Exact triangles for SO(3) instanton homology of webs

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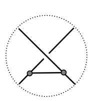
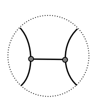
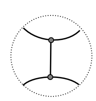
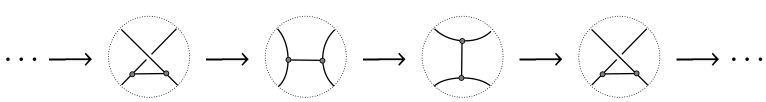
1 Introduction

Let $K \subset \mathbb{R}^3$ be an unoriented web, i.e. an embedded trivalent graph whose local model at the vertices is that of three arcs meeting with distinct tangent directions. In a previous paper [6], the authors defined an invariant $J^\sharp(K)$ for such webs, as an $SO(3)$ instanton homology with coefficients in the field $\mathbb{F} = \mathbb{Z}/2$. This instanton homology is functorial for *foams*, which are singular cobordisms between webs. The construction of $J^\sharp(K)$ closely resembles an invariant $I^\sharp(K)$ defined earlier for knots and links in [8]. Knots and links are webs without vertices; but even for these, $I^\sharp(K)$ and $J^\sharp(K)$ are different, because $I^\sharp(K)$ was defined using $SU(2)$ representation varieties, while $J^\sharp(K)$ uses $SO(3)$. Our conventions and definitions are briefly recalled in section 2.

This paper is a continuation of [6] and establishes a type of skein relation (an exact triangle) for $J^\sharp$. The main result concerns three webs L_2, L_1, L_0 which differ only inside a ball, as shown:

$L_2 =$ , $L_1 =$ , $L_0 =$ . (1)

There are standard foam cobordisms between these (see section 3 for a fuller description):



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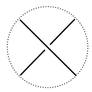
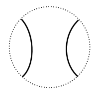
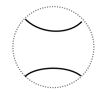
We then have:

Theorem 1.1. *The sequence of $\mathbb{F}$-vector spaces obtained by applying $J^\sharp$ to the above sequence of webs and foams is exact:*

$$\cdots \longrightarrow J^\sharp(L_2) \longrightarrow J^\sharp(L_1) \longrightarrow J^\sharp(L_0) \longrightarrow J^\sharp(L_2) \longrightarrow \cdots$$

There is a variant of this exact triangle. Consider three webs differing in the ball as in the following diagrams:

$K_2 =$  , $K_1 =$  , $K_0 =$  .

Again, there are standard cobordisms between these. In [8], we established an exact triangle in $I^\sharp$ relating these three. The corresponding sequence of vector spaces $J^\sharp(K_i)$ do *not* form an exact triangle. Instead, there is an exact triangle involving L_{i+2}, K_{i+1} and K_i, for each i (with the indices interpreted cyclically modulo 3). Thus:

Theorem 1.2. *For each $i = 0, 1, 2$, we have an exact sequence of $\mathbb{F}$-vector spaces,*

$$\cdots \longrightarrow J^\sharp(L_{i+2}) \longrightarrow J^\sharp(K_{i+1}) \longrightarrow J^\sharp(K_i) \longrightarrow J^\sharp(L_{i+2}) \longrightarrow \cdots ,$$

in which the maps are obtained by applying $J^\sharp$ to standard foam cobordisms.

The four exact triangles contained in the two theorems above can be arranged as four of the triangular faces in an octahedral diagram, a particular realization of the diagram for the octahedral axiom for a triangulated category. In the diagram, Figure 1, the top vertex L_2' has a crossing of a different sign from the picture of L_2. The exact triangles in this octahedron are

$$\begin{aligned}
&\cdots \longrightarrow J^\sharp(K_2) \longrightarrow J^\sharp(K_1) \longrightarrow J^\sharp(L_0) \longrightarrow J^\sharp(K_2) \longrightarrow \cdots \\
&\cdots \longrightarrow J^\sharp(K_2) \longrightarrow J^\sharp(L_1) \longrightarrow J^\sharp(K_0) \longrightarrow J^\sharp(K_2) \longrightarrow \cdots \\
&\cdots \longrightarrow J^\sharp(K_0) \longrightarrow J^\sharp(K_1) \longrightarrow J^\sharp(L_2') \longrightarrow J^\sharp(K_0) \longrightarrow \cdots \\
&\cdots \longrightarrow J^\sharp(L_0) \longrightarrow J^\sharp(L_1) \longrightarrow J^\sharp(L_2') \longrightarrow J^\sharp(L_0) \longrightarrow \cdots .
\end{aligned}$$

The last two are duals of exact triangles in Theorems 1.2 and 1.1 respectively. The other four faces of octahedron become commutative diagrams of $\mathbb{F}$-vector spaces on applying $J^\sharp$. For example, the triangle

$$\begin{array}{ccc}
J^\sharp(K_1) & \longleftarrow & J^\sharp(K_0) \\
\nwarrow & & \swarrow \\
& J^\sharp(K_2) &
\end{array}$$

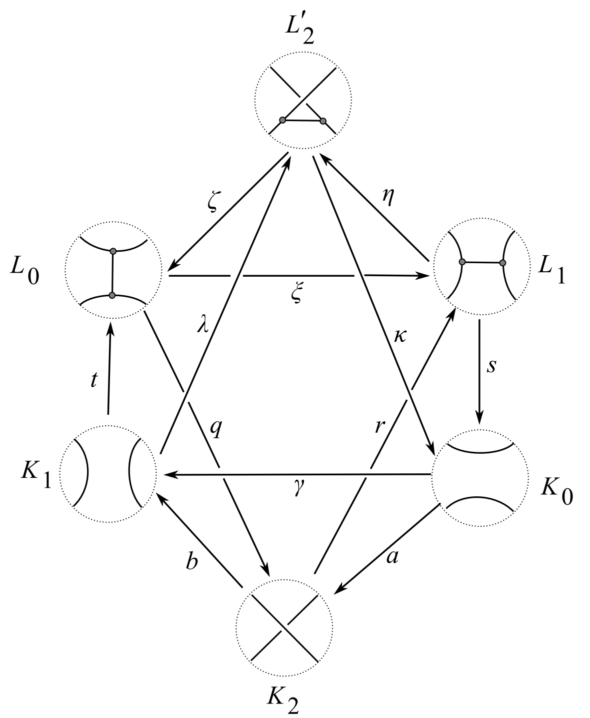
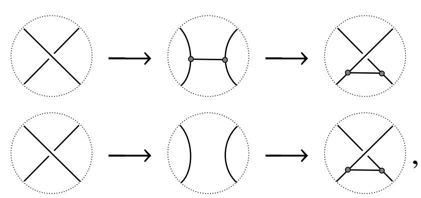


Figure 1: An octahedral diagram whose faces are four exact triangles and four commutative triangles.

is a commutative diagram. Finally, the two different composites from K_2 to L_2',



give the same map $J^\sharp(K_2) \to J^\sharp(L_2')$, with a similar (and equivalent) statement about the two composites from L_2' to K_2.

Fuller versions of these results are stated in section 3, where we also broaden the scope of the theorems a little by discussing webs embedded in arbitrary oriented 3-manifolds, rather than in $\mathbb{R}^3$.

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2 Review of $SO(3)$ instanton homology

We briefly recall some of the constructions which are described more fully in [6]. If Z is an n-dimensional orbifold, and $z \in Z$, then we write H_z for the local stabilizer group at z. All our orbifolds will be orientable, so H_z is a subgroup of $SO(n)$ acting effectively on $\mathbb{R}^n$.

Bifolds. For $m \leq n$, let $H_m \subset SO(m)$ be the elementary abelian 2-group of order 2^{m-1} consisting of diagonal matrices of determinant $+1$ whose diagonal entries are ± 1. Regard H_m also as a subgroup of $SO(n)$ for $m \leq n$. We call Z an n-dimensional *bifold* if its local stabilizer groups $H_z \subset SO(n)$ are conjugate to H_m for some $m \leq n$. All our bifolds will be equipped with Riemannian metrics, in the orbifold sense.

Webs and foams. The underlying topological space of a bifold is a manifold X, and the set of points with non-trivial local stabilizer is a codimension-2 subcomplex of X. In the case of dimension 2, this subcomplex is a set of points. In the case $n = 3$, it is a trivalent graph, which we refer to as a web.

In the case $n = 4$, the points with $H_z \neq 1$ form a 2-complex which we call a foam. A foam can have tetrahedral points, where the local stabilizer is H_4. The set of points with $H_z \cong H_3$ is a union of arcs and circles: these are the seams, which together with the tetrahedral points comprise a 4-valent graph. The remainder of the foam is a 2-manifold whose components are the *faces*.

A pair (Y, K) consisting of a smooth 3-manifold and smoothly embedded web can be used to construct a corresponding bifold $\check{Y}$. The same is true for a pair (X, Φ) consisting of a 4-manifold and an embedded foam: we may write the corresponding bifold as $\check{X}$. There is a cobordism category in which the objects are closed, oriented 3-dimensional bifolds $\check{Y}$ with bifold metrics, and in which the morphisms are isomorphism classes of oriented 4-dimensional bifolds $\check{X}$ with boundary. Equivalently, we have a category in which the objects are 3-manifolds (Y, K) with embedded webs, and the morphisms are 4-manifolds (X, Φ) with embedded foams.

Bifold connections. By a *bifold connection* over a bifold $\check{X}$, we mean an $SO(3)$ orbifold vector bundle $E \to \check{X}$ equipped with an orbifold $SO(3)$ connection A, subject to the constraint that at each point x where H_x has order 2, the local action of H_x on the $SO(3)$ fiber is non-trivial. This condition determines the local model uniquely at other orbifold points. In particular, if $\check{X}$ is 4-dimensional and x belongs to a seam of the corresponding foam, so that H_x is the Klein 4-group, then the representation of H_x on the $SO(3)$ fiber is the inclusion of the standard Klein 4-group $V \subset SO(3)$.

Marking data. Bifold connections may have non-trivial automorphisms. For example, if the monodromy group of the connection is the 4-group V, then the automorphism group is also V. In order to have objects without automorphisms, we introduce *marked* bifold connections.

By *marking data* μ on a bifold $\check{X}$, we mean a pair (U_μ, E_μ) consisting of an open set U_μ and an $SO(3)$ bundle $E_\mu \to U_\mu \cap \check{X}^o$ (where $\check{X}^o$ is the locus of non-orbifold points). A *marked* bifold connection is a bifold connection (E, A) on $\check{X}$ together with a choice of an equivalence class of an isomorphism σ from $E|_{U_\mu \cap \check{X}^o}$ to $E_\mu|_{U_\mu \cap \check{X}^o}$. Two isomorphism σ_1 and σ_2 are equivalent if $\sigma_1 \circ \sigma_2^{-1} : E_\mu \to E_\mu$ lifts to the determinant-1 gauge group, i.e. a section of the associated bundle with fiber $SU(2)$. The marking data is *strong* if the automorphism group of every μ-marked bifold connection is trivial. In dimension 3, a sufficient condition for μ to be strong is that U_μ contains a point x with $H_x = V$ (a vertex of the corresponding web), or that $U_\mu \check{X}^o$ contains a torus on which $w_2(E_\mu)$ is non-zero.

Instanton homology. Let $\check{Y}$ be a closed, connected, oriented 3-dimensional bifold with strong marking data μ. The set of isomorphism classes of μ-marked bifold connections of Sobolev class L^2_k, for large enough k, is parametrized by a Hilbert manifold $\mathcal{B}_k(\check{Y}; \mu)$. Using the perturbed Chern-Simons functional, one constructs a Morse complex, whose homology we call the $SO(3)$ instanton homology. It is defined with coefficients $\mathbb{F} = \mathbb{Z}/2$. We use the notation $J(\check{Y}; \mu)$. If (Y, K) is a pair consisting of a 3-manifold and an embedded web, we similarly write $J(Y, K; \mu)$.

Let $\check{X}$ be an oriented bifold cobordism from $\check{Y}_1$ to $\check{Y}_2$, let ν be marking data for $\check{X}$, and let μ_i be the restriction of ν to $\check{Y}_i$. If μ_i is strong, for $i = 1, 2$, then $(\check{X}, \nu)$ gives rise to a linear map

$$J(\check{X}; \nu) : J(\check{Y}_1; \mu_1) \to J(\check{Y}_2; \mu_2).$$

In general, the map which J assigns to composite cobordism may not be the composite map. However, the composition law does hold if the marking data ν on the two cobordisms satisfies an extra condition. In this paper, our cobordisms will always contain product cobordisms in the neighborhoods of the marking data, and ν will always be a product $[0,1] \times \mu_1$. This restriction is sufficient to ensure that the composition law holds.

The construction of $J^\sharp$. Let K be a compact web in $\mathbb{R}^3$. From K, we form a new web $K^\sharp \subset S^3$ as the disjoint union of K and a Hopf link H contained in a ball near the point at infinity. As marking data for $(S^3, K^\sharp)$, we take U_μ to be the ball containing H, disjoint from K, and we take E_μ to have $w_2 \neq 0$ on the torus which separates the two components of the Hopf link. This marking data is strong. We define

$$J^\sharp(K) = J(S^3, K^\sharp; \mu).$$

Given a foam cobordism $\Phi \subset [0,1] \times \mathbb{R}^3$ from K_1 to K_2, we similarly construct a new foam $\Phi^\sharp$ as $\Phi \cup ([0,1] \times H)$, with marking data $\nu = [0,1] \times \mu$. In this way, Φ gives rise to a linear map

$$J^\sharp(\Phi) : J^\sharp(K_1) \to J^\sharp(K_2).$$

In this way we obtain a functor $J^\sharp$ with values in the category of $\mathbb{F}$-vector spaces, from a category whose objects are webs in $\mathbb{R}^3$ and whose morphisms are isotopy classes of foams with boundary in intervals $[a,b] \times \mathbb{R}^3$.

3 Statement of the results

We state now the version of the Theorem 1.1 that we shall prove. Instead of $\mathbb{R}^3$, we consider a closed, oriented 3-manifold Y, and three webs L_2, L_1, L_0 in Y which are identical outside a standard ball $B \subset Y$. As in Theorem 1.1, we suppose that, inside the ball B, they look as shown in (1). As with other variants of Floer's exact triangle, there is more symmetry between the three pictures than immediately meets the eye. The same pictures are drawn from a different point of view in the bottom row of Figure 2, to exhibit the cyclic symmetry between the three. We write K_i (as in the introduction) for the web obtained from L_i by forgetting the two vertices inside the ball and deleting the edge joining them. Similar picture of the these webs are shown in Figure 3. Let $\mu = (U_\mu, E_\mu)$ be strong marking data with U_μ disjoint from B. We may regard μ as marking data for all three of the pairs (Y, L_i) and all three of the pairs (Y, K_i).

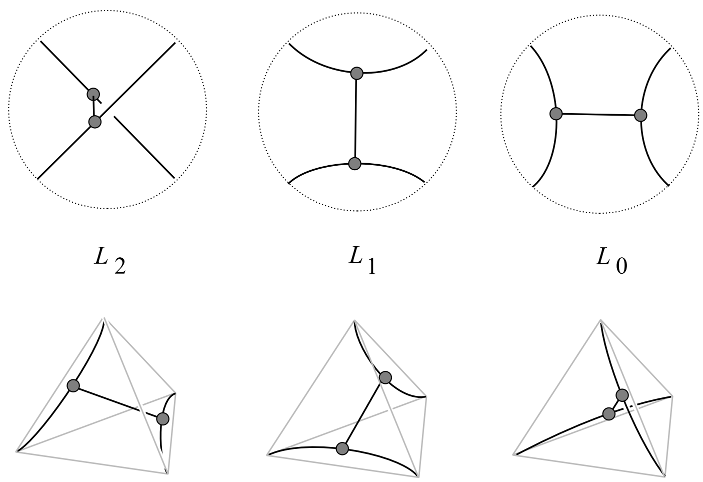


Figure 2: The three webs L_i, from two different points of view.

For each i, there is a standard cobordism from K_{i+1} to K_i given by a foam $\Sigma(K_{i+1}, K_i)$ in $I \times Y$. (The index i is to be interpreted cyclically.) The cobordism in each case is the addition of a standard 1-handle. There are also standard cobordisms to and from the L_i, which we write as $\Sigma(L_{i+1}, K_i)$, $\Sigma(K_{i+1}, L_i)$, and $\Sigma(L_{i+1}, L_i)$. These are all obtained from $\Sigma(K_{i+1}, K_i)$ by adding one or two disks. Pictures of $\Sigma(L_1, K_0)$ and $\Sigma(L_1, L_0)$ are given in Figure 4. The latter foam has a single tetrahedral point. In the picture of the cobordism from L_1 to L_0, we have labeled as δ_1 and δ_0 the edges of the webs L_1 and L_0 which are contained in the interior of the ball. These edges appear on the boundary of disks Δ_1^+ and Δ_0^- in the foam $\Sigma(L_1, L_0)$. The tetrahedral point is the unique intersection point $\Delta_1^+ \cap \Delta_0^-$.

The standard cobordisms give maps such as

$$J(I \times Y, \Sigma(L_{i+1}, L_i); \nu) : J(Y, L_{i+1}; \mu) \to J(Y, L_i; \mu)$$

where the marking data ν is the product $I \times \mu$. Thus we have a sequence of maps with period 3,

$$\cdots \to J(Y, L_2; \mu) \to J(Y, L_1; \mu) \to J(Y, L_0; \mu) \to J(Y, L_2; \mu) \to \cdots \qquad (2)$$

Theorem 3.1. *The above sequence is exact.*

As a special case, we can consider webs in $\mathbb{R}^3$ and apply the functor $J^\sharp$, in which case we deduce the version in the introduction, Theorem 1.1.

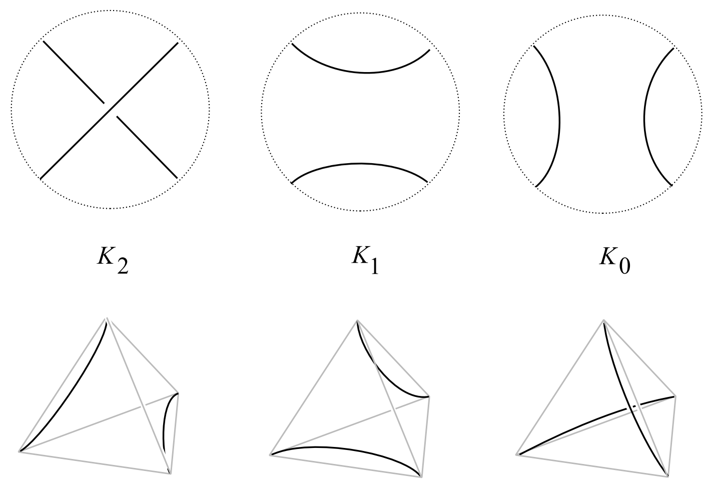


Figure 3: The three webs in Y obtained from the L_i by removing an edge.

There is a similar generalization of Theorem 1.2 in the setting of foams in a 3-manifold Y with strong marking, whose statement is easily formulated. It will turn out that there is an argument that allows Theorem 1.2 to be deduced from Theorem 1.1 (or, in the more general form, from Theorem 3.1). We will therefore focus on Theorem 3.1 to begin with.

4 Calculations for some connected sums

The quotient of $-\mathbb{CP}^2$ by the action of complex conjugation, $[z_1, z_2, z_3] \mapsto [\bar{z}_1, \bar{z}_2, \bar{z}_3]$, is an orbifold S^4 containing as branch locus the image $R \subset S^4$ of $\mathbb{RP}^2 \subset -\mathbb{CP}^2$. If L is a complex line in $-\mathbb{CP}^2$ defined by real linear equation in the homogeneous coordinates, then the image of L in the quotient is a disk D whose boundary is a real projective line in the branch locus R. Given n such lines in $-\mathbb{CP}^2$, say

$$L_1, \ldots, L_n,$$

we obtain n disks $D_1, \ldots, D_n$ in S^4 whose interiors are disjoint and whose boundaries meet in pairs at points of R. The union

$$\Psi_n = R \cup D_1 \cup \cdots \cup D_n \tag{3}$$

is a foam in S^4. This description does not specify the topology of Ψ_n uniquely when n is large, because there are different combinatorial configurations of real

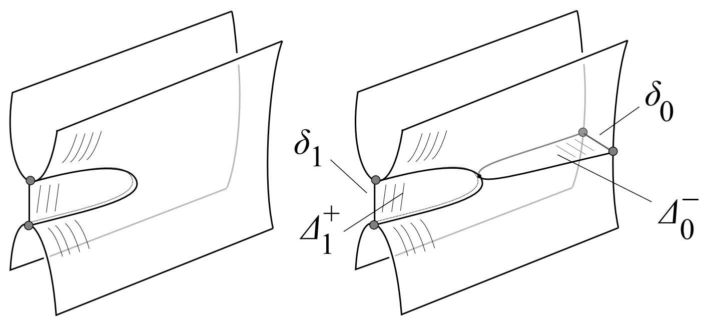


Figure 4: The cobordisms from L_1 to K_0 (left) and from L_1 to L_0 (right).

projective lines. The cases of most interest to us are Ψ_0 (which is just the real projective plane R in S^4, with $R \cdot R = 2$), and the foams Ψ_1, Ψ_2 and Ψ_3. The foam Ψ_2 has a single tetrahedral point where ∂D_1 meets ∂D_2 in R, while Ψ_3 has three tetrahedral points.

Lemma 4.1. *The formal dimension of the moduli space of anti-self-dual bifold connections of action κ on (S^4, Ψ_n) is given by*

$$8\kappa - (1 - n/2)^2.$$

In particular, we have

(a) $8\kappa - 1$ *for* Ψ_0*;*

(b) $8\kappa - 1/4$ *for* Ψ_1*;*

(c) 8κ *for* Ψ_2*;*

(d) $8\kappa - 1/4$ *for* Ψ_3*;*

Proof. The dimension formula in general is given in [6, Proposition 2.6], and for foams in S^4 it reads

$$8\kappa + \chi(\Psi_n) + \frac{1}{2}(\Psi_n \cdot \Psi_n) - \tau(\Psi_n) - 3,$$

where the self-intersection number $\Psi_n \cdot \Psi_n$ is computed face-by-face using the framing of the double-cover of the boundary obtained from the seam [6]. For Ψ_n, there is a contribution of $+2$ from $R \cdot R$ and $-1/2$ from each disk D_i, so $\Psi_n \cdot \Psi_n = 2 - n/2$. The term τ is the number of tetrahedral points, which is $n(n-1)/2$, and the Euler number χ is $1+n$. The formula in the lemma follows. □

For each of the cases in the previous lemma, we can now consider the non-empty moduli spaces of smallest possible action. For appropriate choices of metrics, we can identify these completely.

Lemma 4.2. *On the bifolds corresponding to* (S^4, Ψ_n)*, the smallest-action non-empty moduli spaces of anti-self-dual bifold connections are as follows, for* $n \leq 2$*.*

(a) *For* $n = 0$ *or* $n = 2$*, there is a unique solution with* $\kappa = 0$*: a flat connection whose holonomy group has order* 2 *for* $n = 0$ *and is the Klein* 4*-group,* V_4*, for* $n = 2$*. The automorphism group of the connection is* $O(2)$ *(respectively,* V_4*), and it is an unobstructed solution in a moduli space of formal dimension* -1 *(respectively, dimension* 0*).*

(b) *For* $n = 1$ *and* $n = 3$*, the smallest non-empty moduli spaces have* $\kappa = 1/32$ *and formal dimension* 0*. In both cases, with suitable choices of bifold metrics, the moduli space consists of a unique unobstructed solution with holonomy group* $O(2)$*.*

Proof. The complement $S^4 \backslash R$ deformation-retracts onto another copy of $\mathbb{RP}^2$ in S^4, which we call R' (the image of R under the antipodal map, in a standard construction of R). The interiors of the disks D_i are the fibers of this retraction. So $S^4 \backslash \Psi_n$ has the homotopy type of R' with n punctures. In particular, the fundamental group is $\mathbb{Z}/2$ for $n = 0$ and $\mathbb{Z} * \mathbb{Z}$ for $n = 2$. For $i = 0$ and 2, the smallest possible action is clearly $\kappa = 0$, and we are therefore looking at representations of $\mathbb{Z}/2$ or $\mathbb{Z} * \mathbb{Z}$ in $SO(3)$ sending the standard generators to involutions. In the second case, the two involutions must be distinct and commuting because of the presence of the tetrahedral point where the disks meet. So the flat connections are as described in the lemma. For these bifold connections A, we can read off H^0_A and H^1_A in the deformation complex by elementary means, and conclude that $H^2_A = 0$ from the dimension formula.

For the case $n = 1$, the corresponding bifold admits a double-cover, branched along R, for which the total space is $-\mathbb{CP}^2$ containing a complex line L as orbifold locus with cone-angle π. According to [1, 3], there exists a conformally anti-self-dual bifold metric on $-\mathbb{CP}^2$ with cone-angle π along L and positive scalar curvature. This metric is invariant under the action of complex conjugation, and it gives rise to a conformally anti-self-dual metric on the bifold $\check{S}^4$ corresponding to (S^4, Ψ_1). For such a metric, the obstruction space H^2_A in the deformation complex of an anti-self-dual bifold connection A is trivial [2, Theorem 6.1], so the moduli space is zero-dimensional and consists of finitely many points, all of which are unobstructed solutions. If A is such a bifold connection, consider its pull-back, $\tilde{A}$, on the double-cover $-\mathbb{CP}^2$, regarded as a bifold with singular locus L. The action of $\tilde{A}$ is $1/16$, and the dimension formula shows that the moduli space containing $\tilde{A}$ on this bifold has formal dimension -1. Since it is unobstructed, the solution must be reducible, and must therefore be an $SO(2)$ connection, with holonomy -1 around the link of $L \subset -\mathbb{CP}^2$. There is a unique such $SO(2)$ solution on $-\mathbb{CP}^2$ with the correct action, and it gives rise to a unique $O(2)$ connection on the original bifold.

In the case $n = 3$, the bifold corresponding to $\Psi_3 \subset S^4$ has a smooth 8-fold cover, which is $-\mathbb{CP}^2$. The covering map is the quotient map for the action of the elementary abelian group of order 8 acting on $-\mathbb{CP}^2$, generated by the action of complex conjugation and the action of the Klein 4-group by projective linear transformations. We equip the quotient bifold with the quotient metric of the Fubini-Study metric, so that (as in the case $n = 1$) all solutions are unobstructed. A solution of action $1/32$ pulls back to a solution of action $1/4$ on the 8-fold cover, which must be the unique instanton with holonomy group $SO(2)$ in the $SO(3)$ bundle with Pontryagin number -1 on $-\mathbb{CP}^2$. This descends to a unique bifold connection on the quotient. □

We use these results about small-action moduli spaces to analyze connected sums in some particular cases. In general, given foams $\Phi \subset X$ and $\Phi' \subset X'$ with tetrahedral points t, t' in each, there is a connected sum

$$(X, \Phi)\#_{t,t'}(X', \Phi') \tag{4}$$

performed by removing standard neighborhoods and gluing together the resulting foams-with-boundary. Similarly, if s and s' are points on seams of Φ and Φ', there is a connected sum

$$(X, \Phi)\#_{s,s'}(X', \Phi'),$$

and there is a connected sum for points f and f' in the interiors of faces of the two foams:

$$(X, \Phi)\#_{f,f'}(X', \Phi'),$$

Although our notation does not reflect this, the connected sum is *not unique* when we summing at a tetrahedral point or a seam. The cause of the non-uniqueness (in the case of the tetrahedral points, for example) is that we have to choose how to identify the 1-skeleta of the two tetrahedra that arise as the links of t and t'.

We consider a connected sum at a tetrahedral point in the case that (X', Φ') is either (S^4, Ψ_2) or (S^4, Ψ_3).

Proposition 4.3. *Let* (X, Σ) *be a foam cobordism with strong marking data* ν*, defining a linear map* $J(X, \Sigma, \nu)$*. Let* t *be a tetrahedral point in* Σ*.*

(a) *If a new foam* $\tilde{\Sigma}$ *is constructed from* Σ *as a connected sum*

$$(X, \Sigma)\#_{t,t_2}(S^4, \Psi_2),$$

where t_2 *is the unique tetrahedral point in* Ψ_2*, then the new linear map* $J(X, \tilde{\Sigma}, \nu)$ *is equal to the old one.*

(b) *If a new foam* $\tilde{\Sigma}$ *is constructed from* Σ *as a connected sum*

$$(X, \Sigma)\#_{t,t_3}(S^4, \Psi_3),$$

where t_3 *is any of the three tetrahedral points in* Ψ_3*, then the new linear map* $J(X, \tilde{\Sigma}, \nu)$ *is zero.*

Proof. Consider a general connected sum at tetrahedral points, as in equation (4). Let A and A' be unobstructed solutions on (X, Φ) and (X', Φ'). Let U_A and $U_{A'}$ be neighborhoods of $[A]$ and $[A']$ in their respective moduli spaces. The limiting holonomy of the connections at the tetrahedral point is the Klein 4-group V, whose commutant in $SO(3)$ is also V. So we have moduli spaces of solutions with framing at t and t' in which $[A]$ and $[A']$ have neighborhoods $\tilde{U}_A$, $\tilde{U}_{A'}$, such that $U_A = \tilde{U}_A/V$ and $U_{A'} = \tilde{U}_{A'}/V$. Gluing theory provides a model for the moduli space on the connected sum with a long neck, of the form

$$\tilde{U}_A \times_V \tilde{U}_{A'}.$$

If the action of V on $\tilde{U}_A$ is free and $U_{A'}$ consists of the single point $[A']$, then this local model is a finite-sheeted covering of U_A with fiber $V/\Gamma_{A'}$, where $\Gamma_{A'} \subset V$ is the automorphism group of the solution A'.

In particular if A' is the smallest energy solution on (S^4, Φ_2), then the fiber is a single point, while for (S^4, Φ_3) the fiber is 2 points. For the case of compact, zero-dimensional moduli spaces on the connected sum, these local models become global descriptions when the neck is long, and we conclude that the moduli space whose point-count defines the map $J(X, \Sigma, \nu)$ is unchanged in the first case and becomes double-covered in the second case. In the second case, the new map is zero because we are working with characteristic 2. □

The next proposition considers similarly the results of connected sum at seam points, where one of the summands is Ψ_1, Ψ_2 or Ψ_3.

Proposition 4.4. *Let* (X, Σ) *be a foam cobordism with strong marking data* ν*, as in the previous proposition. Let* s *be a point in a seam of* Σ*. For* $n = 1, 2, 3$*, let* s_n *be a point on a seam of* Ψ_n*. If a new foam* $\tilde{\Sigma}_n$ *is constructed from* Σ *as the connected sum*

$$(X, \Sigma)\#_{s,s_n}(S^4, \Psi_n),$$

then the new linear map $J(X, \tilde{\Sigma}_n, \nu)$ *is equal to the old one in the case* $n = 2$*, and is zero in the case that* $n = 1$ *or* $n = 3$*.*

Proof. The proofs are standard, modeled on the proofs in the previous proposition. □

Next we have a proposition about connected sums at points interior to faces of the foams.

Proposition 4.5. *Let* (X, Σ) *be a foam cobordism with strong marking data* ν*, as in the previous propositions. Let* f *be a point in the interior of a face of* Σ*. Let* f_n *be a point in a face of* Ψ_n*. If a new foam* $\tilde{\Sigma}_n$ *is constructed from* Σ *as the connected sum*

$$(X, \Sigma)\#_{f,f_n}(S^4, \Psi_n),$$

then the new linear map $J(X, \tilde{\Sigma}_n, \nu)$ *is equal to the old one in the case* $n = 0$*, and is zero in all other cases.*

Proof. Again, this is now straightforward. □

We consider next a different type of connect sum. Let Ψ_2^- be the mirror image of Ψ_2. It has self-intersection number -1, Euler number 2, and one tetrahedral point. In the description of Ψ_2 in (3), the surface R is divided into two components by the seams of the foam. Let $f_2 \in R$ be a point in one of those two components.

Proposition 4.6. *Let* (X, Σ) *be a foam cobordism with strong marking data* ν*. Let* f *be a point in the interior of a face of* Σ*. Let* $f_2 \in \Psi_2^-$ *be as above. If a new foam* $\tilde{\Sigma}$ *is constructed from* Σ *as the connected sum*

$$(X, \Sigma)\#_{f,f_2}(S^4, \Psi_2^-),$$

then the new linear map $J(X, \tilde{\Sigma}, \nu)$ *is equal to the old one.*

Proof. Consider a 0-dimensional moduli space on $(X, \tilde{\Sigma})$. Let κ be the action of solutions in this moduli space. If we stretch the neck at the connected sum, and if we obtain in the limit a solution on (X, Σ) and a solution on (S^4, Ψ_2^-), then the action of the solutions on these two summands must be (respectively) κ and 0. The moduli space on (X, Σ) with action κ is zero-dimensional. The moduli space on (S^4, Ψ_2^-) with action 0 consists of a unique V-connection A_V, but the formal dimension of the corresponding moduli space is -1, because there is a 1-dimensional obstruction space $H^2(A_V)$ in the deformation complex. The gluing parameter is $O(2)/V = S^1$, so the description of the moduli space on (X, Σ) is as the zero-set of a real line bundle h over an S^1 bundle over the moduli space associated to (X, Σ). From this description, we see that

$$J(X, \tilde{\Sigma}, \nu) = \epsilon J(X, \Sigma, \nu)$$

where $\epsilon \in \{0, 1\}$ is the evaluation of $w_1(h)$ on the S^1 fibers.

To describe $w_1(h)$, let us write the $SO(3)$ vector bundle E for the connection A_V as a sum of three line bundles,

$$E = L_0 \oplus L_1 \oplus L_2.$$

In this decomposition, let i, j, k be the non-identity elements of V, chosen to have the form

$$i = \mathrm{diag}(1, -1, -1), \quad j = \mathrm{diag}(-1, 1, -1), \quad k = \mathrm{diag}(-1, -1, 1).$$

Up to conjugacy, the monodromy of A_V around the link of the disks $D_1, D_2 \subset \Psi_2^-$ is i. The monodromies around the links of the two faces $R \setminus (D_1 \cup D_2)$ are j and k. The 1-dimensional vector space $H^2(A_V)$ is spanned a by an E-valued form ω whose values lie in the summand $L_0 \subset E$ (as follows from the fact that the branched double cover of S^4 branched over R has $b^+ = 1$). When we form connected sum at a point f_2 in a face of Ψ_2^-, the obstruction bundle h will be non-trivial on S^1 if and only if the monodromy of the link at f_2 acts non-trivially on the summand L_0 in which ω lies. Thus h is non-trivial if the monodromy at f_2 is j or k, but h is trivial if the monodromy is i. Under the hypotheses of the proposition, f_2 belongs to one of the faces where the monodromy if j or k, so the result follows. □

There is a variant of Proposition 4.6 which we will apply in section 8. Consider S^4 as the union of two standard balls $B^4_+ \cup B^4_-$. Let $M \subset S^3$ be a standard Möbius band whose boundary is an unknot U. Let D^+ and D^- be standard disks in B^4_+ and B^4_- whose boundaries are both U. These two disks together with M make a foam,

$$\Psi = D^+ \cup D^- \cup M.$$

If the half-twist in M has the appropriate sign, the self-intersection of the surface real projective plane $R = D^+ \cup M$ in S^4 will be -2, while $D^- \cup M$ has self-intersection $+2$. We can realize (S^4, Ψ), if we wish, as the V-quotient of $S^2 \times S^2$, where one generator of V interchanges the two factors and another generator acts as a reflection on each S^2 (so that the fixed set of the second generator is $S^1 \times S^2$). Let $f_\pm$ be interior points in the faces $D^\pm$ of Ψ.

Proposition 4.7. *Let (X, Σ) be a foam cobordism with strong marking data ν. Let f be a point in the interior of a face of Σ. Let Ψ be the foam just described, and let $g \in \Psi$ be an interior point of one of the faces of Ψ. Let a new foam $\tilde{\Sigma}$ be constructed from Σ as the connected sum*

$$(X, \Sigma) \#_{f,g} (S^4, \Psi).$$

Then the new linear map $J(X, \tilde{\Sigma}, \nu)$ is equal to the old one if g belongs to D^+ or to M. If g belongs to D^-, then $J(X, \tilde{\Sigma}, \nu)$ is zero.

Proof. This is an obstructed gluing problem of just the same sort as in the previous proposition. Once again, the branched doubee cover of S^4 over the surface $R = D^+ \cup M$ has $b^+ = 1$, and the calculation proceeds as before. (One can alternatively derive this proposition from the previous one by showing that Ψ is a connect sum $\Psi_2^- \#_{t,t} \Psi_2$ at the tetrahedral points.) □

5 Topology of the composite cobordisms

As stated in the introduction, the proof of the exact triangle (Theorem 3.1) is very little different from the proof of a corresponding result for the $SU(2)$ instanton knot homology $I^\sharp$. The first step is to understand the topology of the composites of the cobordisms $\Sigma(L_{i+1}, L_i)$. We shall abbreviate the notation for these cobordisms to just $\Sigma_{i+1,i}$.

Figure 5 shows (schematically) the composite of three consecutive foam cobordisms, from L_3 to L_0. The indices are interpreted cyclically, so L_3 and L_0 are the same web in Y. To explain the picture, in the foam $\Sigma_{1,0}$ pictured in Figure 4, there are two half disks whose removal would leave a standard saddle cobordism from K_1 to K_0. When $\Sigma_{2,1}$ and $\Sigma_{1,0}$ are concatenated, two half-disks are joined to form a single disk $\Delta_1 = \Delta_1^- \cup \Delta_1^+$. The triple composite $\Sigma_{3,2} \cup \Sigma_{2,1} \cup \Sigma_{1,0}$ contains two such disks Δ_2 and Δ_1, as well as two half-disks Δ_3^+ and Δ_0^-. These disks are shown shaded in the figure, and in a somewhat schematic manner (because the foams do not embed in $\mathbb{R}^3$). If we remove the interiors of these disks from the foam, what remains is a plumbing of Möbius bands, which can be interpreted as a composite cobordism from K_3 to K_0 and which appears also as Figure 10 of [8]. We write $\Phi_{3,0}$ for this composite cobordism from L_3 to L_0, and $\Phi'_{3,0}$ for the complement of the interiors of the disks, as a cobordism from K_3 to K_0.

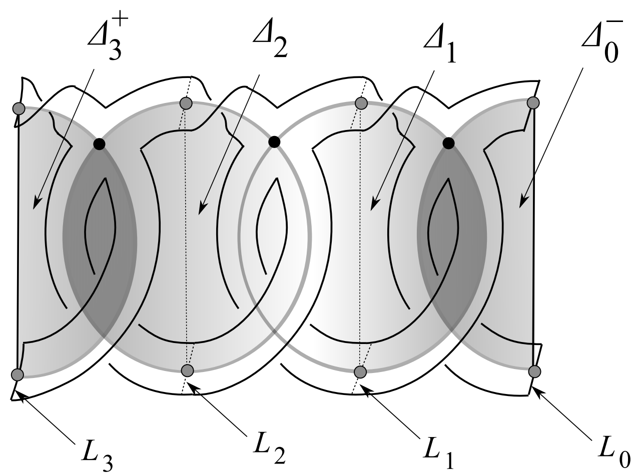


Figure 5: The composite cobordism $\Sigma_{3,2} \cup \Sigma_{2,1} \cup \Sigma_{1,0}$ (portrayed schematically because it is not embedded in $\mathbb{R}^3$). The black dots are tetrahedral points. The gray dots are vertices of the webs L_i.

As explained in [8], a regular neighborhood B_1 of Δ_1 meets $\Phi'_{3,0}$ in a Möbius band M. The same regular neighborhood meets $\Phi_{3,0}$ in the union

$$M \cup \Delta_2^+ \cup \Delta_1 \cup \Delta_0^-. \tag{5}$$

This union is a foam in the 4-ball B_1, and its boundary is the web consisting of the boundary of M and an arc on the boundary of the half-disks Δ_2^+ and Δ_0^-. This web is isomorphic to the 1-skeleton of a tetrahedron, and the foam (5) is the complement of the neighborhood of a tetrahedral point t_3 in $\Psi_3 = R \cup D_1 \cup D_2 \cup D_3$. So we have an isomorphism of pairs,

$$(B_1, \Phi_{3,0} \cap B_1) = (S^4 \backslash N_{t_3}, \Psi_3 \backslash N_{t_3}), \tag{6}$$

where $N_{t_3} \subset S^4$ is a regular neighborhood of t_3. In particular, we have the following counterpart to Lemma 7.2 of [8].

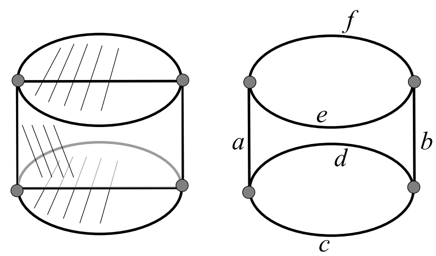


Figure 6: The foam Ω, comprising two disks and a rectangle, and the web Γ which is its boundary.

Lemma 5.1. *The composite cobordism from (Y, L_2) to (Y, L_0) formed from the union of the foams $\Sigma_{2,1}$ and $\Sigma_{1,0}$ has the form*

$$(I \times Y, V_{2,0})\#_{t,t_3}(S^4, \Psi_3),$$

where $V_{2,0}$ is a foam cobordism from L_2 to L_0 with a single tetrahedral point t.

Consider next the regular neighborhood $B_{2,1}$ of the union of two disks, $\Delta_2 \cup \Delta_1$. The regular neighborhood meets $\Phi'_{3,0}$ in a plumbing of two Möbius bands, which is a twice-punctured $\mathbb{RP}^2$. There is an isomorphism of pairs,

$$(B_{2,1}, \Phi_{3,0} \cap B_{2,1}) = (S^4 \backslash N_\delta, \Psi_3 \backslash N_\delta) \tag{7}$$

where Ψ_3 is as in (3) as before, and N_δ is the regular neighborhood of an arc $\delta \subset D_3$ which joins the two points $p, q \in \partial D_3$ in Ψ_3. The points p and q lie in the interior of the two arcs which into which ∂D_3 is divided by the two tetrahedral points.

To examine the picture of (7) further, we note the web that arises as the boundary of $\Phi_{3,0} \cap B_{2,1}$ is also the boundary of the foam $\Omega = N_\delta \cap \Psi_3$. The foam Ω consists of two disks which comprise $N_\delta \cap R$ together with the rectangle $N_\delta \cap D_3$. See Figure 6. The foam formed from $\Phi_{3,0}$ removing $\Phi_{3.0} \cap B_{2,1}$ and replacing it with the foam Ω is isomorphic to the product foam $I \times L_3 = I \times L_0$. Stating this the other way round, we have the following counterpart to Lemma 7.4 of [8].

Lemma 5.2. *The foam $\Phi_{3,0}$ in $I \times Y$ is obtained from the product foam $[0, 1] \times L_0$ by removing a neighborhood N of the arc $\{1/2\} \times \delta_0$ and replacing it with the foam $\Psi_3 \backslash N_\delta$.*

6 The chain-homotopies

In order to continue our argument, we streamline our notation. We will then follow closely the argument in [8]. We write simply Y_i for the bifold corresponding to the pair (Y, L_i). We write $X_{1,0}$ for the bifold cobordism from Y_1 to Y_0 etc., and we write $X_{3,0}$ (for example) for the composite cobordism from Y_3 to Y_0, corresponding to the foam $\Phi_{3,0}$ in $I \times Y$. We write $B_{2,1}$ again for the regular neighborhood of $\Delta_3 \cup \Delta_1$ which we now regard as a bifold ball, contained in the interior of $X_{3,0}$. We also have B_1, the bifold regular neighborhood of Δ_1, which arrange to be contained in the interior of $B_{2,1}$. We similarly have B_2, the regular neighborhood of Δ_2. We write S_i for the boundary of B_i and $S_{2,1}$ for the regular neighborhood of $B_{2,1}$.

In all, the interior of $X_{3,0}$ contains five 3-dimensional bifolds,

$$Y_2, S_2, S_1, Y_1, S_{2,1}. \tag{8}$$

Arranged cyclically in the above order, each of these five bifolds intersects the ones before and after it, but not the other two. We equip $X_{3,0}$ with a bifold metric which is a product in the two-sided collar of each of the five bifolds, and arrange also that the bifolds meet orthogonally where they intersect. Given a 4-dimensional bifold Z with boundary, having a metric which is a product metric on a collar of ∂Z, we will write Z^+ for the complete bifold obtained by attaching cylindrical ends to the boundary components:

$$Z^+ = Z \cup [0, \infty) \times \partial Z.$$

After choosing perturbations, we have a chain complex (C_i, D_i) associated to Y_i, whose homology is the instanton Floer homology group $J(Y_i; \mu)$. From each $X^+_{i+1,i}$, we obtain chain maps,

$$F_{i+1,i} : C_{i+1} \to C_i.$$

We will just write F for $F_{i+1,i}$ and D for D_i, and so write the chain condition (mod 2) as

$$FD + DF = 0.$$

Combining Lemma 5.1 with Proposition 4.3, we learn that the composite cobordism $X_{2,0}$ gives rise to the zero map from $J(Y_2; \mu)$ to $J(Y_0; \mu)$. So $F \circ F$ induces the zero map in homology. The proof supplies an explicit chain-homotopy $J = J_{2,0}$ (or $J_{i+2,i}$ in general), so that

$$FF + DJ + JD = 0. \tag{9}$$

The chain-homotopy J is defined by counting instantons over a 1-parameter family of bifold metrics g_t on $X^+_{2,0}$. For $t = 0$, the metric is the restriction of our chosen metric on $X_{3,0}$. For $t < 0$, the metric is stretched across the collar of $Y_1 \subset X_{2,0}$, and for $t > 0$ the metric is stretched along the collar of $S_1 \subset X_{2,0}$.

We have learned here that the composite of any two consecutive maps in the sequence (2) is zero. To prove exactness, following the argument of [9, Lemma 4.2], it suffices to find chain-homotopies

$$K_{i+3,i} : C_{i+3} \to C_i$$

for all i, such that

$$FJ + JF + DK + KD : C_{i+3} \to C_i \tag{10}$$

is an isomorphism.

As in [8] and [4], the map K is constructed as follows. For each pair of *non-intersecting* bifolds among the five bifolds (8), we can construct a family of metrics on $X_{3,0}$ parametrized by the quadrant $[0,\infty) \times [0,\infty)$, by stretching in the collars of both of the bifolds. There are five such non-intersecting pairs, and the corresponding five quadrants of metrics fit together to form a family of metrics parametrized by an open disk P. The map K is defined by counting points in zero-dimensional moduli spaces over the family of metrics parametrized by P, on $X^+_{3,0}$.

The family of metrics P has a natural closure $\bar{P}$ which is a closed pentagon whose paramatrize certain broken metrics on $X^+_{3,0}$, i.e. metrics where one (or more) of the collars has been stretched to infinity, and we regard the limiting space has having two (or more) new cylindrical ends. Each side of the pentagon corresponds to a family of metrics which is broken along one of the five 3-dimensional bifolds, and the vertices correspond to metrics which are broken along two of them (a pair of bifolds which do not intersect). We write

$$\partial P = Q_{Y_2} \cup Q_{S_2} \cup Q_{S_1} \cup Q_{Y_1} \cup Q_{S_{2,1}}. \tag{11}$$

To prove (10), one considers one-dimensional moduli spaces on the bifold $X_{3,0}$ over the parameter space P, and applies as usual the principal that a one-manifold has an even number of ends. The compactification of such a one-dimensional moduli space contains points of a type that did not arise in the argument in [8], namely those corresponding to the bubbling off of an instanton at a tetrahedral point, which is a codimension-1 phenomenon. However, as in [6, Section 3.4], the number of endpoints of the moduli space which are accounted for by such bubbling is even, so there is no new contribution from these endpoints. We arrive at a standard formula,

$$DK + KD + W = 0,$$

where W is a linear maps defined by counting the number of endpoints of the compactified moduli space which lie over ∂P. Following the description of ∂P as the union of five parts in (11), we write U as a sum of five corresponding terms:

$$DK + KD + U_{Y_2} + U_{S_2} + U_{S_1} + U_{Y_1} + U_{S_{2,1}} = 0. \tag{12}$$

In the equation (12), the terms U_{Y_2} and U_{Y_1} are respectively JF and FJ. The terms U_{S_1} and U_{S_2} are both zero, because they correspond to a connect sum decomposition at a tetrahedral point, when one of the summands is Ψ_3. So the formula reads

$$DK + KD + JF + FJ = U_{S_{2,1}}$$

and we must show that $U_{S_{2,1}}$ is chain-homotopic to the identity.

7 Completing the proof

The term $U_{S_{2,1}}$ counts endpoints of the compactified moduli space of $X^+_{3,0}$ that arise as limit points when the length of the collar of $S_{2,1}$ is stretched to infinity. As in [8], identifying the number of such endpoints is a gluing problem, for gluing along $S_{2,1}$.

The two orbifolds that are being glued in this case are as follows. The first piece is the regular neighborhood

$$W \supset S_1 \cup S_2$$

of the union $S_1 \cup S_2$ in $X^+_{3,0}$. The second piece is the complement of W. If we adapt Lemma 5.2 from the language of foams to the language of orbifolds, we obtain a description of these two orbifolds. The complement of W in $X^+_{3,0}$ is isomorphic to the complement of the arc δ_0 in the cylindrical orbifold $\mathbb{R} \times Y_0$. Meanwhile, W is isomorphic to the complement of an arc δ in the orbifold 4-sphere Z_3 which corresponds to the foam $\Psi_3 \subset S^4$.

We write W^+ for the cylindrical-end orbifold obtained by attaching $\mathbb{R}^+ \times S_{2,1}$ to W. The arc $Q_{S_{2,1}} \subset \partial P$ parametrizes a 1-parameter family of metrics on W^+. The main step now is to understand the 1-dimensional moduli space M_W of solutions on W^+, lying over this 1-parameter family of metrics, and to understand the map M_W to the representation variety of the end $S_{2,1}$.

Proposition 7.1. *Let $R(S_{2,1})$ denote the representation variety parametrizing flat bifold connections on $S_{2,1}$. Let G denote the one-parameter family of metrics on W^+ corresponding the interior of the interval $Q(S_{2,1})$. Let M_W denote the 1-dimensional moduli space of solutions on W^+ over the family of metrics G, and let r be the restriction map to the end:*

$$r : M_W \to R(S_{2,1})$$

Then $R(S_{2,1})$ is a closed interval, and for generic choice of metric perturbations, r maps M_W properly and surjectively to the interior of the interval $R(S_{2,1})$ with degree 1 mod 2.

Proof. The orbifold $S_{2,1}$ corresponds to the web $\Gamma \subset S^3$ show in Figure 6. In an $SO(3)$ representation corresponding to a flat bifold connection on $S_{2,1}$, the generators corresponding to the edges a and b will map to the same involution (say i) in $SO(3)$. The generators corresponding to the remaining four edges map to involutions which are rotations about axes orthogonal to the axis of i. Up to conjugacy, the representation is determined by a single angle

$$\theta \in [0, \pi/2]$$

which is the angle between the axes of rotation corresponding to the edges c and e. Thus $R(S_{2,1})$ is an interval.

The solutions belonging to the 1-dimensional moduli space M_W have action $\kappa = 1/32$. Since the smallest action that can occur at a bubble is $1/8$, there is no possibility of non-compactness due to bubbling, nor can action be lost from the cylindrical end. The moduli space M_W is therefore proper over the interior of G.

The two limit points of the 1-parameter family of metrics G correspond to pulling out a neighborhood of either Δ_1 or Δ_2 from W. (See Figure 5.) In either case, this is a connect-sum decomposition of W^+, in which the summand that is being pulled off is a copy of Z_3 (the orbifold corresponding to Ψ_3) and the sum is at a tetrahedral point. Thus, in both cases, we see connect-sum decompositions,

$$W = Z_3 \#_{t,t'} W'_1$$
$$W = Z_3 \#_{t,t'} W'_2$$

corresponding to pulling out a neighborhood of Δ_1 or Δ_2 respectively. The orbifolds W'_1 and W'_2 correspond to the foams F_1 and F_2 shown in Figure 7, regarded as a foams in B^4 with boundary Γ. These two foams are isomorphic, but not by

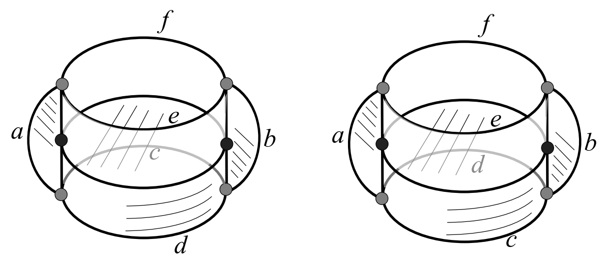


Figure 7: Two isomorphic foams F_1 (left) and F_2 (right), corresponding to the bifolds W'_1 and W'_2. Their boundaries are identified with Γ in two different ways.

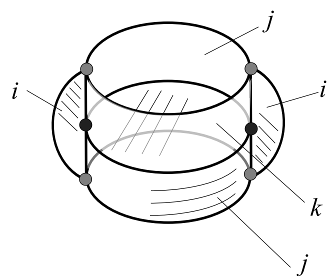


Figure 8: The V_4-connection for the bifolds W'_i.

an isomorphism which is the identity on the boundary. Thus, as drawn, F_1 and F_2 are the same foam, with two different identifications of the boundary with Γ. The identifications are indicated by the labeling of the edges in the figure.

Since the smallest action of a moduli space on Z_3 is $1/32$, the limiting solution on W'_1 or W'_2 in either case must have action 0, and must therefore be flat. For each of the (isomorphic) bifolds W'_1 and W'_2, there is just one flat bifold connection up to conjugacy. This unique connection is a V_4-connection, in which the links of the faces are mapped to the non-trivial elements $\{i, j, k\}$ of V_4 as shown in Figure 8. Inspecting Figure 7, we see that, in the case of W'_1, the links of the edges c and e for Γ are both mapped to j, while for W'_2, one is mapped to j and the other to k. That is to say, the unique bifold connection on W'_1 (respectively, W'_2) restricts to the endpoint 0 (respectively $\pi/2$) in the representation variety $R(S_{2,1}) \cong [0, \pi/2]$.

To summarize this, if we divide the ends of M_W into two classes, according to which end of G they lie over, then the restriction map r maps the ends of M_W belonging to these two classes properly to the two different ends of the interval $(0, \pi/2)$. An analysis of the glueing problem for the connected sum shows that there is just one end in each class. Indeed, the connected sum is the same as the second case of Proposition 4.3, but now the gluing parameter V extends as the group of automorphisms of the bifold connection on W_1' (or W_2'), so we have just one end instead of two. The proposition follows. □

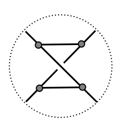
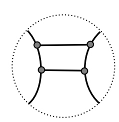
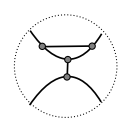
The remainder of the proof that $U_{S_{2,1}}$ is chain-homotopic to the identity follows the argument in [8]; and Theorem 3.1 follows.

8 Deducing the other exact triangles

We have now proved Theorem 3.1, which becomes Theorem 1.1 when applied to webs in $\mathbb{R}^3$. We now turn to Theorem 1.2. Because of the cyclic symmetry, it is only necessary to treat one of the three cases, so we take $i = 1$. A proof can be constructed by following the same general outline as the previous sections. The composite cobordisms have the same description as shown in Figure 5, except that the disks Δ_0^-, Δ_2 and Δ_3^+ are absent. The proof proceeds as before, except that the role of Ψ_2 will now be played by Ψ_0 and the role of Ψ_3 by Ψ_1. We also lose some of the symmetry between the three webs, so the proof needs to treat three cases.

Rather than repeat the details, we give here an alternative argument, showing how to deduce Theorem 1.2 from Theorem 1.1 by applying the basic properties of $J^\sharp$ and the results of section 4.

Let L_2, L_1 and L_0 be three webs which again differ only inside a ball, as indicated in (1). Let $\tilde{L}_i$ be obtained from L_i by attaching an extra edge, as shown in the following diagram:

$\tilde{L}_2 =$ , $\quad \tilde{L}_1 =$ , $\quad \tilde{L}_0 =$ . (13)

In each case, the added edge is the top edge in the diagram, which we call e, so

$$\tilde{L}_i = L_i \cup e.$$

We have the standard cobordisms $\Sigma_{i,i-1}$ from L_i to L_{i-1} as before, and these give rise to cobordisms

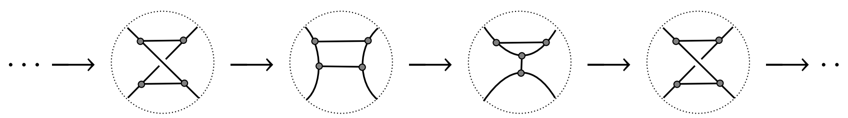
$$\tilde{\Sigma}_{i,i-1} = \Sigma_{i,i-1} \cup [0,1] \times e$$

from $\tilde{L}_i$ to $\tilde{L}_{i-1}$.

Theorem 1.1 tells us that we have an exact sequence

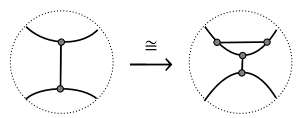
$$\cdots \longrightarrow J^\sharp(\tilde{L}_2) \longrightarrow J^\sharp(\tilde{L}_1) \longrightarrow J^\sharp(\tilde{L}_0) \longrightarrow J^\sharp(\tilde{L}_2) \longrightarrow \cdots$$

where the maps are those arising from the cobordisms $\tilde{\Sigma}_{i,i-1}$, or in pictures:

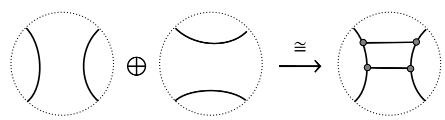


(14)

where the application of $J^\sharp$ to these terms is implied. If we apply the "triangle relation" [6, Proposition 6.6] to $\tilde{L}_0$ we see that there is an isomorphism on $J^\sharp$ between L_0 and $\tilde{L}_0$:

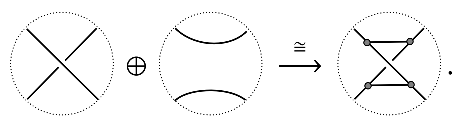


From the square relation [6, Proposition 6.8], we obtain isomorphisms



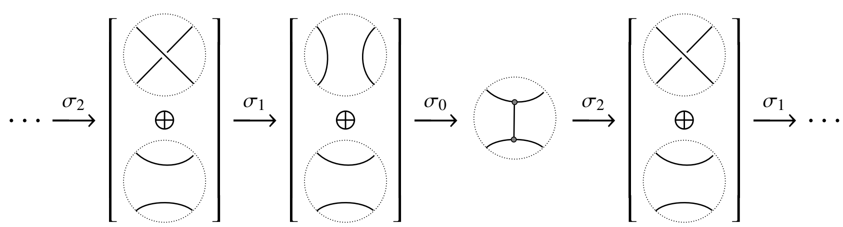
(15)

and



(16)

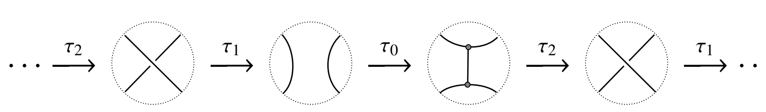
Using these isomorphism to substitute for the terms in the exact sequence (14), we obtain an isomorphic exact sequence



(17)

The maps σ_i in this sequence are obtained from the foam cobordisms $\tilde{\Sigma}_{i+1,i}$ and the isomorphisms from the triangle and square relations.

The claim in Theorem 1.2 (for $i = 0$) is that there is an exact sequence



(18)

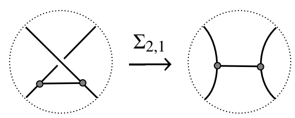
where the maps τ_i arise from the standard cobordisms. The exactness of this sequence will follow if we can show the following relations between the maps:

Proposition 8.1. *The maps σ_i and τ_i in the above diagrams are related by*

$$\sigma_1 = \begin{bmatrix} \tau_1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_0 = \begin{bmatrix} \tau_0 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} \tau_2 \\ 0 \end{bmatrix}.$$

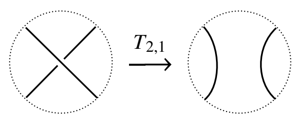
For the proof of the proposition, we need the following lemma.

Lemma 8.2. *Let $\Sigma_{2,1}$ be the standard cobordism with a single tetrahedral point,*



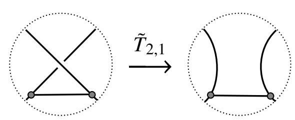
.

Let $T_{2,1}$ be the standard cobordism from K_2 to K_1,



,

and let $\tilde{T}_{2,1}$ be the union of $T_{2,1}$ with a product $[0,1] \times f$, where f is an extra edge:



.

Then $\Sigma_{2,1}$ and $\tilde{T}_{2,1}$ give rise to the same map on $J^\sharp$.

Proof. The cobordism $\tilde{T}_{2,1}$ is isomorphic to a connect sum of $\Sigma_{2,1}\#_{t,t_2}\Psi_2$, where t is the tetrahedral point. The result follows from Proposition 4.3. □

Proof of Proposition 8.1. We illustrate the arguments with one case, showing that the top left entry in the matrix for σ_1 is equal to τ_1. Consider the foam described by the movie in Figure 9. The first three frames of the movie realize the first component of the isomorphism (16). Frames 3–5 are the addition of a standard 1-handle, which realize the same map as $\tilde{\Sigma}_{2,1}$, by the lemma above. Frames 5–7 realize the first component of the inverse of the isomorphism (15). Taken together, the foam described by all seven frames gives a map equal to the top-left component of σ_1.

Regard the movie as defining a foam S in the 4-ball $[1,7] \times B^3$. The boundary of this foam is an unknot consisting of the two arcs at $t = 1$, the two arcs at $t = 7$, and the product $[1,7] \times \{\text{four points on the boundary}\}$. Let $\bar{S}$ be closed foam in $\mathbb{R}^4$

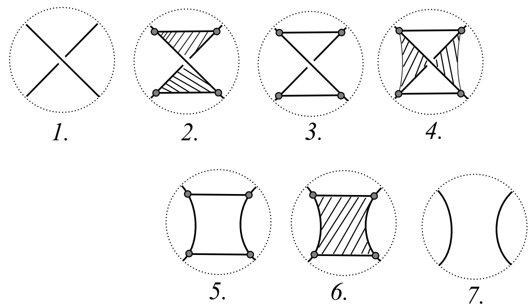


Figure 9: A movie for a foam equivalent to the top-left entry of σ_1.

obtained by attaching a disk D to this unknot, on the outside of the ball. Then, tautologically,

$$S = T_{2,1} \# \bar{S},$$

where $T_{2,1}$ is the standard 1-handle cobordism from K_2 to K_1, and the connect sum with $\bar{S}$ is made at a point of $D \subset \bar{S}$. The claim is therefore that $T_{2,1}$ and $T_{2,1}\#\bar{S}$ define the same map. An examination of the movie shows that

$$\bar{S} \cong \Psi,$$

where Ψ is the foam that appears in Proposition 4.7, in such a way that the disk D in $\bar{S}$ corresponds to the disk D^+ in Ψ. So the present claim follows from that proposition. □

9 The octahedral diagram

We now turn to the diagram in Figure 1, and will verify the properties discussed in the introduction. They are summarized in the following theorem.

Theorem 9.1. *In the diagram of standard cobordisms pictured in Figure 1, the triangles involving*

(a) L_0, K_2, K_1,

(b) L_1, K_0, K_2,

(c) L_2', K_0, K_1, *and*

(d) L_2', L_0, L_1

become exact triangles on applying $J^\sharp$. The faces

(e) K_0, K_2, K_1,

(f) L_0, K_2, L_1,

(g) K_1, L'_2, L_0, *and*

(h) L_1, L'_2, K_0

become commutative diagrams. And finally,

(i) *the composites $K_2 \to K_1 \to L'_2$ and $K_2 \to L_1 \to L'_2$ give the same map on $J^\sharp$, and*

(j) *the composites $L'_2 \to K_0 \to K_2$ and $L'_2 \to L_0 \to K_2$ give the same map on $J^\sharp$.*

Proof. The first two items are verbatim restatements of cases of Theorem 1.2. The second two cases are cases of Theorem 1.2 and Theorem 1.1. (The pictures are rotated a quarter turn relative to the standard pictures. Alternatively, these pictures portray the dual of the standard triangles.)

The composite cobordism $K_0 \to K_2 \to K_1$ is equal to the connect sum $\Sigma \# \Psi_0$, where Σ is the standard cobordism from K_0 to K_1 [8]. So the commutativity in case (e) follows from Proposition 4.5. The next three cases of the theorem are similar, except that the connect sums are with Ψ_2 at a tetrahedral point in case (f), and with Ψ_2 at a seam point in cases (g) and (h). So Propositions 4.3 and 4.4 deal with these cases.

In each of the final two cases of the theorem, the first composite cobordism is obtained from the second composite by forming a connect sum with Ψ_2 at a tetrahedral point. So these cases are also consequences of Proposition 4.3. □

10 Equivalent formulations of the Tutte relation

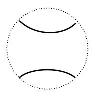
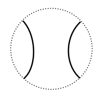
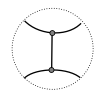
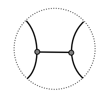
The authors conjectured in [6] that if K is a planar web (i.e. is contained in $\mathbb{R}^2 \subset \mathbb{R}^3$), then the dimension of $J^\sharp(K)$ is equal to the number of Tait colorings of the underlying abstract graph. (See Conjecture 1.2 in [6].) As explained there, confirming this conjecture would provide a new proof of Appel and Haken's four-color theorem. If we write $\tau(K)$ for the number of Tait colorings of K, then τ is uniquely characterized, for planar webs, by the following properties:

(a) $\tau(K) = 3$ if K is a circle;

(b) $\tau(K) = 0$ if K has a bridge;

(c) τ is multiplicative for disjoint unions of planar webs;

(d) τ satisfies the "Tutte relation", namely if K_0, K_1, L_0, L_1 are planar webs which differ only in a ball, in the following manner,

$K_0 =$ , $K_1 =$ , $L_0 =$ , $L_1 =$ ,

then

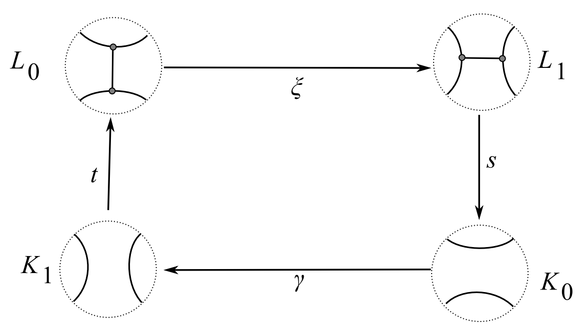
$$\tau(K_0) - \tau(K_1) + \tau(L_0) - \tau(L_1) = 0. \tag{19}$$

The first three of these properties hold also for the quantity $\dim J^\sharp(K)$, for planar webs K. They are proved in [6]. So the question of whether $\dim J^\sharp(K)$ is equal to the number of Tait colorings is equivalent to the following conjecture:

Conjecture 10.1. *If K_0, K_1, L_0, L_1 are three planar webs differing only in a ball as shown above, then*

$$\dim J^\sharp(K_0) - \dim J^\sharp(K_1) + \dim J^\sharp(L_0) - \dim J^\sharp(L_1) = 0. \tag{20}$$

The four webs that appear in this conjecture appear also as the four vertices of the central rectangle in the octahedral diagram, Figure 1. We reproduce that part of the diagram here:



(21)

Lemma 10.2. *In the rectangle above, the composite of any two consecutive foams is the zero map on $J^\sharp$. So the vector spaces $J^\sharp(K_0)$, $J^\sharp(K_1)$, $J^\sharp(L_0)$, $J^\sharp(L_1)$, together with the maps between them, form a chain complex which is periodic mod* 4.

Proof. Referring to Figure 1 and Theorem 9.1, we see that (with the application of $J^\sharp$ implied), $t \circ \gamma = t \circ b \circ a$, because $\gamma = b \circ a$. On the other hand, $t \circ b = 0$, because these are two sides of an exact triangle. This shows that $t \circ \gamma = 0$, and essentially the same argument deals with the composites at the other three vertices. □

We can now interpret Conjecture 10.1 as asserting that the Euler characteristic of the 4-periodic complex (21) is zero. Since the Euler characteristic can be computed equally as the alternating sum of the dimensions of the chain groups or as the alternating sum of the dimensions of the homology groups, the left-hand side of (20) can be expressed also as

$$\left|\frac{\ker(\gamma)}{\operatorname{im}(s)}\right| - \left|\frac{\ker(t)}{\operatorname{im}(\gamma)}\right| + \left|\frac{\ker(\xi)}{\operatorname{im}(t)}\right| - \left|\frac{\ker(s)}{\operatorname{im}(\xi)}\right|, \tag{22}$$

where $|V|$ denotes the dimension of the vector space V, and $J^\sharp$ is understood.

Lemma 10.3. *In the* 4*-periodic complex* (21)*, the homology groups at diametrically opposite corners are equal. Furthermore, the Euler characteristic* (22) *is equal to*

$$2(\operatorname{rank}(a \circ \kappa) - \operatorname{rank}(\lambda \circ b)),$$

and also equal to

$$2(\operatorname{rank}(a) - \operatorname{rank}(b)).$$

Here a, b, κ and λ are the maps corresponding to the standard cobordisms in Figure 1, and the application of $J^\sharp$ is implied.

Proof. From the exactness of the triangles of maps $(\lambda, \kappa, \gamma)$ and (a, r, s), we have $\ker(\gamma) = \operatorname{im}(\kappa)$ and $\operatorname{im}(s) = \ker(a)$. So

$$\frac{\ker(\gamma)}{\operatorname{im}(s)} = \frac{\operatorname{im}(\kappa)}{\ker(a)}.$$

At the same time, we learn that $\ker(a) \subset \operatorname{im}(\kappa)$, so that the dimension of $\operatorname{im}(\kappa)/\ker(a)$ is equal to the rank of $a \circ \kappa$. Thus

$$\left|\frac{\ker(\gamma)}{\operatorname{im}(s)}\right| = \operatorname{rank}(a \circ \kappa).$$

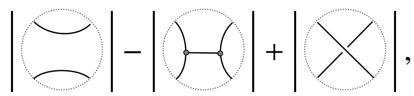
A similar discussion can be applied to each of the four terms in (22), so the dimensions of the four quotients that appear there are respectively

$$\operatorname{rank}(a \circ \kappa), \quad \operatorname{rank}(\lambda \circ b), \quad \operatorname{rank}(q \circ \zeta), \quad \operatorname{rank}(\eta \circ r).$$

The last two parts of Theorem 9.1 say that $\lambda \circ b = \eta \circ r$ and $a \circ \kappa = q \circ \zeta$. So alternate terms of the above four are equal. This verifies the first assertion in the lemma, and also shows that the Euler characteristic is equal to $2(\mathrm{rank}(a \circ \kappa) - \mathrm{rank}(\lambda \circ b))$.

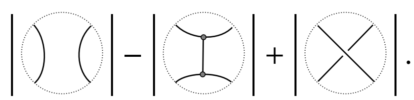
From the exact triangle (a, r, s), we have

$$2\,\mathrm{rank}(a) =$$



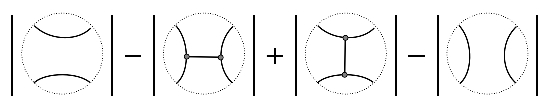
while from the exact triangle (b, t, q), we have

$$2\,\mathrm{rank}(b) =$$



Taking the difference of these last two equalities, we obtain

$$2\,\mathrm{rank}(a) - 2\,\mathrm{rank}(b) =$$



This is the last assertion of the lemma. □

Remarks. Since Conjecture 10.1 asserts the vanishing of an Euler characteristic for planar webs, it is natural to ask whether something stronger is true, namely that the 4-periodic sequence of maps (21) is exact. By the proof of the above lemma, this would be equivalent to the vanishing of the composites $a \circ \kappa$ and $\lambda \circ b$. Although this holds in simple cases, it does not appear to be true in general. Calculations for the case that L_0 is the 1-skeleton of a dodecahedron (in the natural planar projection) suggest that the rank of $a \circ \kappa$ is 5 in this case [5]. From this it follows (by the lemma) that the Euler characteristic of the complex is at most 10. The webs K_0, K_1 and L_1 in this example are "simple webs" in the sense of [6], so $J^\sharp$ is easily computed for these three. The inequality on the Euler characteristic then tells us that the dimension of $J^\sharp$ of the dodecahedral graph is at most 70. In the other direction, a different calculation [5] leads to a lower bound of 58 on the dimension. So we have

$$58 \leq \dim J^\sharp(\text{Dodecahedron}) \leq 70.$$

The number of Tait colorings, on the other hand, is 60.

A slightly sharper upper bound for the dimension of $J^\sharp$ of the dodecahedron arises by a different argument. In [7], the representation variety $\mathcal{R}^\sharp(K)$ is described for the dodecahedron: it consists of 10 copies of the flag manifold and two copies of $SO(3)$. The Chern-Simons functional is Morse-Bott along $\mathcal{R}^\sharp(K)$, and it follows that there is a perturbation of the Chern-Simons functional having exactly 68 critical points, leading to a bound of 68 on the rank of $J^\sharp$. From this point of view, the question of whether the rank is strictly less than 68 is therefore the question of whether there are any non-trivial differentials in the complex, for this particular perturbation.

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