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Numerical modeling of a time-fractional Burgers equation

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Abstract

A fractional time derivative is introduced into Burgers equation to model losses of nonlinear waves arising in acoustics. A diffusive representation of the fractional derivative replaces the nonlocal operator by a continuum of memory variables that satisfy local ordinary differential equations. A quadrature formula yields a system of local partial differential equations. The quadrature coefficients are computed by optimization with a positivity constraint. One resolves the hyperbolic part by a shock-capturing scheme, and the diffusive part exactly. Extensive details can be found in [3].

Keywords: fractional derivatives, diffusive representation, nonlinear acoustics, Strang splitting

1 Introduction

We investigate Burgers equation with a fractional time derivative D_t^α ($\varepsilon \geq 0$, $0 < \alpha < 1$):

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(a u + b \frac{u^2}{2} \right) = -\varepsilon D_t^\alpha u, \quad (1)$$

For a causal function $u(t)$, $D_t^\alpha u$ refers to the Caputo fractional derivative in time of order α :

$$D_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u}{\partial \tau}(x, \tau) d\tau, \quad (2)$$

where Γ is the Gamma function. The l.h.s. of (1) is a standard transport equation, with linear advection at constant speed a and a nonlinear quadratic term with coefficient b . The r.h.s. of (1) models linear losses and memory effects along the propagation. Since $\alpha < 1$, the hyperbolic nature of Burgers equation is preserved.

Various physical configurations are described by (1). Particular values of ε and α enable to recover Chester's equation describing propagation of finite-amplitude sound waves in tubes, up to $\mathcal{O}(\varepsilon^2)$ terms. This equation is widely used to model brass instruments (trombones, trumpets): the transport terms describe the steepening of waves, yielding the typical "brassy"

effect, and the fractional term models the viscothermal losses at the wall of the duct. Moreover, the linear Lokshin equation can be seen as the superposition of two one-way fractional transport equations of this type [1]. Other applications of (1) concern viscoelasticity, propagation in elastic-walled tubes, or more generally wave propagation in media with memory and complex rheological properties.

2 Diffusive approximation

The convolution product (2) can be recast as

$$D_t^\alpha u = \int_0^{+\infty} \phi(x, t, \theta) d\theta, \quad (3)$$

with the diffusive variable ϕ given by

$$\phi(x, t, \theta) = \gamma_\alpha \theta^{2\alpha-1} \int_0^t \frac{\partial u}{\partial \tau}(x, \tau) e^{-(t-\tau)\theta^2} d\tau, \quad (4)$$

with $\gamma_\alpha = \frac{2\sin(\pi\alpha)}{\pi} > 0$. From equation (4), ϕ satisfies the following first-order ordinary differential equation (ODE):

$$\frac{\partial \phi}{\partial t} = -\theta^2 \phi + \gamma_\alpha \theta^{2\alpha-1} \frac{\partial u}{\partial t}. \quad (5)$$

The integral in (3) is approximated by a quadrature formula on L points, where the diffusive variables ϕ_ℓ satisfy an ODE deduced from (5):

$$\begin{cases} D_t^\alpha u(x, t) \approx \sum_{\ell=1}^L \mu_\ell \phi(x, t, \theta_\ell) \equiv \sum_{\ell=1}^L \mu_\ell \phi_\ell(x, t), \\ \frac{\partial \phi_\ell}{\partial t} = -\theta_\ell^2 \phi_\ell + \gamma_\alpha \theta_\ell^{2\alpha-1} \frac{\partial u}{\partial t}, \quad \ell = 1, \dots, L. \end{cases} \quad (6)$$

An adequate choice of the weights μ_ℓ and nodes θ_ℓ is crucial for the efficiency and accuracy of the diffusive approximation, see e.g. [3] and references therein. Injecting (6) into (1) yields

$$\frac{\partial}{\partial t} \mathbf{U} + \frac{\partial}{\partial x} \mathbf{F}(\mathbf{U}) = \mathbf{S} \mathbf{U}, \quad (7)$$

with $\mathbf{U}(x, t) = (u, \phi_1, \dots, \phi_L)^T$. The energy of \mathbf{U} decreases if $\mu_\ell > 0$ and $\theta_\ell > 0$. A Strang

splitting is used to solve (7). The propagative part is solved by a finite-volume scheme with flux-limiters [2], whereas the diffusive part is solved exactly. The CFL condition of stability is the same as for the inviscid Burgers equation.

3 Numerical experiments

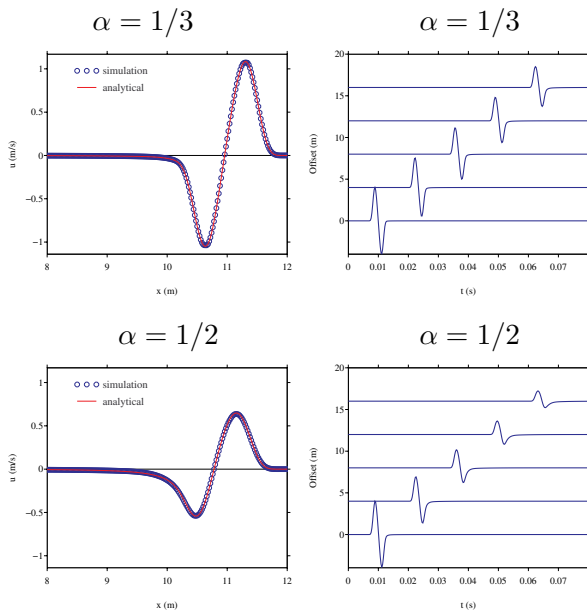


Figure 1: linear fractional advection, for various fractional order α . Left row: snapshots of the numerical and exact solutions. Right row: seismograms.

First, we consider linear advection ($b = 0$). A smooth truncated sinusoid is injected at the left boundary of the domain. Closed-form analytical solutions are known for $\alpha = 1/3$ and $\alpha = 1/2$. These cases are illustrated in figure 1. In the left row, one compares snapshots of the numerical and exact solutions. Greater values of α yield a greater attenuation and a slower propagation, as predicted by the dispersion analysis [3]. The right row illustrates the time and space evolution of the waves.

Second, we consider both nonlinear propagation and fractional attenuation (figure 2). The initial data is a rectangular pulse. Without attenuation, classical phenomena are observed: the pulse splits into a left rarefaction wave and a right shock (a), which collide (b). With attenuation, the right-going shock smears and even disappears for sufficiently large ε .

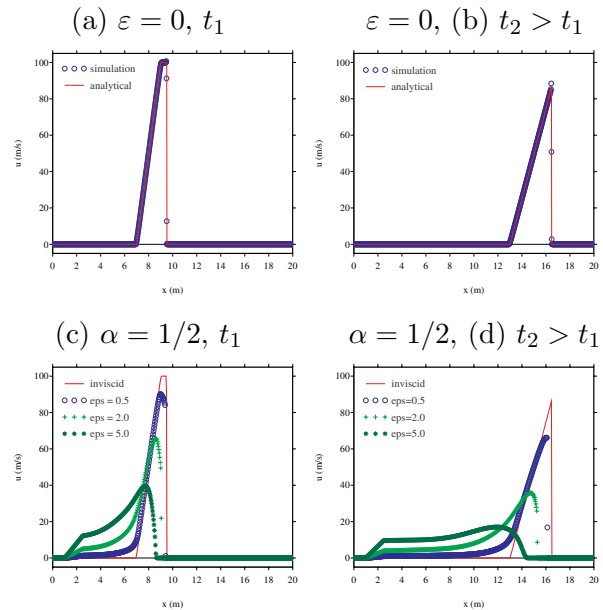


Figure 2: nonlinear advection and fractional attenuation. (a-b): numerical and exact solutions without attenuation. (c-d): numerical solutions for $\alpha = 1/2$ and various values of ε .

4 Conclusion

This article is an attempt for better understanding the competition between nonlinear effects and nonlocal relaxation. Many theoretical questions remain to be addressed. In particular, it seems that the emergence of shocks is conditional (unlike the inviscid Burgers equation). This question requires a deeper analysis to confirm / infirm the numerical observations.

References

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