The onset of thermal convection in an infinite Prandtl number, variable-viscosity, compressible Earth's mantle by the mean-field approximation

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Summary. — The values of the Rayleigh number critical for the onset of thermal instability in an infinite Prandtl number, compressible Earth's mantle are determined against the aspect ratio (the ratio between the width and the depth of the cell) for constant and for pressure- and temperature-dependent viscosity. The procedure adopted follows the experimental approach which determines the critical Rayleigh number as the value at which the heat flux across the fluid exceeds the conductive one. We solve the single-mode mean-field equations for the conservation of mass, momentum and energy and evaluate the Nusselt number, *i.e.* the ratio between the actual heat transfer and the heat which would be transported only by conduction. The value of the Rayleigh number for which the Nusselt number exceeds unity can be identified as the critical value. Between incompressible and compressible fluids with constant viscosity the critical Rayleigh numbers differ at most by 10–20% for cells aspect ratios between 0.1 to about 30, while at larger aspect ratios compressibility yields critical Rayleigh numbers one or more orders of magnitude larger. For variable viscosity, since a more viscous interior prevents motion, a much higher Rayleigh number is required for the onset of convection. It is also shown how viscous dissipation of heat increases the critical Rayleigh number, while compressibility favours the onset of convection

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1. – Introduction

In the recent years it has been shown that a satisfactory picture of the thermal convection in the Earth's mantle cannot be achieved by modelling the terrestrial mantle fluid as a simple incompressible, Boussinesq fluid, which neglects any density variation except for the buoyancy term. In fact, in addition to large viscosity variations due to temperature and pressure dependence, viscous dissipation and adiabatic heating phenomena certainly occur in the mantle, and it has been demonstrated that their effects are so relevant that they cannot be disregarded [1-4]. The study of viscous dissipation and adiabatic heating processes in fluids endowed with temperature- and pressure-dependent viscosity has been shown to produce a strong non-linear coupling between the thermodynamic and creep parameters, which is of foremost interest in geophysical studies. In fact, this trade-off allows to impose bounds on the activation volume and activation energy which are not well constrained by experimental measures, due to the lack of laboratory data at high temperature and pressure. Because of the complexity of the fluid flow, numerical methods are then the most direct and appropriate approach to mimic the basic physics of mantle convection.

In spite of the fact that a good deal of numerical research has been conducted for high Rayleigh number convective patterns in variable viscosity and non-Boussinesq fluids [3-5], a study of the onset of convection in mantle-like fluids, *i.e.* infinite Prandtl number, compressible fluids with temperature- and pressure-dependent viscosity, has not been treated extensively. The traditional procedure for the determination of the marginal states of a hydrodynamic stability grounds on the so-called linear stability theory [6], which has been widely applied for Rayleigh-Bénard convection, in simple cases as well as in the presence of rotation and of a magnetic field [6]. The approach we adopt here is in principle analogous to the observation of the instability made in laboratory experiments by Schmidt and Milverton [7], Saunders *et al.* [8], Schmidt and Saunders [9], and later with more accuracy by Silveston [10]. In fact, for fixed aspect ratios, we repeat solving the coupled equations for the conservation of mass, momentum, and energy, each time increasing the Rayleigh number from a subcritical value, until we "observe" the onset of convection. We adopt the experimental principle of Schmidt and his coworkers for the detection of the onset of thermal instability and thus for the determination of the critical Rayleigh number. This criterion is based on the measure of the Nusselt number, *i.e.* the ratio of the heat actually transported and the heat transferred only by conduction when the fluid is at rest. In the absence of convection, the Nusselt number is equal to 1 and, as the Rayleigh number reaches its critical value, it assumes values grater than unity quite abruptly.

Since our procedure may require several numerical solutions, the conservation equations are written in the single-mode, mean-field approximation, which has been widely shown to produce correct physical results for fully developed convection in incompressible fluids with constant [11] and variable viscosity [12-15] and in variable-viscosity anelastic fluids [2]. Furthermore, in this paper, its reliability is also ascertained for low Rayleigh numbers comparing the mean-field and the analytical stability curves for a constant-viscosity, incompressible fluid.

The critical Rayleigh numbers *vs.* the aspect ratio are then determined for a constant-viscosity and for variable-viscosity compressible fluid.

2. – The single-mode mean-field equations

The mean-field approximation for thermal convection was first developed by Herring [16] and by Roberts [17] for finite Prandtl number fluids. It has been widely used by astrophysicists for studying stellar convection [18-21], since it allows a much simpler and faster numerical simulation of convective processes than three- or even two-dimensional models, which can be fairly time consuming.

The basic feature of the mean-field approximation is the assumption that the temperature field may be decomposed into a horizontally averaged component, $\overline{T}$, and a fluctuating component, θ. This thermal component, θ, and the velocity field, $\vec{v}$, are in turn expanded in terms of a complete set of functions specifying the horizontal planform, which has to be prescribed *a priori*. The requirement of mass conservation imposed by the continuity equation must be satisfied by the mean-field variables. The single-mode, mean-field approximation consists in retaining only the first term of the expansion. The derivation of the single-mode, mean-field equation for a Boussinesq fluid can be found in the papers by Quareni and Yuen [11] and by Quareni *et al.* [14], for constant and variable viscosity, respectively.

The following conservation equations govern the steady-state thermal convection for a layer of thickness d of an infinite Prandtl number, compressible fluid in a system of Cartesian coordinates, with the vertical z-axis pointing downward [6]:

$$\nabla(\varrho \vec{v}) = 0 , \tag{1}$$

$$\varrho \vec{v} \nabla \vec{v} = - \vec{\nabla} P + \varrho \vec{g} + \frac{\partial \tau_{ij}}{\partial x_j} , \tag{2}$$

$$\varrho C_p \vec{v} [\vec{\nabla} T - (\vec{\nabla} T)_S] = \nabla(k \vec{\nabla} T) + \tau_{ij} \frac{\partial v_i}{\partial x_j} , \tag{3}$$

where $\vec{v}$ is the velocity, T the temperature, τ_{ij} the stress tensor, P the pressure, $\vec{g}$ the acceleration of gravity, ϱ the reference state density (hydrostatic pressure and adiabatic temperature distribution), k the thermal conductivity, and C_p the isoentropic heat capacity.

The conservation equations (1)-(3) are then set non-dimensional according to the following scheme:

$$\begin{cases} (x', z') = (x, z)/d \,; & T' = (T - T_0)/\Delta T \,; & \vec{v}\,' = \vec{v}\, d/k_0; \\ P' = P d^2 \varrho_0 C_p / \eta_\infty k_0; & \tau'_{ij} = \tau_{ij} d^2 \varrho_0 C_p / \eta_\infty k_0; & k' = k/k_0; \\ \varrho' = \varrho/\varrho_0; & \alpha' = \alpha/\alpha_0; & K'_T = K_T/K_{T0}, \end{cases} \tag{4}$$

where primes denote the non-dimensional quantities and will be dropped in the following, the subscript "0" indicates values taken at the top of the layer, x and z are the horizontal and the vertical coordinate, respectively, ΔT is the temperature difference across the layer, α the coefficient of thermal expansion, K_T the isothermal bulk modulus, η the dynamic viscosity, and η_∞ its characteristic value.

Since the mean-field theory can handle only linear stress-strain relationships, we consider a Newtonian rheology:

$$\tau_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right). \tag{5}$$

In the single-mode, mean-field approximation the steady-state temperature field and the vertical and horizontal components of velocity, w and u, respectively, are written as

$$T(x, z) = \overline{T}(z) + \theta(z)\, f(x) , \tag{6}$$

(7) $$w = \frac{1}{\varrho(z)} W(z)\, f(x)\,,$$

(8) $$u = \frac{1}{k_A^2} \frac{1}{\varrho(z)} \frac{\mathrm{d}W}{\mathrm{d}z} \frac{\mathrm{d}f}{\mathrm{d}x}\,,$$

where $\varrho(z)$ is the depth-dependent density, and the planform function $f(x)$ is $\sqrt{2}\cos(k_A x)$ for rolls along the y-direction of aspect ratio $a = \pi/k_A$ (the aspect ratio being defined as the ratio between the width and the depth of the cell, and k_A being the wave number). The continuity equation (1) is satisfied by the choice of (7) and (8).

The mean-field variables, $\overline{T}$, θ, and W, are substituted into the non-dimensional form of the conservation equations (2)-(3). After horizontal averaging and after eliminating the pressure and horizontal components of velocity, we obtain a fourth-order momentum equation and two second-order energy equations:

(9) $$\begin{aligned} Y' + \frac{\eta'}{\eta} Y + \mathrm{Ra}_0\, \alpha k_A^2 \theta \frac{\varrho^2}{\eta} + k_A^2 W - k_A^2 C \frac{\eta'}{\eta} W + \frac{1}{3} k_A^2 C \frac{\varrho'}{\varrho} W - \\ -2k_A^2 \frac{\varrho'^2}{\varrho^2} W + k_A^2 C W' - 3k_A^2 \frac{\eta'}{\eta} W' + 3k_A^2 \frac{\varrho'}{\varrho} W' + C \frac{\varrho'}{\varrho} \frac{\eta'}{\eta} W' - \\ -2C \frac{\varrho'^2}{\varrho^2} W' + 3 \frac{\eta'}{\eta} \frac{\varrho'^2}{\varrho^2} W' - 6 \frac{\varrho'^3}{\varrho^3} W' + k_A^2 \frac{\varrho''}{\varrho} W + C \frac{\varrho''}{\varrho} W' - \\ - \frac{\varrho''}{\varrho} \frac{\eta'}{\eta} W' + 6 \frac{\varrho' \varrho''}{\varrho^2} W' - 2k_A^2 W'' + C' W'' + 2C \frac{\varrho'}{\varrho} W'' + \\ +3 \frac{\varrho'^2}{\varrho^2} W'' - \frac{\varrho'''}{\varrho} W' = 0\,, \end{aligned}$$

(10) $$(kT')' = C_p(W'\theta + \theta' W) - D\alpha\theta W - \Phi\,,$$

(11) $$(k\theta')' = C_p W T' - D\alpha(T + T_0)\, W + k_A^2 k\theta\,,$$

where primes denote derivatives with respect to z, and

(12) $$Y = W''' + \left(-C - 3\frac{\varrho'}{\varrho} + \frac{\eta'}{\eta} \right) W'' - \frac{\varrho'}{\varrho} \frac{\eta'}{\eta} W' + k_A^2 \frac{\eta'}{\eta} W\,,$$

(13) $$C = \frac{D}{\gamma} \frac{\varrho(z)}{K_T(z)}\,,$$

(14) $$\begin{aligned} \Phi = \frac{D\eta}{\mathrm{Ra}_0\, \varrho^2} \Bigg[4W'^2 - 6W'W \frac{\varrho'}{\varrho} + \frac{W''^2}{k_A^2} + \frac{W'^2}{k_A^2} \frac{\varrho'^2}{\varrho^2} + \\ + W^2 k_A^2 - \frac{2W''W'}{k_A^2} \frac{\varrho'}{\varrho} + 2W''W + \frac{4}{3} W^2 \frac{\varrho'^2}{\varrho^2} \Bigg], \end{aligned}$$

where $D = dg\alpha_0/C_p$ is the dissipation number, γ the Grüneisen parameter, and T_0 the adimensional value of the top temperature (scaled with respect to ΔT) which is an intrinsic parameter in convection with adiabatic heating [22]. The absolute temperature at the top surface has to be intended as the temperature at the basis of the lithosphere, which is not included in the model in order to avoid stiffness problems. In this paper, T_0 corresponds to a dimensional value of 1100 K, for the sake of consistency with previous works, where it has been used to avoid the stiffness produced by a cold lithosphere. However, Quareni and Yuen [5] demonstrate that even halving the value of T_0 does not lead to significant differences. The term Φ defined in eq. (14) represents the mean-field expression for viscous dissipation. The Rayleigh number, Ra_0, is defined as

$$\mathrm{Ra}_0 = \frac{\varrho_0 g \alpha_0 d^3 \Delta T}{k_0^2 \eta_\infty}, \tag{15}$$

where η_∞ is a reference value of the viscosity.

Impermeability and stress-free boundary conditions are imposed at the top and bottom of the convecting layer by

$$\begin{cases} W|_{z=0} = 0, & W|_{z=1} = 0, \\ W''|_{z=0} = 0, & W''|_{z=1} = 0. \end{cases} \tag{16}$$

The thermal boundary conditions prescribe the top temperature and the heat flux across the bottom:

$$\begin{cases} T|_{z=0} = 0, & \theta|_{z=0} = 0, \\ kT'|_{z=1} = 1, & \theta'|_{z=1} = 0. \end{cases} \tag{17}$$

The efficiency of convection in transporting heat is measured by the Nusselt number defined as

$$\mathrm{Nu} = \frac{\langle T\rangle_\mathrm{c}}{\langle T\rangle}, \tag{18}$$

where brackets indicate average over the layer and the subscript "*c*" denotes the conductive state.

The mean-field conservation equations can be considered as two coupled fourth-order boundary value problems: one consists of the momentum equation (9) with the boundary conditions (16), the other of the energy equations (10) and (11) with the boundary conditions (17). In order to solve the momentum equation, it is useful to introduce the variable Y which, according to eq. (12), allows to eliminate the second derivative of the viscosity, leads to greater numerical stability, and speeds up the computations. The two boundary value problems are solved by a variable order, variable step-size finite-difference method, according to an iterative scheme, with the momentum equation leading [14].

3. – Rheology and density

The definition of a rheological law appropriate for terrestrial mantle is still an open problem [23-27]. From micro-rheological models a power law relating the strain rate and the stress, $\dot{\varepsilon} \propto \tau^n$, is often proposed, with n from 1 to 5. Since mean-field approximation can only handle linear rheologies, we choose a Newtonian behaviour and consider the convective flow as a steady-state creep phenomenon.

The relationship between stress and strain rate then reads:

$$\tau_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k} \right) + \zeta \delta_{ij} \frac{\partial v_k}{\partial x_k}\,, \tag{18}$$

where η is the dynamic viscosity, which is a function of temperature and pressure, and ζ is the secondary bulk viscosity, which depends on the chemical nature of fluid as well as on temperature and pression. However, ζ, which is related with isotropic compression, can be neglected with respect to η, since it is 6 to 7 orders of magnitude smaller for mantle type of materials.

The thermally activated process in mantle creep is heavily conditioned by temperature and pression conditions [25]. This results in a strong dependence of the dynamic viscosity on temperature and pressure, which can be expressed by means of the Arrhenius formula for kinetics, placing constraints on the interior viscosity to set it at a given value η_∞ for a given reference temperature T_∞:

$$\eta(T, P) = \eta_\infty \exp\left[\frac{Q^* + P_H V^*}{RT} - \frac{Q^*}{RT_\infty} \right]. \tag{19}$$

R is the gas constant, $P_H = \varrho(z)\, gz$ the hydrostatic pressure, Q^* the activation energy, and V^* the activation volume. While the values of the rheological parameters of candidate lower mantle substances are still uncertain, a strong coupling between rheological parameters and the thermodynamic quantities governing adiabatic and viscous heating has been found for compressible convection both in a fluid endowed with all thermodynamic quantities constant across the layer [2, 22], and in a fluid with variable thermal expansion, thermal conductivity and bulk modulus [4]. For the Earth's mantle, taking into account the increase in temperature for adiabatic compression, this trade-off can in turn be used as a constraint for the parameters: an activation volume too large, greater than 6-8 cm^3/mole, and an activation energy over 50 kcal/mole, would produce unreasonably high temperatures at the core-mantle boundary [2, 4]). The suggested ranges for Q^* and V^* lie well below the values classically preferred in the literature (10 to 15 cm^3/mole and 120 to 150 kcal/mole, respectively). This discrepancy can be explained by the findings of Christensen [27] who showed that, for an incompressible fluid, Newtonian convection can mimic closely the physical properties of a non-Newtonian flow, provided that the activation energy and volume in the temperature- and pressure-dependent viscosity of the former are reduced by a factor equal to the exponent of the stress-strain relationship of the latter. Schmeling [3] demonstrates that this correspondence holds to a satisfactory degree also for compressible fluids.

The variation of the density $\bar{\varrho}$ with temperature and pressure is related to the equation of state. As far as small perturbations are produced by the fluid motion, the equation of state can be expanded as a linear Taylor series of the density $\bar{\varrho}(T, p)$ about the reference state density distribution ϱ [28, 29]:

$$\bar{\varrho}(T, P) = \varrho(z)[1 - \alpha(T - T_S) + K_T^{-1}(P - P_H)]\,, \tag{20}$$

where T_S and P_H denote the adiabatic temperature distribution and the hydrostatic pressure assumed as a reference state.

Provided that i) the temperature has an adiabatic distribution across the fluid, ii) there is no phase change across the fluid, iii) the fluid is chemically homogeneous, the reference density distribution, which actually appears in the mean-field equations (7)-(14), can be readily derived from the Adams-Williamson relation [30], which in adimensional units is written as

$$\frac{1}{\varrho}\frac{\mathrm{d}\varrho}{\mathrm{d}z} = \frac{D}{\gamma}\,. \tag{21}$$

So far, all the models proposed for convection in compressible fluids have used an exponential depth-dependence of density obtained from the integration of eq. (21), and most of them (*e.g.* [1]—for constant viscosity—and [2, 3, 5, 30]—for variable viscosity) studied compressible convection under the assumption that the ratio D/γ on the right-hand side of eq. (21) is constant across the layer.

4. – Analysis of stability: results and discussion

We determine the critical values of the Rayleigh number for the onset of thermal convection *vs.* the aspect ratio by successive approximations solving numerically the single-mode, mean-field equations for rolls planform (8)-(10). Our procedure is the numerical counterpart of the laboratory experiments of Schmidt and Milverton [7] devoted to the observation of the instability in a fluid heated from below. For each aspect ratio, we solve the governing equations and estimate by means of successive trials the range of the Rayleigh number values which yield $\mathrm{Nu} = 1 \pm 0.001$. Since the transition from a stable situation to convection is found to be sharp [7, 8], its central value provides a good estimate of the critical Rayleigh number, typically with a precision of 1 part over 10^3. The procedure may require several iterations, but since the mean-field approximation allows to obtain the solution in a very short time (3 to 20 s of CPU on a MICROVAX are required for a run with $\mathrm{Ra} \simeq \mathrm{Ra}_c$), the total effort and computer resources spent are still limited.

We evaluate the stability curves for different values of the dissipation number, D, Grüneisen parameter, γ, activation energy Q^* and volume V^*. The boundary conditions prescribe rigid or stress-free top and stress-free bottom, and fixed temperature at top and bottom.

4·1. *Onset of convection in an incompressible, constant-viscosity fluid.* – First we check the reliability of the mean-field approximation as far as the study of marginal stability is concerned. The mean-field approximation has been proven to be a very useful tool in the study of fully developed convection, since it is able to capture the physical features for incompressible fluids with constant [11] and variable viscosity [12-15] as well as for variable-viscosity anelastic fluids [2]. In particular, the mean-field approximation describes very well the relationship between the Nusselt number and the Rayleigh number in the case of well-developed convection, *i.e.* at high Rayleigh numbers, but no extensive studies have been carried out about the capability of the mean-field approximation to predict the onset of convection. We then compare the mean-field results with the stability curves available in the literature, which are for constant-viscosity incompressible fluids, before we examine the stability of variable-viscosity, compressible fluids.

In fig. 1 we compare the analytical stability curve [6] with the mean-field results for

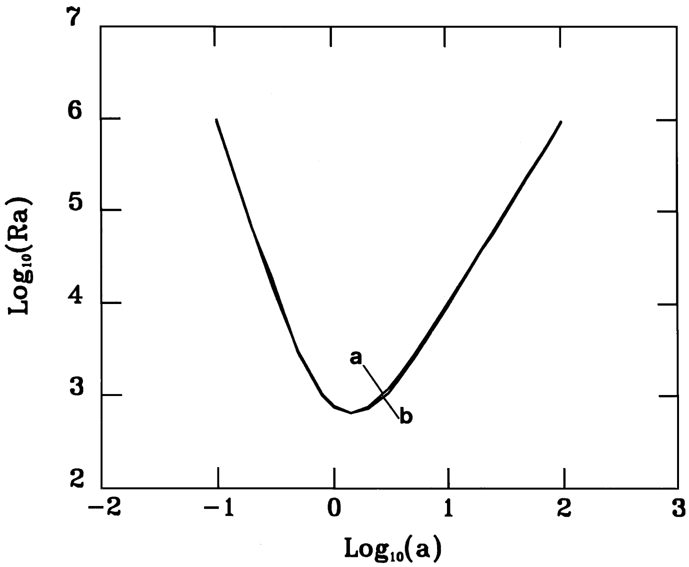

Fig. 1. – Theoretical (a) and numerical (b) marginal curves for constant-viscosity, incompressible fluid, with no dissipation, no internal heat generation, stress-free and fixed temperature boundary conditions at top and bottom.

an incompressible, constant-viscosity fluid. The agreement between the two curves is very good, being the difference contained within 10% and even lower than 5% for aspect ratios ranging from 0.5 to 5.

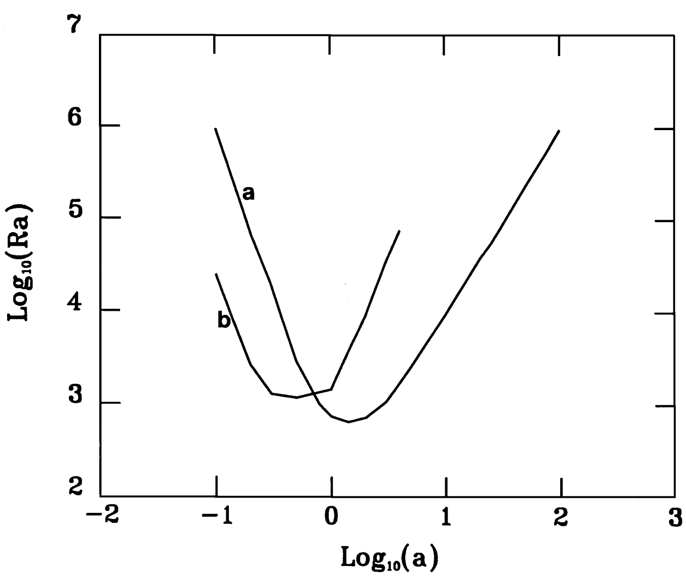

Fig. 2. – Marginal curves for incompressible fluid ($\gamma \rightarrow \infty$) with constant viscosity, no dissipation ($D = 0$) with stress-free boundary conditions (a) and with rigid top and stress-free bottom (b).

To analyze the effect of boundary conditions, we evaluate the stability curve in the case of stress-free and rigid top, keeping the bottom surface free of stress (fig. 2). In the case of stress-free top the minimum value of the critical Rayleigh number is 645 which is achieved for an aspect ratio of about 1.4 against the value 657.5 at $a = \sqrt{2}$ from the linear stability theory, while the minimum in the case of rigid top is 1175 and is obtained for an aspect ratio 0.8 against the theoretical result 1101 at $a = 1.2$. The good agreement between numerical and analytical results demonstrates the reliability of the mean-field approach and supports the calculations for the stability curves for variable-viscosity, compressible fluids.

4·2. *Onset of convection in a compressible, constant-viscosity fluid.* – The stability curves for a constant-viscosity, compressible fluid with dissipation number $D = 0.6$ and Grüneisen parameter $\gamma = 1$ and 2 are plotted in fig. 3; the curve for an incompressible fluid, $\gamma \to \infty$, is also displayed for comparison. The results show that lower values of the Grüneisen parameter, *i.e.* higher compressibility, favour the onset of convection, yielding values of the critical Rayleigh number for $\gamma = 2$ about 40% higher than for $\gamma = 1$.

Figure 4 shows the stability curves for $\gamma = 1$ and for different amounts of viscous dissipation, namely for $D = 0.3$, 0.6, and 1.1. The critical Rayleigh number increases with the dissipation number, as viscous dissipation behaves as a source of heat, thus making the heat transport less efficient.

Studies of well-developed convective motions in compressible fluids [1, 2, 5, 22] demonstrate the existence of non-linear effects related to the Grüneisen parameter and to the dissipation number. These non-linearities also affect the stability of the system, since the parameters γ and D both rule the non-linear terms in the governing equations. In fact, for fixed D (fig. 3) the difference between the critical Rayleigh numbers for $\gamma = 2$ and for $\gamma \to \infty$ is about 30% and the difference between those for

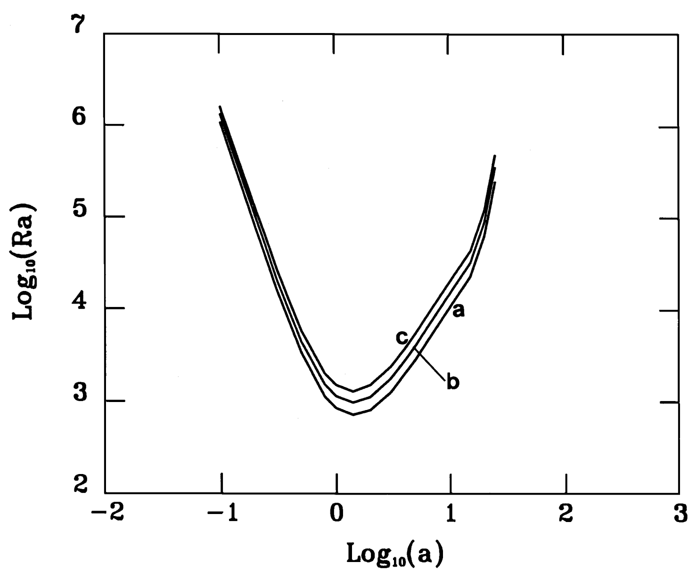

Fig. 3. – Marginal curves for a constant-viscosity, compressible fluid with $D = 0.6$, stress-free top and bottom for different values of the Grüneisen parameter: $\gamma = 1$ (a), $\gamma = 2$ (b), and $\gamma \to \infty$ (c).

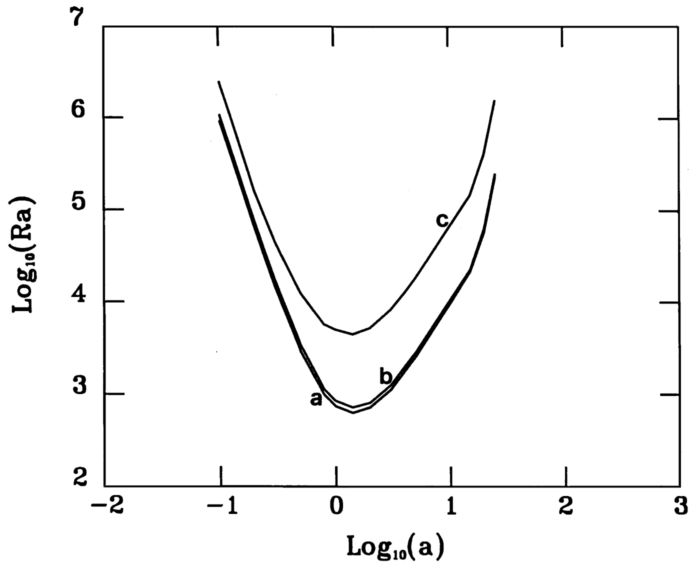

Fig. 4. – Marginal curves for a constant-viscosity, compressible fluid with $\gamma = 1$ stress-free top and bottom for different values of the dissipation number: $D = 0.3$ (a), $D = 0.6$ (b), and $D = 1.1$ (c).

$\gamma = 10$ (not shown in the figure) and for $\gamma \to \infty$ is only about 10%, while, for fixed γ (fig. 4), the difference of Ra_c is about 10% doubling the value of D from 0.3 to 0.6 and about 20% between the cases with $D = 0.6$ and $D = 1.1$.

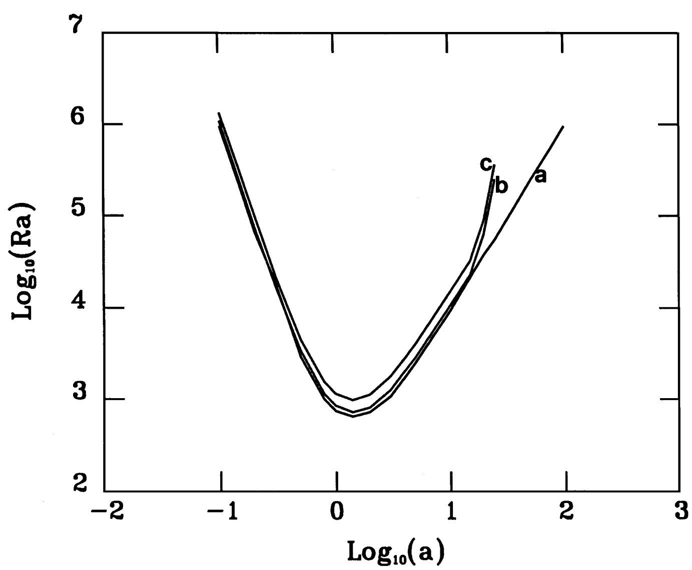

Fig. 5. – Marginal curves for an incompressible fluid with $D = 0$ (a), and for a compressible fluid with $D = 0.6$ and $\gamma = 1$ (b), and $\gamma = 2$ (c). Viscosity is constant in all cases.

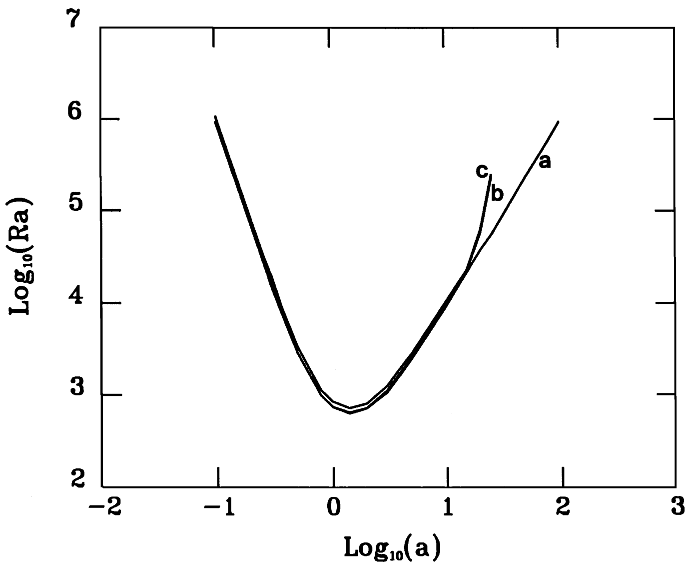

Fig. 6. – Marginal curves for an incompressible fluid with $D = 0$ (a), and for a compressible fluid with $\gamma = 1$ and $D = 0.3$ (b), and $D = 0.6$ (c). Viscosity is constant in all cases.

In fig. 5 we compare the stability curve for an incompressible, non-dissipative fluid ($\gamma \rightarrow \infty$ and $D = 0$) with the stability curves for a compressible fluid with $D = 0.6$ and $\gamma = 1$ and 2. For aspect ratios of the order of 1 to 10 the lowest value of the critical Rayleigh number is attained for $D = 0.6$ and $\gamma = 1$, while at higher aspect ratios the incompressible, non-dissipative fluid has the lowest value of Ra_c.

Together with the case of the incompressibile, non-dissipative fluid, fig. 6 shows the stability curves for fluids endowed with $\gamma = 1$ and $D = 0.3$ and 0.6. The differences among the values of Ra_c for aspect ratios smaller than 30 are contained within 5%.

We can conclude that the critical conditions for the onset of convection in constant-viscosity, incompressible and compressible fluids are very similar for cells aspect ratios between 0.1 to about 30 (the Ra_c's differ at most by 10–20%), while at larger aspect ratios the critical Rayleigh numbers for compressible fluids become one or more orders of magnitude larger than those for incompressible fluids (cf. figs. 5 and 6). From the analysis of the stability curves we can also notice that compressibility and viscous dissipation, described in terms of Grüneisen parameter γ and dissipation number D, produce non-linear effects (cf. figs. 3 and 4).

4·3. *Onset of convection in a compressible, variable-viscosity fluid.* – In this section we study the onset of convection in a fluid endowed with strongly non-linear temperature- and pressure-dependent viscosity, according to the exponential behaviour prescribed in eq. (19), and the effects of varying the creep activation energy, Q^*, and activation volume, V^*. Stress-free boundary conditions are assumed at top and bottom and a base-heated case is analyzed. Grüneisen parameter and dissipation number are taken equal to 1.1 and 0.6, respectively, which are realistic values for the whole mantle.

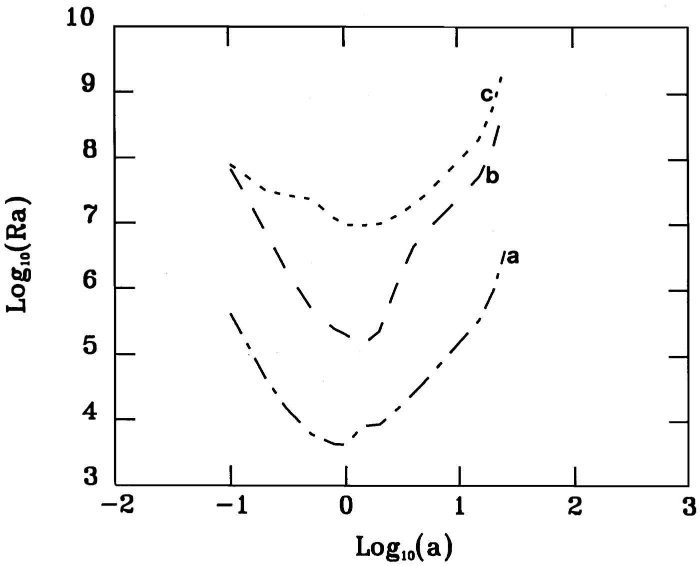

Fig. 7. – Marginal curves for a variable-viscosity compressible fluid with $\gamma = 1.1$, $D = 0.6$, and activation energy $Q^* = 30$kcal/mol for different values of the activation volume $V^* = 0$ (a), 1 (b), and 2 cm^3/mol (c).

In order to convect, the fluid at the bottom of the cell has to be heated up enough to overcome the viscous drag opposed to buoyancy. The onset of convection is hindered by

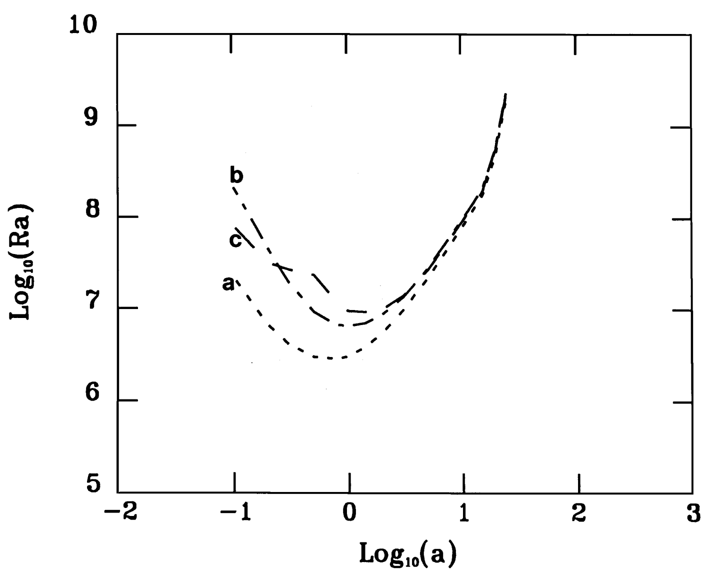

Fig. 8. – Marginal curves for a variable-viscosity compressible fluid with $\gamma = 1.1$, $D = 0.6$, and activation volume $V^* = 2$ cm^3/mol for different values of the activation energy $Q^* = 0$ (a), 15 (b), and 30 kcal/mol (c).

pressure dependence of the viscosity, since a fluid endowed with a finite activation volume is more viscous close to the bottom (cf. eq. (19)). Figure 7 shows the stability curves for $Q^* = 30$ kcal/mole, and for three different values of V^*, namely 0, 1, and 2 cm^3/mole. For aspect ratios of order of 1, the critical Rayleigh numbers is increased by 2 orders of magnitude from $V^* = 0$ and 1 cm^3/mole and from 1 to 2 cm^3/mole.

Figure 8 shows the effect of varying the activation energy, considering three values for Q^*, namely 0, 15, and 30 kcal/mole, in the case of $V^* = 2$ cm^3/mole. For large aspect ratios ($\geqslant O(10)$) the three stability curves merge together, while for aspect ratios of order of 1 or smaller, the differences in Ra_c are contained within one order of magnitude.

It has also to be noted that for a variable-viscosity, compressible fluid the minimum Ra_c is shifted to larger aspect ratios for higher values of the rheological parameters, *i.e.* for a stronger dependence of the viscosity on temperature and pressure. Furthermore, the stability curves may exhibit secondary minima, as in the cases displayed in fig. 7.

5. – Conclusions

The onset of convection is studied in a simple Boussinesq, constant-viscosity fluid, and in a compressible constant- and variable-viscosity fluid, by solving directly the momentum and energy equations, rather than through the usual perturbation analysis. This approach has been chosen because of its simplicity, avoiding the derivation of the eigenvalue differential equations, which makes it a powerful method to investigate the marginal stability in fluids endowed with pressure- and temperature-dependent parameters, as is the case of the Earth mantle material.

The single-mode mean-field approximation is used, since it is much less time consuming than the full approach and it is known to provide a very satisfactory agreement with the solutions of the full system of equations for incompressible fluids with constant [11] and variable viscosity [12-15] and in variable-viscosity anelastic fluids [2]. The equations are solved at a subcritical Rayleigh number and the Nusselt number is then evaluated, then the Rayleigh number is increased and the procedure is repeated until the Nusselt number exceeds unity, thus indicating the onset of convection. The reliability of the method is checked with the results of the standard stability problem, namely for an incompressible fluid with constant viscosity, with stress-free or rigid top boundary conditions, and a very good agreement is found.

Table I summarizes the minima of the Rayleigh numbers critical for the onset of convection in all the cases considered. It shows that the minimum Rayleigh number for the onset of thermal convection is significantly affected by the compressible nature of the fluid, by viscous dissipation, and by the temperature- and pressure-dependence of viscosity, and is found to increase of several orders of magnitude with respect to the simplest case of a Boussinesq, constant-viscosity fluid. The preferred aspect ratios of the convective cells are also given in table I, which shows that their dependance on the physical features of the fluid is not very important, and that for realistic values of the thermodynamic and reological parameters the value 1.4 typical of the simplest Boussinesq, constant-viscosity fluid is returned.

Viscous dissipation of heat in the fluid layer, which is measured by the dissipation number D, acts as a source of internal heat and diminishes the efficiency of convection since more heat has to be carried out, thus inhibiting the onset of convection, resulting in an increase of Ra_c with increasing D. On the contrary, the compressibility, which is proportional to the inverse of the Grüneisen parameter γ favours it (Ra_c increases with

TABLE I. – *Values of Rayleigh numbers,* Ra_c, *and aspect ratios,* a_c, *critical to the onset of convection in a Boussinesq and in a compressible fluid layer with fixed top and bottom temperatures, in the case of constant or variable viscosity.*

		a_c	Ra_c
$\gamma \to \infty$; η = constant	Free-free b.c.	1.4	645
	Rigid-free b.c.	0.8	1175
$D = 0.6$; η = constant	$\gamma = 1$	1.4	725
	$\gamma = 2$	1.4	977
	$\gamma \to \infty$	1.4	977
$\gamma = 1$; η = constant	$D = 0.3$	1.4	630
	$D = 0.6$	1.4	725
	$D = 1.1$	1.4	4467
$\gamma = 1$; $D = 0.6$; $Q^* = 30$ kcal/mol	$V^* = 0$ cm^3/mol	1.0	4265
	$V^* = 1$ cm^3/mol	1.4	1.4×10^5
	$V^* = 2$ cm^3/mol	1.4	9.3×10^6
$\gamma = 1$; $D = 0.6$; $V^* = 2$ cm^3/mol	$Q^* = 0$ kcal/mol	0.8	2.9×10^6
	$Q^* = 15$ kcal/mol	1.0	6.6×10^6
	$Q^* = 30$ kcal/mol	1.4	9.3×10^6

decreasing γ), although for the geophysically relevant values of D and γ ($D = 0.6$ and $\gamma = 1$) the viscous dissipation effect prevails. The dependence of the viscosity on temperature and pressure stiffens the interior of the fluid layer and prevents motion, thus requiring a much higher degree of buoyancy, *i.e.* a larger value of the critical Rayleigh number, for the onset of convection.

If the Earth's mantle is modeled as a simple Boussinesq, constant-viscosity fluid, which is characterized by a critical Rayleigh number of the order of 10^3, then a very vigorous thermal convection should take place, since the whole mantle Rayleigh number can be estimated of the order of 10^7 to 10^9, depending on the hypothesis on the reference viscosity and on the core/mantle boundary temperature. Such a strong convective motion is likely to produce velocities which are orders of magnitude higher than the observed plate velocities. The analysis of stability carried out in this paper shows that a more realistic fluid, which can dissipate heat, which is compressible, and endowed with a temperature- and pressure-dependent viscosity, is not as much supercritical for the onset of thermal convection, and suggests the picture of a slowly convecting mantle which fits better the geophysical observables.

* * *

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