Matrix-valued Impedances with Fractional Derivatives and Integrals in Boundary Feedback Control: a port-Hamiltonian approach

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Abstract: This paper discusses the passivity of the port-Hamiltonian formulation of a multivariable impedance matching boundary feedback of fractional order, expressed through diffusive representation. It is first shown in the 1D-wave equation case that the impedance matching boundary feedback can be written as a passive feedback on the boundary port variables. In the Euler-Bernoulli case, the impedance matching feedback matrix involves fractional derivatives and integrals. It is shown that the usual diffusive representation of such feedback is not formally a dissipative port-Hamiltonian system, even if from a frequency point of view this feedback proves passive.

Keywords: fractional differential equations, diffusive systems, pseudo-differential operators, hereditary mechanics, stability, numerical methods, boundary control of PDEs.

1. INTRODUCTION

Boundary control of distributed parameter systems is of great interest for engineering applications, since actuators and sensors are generally distributed over the boundary of the spatial domain. It is especially the case for the control of compliant structures, acoustic systems and fluidic systems. An appealing control strategy when travelling waves are involved in the dynamics of the system is the impedance matching approach (von Flotow and Schäfer, 1986; Matsuda and Fujii, 1993). It consists in designing in the frequency domain a controller that is able to attenuate the wave propagation in one direction. Such a control strategy has been succesfully applied in the context of 1D Single Input Single Output (SISO) wave propagation (Mbodje and Montseny, 1995), and also 1D Multi Input Multi Output (MIMO) Euler-Bernoulli beam equation (Montseny et al., 1997). Extensions to the 2D wave propagation case have been proposed for different geometries, either rectangular (Montseny et al., 2000) or circular (Levadoux and Montseny, 2003).

In this paper, we discuss the port-Hamiltonian (pHs) formulation of this impedance matching boundary control of the wave equation in the one dimensional spatial domain. In the 1D case, a simple frequency analysis allows to derive a scalar absorbing boundary control. This well-known result can be recast in the port-Hamiltonian framework, as soon as the boundary port-variables are adequately defined. It is then straightforward to show that this feedback is passive. In the case of the Euler Bernoulli beam equation, absorbing boundary control induces the use of a MIMO feedback with fractional integral and derivative operators of order $\frac{1}{2}$. As long as the analysis of solutions is concerned, the use of diffusive input-output realization of Abel fractional integral operator allows to represent the system as an abstract linear system with infinitesimal generator of a semigroup on a convenient Hilbert space.

In this latter case, the passivity of the diffusive realization of the feedback has been admitted and the stability analysis could be carried out, at least formally, only when the cross-coupling terms in the feedback were neglected (Montseny et al., 1997). In the present paper we show that these cross-coupling terms are important and that their impact on the passivity of the diffusive realization cannot be neglected.

The paper is organized as follows. In Section 2, the 1D wave equation with impedance matching boundary control is treated. In Section 3, Euler-Bernoulli beam model is detailed, and the validity of scalar diffusive representations of the feedback with respect to passivity is discussed. In Section 4, some conclusions and perspectives are given.

2. THE 1D WAVE EQUATION

We first consider the boundary controlled 1D wave equation:

$$\begin{cases} \partial_{t^2}^2 \theta - \partial_{z^2}^2 \theta = 0, & z \in [0,1] \\ \theta(z,0) = \theta_0(z) \\ \partial_t \theta(z,0) = \theta_1(z) \\ \\ \partial_t \theta(0,t) = 0 \\ \partial_z \theta(1,t) = u_1(t) \end{cases}$$

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with observation y_1 at the boundary:

$$y_1(t) = \partial_t \theta(1,t)\,.$$

The total energy is given by:

$$E(t) := \frac{1}{2}\int_0^1 \left((\partial_t\theta)^2 + (\partial_z\theta)^2 \right)\,\mathrm{d}z\,,$$

and the energy balance by:

$$\frac{dE}{dt} = y_1(t)\,u_1(t)\,.$$

The objective below is to derive the classical scalar impedance relations and to design in the frequency domain a boundary feedback ensuring no forward propagation waves.

2.1 Model and derivation of the matched impedance: SISO case

Looking for harmonic solutions of the form $\exp(i(\omega t - kz))$ for the wave equation alone, without taking into account any boundary condition, one can find the necessary *dispersion* relation:

$$\omega^2 = k^2\,,$$

with two solutions, $k(\omega) = \pm\omega$. The elementary solutions are *travelling* waves, either forward $\exp(i\omega(t-z))$ or backward $\exp(i\omega(t+z))$.

From this preliminary analysis, we can compute the matched impedance $Z(\omega)$, which ensures that no wave be transmitted at the boundary $z = 1$. Let us suppose the following structure for the solution:

$$\theta = 1.e^{i(\omega t - \omega z)} + E e^{(i\omega t + \omega z)}\,. \tag{1}$$

where E is a scalar to be determined from the boundary conditions. After some straightforward computation, the input and output of the system can be written:

$$y_1(t) = \partial_t\theta(1,t) = i\omega\left(-e^{-i\omega} + E e^{i\omega}\right)e^{i\omega t}\,,$$
$$u_1(t) = \partial_z\theta(1,t) = i\omega\left(e^{-i\omega} + E e^{i\omega}\right)e^{i\omega t}\,.$$

We look for a scalar feedback impedance of the form $Z(\omega)$, such that $y = Zu$. The goal now is to compute this unique frequency-dependent coefficient, in order to cancel the forward traveling wave, *i.e.* to ensure $E = 0$ in (1), whatever the value of the input amplitude 1. Defining $E_1 := E e^{2i\omega}$, a straightforward computation leads to:

$$Z(\omega) := \frac{E_1 - 1}{E_1 + 1}\,.$$

Thus, it is easy to see that $E_1 = 0 \Leftrightarrow Z = -1$, $\forall\omega \in \mathbb{R}$. Then, there are no forward travelling waves.

In the frequency domain, $y_1 = -u_1$, $\forall\omega\in\mathbb{R}$, whereas in the time domain $y_1 = -u_1$, that is $\partial_t\theta(1,t) = -\partial_z\theta(1,t)$.

2.2 Some useful properties

First note that the feedback is constant or of differential nature, so it is very easy to implement! Moreover, one can notice that if absorbing boundary conditions are searched for (i.e. not necessarily matched impedance), they are easily characterized as follows:

$$|E| \le 1 \Leftrightarrow \Re e(Z) \le 0\,.$$

Once again, the relation reads in the time domain $\partial_t\theta(1,t) = Z\,\partial_x\theta(1,t)$, with $Z \le 0$. Two limiting cases can be easily inspected: either $Z = 0 \Leftrightarrow E = 1$, or $Z = -\infty \Leftrightarrow E = -1$.

2.3 A port-Hamiltonian formulation

An interesting idea is thus to rewrite the previous feedback in the pHs setting. The system can be written

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial z} & 0 \end{pmatrix} x = P_1 \frac{\partial}{\partial z} x\,,$$

with $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} \partial_z\theta \\ \partial_t\theta \end{pmatrix}$. The boundary port-variables can be defined from (Le Gorrec et al., 2005):

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}\begin{pmatrix} x(1) \\ x(0) \end{pmatrix}\,,$$

or equivalently

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \partial_t\theta(1) - \partial_t\theta(0) \\ \partial_z\theta(1) - \partial_z\theta(0) \\ \partial_z\theta(1) + \partial_z\theta(0) \\ \partial_t\theta(1) + \partial_t\theta(0) \end{pmatrix}\,.$$

One can choose as input

$$u = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = W \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}\,,$$

and as output

$$y = \begin{pmatrix} y_1 \\ y_0 \end{pmatrix} = \widetilde{W}\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}\,.$$

We have

$$W\Sigma W^T = 0 = \widetilde{W}\Sigma\widetilde{W}^T \quad\text{and}\quad W\Sigma\widetilde{W}^T = I = \widetilde{W}\Sigma W^T\,,$$

and then from (Le Gorrec et al., 2005):

$$\frac{dE}{dt} = \frac{d}{dt}\|x\|^2 = y^T u\,.$$

In this case, the matched impedance expressed from:

$$\partial_z\theta(1) = -\partial_t\theta(1)\,,$$

considered with $\partial_t\theta(0) = 0$ is equivalent to $u_1 = -y_1$ and $u_0 = 0$ then

$$\frac{d}{dt}\|x\|^2 = -y_1^2 \le 0\,.$$

3. EULER-BERNOULLI BEAM

We consider now the boundary controls $u_1 = (u_{11}, u_{12})^T$ of the beam equation:

$$\begin{cases} \partial_{t^2}^2\theta + \partial_{z^4}^4\theta = 0, & z\in[0,1] \\ \theta(z,0) = \theta_0(z) \\ \partial_t\theta(z,0) = \theta_1(z) \\ \\ \theta(0,t) = 0 \\ \partial_z\theta(0,t) = 0 \\ \partial_z^2\theta(1,t) = u_{11}(t) \\ \partial_z^3\theta(1,t) = u_{12}(t) \end{cases} \tag{2}$$

with observations $y_1 = (y_{11}, y_{12})^T$ at the boundary:

$$y_1 = \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = \begin{pmatrix} \partial_t\partial_z\theta(1,t) \\ -\partial_t\theta(1,t) \end{pmatrix}\,.$$

The total energy is given by:

$$E(t) = \frac{1}{2}\int_0^1 \left((\partial_t\theta)^2 + \left(\partial_{z^2}^2\theta\right)^2 \right)\,\mathrm{d}z\,,$$

and the energy balance by

$$\frac{dE}{dt}(t) = y_1^T(t)\,u_1(t)\,.$$

In § 2 for waves, it was said that the objective was to derive the classical scalar impedance relations and to design in the frequency domain a boundary feedback ensuring no forward propagation waves: it was quite straightforward, and we ended up with a scalar impedance, which was constant coefficient; hence, going back to the time domain was an easy task.
Now in § 3 for beams, the analogous objective below is to derive the not so classical impedance matrix relations in order to define in the frequency domain a dynamic boundary feedback that is able to eliminate the forward wave propagation (exact impedance matching), or to damp it (passivation): the novelty will be first that this feedback is a matrix-valued impedance, and second that it involves non-integer powers of the frequency variable ω; hence converting it back into the time domain will give rise to fractional differential operators, which are a special subclass of pseudo-differential operators.

Thus, the outline of this more technically involved section is as follows. In § 3.1, the goal is to compute the matched impedance: to this end, we decompose the general harmonic solution of the beam PDE into four elementary components, and we look for those solutions which lead to no transmitted wave at the boundary $z = 1$, whatever the amplitudes of the incoming waves (both travelling and nearfield). In § 3.2, we study the properties of the obtained impedance matrix, and point out some of it specificities. Finally, § 3.3 is devoted to an equivalent port-Hamiltonian formulation of the global system, i.e. beam together with the matched impedance.

3.1 Model and derivation of the matched impedance: MIMO case

Looking for harmonic solutions of the form $\exp(i(\omega t - kz))$ for the beam equation alone, without taking into account any boundary condition, one can find the necessary *dispersion* relation :

$$\omega^2 = k^4 ,$$

with four solutions, $k(\omega) = \pm\sqrt{\omega}$ and $k(\omega) = \pm i\sqrt{\omega}$. The first two elementary solutions are *travelling* waves, either forward $\exp(i(\omega t - \sqrt{\omega} z))$ or backward $\exp(i(\omega t + \sqrt{\omega} z))$, whereas the last two are *nearfield* waves, either forward $\exp(i\omega t - \sqrt{\omega} z)$ or backward $\exp(i\omega t + \sqrt{\omega} z)$.

From this preliminary analysis, we compute the impedance matrix $Z(\omega)$ of the boundary feedback which ensures that no wave is transmitted at the boundary $z = 1$. Let us suppose the following structure for the solution:

$$\theta = e^{i(\omega t - \sqrt{\omega} z)} + \sigma e^{i\omega t - \sqrt{\omega} z} + E e^{i(\omega t + \sqrt{\omega} z)} + F e^{i\omega t + \sqrt{\omega} z} .$$

where σ, E and F are scalars to be determined from the boundary conditions. After some straightforward computation, the input and output of the system can be written:

$$\begin{cases} u_{11} = \partial^2_{z^2}\theta(1) = \omega\left(-e^{-i\sqrt{\omega}} + \sigma e^{-\sqrt{\omega}} - E e^{i\sqrt{\omega}} + F e^{\sqrt{\omega}}\right) e^{i\omega t} , \\ u_{12} = \partial^3_{z^3}\theta(1) = \omega^{\frac{3}{2}}\left(i e^{-i\sqrt{\omega}} - \sigma e^{-\sqrt{\omega}} - iE e^{i\sqrt{\omega}} + F e^{\sqrt{\omega}}\right) e^{i\omega t} , \\ y_{11} = \partial_t \partial_z \theta(1) = i\omega\sqrt{\omega}\left(-i e^{-i\sqrt{\omega}} - \sigma e^{-\sqrt{\omega}} + iE e^{i\sqrt{\omega}} + F e^{\sqrt{\omega}}\right) e^{i\omega t} \\ y_{12} = -\partial_t \theta(1) = -i\omega\left(e^{-i\sqrt{\omega}} + \sigma e^{-\sqrt{\omega}} + E e^{i\sqrt{\omega}} + F e^{\sqrt{\omega}}\right) e^{i\omega t} . \end{cases}$$

We look for a 2×2 feedback matrix $Z(\omega)$ of the form

$$\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} .$$

The goal now is to compute these four frequency-dependent coefficients in order to prevent waves from propagating forward, *i.e.* in order to ensure that both outgoing wave amplitudes $E = 0$ and $F = 0$, whatever the value of the incoming wave amplitudes, 1 and σ. First, we get

$$\begin{pmatrix} \omega\left(-e^{-i\sqrt{\omega}} + \sigma e^{-\sqrt{\omega}} - E e^{i\sqrt{\omega}} + F e^{\sqrt{\omega}}\right) \\ \omega^{\frac{3}{2}}\left(i e^{-i\sqrt{\omega}} - \sigma e^{-\sqrt{\omega}} - iE e^{i\sqrt{\omega}} + F e^{\sqrt{\omega}}\right) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} i\omega\sqrt{\omega}\left(-i e^{-i\sqrt{\omega}} - \sigma e^{-\sqrt{\omega}} + iE e^{i\sqrt{\omega}} + F e^{\sqrt{\omega}}\right) \\ -i\omega\left(e^{-i\sqrt{\omega}} + \sigma e^{-\sqrt{\omega}} + E e^{i\sqrt{\omega}} + F e^{\sqrt{\omega}}\right) \end{pmatrix} \quad (3)$$

Second, using a change of unknowns: $\tilde{A} := A\omega^{\frac{1}{2}}, \tilde{D} := D\omega^{-\frac{1}{2}}$, $E_1 := E e^{2i\sqrt{\omega}}$, $F_1 := F e^{\left(\sqrt{\omega} + i\sqrt{\omega}\right)}$, and $\sigma_1 := \sigma e^{\left(-\sqrt{\omega} + i\sqrt{\omega}\right)}$, we have:

$$\begin{bmatrix} 1 - \tilde{A} - iB & -1 + i\tilde{A} - iB \\ i - C - i\tilde{D} & -1 + iC - i\tilde{D} \end{bmatrix} \begin{bmatrix} E_1 \\ F_1 \end{bmatrix} = \begin{bmatrix} -1 - \tilde{A} + iB & 1 + i\tilde{A} + iB \\ i - C + i\tilde{D} & -1 + iC + i\tilde{D} \end{bmatrix} \begin{bmatrix} 1 \\ \sigma_1 \end{bmatrix} . \quad (4)$$

We are now looking for values of $(\tilde{A}, B, C, \tilde{D})$ that ensure that (E_1, F_1) equal zero $\forall \omega \in \mathbb{R}$. For that purpose, we compute $(\tilde{A}, B, C, \tilde{D})$ such that the right hand side of (4) vanishes and check that in this case the premultiplying matrix of (E_1, F_1) is full rank. It is the case if we choose:

$$\tilde{A} = -(1 - i) , \; B = 1, C = 1, \; \tilde{D} = -(1 + i)$$

Indeed, in this case

$$\begin{bmatrix} 1 - \tilde{A} - iB & -1 + i\tilde{A} - iB \\ i - C - i\tilde{D} & -1 + iC - i\tilde{D} \end{bmatrix} \begin{bmatrix} E_1 \\ F_1 \end{bmatrix} = 0 ,$$

and

$$\begin{bmatrix} 1 - i & -i - 1 \\ i - 1 & i - 1 \end{bmatrix} \begin{bmatrix} E_1 \\ F_1 \end{bmatrix} = 0 ,$$

then, there are no forward nearfield nor travelling waves.

The corresponding feedback is given by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -(1 - i)\,\omega^{-\frac{1}{2}} & 1 \\ 1 & -(1 + i)\,\omega^{\frac{1}{2}} \end{pmatrix} .$$

Since $(1 - i) = \sqrt{2} e^{-i\frac{\pi}{4}}$, and $(1 + i) = \sqrt{2} e^{i\frac{\pi}{4}}$, then we get in the frequency domain:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\sqrt{2} e^{-i\frac{\pi}{4}} \omega^{-\frac{1}{2}} & 1 \\ 1 & -\sqrt{2} e^{i\frac{\pi}{4}} \omega^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} -\dfrac{\sqrt{2}}{\sqrt{i\omega}} & 1 \\ 1 & -\sqrt{2}\sqrt{i\omega} \end{pmatrix} := Z(\omega) ,$$

and in the time domain:

$$\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} = \begin{pmatrix} -\sqrt{2} I^{\frac{1}{2}} & 1 \\ 1 & -\sqrt{2} D^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} . \quad (5)$$

The above notation is: I^β is Riemann-Liouville fractional integral operator of order $\beta \in (0, 1)$, the Fourier transform of which is $(i\,\omega)^{-\beta}$, and D^α is Riemann-Liouville fractional derivative operator of order $\alpha \in (0, 1)$, the Fourier transform of which is $(i\,\omega)^\alpha$, see e.g. (Oldam and Spanier, 1985; Matignon, 1998).

3.2 Some useful properties

First note that the feedback is of pseudo-differential nature, involving non-local operators in time. Note also, that a strong coupling occurs between the variables, due to the presence of 1 in positions Z_{12} and Z_{21}: since they have the same sign, symmetrising will not make them disappear and will not give rise to a diagonal matrix, with fractional integrals and derivatives only. Namely:

$$R(\omega) := \frac{Z+Z^H}{2} = \begin{pmatrix} -\sqrt{2}|\omega|^{-\frac{1}{2}}\cos(\frac{\pi}{4}) & 1 \\ 1 & -\sqrt{2}|\omega|^{+\frac{1}{2}}\cos(\frac{\pi}{4}) \end{pmatrix}.$$

The real symmetric matrix $R(\omega)$ is negative, since its eigenvalues are $r_0 = 0$ and $r_1 = -(|\omega|^{+\frac{1}{2}} + |\omega|^{-\frac{1}{2}})$. Hence, this matched impedance is an absorbing feedback. This result is in adequation with the more general result stated in (Haddar and Matignon, sept 2004) for an impedance matrix with fractional integrals and derivatives of arbitrary order α and arbitraty numerical coefficients.

In the following we propose to study the diffusive representation of the feedback (5).

3.3 A port-Hamiltonian formulation

This last subsection aims at finding an equivalent port-Hamiltonian formulation of the global system, i.e. beam together with the matched impedance. To this end, in § 3.3.1, we recall the natural port variables associated to Euler-Bernoulli beam equation. And since fractional integrals and derivatives are to be found in the expression of the matched impedance, an equivalent so-called diffusive representation is needed, and presented in § 3.3.2.

3.3.1. Euler-Bernoulli beam By choosing as state variables the energy variables $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \partial_z^2\theta \\ \partial_t\theta \end{pmatrix}$, the system can be recast into the port-Hamiltonian framework, as

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \frac{\partial^2}{\partial z^2} \\ -\frac{\partial^2}{\partial z^2} & 0 \end{pmatrix} x = P_2 \frac{\partial^2}{\partial z^2} x\,,$$

with $P_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then the boundary port-variables can be defined from:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} Q & -Q \\ I & I \end{pmatrix} \begin{pmatrix} x(1) \\ \partial_z x(1) \\ x(0) \\ \partial_z x(0) \end{pmatrix},$$

with

$$Q = \begin{pmatrix} P_1 & P_2 \\ -P_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_z x_2(1) - \partial_z x_2(0) \\ -\partial_z x_1(1) + \partial_z x_1(0) \\ -x_2(1) + x_2(0) \\ x_1(1) - x_1(0) \\ x_1(1) + x_1(0) \\ x_2(1) + x_2(0) \\ \partial_z x_1(1) + \partial_z x_1(0) \\ \partial_z x_2(1) + \partial_z x_2(0) \end{pmatrix}.$$

Let us now consider the boundary control system defined by:

$$u = W \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}, \quad y = \widetilde{W} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix},$$

with

$$W = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\widetilde{W} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

In this case:

$$u = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} \partial_z^2\theta(1) \\ \partial_z^3\theta(1) \\ \partial_t\theta(0) \\ \partial_z\partial_t\theta(0) \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_0 \end{pmatrix} = \begin{pmatrix} \partial_z\partial_t\theta(1) \\ -\partial_t\theta(1) \\ \partial_z^3\theta(0) \\ -\partial_z^2\theta(0) \end{pmatrix},$$

and

$$W\Sigma W^T = 0 = \widetilde{W}\Sigma\widetilde{W}^T \text{ and } W\Sigma\widetilde{W}^T = I = \widetilde{W}\Sigma W^T,$$

then

$$\frac{d}{dt}\|x\|^2 = y^T u\,.$$

3.3.2. Impedance matching boundary control: a diffusive representation The boundary control feedback is of the form

$$\begin{pmatrix} y_{c1} \\ y_{c2} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}I^{\frac{1}{2}} & 1 \\ 1 & -\sqrt{2}\partial_t^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} u_{c1} \\ u_{c2} \end{pmatrix}. \qquad (6)$$

The idea is to use the diffusive representation of fractional derivative and integrals to express the feedback control and the closed-loop system under an abstract system formulation. Then, it is possible to apply semigroup theory and associated tools in order to derive existence of solution and stability properties, see e.g. (Matignon and Prieur, 2014). It is usually admitted that the controller expressed using a diffusive representation is passive. We shall see from its port-Hamiltonian formulation that this property is not straightforward at all.

In what follows, we use for the fractional integral $y_\varphi = I^{\frac{1}{2}} u_{c1}$ the diffusive representation:

$$\begin{cases} y_\varphi = \int_0^\infty \mu(\xi)\,\varphi\,\mathrm{d}\xi, \quad \mu(\xi) = \frac{2}{\sqrt{\xi}}, \\ \partial_t\varphi = -\xi\varphi + u_{c1}\,. \end{cases}$$

For the fractional derivative of $\widetilde{y} = \partial_t^{\frac{1}{2}} u_{c2}$:

$$\begin{cases} \widetilde{y_{\widetilde{\varphi}}} = \int_0^\infty \nu(\xi)\,\partial_t\widetilde{\varphi}\,\mathrm{d}\xi = \int_0^\infty \left(-\nu(\xi)\,\xi\,\widetilde{\varphi} + \nu(\xi)\,u_{c2}\right)\,\mathrm{d}\xi\,, \\ \partial_t\widetilde{\varphi} = -\xi\widetilde{\varphi} + u_{c2}\,, \end{cases} \qquad (7)$$

where $\nu(\xi) \propto \xi^{-1/2}$, like μ.

The energy associated to this system is defined by:

$$E_c = \frac{1}{2}\int_0^{+\infty} \left(\sqrt{2}\mu(\xi)\,\varphi^2 + \sqrt{2}\xi\nu(\xi)\,\widetilde{\varphi}^2\right)\,\mathrm{d}\xi\,.$$

Following e.g. (Le Gorrec and Matignon, 2013), the overall feedback can be written:

$$
\underbrace{\begin{pmatrix} y_{c1} \\ y_{c2} \\ \partial_t \varphi \\ \partial_t \widetilde{\varphi} \end{pmatrix}}_{f} =
$$

$$
\underbrace{\begin{pmatrix} 0 & 1 & -\int_0^\infty (.)\,\mathrm{d}\xi & 0 \\ 1 & -\sqrt{2}\int_0^\infty \nu(\xi)\,(.)\,\mathrm{d}\xi & 0 & \int_0^\infty (.)\,\mathrm{d}\xi \\ 1 & 0 & -\frac{\xi}{\sqrt{2}\mu(\xi)} & 0 \\ 0 & 1 & 0 & -\frac{1}{\sqrt{2}\nu(\xi)} \end{pmatrix}}_{\mathcal{J}-\mathcal{R}} \underbrace{\begin{pmatrix} u_{c1} \\ u_{c2} \\ \sqrt{2}\mu(\xi)\varphi \\ \sqrt{2}\xi\nu(\xi)\widetilde{\varphi} \end{pmatrix}}_{e} \tag{8}
$$

with $e \in \mathcal{E}$ and $f \in \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are two real Hilbert spaces, the efforts and the flows respectively. The bond space $\mathcal{B}$ defined by $\mathcal{B} = \mathcal{E} \times \mathcal{F}$ is equipped with the natural power product:

$$
\langle (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4) \rangle = \int_0^1 \left(e_1 f_1 + e_2 f_2 + \int_0^\infty (e_3 f_3 + e_4 f_4)\,\mathrm{d}\xi \right) \mathrm{d}z\,. \tag{9}
$$

The differential operator $\mathcal{J}$ is the skew-symmetric part of the differential operator defined by (8), and $\mathcal{R}$ its symmetric part. The passivity of the boundary control feedback is associated with the positivity of $\mathcal{R}$. To check the positivity of $\mathcal{R}$, one has to check the sign of $\langle e, \mathcal{R}e \rangle, \forall e \in \mathcal{E}$:

$$
\begin{aligned}
\langle e, \mathcal{R}e \rangle &= \int_0^1 \int_0^\infty -2e_1e_2 + b\nu(\xi)e_2^2 \,\mathrm{d}\xi\,\mathrm{d}z \\
&- 2\int_0^1 \int_0^\infty e_2 e_4 \,\mathrm{d}\xi\,\mathrm{d}z + \int_0^1 \int_0^\infty \left(\frac{\xi}{\mu(\xi)a} e_3^2 + \frac{1}{\nu(\xi)b} e_4^2 \right) \mathrm{d}\xi\,\mathrm{d}z\,.
\end{aligned} \tag{10}
$$

Then

$$
\begin{aligned}
\langle e, \mathcal{R}e \rangle &= \int_0^1 \int_0^\infty -2e_1e_2 \,\mathrm{d}\xi\,\mathrm{d}z \\
&+ \int_0^1 \left(\frac{\xi}{\mu(\xi)a} e_3^2 + \left(\sqrt{\nu(\xi)b}\,e_2 - \frac{1}{\sqrt{\nu(\xi)b}} e_4 \right)^2 \right) \mathrm{d}\xi\,\mathrm{d}z\,.
\end{aligned} \tag{11}
$$

We can see that in (11) the first right-hand side term has no determined sign, and nothing can be concluded regarding the positivity of $\mathcal{R}$.

We now consider the interconnection of the system with the boundary control feedback:

$$
u_1 = y_c; \quad u_c = y_1
$$

with

$$
u_0 = \begin{pmatrix} \partial_t \theta(0) \\ \partial_z \partial_t \theta(0) \end{pmatrix} = 0
$$

One can check that the total energy balance is given by:

$$
\frac{dE_t}{dt} = \frac{dE}{dt} + \frac{dE_c}{dt} = -e^T \mathcal{R} e \leq 2e_1e_2 = -\left(\partial_t \partial_x \theta(1)\right)\left(\partial_t \theta(1)\right)\,.
$$

Once again nothing can be said on the sign of $\frac{dE_t}{dt}$. It is important to notice that all the analysis that has been provided by using the diffusive representation of the boundary feedback, in (Montseny et al., 1997) for example, has been done considering the non-diagonal terms both equal to zero. In this latter case the port-Hamiltonian formulation can be simplified, and the new $\mathcal{R}$ term is such that:

$$
\langle e, \mathcal{R}e \rangle = \int_0^1 \left(\frac{\xi}{\mu(\xi)a} e_3^2 + \left(\sqrt{\nu(\xi)b}\,e_2 - \frac{1}{\sqrt{\nu(\xi)b}} e_4 \right)^2 \right) \mathrm{d}\xi\,\mathrm{d}z\,. \tag{12}
$$

In this latter case, it is clear that the system is passive and classical analysis tools can be applied. It does not seem to be the case when exact matching conditions are considered.

Remark. Something to notice is that Diffusive Representations only give *sufficient* stability conditions, even in very elementary cases (Linear Fractional Differential Equations with constant coefficients), as already pointed out in (Matignon, 1998). Hence, if the use of DR does not help for stability analysis in the MIMO case of interest here, it will not be a proof that the closed-loop system is not stable.

4. CONCLUSION AND PERSPECTIVES

In this paper we discussed the boundary feedback control of some wave equation systems using impedance matching strategies. The port-Hamiltonian formulation has been used to check the passivity properties of the resulting feedbacks. In the SISO case, the matching feedback is trivial and has some passivity properties. It is not the case for MIMO systems like for the Euler Bernoulli beam model. In this case the feedback is made up with direct transmission terms and fractional order integrals and derivatives. This feedback is usually expressed using a diffusive representation in order to use the Hilbert spaces properties and semigroup theory. We show by using the port-Hamiltonian framework that passivity properties are not trivial, at least in the exact impedance matching case. It seems that some properties of the MIMO fractional order feedback are lost by the diffusive representation. Thus, as a first perspective, port-Hamiltonian framework could definitely help in modifying the representation of the controller to guarantee the preservation of its natural properties.

An interesting second perspective would be to go to more realistic 2D-systems: so far, two such examples have been treated in the literature, on the wave equation:

- In (Montseny et al., 2000), the perfectly absorbing feedback for the wave equation is defined by the pseudo-differential operator $\mathcal{K} = -\left(\sqrt{\partial_t^2 - \partial_y^2}\right)^{-1}$: diffusive representations can be adapted to tackle this problem.
- In (Levadoux and Montseny, 2003), the perfectly absorbing feedback has a more involved formulation, but can be represented exactly, using Fourier series in the angular variables and Hankel functions: diffusive representations can be adapted to tackle this problem.

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