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Technical Report

The MPA graph problem

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Abstract

Given an undirected graph G whose vertices are associated to different subsets of colors, the MPA problem asks for partitioning G into a minimal set of monochromatic connected subgraphs. We prove that MPA is NP-hard as well as its extensions BDMPA where the vertices of G have a bounded maximum degree, PMPA where G is planar, BDPMPA where the maximum vertex degree is bounded and G is planar, and GMPA where G consists of a $p \times q$ grid.

Let $G = (V, E)$ be an undirected graph with $|V| = n$ and $|E| = m$; and let C be a set of k colors such that each vertex $v_i \in V$ is associated to (or "contains") a non-void subset C_i of C. We set:

MPA Problem. *Assign to each vertex v_i of G a single color from among the ones in C_i such that the number σ of maximal connected subgraphs of $G_1, \ldots, G_\sigma$ of G whose vertices have the same color is minimal. One such color assignment, and the corresponding family of subgraphs, will be called an* MPA *for* minimal partition (color) assignment.

See the sample graph of Figure **??** with its (unique) MPA.

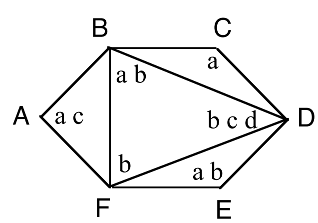

Figure 1: A graph with six vertices A, B, C, D, E, F and four colors a, b, c, d. The subsets of colors are indicated at the vertices. In this case Problem 1 has exactly one MPA with $\sigma = 2$, where G_1 contains vertices A, B, C with assigned color a, and G_2 contains vertices D, E, F with assigned color b.

Observation 1. *Let $M \subseteq C$ be a minimum subset of colors such that the total number of their occurrences in the vertices of G is $\geq n$. Then in any MPA we have $\sigma \geq |M|$.*

In Figure **??** the minimal subsets are $\{a, c\}$, $\{b, c\}$, whose colors occur in total $\geq 6 = n$ times in the vertices. So $\sigma \geq 2$, although the MPA contains colors a, b. Observation 1, whose validity is obvious, suggests that the subgraphs in an MPA tend to be as large as possible. However a largest possible mono-colored subgraph of G may not occur in any MPA. In Figure 1 the subgraph of four vertices B, D, E, F could be colored with b, but the vertices A, C should then be assigned to two different subsets and the partition would not be minimal.

Observation 2. *The subsets of vertices of any two subgraphs G_i, G_j of an MPA are mutually disjoint.*

In fact if G_i, G_j contained a common vertex v with color c, all the vertices of the two subgraphs would have color c and G_i, G_j should be merged into a unique subgraph. Letting $|G_i| = n_i$, $1 \leq i \leq \sigma$, we then have $n_1 + \ldots n_\sigma = n$. Note that two subsets G_i, G_j of an MPA may be associated with the same color, but in this case they must not be connected with an edge of G. We have:

Theorem 1. *The MPA Problem is NP-hard.*

Proof. Reduction from Set Cover. Let $X = \{x_1, \ldots, x_n\}$ be a set of n integers and $\{Y_1, \ldots, Y_k\}$ be a set of subsets of X. Build an MPA Problem where G is a *complete graph* whose vertices $v_1, \ldots, v_n$ correspond to the elements $x_1, \ldots, x_n$ of X; C contains k colors $c_1, \ldots, c_k$ corresponding to $Y_1, \ldots, Y_k$; if x_i is contained in $Y_{i_1}, \ldots, Y_{i_s}$, vertex v_i is associated to the colors $c_{i_1}, \ldots, c_{i_s}$.

A polynomial time algorithm for solving the MPA Problem would produce a minimal number of disjoint subgraphs $G_1, \ldots, G_\sigma$ of G, where each G_j is associated with a color c_{i_j} and its vertices correspond to (possibly not all) the elements of Y_{i_j}: in fact note that the subsets $\{Y_1, \ldots, Y_k\}$ are not necessarily disjoint. The solution of Set Cover on X consists of the collection of subsets $Y_{i_1}, \ldots, Y_{i_\sigma}$. □

Note that MPA is NP-hard in its general form, but its complexity could be lower for particular classes of graphs. We now show that the following versions of the problem are also NP-hard.

BDMPA Problem (for Bounded-Degree MPA), that is the MPA Problem where the maximum vertex degree (number of incident arcs) is a constant $d \geq 3$.

PMPA Problem (for Planar MPA), that is the MPA Problem where G is planar.

BDPMPA Problem (for Bounded-Degree PMPA), that is the PMPA Problem where the maximum degree of the vertices is a constant $d \geq 3$.

GMPA Problem (for Grid MPA), that is the MPA Problem where G is a $p \times q$. This is a special case of BDPMPA because the grid is a planar graph with bounded vertex degree four.

PMPA, BDPMPA, and GMPA arise in the field of innovative chip layout where the colors represent input variables each of which has to be connected to many terminals of a chip.

Theorem 2. *The BDMPA Problem is NP-hard.*

Proof. Reduction from MPA, by insertion of new vertices to reduce all vertex degrees to at most 3. If edges $(a, b), (a, c), (a, d), (a, e)$ and possibly other edges (a, x) exist, i.e.

deg(a)> 3, insert a new vertex z with $C_z = C_a \cup C_b \cup C_c$, delete edges $(a, b), (a, c)$, and insert new edges $(a, z), (z, b), (z, c)$. Note that the degree of a decreases by 1, the degrees of b, c are unchanged, and z has degree 3. Continue until each vertex has degree ≤ 3. The solution for the new graph, i.e. the connected mono-colored subgraphs, coincides with a solution for MPA if the new vertices z are deleted and the original edges are restored. □

Theorem 3. *The PMPA Problem is NP-hard.*

Proof. Reduction from planar graph 3-coloring. Let H be an arbitrary planar graph to be 3-colored with colors $1, 2, 3$, and let G be a corresponding PMA to be built. The set of colors of G is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For each edge $e = (u, v)$ of H there are nine vertices $u, e, v, x_1, x_2, x_3, y_1, y_2, y_3$ in G connected as shown in Figure **??**, with subsets of colors:

$$C_u = \{1, 2, 3\},\ C_v = \{1, 2, 3\},\ C_e = \{4, 5, 6, 7, 8, 9\},$$

$$C_{x_1} = \{1, 6, 7, 8, 9\},\ C_{x_2} = \{2, 4, 5, 8, 9\},\ C_{x_3}\{3, 4, 5, 6, 7\},$$

$$C_{y_1} = \{1, 4, 5, 6, 9\},\ C_{y_2} = \{2, 5, 6, 7, 8\},\ C_{y_3}\{3, 4, 7, 8, 9\}.$$

Consider a minimal collection of monochromatic connected subgraphs $G_1, \ldots, G_\sigma$ of G. We have: (1) the vertices u, v must belong to two distinct subgraphs G_i, G_j and the vertex e cannot belong to G_i or to G_j because $C_e \cap C_u$ and $C_e \cap C_v$; (2) at most one of the vertices x_1, x_2, x_3 may belong to G_i and most one of the vertices y_1, y_2, y_3 may belong to G_j due to their colors; (3) at most two of the vertices x_1, x_2, x_3 and most two of the vertices y_1, y_2, y_3 may belong to the same subgraph of e implying that the colors assigned to u and v must be different due to the colors of all the vertices involved. Letting $G = (V, E)$, from the points 1, 2, and 3 we have $\sigma \geq |V| + |E|$ and equality is met if and only if different colors can be assigned to u and v, depending on the color constraint imposed by the other vertices to which u and v are adjacent in H. That is, H can be 3-colored if and only if MPA can be solved on G with $\sigma = |V| + |H|$. □

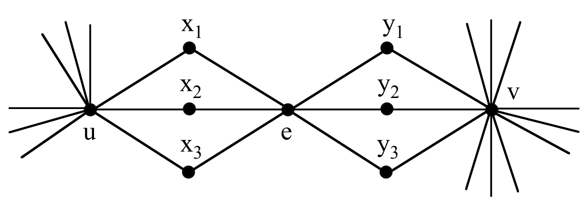

Figure 2: Portion of graph G corresponding to the edge $e = (u, v)$ of graph H.

Theorem 4. *The BDPMPA Problem is NP-hard.*

Proof. Reduction from PMPA with the same transformation used in the proof of Theorem **??**. Note that the graph resulting after transformation is planar. □

Theorem 5. *The GMPA Problem is NP-hard.*

Proof. By reduction from BDPMPA.

1. A result of Leslie G. Valiant (Theorem 2 of [**?**]), state that a planar graph G of n vertices with degree at most four admits a planar embedding in an $O(n \times n)$ grid Γ. Of the

$O(n^2)$ cells of Γ, obviously only n are used in the embedding for the vertices of G, while many of the others are used for embedding the edges of G as non intersecting sequences of cells in i, j directions. In [**?**] was then shown that one such embedding can be built where all edges are just straight line segments.

2. Build the embedding on Γ, and extend the grid to Γ' a follows. If two horizontal sequences of cells representing two edges of G lie in two rows $i, i+1$ and part of these sequences share the same columns (i.e. the two sequences are partly adjacent), insert a new empty row between i and $i+1$, using its cells where needed to fix vertical sequences possibly interrupted by the new row. Repeat the operation for any pair of partly adjacent sequences. Repeat the process on the columns, inserting new columns until no vertical sequences are partly adjacent. Note that the construction of Γ' has been done in time and space polynomial in n.

3. If two adjacent vertices a, b of G are embedded in two non adjacent cells of Γ', assign the set of colors $C_a \cap C_b$ to the cells of the sequence representing the edge (a, b). Repeat for all pairs of adjacent vertices. Assign a new color $c \notin C$ to all the grid cells not corresponding to the vertices and and to the edges of G.

4. Solve GMPA on Γ' considering all the cells as vertices of a new larger graph. Discard the subsets of cells with color c, and in any other subset take only the cells corresponding to original vertices of G. These subsets constitute a solution for BDPMPA. □

Two other possibly important versions of MPA are the following. Their complexity has still to be investigated.

BCMPA Problem (for Bounded-color MPA), that is the MPA Problem where the number of colors is a constant $k \geq 2$.

LMPA Problem (for power-Law MPA), that is the MPA Problem where vertex degrees follow a power law.

BCMPA has to do with networks where a unique leader (color) among a bounded set of candidates has to be assigned to a cluster of vertices. LMPA may be connected to the diffusion of information in social and other networks.

References

[1] H. da Fraysseix, J. Pach, and R. Pollack. Small sets supporting Fary embeddings of planar graphs. *Proc. STOC '88*, 426-433 (1988).

[2] L.G. Valiant. Universality considerations in VLSI circuits. *IEEE Trans. on Computers* C-30, 135-140 (1981).