Asymptotic stability of the linearised Euler equations with long-memory impedance boundary condition

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Abstract

This work focuses on the well-posedness and stability of the linearised Euler equations (1) with impedance boundary condition (2,3). The first part covers the *acoustical* case ($\boldsymbol{u_0} = \boldsymbol{0}$), where the complexity lies solely in the chosen impedance model. The existence of an asymptotically stable C_0-semigroup of contractions is shown when the passive impedance admits a dissipative realisation; the only source of instability is the time-delay τ. The second part discusses the more challenging *aeroacoustical* case ($\boldsymbol{u_0} \neq \boldsymbol{0}$), which is the subject of ongoing research. A discontinuous Galerkin discretisation is used to investigate both cases.

Keywords: impedance boundary condition, diffusive representation, stability, discontinuous Galerkin

Introduction

This work focuses on the (dimensionless) homentropic linearised Euler equations (LEEs)

$$\begin{cases} \partial_t p + \nabla\cdot\boldsymbol{u} + \boldsymbol{u_0}\cdot\nabla p + \gamma\, p\nabla\cdot\boldsymbol{u_0} = 0 \\ \partial_t\boldsymbol{u} + \nabla p + [\boldsymbol{u_0}\cdot\nabla]\boldsymbol{u} + [\boldsymbol{u}\cdot\nabla]\boldsymbol{u_0} + p[\boldsymbol{u_0}\cdot\nabla]\boldsymbol{u_0} = \boldsymbol{0}, \end{cases} \quad (1)$$

defined on $(0,\infty)\times\Omega$, where $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz open subset, p ($\boldsymbol{u}$) is the acoustical pressure (velocity), $\boldsymbol{u_0}\in\mathcal{C}^\infty(\overline{\Omega})^n$ is the (given) base flow, and $\gamma > 1$ is the specific heat ratio. On the boundary $\Gamma := \partial\Omega$ (with outward normal $\boldsymbol{n}$), a so-called acoustical *impedance boundary condition* is prescribed :

$$p(t,x) = \left[z \underset{t}{\star} \boldsymbol{u}\cdot\boldsymbol{n}(\cdot,x)\right](t) \quad (x\in\Gamma := \partial\Omega), \quad (2)$$

where the *impedance* ($z\in\mathcal{D}'_+(\mathbb{R})\cap\mathcal{S}'(\mathbb{R})$, causal convolution kernel) models a mono-dimensional medium as a continuous linear time-invariant system.

A recent analysis of acoustical models in the time domain [6] has shown that a wide range of sound absorbing materials and ground layers, assumed locally-reacting, can be modelled by kernels such as ("$'$" is the weak derivative, a_0,$a_1\geq 0$):

$$z = a_0\delta + a_1\delta' + D_2' + D_3(\cdot-\tau), \quad (3)$$

where $\tau\geq 0$ and $D_i\in L^1_{\text{loc}}(0,\infty)$ is a causal *oscillatory-diffusive* kernel ($I_i\subset\mathbb{Z}$ countable, poles $\Re[s_{n,i}]<0$, $r_{n,i}>0$, μ_i positive Borel measure):

$$D_i(t) = \underbrace{\sum_{n\in I_i} r_{n,i}e^{s_{n,i}t}}_{\text{oscillatory}} + \underbrace{\int_0^\infty e^{-\xi t}\,\mathrm{d}\mu_i(\xi)}_{\text{diffusive}}, \quad (4)$$

which models resonances and visco-thermal losses (e.g. fractional kernel $D_2\propto t^{-1/2}$). A key feature of such *positive real* kernels is that they can be realised (in the sense of systems theory) by a *diagonal*, *dissipative*, infinite-dimensional dynamical system. Note that, if $\tau>0$ in (3), then (2) is a *delayed* boundary condition, which models wave reflections. The two sources of instability in (1,2) are the base flow $\boldsymbol{u_0}$ and the impedance z.

1 Acoustical case

The acoustical assumption ($\boldsymbol{u_0} = 0$) removes hydrodynamic instabilities, but leaves room for purely acoustical ones triggered by the impedance boundary condition (2,3). Below, the delayed ($\tau = 0$) and undelayed ($\tau > 0$) cases are successively investigated by recasting the PDE (1,2) into a Cauchy problem on a Hilbert space $\mathcal{H}$:

$$\dot{X}(t) = \mathcal{A}\,X(t), \quad X(0) = X_0\in\mathcal{D}(\mathcal{A}). \quad (5)$$

To express $\mathcal{A}$, a time-domain realisation of z in a state-space Θ is needed. The given asymptotic stability results (see Thms. 3 and 5), crucially rely on the *dissipativity* of this realisation.

1.1 Undelayed impedance ($\tau = 0$)

Impedances z of increasing complexity can be considered, with Θ either finite or infinite-dimensional: proportional ($z = a_0\delta$), for which no realisation is required; derivative ($z = a_1\delta'$), for which $\Theta=\mathbb{C}$. For the sake of brevity and clarity, only two simplified examples (compared with (3)) are given below before the statement of the general result.

Example 1. Let $\hat{z}(s)$ be a real rational function, bounded for $\Re[s]\geq 0$. If $\Re[\hat{z}(s)]\geq 0$ (passivity), then it can be realised by a dissipative ODE in $\Theta = \mathbb{R}^N$, with a suitable energy norm (positive real lemma, see [4, § 3.1]). Eq. (5) is then defined on $\mathcal{H} = L^2(\Omega) \times (L^2(\Omega))^n \times L^2(\Gamma;\Theta)$.

Example 2. Let $z = a_0\delta + D_2$ (*not* $D_2^{'}$), and define the weighted spaces $\Phi_{\alpha(\xi)} = L^2(0,\infty;\alpha(\xi)\,\mathrm{d}\mu_2)$. The diagonal, dissipative, infinite-dimensional realisation of D_2 in $\Theta = \Phi_1$ leads to $\mathcal{H} = L^2(\Omega) \times (L^2(\Omega))^n \times L^2(\Gamma;\Phi_1)$, and (5) then reads:

$$\mathcal{A}X = \mathcal{A}\begin{pmatrix} p \\ \boldsymbol{u} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\nabla\cdot\boldsymbol{u} \\ -\nabla p \\ A\,\varphi + B\,\boldsymbol{u}\cdot\boldsymbol{n} \end{pmatrix}$$

$$V = H^1 \times (H^1(\mathrm{div}) \cap (H^{1/2})^n) \times L^2(\Gamma;\Phi_{1+\xi})$$

$$\mathcal{D}(\mathcal{A}) = \left\{ X \in V \,\middle|\, \begin{array}{l} [A\,\varphi + B\,\boldsymbol{u}\cdot\boldsymbol{n}] \in L^2(\Gamma;\Phi_1) \\ p_{|\Gamma} = a_0\boldsymbol{u}\cdot\boldsymbol{n} + C\varphi\;(\text{in } L^2(\Gamma)) \end{array} \right\},$$

where, formally, $(A\,\varphi)(x,\xi) = -\xi\varphi(x,\xi)$ (state operator), $(B\,\boldsymbol{u}\cdot\boldsymbol{n})(x,\xi) = \mathbb{1}(\xi)\,\boldsymbol{u}\cdot\boldsymbol{n}(x)$ (control), and $(C\,\varphi)(x) = \int_0^\infty \varphi(x,\xi)\,\mathrm{d}\mu_2(\xi)$ (observation).

Theorem 3. *Assume that $\tau = 0$ in (3). If $\Re[a_0] > 0$, $a_1 \geq 0$, $\Re[s_{n,i}] < 0$, $r_{n,i} > 0$ and μ_i is a positive Borel measure, then z admits a dissipative realisation, and (5) has a unique strong solution X, such that $\|X(t)\| \leq \|X_0\|$ for $t \geq 0$ and $\|X(t)\| \xrightarrow[\infty]{t} 0$.*

Proof (Sketch). We follow [4]. The dissipativity of the realisation of z implies that of $\mathcal{A}$. Well-posedness follows from the m-dissipativity of $\mathcal{A}$. With the Fredholm alternative, we show that $\rho(\mathcal{A}) \supset i\mathbb{R}^*$ (we use that $H^s(\Omega) \subset L^2(\Omega)$, $s > 0$, is a compact embedding). Since $0 \notin \sigma_p(\mathcal{A})$, asymptotic stability then follows from the Arendt-Batty theorem. □

Remark 4. With an infinite-dimensional realisation of z, the embedding $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$ may *not* be compact, hence the need to finely inspect $\rho(\mathcal{A})$, as the pre-compactness condition of LaSalle's invariance principle is not straightforward to verify.

1.2 Delayed impedance ($\tau > 0$)

The delayed case ($\tau > 0$) can also be recast into (5) using a *hyperbolic* realisation of the delay through a transport equation, which leads to an additional extension: $\widetilde{\mathcal{H}} = \mathcal{H} \times L^2(\Gamma; L^2(0,\tau;\Theta))$. Asymptotic stability then becomes delay-dependent, which is typical of time-delayed linear systems (see [5] and references therein). The energy method of Thm. 3 leads to a sufficient stability condition for the *pure* delay case (i.e. $D_3 = a_\tau\delta$, *not* a diffusive kernel).

Theorem 5. *Let $a_\tau \in \mathbb{C}$ and $a_1 > 0$. If $\Re[a_0] > |a_\tau|$, then the result of theorem 3 extends to the case $z = a_0\delta + a_1\delta^{'} + D_2^{'} + a_\tau\delta(\cdot - \tau)$.*

Proof (Sketch). Similar to Thm. 3. The energy norm on the hyperbolic variables, $\|\cdot\|_{L^2(\Gamma;L^2(0,\tau;\mathbb{C}))}$ (here, $\Theta = \mathbb{C}$), is tuned so that $\mathcal{A}$ is dissipative. [5] □

2 Aeroacoustical case

The aeroacoustical assumption is $\boldsymbol{u_0} \neq 0$ in (1). In the case of a subsonic base flow ($|\boldsymbol{u_0}| < 1$), and under stringent assumptions on $\boldsymbol{u}$ and $\boldsymbol{u_0}$ (which must be, in particular, potential), the energy functional of Cantrell and Hart [1, Eq. (64)] can be used to construct a contraction C_0-semigroup. Without these assumptions, however, there is no energy balance, and the dissipativity of $\mathcal{A}$ is lost: well-posedness can only be achieved in a space like "$e^{-\mu t}L^2(\Omega)$", for some $\mu > \omega_0(\mathcal{A}) > 0$, where $\omega_0(\mathcal{A})$ is the growth rate of $\mathcal{A}$. (This constitutes a difficulty of the LEEs, compared to e.g. the Galbrun equation, see [1].) Current research focuses on the identification of instabilities with (2,3), see e.g. [3].

3 Numerical method

Insights into the stability of (5) can be gained by a numerical approximation of the temporal growth rate $\omega_0(\mathcal{A})$. A nodal discontinuous Galerkin method [2] is used to formulate $\dot{X}^h = \mathcal{A}^h X^h + \mathcal{B}^h X(\cdot - \tau)$, with $X^h = (p^h, \boldsymbol{u}^h, \varphi^h)$. The time-domain impedance boundary condition (2,3) is enforced through a centred numerical flux that couples the acoustical unknowns $(p^h, \boldsymbol{u}^h)$ with the memory variables φ^h. If $\tau > 0$, finite-dimensional criteria, which rely on e.g. linear matrix inequalities (LMIs) or spectral conditions, are used to assess stability.

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