Let ι be fixed; the problem is now to describe the local extension $\widetilde{\boldsymbol{v}}_\iota$ in the annular domain $\widetilde{\Gamma}_\iota \setminus \Gamma_\iota$. The divergence-free property of the space V_0 prompts us to seek it as a solution to the following Stokes problem: Find $\boldsymbol{w}_\iota \in [H^1(\widetilde{\Gamma}_\iota \setminus \Gamma_\iota)]^N$ such that

$$\begin{cases} \nabla p - \Delta \boldsymbol{w}_\iota = \boldsymbol{0} & \text{in } \widetilde{\Gamma}_\iota \setminus \Gamma_\iota, \\ \nabla \cdot \boldsymbol{w}_\iota = 0 & \text{in } \widetilde{\Gamma}_\iota \setminus \Gamma_\iota, \\ \boldsymbol{w}_\iota = \boldsymbol{v} & \text{on } \partial\Gamma_\iota, \\ \boldsymbol{w}_\iota = 0 & \text{on } \partial\widetilde{\Gamma}_\iota. \end{cases} \tag{40}$$

Because $\boldsymbol{v}$ itself is divergence-free, the boundary conditions clearly are compatible. Hence the problem (40) has a solution by the standard theory for the Stokes equations; see [20, p. 31]. We now have a candidate for the extension about a single hole Γ_ι:

$$\widetilde{\boldsymbol{v}}_\iota = \begin{cases} \boldsymbol{v} & \text{in } \Gamma_\iota, \\ \boldsymbol{w}_\iota & \text{in } \widetilde{\Gamma}_\iota \setminus \Gamma_\iota, \\ \boldsymbol{0} & \text{in } \Omega_\kappa^n \setminus \widetilde{\Gamma}_\iota \end{cases} \tag{41}$$

for each $\iota \in \mathcal{J} = \{i \in \mathcal{I} : \Gamma_i \subset \Omega_\kappa^n\}$.

To verify that the newfound extension (41) is of zero mean on Ω_κ^n, it is enough to look at $\widetilde{\Gamma}_\iota$. The function $\widetilde{\boldsymbol{v}}_\iota$ lies in the space $[H_0^1(\widetilde{\Gamma}_\iota)]^N$, hence it is in $H_0^1(\mathrm{div}; \widetilde{\Gamma}_\iota)$ as well by [12, Lemma 35, p. 35]. Now, Lemma 26 in [18, p. 24] states, among other things, that there exists a $\boldsymbol{\psi}$ such that

$$\widetilde{\boldsymbol{v}}_\iota = \nabla \times \boldsymbol{\psi} \quad \text{and} \quad \boldsymbol{\psi} \in H_0^1(\mathrm{curl}; \widetilde{\Gamma}_\iota).$$

Integrating by parts, we obtain for $i = 1, \ldots, N$

$$\int_{\widetilde{\Gamma}_\iota} \widetilde{\boldsymbol{u}}_\iota \cdot \boldsymbol{e}_i = \int_{\widetilde{\Gamma}_\iota} (\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{e}_i = \int_{\widetilde{\Gamma}_\iota} \boldsymbol{\psi} \cdot (\nabla \times \boldsymbol{e}_i) - \int_{\partial\widetilde{\Gamma}_\iota} (\boldsymbol{\psi} \times \boldsymbol{n}) \cdot \boldsymbol{e}_i = 0,$$

where the boundary term vanishes because the normal trace of $\boldsymbol{\psi} \in H_0^1(\mathrm{curl}; \widetilde{\Gamma}_\iota)$ does. Property (i) is hereby proven.

Finally, the desired extension $\widetilde{\boldsymbol{v}}$ to Ω_κ^n is obtained by summing up the contributions from each hole: $\widetilde{\boldsymbol{v}} = \sum_{\iota \in \mathcal{J}} \widetilde{\boldsymbol{v}}_\iota$. The norm estimate (ii) follows from the trace theorem and its inverse (Theorems A.5 and A.6), noticing that the trace of $\widetilde{\boldsymbol{v}}$ vanishes on $\cup_\iota(\partial\widetilde{\Gamma}_\iota)$:

$$\begin{aligned} \|\widetilde{\boldsymbol{v}}\|_{H^1(\Omega_\kappa^n)} &= \|\mathcal{E}\boldsymbol{v}\|_{H^1(\cup_\iota(\widetilde{\Gamma}_\iota \setminus \Gamma_\iota))} + \|\mathcal{E}\boldsymbol{v}\|_{H^1(\cup_\iota \Gamma_\iota)} \\ &\leq C(\Omega_\kappa^n) \, \|\mathrm{tr}\, \widetilde{\boldsymbol{v}}\|_{H^{\frac{1}{2}}(\cup_\iota(\partial\Gamma_\iota))} + C(\Omega_\kappa^n) \, \|\mathrm{tr}\, \boldsymbol{v}\|_{H^{\frac{1}{2}}(\cup_\iota(\partial\Gamma_\iota))} \\ &\leq C(\Omega_\kappa^n) \, \|\boldsymbol{v}\|_{H^1(\cup_\iota \Gamma_\iota)} \, . \end{aligned}$$

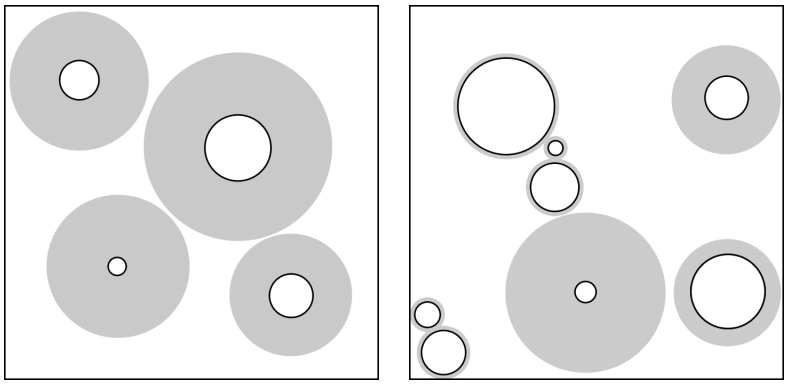
□

Remark 3.4. We may even choose $n = 0$ in the previous lemma and apply it in each partition set Ω_κ to obtain extensions $\widetilde{\boldsymbol{v}}_\kappa \in V_0(\Omega_\kappa)$. Summing these up, we have a function $\widetilde{\boldsymbol{v}} \in W_f$ and the estimate

$$\|\widetilde{\boldsymbol{v}}\|_{H^1(\Omega)} \leq \max_{i \in \mathcal{I}} C_i \cdot \|\boldsymbol{v}\|_{H^1(\cup_i \Gamma_i)} \, .$$

Scaling. The above estimate provides no information about the constant $C(\Omega_\kappa^n)$ originating from the trace theorem. One may reason that this constant will be comparatively large for cells where two or more holes happen to be close to each other or the patch boundary, because the neighborhoods $\widetilde{\Gamma}_\iota \setminus \Gamma_\iota$ will necessarily be small in order to remain disjoint. This, in turn, will increase the gradient of the extension, as the flow must complete a loop in a small space.

Even without looking any deeper into the trace constant, we may readily estimate its dependence on the diameter of the set $\widetilde{\Gamma}_\iota$ by a scaling argument.



(a) Good (b) Unfortunate

Figure 6: Holes $\{\Gamma_\iota\}_{\iota\in\mathcal{J}}$ (depicted as circular for convenience only) in a patch Ω^n_κ (square for convenience only) and their neighborhoods $\left\{\widetilde{\Gamma}_\iota \setminus \Gamma_\iota\right\}_{\iota\in\mathcal{J}}$ (gray). The configuration (a), with neighborhoods reasonably spacious both in relation to the patch and the holes themselves, is expected to yield a good constant $C(\Omega^n_\kappa)$; not so for the case (b).

Lemma 3.4. *The norm estimate (ii) given by Lemma 3.3 holds with*

$$C(\Omega^n_\kappa) = C_\gamma \max_{\iota\in\mathcal{J}} \left(\operatorname{diam} \widetilde{\Gamma}_\iota\right)^{-1},$$

where the constant C_γ only depends on the shape and size of the holes $\Gamma_\iota \subset \Omega^n_\kappa$ in relation to their respective supersets $\widetilde{\Gamma}_\iota$, and not on their absolute diameter.

Proof. For the sake of argument, let Ω be a patch containing a hole Γ, and $\widetilde{\boldsymbol{v}}$ the extension of a function $\boldsymbol{v} \in V_0(\Gamma)$ about this hole. The extension is supported on the set $\widetilde{\Gamma}$, whose diameter is of order 1. To observe the behaviour of the constant, we shrink the support into a set $\widetilde{\Gamma}_\varepsilon$ of diameter ε, on which we define the function $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{v}(\varepsilon^{-1}\boldsymbol{x})$. This operation yields

$$\begin{aligned}
\|\widetilde{\boldsymbol{v}}\|^2_{H^1(\Omega)} \leq C^2 \|\boldsymbol{v}\|^2_{H^1(\Gamma_1)} &= C^2 \left(\int_\Gamma |\boldsymbol{v}(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} + \int_\Gamma |\nabla_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \right) \\
&\overset{\boldsymbol{\xi}=\varepsilon\boldsymbol{x}}{=} C^2 \left(\int_{\Gamma_\varepsilon} |\boldsymbol{w}(\boldsymbol{\xi})|^2 \cdot \varepsilon^{-1} \,\mathrm{d}\boldsymbol{\xi} + \int_{\Gamma_\varepsilon} \varepsilon^2 |\nabla_{\boldsymbol{\xi}} \boldsymbol{w}(\boldsymbol{\xi})|^2 \cdot \varepsilon^{-1} \,\mathrm{d}\boldsymbol{\xi} \right) \\
&= C^2 \left(\varepsilon^{-1} \|\boldsymbol{w}\|^2_{L^2(\Gamma_\varepsilon)} + \varepsilon \, |\boldsymbol{w}|^2_{H^1(\Gamma_\varepsilon)} \right). \qquad (42)
\end{aligned}$$

On the other hand, by performing the same change of variable on the left-hand side,

$$\|\widetilde{\boldsymbol{v}}\|^2_{H^1(\Omega)} = \varepsilon^{-1} \|\widetilde{\boldsymbol{w}}\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon \, |\widetilde{\boldsymbol{w}}|^2_{H^1(\Omega_\varepsilon)}. \qquad (43)$$

Combining (42) and (43), we obtain when $\varepsilon < 1$

$$\begin{aligned}
\|\widetilde{\boldsymbol{w}}\|^2_{H^1(\Omega_\varepsilon)} &\leq \varepsilon^{-2} \|\widetilde{\boldsymbol{w}}\|^2_{L^2(\Omega_\varepsilon)} + |\widetilde{\boldsymbol{w}}|^2_{H^1(\Omega_\varepsilon)} \leq C^2 \left(\varepsilon^{-2} \|\boldsymbol{w}\|^2_{L^2(\Gamma_\varepsilon)} + |\boldsymbol{w}|^2_{H^1(\Gamma_\varepsilon)} \right) \\
&\leq C^2 \varepsilon^{-2} \|\boldsymbol{w}\|^2_{H^1(\Gamma_\varepsilon)},
\end{aligned}$$

which is the desired result. □

3.4 Localisation

Let us recall the fast problem (38): Find $(\boldsymbol{u}_f, \boldsymbol{\lambda}) \in W_f \times V_0(\cup_i \Gamma_i)$, such that

$$\begin{cases} (\nabla \boldsymbol{u}_f, \nabla \boldsymbol{v}_f)_{L^2(\Omega)} + (\boldsymbol{\lambda}, \boldsymbol{v}_f)_{H^1(\cup_i \Gamma_i)} = 0 & \text{for all } \boldsymbol{v}_f \in W_f, \\ (\boldsymbol{u}_f, \boldsymbol{\mu})_{H^1(\cup_i \Gamma_i)} = (-\boldsymbol{u}_s, \boldsymbol{\mu})_{H^1(\cup_i \Gamma_i)} & \text{for all } \boldsymbol{\mu} \in V_0(\cup_i \Gamma_i). \end{cases} \tag{44}$$

In practice, we would like to approximate the fast solution $\boldsymbol{u}_f$ locally by solving this problem in each partition set or "cell" Ω_κ. The correct conditions at the cell boundary are difficult to determine. Cutting corners, we impose a zero Dirichlet condition, which clearly adds to the error we are making. However, it turns out that the effect of the artificial boundary condition diminishes quickly (indeed, exponentially) when moving away from the boundary. Thanks to this *localisation property*, we may solve the cell problem (44) on a larger set with a zero Dirichlet condition and thereby obtain a good approximation of the original solution on Ω_κ, provided that we extend enough. In the following, we will attempt to quantify how much is "enough", given that we want our approximation to be "good".

Instead of problem (44), we would prefer to solve the following truncated version on Ω_κ^n for some n, with a zero Dirichlet condition on the patch boundary: Find $(\widehat{\boldsymbol{u}}_f, \widehat{\boldsymbol{\lambda}}) \in \left[W_f \cap [H_0^1(\Omega_\kappa^n)]^N\right] \times V_0(\cup_\iota \Gamma_\iota)$ such that

$$\begin{cases} (\nabla \widehat{\boldsymbol{u}}_f, \nabla \widehat{\boldsymbol{v}}_f)_{L^2(\Omega_\kappa^n)} + (\widehat{\boldsymbol{\lambda}}, \widehat{\boldsymbol{v}}_f)_{H^1(\cup_\iota \Gamma_\iota)} = 0 & \text{for all } \widehat{\boldsymbol{v}}_f \in W_f \cap [H_0^1(\Omega_\kappa^n)]^N, \\ (\widehat{\boldsymbol{u}}_f, \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)} = (-\boldsymbol{u}_s, \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)} & \text{for all } \boldsymbol{\mu} \in V_0(\cup_\iota \Gamma_\iota) \end{cases} \tag{45}$$

where $\cup_\iota \Gamma_\iota$ denotes the holes contained in Ω_κ^n. If $\boldsymbol{u}_f$ and $\widehat{\boldsymbol{u}}_f$ are the solutions of problems (44) and (45) respectively, then the restriction of $\boldsymbol{u}_f$ to Ω_κ^n is equal to the solution of (45) with the actual boundary conditions and will serve as reference.

Reformulating the problem. To extract information about the error $\boldsymbol{e} = \boldsymbol{u}_f - \widehat{\boldsymbol{u}}_f$, we have to operate in a space of functions that are divergence-free, of zero mean on Ω_κ^n, and respect the Dirichlet condition on the hole as well as domain boundaries. The direct construction of test functions in this space would involve a lot of technical tinkering; the case of the Poisson equation in [14, pp. 10ff] serves as an admonishment. We adopt an alternative strategy and use the extension operator introduced in Lemma 3.3 to generate functions with the desired properties, permitting us to express the problems (44) and (45) in a comparable form.

On Ω_κ^n, let $\widetilde{\boldsymbol{u}} \in W_f$ be the extension of $\boldsymbol{u}_s|_{\cup_\iota \Gamma_\iota}$ provided by Lemma (3.3). The solutions to problems (44) and (45) can be decomposed as follows:

$$\begin{cases} \boldsymbol{u}_f = \boldsymbol{u}_f^0 + \widetilde{\boldsymbol{u}}, \\ \widehat{\boldsymbol{u}}_f = \widehat{\boldsymbol{u}}_f^0 + \widetilde{\boldsymbol{u}}. \end{cases}$$

By construction, the functions $\boldsymbol{u}_f^0$ and $\widehat{\boldsymbol{u}}_f^0$ vanish on any hole boundaries within their respective domains of definition. Using the above decomposition, the problem (44) can be written as

$$\begin{cases} (\nabla(\boldsymbol{u}_f^0 + \widetilde{\boldsymbol{u}}), \nabla \boldsymbol{v}_f)_{L^2(\Omega)} + (\boldsymbol{\lambda}, \boldsymbol{v}_f)_{H^1(\cup_i \Gamma_i)} = 0 & \text{for all } \boldsymbol{v}_f \in W_f, \\ (\boldsymbol{u}_f^0 + \widetilde{\boldsymbol{u}}, \boldsymbol{\mu})_{H^1(\cup_i \Gamma_i)} = (-\boldsymbol{u}_s, \boldsymbol{\mu})_{H^1(\cup_i \Gamma_i)} & \text{for all } \boldsymbol{\mu} \in V_0(\cup_i \Gamma_i). \end{cases}$$

Now, the second equation merely states that $(\boldsymbol{u}_f^0, \boldsymbol{\mu})_{H^1(\cup_i \Gamma_i)} = 0$, allowing us to reformulate the first one:

$$(\nabla(\boldsymbol{u}_f^0 + \widetilde{\boldsymbol{u}}), \nabla \boldsymbol{v}_f)_{L^2(\Omega)} = 0 \text{ for all } \boldsymbol{v}_f \in W_f \cap \left[H_0^1(\cup_i \Gamma_i)\right]^N. \tag{46}$$

An analogous treatment of problem (45) results in

$$(\nabla(\widehat{\boldsymbol{u}}_f^0 + \widetilde{\boldsymbol{u}}), \nabla \widehat{\boldsymbol{v}}_f)_{L^2(\Omega_\kappa^n)} = 0 \text{ for all } \widehat{\boldsymbol{v}}_f \in W_f \cap \left[H_0^1(\Omega_\kappa^n)\right]^N \cap \left[H_0^1(\cup_\iota \Gamma_\iota)\right]^N. \tag{47}$$

Remark 3.5. Introducing the extension from Lemma 3.3 in both (46) and (47), we have reduced the Stokes problem to an energy minimisation problem of the type

$$\min_{\boldsymbol{v} \in V} J(\boldsymbol{v}) = \min_{\boldsymbol{v} \in V} \left[\frac{1}{2} \|\nabla \boldsymbol{v}\|_{L^2(\omega)}^2 + (\nabla \widetilde{\boldsymbol{u}}, \nabla \boldsymbol{v})_{L^2(\omega)} \right].$$

Here $V \subset [H_0^1(\omega)]^N$ is a closed subspace, in which a minimiser definitely exists.

So far, so good: we have exterminated the extra terms in problems (44) and (45). To compare the solutions $\boldsymbol{u}_f$ and $\widehat{\boldsymbol{u}}_f$ (or, equivalently, $\boldsymbol{u}_f^0$ and $\widehat{\boldsymbol{u}}_f^0$) on Ω_κ^n, we still need a test function that restricts the problem (46) onto Ω_κ^n. To this effect, let $\eta \in C_0^\infty(\Omega; [0,1])$ be the scalar-valued cut-off function obtained by mollification with radius $R/3$ (see [14, p. 11] for the exact construction) such that

$$\eta(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \in \partial\Omega_\kappa^n, \\ 1, & \boldsymbol{x} \in \Omega_\kappa^{n-1} \end{cases} \quad \text{and} \quad \|\nabla \eta\| \leq \frac{3}{R} \tag{48}$$

in the Euclidean norm $\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$. The function $\eta \boldsymbol{e}$ now lies in the spaces $[H_0^1(\Omega)]^N \cap [H_0^1(\cup_i \Gamma_i)]^N$ as well as $[H_0^1(\Omega_\kappa^n)]^N \cap [H_0^1(\cup_\iota \Gamma_\iota)]^N$; the solutions $\boldsymbol{u}_f$ and $\widehat{\boldsymbol{u}}_f$ agree at the hole boundaries where defined. One more move is needed to ensure that our test function is in the space W_f. Let the projection operators

$$\begin{aligned} P_f &: \left[H_0^1(\Omega)\right]^N \cap \left[H_0^1(\cup_i \Gamma_i)\right]^N \to W_f, \\ \widehat{P}_f &: \left[H_0^1(\Omega_\kappa^n)\right]^N \cap \left[H_0^1(\cup_\iota \Gamma_\iota)\right]^N \to W_f \end{aligned}$$

be defined analogously to (29), that is, orthogonal with respect to the gradient inner product. Letting $\boldsymbol{v}_f = P_f(\eta \boldsymbol{e})$ in (46) and $\widehat{\boldsymbol{v}}_f = \widehat{P}_f(\eta \boldsymbol{e})$ in (47), we have

$$\begin{cases} (\nabla(\boldsymbol{u}_f^0 + \widetilde{\boldsymbol{u}}), \nabla\left[P_f(\eta \boldsymbol{e})\right])_{L^2(\Omega)} = 0 \\ (\nabla(\widehat{\boldsymbol{u}}_f^0 + \widetilde{\boldsymbol{u}}), \nabla[\widehat{P}_f(\eta \boldsymbol{e})])_{L^2(\Omega_\kappa^n)} = 0 \end{cases} \iff \begin{cases} (\nabla(\boldsymbol{u}_f^0 + \widetilde{\boldsymbol{u}}), \nabla(\eta \boldsymbol{e}))_{L^2(\Omega_\kappa^n)} = 0 \\ (\nabla(\widehat{\boldsymbol{u}}_f^0 + \widetilde{\boldsymbol{u}}), \nabla(\eta \boldsymbol{e}))_{L^2(\Omega_\kappa^n)} = 0, \end{cases} \tag{49}$$

since $\operatorname{supp}(\eta \boldsymbol{e}) \in \Omega_\kappa^n$ and $(\nabla \boldsymbol{w}_f, \nabla\left[P_f \boldsymbol{v}\right]) = (\nabla \boldsymbol{w}_f, \nabla \boldsymbol{v})$ for all $\boldsymbol{w}_f \in W_f$.

We are now set to estimate the error.

Proposition 3.5 (Localisation property)**.** *Let $\boldsymbol{e} = \boldsymbol{u}_f - \widehat{\boldsymbol{u}}_f$. There exist $n \in \mathbb{Z}_+$ and $R > 0$ such that*

$$\|\nabla \boldsymbol{e}\|_{L^2(\Omega_\kappa^n)} \leq 2 C_\gamma \|\boldsymbol{u}_s\|_{H^1(\Omega)},$$

where $\Omega_\kappa^n = \bigcup_{i \in \mathcal{I}} \{\Omega_i : \operatorname{dist}(\Omega_i, \Omega_\kappa^{n-1}) < R\}$ is the extended patch defined by (39) and C_γ the constant given by Lemma 3.4.

Proof. As above, suppose we have at hand the solutions $\boldsymbol{u}_f$ and $\widehat{\boldsymbol{u}}_f$ on Ω_κ^n. We would like to estimate the error on the previous patch Ω_κ^{n-1}. To begin with, consider the difference of the equations (49):

$$\begin{aligned}(\nabla \boldsymbol{e}, \nabla(\eta \boldsymbol{e}))_{L^2(\Omega_\kappa^n)} &= \int_{\Omega_\kappa^n} \nabla \boldsymbol{e} : \eta \nabla \boldsymbol{e} + \int_{\Omega_\kappa^n} \nabla \boldsymbol{e} : \left[\boldsymbol{e} (\nabla \eta)^\top\right] \\ &= \int_{\Omega_\kappa^n} \eta \nabla \boldsymbol{e} : \nabla \boldsymbol{e} + \int_{\Omega_\kappa^n} \boldsymbol{e}^\top \nabla \boldsymbol{e} \nabla \eta = 0.\end{aligned}$$

The first term gives an upper bound for the gradient of the error on the smaller patch:

$$\begin{aligned}\|\nabla \boldsymbol{e}\|^2_{L^2(\Omega_\kappa^{n-1})} &\leq \int_{\Omega_\kappa^n} \eta \nabla \boldsymbol{e} : \nabla \boldsymbol{e} = -\int_{\Omega_\kappa^n} \boldsymbol{e}^\top \nabla \boldsymbol{e} \nabla \eta \leq \int_{\Omega_\kappa^n} \left|\boldsymbol{e}^\top \nabla \boldsymbol{e} \nabla \eta\right| \\ &\leq \int_{\Omega_\kappa^n} \|\boldsymbol{e}\| \, \|\nabla \boldsymbol{e} \nabla \eta\| \leq \int_{\Omega_\kappa^n} \|\boldsymbol{e}\| \, \|\nabla \boldsymbol{e}\|_{\mathrm{F}} \, \|\nabla \eta\| \\ &\leq \sup_{\boldsymbol{x} \in \Omega_\kappa^n} \|\nabla \eta\| \cdot \left(\int_{\Omega_\kappa^n} \|\boldsymbol{e}\|^2\right)^{1/2} \left(\int_{\Omega_\kappa^n} \|\nabla \boldsymbol{e}\|^2_{\mathrm{F}}\right)^{1/2} \\ &\leq \frac{3}{R} \|\boldsymbol{e}\|_{L^2(\Omega_\kappa^n)} \|\nabla \boldsymbol{e}\|_{L^2(\Omega_\kappa^n)} \leq \frac{3}{R} \cdot \frac{h_\kappa}{\pi} \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega_\kappa^n)} \\ &\leq \frac{3h}{\pi R} \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega_\kappa^n)}.\end{aligned}$$

Here $\|\boldsymbol{A}\|^2_{\mathrm{F}} = \boldsymbol{A} : \boldsymbol{A}$ denotes the Frobenius norm of the matrix. Repeating the process n times, we obtain

$$\|\nabla \boldsymbol{e}\|^2_{L^2(\Omega_\kappa)} \leq \left(\frac{3h}{\pi R}\right)^n \|\nabla \boldsymbol{e}\|^2_{L^2(\Omega_\kappa^n)}. \tag{50}$$

Choosing $R = \frac{6h}{\pi}$, we have $\frac{3h}{\pi R} = \frac{1}{2}$. Then, for $n > -\frac{2 \log \varepsilon}{\log 2}$, it holds that

$$\begin{aligned}\|\nabla \boldsymbol{e}\|_{L^2(\Omega_\kappa)} &\leq \varepsilon \|\nabla \boldsymbol{e}\|_{L^2(\Omega_\kappa^n)} \leq \varepsilon \left\|\nabla\left(\boldsymbol{u}_f^0 - \widehat{\boldsymbol{u}}_f^0\right)\right\|_{L^2(\Omega_\kappa^n)} \\ &\leq \varepsilon \cdot 2 \|\nabla \widetilde{\boldsymbol{u}}\|_{L^2(\Omega_\kappa^n)} \qquad (51) \\ &= \varepsilon \cdot 2 \|\nabla \mathcal{E} \boldsymbol{u}_s\|_{L^2(\Omega_\kappa^n)} \leq \varepsilon \cdot 2C \|\mathcal{E} \boldsymbol{u}_s\|_{H^1(\Omega)} \\ &\leq \varepsilon \cdot 2C_\gamma \varepsilon^{-1} \|\boldsymbol{u}_s\|_{H^1(\Omega)} \qquad (52) \\ &= 2C_\gamma \|\boldsymbol{u}_s\|_{H^1(\Omega)}.\end{aligned}$$

In the above, (51) follows from the stability of the problems (46) and (47). In (52), the estimate from Lemma 3.4 has been applied. □

3.5 The homogenised problem

We proceed to give an approximate solution to the fast problem in each partition set Ω_κ, and insert these back into the original problem to arrive at the homogenised form.

Solving the fast (cell) problem. Let Ω_κ be fixed. By virtue of the localisation property, we can approximate the fast solution on Ω_κ by solving (44) on an extended patch Ω_κ^n, the extent of which is determined by Proposition 3.5.

For all $\Omega_\kappa \in \{\Omega_i\}_{i\in\mathcal{I}}$, we seek $(\boldsymbol{u}_f, \boldsymbol{\lambda}) \in W_f \times V_0(\Omega_\kappa^n)$ such that

$$\begin{cases} (\nabla \boldsymbol{u}_f, \nabla \boldsymbol{v}_f)_{L^2(\Omega_\kappa^n)} + (\boldsymbol{\lambda}, \boldsymbol{v}_f)_{H^1(\cup_\iota \Gamma_\iota)} = 0 & \text{for all } \boldsymbol{v}_f \in W_f, \\ (\boldsymbol{u}_f, \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)} = (-\boldsymbol{u}_s, \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)} & \text{for all } \boldsymbol{\mu} \in V_0(\cup_\iota \Gamma_\iota), \end{cases} \tag{53}$$

in which $\cup_\iota \Gamma_\iota$ denotes the holes contained in Ω_κ^n. We approximate the load vector by interpolating the slow solution by a constant (recall that $[\boldsymbol{v}]_i$ denotes the i:th component of the vector):

$$(\boldsymbol{u}_s, \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)} \approx \sum_{i=1}^{N} \pi_0 \left([\boldsymbol{u}_s]_i\right) (\boldsymbol{e}_i, \boldsymbol{\mu})_{L^2(\cup_\iota \Gamma_\iota)} + \sum_{i,j=1,2}^{N} \pi_0 \left(\partial_j [\boldsymbol{u}_s]_i\right) (\boldsymbol{e}_i, \nabla \boldsymbol{\mu} \boldsymbol{e}_j)_{L^2(\cup_\iota \Gamma_\iota)}.$$

Then, the approximate fast solution is found by solving the second equation of the system (53) for the load terms

$$(\boldsymbol{e}_i, \boldsymbol{\mu})_{L^2(\cup_\iota \Gamma_\iota)} \text{ and } (\boldsymbol{e}_i, \nabla \boldsymbol{\mu} \boldsymbol{e}_j)_{L^2(\cup_\iota \Gamma_\iota)} \text{ with } i, j = 1, \dots, N,$$

that is, $N(N+1)$ problems in total. Denoting the obtained solutions by $\mathcal{X}_i^{(0)}$ and $\mathcal{X}_{ij}^{(1)} \in W_f$ respectively, the total approximate fast solution on Ω_κ is

$$\boldsymbol{u}_f \approx -\left(\sum_{i=1}^{N} \pi_0 \left([\boldsymbol{u}_s]_i\right) \mathcal{X}_i^{(0)} + \sum_{i,j=1,2}^{N} \pi_0 \left(\partial_j [\boldsymbol{u}_s]_i\right) \mathcal{X}_{ij}^{(1)} \right)$$

or, in matrix form,

$$\boldsymbol{u}_f \approx -\boldsymbol{B} \pi_0 \left(\mathrm{D}\boldsymbol{u}_s\right). \tag{54}$$

Here $\pi_0(\cdot)$ denotes componentwise interpolation by a constant over Ω_κ^n, while $\mathrm{D}\boldsymbol{u}_s$ and $\boldsymbol{B}$ are defined

$$\mathrm{D}\boldsymbol{u}_s = \begin{bmatrix} [\boldsymbol{u}_s]_1 & \dots & [\boldsymbol{u}_s]_N & \frac{\partial [\boldsymbol{u}_s]_1}{\partial x_1} & \frac{\partial [\boldsymbol{u}_s]_1}{\partial x_2} & \dots & \dots & \frac{\partial [\boldsymbol{u}_s]_N}{\partial x_{N-1}} & \frac{\partial [\boldsymbol{u}_s]_N}{\partial x_N} \end{bmatrix}^\top_{N(N+1)},$$

$$\boldsymbol{B} = \begin{bmatrix} \mathcal{X}_1^{(0)} & \dots & \mathcal{X}_N^{(0)} & \mathcal{X}_{11}^{(1)} & \mathcal{X}_{12}^{(1)} & \dots & \dots & \mathcal{X}_{NN-1}^{(1)} & \mathcal{X}_{NN}^{(1)} \end{bmatrix}_{N \times N(N+1)}.$$

The functions $\mathcal{X}_i^{(0)}$ and $\mathcal{X}_{ij}^{(1)}$ represent the approximate, componentwise contribution of the small-scale inhomogeneities to the total solution, and can hence be spoken of as *correctors* to the slow field. However, we performed constant interpolation on the the load side, so this neat expression comes at a price:

Lemma 3.6. *Let $\boldsymbol{u}_f$ be the solution of the localised problem (53), and $\widehat{\boldsymbol{u}}_f$ the solution to the same problem with averaging of the load term, that is, the second equation replaced by $(\boldsymbol{u}_f, \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)} = (-\pi_0 \boldsymbol{u}_s, \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)}$ for all $\boldsymbol{\mu} \in V_0(\cup_\iota \Gamma_\iota)$. The error $\boldsymbol{e} = \boldsymbol{u}_f - \widehat{\boldsymbol{u}}_f$ satisfies*

$$\|\nabla \boldsymbol{e}\|_{L^2(\Omega_\kappa^n)} \leq C_\gamma \varepsilon^\alpha,$$

where ε and α are the two-scale parameters (32), and C_γ the constant from the extension lemma 3.4.

Proof. The problem we solve for each Ω_κ is (53), with the actual load $\boldsymbol{u}_s$ replaced by $\pi_0\boldsymbol{u}_s$. The problem satisfied by the resulting error $\boldsymbol{e} \in W_f$ is

$$\begin{cases} (\nabla \boldsymbol{e}, \nabla \boldsymbol{v}_f)_{L^2(\Omega_\kappa^n)} + (\boldsymbol{\lambda}_{\boldsymbol{e}}, \boldsymbol{v}_f)_{H^1(\cup_\iota \Gamma_\iota)} = 0 & \text{for all } \boldsymbol{v}_f \in W_f, \\ (\boldsymbol{e}, \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)} = (-(\boldsymbol{u}_s - \pi_0 \boldsymbol{u}_s), \boldsymbol{\mu})_{H^1(\cup_\iota \Gamma_\iota)} & \text{for all } \boldsymbol{\mu} \in V_0(\cup_\iota \Gamma_\iota). \end{cases} \tag{55}$$

The proof gives us occasion to brandish all the tools we have devised for the analysis of the homogenised problem so far. Yet again, we begin by restating the problem using the extension operator from Lemma 3.3. Let $\boldsymbol{q} \in V_0(\cup_\iota \Gamma_\iota)$ be the Riesz representant of $l(\boldsymbol{v}) = (-(\boldsymbol{u}_s - \pi_0 \boldsymbol{u}_s), \boldsymbol{v})_{H^1(\cup_\iota \Gamma_\iota)}$. Now, the second equation of (55) states that $\boldsymbol{e}|_{\cup_\iota \Gamma_\iota} = \boldsymbol{q}$. Decompose the error into $\boldsymbol{e} = \mathcal{E}\boldsymbol{q} + \boldsymbol{e}^0$, where $\boldsymbol{e}^0 \in H_0^1(\cup_\iota \Gamma_\iota)$. The problem (55) may now be rewritten analogously to (46) – (47):

$$(\nabla \boldsymbol{e}^0, \nabla \boldsymbol{v}_f)_{L^2(\Omega_\kappa^n)} = (-\nabla[\mathcal{E}\boldsymbol{q}], \nabla \boldsymbol{v}_f)_{L^2(\Omega_\kappa^n)} \quad \text{for all} \quad \boldsymbol{v}_f \in W_f \cap H_0^1(\cup_\iota \Gamma_\iota).$$

The stability of this problem together with the extension lemma imply that

$$\left\|\nabla \boldsymbol{e}^0\right\|_{L^2(\Omega_\kappa^n)} \leq \|\nabla(\mathcal{E}\boldsymbol{q})\|_{L^2(\Omega_\kappa^n)} \leq C_\gamma \max_{\iota \in \mathcal{J}} \left(\operatorname{diam} \widetilde{\Gamma}_\iota\right)^{-1} \cdot \|\boldsymbol{q}\|_{H^1(\cup_\iota \Gamma_\iota)}. \tag{56}$$

To proceed, we need an estimate for $\|\boldsymbol{q}\|_{H^1(\cup_\iota \Gamma_\iota)}$. Beginning with Riesz' theorem A.7,

$$\begin{aligned} \|\boldsymbol{q}\|^2_{H^1(\cup_\iota \Gamma_\iota)} &= l(\boldsymbol{q}) \\ &= (-(\boldsymbol{u}_s - \pi_0 \boldsymbol{u}_s), \boldsymbol{q})_{L^2(\cup_\iota \Gamma_\iota)} + (-(\nabla \boldsymbol{u}_s - \pi_0 \nabla \boldsymbol{u}_s), \nabla \boldsymbol{q})_{L^2(\cup_\iota \Gamma_\iota)} \\ &\leq \|\boldsymbol{u}_s - \pi_0 \boldsymbol{u}_s\|_{L^2(\cup_\iota \Gamma_\iota)} \|\boldsymbol{q}\|_{L^2(\cup_\iota \Gamma_\iota)} + \|\nabla \boldsymbol{u}_s - \pi_0 \nabla \boldsymbol{u}_s\|_{L^2(\cup_\iota \Gamma_\iota)} \|\nabla \boldsymbol{q}\|_{L^2(\cup_\iota \Gamma_\iota)} \\ &\leq C\varepsilon^{1+\alpha} \|\boldsymbol{q}\|_{H^1(\cup_\iota \Gamma_\iota)}. \end{aligned}$$

Here, we have applied the two-scale assumption (32). With this information, (56) becomes

$$\begin{aligned} \left\|\nabla \boldsymbol{e}^0\right\|_{L^2(\Omega_\kappa^n)} &\leq C_\gamma \max_{\iota \in \mathcal{J}} \left(\operatorname{diam} \widetilde{\Gamma}_\iota\right)^{-1} \cdot \varepsilon^{1+\alpha} \\ &\leq C_\gamma \varepsilon^\alpha, \end{aligned}$$

whenever $\alpha > 0$ and $\operatorname{diam} \widetilde{\Gamma}_\iota = \mathcal{O}(\varepsilon)$. The estimate for $\|\nabla \boldsymbol{e}\|_{L^2(\Omega_\kappa^n)}$ follows by a simple application of the triangle inequality. □

Remark 3.6. The above prooflet is sneaking another assumption on the hole distribution in through the back door. Namely, the diameters of the hole sets $\widetilde{\Gamma}_\iota$ may not be smaller than $\mathcal{O}(\varepsilon)$: there are no "tiny holes in tight spaces".

The two-scale assumption revisited. The error estimate of Proposition 3.6 presupposes that the base set Ω^* where the two-scale assumption (32) is made are the holes $\cup_{\iota \in \mathcal{J}} \Gamma_\iota$ on the minimal patch Ω_i^n determined by the localisation estimate (Proposition 3.5), while the constant interpolation is taken to happen over the said patch. This statement has the spherical charm of a circular argument and calls for a

better understanding of the two-scale property, which we unfortunately do not have at the moment. With a tip of the hat to classical homogenisation, we assume that *the holes are small compared to the patch*: $|\Gamma_\iota| = \mathcal{O}(\varepsilon^{2+2\alpha})$. Accepting this for the moment, we may estimate at least heuristically

$$\begin{aligned}\|\boldsymbol{u}_s - \pi_0 \boldsymbol{u}_s\|^2_{L^2(\cup_\iota \Gamma_\iota)} \leq \|\boldsymbol{u}_s\|^2_{L^2(\cup_\iota \Gamma_\iota)} &= \sum_{\iota \in \mathcal{J}} \int_{\Gamma_\iota} |\boldsymbol{u}_s|^2 \\ &\leq \sum_{\iota \in \mathcal{J}} \|\boldsymbol{u}_s\|_{L^\infty(\Gamma_\iota)} |\Gamma_\iota| \leq C\varepsilon^{2+2\alpha}.\end{aligned}$$

Combined with remark 3.6, we have actually conceded that for every hole Γ_ι, its neighborhood $\widetilde{\Gamma}_\iota$ is asymptotically larger than the hole itself! "Asymptotics" may be a bad word considering that the notions of limits or convergence have hardly surfaced in our discussion of the energy decomposition so far, but should serve to pinpoint the problem.

What is more, letting $\boldsymbol{u}_s \approx \pi_0 \boldsymbol{u}_s$ is likely not the correct approach when the holes are allowed to intersect with the boundaries of the partition sets. The problem could be observed numerically. Of course, nothing prevents us from choosing the partition sets so that intersections do not happen, but this may be difficult in practice. Placing constraints on the distribution of the holes in Remark 3.3, we are trading generality for ease.

Derivation of the homogenised problem. Suppose that we have at hand an approximate fast solution (54) on each Ω_κ. We arrive at the homogenised problem by plugging these back into the energy functional:

Proposition 3.7. *The approximate homogenised form of problem* (31) *is*

$$\min_{\boldsymbol{u} \in V_0(\Omega_{\mathrm{H}})} J(\boldsymbol{u}) = \min_{\boldsymbol{u}_s \in W_s} \left[\frac{1}{2} \int_\Omega \nabla \boldsymbol{u}_s : \nabla \boldsymbol{u}_s + \frac{1}{2} \sum_{\kappa \in \mathcal{I}} \int_{\Omega_\kappa} \mathrm{D}\boldsymbol{u}_s : \pi_0 \boldsymbol{C}_\kappa : \mathrm{D}\boldsymbol{u}_s - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_s \right], \tag{57}$$

where the $(\boldsymbol{C}_\kappa)_{N \times N(N+1) \times N}$ *are three-dimensional hypermatrices with elements given by*

$$[\boldsymbol{C}_\kappa]_{ijk} = \left(\frac{\partial [\boldsymbol{B}_\kappa]_{ij}}{\partial x_k} \right)^2.$$

Remark 3.7. The product $\boldsymbol{C} : \boldsymbol{u}$ here denotes a vector with the elements

$$[\boldsymbol{C} : \boldsymbol{u}]_j = \sum_{i,k} [\boldsymbol{u}]_j [\boldsymbol{C}]_{ijk}.$$

This *ad hoc* choice of notation permits us to write $\boldsymbol{u} : \boldsymbol{C} : \boldsymbol{u} = \sum_{i,j,k} [\boldsymbol{u}]_j^2 [\boldsymbol{C}]_{ijk}$. In particular, the terms $\int_{\Omega_\kappa} \mathrm{D}\boldsymbol{u}_s : \pi_0 \boldsymbol{C}_\kappa : \mathrm{D}\boldsymbol{u}_s$ in (57) are all nonnegative.

Proof of Proposition 3.7. Consider the fast minimisation problem (36) and recall that the load term was dropped as negligible. Inserting the solutions $(\boldsymbol{u}_f)_\kappa \approx -\boldsymbol{B}_\kappa \pi_0 (\mathrm{D}\boldsymbol{u}_s)$

into the functional to be minimised, we obtain

$$\begin{aligned} J(\boldsymbol{u}_f) &\approx \frac{1}{2}(\nabla \boldsymbol{u}_f, \nabla \boldsymbol{u}_f)_{L^2(\Omega)} \\ &\approx \frac{1}{2}\sum_{\kappa\in\mathcal{I}}(-\nabla \boldsymbol{B}_\kappa \pi_0(\mathrm{D}\boldsymbol{u}_s), -\nabla \boldsymbol{B}_\kappa \pi_0(\mathrm{D}\boldsymbol{u}_s))_{L^2(\Omega_\kappa)} \\ &= \frac{1}{2}\sum_{\kappa\in\mathcal{I}}\int_{\Omega_\kappa} \pi_0(\mathrm{D}\boldsymbol{u}_s) : \boldsymbol{C}_\kappa : \pi_0(\mathrm{D}\boldsymbol{u}_s) \\ &= \frac{1}{2}\sum_{\kappa\in\mathcal{I}}\int_{\Omega_\kappa} \pi_0(\mathrm{D}\boldsymbol{u}_s) : \pi_0\boldsymbol{C}_\kappa : \pi_0(\mathrm{D}\boldsymbol{u}_s) \\ &\approx \frac{1}{2}\sum_{\kappa\in\mathcal{I}}\int_{\Omega_\kappa} \mathrm{D}\boldsymbol{u}_s : \pi_0\boldsymbol{C}_\kappa : \mathrm{D}\boldsymbol{u}_s. \end{aligned} \tag{58}$$

Inserting the fast functional (58) back into the overall minimisation problem (31), we obtain (57). □

Remark 3.8. The final step 58 asks more regularity of the slow solution than we know of at the moment. The penultimate integral may be written

$$\begin{aligned} \int_{\Omega_\kappa} \pi_0(\mathrm{D}\boldsymbol{u}_s) : \pi_0\boldsymbol{C}_\kappa : \pi_0(\mathrm{D}\boldsymbol{u}_s) = \int_{\Omega_\kappa} &(\pi_0(\mathrm{D}\boldsymbol{u}_s) : \pi_0\boldsymbol{C}_\kappa : (\pi_0(\mathrm{D}\boldsymbol{u}_s) - \mathrm{D}\boldsymbol{u}_s)) \\ &+ (\pi_0(\mathrm{D}\boldsymbol{u}_s) - \mathrm{D}\boldsymbol{u}_s) : \pi_0\boldsymbol{C}_\kappa : \mathrm{D}\boldsymbol{u}_s + \mathrm{D}\boldsymbol{u}_s : \pi_0\boldsymbol{C}_\kappa : \mathrm{D}\boldsymbol{u}_s)\,. \end{aligned}$$

If we were able to show that $\boldsymbol{u}_s$ belongs to H^2, the extra terms could be controlled by

$$\|\mathrm{D}\boldsymbol{u}_s - \pi_0(\mathrm{D}\boldsymbol{u}_s)\|_{L^2(\Omega_\kappa)} \le C\,\mathrm{diam}(\Omega_\kappa)\,\|\boldsymbol{u}_s\|_{H^2(\Omega_\kappa)}\,,$$

where $\mathrm{diam}(\Omega_\kappa) = \mathcal{O}(\varepsilon)$. Higher regularity of the slow solution seems plausible, but is not guaranteed by the standing two-scale assumption. Again, the issue calls for a better understanding of the two-scale structure.

The homogenised problem has the appearance of a Brinkman-type equation, but a more detailed analysis of the underlying assumptions would be needed to compare it to the classical Theorem 2.1.

In the general (possibly aperiodic) setting, we indeed need to solve a fast problem (53) for each partition set in $\{\Omega_i\}_{i\in\mathcal{I}}$. To solve a flow in a real-life two-scale material, we could take samples of the material and form a probability distribution describing its properties, notably the size and distribution of the pores with respect to the matrix. Drawing parameters from this distribution, the $\{\pi_0\boldsymbol{C}_\kappa\}_{\kappa\in\mathcal{I}}$ could then be computed in parallel.

In the periodic case, the obvious choice is to partition the domain Ω into identical cells, comparable to the classical approach. In this case, the fast solutions (54) (and thus the hypermatrices $\pi_0\boldsymbol{C}_\kappa$) will be identical on each Ω_κ. It is enough to solve the fast problem once, and the energy functional in (57) takes a simplified form without the sum:

$$J(\boldsymbol{u}) = \frac{1}{2}\int_\Omega (\nabla \boldsymbol{u}_s : \nabla \boldsymbol{u}_s + \mathrm{D}\boldsymbol{u}_s : \pi_0\boldsymbol{C} : \mathrm{D}\boldsymbol{u}_s) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}_s.$$

Getting rid of W_s. Let there be a symmetric, positive, bilinear form defined by the homogeneous problem:

$$\widehat{a}(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v} + \sum_{\kappa\in\mathcal{I}} \int_{\Omega_\kappa} \mathrm{D}\boldsymbol{u} : \pi_0\boldsymbol{C}_\kappa : \mathrm{D}\boldsymbol{v}. \tag{59}$$

The variational formulation of the homogenised problem (57) is: Find $\boldsymbol{u}_s \in W_s$ such that

$$\widehat{a}(\boldsymbol{u}_s,\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_{L^2(\Omega)} \text{ for all } \boldsymbol{v}\in W_s. \tag{60}$$

The solution lives exclusively in the slow space W_s. The explicit construction of this space is computationally expensive, which is why we would prefer to solve the problem in the entire $V_0(\Omega)$ instead: Find $\boldsymbol{u}_0 \in V_0$ such that

$$\widehat{a}(\boldsymbol{u}_0,\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v})_{L^2(\Omega)} \text{ for all } \boldsymbol{v}\in V_0(\Omega). \tag{61}$$

The error resulting from this simplification is readily estimated in the *energy norm*. Let $(\boldsymbol{u},\boldsymbol{v})_{\mathrm{E}}$ be the inner product defined by the bilinear form (59) associated with the homogenised problem (57) and $\|\cdot\|_{\mathrm{E}}$ the induced norm:

$$(\boldsymbol{u},\boldsymbol{v})_{\mathrm{E}} = \widehat{a}(\boldsymbol{u},\,\boldsymbol{v}), \qquad \|\boldsymbol{u}\|_{\mathrm{E}}^2 = (\boldsymbol{u},\boldsymbol{u})_{\mathrm{E}}.$$

Let $\boldsymbol{u}_s \in W_s$ be the solution of (60). We have Galerkin orthogonality:

$$\widehat{a}(\boldsymbol{u}_0-\boldsymbol{u}_s,\boldsymbol{v}) = 0 \quad \text{for all } \boldsymbol{v}\in W_s. \tag{62}$$

Let $P_f^{\mathrm{E}} : V_0(\Omega) \to W_f$ be the orthogonal projection to W_f with respect to $(\cdot,\cdot)_{\mathrm{E}}$, that is,

$$(P_f^{\mathrm{E}}\boldsymbol{u}_0,\boldsymbol{v})_{\mathrm{E}} = 0 \quad \text{for all } \boldsymbol{v}\in W_s.$$

Proposition 3.8. *Let $\boldsymbol{u}_s \in W_s$ and $\boldsymbol{u}_0 \in V_0(\Omega)$ be the solutions of (60) and (61) respectively. We have*

$$\|\boldsymbol{u}_0-\boldsymbol{u}_s\|_{\mathrm{E}} \le Ch\,\|\boldsymbol{f}\|_{L^2(\Omega)},$$

where $h = \max_{i\in\mathcal{I}} h_i$.

Proof. By Galerkin orthogonality (62), it holds that $\boldsymbol{u}_0-\boldsymbol{u}_s = P_f^{\mathrm{E}}\boldsymbol{u}_0$. We estimate

$$\begin{aligned}
\|\boldsymbol{u}_0-\boldsymbol{u}_s\|_{\mathrm{E}}^2 &= \widehat{a}(\boldsymbol{u}_0-\boldsymbol{u}_s,\boldsymbol{u}_0-\boldsymbol{u}_s)\\
&= \widehat{a}(\boldsymbol{u}_0,\boldsymbol{u}_0-\boldsymbol{u}_s)\\
&= \widehat{a}(\boldsymbol{u}_0,P_f^{\mathrm{E}}\boldsymbol{u}_0) = (\boldsymbol{f},P_f^{\mathrm{E}}\boldsymbol{u}_0)_{L^2(\Omega)} \qquad (63)\\
&\le \left\|P_f^{\mathrm{E}}\boldsymbol{u}_0\right\|_{L^2(\Omega)} \|\boldsymbol{f}\|_{L^2(\Omega)} \le Ch\left\|\nabla(P_f^{\mathrm{E}}\boldsymbol{u}_0)\right\|_{L^2(\Omega)} \|\boldsymbol{f}\|_{L^2(\Omega)}\\
&\le Ch\left\|P_f^{\mathrm{E}}\boldsymbol{u}_0\right\|_{\mathrm{E}} \|\boldsymbol{f}\|_{L^2(\Omega)} \qquad (64)\\
&= Ch\,\|\boldsymbol{u}_0-\boldsymbol{u}_s\|_{\mathrm{E}}\,\|\boldsymbol{f}\|_{L^2(\Omega)}.
\end{aligned}$$

In addition, whenever $\boldsymbol{f}$ is orthogonal to W_f in $L^2(\Omega)$, (63) amounts to zero. The inequality (64) follows from Remark 3.7. □

3.6 Derivation of an error estimate

Finally, we estimate the the total error in passing to the homogeneous problem. Recall the bilinear form $\widehat{a}(\boldsymbol{u},\boldsymbol{v})$ (59). Analogously, we define the bilinear form $a(\boldsymbol{u},\boldsymbol{v})$ associated with the original problem, which leaves us with

$$\begin{aligned}\widehat{a}(\boldsymbol{u},\boldsymbol{v}) &= \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v} + \sum_{\kappa\in\mathcal{I}} \int_{\Omega_\kappa} \mathrm{D}\boldsymbol{u} : \pi_0\boldsymbol{C}_\kappa : \mathrm{D}\boldsymbol{v},\\ a(\boldsymbol{u},\boldsymbol{v}) &= \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v} + \int_\Omega \nabla(\mathcal{Q}\boldsymbol{u}) : \nabla(\mathcal{Q}\boldsymbol{v}).\end{aligned}$$

In the above, $\mathcal{Q}$ denotes the solution operator of the original fast problem (38) such that $\mathcal{Q}\boldsymbol{u}_s = \boldsymbol{u}_f$. We are solving the homogenised problem (60) involving the approximate bilinear form $\widehat{a}$, while the exact problem would concern a instead; let us denote the respective solutions by $\widehat{\boldsymbol{u}}_s$ and $\boldsymbol{u}_s$. The natural habitat of both fields is the slow space W_s. The passage to $V_0(\Omega)$ can be treated separately as in Proposition 3.8, and is not presupposed by the following discussion. It is fairly simple to obtain a patchwise estimate for the error in the energy norm.

Proposition 3.9. *The error $\boldsymbol{e} = \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s$ per patch in the energy norm is*

$$\|\boldsymbol{e}\|_{E(\Omega_\kappa^n)} \leq C_\gamma \varepsilon^{\alpha/2} \|\boldsymbol{f}\|_{L^2(\Omega_\kappa^n)},$$

where C_γ is the constant from the extension lemma 3.4.

Proof. An exercise in adding, subtracting, and the triangle inequality follows.

$$\begin{aligned}&\widehat{a}(\boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s, \boldsymbol{v})|_{\Omega_\kappa^n}\\
&= \int_{\Omega_\kappa^n} \mathrm{D}\boldsymbol{u}_s : \pi_0\boldsymbol{C}_\kappa : \mathrm{D}\boldsymbol{v} - \int_{\Omega_\kappa^n} \nabla(\mathcal{Q}\boldsymbol{u}_s) : \nabla(\mathcal{Q}\boldsymbol{v})\\
&= \int_{\Omega_\kappa^n} \boldsymbol{B}\pi_0(\mathrm{D}\boldsymbol{u}_s) : \boldsymbol{B}\pi_0(\mathrm{D}\boldsymbol{v}) - \int_{\Omega_\kappa^n} \nabla(\mathcal{Q}\boldsymbol{u}_s) : \nabla(\mathcal{Q}\boldsymbol{v})\\
&= \int_{\Omega_\kappa^n} \boldsymbol{B}\pi_0(\mathrm{D}\boldsymbol{u}_s) : (\boldsymbol{B}\pi_0(\mathrm{D}\boldsymbol{v}) - \nabla(\mathcal{Q}\boldsymbol{v})) - \int_{\Omega_\kappa^n} (\nabla(\mathcal{Q}\boldsymbol{u}_s) - \boldsymbol{B}\pi_0\mathrm{D}\boldsymbol{u}_s) : \nabla(\mathcal{Q}\boldsymbol{v})\\
&\leq \|\boldsymbol{B}\pi_0(\mathrm{D}\boldsymbol{u}_s)\|_{L^2(\Omega_\kappa^n)} \|\boldsymbol{B}\pi_0(\mathrm{D}\boldsymbol{v}) - \nabla(\mathcal{Q}\boldsymbol{v})\|_{L^2(\Omega_\kappa^n)}\\
&\qquad + \|\nabla(\mathcal{Q}\boldsymbol{u}_s) - \boldsymbol{B}\pi_0\mathrm{D}\boldsymbol{u}_s\|_{L^2(\Omega_\kappa^n)} \|\nabla(\mathcal{Q}\boldsymbol{v})\|_{L^2(\Omega_\kappa^n)}.\end{aligned}$$

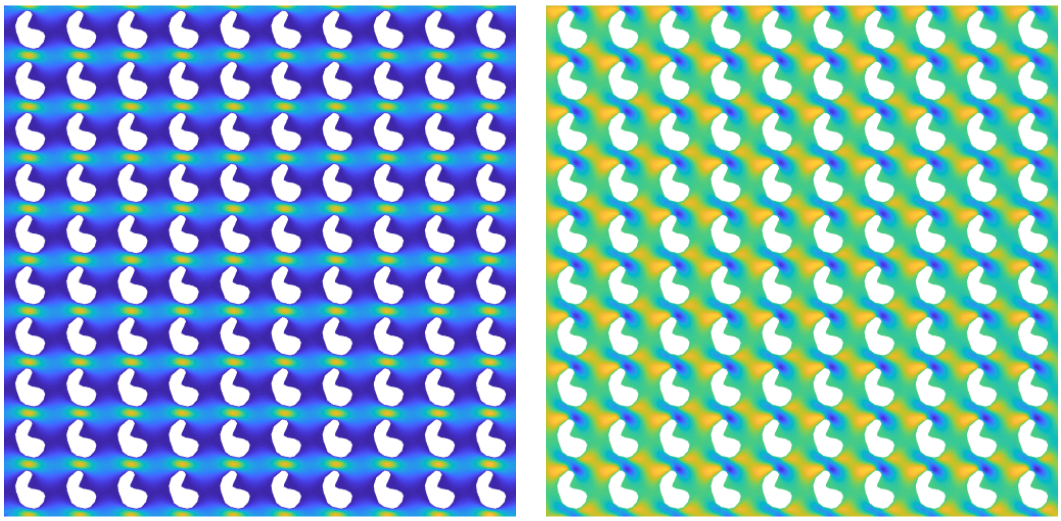
Testing with $\boldsymbol{e} = \boldsymbol{u}_s - \widehat{\boldsymbol{u}}_s$, we have

$$\begin{aligned}&\|\nabla(\mathcal{Q}\boldsymbol{e})\|_{L^2(\Omega_\kappa^n)}\\
&\qquad \leq \|\nabla(\mathcal{Q}\boldsymbol{u}_s)\|_{L^2(\Omega_\kappa^n)} + \|\nabla(\mathcal{Q}\widehat{\boldsymbol{u}}_s)\|_{L^2(\Omega_\kappa^n)}\\
&\qquad \leq \|\nabla(\mathcal{Q}\boldsymbol{u}_s)\|_{L^2(\Omega_\kappa^n)} + \|\nabla(\mathcal{Q}\widehat{\boldsymbol{u}}_s) - \boldsymbol{B}\pi_0\mathrm{D}\widehat{\boldsymbol{u}}_s\|_{L^2(\Omega_\kappa^n)} + \|\boldsymbol{B}\pi_0\mathrm{D}\widehat{\boldsymbol{u}}_s\|_{L^2(\Omega_\kappa^n)}\\
&\qquad \leq \|\boldsymbol{f}\|_{L^2(\Omega_\kappa^n)} + C_\gamma\varepsilon^\alpha + \|\boldsymbol{f}\|_{L^2(\Omega_\kappa^n)},\end{aligned}$$

where we have applied the error estimate of Lemma 3.6, as well as the stability of the original problems. The tacit assumption here is that the homogenised solution $\widehat{\boldsymbol{u}}_s$ likewise has a two-scale structure, i.e. (32) holds.

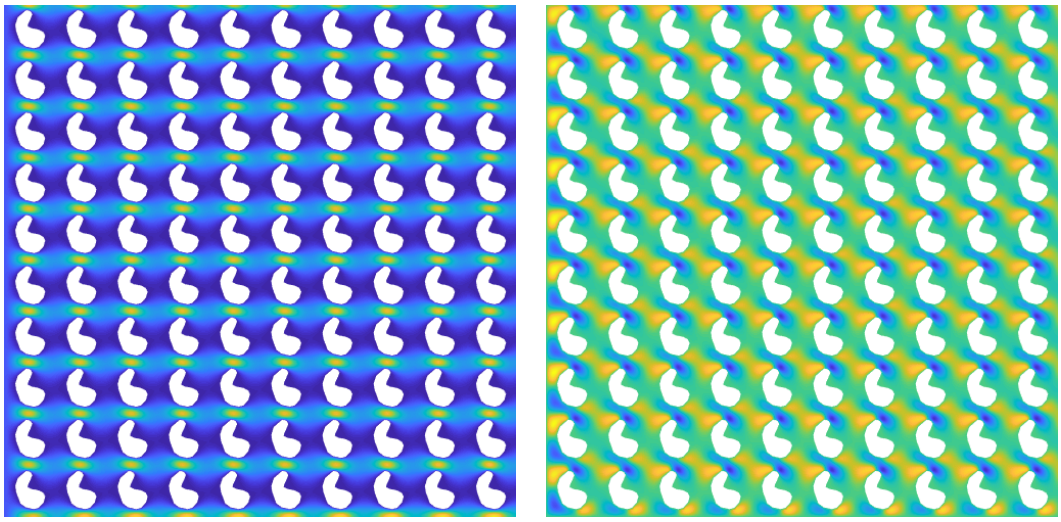
$$
\begin{aligned}
\|\boldsymbol{e}\|^2_{E(\Omega^n_\kappa)} &= \widehat{a}(\boldsymbol{e},\boldsymbol{e})|_{\Omega^n_\kappa} \\
&\leq C_\gamma \varepsilon^\alpha \left(\|\boldsymbol{B}\pi_0(\mathrm{D}\boldsymbol{u}_s)\|_{L^2(\Omega^n_\kappa)} + \|\nabla(\mathcal{Q}\boldsymbol{e})\|_{L^2(\Omega^n_\kappa)} \right) \\
&\leq C_\gamma \varepsilon^\alpha \left(\|\boldsymbol{B}\pi_0(\mathrm{D}\boldsymbol{u}_s) - \nabla(\mathcal{Q}\boldsymbol{u}_s)\|_{L^2(\Omega^n_\kappa)} + \|\nabla(\mathcal{Q}\boldsymbol{u}_s)\|_{L^2(\Omega^n_\kappa)} + \|\nabla(\mathcal{Q}\boldsymbol{e})\|_{L^2(\Omega^n_\kappa)} \right) \\
&\leq C_\gamma \varepsilon^\alpha \left(C_\gamma \varepsilon^\alpha + \|\boldsymbol{f}\|^2_{L^2(\Omega^n_\kappa)} + C_\gamma \varepsilon^\alpha + 2\,\|\boldsymbol{f}\|^2_{L^2(\Omega^n_\kappa)} \right) \\
&\leq C_\gamma \varepsilon^\alpha \, \|\boldsymbol{f}\|^2_{L^2(\Omega^n_\kappa)} \,,
\end{aligned}
$$

which is the desired result. □



(a) $\widehat{\boldsymbol{u}}_1$ (b) $\widehat{\boldsymbol{u}}_2$

Figure 7: Components of the homogenised velocity field $\widehat{\boldsymbol{u}}$.



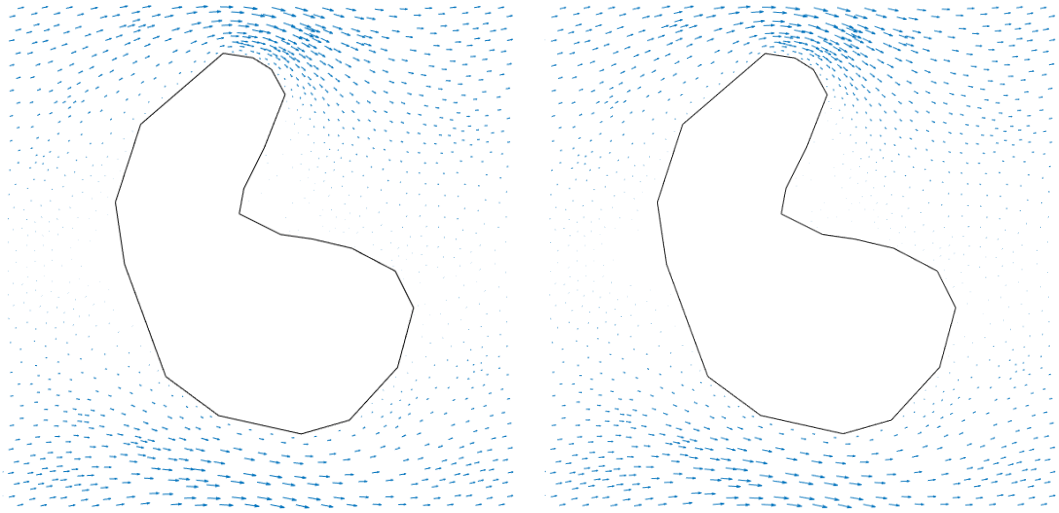
(a) $\boldsymbol{u}_1$ (b) $\boldsymbol{u}_2$

Figure 8: Components of the reference velocity field $\boldsymbol{u}$.

4 A numerical experiment

To illustrate the method and to convince ourselves of the error results, we solve an example Stokes problem in a "porous medium". We are interested in seeing the homogenised formalism in action, less so in the technicalities of the implementation. Preexisting FEM code by Antti Hannukainen was used for the task; all computations were done using the Taylor-Hood $\mathcal{P}_2$ - $\mathcal{P}_1$ element with continuous pressure.

We have previously shown that the H^1 inner product is essential to the stability of the fast problem; see Proposition 3.2. However, the FEM solution lives in a finite-dimensional space, where the problem ceases to be. Profiting from this fact, we



(a) Homogenised $\widehat{\boldsymbol{u}}$ (b) Reference $\boldsymbol{u}$

Figure 9: The velocity field in a single cell.

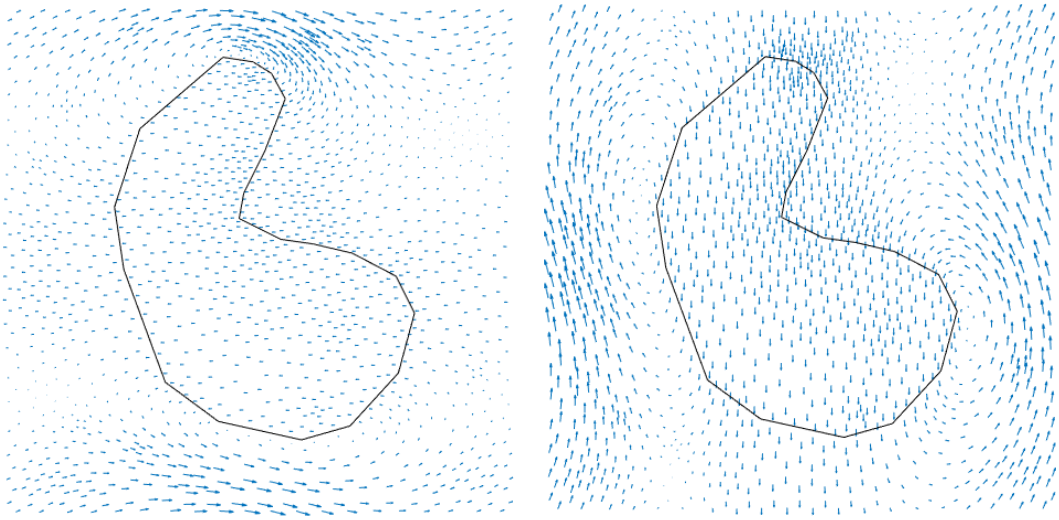
may replace the H^1 inner product with its L^2 counterpart to define the fast problem in a FEM context. This means we only use the zeroth-degree correctors $\mathcal{X}_i^{(0)}$ to derive the homogenised problem. As a consequence, the vector $\mathrm{D}\boldsymbol{u}_s$ and the matrix $\boldsymbol{B}$ are truncated compared to their complete counterparts derived in Chapter 3.5:

$$\begin{aligned} \mathrm{D}\boldsymbol{u}_s &= \boldsymbol{u}_s, \\ \boldsymbol{B} &= \begin{bmatrix} \mathcal{X}_1^{(0)} & \dots & \mathcal{X}_N^{(0)} \end{bmatrix}_{N\times N}. \end{aligned}$$

Likewise for the sake of simplicity, our test domain $\Omega \subset [0,1]^2$ is a periodic grid, consisting of $n^2 = 100$ copies of a reference cell with a kidney-shaped hole drawn freehand in the middle. The side length of a single cell is $\varepsilon = n^{-1} = 0.1$. In the absence of any external forces, the flow is driven by an inflow condition on the left and kept parallel to the top and bottom boundaries. From the right, it may exit as it will:

$$\begin{cases} \boldsymbol{u} = \boldsymbol{e}_1 & \text{on } \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{x}_1 = 0\}, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 & \text{on } \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{x}_2 = 0 \text{ or } \boldsymbol{x}_2 = 1\}. \end{cases}$$

Figure 7 represents the solution $\widehat{\boldsymbol{u}}$ of the homogenised problem (see Proposition 3.7) with these boundary conditions. The reference solution $\boldsymbol{u}$, shown in Figure 8, was computed directly using the same elements. Figure 9 shows both solutions as vector fields in a single cell. To minimize any error due to the domain boundary, the cell $[0.5, 0.6]^2$ in the middle of the domain was chosen. Finally, Figure 10 shows the first two correctors $\mathcal{X}_i^{(0)}$, $i = 1, 2$, modifying the slow solution (see Chapter 3.5).



(a) $\mathcal{X}_1^{(0)}$ (b) $\mathcal{X}_2^{(0)}$

Figure 10: The two correctors modifying the slow solution $\widehat{\boldsymbol{u}}_s$.

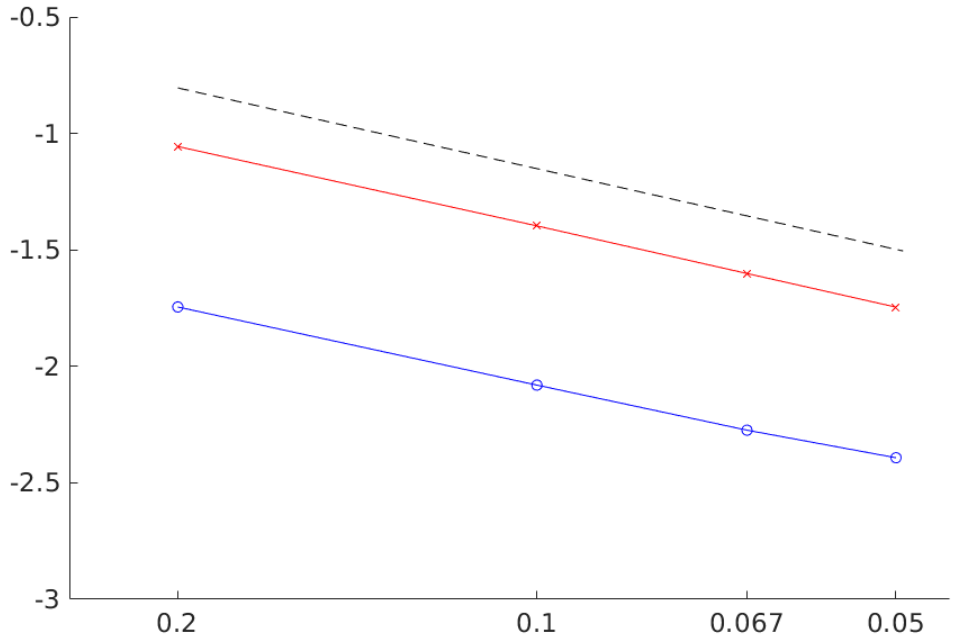


Figure 11: Convergence with decreasing ε. The error has been plotted as a function of $\log \varepsilon$, with the corresponding values of ε on the x-axis. Both the H^1 seminorm (red/crosses) and the L^2 norm (blue/circles) are shown; see (65). The slope of the dashed black line is 1/2.

Verifying the error estimate. To be sure, no "convergence" ever happens in the process of homogenisation by energy decomposition, just as it does not happen in the vast majority of situations involving real-life porous media. However, to investigate the behaviour of the error, it is possible to make it happen. In Proposition 3.9, we derived an estimate of the type

$$\|\boldsymbol{e}\|_{E(\Omega)} \leq C\varepsilon^{\alpha/2} \|\boldsymbol{f}\|_{L^2(\Omega)}.$$

In other words, the error should behave as $C\sqrt{\varepsilon^{\alpha}}$ for some $\alpha > 0$, whose value depends on the particularities of the medium that have been embedded in the two-scale assumption.

In practice, we solve the homogenised and reference problems in a series of square domains consisting of n^2 copies of the reference cell for $n = 5, 10, 15, 20$. When the result is mapped back into the unit square, the result is equivalent to decreasing ε in the original domain. The normalization is easily accomplished by dividing with the norm of the reference solution in the corresponding domain:

$$\|\boldsymbol{e}\|^*_{L^2} = \frac{\|\boldsymbol{u} - \widehat{\boldsymbol{u}}\|_{L^2(\Omega_n)}}{\|\boldsymbol{u}\|_{L^2(\Omega_n)}}, \qquad |\boldsymbol{e}|^*_{H^1} = \frac{\|\nabla(\boldsymbol{u} - \widehat{\boldsymbol{u}})\|_{L^2(\Omega_n)}}{\|\nabla \boldsymbol{u}\|_{L^2(\Omega_n)}}. \tag{65}$$

These expressions for the error have been plotted in Figure 11, which shows convergence of the order $\mathcal{O}(\varepsilon^{1/2})$, at least in the few cases we have computed; the direct solution of the problem quickly becomes difficult with decreasing ε. We have no means to estimate *a priori* the size of α, but it appears to be close to one in our test case. This makes sense in view of the very regular distribution of the inhomogeneities, which would seem to guarantee that the "slow" and "fast" scales are neatly separated.

A Toolbox

Definition A.1 (Weak convergence)**.** Let E be a real Banach space. A sequence $\{x_n\} \in E$ is said to *converge weakly to* x if and only if

$$\langle x', x_n\rangle_{E',E} \to \langle x', x\rangle_{E',E} \quad \forall x' \in E'.$$

Then we denote $x_n \rightharpoonup x$ *weakly.*

Proposition A.1 (Weak–strong convergence)**.** *Let* $\{x_n\} \in E$ *and* $\{y_n\} \in E'$ *such that*

$$\begin{aligned} x_n &\rightharpoonup x \text{ weakly in } E, \\ y_n &\to y \text{ strongly in } E'. \end{aligned}$$

Then $\lim_{n\to\infty}\langle y_n,\, x_n\rangle_{E',\,E} = \langle y,\, x\rangle_{E',\,E}$. [7, p. 16] □

Definition A.2 (Sobolev spaces of negative order)**.** The Banach space $W^{-m,p'}(\Omega)$ is defined for all $m \in \mathbb{N},\ p \in [1,\infty)$ as the dual Sobolev space

$$W^{-m,p'}(\Omega) = \left[W_0^{m,p}(\Omega)\right]',$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and equipped with the dual norm

$$\|u\|_{W^{-m,p'}(\Omega)} = \sup\left\{\langle u, v\rangle : v \in W_0^{m,p}(\Omega),\ \|v\|_{W^{m,p}(\Omega)} = 1\right\}.$$

Lemma A.2. *Let* $\Omega \in \mathbb{R}^N$ *be a bounded, connected, Lipschitz domain and* $p \in \mathcal{D}(\Omega)$ *such that* $\nabla p \in \left[H^{-1}(\Omega)\right]^N$. *Then* $p \in L^2(\Omega)\,/\,\mathbb{R}$ *and*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \le C\,\|\nabla p\|_{H^{-1}(\Omega)},$$

where the constant C *depends only on* Ω. [12, p. 20] □

Theorem A.3 (Babuška–Lax–Milgram)**.** *Let* X_1 *and* X_2 *be two Hilbert spaces with their respective inner products* $(\cdot,\cdot)_{X_i}$ *and induced norms* $\|\cdot\|_{X_i}$. *Let* $b : X_1 \times X_2 \to \mathbb{R}$ *be a bilinear form satisfying*

(i) continuity:

$$b(u,v) \le C\,\|u\|_{X_1}\,\|v\|_{X_2} \quad \text{for all } (u,v) \in X_1 \times X_2;$$

(ii) the inf-sup *condition:*

$$\sup_{u\in X_1,\, u\neq 0} \frac{b(u,v)}{\|u\|_{X_1}} \ge \alpha\,\|v\|_{X_2} \quad \text{for all } v \in X_2;$$

(iii) the condition

$$\sup_{v\in X_2} b(u,v) \neq 0 \quad \text{for all } u \in X_1, u \neq 0$$

with $C < \infty$ and $\alpha > 0$. Then, the variational problem: Find $u \in X_1$ such that

$$b(u, v) = l(v) \quad \text{for all } v \in X_2$$

has exactly one solution depending continuously on the data:

$$\|u\|_{X^1} \leq \alpha^{-1} \|l\|_{X_2'} .$$

[5] □

Lemma A.4 (Poincaré constant in a convex domain)**.** *Let $\Omega \in \mathbb{R}^N$ be a bounded, convex, Lipschitz domain. We have*

$$\|\boldsymbol{u} - \pi_0 \boldsymbol{u}\|_{L^2(\Omega)} \leq \pi^{-1} \operatorname{diam}(\Omega) \|\nabla \boldsymbol{u}\|_{L^2(\Omega)} .$$

[16] □

Definition A.3 (The spaces $H^1(\mathrm{curl})$ and $H_0^1(\mathrm{curl})$)**.** For a bounded, simply connected, Lipschitz domain $\Omega \subset \mathbb{R}^N$, we define

$$H^1(\mathrm{curl}; \Omega) = \left\{ \boldsymbol{v} \in [L^2(\Omega)]^N : \nabla \times \boldsymbol{v} \in [L^2(\Omega)]^N \right\} .$$

This is a Hilbert space for the norm $\|\boldsymbol{v}\|^2_{H^1(\mathrm{curl};\Omega)} = \|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\nabla \times \boldsymbol{v}\|^2_{L^2(\Omega)}$. The space $H_0^1(\mathrm{curl}; \Omega)$ can be described as

$$H_0^1(\mathrm{curl}; \Omega) = \left\{ \boldsymbol{v} \in H^1(\mathrm{curl}; \Omega) : \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\Omega \right\} .$$

See [12, pp. 32–35].

Theorem A.5 (Trace theorem)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set with a Lipschitz boundary. There exists a continuous linear map $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, with $\operatorname{tr} \boldsymbol{u} = \gamma \boldsymbol{u} = \boldsymbol{u}|_{\partial\Omega}$, such that*

$$\|\gamma \boldsymbol{u}\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(\Omega) \|\boldsymbol{u}\|_{H^1(\Omega)}$$

for all $\boldsymbol{u} \in H^1(\Omega)$. □

Theorem A.6 (Inverse trace theorem)**.** *The trace operator given by Theorem A.5 has a continuous right inverse $\mathcal{E} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$, with $(\gamma \circ \mathcal{E})\boldsymbol{u} = \boldsymbol{u}$, such that*

$$\|\mathcal{E} \boldsymbol{u}\|_{H^1(\Omega)} \leq C(\Omega) \|\boldsymbol{u}\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

for all $\boldsymbol{u} \in H^1(\Omega)$. □

Theorem A.7 (Riesz' representation theorem)**.** *Let X be a Hilbert space and $\mathcal{L}$ a bounded linear functional on X. Then there exists a unique element $y \in X$ such that $\mathcal{L}(x) = (x, y)_X$ for all $x \in X$, and $\|\mathcal{L}\|_{X'} = \|y\|_X$.* □

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