Could the rotation of the Earth's inner core be the cause of a dipolar magnetic field generation?

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Summary. — Beginning with the comment of S. Marinov (*Nuovo Cimento C*, **19** (1966) 215) article, we studied two processes that could generate a dipolar magnetic field: the electron inertia in metals and the Barnett effect, and calculated the field generated by such processes. Then, we solved analytically the hydrodynamic problem of the fluid motion in the spherical shell when the inner and outer spheres rotate with different angular velocities and considered an analogue model for the liquid outer core of the Earth. Putting this solution into the kinematics equation of the dynamo theory, we discussed about its solution and about the possibility of modifying and amplifying the dipolar field by the fluid motion.

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Introduction

The starting point of this work was the article: *Earth's rotation is the cause for its magnetization* by S. Marinov [1]. According to this article, the known Barnett effect (where the magnetization of a rotating body is proportional to the angular rotational velocity) should be the Monstein-Barnett effect. Referring to Monstein's experiments, the author concluded that the magnetization of rotating cylinders (made of a non-ferromagnetic material, brass or aluminum) was proportional to the linear rotational velocity. The magnetic field measured on periphery or at different distances from the axis was [1]

$$B = (8 - 10)\, V \mu T l \, ,$$

where V is the linear rotational velocity in m/s.

He claimed the explanation of such effect (called "Barnett-Monstein" effect) with the reasoning based on fig. 5 of [1], where one has a big disk (with radius R) rotating about its axis with angular velocity Ω and on its periphery numerous pairs of small

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disks (with radius r) carrying negative electric charges on their peripheries (it seems that they model the atoms). The pairs of small disks rotate in mutually opposite direction with the same angular velocity ω about their own axes.

Adding the velocities of negative charges that rotate in positive sense of ω and that rotate in negative sense of ω [1]:

$$\begin{cases} V^{\text{pos}}_{R+r} = (R+r)(\Omega+\omega)\,, & V^{\text{pos}}_{R-r} = (R-r)(\Omega-\omega)\,, \\ V^{\text{neg}}_{R+r} = (R+r)(\Omega-\omega)\,, & V^{\text{neg}}_{R-r} = (R-r)(\Omega+\omega)\,, \end{cases} \tag{1}$$

he found a "net velocity" of negative charges: $V^{\text{net}} = 4 \cdot R \cdot \Omega$.

Multiplying such V^{net} with half of atomic density, he got an electrical "net current" density which will generate a magnetic field proportional to the linear velocity.

The author claimed that on the base of the above effect the magnetism of the planets, especially the magnetism of the Earth, could be explained.

I think that the author's reasoning for the "net velocity" of the charges was wrong. The linear velocities of negative charges that rotate in positive and negative sense should be:

$$\begin{cases} V^{\text{pos}}_{R+r} = (R+r)\;\Omega + r\omega\;, & V^{\text{pos}}_{R-r} = (R+r)\;\Omega - r\omega\;, \\ V^{\text{neg}}_{R+r} = (R-r)\;\Omega - r\omega\;, & V^{\text{neg}}_{R-r} = (R-r)\;\Omega + r\omega\;. \end{cases} \tag{2}$$

Formula (1) (or (5) and (6) of [1]) would not be right even in the approximation when $\Omega \ll \omega$. Nevertheless, the exact formulas (2) give as the sum of 4 velocities the same result $V^{\text{net}} = 4 \cdot R \cdot \Omega$. So, the result does not depend on ω and is valid even for $\omega = 0$. That is to say even for the positive charges (the atom nucleus) that have $\omega = 0$, we get the same "net velocity" $V^{\text{net}} = 4 \cdot R \cdot \Omega$ and as a result we get a "net current" equal to zero and not any magnetization effect could be explained in this way.

So, the explanation of so-called "Barnett-Monstern effect" is wrong. Does a "Barnett-Monstern effect" really exist?

We could not answer such a question without knowing how the Monstein experiments are carried out: how the magnetic field was measured, how the Earth's magnetic field was compensated, etc. Such details we could not find and we are doubtful about the results of Monstein's experiments.

If we accepted the results of Monstein's experiments, then we have to find out some other effect that causes such an observed magnetic field. For example, one could be the effect of the thin fluid layer that corotates with the body and although having a small electrical conductivity, perhaps could amplify the magnetic field of a body magnetized by rotation, or some other secondary effect. But I think whatever be the effect that could explain the magnetization of the rotating body, we could not find a proportionality between the magnetization and the linear rotational velocity. I have imagined two processes analyzed below that could give the magnetization effect from the rotation of a body.

1. – Magnetization caused by inertia of free electrons in metals

The rotation of a metallic body with a constant angular velocity ω oriented parallel to the Z-axis causes a redistribution of the free electrons in such way that the electric field created by this redistribution should compensate the action of inertial force on free electrons. Thus, we have

$$m_e \omega^2 r = eE \,, \tag{3}$$

where r is the distance from the axis; m_e, e are, respectively, the mass and electrical charge of the electron; E is the intensity of the electrical field. The electrical field created by this distribution of charges is perpendicular to the rotation axis:

$$E = E_r = \frac{m_e \omega^2 r}{e} \,. \tag{4}$$

From the Maxwell equation

$$\operatorname{div} \mathbf{E} = \frac{\varrho}{\varepsilon_0} \tag{5}$$

we can find the volume distribution density ϱ of electrical charges which creates such field:

$$\varrho = \varepsilon_0 \operatorname{div} \mathbf{E} = \varepsilon_0 \left[\frac{1}{r} \frac{\partial (r E_r)}{\partial r} + \frac{1}{r} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \right],$$

where the divergence operator is written in the cylindrical coordinates. As $E_\phi = E_z = 0$, we can find

$$\varrho = \varepsilon_0 \cdot \frac{2 m_e}{e} \cdot \omega^2 \,. \tag{6}$$

So, the density ϱ results constant, positive and the same result could be found for any geometrical shape of the rotating body.

There are surface charges that compensate the volume charges. From the condition of electrical neutrality, for a cylindrical body with radius R we can find the surface charge density:

$$\sigma = -\frac{\varrho R}{2} = -\frac{\varepsilon_0 m_e \omega^2 R}{e} \,, \tag{7}$$

that is negative and constant. These charges, being in rest relative to the body, rotate together with the body creating electrical currents with volume density

$$\mathbf{j} = \varrho \mathbf{v} \,; \qquad j = j_\phi = \varrho v_\phi = \varrho \omega r = \frac{2 \varepsilon_0 m_e \omega^3 r}{e} \,, \tag{8}$$

and surface density

$$\mathbf{j}_S = \sigma \mathbf{v} \,; \qquad j_S = j_{S\phi} = \sigma v_\phi = \sigma \omega R = -\frac{\varepsilon_0 m_e \omega^3 R^2}{e} \,. \tag{9}$$

The magnetic field corresponding to these electrical current satisfies the Maxwell equations:

$$\text{(10)} \qquad \operatorname{rot}\mathbf{B} = \begin{cases} \mu_0 \mathbf{j}, & \text{if } r < R, \\ 0, & \text{if } r > R, \end{cases} \qquad \operatorname{div}\mathbf{B} = 0$$

and the boundary condition in jump discontinuity:

$$\text{(11)} \qquad \mathbf{B}^+ - \mathbf{B}^- = -\mu_0 \mathbf{n} \times \mathbf{j}_{\mathrm{S}}.$$

Writing eqs. (10) in cylindrical coordinates, we have

$$\left(\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\mathbf{e}_\phi + \left(\frac{1}{r}\frac{\partial (rB_\phi)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \phi}\right)\mathbf{e}_z = j_\phi \mathbf{e}_\phi, \quad \text{if } r<R,$$

$$\left(\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\mathbf{e}_\phi + \left(\frac{1}{r}\frac{\partial (rB_\phi)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \phi}\right)\mathbf{e}_z = 0, \quad \text{if } r>R.$$

If we consider the cylindrical body infinitely long in the Z-direction, then from the symmetry we have

$$B_r = B_\phi = 0, \qquad B = B_z;$$

$$\text{(12)} \qquad \begin{cases} \dfrac{\partial B_z}{\partial r} = \mu_0 j \Rightarrow \dfrac{\mathrm{d}B}{\mathrm{d}r} = -\dfrac{2\mu_0 \varepsilon_0 m_{\mathrm{e}} \omega^3 r}{e}, & \text{if } r < R, \\ \dfrac{\partial B_z}{\partial r} = 0 \Rightarrow \dfrac{\mathrm{d}B}{\mathrm{d}r} = 0, & \text{if } r > R, \\ B = B_z = -\dfrac{\mu_0 \varepsilon_0 m_{\mathrm{e}} \omega^3 r^2}{e}, & \text{if } r < R \end{cases}$$

(considering $B = 0$ at $r = 0$) and

$$\text{(13)} \qquad B^- = -\frac{\mu_0 \varepsilon_0 m_{\mathrm{e}} \omega^3 R^2}{e}.$$

From the boundary condition (11), we have

$$B^+ - B^- = -\mu_0 j_{\mathrm{S}} = \frac{\mu_0 \varepsilon_0 m_{\mathrm{e}} \omega^3 R^2}{e};$$

substituting here B^- from (13), we find $B^+ = 0$, as $\mathrm{d}B/\mathrm{d}r = 0$ for $r > R$, then $B = 0$ for $r > 0$.

So, the magnetic field created by the inertia of electrons is zero outside the body and is proportional to ω^3 and r^2 inside the body. Its orientation is opposite to the angular velocity vector ω. The maximum value of this field would be

$$B_{\max} = -\frac{\mu_0 \varepsilon_0 m_{\mathrm{e}} \omega^3 R^2}{e}.$$

For the Monstein experiment data [1], the quantities of this formula are: $\omega \sim 250\ \mathrm{s}^{-1}$, $R \sim 0,\ 0.022\ \mathrm{m}$, $m_{\mathrm{e}}/e \sim 10^{-12}$, $\varepsilon_0 \mu_0 \sim 10^{-16}$; and the magnetic field would be in order: $B_{\max} \sim 10^{-26}\ \mathrm{T}$. Such field is negligible in comparison to the result of Monstein experiment [1]: $B \sim 10^{-5}\ \mathrm{T}$.

So, the magnetic field caused by the effect of electron inertia is so small inside the body and zero outside the body that could not be observed in the experiment.

2. – The Barnett effect

The magnetomechanic effect is observed for the first time in the Einstein and De Haas experiment (1915): During the magnetization of an iron cylinder in the direction of its axis, a rotation of the cylinder about its axis was noticed. The effect was very small but observable; for a cylinder with diameter of some millimeter, settled in the magnetic field with intensity $H \sim 10^4$ A/m, an angular velocity of the order $10^{-3}\ \mathrm{s}^{-1}$ was noticed.

The opposite effect is the magnetization of a body that rotates, the so-called Barnett effect, observed for the first time in 1936 [2]. The order of the observed effect for an iron cylinder was such that the magnetization for an angular velocity $3 \cdot 10^2\ \mathrm{s}^{-1}$, was equivalent to that caused by a magnetic field of intensity of order 10^{-2} A/m.

The Barnett effect was explained with the gyroscopic effect of elementary gyrostats that model the orbital momentum or spin of electrons. The gyromagnetic ratio (the ratio between the kinetic momentum l and magnetic momentum μ) is $g = 2m_{\mathrm{e}}/e$ for the orbital motion of the electron and $g = m_{\mathrm{e}}/e$ for the spin of the electron. Supposing that any electron has a kinetic momentum $\mathbf{l}$ that is inclined with an angle θ to the rotation axis of the body which rotates with angular velocity ω, then the torque $\omega \times \mathbf{l}$ would be exerted on the electron. This torque tends to turn the kinetic momentum $\mathbf{l}$ parallel to the rotation axis causing a magnetization parallel to the rotation axis of the body. Let B_{eq} be the magnetic field parallel to the rotation axis that would cause the same magnetization as that caused by the rotation, then we have

$$\mu B_{\mathrm{eq}} \sin\theta = -\omega l \sin\theta \Rightarrow B_{\mathrm{eq}} = -\frac{\omega l}{\mu} = -\omega g\,.$$

In the Barnett experiment just B_{eq} and ω were measured and the gyromagnetic ratio g was determined. For different materials like iron, steel, nickel, cobalt, it was $g \approx$ 1.08–1.071 m_{e}/e. It seems that the Barnett experiment served mostly to prove that the magnetic effects are caused more by the electron spins than by the electron orbital momentum.

We will present below a calculation of the magnetization caused by the rotation that is analogue to the calculation of the paramagnetic effect [3].

The kinetic momentum tends to be oriented along the rotation axis. Such orientation is obstructed by the thermal motion. For any orientation of a kinetic momentum, there is the potential energy

$$W = -\omega \cdot \mathbf{l} = -\omega l \cos\theta\,,$$

which is minimum when $\theta = 0$.

This tendency of orientation of the kinetic momentum (orbital or spin) and of the magnetic momentum that is anti-parallel to the kinetic momentum, causes a magnetization anti-parallel to ω, with the module [3]

$$M = N\mu L(a),$$

where N is the number of momentum for volume unit and $L(a)$ is the Langevin function:

$$L(a) = \cosh(a) - \frac{1}{a} \qquad \text{and} \qquad a = \frac{\omega l}{kT} .$$

For $a \ll 1$ the Langevin function became $L(a) \approx a/3 = \omega l/(3kT)$.

Such approximation is satisfied for temperatures of the order $T \sim 10^2$ K when the angular velocity is of order $\omega \ll 10^{13}\ \mathrm{s}^{-1}$, that is to say even in the case of Monstein experiments even in the case of the Earth's inner core this approximation could be applied. Then the magnetization caused by the rotation would be

$$\text{(14)} \qquad \begin{cases} M = \dfrac{N\mu\omega l}{(3kT)} , & \left(l = \mu g = \dfrac{\mu m_{\mathrm{e}}}{e} \right), \\ M = \dfrac{N l^2 \omega e}{3kTm_{\mathrm{e}}} . \end{cases}$$

It results that the magnetization is proportional to the angular velocity and depends on the temperature. It does not depend on the rotating body shape. Applying this formula to the case of the Monstein experiments [1], where $\omega \sim 10^3\ \mathrm{s}^{-1}$, $(l \sim 10^{-34}\ \mathrm{J\,s})$, we find

$$M \sim 10^{28} 10^{-68} 10^{3} 10^{-19} / (10^{-20} 10^{-31}) \sim 10^{-5}\ \mathrm{A/m} .$$

Such magnetization would create a magnetic field of order

$$B = \mu_0 M \sim 4\pi 10^{-7} 10^{-5} \sim 10^{-11}\ \mathrm{T} ,$$

while according to Marinov [1], the observed field was of order $10\mu\mathrm{T} = 10^{-5}\ \mathrm{T}$. Applying the result to the case of the Earth's inner core, we would find the dipolar magnetic momentum caused by the rotation ($\omega \sim 10^{-4}\ \mathrm{s}^{-1}$):

$$M_{\mathrm{dip}} = M \cdot V = N l^2 (\omega/3kT)(e/m_{\mathrm{e}})(4/3\pi R^3) \sim 10^6\ \mathrm{A/m^2} .$$

While it is known from the Earth's magnetic field observation that the dipolar magnetic momentum of the Earth is $M_{\mathrm{dip}} \approx 8 \cdot 10^{22}\ \mathrm{A/m^2}$. This value could be not achieved by the above analyzed effect.

So, both mechanisms of the magnetic field generation cannot explain either the Monstein experiment results (if they are exact) or the origin of the dipolar magnetic field of the Earth. But those effects, mostly the Barnett effect, could incite a dipolar magnetic field which is strengthened by the fluid flow in the outer core of the Earth. According to the dynamo theory of the Earth's magnetic field generation, a small seed magnetic field would be sufficient for the dynamo action: the observed magnetic field could be generated and maintained by the fluid flow in the outer core [4, 5].

Before entering the dynamo theory problems, let us study first the fluid flow in a spherical shell simply without the presence of magnetic field, which is a hydrodynamic problem.

3. – The fluid motion between two rotating spheres

a) *The spheres rotate about the same axis.*

Firstly, let us consider that the inner solid sphere with radius R_1 rotates with angular velocity Ω_1 about the Z-axis and the outer solid sphere $r > R_2$ rotates with angular velocity Ω_2 about the same axis Z. If the fluid which fills the spherical shell $R_1 < r < R_2$, were ideal (the viscosity coefficient $\nu = 0$) then it would be not excited by such a rotation. We will consider the fluid incompressible but having a viscosity. Then it will be driven in motion with boundary conditions that in spherical coordinates are

$$\begin{cases} \text{in } r = R_1: & v_r = 0,\ v_\theta = 0; \quad v = v_\phi = \Omega_1 R_1 \sin\theta, \\ \text{in } r = R_2: & v_r = 0,\ v_\theta = 0; \quad v = v_\phi = \Omega_2 R_2 \sin\theta. \end{cases} \tag{15}$$

From such boundary conditions there results that $v_r = 0$, $v_\theta = 0$ in the whole fluid volume. From the symmetry we have: $\partial v_\phi / \partial\phi = \partial v / \partial\phi = 0$. Writing the Navier-Stokes equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\varrho}\nabla p + \nu\,\Delta\mathbf{v} \tag{16}$$

in spherical coordinates [6]:

$$\begin{aligned} &\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{(v_\theta^2 + v_\phi^2)}{r} = \\ &\quad = -\frac{1}{\varrho}\frac{\partial p}{\partial r} + \nu\left[\frac{1}{r}\frac{\partial^2 (r v_r)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} + \right.\\ &\qquad \left. + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi} - \frac{2v_r}{r^2} - \frac{2\cot\theta}{r^2}v_\theta\right], \end{aligned} \tag{17}$$

$$\begin{aligned} &\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r} = \\ &\quad = -\frac{1}{\varrho r}\frac{\partial p}{\partial\theta} + \nu\left[\frac{1}{r}\frac{\partial^2 (r v_\theta)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \right.\\ &\qquad \left. + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2\sin^2\theta}\right], \end{aligned} \tag{18}$$

$$\begin{aligned} &\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi\cot\theta}{r} = \\ &\quad = -\frac{1}{\varrho r\sin\theta}\frac{\partial p}{\partial\phi} + \nu\left[\frac{1}{r}\frac{\partial^2 (r v_\phi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} + \right.\\ &\qquad \left. + \frac{\cot\theta}{r^2}\frac{\partial v_\phi}{\partial\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{v_\phi}{r^2\sin^2\theta}\right], \end{aligned} \tag{19}$$

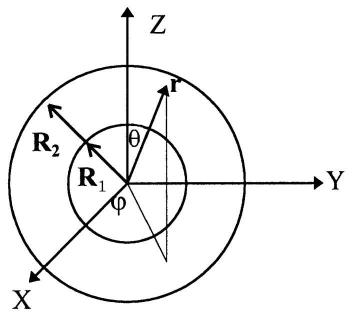

Fig. 1. – Two spheres rotate about the same axis.

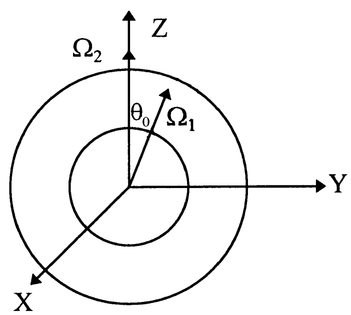

Fig. 2. – Two spheres rotate about different axes.

and the continuity equation

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{2v_r}{r} + \frac{v_\theta \cot\theta}{r} = 0\,, \tag{20}$$

then for a stationary flow (from eqs. (17) and (18)) we have

$$\frac{v^2}{r} = -\frac{1}{\varrho}\frac{\partial p}{\partial r} \quad \text{and} \quad \frac{v^2\cot\theta}{r} = -\frac{1}{\varrho r}\frac{\partial p}{\partial \theta}\,.$$

From these equations, the pressure field can be found when the velocity field is known. While eq. (19) when $\partial p/\partial\phi = 0$ became

$$\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_\phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v_\phi}{\partial \theta} - \frac{v_\phi}{r^2\sin^2\theta} = 0\,. \tag{21}$$

We have searched the solution of this equation in the form $v = v_\phi = cr^n\sin\theta$, where c is a constant. Putting this solution into eq. (21), we found $n = 1$, $n = -2$.

So, the solution of eq. (21) should be

$$v = \frac{a\sin\theta}{r^2} + br\sin\theta\,, \tag{22}$$

where the constants a and b could be determined by the boundary conditions (15):

$$a = \frac{(\Omega_1 - \Omega_2) R_1^3 R_2^3}{R_2^3 - R_1^3}\,, \qquad b = \frac{\Omega_2 R_2^3 - \Omega_1 R_1^3}{R_2^3 - R_1^3}\,.$$

The solution (22) for the velocity is

$$v = v_\phi = \frac{(\Omega_1 - \Omega_2) R_1^3 R_2^3}{R_2^3 - R_1^3} \frac{\sin\theta}{r^2} + \frac{\Omega_2 R_2^3 - \Omega_1 R_1^3}{R_2^3 - R_1^3} r \sin\theta\,. \tag{23}$$

By this way, the velocity field in the spherical shell in the stationary conditions has only an azimuth component that depends only on the coordinates r and θ. This field is axisymmetric, *i.e.* it has the symmetry of rotation about the Z-axis.

b) *The spheres rotate about different axes*: Ω_1 *and* Ω_2.

Choosing Ω_2 along the Z-axis, then the boundary condition at $r = R_2$ could be written as

$$v_r = 0\,, \qquad v_\theta = 0\,, \qquad v = v_\phi = \Omega_2 R_2 \sin\theta\,. \tag{24}$$

While at $r = R_1$, we have: $\mathbf{v} = \mathbf{\Omega}_1 \times \mathbf{R}_1$, or in spherical coordinates, when $\mathbf{\Omega}_1$ is in the YZ plane with polar angle θ_0, we have

$$v_r = 0\,; \quad v_\theta = \Omega_1 R_1 \sin\theta_0 \cos\phi\,; \quad v_\phi = \Omega_1 R_1 (\cos\theta_0 \sin\theta - \sin\theta_0 \cos\theta \sin\phi)\,. \tag{25}$$

As $v_r = 0$ at both boundaries, we can search the solution with $v_r = 0$ in the whole volume. Also, as v_θ does not depend on θ at both boundaries, we can consider $\partial v_\theta / \partial\theta = 0$ in the whole volume. Although Navier-Stokes equations are not linear, we will find the solution imagining that the rotation with angular velocity Ω_1 could be divided into one rotation with the angular velocity $\Omega_1 \cos\theta_0$ about the Z-axis (parallel to Ω_2) and one rotation with angular velocity $\Omega_1 \sin\theta_0$ about the Y-axis (perpendicular to Ω_2). The first rotation brings only azimuth velocity (at $r = R_1$):

$$v_{\phi 1} = \Omega_1 R_1 \cos\theta_0 \sin\theta\,.$$

While the second rotation brings even azimuth velocity:

$$v_{\phi 2} = -\Omega_1 R_1 \sin\theta_0 \cos\theta \sin\phi\,,$$

and even meridian velocity, at $r = R_1$:

$$v_\theta = \Omega_1 R_1 \sin\theta_0 \cos\phi\,.$$

The rotations with angular velocities $\Omega_1 \cos\theta_0$ and Ω_2 about the same axis bring on the solution found in the case a). Substituting to the solution (23) Ω_1 with $\Omega_1 \cos\theta_0$, then we find the solution:

$$v_{\phi 1} = \frac{(\Omega_1 \cos\theta_0 - \Omega_2) R_1^3 R_2^3}{R_2^3 - R_1^3} \frac{\sin\theta}{r^2} + \frac{\Omega_2 R_2^3 - \Omega_1 \cos\theta_0 R_1^3}{R_2^3 - R_1^3} r \sin\theta\,,$$

that satisfies the boundary conditions at $r = R_1$ and at $r = R_2$. While the rotation with angular velocity $\Omega_1 \sin\theta_0$ about the axis perpendicular to Ω_2 has the boundary conditions

$$\text{at} \quad r = R_1, \quad v_{\phi 2} = -\Omega_1 R_1 \sin\theta_0 \cos\theta \sin\phi ,$$
$$\text{and at} \quad r = R_2, \quad v_{\phi 2} = 0 .$$

Therefore we search the solution $v_{\phi 2}$ in the form

$$v_{\phi 2} = -\frac{\Omega_1 R_1^3 R_2^3}{R_2^3 - R_1^3}\left(\frac{1}{r^2} - \frac{r}{R_2^3}\right)\sin\theta_0 \cos\theta \sin\phi$$

and the whole solution in the form

$$v_\phi = \frac{(\Omega_1 \cos\theta_0 - \Omega_2)\, R_1^3 R_2^3}{R_2^3 - R_1^3}\,\frac{\sin\theta}{r^2} + \frac{\Omega_2 R_2^3 - \Omega_1 \cos\theta_0 R_1^3}{R_2^3 - R_1^3}\, r \sin\theta - \\ - \frac{\Omega_1 R_1^3 R_2^3}{R_2^3 - R_1^3}\left(\frac{1}{r^2} - \frac{r}{R_2^3}\right)\sin\theta_0 \cos\theta \sin\phi . \tag{26}$$

From the continuity equation (20), after substituting $\partial v_\theta / \partial\theta = 0$, we find

$$\frac{1}{r\sin\theta}\,\frac{\partial v_\phi}{\partial\phi} + \frac{v_\theta \cot\theta}{r} = 0 .$$

Substituting here the derivative of v_ϕ from (26) we found the solution of v_θ:

$$v_\theta = \frac{\Omega_1 R_1^3 R_2^3}{R_2^3 - R_1^3}\left(\frac{1}{r^2} - \frac{r}{R_2^3}\right)\sin\theta_0 \cos\phi \tag{27}$$

which satisfies both boundary conditions at $r = R_1$ and at $r = R_2$. As the Navier-Stokes equations with given boundary conditions have the unique solution [7], then we can say that the found solution is the unique solution. The same result is taken by [6] in another way supposing the linearity of Navier-Stokes equation (neglecting the second term of the left side of eq. (16)). Somehow even here is used the superposition of two motions, which is valid under the conditions of linearity.

4. – A similar model of the liquid core of the Earth

From the above-analyzed models, which one would be suitable for the liquid core of the Earth?

Up to now it was considered that the inner core and the mantle of the Earth rotate about the same axis with the same angular velocity. In the last years, the seismic data brought to the supposition of the existence of a difference between the angular velocities of inner core and mantle. According to the evaluation of Song and Richard [8], the rotation of inner core relative to the mantle is 1.1° per year, while Su *et al.* [8] have evaluated it to be about 3° per year and Glatzmaier and Roberts [9] have calculated it ≈2.6° per year.

The rotation of the inner core relative to the mantle explained the "westward drift" of the non-dipolar geomagnetic field observed on the Earth's surface which is evaluated to be 0.2° per year [6]. It seems as if the outer liquid core not only could reinforce the magnetic field generated by the rotation of the inner core but could transmit up to core-mantle boundary the properties of this field: the dipolar property and "westward drift" relative to the mantle.

Another interesting fact found by the seismic observations, is that the inner core has an anisotropy in three dimensions, having an axis of symmetry which is tilted by $10.5° \pm 1°$ from the Earth's rotation axis in the direction $160°\mathrm{E} \pm 5°$ in the Northern hemisphere [10]. The interesting thing is that the tilt of geomagnetic dipolar axis from the Earth's rotation axis is near this value. According to Barton [11], the geomagnetic axis was tilted by 11.54° N from the geographic axis in 1955, was tilted by 10.7° N in 1995 and is projected to be tilted by 10.5° in the year 2000. But the geographic longitude (relative to the mantle) of the geomagnetic axis is 289° E, while that of symmetry axis of the inner core is evaluated to be $160°\ \mathrm{E} \pm 5°$ [10], so both axes are quite opposite in longitude. A strong correlation was noticed between the non-dipole magnetic potential and the potential of gravitational fluctuations over the Earth surface (Hide and Malin, 1965, according to [5]) and the shift of the geomagnetic potential required to maximize the correlation was 160° to the East. Such a shift increases in time at a rate of about 2.27° per year [5], then it would be about 227° E in the year 1995, while now the longitude shift of the geomagnetic axis relative to symmetry axis of the inner core is about 129°.

If we suppose that the rotation of the inner core is about its axis of symmetry and that the dipolar moment is directed along this axis, then how would such a movement toward East (about 130°) of the North Pole during the pass from the bottom boundary of the liquid core to its upper boundary (core-mantle boundary) be explained? Is it possible that the fluid flow in the outer core be able to carry such a movement? It seems too difficult to answer such questions, therefore I have chosen the model of case a) as more realistic for the liquid core of the Earth.

5. – Discussion about the solution of the kinematic equation

In the dynamo theory it is proved that the fluid flow in the outer core of the Earth can regenerate the magnetic field [4, 5]. Nevertheless, there is not an analytical solution of the kinematic dynamo equation [4, 5]:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta\, \Delta \mathbf{B}\,. \tag{28}$$

It seems that when the conducting fluid flows across the magnetic lines of force, the resulting electromotive force $\mathbf{v} \times \mathbf{B}$ will drive a current against Ohmic losses (the second term of eq. (28)). Under the conditions of high conductivity σ of the liquid core ($\eta = 1/\mu_0 \sigma$, $\eta \ll 1$) (the "frozen flux hypotheses") and appropriate conditions concerning the symmetry of the turbulent motions, the small-scale motions generate a large-scale electromotive force parallel to the large-scale magnetic field $\langle \mathbf{B} \rangle$ and proportional to it (the so-called dynamo α-effect, where the e.m.f. is: $\varepsilon = \langle \mathbf{v} \times \mathbf{B} \rangle = \alpha \langle \mathbf{B} \rangle$) [4].

To such results came the research on dynamo theory, as there was a restriction derived from the Cowling theorem: an axisymmetric magnetic field cannot be maintained by a self-sustaining dynamo and from the Bullard and Gellman antidynamo theorem according to which in an incompressible fluid ($\operatorname{div} \mathbf{v} = 0$) with the radial component of the velocity zero ($v_r = 0$) and with magnetic diffusivity η constant,the radial component B_r of magnetic field will die away.

Here, I will discuss about the solution of kinematic and dynamic equations, without claiming the exact solution of such equations.

For the model of liquid core analogue to the case a), we will analyze eq. (28) when the term $-\eta\,\Delta\mathbf{B}$ is negligible, *i.e.* in the conditions of "frozen flux hypothesis" when the magnetic lines of force are "frozen" to the fluid and move with it. Then the eq. (28) became

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{29}$$

The presence of gravitational field and magnetic field changes the hydrodynamic eq. (16) to

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{\nabla p}{\varrho} + \nu\,\Delta\mathbf{v} + \mathbf{g} + \frac{1}{\varrho\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}\,. \tag{30}$$

As $\mathbf{g}$ has only the radial component $g = g_r = g_0(r/R_{\mathrm{E}})$ (where g_0 is the acceleration of the free fall on the Earth surface $r = R_{\mathrm{E}}$), it will contribute only to the radial dependence of the pressure but will not change the solution (23), while the Lorentz force term $(1/\varrho\mu_0(\nabla \times \mathbf{B}) \times \mathbf{B}$ could be considered too small (for the magnetic field in the liquid core that is thought to have gradients of order 10^{-3}T, the gradient of magnetic pressure B^2/μ_0 divided by ϱ is of order 10^{-3}, that is too small in comparison to g). Therefore I will consider that in the liquid core the solution (23) for $\mathbf{v}$ is valid, then the cross product $\mathbf{v} \times \mathbf{B}$ in spherical coordinates, has the components

$$(\mathbf{v} \times \mathbf{B})_r = -v_\phi B_\theta\,; \quad (\mathbf{v} \times \mathbf{B})_\theta = v_\phi B_r\,; \quad (\mathbf{v} \times \mathbf{B})_\phi = 0$$

and the projections of the eq. (29) in spherical coordinates are

$$\begin{cases} \dfrac{\partial B_r}{\partial t} = -\dfrac{1}{r\sin\theta}\,\dfrac{\partial(v_\phi B_r)}{\partial\phi}\,, \\[2ex] \dfrac{\partial B_\theta}{\partial t} = \dfrac{1}{r\sin\theta}\,\dfrac{\partial(-v_\phi B_\theta)}{\partial\phi}\,, \\[2ex] \dfrac{\partial B_\phi}{\partial t} = -\dfrac{1}{r}\left[\dfrac{\partial(r v_\phi B_r)}{\partial r} - \dfrac{\partial(-v_\phi B_\theta)}{\partial\theta}\right]. \end{cases} \tag{31}$$

In order to find the solution of eqs. (31) we have to know the boundary conditions at $r = R_1$ and at $r = R_2$. If we approve the idea that the magnetic field in the solid inner core is caused by the rotation (the Barnett effect) and that there are not superficial electric currents at $r = R_1$ (there is continuity of B_r, B_θ, B_ϕ) then the boundary conditions at $r = R_1$ can be taken as

$$B_r = -\mu_0 M\cos\theta\,; \quad B_\theta = \mu_0 M\sin\theta\,; \quad B_\phi = 0\,, \tag{32}$$

where M is the known magnetization (14). If such field were continued outside the sphere R_1 it would present a typical poloidal field. According to the Backus theorem [4]: If no current crosses a spherical surface, then the magnetic field on that surface is purely poloidal.

We have also considered that the magnetic field inside the inner core is not influenced by the flow in the outer core. Although there is a strong magnetic coupling between inner and outer cores [12], it is considered that the field of the outer core cannot penetrate deep into the inner core.

The boundary conditions at $r = R_2$ could be found by the downward continuation of the observed field on the Earth surface. Such continuation up to mantle-core boundary could be done if there are not superficial currents in this boundary and if the mantle is non-conductive. The magnetic field on the Earth surface can be expressed as the sum of spherical harmonics [4]. Taking only the dipolar term of spherical harmonics expansion, then at $r = R_2$, an observer on the Earth has

$$\left\{\begin{aligned} B_r &= 2\left(\frac{R}{R_2}\right)^3 (g_1^0 \cos\theta + g_1^1 \sin\theta\cos\phi + h_1^1 \sin\theta\sin\phi)\,, \\ B_\theta &= \left(\frac{R}{R_2}\right)^3 (g_1^0 \sin\theta - g_1^1 \cos\theta\cos\phi - h_1^1 \cos\theta\sin\phi)\,, \\ B_\phi &= \left(\frac{R}{R_2}\right)^3 (g_1^1 \sin\phi - h_1^1 \cos\phi)\,, \end{aligned}\right. \tag{33}$$

where R is the outer radius of the mantle ($R = 6371.2$ km), R_2 is the outer radius of the liquid core ($R_2 \approx 3477$–3550 km, see the discussion of [12]); g_1^0, g_1^1, h_1^1 are the Gauss coefficients of degree $n = 1$ [4].

While for an observer on the rest referential, instead of the angle ϕ there should be $\phi + \Omega_2 t$, where t is the time and Ω_2 is the angular velocity of the mantle.

Without searching for the analytic solution, we can find some qualitative conclusion. In the case of axisymmetric field $\partial B_\phi/\partial\phi = 0$), from the equation div $\mathbf{B} = 0$ in spherical coordinates,we find

$$\frac{1}{r}\,\frac{\partial(r^2 B_r)}{\partial r} + \frac{1}{\sin\theta}\,\frac{\partial(\sin\theta B_\theta)}{\partial\theta} = 0\,. \tag{34}$$

In the stable state ($\partial\mathbf{B}/\partial t = 0$), having the velocity field (23), from the two first eqs. (31) we find $\partial B_r/\partial\phi = 0$ and $\partial B_\theta/\partial\phi = 0$; while, from the third eq. (31) and eq. (34), we find

$$0 = -2B_r\,\frac{(\Omega_1 - \Omega_2)\,R_1^3 R_2^3}{R_2^3 - R_1^3}\,\frac{\sin\theta}{r^2}\,,$$

that means $B_r = 0$. This conclusion is in agreement with the Cowling theorem, so an axisymmetrical magnetic field could not be stable.

In the condition of non-stability, eqs. (31) where the velocity (23) is substituted, lead to the equations from which one can see that the poloidal field can generate the toroidal field simply by the differential rotation in the liquid core. Such conclusion is now approved in the dynamo theory [4, 5, 9, 13, 14]. But these equations do not tell us that the poloidal field could be generated by the toroidal field.

Our model could not explain the generation of a poloidal field from a toroidal field, because we have not considered the thermal and compositional buoyancy, which incite the radial component of fluid velocity in the liquid core. Such velocity components should be zero at the boundary of the spherical shell.

In the presence of convection, the continuity equation (19) should be

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0 , \tag{35}$$

where $\operatorname{div}\mathbf{v} \neq 0$, and in the stable conditions and spherical coordinates it would be

$$\varrho \frac{\partial v_r}{\partial r} + \frac{\varrho}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\varrho}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{2\varrho v_r}{r} + \frac{\varrho v_\theta \cot\theta}{r} + v_r\frac{\partial \varrho}{\partial r} + \frac{v_\phi}{r\sin\theta}\frac{\partial \varrho}{\partial \phi} = 0 . \tag{36}$$

Now, in this equation only derivatives of ϱ relative to θ and ϕ could be zero but not its derivative relative to r, therefore the equations would be much more complicated.

The recent models of dynamo theory accept the differential rotation in the liquid core, but they consider it as a consequence of magnetic interaction between core and mantle or as a consequence of anisotropy of convective flow in the outer core [14]. The convection in the spherical shell should be non-axisymmetrical in order to generate a magnetic field with strong axisymmetrical components [14]. The models of convective flow must always have radial component zero at the boundaries of the spherical shell, like in the model of Matsushima [15]. Boundary conditions maintain the buoyancy gradients that would drive the dynamo action. For a dynamo driven by buoyancy force, the flow must have a radial component for the buoyancy force to be able to work on it. Busse [14] has derived a lower bound of this radial component necessary for dynamo action.

As concerns the amplification of magnetic field from the fluid motion, although we have not got the calculations, we would like to mention the Hollerbach reasoning [16] that the **B** lines may be stretched and thereby amplified. The tension in the field lines resists the kinematic stretching and by doing work against this elastic tension the flow pumps energy into the field [16]. It is also possible that the differential azimuth motion of fluid in outer core, stretching the poloidal field lines could amplify the field realizing the transform of mechanic energy into the magnetic energy.

I think that the generation of the magnetic field from the rotation of the inner core could be a continuous source of magnetic field and energy that would compensate the Ohmic losses of energy in the liquid core. Knowing that the Ohmic losses in the outer core are of order 10^{11} J/s and that the mechanic energy of rotation is of order 10^{28} J, it would need a time of order $10^{17}\ \mathrm{s} \approx 10^9$ years for the Earth's rotation to die away. This time is of the order of the Earth's formation, so we are sure that it would not have happened by this energetic balance.

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