Inverting the Hopf map

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Abstract

We calculate the η-localization of the motivic stable homotopy ring over $\mathbb{C}$, confirming a conjecture of Guillou and Isaksen. Our approach is via the motivic Adams-Novikov spectral sequence. In fact, work of Hu, Kriz, and Ormsby implies that it suffices to compute the corresponding localization of the classical Adams-Novikov E_2-term, and this is what we do. Guillou and Isaksen also propose a pattern of differentials in the localized motivic classical Adams spectral sequence, which we verify using a method first explored by Novikov.

Dedicated to the memory of Goro Nishida (1943-2014)

1 Introduction

1.1 Overview

The chromatic approach to stable homotopy theory [11] rests on the fact that non-nilpotent graded endomorphisms of finite complexes can be essentially classified. They are always detected by MU (or, working locally at a prime, by BP), and up to taking p^{th} powers every graded BP_*BP-comodule endomorphism survives the Adams-Novikov spectral sequence. At the prime 2, for example, the Hopf map $\eta \in \pi_1(\mathbb{S})$ lies in filtration one, and the celebrated nilpotence theorem of Nishida [23] already guarantees that η is nilpotent; in fact, we know that $\eta^4 = 0$.

On the other hand, the element $\overline{\alpha}_1$ that detects η in the Adams-Novikov E_2-term $E_2(\mathbb{S}; BP)$ is non-nilpotent. This immediate failure of the Adams-Novikov E_2-term to accurately reflect 2-primary stable homotopy even in low degrees has lessened its attractiveness as a computational tool at the prime 2. One can nevertheless hope to calculate $\overline{\alpha}_1^{-1}E_2(\mathbb{S}; BP)$ and discover the range in which the localization map is an isomorphism, and this is our first main result.

Some $\overline{\alpha}_1$-free elements have been known for many years. The group $E_2^{1,2n}(\mathbb{S}; BP)$ is cyclic for $n \geq 1$, generated by elements $\overline{\alpha}_n$ closely related to the image of J [24, Theorem 11.2], [20, Corollary 4.23]; in particular, $\overline{\alpha}_4$ detects $\sigma \in \pi_7(\mathbb{S})$. In [20] it was shown that for $n \neq 2$, $\overline{\alpha}_1^k\overline{\alpha}_n \neq 0$ for all $k \geq 0$. The Adams-Novikov differentials on these classes are also well-known and due to Novikov [24], [25, p. 171]. In this paper we show that there are no other $\overline{\alpha}_1$-free generators. A more precise statement is that

$$\overline{\alpha}_1^{-1}E_2(\mathbb{S}; BP) = \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_4^2). \tag{1.1.1}$$

The complete structure of the localized Novikov spectral sequence now follows from the differential

$$d_3\overline{\alpha}_3 = \overline{\alpha}_1^4\,. \tag{1.1.2}$$

Since $\overline{\alpha}_1$ is a unit, this differential terminates the localized spectral sequence.

We also find that the localization map $E_2(\mathbb{S}; BP) \longrightarrow \overline{\alpha}_1^{-1} E_2(\mathbb{S}; BP)$ is an isomorphism above a line of slope $1/5$ when we plot the Adams-Novikov spectral sequence in the usual manner. This resolves a question raised by Zahler [28] at the dawn of the chromatic era.

This result is of secondary interest from the perspective of the classical homotopy groups of spheres because we already know that $\eta^4 = 0$. However, the advent of motivic homotopy theory has led to interesting related questions. There is a ground field in motivic homotopy theory, which for us will be the complex numbers, and homotopy is bigraded. It is known that the motivic $\eta \in \pi_{1,1}(\mathbb{S}_{\mathrm{Mot}})$ is non-nilpotent, and one may ask (as Dugger and Isaksen did [6]) to calculate $\eta^{-1}\pi_{*,*}(\mathbb{S}_{\mathrm{Mot}})$. The determination of this localization is our second major result. We will in fact compute $\pi_{*,*}(\eta^{-1}((\mathbb{S}_{\mathrm{Mot}})^\wedge_2))$, but a simple argument (Lemma 8.2) shows that $\eta^{-1}\mathbb{S}_{\mathrm{Mot}} \longrightarrow \eta^{-1}((\mathbb{S}_{\mathrm{Mot}})^\wedge_2)$ is an equivalence so we get an uncompleted result as well.

We can describe the result in terms of elements in $\pi_{*,*}(\mathbb{S}_{\mathrm{Mot}})$. There are motivic "Hopf maps"

$$\eta \in \pi_{1,1}(\mathbb{S}_{\mathrm{Mot}}), \quad \sigma \in \pi_{7,4}(\mathbb{S}_{\mathrm{Mot}})$$

as well as a unique nonzero class $\mu_9 \in \pi_{9,5}(\mathbb{S}_{\mathrm{Mot}})$ ([12], 2.10). Then

$$\eta^{-1}\pi_{*,*}(\mathbb{S}_{\mathrm{Mot}}) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\sigma^2)$$

This extends work of Morel [22], who verified this calculation (and extended it to a general ground field) for coweight zero, and of Hu, Kriz, and Ormsby [14], who showed that the localization is at least this big, and it verifies a conjecture of Guillou and Isaksen [9]. In subsequent work [10] these authors carry out the analogous computation over the reals, starting with the result over $\mathbb{C}$ obtained here.

1.2 Methods

In our approach to (1.1.1) we follow Novikov [24] as interpreted in [19]: we filter the BP cobar construction by powers of the kernel of the augmentation $BP_* \to \mathbb{F}_2$. The resulting "algebraic Novkiov spectral sequence" has the form

$$H^*(P; Q) \implies E_2(\mathbb{S}; BP)$$

where P is the Hopf subalgebra of squares in the dual Steenrod algebra A and

$$Q = \mathrm{gr}^* BP_* = \mathbb{F}_2[q_0, q_1, \ldots]$$

is the associated graded of BP_*; q_n is the class of the Hazewinkel generator v_n. In this spectral sequence the element $\overline{\alpha}_1$ is represented by the class of $[\xi_1^2]$, which, following the notational conventions in force at an odd prime, we denote by h_0. By inverting h_0 we obtain a "localized algebraic Novikov spectral sequence" converging to $\overline{\alpha}_1^{-1} E_2(\mathbb{S}; BP)$. Our main result is a complete description of this spectral sequence.

The E_1-page of the localized algebraic Novikov spectral sequence is given by $h_0^{-1}H^*(P; Q)$. The computation of this object parallels the well-known fact [17, 18] that if M is a bounded below comodule over A then the localized cohomology of A with coefficients in M can be computed in terms of the homology of M with respect to the differential Sq^1:

$$q_0^{-1}H^*(A; M) = H(M; \mathrm{Sq}^1) \otimes \mathbb{F}_2[q_0^{\pm 1}]$$

where $q_0 \in H^{1,1}(A)$ is the class dual to Sq^1. The result we obtain is

$$h_0^{-1}H^*(P;Q) = \mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots]\,.$$

As in [19], the differentials in the localized algebraic Novikov spectral sequence are calculated by exploiting the connection between $E_2(\mathbb{S};BP)$ and the theory of formal groups. The facts we rely on are easier to obtain than those used in [19] and date back to [21]. We prove that

$$d_1 q_{n+1} = q_n^2 h_0 \quad \text{for } n \geq 2\,. \tag{1.2.1}$$

This is the last possible differential, and we arrive at (1.1.1). It follows, incidentally, that for $m, n \geq 0$, $\overline{\alpha}_{2m}\overline{\alpha}_{2n}$ is $\overline{\alpha}_1$-torsion, and that modulo $\overline{\alpha}_1$-torsion the classes $\overline{\alpha}_{2m+1}\overline{\alpha}_{2n+1}$ and $\overline{\alpha}_{2m+1}\overline{\alpha}_{2n}$ depend only on the sum $m+n$.

This computation leads without difficulty to the motivic result. Hu, Kriz and Ormsby ([13,14]; see also [6]) study a motivic analogue of the Adams-Novikov spectral sequence. To circumvent incomplete understanding of the motivic analogue of BP, they work with its 2-adic completion, which we will denote by BPM. They show that its E_2-term is

$$E_2(\mathbb{S}_{\mathrm{Mot}};BPM) = E_2(\mathbb{S};BP) \otimes \mathbb{Z}_2[\tau]\,,$$

where τ detects a certain class in $\pi_{0,-1}((\mathbb{S}_{\mathrm{Mot}})^\wedge_2)$. Inverting $\overline{\alpha}_1$ results in a spectral sequence which we show converges to $\pi_{*,*}(\eta^{-1}\mathbb{S}_{\mathrm{Mot}}) = \pi_{*,*}((\mathbb{S}_{\mathrm{Mot}})^\wedge_2)$. The unique nonzero differential is determined by the motivic analogue of (1.1.2), namely

$$d_3\overline{\alpha}_3 = \tau\overline{\alpha}_1^4\,, \tag{1.2.2}$$

and (1.1) follows.

Our calculation of $\eta^{-1}\pi_{*,*}(\mathbb{S}_{\mathrm{Mot}})$ verifies the conjecture of Guillou and Isaksen [9]). Those authors approached this calculation by means of the motivic classical Adams spectral sequence. Using the motivic May spectral sequence they found that

$$h_0^{-1}E_2(\mathbb{S}_{\mathrm{Mot}};H) = \mathbb{F}_2[h_0^{\pm 1}, v_1^4, v_2, v_3, \ldots]\,. \tag{1.2.3}$$

Based on extensive computations they conjectured that in the localized motivic Adams spectral sequence

$$d_2 v_{n+1} = v_n^2 h_0\,, n \geq 2\,. \tag{1.2.4}$$

In the last part of this paper we recover (1.2.3) using a localized Cartan-Eilenberg spectral sequence. We then use the methods of [19] to show that the Adams differentials (1.2.4) follow from the differentials (1.2.1) in the algebraic Novikov spectral sequence.

1.3 Organization

In Section 2 we recall the form of the Adams-Novikov E_2 term and the definition of the elements $\overline{\alpha}_i \in E_2^{1,2i+1}(\mathbb{S};BP)$. Next we recall the algebraic Novikov spectral sequence, and in Section 4 we compute the localization of its initial term and verify vanishing lines required to check its convergence. These vanishing lines give us a range of dimensions in which the localization map is an isomorphism. In Section 6 we construct some permanent cycles, and then bring the theory

of formal groups into play to compute the differential in the localized algebraic Novikov spectral sequence and verify (1.1.1).

In Section 8 we recall the basic setup for motivic stable homotopy theory over $\mathbb{C}$, and prove (1.1). In the final section, we provide an improved statement and proof of the comparison of differentials in the Adams and algebraic Novikov spectral sequences ([19,24]), and observe that the local algebraic Novikov differentials then imply the local motivic Adams differentials conjectured by Guillou and Isaksen.

1.4 Acknowledgments

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2 The Adams-Novikov spectral sequence

In this paper we follow [20] and work with *right* comodules. Given a Hopf algebroid (A,Γ) and a right Γ-comodule M, the *cobar construction* $\Omega^*(\Gamma;M)$ has $\Omega^s(\Gamma;M) = M\otimes_A \overline{\Gamma}\otimes_A\cdots\otimes_A\overline{\Gamma}$ with s copies of $\overline{\Gamma} = \ker(\epsilon:\Gamma\to A)$, and is equipped with a natural differential [20, (1.10)] of degree 1. $H^*(\Gamma;M)$ denotes the cohomology of this complex. If Γ and M are graded then $H^*(\Gamma;M)$ becomes bigraded; the first index is the cohomological grading and the second is inherited from the gradings on Γ and M.

Recall (e.g. [21, §2]) the Hopf algebroid given to us by the p-typical factor of complex cobordism: (BP_*, BP_*BP). We will work at the prime $p=2$ throughout this paper. The Adams-Novikov spectral sequence for a connective spectrum X takes the following form:

$$E_2^{s,u} = H^{s,u}(BP_*BP; BP_*(X)) \Longrightarrow \pi_{u-s}(X)\otimes\mathbb{Z}_{(2)},\ d_r : E_r^{s,u}\longrightarrow E_r^{s+r,u+r-1}\,.$$

The coefficient ring of BP is a polynomial algebra,

$$BP_* = \mathbb{Z}_{(2)}[v_1,v_2,\ldots]\,,\quad |v_i| = 2(2^i-1)\,.$$

As in [20], we have a short exact sequence of BP_*BP-comodules

$$0\longrightarrow BP_*\longrightarrow 2^{-1}BP_*\longrightarrow BP_*/2^\infty\longrightarrow 0\,,$$

which gives rise to a connecting homomorphism

$$\delta : H^{0,*}(BP_*BP; BP_*/2^\infty)\longrightarrow H^{1,*}(BP_*BP; BP_*)\,.$$

Following [20], define $x = v_1^2 - 4v_1^{-1}v_2 \in 2^{-1}v_1^{-1}BP_*$. When $s \geq 2$, the image of $x^s/8s$ in $v_1^{-1}BP_*/2^\infty$ lies in the subgroup $BP_*/2^\infty$. We define

$$\overline{\alpha}_s = \begin{cases} \delta(v_1^s/2) & s \geq 1 \text{ and odd} \\ \delta(v_1^2/4) & s = 2 \\ \delta(x^{s/2}/4s) & s \neq 2 \text{ and even}\,. \end{cases}$$

Proposition 2.1 ([20])**.** *These classes generate $H^{1,*}(BP_*BP)$ and are of the same 2-order as the denominator of the element to which δ was applied.*

The element $\overline{\alpha}_1$ is a permanent cycle detecting $\eta \in \pi_1(\mathbb{S})$. In Lemma 7.2 we will show that for $s \neq 2$, $\overline{\alpha}_s$ is not killed by any power of $\overline{\alpha}_1$. This is [20, Corollary 4.23], but our proof is different. We will analyze the images of these elements in $\overline{\alpha}_1^{-1}H^*(BP_*BP)$, an object we will compute explicitly. We show that these classes, and 1, generate $\overline{\alpha}_1^{-1}H^*(BP_*BP)$ as an $\mathbb{F}_2[\overline{\alpha}_1^{\pm 1}]$-vector space.

3 The algebraic Novikov spectral sequence

One of the best tools for gaining information about the E_2-page of the Adams-Novikov spectral sequence is the algebraic Novikov spectral sequence [19, 24], which arises from filtering the cobar complex $\Omega^*(BP_*BP)$ by powers of the augmentation ideal

$$I = \ker(BP_* \to \mathbb{F}_2)\,.$$

In this paper we will write A for the dual of the Steenrod algebra and P for the Hopf subalgebra of squares in A. Write

$$\zeta_n = \overline{\xi}_n^2$$

for the square of the conjugate of the Milnor generator ξ_n, so that

$$P = \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$$

with diagonal

$$\Delta\zeta_n = \sum_{i+j=n} \zeta_i \otimes \zeta_j^{2^i}\,.$$

Define the graded algebra in P-comodules

$$Q = \mathbb{F}_2[q_0, q_1, \ldots]\,, \quad |q_i| = (1, 2(2^i - 1))\,,$$

with coaction defined by

$$q_n \longmapsto \sum_{i+j=n} q_i \otimes \zeta_j^{2^i}\,.$$

Write Q^t for the component of Q with first gradation t; this is the "Novikov degree." Then [19]

$$\mathrm{gr}^t\Omega^s(BP_*BP) = \Omega^s(P; Q^t) \tag{3.1}$$

and the algebraic Novikov spectral sequence takes the form

$$E_1^{s,t,u} = H^{s,u}(P; Q^t) \Longrightarrow H^{s,u}(BP_*BP)\,, \quad d_r : E_r^{s,t,u} \longrightarrow E_r^{s+1,t+r,u}\,. \tag{3.2}$$

Certain elements will be important for us. We will abbreviate the element in $H^{0,0}(P;Q^1)$ represented by $q_0[\,]$ to q_0, and write h_n for the element in $H^{1,2^{n+1}}(P;Q^0)$ represented by $[\zeta_1^{2^n}]$.

The algebra $H^*(P;Q)$ is not only the E_1-page for the algebraic Novikov spectral sequence, but also the E_2-page for the Cartan-Eilenberg spectral sequence associated to the extension of Hopf algebras

$$P \longrightarrow A \longrightarrow E,$$

where E denotes the exterior algebra $E[\overline{\xi}_1, \overline{\xi}_2, \ldots]$. The E_2 term has the form $H^*(P;H^*(E))$, and

$$H^*(E) = \mathbb{F}_2[q_0, q_1, \ldots] = Q\,, \quad q_{n-1} = [\overline{\xi}_n]\,.$$

The coaction of P on Q coincides with the action described above. The extension spectral sequence converges to the E_2-page of the Adams spectral sequence:

$$E_2^{s,t,u} = H^{s,u}(P;Q^t) \Longrightarrow H^{s+t,u+t}(A),\ d_r : E_r^{s,t,u} \longrightarrow E_r^{s+r,t-r+1,u+r-1}\,. \tag{3.3}$$

This leads to the following square of spectral sequences [19, 24]:

$$\begin{array}{ccc} H^{s,u}(P;Q^t) & \xRightarrow{\text{CESS}} & E_2^{s+t,u+t}(\mathbb{S};H) \\ \Big\Downarrow{\scriptstyle \text{ANSS}} & & \Big\Downarrow{\scriptstyle \text{ASS}} \\ E_2^{s,u}(\mathbb{S};;BP) & \xRightarrow{\text{NSS}} & \pi_{u-s}(\mathbb{S}) \end{array} \tag{3.4}$$

It is useful to keep in mind two projections of the trigraded object $H^*(P;Q^*)$, corresponding to the two spectral sequences of which it is an initial term. The "Novikov projection" displays $(u-s,s)$ and suppresses the filtration grading in the algebraic Novikov spectral sequence (Novikov weight); it presents the spectral sequence as it will appear in $E_2(\mathbb{S};BP)$. The "Adams projection" displays $(u-s,s+t)$, and suppresses the filtration grading s in the Cartan-Eilenberg spectral sequence; it presents the spectral sequence as it will appear in $E_2(\mathbb{S};H)$.

In Figure 1 we have shown the low-dimensional part of these two projections. Vertical black lines indicate multiplication by q_0. The vertical blue arrow indicates a q_0 tower which continues indefinitely. Black lines of slope one indicate multiplication by h_0. The blue arrows of slope one indicate h_0 towers which continue indefinitely. Green arrows denote algebraic Novikov differentials and red arrows denote Cartan-Eilenberg differentials. On the first chart square nodes denote multiple basis elements connected by q_0-multiplication; the number to the upper left indicates how many such basis elements.

4 Localizing the algebraic Novikov E_1-page.

We wish to localize the algebraic Novikov spectral sequence by inverting h_0. We begin with a well-known localization theorem, dealing with comodules over the dual Steenrod algebra A. In working with ordinary homology we will work with left comodules, to be consistent with our presentation of the proof of Theorem 9.3.3 below, though of course the categories of left comodules and right comodules are equivalent via the anti-automorphism in the Hopf algebra A. Write q_0 for the class of $[\xi_1]$ in $H^{1,1}(A)$. It acts on $H^*(A;M)$ for any A-comodule M. Write E for the quotient Hopf algebra

$$E = A/(\xi_1^2, \xi_2, \xi_3, \ldots)\,.$$

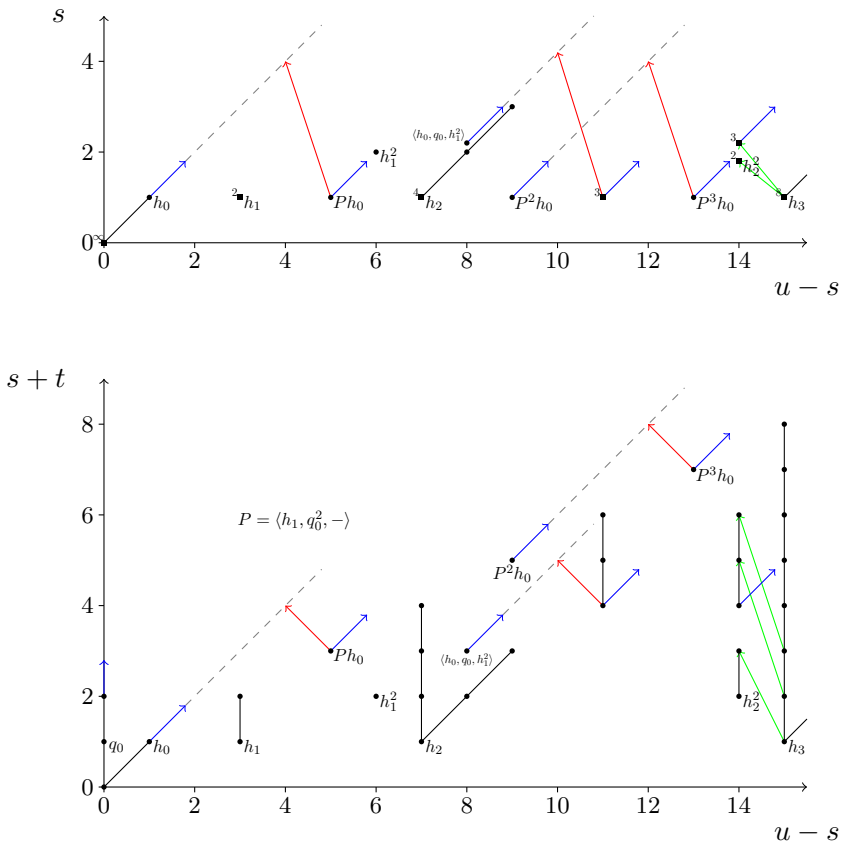


Figure 1: $H^*(P;Q)$ in Novikov and classical Adams projections. Green arrows are algebraic Novikov differentials and red arrows are Cartan-Eilenberg differentials.

It is the exterior algebra generated by the image of ξ_1. Any A-comodule M becomes an E-comodule, and an E-comodule structure on M is equivalent to a degree -1 differential Sq^1 on M given in terms of the coaction by

$$x \longmapsto 1 \otimes x + \xi_1 \otimes x\mathrm{Sq}^1 .$$

Proposition 4.1. *Let M be an A-comodule such that $M_u = 0$ whenever $u < 0$, and consider the following diagram.*

$$\begin{array}{ccc} H^*(A;M) & \longrightarrow & H^*(E;M) \\ \downarrow & & \downarrow \\ q_0^{-1}H^*(A;M) & \longrightarrow & q_0^{-1}H^*(E;M) \end{array}$$

The top map is surjective in bidegrees (s,u) with $u-s < 2s-2$ and an isomorphism in bidegrees with $u-s < 2s-5$, i.e. above a line of slope $1/2$ in the usual $(u-s,s)$ plot. The bottom map is an isomorphism and the right map is an isomorphism for bidegrees (s,u) with $s > 0$. Moreover,

$$q_0^{-1}H^*(E;M) = H(M;Sq^1) \otimes \mathbb{F}_2[q_0^{\pm 1}] .$$

Proof. The cotensor product $A\Box_E M$ is a submodule of $A \otimes M$. Since the coaction map $M \longrightarrow A \otimes M$ is associative, it factors through a map $i : M \longrightarrow A\Box_E M$. Define L by the following short exact sequence of A-comodules.

$$0 \longrightarrow M \xrightarrow{i} A\Box_E M \longrightarrow L \longrightarrow 0 \tag{4.2}$$

We claim that $H^{s,u}(A;L) = 0$ whenever $u - s < 2s - 2$.

If $M = \mathbb{F}_2$, the middle comodule $A\Box_E\mathbb{F}_2$ is the homology of the integral Eilenberg Mac Lane spectrum. It is well known, in that case, that the map i induces an isomorphism in Sq^1-homology. (One way to see this is to think about the dual: left multiplication by Sq^1 gives a bijection between the Cartan-Serre basis elements for $H^*(H\mathbb{Z})$ with even leading entry and those with odd leading entry, with the exception of the basis element 1 in dimension 0.) Filtering the general comodule M by dimension shows that the same is true in general. We deduce that L is Sq^1-acyclic and so we can apply [1, Theorem 2.1] or [2, Theorem 1.1] to give the claimed vanishing line for $H^*(A;L)$.

Under the identification $H^*(A;A\Box_E M) = H^*(E;M)$, the map induced by applying $H(A;-)$ to i is the top map in the proposition statement and so the first statement in the proposition follows from the cohomology long exact sequence associated to (4.2) and the vanishing line just proved.

Since q_0 acts vertically in $(u-s,s)$ coordinates we find that the bottom map is an isomorphism. The remaining statements follow from the identification

$$H^*(E;M) = \frac{\ker(\mathrm{Sq}^1) \otimes \mathbb{F}_2[q_0]}{\mathrm{im}(\mathrm{Sq}^1) \otimes (q_0)}.$$

□

Thus the localization of the Adams E_2-page coincides with the E_2-page of the Bockstein spectral sequence. In fact [17,19] the two spectral sequences coincide from E_2 onwards, giving a qualitative strengthening of Serre's observation that $\pi_*(X) \otimes \mathbb{Q} \cong H_*(X;\mathbb{Q})$.

By doubling degrees we obtain a parallel result for the Hopf subalgebra P of A. Now E will be the quotient Hopf algebra $P/(\zeta_1^2, \zeta_2, \ldots)$. Any P-comodule M becomes an E-comodule, and just as

we wrote Sq^1 above we will write P^1 for the operator on a right E-comodule corresponding to ζ_1. A P-comodule splits naturally into even and odd parts, which one can handle separately to prove the following result.

Proposition 4.3. *Let M be a P-comodule such that $M_u = 0$ whenever $u < 0$, and consider the following diagram.*

$$\begin{array}{ccc} H^*(P;M) & \longrightarrow & H^*(E;M) \\ \downarrow & & \downarrow \\ h_0^{-1}H^*(P;M) & \longrightarrow & h_0^{-1}H^*(E;M) \end{array}$$

The top map is surjective in bidegrees (s,u) with $u-s < 5s-4$ and an isomorphism in bidegrees with $u-s < 5s-10$, i.e. above a line of slope $1/5$ in the usual $(u-s,s)$ plot. The bottom map is an isomorphism and the right map is an isomorphism for bidegrees (s,u) with $s > 0$. Moreover,

$$h_0^{-1}H^*(E;M) = \mathbb{F}_2[h_0^{\pm 1}] \otimes H(M;P^1)\,.$$

As an application, we obtain a calculation of the h_0-localization of the E_1-page of the algebraic Novikov spectral sequence together with the range in which the localization map is an isomorphism.

Corollary 4.4. *For any t, the localization map $H^*(P;Q^t) \longrightarrow h_0^{-1}H^*(P;Q^t)$ is surjective in bidegrees (s,u) with $u-s < 5s-4$ and an isomorphism in bidegrees with $u-s < 5s-10$. Moreover,*

$$h_0^{-1}H^*(P;Q) = \mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots]\,.$$

Proof. It is enough to note that $q_1P^1 = q_0$, $\ker P^1 = \mathbb{F}_2[q_0, q_1^2, q_2, q_3, \ldots]$, and $\operatorname{im} P^1 = (q_0)$. □

In order to check convergence of the localized algebraic Novikov spectral sequence we will require some basic vanishing lines, which are suggested by the diagrams in Figure 1. The first one is easy:

Lemma 4.5. *$H^{s,u}(P;Q^t) = 0$ when $u-s < s$.*

Proof. The cobar construction has the form

$$\Omega^{s,u}(P;M) = (\overline{P}^{\otimes s} \otimes M)_u$$

where $\overline{P}$ denotes the positive-dimensional part of P. Since $\overline{P}_u = 0$ for $u < 2$, $\Omega^{s,u}(P;M) = 0$ for $u < m + 2s$ if $M_u = 0$ for $u < m$.

For any t, $Q^{t,u} = 0$ for $u < 0$. Thus $\Omega^{s,u}(P;Q^t) = 0$ for $u < 2s$. □

This crude vanishing line can be improved when $u > s$:

Lemma 4.6. *$H^{s,u}(P;Q^t) = 0$ when $0 < u-s < s+t$.*

Proof. The cobar complex itself vanishes if u, s, or t is negative. The groups $H^{0,*}(P;Q^t)$ constitute the primitives of Q^t, which are generated as a vector space by q_0^t. The constraint $0 < u-s$ avoids these classes.

Define an algebra map $\varphi : Q \longrightarrow P$ by sending q_n to ζ_n (so q_0 maps to 1). Restricting to Novikov degree t gives us an embedding of P-comodules, and, by the vanishing of $H^*(P;P)$, the boundary map $H^{s-1,u}(P;P/Q^t) \longrightarrow H^{s,u}(P;Q^t)$ in the associated long exact sequence is surjective as long as $s > 0$. Now $(P/Q^t)_u = 0$ if $u < 2(t+1)$, since the first element not in the image of $\varphi|_{Q^t}$ is ζ_1^{t+1}. Thus $H^{s-1,u}(P;P/Q^t)$ is zero provided that $u < 2(s-1)+2(t+1) = 2s+2t$. Since $Q^t = 0$ for $t < 0$, we may assume $t \geq 0$, and then $u-s < s+t$ implies $u < 2s+2t$. □

5 The localized algebraic Novikov spectral sequence

The algebraic Novikov spectral sequence is multiplicative since it is obtained by filtering the DG algebra $\Omega^*(BP_*BP)$ by powers of a differential ideal. Since the class $h_0 \in E_1^{1,0,2}$ is a permanent cycle and $H^*(BP_*BP)$ is commutative, inverting h_0 gives a new multiplicative spectral sequence.

In forming this localization we may lose convergence; this is the issue at stake in the "telescope conjecture" of chromatic homotopy theory. Here we are lucky, however. Convergence is preserved because, as was the case in [19], the operator we are inverting acts parallel to a vanishing line. This vanishing line is visible in the lower diagram in figure 1 and is the content of Proposition 4.6.

The vanishing line has the following two implications for the algebraic Novikov spectral sequence.

1. If $x \in H^*(BP_*BP)$ is not killed by any power of $\overline{\alpha}_1$, then for some k, $\overline{\alpha}_1^k x$ has a representative at E_1 that is not killed by any power of h_0.

2. For any $a \in E_1$, $h_0^k a$ is a permanent cycle for all sufficiently large k (though it may be zero).

The first fact tells us that we detect everything we are supposed to; the second fact tells us that we do not detect more than we are supposed to. A more thorough account of similar convergence issues may be found in [3].

Coupled with the isomorphism range in Corollary 4.4, the natural map of spectral sequences from the algebraic Novikov spectral sequence to its localized counterpart implies an isomorphism range for the $\overline{\alpha}_1$-localization of the Adams-Novikov E_2-page for the sphere.

Proposition 5.1. *The localization map*

$$H^*(BP_*BP) \longrightarrow \overline{\alpha}_1^{-1} H^*(BP_*BP)$$

is surjective in bidegrees (s,u) for which $u-s < 5s-4$ and an isomorphism in bidegrees for which $u-s < 5s-10$, i.e. above a line of slope $1/5$ in the usual $(u-s,s)$ plot.

Proof. We use the natural map of spectral sequences from the algebraic Novikov spectral sequence to its localized counterpart.

First, Lemma 4.6 implies that in each bidegree (s,u) other than $(0,0)$,

$$E_1^{s,t,u} = 0 \quad \text{for} \quad t > u - 2s\,.$$

The same then holds for E_r for all $r \geq 0$, so by convergence of the spectral sequence

$$F^{u-2s} H^{s,u}(BP_*BP) = 0 \quad \text{for} \quad (s,u) \neq (0,0)\,.$$

Multiplication by h_0 preserves $u-2s$ and t, so the same facts hold for the localized spectral sequence and for the filtration of $\overline{\alpha}_1^{-1} H^*(BP_*BP)$.

Next, observe that Lemma 4.6 implies that

$$d_r(E_r^{s,t,u}) = 0 \quad \text{for} \quad r > u - 2(s+1) - t$$

and the fact that $Q^t = 0$ for $t < 0$ implies that

$$E_r^{s,t,u} \supseteq \mathrm{im}(d_r) = 0 \quad \text{for} \quad r > t\,,$$

so in each tridegree the spectral sequence terminates at a finite stage.

Finally, we claim that for each $r \geq 1$ the map at E_r is surjective in bidegrees (s,u) for which $u-s < 5s-4$ and an isomorphism in bidegrees for which $u-s < 5s-10$. We know this to be true at the E_1-page by Proposition 4.3; suppose it is true for the E_r-page. A d_r-differential in the algebraic Novikov spectral sequence has (s,u) bidegree $(1,0)$. Because $u-s < 5s-4$ if and only if $u-(s+1) < 5(s+1)-10$, it has source in the surjective region if and only if it has target in the isomorphism region. Thus, we can deduce the result for the E_{r+1}-page using the following simple observation: if a map of cochain complexes is a surjection in degree n and an isomorphism in higher degrees, then the same is true of the map induced in cohomology.

The result now follows by an induction on the filtration. □

6 Computing the localized algebraic Novikov spectral sequence

We begin by identifying some permanent cycles in the algebraic Novikov spectral sequence. Recall that

$$BP_*BP = BP_*[t_1, t_2, \ldots], \quad |t_i| = 2(2^i - 1).$$

Lemma 6.1. *The following elements are cocycles in the cobar construction $\Omega^*(BP_*BP)$:*

1. $[\,]$ *and* $[t_1]$*;*
2. $v_1^2[t_1] + 2v_1[t_1^2] + \frac{4}{3}[t_1^3]$*;*
3. $v_2[t_1|t_1] + v_1[t_1|t_1^3] - v_1[t_1^2|t_1^2] + v_1[t_1^3|t_1] - 3v_1[t_1|t_2] + 2[t_1|t_1t_2] + 2[t_1^2|t_1^3] - 2[t_1^2|t_2] + 2[t_1t_2|t_1]$.

Proof. Direct calculation. □

Corollary 6.2. *The following elements are cocycles in the cobar construction $\Omega^*(P;Q)$:*

1. $[\,]$ *and* $[\zeta_1]$*;*
2. $q_1^2[\zeta_1] + q_0q_1[\zeta_1^2] + q_0^2[\zeta_1^3]$*;*
3. $q_2[\zeta_1|\zeta_1]+q_1[\zeta_1|\zeta_1^3]+q_1[\zeta_1^2|\zeta_1^2]+q_1[\zeta_1^3|\zeta_1]+q_1[\zeta_1|\zeta_2]+q_0[\zeta_1|\zeta_1\zeta_2]+q_0[\zeta_1^2|\zeta_1^3]+q_0[\zeta_1^2|\zeta_2]+q_0[\zeta_1\zeta_2|\zeta_1]$.

Moreover, these elements define the classes 1, h_0, $\langle h_1, q_0^2, h_0\rangle$ *and* $\langle h_0, q_0, h_1^2\rangle$ *in* $H^*(P;Q)$ *and they are permanent cycles in the algebraic Novikov spectral sequence.*

Proof. Checking the first Massey product is straightforward. The second stops being difficult once one realizes that $d\left(q_2[\zeta_1] + q_1[\zeta_2 + \zeta_1^3] + q_0[\zeta_1\zeta_2]\right) = q_0[\zeta_1^2|\zeta_1^2]$. □

The main result of this section is the following proposition, which completely describes the localized algebraic Novikov spectral sequence.

Proposition 6.3. *In the localized algebraic Novikov spectral sequence*

$$h_0^{-1}H^*(P;Q) = \mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots] \implies \overline{\alpha}_1^{-1}E_2(\mathbb{S};BP),$$

the elements 1, h_0, q_1^2 *and* q_2 *are permanent cycles, while* $d_1q_{n+1} = q_n^2h_0$ *for* $n \geq 2$.

Proof. The images in $\Omega^*(E[\zeta_1];Q)$ of the elements in Corollary 6.2 are $[\ ]$, $[\zeta_1]$, $q_1^2[\zeta_1]$, $q_2[\zeta_1|\zeta_1]$, respectively. So there are permanent cycles in the algebraic Novikov E_2-term that map to 1, h_0, $q_1^2h_0$, and $q_2h_0^2$ in the localized E_2-term. We are left with proving the differential, so suppose that $n \geq 2$ and write E for $E[\zeta_1]$.

By Proposition 4.3, there exists a positive integer N such that $q_{n+1}h_0^N$ is in the image of $H^*(P;Q^1) \to H^*(E;Q^1)$. Pick an X in $H^{N,*}(P;Q^1)$ mapping to $q_{n+1}h_0^N$. To complete the proof of the proposition it is enough to calculate d_1X in the unlocalized algebraic Novikov spectral and check that its image under $H^*(P;Q) \to H^*(E;Q)$ is $q_n^2h_0^{N+1}$.

Since $\Omega^*(P;Q) \to \Omega^*(E;Q)$ is surjective, we can find a cocycle in $\Omega^*(P;Q)$ that represents X and maps to $q_{n+1}[\zeta_1]^N$ in $\Omega^*(E;Q)$. Now all elements of the monomial basis for $\Omega^*(P;Q)$ that include a tensor factor containing some monomial in P other than ζ_1 map to zero in $\Omega^*(E;Q)$. This means that when we write our cocycle in this monomial basis it must contain the term $q_{n+1}[\zeta_1]^N$. We write the cocyle representing X as $q_{n+1}[\zeta_1]^N + x$, where x is a linear combination of other basis elements.

By (3.1) we have a surjection

$$I\Omega^*(BP_*BP) \longrightarrow \Omega^*(P;Q^1)\,.$$

We will make use of the set-theoretic splitting that in each term of a linear combination of monomial basis elements replaces each ζ_i by t_i and each q_i by v_i. (Remember that $v_0 = 2$; this splitting is not linear.) With this choice of splitting, $v_{n+1}[t_1]^N + y$ is selected to map to our cocycle representing X, where y is a linear combination of terms, each of which involves, as a tensor factor, some monomial in the t_i's other than the monomial t_1, and such that each nonzero coefficient is v_i for some i.

Since $q_{n+1}[\zeta_1]^N + x \in \Omega^*(P;Q)$ is a cocycle, $d(v_{n+1}[t_1]^N + y) \in I^2\Omega^*(BP_*BP)$. Mapping to $\mathrm{gr}^2\Omega^*(BP_*BP) = \Omega^*(P;Q^2)$ gives an element representing $d_1X \in H^*(P;Q)$. As explained at the start of the proof, we wish to understand the image of this element in $H^*(E;Q)$.

To do this we will consider the BP_*-basis of the cobar construction given by placing a monomial in the t_i's in each tensor factor. Any element of $I^2\Omega^*(BP_*BP)$ is uniquely a linear combination of these elements with coefficients in I^2. Of these terms, only those of the form $\alpha[t_1]^j$ with $\alpha \notin I^3$ map nontrivially to $\Omega^*(E;Q^2)$. The elements $d(v_{n+1}[t_1]^N)$ and dy are linear combinations of these basis elements with coefficients in BP_*. Since $q_{n+1}[\zeta_1]^N$ is not a cocycle, neither set of coefficients by themselves need to lie in I^2, though their sums do. First, we look at the contribution from $d(v_{n+1}[t_1]^N)$.

Lemma 6.4. *For $n \geq 1$, the coefficient of $[t_1]^{N+1}$ in $d(v_{n+1}[t_1]^N)$ is v_n^2 mod I^3.*

Proof. Because t_1 is primitive it is enough to investigate the coefficient of t_1 in $\eta_R v_{n+1}$. Since the elements $\eta_R v_{n+1}$ and $v_{n+1} = \eta_L v_{n+1}$ have the same augmentation, we have

$$\eta_R(v_{n+1}) \equiv v_{n+1} + ct_1 \mod (t_1^2, t_2, t_3, \ldots)$$

for some $c \in BP_{2(2^{n+1}-2)}$. The only monomial in the v_i's of the degree of c that is not in I^3 is v_n^2. Moreover, $2v_n^2t_1 \in I^3$ so that

$$\eta_R(v_{n+1}) \equiv v_{n+1} + bv_n^2t_1 \mod I^3 + (t_1^2, t_2, t_3, \ldots)$$

where $b = 0$ or 1. Since [21, 5.1]

$$\eta_R(v_{n+1}) \equiv v_{n+1} + v_nt_1^{2^n} - v_n^2t_1 \mod (2, v_1, \ldots, v_{n-1})$$

we must have $b = 1$. □

Mapping $v_n^2[t_1]^{N+1} \in I^2\Omega^*(BP_*BP)$ to $\mathrm{gr}^2\Omega^*(BP_*BP) = \Omega^*(P;Q^2)$ gives $q_n^2[\zeta_1]^{N+1}$. Mapping further to $\Omega^*(E;Q)$, gives a cocycle representing $q_n^2 h_0^{N+1}$. In order to complete the proof it suffices to show that the coefficient of $[t_1]^{N+1}$ in dy is zero.

Recall that y is a linear combination of terms, each of which involves, as a tensor factor, some monomial in the t_i's other than the monomial t_1. The differential in the cobar complex makes use of the right unit and the diagonal map in BP_*BP. When evaluating dy, the right unit is used on the coefficients of the terms in y. Since none of the monomials occurring in y are of the form $[t_1]^N$, BP_* multiples of $[t_1]^{N+1}$ cannot arise from this part of the differential. Thus we just need to consider terms coming from the diagonal map. The following simple lemma is crucial.

Lemma 6.5. *The only monomials in the t_i's that contain a nonzero BP_*-multiple of $t_1 \otimes t_1$ in their diagonal are t_1^2 and t_2.*

Proof. Recall that we have an inclusion $BP_* \subset H_*(BP) = \mathbb{Z}_{(2)}[m_1, m_2, \ldots]$ given by the Hurewicz homomorphism. Thus, we can compute in $H_*(BP \wedge BP) = H_*(BP)[t_1, t_2, \ldots]$ and there we have (see [21]) an inductive formula for the diagonal of t_n.

$$\Delta t_n = \sum_{i+j+k=n} m_i t_j^{2^i} \otimes t_k^{2^{i+j}} - \sum_{i=1}^{n} m_i (\Delta t_{n-i})^{2^i}.$$

The first sum does not contain a term $t_1 \otimes t_1$. Moreover, the only terms in Δt_{n-i} with a 1 on one side or the other are $t_{n-i} \otimes 1$ and $1 \otimes t_{n-i}$. Thus, in the expression of $(\Delta t_{n-i})^{2^i}$, the only way one can achieve $t_1 \otimes t_1$ is with $i = 1$ and $n - i = 1$ so that $n = 2$.

The diagonal is multiplicative and so one can achieve $t_1 \otimes t_1$ in the diagonal of a monomial only in the cases t_2 and t_1^2. □

Now consider a tensor product of monomials, a basis element in the cobar construction. The differential is computed by applying the reduced diagonal to each factor and taking the alternating sum. One receives a term that is a BP_*-multiple of $[t_1]^{N+1}$ only by starting with a tensor product of monomials in which all but one term is t_1, and the remaining term is either t_1^2 or t_2.

We should call attention to a subtlety here. When the reduced diagonal is applied to a monomial, the result is a BP_*-linear combination of monomials. Given a basis element of the cobar complex, to express the value of the differential on it as a BP_*-linear combination of tensor products of such monomials, one needs to pull coefficients outside the tensor products. This operation is nontrivial since the tensor products, while formed over BP_*, use the left and right actions on the right and left factors, respectively. In particular, $t \otimes vt' = \eta_R(v)t \otimes t'$. The element $\eta_R(v)$ will itself be a linear combination of monomials in the t_i's (where we now include 1 as t_0) so if the expression involves more than $[t_1]$'s before this maneuver, it will continue to involve more than $[t_1]$'s afterwards as well.

Now recall that y has internal dimension $2(2^{n+1} - 1) + 2N$. The internal dimensions of $[t_1]^{N-i}[t_1^2][t_1]^{i-1}$ and $[t_1]^{N-i}[t_2][t_1]^{i-1}$ are $2(N + 1)$ and $2(N + 2)$, respectively and so the coefficients of these basis elements in y must have internal dimensions $2(2^{n+1} - 2)$ and $2(2^{n+1} - 3)$, respectively. But recall that the coefficient of each term appearing in y is a v_i. The first dimension does not occur as the dimension of a v_i, and the second occurs only for $n = 1$.

We note that when $n = 1$ such terms do occur, as we see in Lemma 6.1, where the third cocycle contains the terms $v_2[t_1|t_1]$ and $-3v_1[t_1|t_2]$. These provide two canceling $v_1^2[t_1|t_1|t_1]$ terms. Of course, this is how we saw $q_2 h_0^2$ was a permanent cycle.

But now we are assuming $n \geq 2$, so the proof of Proposition 6.3 is complete. □

We see immediately from the proposition that the E_2-page of the localized algebraic Novikov spectral sequence consists of permanent cycles and so we obtain the following corollary.

Corollary 6.6. *The E_∞-page of the localized algebraic Novikov spectral sequence is*

$$\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2]/(q_2^2).$$

7 What happens to $\overline{\alpha}_s$?

We now return to the elements $\overline{\alpha}_s$ of Section 2. In order to say what happens to them under the localization map it is convenient to consider the mod 2 Moore spectrum analogues of our results for $\mathbb{S}$, which are of interest in their own right. We write $\mathbb{S}/2$ for the mod 2 Moore spectrum. Just as before, when we filter $\Omega^*(BP_*BP; BP_*(\mathbb{S}/2)) = \Omega^*(BP_*BP/2)$ by powers of the augmentation ideal $BP_*BP/2 \longrightarrow \mathbb{F}_2$, we arrive at the "mod 2 algebraic Novikov spectral sequence"

$$E_1^{s,t,u} = H^{s,u}(P; [Q/(q_0)]^t) \Longrightarrow H^{s,u}(BP_*BP/2)\,, \quad d_r : E_r^{s,t,u} \longrightarrow E_r^{s+1,t+r,u}\,.$$

Again $h_0 \in E_1^{1,0,2}$ is a permanent cycle, and, continuing to follow the argument above, we arrive at the following result.

Proposition 7.1. *The localized mod 2 algebraic Novikov spectral sequence converges, and has*

$$E_1 = h_0^{-1}H^*(P; Q/(q_0)) = \mathbb{F}_2[h_0^{\pm 1}, q_1, q_2, \ldots]\,.$$

The elements q_1 and q_2 are permanent cycles in this spectral sequence and we have

$$d_1 q_{n+1} = q_n^2 h_0 \ \textit{for } n \geq 2\,.$$

The map $\mathbb{S} \longrightarrow \mathbb{S}/2$ induces a map between the localized algebraic Novikov spectral sequences. At the E_1-page, the map is given by the inclusion

$$\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots] \longrightarrow \mathbb{F}_2[h_0^{\pm 1}, q_1, q_2, \ldots]\,.$$

At the E_∞-page, it is given by the inclusion

$$\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2]/(q_2^2) \longrightarrow \mathbb{F}_2[h_0^{\pm 1}, q_1, q_2]/(q_2^2)\,.$$

We now locate the elements $\overline{\alpha}_s$ in the localized algebraic Novikov spectral sequence.

Lemma 7.2. *For $s \neq 2$, $\overline{\alpha}_s$ is $\overline{\alpha}_1$-free and its image in $\overline{\alpha}_1^{-1}H^*(BP_*BP)$ is detected by $q_1^{s-1}h_0$ when s is odd and by $q_1^{s-4}q_2h_0$ when s is even.*

Proof. When s is odd $\overline{\alpha}_s$ has a cocycle representative with leading term $sv_1^{s-1}[t_1]$ because $\eta_R v_1 = v_1 + 2t_1$. All other terms have the same filtration and involve higher powers of t_1. Thus, $\overline{\alpha}_s$ is detected by $q_1^{s-1}h_0$.

Suppose s is even and bigger than 2. Because the map induced by $\mathbb{S} \to \mathbb{S}/2$ between our localized algebraic Novikov spectral sequences is injective at each page, it suffices to check the result for the image of $\overline{\alpha}_s$ in $H^*(BP_*BP; BP_*/2)$. For $s > 4$ one can find (see [25, 4.4.35], for instance) an explicit cocycle representative for $\overline{\alpha}_s$:

$$v_1^{s-4}v_2[t_1] + v_1^{s-3}[t_2] + v_1^{s-3}[t_1^3]\,.$$

This element is detected by $q_1^{s-4}q_2h_0$ in the localized algebraic Novikov spectral sequence.

For $s = 4$ one finds, by direct computation, the following cocycle respresentative for $\overline{\alpha}_4$:

$$[t_1^4] + v_2[t_1] + v_1[t_2] + v_1[t_1^3] + v_1^2[t_1^2]\,.$$

Upon multiplying by $\overline{\alpha}_1^3$ we have leading term $[t_1^4|t_1|t_1|t_1]$. It is classical that $h_0^3h_2 = 0$ in $H^*(P)$. Find $y \in \Omega^3 P$ with $dy = [\zeta_1^4|\zeta_1|\zeta_1|\zeta_1]$. Then obtain $y' \in \Omega^3(BP_*BP)$ by replacing ζ's by t's. Using Lemma 6.5 we see that dy' cannot contain $v_2[t_1|t_1|t_1|t_1]$. Thus, picking off the elements of filtration 1 with only single powers of t_1's appearing in

$$\left([t_1^4] + v_2[t_1] + v_1[t_2] + v_1[t_1^3] + v_1^2[t_1^2]\right) \cdot [t_1|t_1|t_1] + dy'$$

gives $v_2[t_1|t_1|t_1|t_1]$. We deduce that $\overline{\alpha}_1^3\overline{\alpha}_4$ is detected by $q_2h_0^4$ in the localized algebraic Novikov spectral sequence. □

We can now obtain an explicit description of the localized Adams-Novikov E_2-page, in terms of elements which exist before localizing.

Corollary 7.3. $\overline{\alpha}_1^{-1}H^*(BP_*BP) = \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_4^2)$.

Proof. Consider the natural map $\mathbb{Z}_{(2)}[\overline{\alpha}_1, \overline{\alpha}_3, \overline{\alpha}_4] \longrightarrow H^*(BP_*BP)$. It can be checked that $\overline{\alpha}_1\overline{\alpha}_4^2 = 0$ in $H^*(BP_*BP)$; this follows for example from Toda's relation $\eta\sigma^2 = 0$ [26] in $\pi_*(\mathbb{S})$. We also have $2\overline{\alpha}_1 = 0$. This map thus factors through a map

$$\mathbb{Z}_{(2)}[\overline{\alpha}_1, \overline{\alpha}_3, \overline{\alpha}_4]/(2\overline{\alpha}_1, \overline{\alpha}_1\overline{\alpha}_4^2) \longrightarrow H^*(BP_*BP)\,.$$

Inverting $\overline{\alpha}_1$ gives a map $f : \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1\overline{\alpha}_4^2) \longrightarrow \overline{\alpha}_1^{-1}H^*(BP_*BP)$.

We have shown that $\overline{\alpha}_3$ and $\overline{\alpha}_4$ have images in $\overline{\alpha}_1^{-1}H^*(BP_*BP)$ detected by $q_1^2h_0$ and q_2h_0, respectively. The E_∞-page of the localized algebraic Novikov spectral sequence is $\mathbb{F}_2[h_0^{\pm 1}, q_1^2, q_2]/(q_2)$; for each bidegree (s, u) there is only one t such that $E_\infty^{s,t,u} \neq 0$ and so the filtration is locally finite. These facts, together with convergence of the localized algebraic Novikov spectral sequence, allow one to check that f is injective and surjective. □

8 The localized motivic Adams-Novikov spectral sequence

Here is our main result about motivic homotopy groups. We are working over an algebraically closed base field of characteristic zero.

Theorem 8.1. *Let $\eta \in \pi_{1,1}(\mathbb{S}_{\mathrm{Mot}})$ and $\sigma \in \pi_{7,4}(\mathbb{S}_{\mathrm{Mot}})$ denote the elements of motivic Hopf invariant 1, and μ_9 the nonzero element in $\pi_{9,5}(\mathbb{S}_{\mathrm{Mot}})$. Then $2\eta = 0$, σ^2 is η-torsion, and*

$$\pi_{*,*}(\eta^{-1}\mathbb{S}_{\mathrm{Mot}}) = \eta^{-1}\pi_{*,*}(\mathbb{S}_{\mathrm{Mot}}) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\sigma^2)\,.$$

The first step is to move to the 2-complete context. We owe the following observation to the referee.

Lemma 8.2. *For any motivic spectrum,*

$$\eta^{-1}X \longrightarrow \eta^{-1}(X_2^\wedge)$$

is an equivalence.

Proof. Since $2\eta = 0$, there is a map $\eta^{-1}X/2 \longrightarrow \eta^{-1}X$, natural in the motivic spectrum X, that splits the map induced by $X \longrightarrow X/2$. On the other hand, the completion map $X/2 \longrightarrow (X/2)^\wedge_2$ is an equivalence. The result follows. □

We now briefly recall the theorems of Voevodsky [14, 27] concerning mod 2 motivic homology and the motivic Steenrod algebra over an algebraically closed field of characteristic 0.

Motivic homotopy and homology are graded by a free abelian group of rank two. One pair of parameters is *dimension* and *weight* and with this bigrading $|\tau| = (0, -1)$. Other parameters are the *coweight* (or "Milnor-Witt degree" [8]) and the Novikov or Chow [7] *degree*; these satisfy the relations

$$\text{cowt} + \text{wt} = \dim \ , \quad \text{cowt} - \text{wt} = \deg .$$

Each of these parameters has its uses. For example, Morel showed that the motivic stable homotopy ring is zero in negative coweight, and given by the Milnor-Witt K-theory (which he defined for the purpose) in coweight zero. Our bigrading will be by dimension and weight.

The coefficient ring of mod 2 motivic homology, written H, is $\mathbb{M}_2 = \mathbb{F}_2[\tau]$.

Hu, Kriz, and Ormsby [14] (see also [6]) describe a "motivic Adams-Novikov spectral sequence." To circumvent the fact that the full structure of the motivic Thom spectrum MGL is unknown they work with the H-completion. They show that this motivic spectrum splits as a wedge of suspensions of a motivic analogue of the Brown-Peterson spectrum, denoted here by BPM, and use it to construct a spectral sequence.

They show that the H-complete motivic analogue of the Hopf algebroid (BP_*, BP_*BP) is simply the classical one tensored with $\mathbb{Z}_2[\tau]$ (where $\mathbb{Z}_2$ denotes the 2-adic integers). It follows that the E_2-page of the motivic Adams-Novikov spectral sequence is obtained from the classical one by completing at 2 and adjoining τ, and that the corresponding algebraic Novikov spectral sequence is obtained by adjoining τ. Thus, our work above has the following consequence.

Corollary 8.3. *Over an algebraically closed field of characteristic zero, the H-complete motivic Adams-Novikov E_2-page localizes to $\overline{\alpha}_1^{-1}E_2 = \mathbb{F}_2[\tau, \overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_4^2)$.*

Dugger and Isaksen observe that the motivic Adams-Novikov spectral sequence converges to $\pi_*((\mathbb{S}_{\text{Mot}})^\wedge_H)$. In [13] this mod 2 homology completion is identified with the 2-adic completion $(\mathbb{S}_{\text{Mot}})^\wedge_2$; so the H-completed motivic Adams-Novikov spectral sequence has the form

$$E_2^{s,u,w} = H^*(BP_*BP) \otimes \mathbb{Z}_2[\tau]^{s,u,w} \stackrel{s}{\Longrightarrow} \pi_{u-s,w}((\mathbb{S}_{\text{Mot}})^\wedge_2) \, , \quad d_r : E_r^{s,u,w} \longrightarrow E_r^{s+r,u+r-1,w} \, .$$

If $x \in H^{s,u}(BP_*BP)$ is nonzero then u is even and $\tau^n x$ defines an element of $H^*(BP_*BP)[\tau]^{s,u,u/2-n}$. We can recover the classical Adams-Novikov spectral sequence by forgetting the weight and setting $\tau = 1$.

Inverting η does not harm convergence of this spectral sequence, since powers of $\overline{\alpha}_1$ constitute the vanishing line at E_2; it converges to $\eta^{-1}\pi_*((\mathbb{S}_{\text{Mot}})^\wedge_2)$.

The classical differential $d_3\overline{\alpha}_3 = \overline{\alpha}_1^4$ appears motivically as

$$d_3\overline{\alpha}_3 = \tau\overline{\alpha}_1^4 \, .$$

When we invert $\overline{\alpha}_1$, this differential has the effect of killing τ, and using the relations above we find that

$$E_\infty = \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3^2, \overline{\alpha}_4]/(\overline{\alpha}_1\overline{\alpha}_4^2) = \mathbb{F}_2[\overline{\alpha}_1^{\pm 1}, \overline{\alpha}_4, \overline{\alpha}_5]/(\overline{\alpha}_4^2) \, .$$

To see what this implies about motivic homotopy groups, note that classically η, σ, μ_9 and $\eta\mu_9 = \mu_{10}$ are detected by $\overline{\alpha}_1$, $\overline{\alpha}_4$, $\overline{\alpha}_5$ and $\overline{\alpha}_1\overline{\alpha}_5 = \overline{\alpha}_3^2$, respectively, in the Adams-Novikov spectral sequence. These facts hold motivically as well [12], and the relation $\eta\sigma^2 = 0$ is true motivically also in the 2-complete sphere [15]. So we receive a map

$$\mathbb{F}_2[\eta, \sigma, \mu_9]/(2\eta, \eta\sigma^2) \longrightarrow \pi_{*,*}((\mathbb{S}_{\mathrm{Mot}}))_2^\wedge .$$

Now η, σ, and μ_9 are detected by $\overline{\alpha}_1$, $\overline{\alpha}_4$, and $\overline{\alpha}_5$ respectively. Convergence of the η-localized H-completed motivic Adams-Novikov spectral sequence then shows that

$$\mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(2\eta, \eta\sigma^2) \longrightarrow \pi_{*,*}((\mathbb{S}_{\mathrm{Mot}})_2^\wedge)$$

is an isomorphism.

This completes the proof of Theorem 8.1.

9 A comparison of spectral sequences

9.1 The diagram

In this final section, we complete the calculation of a square of spectral sequences, a localized version of the following square.

$$\begin{array}{ccc} H^*(P;Q)[\tau] & \xRightarrow{\text{CESS}} & E_2(\mathbb{S}_{\mathrm{Mot}};H) \\ {\scriptstyle \text{ANSS}[\tau]}\Downarrow & & \Downarrow{\scriptstyle \text{MASS}} \\ E_2(\mathbb{S}_{\mathrm{Mot}};BPM) & \xRightarrow{\text{MNSS}} & \pi_{*,*}(\mathbb{S}_{\mathrm{Mot}}) \end{array}$$

The right spectral sequence is the motivic Adams spectral sequence as studied in [6, 14]. The bottom spectral sequence is the motivic Adams-Novikov spectral sequence described above, which was first studied in [14]. The left spectral sequence is the *motivic algebraic Novikov spectral sequence*, obtained by filtering $\pi_*(BPM) = BP_*[\tau]$ by powers of the kernel of the augmentation $\pi_*(BPM) \longrightarrow \mathbb{F}_2[\tau]$. By the results of Hu, Kriz, and Ormsby [14] this is simply the algebraic Novikov spectral sequence described in Section 3 extended by adjoining τ. The grading of $H^*(P;Q)[\tau]$ follows that of $H^*(BP_*BP)[\tau]$. If $x \in H^{s,u}(P;Q^t)$ is nonzero then u is even and $\tau^n x$ defines an element of $H^*(P;Q)[\tau]^{s,t,u,u/2-n}$. The top spectral sequence is the Cartan-Eilenberg spectral sequence associated to the extension of Hopf algebras

$$\mathbb{M}_2 \otimes P \longrightarrow A_{\mathrm{Mot}} \longrightarrow \mathbb{M}_2 \otimes E. \tag{9.1.1}$$

This motivic Cartan-Eilenberg spectral sequence is indexed just as in (3.3), but with the additional weight grading that is preserved by differentials. The vanishing lines of (4.5) and (4.6) ensure that we can localize all the spectral sequences to obtain a square of convergent spectral sequences. The behavior of these spectral sequences is summarized in the following diagram.

$$\begin{array}{ccc} \mathbb{F}_2[\tau, h_0^{\pm 1}, q_1^2, q_2, q_3, \ldots] & \xRightarrow{d_3 q_1^2 = \tau h_0^3} & \mathbb{F}_2[h_0^{\pm 1}, v_1^4, v_2, v_3, \ldots] \\ {\scriptstyle d_1 q_{n+1} = q_n^2 h_0,\ n \geq 2}\Downarrow & & \Downarrow{\scriptstyle d_2 v_{n+1} \equiv v_n^2 h_0,\ n \geq 2} \\ \mathbb{F}_2[\tau, \overline{\alpha}_1^{\pm 1}, \overline{\alpha}_3, \overline{\alpha}_4]/(\overline{\alpha}_1\overline{\alpha}_4^2) & \xRightarrow{d_3\overline{\alpha}_3 = \tau\overline{\alpha}_1^4} & \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta\sigma^2) \end{array}$$

We have calculated the left spectral sequence and the bottom one in the earlier sections of this paper. Guillou and Isaksen calculated the E_2-page of the localized motivic Adams spectral sequence in [9]. In the next section, we will give a different proof of their result by calculating the top spectral sequence. In the final section, we will use the techniques of [19] to determine the differentials in the localized motivic Adams spectral sequence, verifying another conjecture of Guillou and Isaksen [9].

9.2 The localized Cartan-Eilenberg spectral sequence

The extension of Hopf algebras (9.1.1) gives rise to a Cartan-Eilenberg spectral sequence, which we may localize by inverting $h_0 \in H(P;Q)[\tau]^{1,0,2,1}$.

Lemma 9.2.1. *In the localized Cartan-Eilenberg spectral sequence we have $d_3 q_1^2 = \tau h_0^3$. The classes q_1^4 and q_n for $n \geq 2$ are permanent cycles and so*

$$E_\infty = \mathbb{F}_2[h_0^{\pm 1}, q_1^4, q_2, q_3, \ldots].$$

Proof. The differential $d_3 q_1^2 = \tau h_0^3$ follows from the unlocalized differential $d_3\langle h_1, q_0^2, h_0\rangle = \tau h_0^4$ and this is forced on us by our limited knowledge of $H^*(A_{\mathrm{Mot}})$. Degree considerations show that q_1^4, and q_n for $n \geq 2$, are permanent cycles. □

We can now prove the following result, established also by Guillou and Isaksen [9]. We note that they follow the classical conventions at $p = 2$ and denote by h_1 the class that we call $h_0 \in E_2^{1,2,1}(\mathbb{S}_{\mathrm{Mot}}; H)$.

Corollary 9.2.2. *There exist classes $v_1^4, v_2, \ldots,$ with $|v_1^4| = (0, 8, 4), |v_n| = (0, 2(2^n - 1), 2^n - 1)$, such that*

$$h_0^{-1} E_2(\mathbb{S}_{\mathrm{Mot}}; H) = \mathbb{F}_2[h_0^{\pm 1}, v_1^4, v_2, v_3, \ldots]\,.$$

Proof. We choose a representative for q_1^4, which we call v_1^4, and for $n \geq 2$ we choose representatives for q_n, which we call v_n. Since the associated graded algebra is free on the classes of these generators, the result follows. □

9.3 Comparing Adams spectral sequences

In this section we will complete the calculation of the localized motivic Adams spectral sequence. By finding representatives, one sees that in the localized motivic Adams spectral sequence for the η-local sphere spectrum the elements v_1^4 and v_2 are permanent cycles. For the other generators, we have the following proposition, which follows from the techniques of [19].

Proposition 9.3.1. *For $n \geq 2$, we have $d_2 v_{n+1} = v_n^2 h_0$ modulo higher Cartan-Eilenberg filtration.*

We will give an improvement, due to the first author, of the statement and the proof of the comparison result of [19] (which, in turn, followed ideas from [24]). The second author is eager to use this opportunity to clarify the proof given in [19], and to fill a gap: Lemma 6.7 is not correct as stated there. What follows is a correct statement that serves the purpose in [19], and which will be used in the proof presented here as well. This lemma relates to the comparison of two boundary maps, and its importance cannot be overstated. It deals with the following situation.

Let $A \longrightarrow B \longrightarrow C$ and $X \longrightarrow Y \longrightarrow Z$ be cofiber sequences. Smash them together to form the following commutative diagram of cofiber sequences.

$$\begin{array}{ccccc} A \wedge X & \longrightarrow & A \wedge Y & \longrightarrow & A \wedge Z \\ \downarrow & & \downarrow & & \downarrow \\ B \wedge X & \longrightarrow & B \wedge Y & \longrightarrow & B \wedge Z \\ \downarrow & & \downarrow & & \downarrow \\ C \wedge X & \longrightarrow & C \wedge Y & \longrightarrow & C \wedge Z \end{array}$$

Let b be an element of $\pi_n(B \wedge Y)$ that maps to 0 in $\pi_n(C \wedge Z)$. Then there is an element $a \in \pi_n(A \wedge Z)$ mapping to the image of b in $\pi_n(B \wedge Z)$, and an element $c \in \pi_n(C \wedge X)$ mapping to the image of b in $\pi_n(C \wedge Y)$.

Lemma 9.3.2 (May [16])**.** *The elements a and c can be chosen so that they have the same image (up to a conventional sign) in $\pi_{n-1}(A \wedge X)$ under the boundary maps associated to the cofiber sequences along the top and the left edge of the diagram.*

This statement is a small part of an elaborate structure enriching the displayed 3×3 diagram. This structure is described in detail and proved by May in [16]. In the founding days of the theory of triangulated categories, Verdier [4] showed that a 2×2 diagram can always be extended to a 3×3 diagram of cofiber sequences. An analysis of his proof reveals that it actually produces precisely the structure verified by May for the specific case in which the 3×3 diagram occurs by smashing together two cofiber sequences.

For clarity, we will work in the non-motivic context, and in the specific case of BP and $H\mathbb{F}_p$ (for any prime p) and the sphere spectrum. We will then indicate the general setting under which the result holds and this will prove the proposition just stated. Write H for the mod p Eilenberg Mac Lane spectrum.

So we have the following square of spectral sequences.

$$\begin{array}{ccc} H^{s,u}(P; Q^t) & \xRightarrow{\text{CESS}} & E_2^{s+t,u+t}(\mathbb{S}; H) \\ \Big\Downarrow{\scriptstyle \text{ANSS}} & & \Big\Downarrow{\scriptstyle \text{ASS}} \\ E_2^{s,u}(\mathbb{S}; BP) & \xRightarrow{\text{NSS}} & \pi_{u-s}(\mathbb{S}) \end{array}$$

The initial two are the algebraic Novikov spectral sequence (3.2) and the Cartan-Eilenberg spectral sequence (3.3); the final two are the (H-based) Adam spectral sequence and the Novikov or BP-based Adams, spectral sequence. Write d_r^H for the Adams differentials, and d_r^{AN} for the differentials in the algebraic Novikov spectral sequence.

Theorem 9.3.3. *Suppose $x \in F_{CE}^s E_2^{s+t,u+t}(\mathbb{S}; H)$. Then the Cartan-Eilenberg filtration of $d_2^H x$ is higher:*

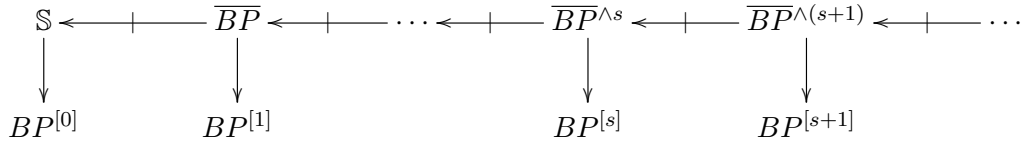
$$d_2^H x \in F_{CE}^{s+1} E_2^{s+t+2,u+t+1}(\mathbb{S}; H)\,.$$

Moreover, if x is detected in the Cartan-Eilenberg spectral sequence by $a \in H^{s,u}(P; Q^t)$ then $d_2^H x$ is detected by $d_1^{AN} a \in H^{s+1,u}(P; Q^{t+1})$.

Proof. The proof depends upon geometric constructions of the two algebraically defined spectral sequences. Both arise from the canonical BP-resolution of $\mathbb{S}$ and so we recall how this resolution is constructed. From the unit map of the ring spectrum BP we can construct a cofiber sequence

$$\mathbb{S} \longrightarrow BP \longrightarrow \overline{BP}. \tag{9.3.4}$$

Smashing this cofiber sequence with various smash-powers of $\overline{BP}$ gives the canonical BP-resolution of $\mathbb{S}$.



(9.3.5)

Here we use the notation

$$BP^{[s]} = \overline{BP}^{\wedge s} \wedge BP,$$

and the marked arrows indicate that they map from a desuspension. The Adams-Novikov spectral sequence for the homotopy of a spectrum X is associated to the exact couple arising by smashing this resolution with X and taking homotopy groups. The maps in the exact couple will be denoted

$$\begin{aligned} i_{BP} &: \pi_{u+s}(X \wedge \overline{BP}^{\wedge(s+1)}) \longrightarrow \pi_{u+s-1}(X \wedge \overline{BP}^{\wedge s}) \\ j_{BP} &: \pi_{u+s}(X \wedge \overline{BP}^{\wedge s}) \longrightarrow \pi_{u+s}(X \wedge BP^{[s]}) \\ k_{BP} &: \pi_{u+s}(X \wedge BP^{[s]}) \longrightarrow \pi_{u+s}(X \wedge \overline{BP}^{\wedge(s+1)}) \end{aligned} \tag{9.3.6}$$

The E_1 term arising from this exact couple is isomorphic, as a complex, to the cobar complex $\Omega^*(BP_*BP)$. The algebraic Novikov spectral sequence arises by filtering this complex by powers of the augmentation ideal in BP_*. In geometric terms, we are filtering $\pi_*(\overline{BP}^{\wedge s})$ by the classical Adams filtration.

Next we set up the Cartan-Eilenberg spectral sequence. Smashing (9.3.4) with a spectrum X and applying mod 2 homology gives a short exact sequence

$$0 \longrightarrow H_*(X) \longrightarrow H_*(BP \wedge X) \longrightarrow H_*(\overline{BP} \wedge X) \longrightarrow 0$$

(by the Künneth formula) and thus a long exact sequence

$$\cdots \longrightarrow E_2^{t,u}(X;H) \longrightarrow E_2^{t,u}(X \wedge BP;H) \longrightarrow E_2^{t,u}(X \wedge \overline{BP};H) \xrightarrow{\delta} E_2^{t+1,u}(X;H) \longrightarrow \cdots . \tag{9.3.7}$$

This means that applying $E_2(-;H)$ to (9.3.5) gives an exact couple and hence a spectral sequence. We index the spectral sequence so that

$$E_1^{s,t,u} = E_2^{t,u+t}(BP^{[s]};H) \implies_s E_2^{s+t,u+t}(\mathbb{S};H).$$

A change of rings theorem identifies $E_2^{t,u+t}(BP^{[s]};H)$ with $(Q^t \otimes \overline{P}^{\otimes s})_u$, so that our E_1-term is isomorphic, as a complex, to $\Omega^*(P;Q)$. Thus $E_2^{s,t,u} = H^{s,u}(P;Q^t)$. This spectral sequence is, in fact, the Cartan-Eilenberg spectral sequence of (3.3). The Cartan-Eilenberg filtration of $E_2(\mathbb{S};H)$ is given by

$$F^s_{\mathrm{CE}} E_2^{s+t,u+t}(\mathbb{S};H) = \operatorname{im}\Big(\delta^s : E_2^{t,u+t}(\overline{BP}^{\wedge s};H) \longrightarrow E_2^{s+t,u+t}(\mathbb{S};H)\Big). \tag{9.3.8}$$

If $x = \delta^s z$ then x is detected in the Cartan-Eilenberg E_1-page by the image of z under the map $j_{BP} : E_2^{t,u+t}(\overline{BP}^{\wedge s}; H) \longrightarrow E_2^{t,u+t}(BP^{[s]}; H)$.

We next recall a construction of the Adams spectral sequence for a spectrum X. We have a cofiber sequence $\mathbb{S} \longrightarrow H \longrightarrow \overline{H}$ and we define

$$H^{[t]} = H \wedge \overline{H}^{\wedge t}.$$

Note that we put the factor of H at the left rather than at the right as in the case of BP; this will let us keep these two types of resolution separate. The cofiber sequences

$$\overline{H}^{\wedge(t+1)} \wedge X \longrightarrow \overline{H}^{\wedge t} \wedge X \longrightarrow H^{[t]} \wedge X$$

link together as in (9.3.5) and the Adams spectral sequence for X is obtained by applying $\pi_*(-)$. We have $E_1^{t,u} = \pi_u(H^{[t]} \wedge X)$ and $E_2^{t,u} = H^{t,u}(A; H_*(X))$. The structure maps in the exact couple will be denoted as in (9.3.6) but subscripted with H.

The map of ring spectra $BP \longrightarrow H$ descends uninque to a map

$$\delta : \overline{BP} \longrightarrow \overline{H}$$

such that $i_H \delta = i_{BP}$. We will denote this map and all the maps it induces by δ, even if they involve the swap map T. For example, we have compatible maps

$$\delta : H^{[t]} \wedge X \wedge \overline{BP} \xrightarrow{1\wedge T} H^{[t]} \wedge \overline{BP} \wedge X \xrightarrow{1\wedge\delta\wedge 1} H^{[t+1]} \wedge X$$
$$\delta : \overline{H}^{\wedge t} \wedge X \wedge \overline{BP} \xrightarrow{1\wedge T} \overline{H}^{\wedge t} \wedge \overline{BP} \wedge X \xrightarrow{1\wedge\delta\wedge 1} \overline{H}^{\wedge(t+1)} \wedge X$$

for any spectrum X, and so a map of spectral sequences

$$\delta : E_r^{t,u}(X \wedge \overline{BP}; H) \longrightarrow E_r^{t+1,u}(X; H)\,.$$

In particular, with $X = \overline{BP}^{\wedge s}$, we have compatible maps

$$\delta : \overline{H}^{\wedge t} \wedge \overline{BP}^{\wedge(s+1)} \longrightarrow \overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge s}$$
$$\delta : H^{[t]} \wedge \overline{BP}^{\wedge(s+1)} \longrightarrow \overline{H}^{[t+1]} \wedge \overline{BP}^{\wedge s}\,.$$

At E_2, this map is the boundary map δ in (9.3.7).

We now address the first claim of the theorem. Suppose that $x \in F^s_{CE} E_2^{s+t,u+t}(\mathbb{S}; H)$, so that $x = \delta^s z$ for some $z \in E_2^{t,u+t}(\overline{BP}^{\wedge s}; H)$. Since δ is a map of spectral sequences, $d_2^H x = \delta^s(d_2^H z)$. Thus in order to show that

$$d_2^H x \in F^{s+1}_{CE} E_2^{s+t+2,u+t+1}(\mathbb{S}; H)$$

it suffices to find an element mapping to $d_2^H z$ under

$$\delta : E_2^{t+1,u+t+1}(\overline{BP}^{\wedge(s+1)}; H) \longrightarrow E_2^{t+2,u+t+1}(\overline{BP}^{\wedge s}; H)\,.$$

To compute $d_2^H z$, begin by picking a representative

$$z' \in \pi_{u+t}(H^{[t]} \wedge \overline{BP}^{\wedge s}) \tag{9.3.9}$$

for z. Since $0 = d_1^H z' = j_H k_H z'$, $k_H z' = i_H y'$ for some $y' \in \pi_{u+t+1}(\overline{H}^{\wedge(t+2)} \wedge \overline{BP}^{\wedge s})$. Then $j_H y' \in \pi_{u+t+1}(H^{[t+2]} \wedge \overline{BP}^{\wedge s})$ is a cocycle representing $d_2^H z$. These maps arise by applying π_* to the bottom row in the following diagram.

$$\begin{array}{ccccccc} & & \overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)} & \xrightarrow{j_H} & H^{[t+1]} \wedge \overline{BP}^{\wedge(s+1)} & & \\ & \overset{-i_{BP}}{\swarrow} & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} & & \\ H^{[t]} \wedge \overline{BP}^{\wedge s} & \xrightarrow{k_H} & \overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge s} & \xleftarrow{i_H} & \overline{H}^{\wedge(t+2)} \wedge \overline{BP}^{\wedge s} & \xrightarrow{j_H} & H^{[t+2]} \wedge \overline{BP}^{\wedge s} \end{array} \tag{9.3.10}$$

We will construct an element $y_0 \in \pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)})$ such that $i_H \delta y_0 = k_H z'$. We can then take $y' = \delta y_0$, so the representative $j_H y'$ for $d_2^H x$ lifts across δ, to the cocyle $j_H y_0$. This proves the first claim of the theorem.

To construct y_0 we first show that $k_H z'$ lifts across i_{BP}, and then that the dotted triangle commutes: $i_H \delta = -i_{BP}$.

The first step is organized by the diagram

$$\begin{array}{ccccccc} & & & & H^{[t]} \wedge \overline{BP}^{\wedge s} & \xrightarrow{j_{BP}} & H^{[t]} \wedge BP^{[s]} \\ & & & & \downarrow{\scriptstyle k_H} & & \downarrow{\scriptstyle k_H} \\ \overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)} & \xrightarrow{i_{BP}} & \overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge s} & \xrightarrow{j_{BP}} & \overline{H}^{\wedge(t+1)} \wedge BP^{[s]} & & d_1^H \\ & & \downarrow{\scriptstyle j_H} & & \downarrow{\scriptstyle j_H} & & \\ & & H^{[t+1]} \wedge \overline{BP}^{\wedge s} & \xrightarrow{j_{BP}} & H^{[t+1]} \wedge BP^{[s]} & & \end{array}$$

We have $d_1^H j_{BP} z' = j_{BP} d_1^H z' = 0$. Since the Adams spectral sequence for $BP^{[s]}$ collapses at E_2, this implies that $k_H j_{BP} z' = 0$, so $j_{BP} k_H z' = 0$ and hence that $k_H z'$ lifts to some $y_0 \in \pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge \overline{BP}^{\wedge(s+1)})$. It is important to note that the only property of y_0 used in this part of the proof is that $i_{BP} y_0 = k_H z'$.

Since $i_H \delta = i_{BP} : \overline{BP} \longrightarrow S^1$, it may appear that the triangle commutes without the sign. But δ has many meanings. The triangle expands to the perimeter of the diagram

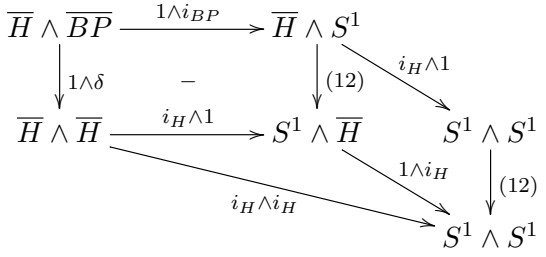
$$\begin{array}{ccc} \overline{H} \wedge Y \wedge X \wedge \overline{BP} & \xrightarrow{1 \wedge i_{BP}} & \overline{H} \wedge Y \wedge X \wedge S^{-1} \\ \downarrow{\scriptstyle (34)} & & \downarrow{\scriptstyle (34)} \\ \overline{H} \wedge Y \wedge \overline{BP} \wedge X & \xrightarrow{1 \wedge i_{BP} \wedge 1} & \overline{H} \wedge Y \wedge S^{-1} \wedge X \quad (134) \\ \downarrow{\scriptstyle 1 \wedge \delta \wedge 1} \quad - & & \downarrow{\scriptstyle (13)} \\ \overline{H} \wedge Y \wedge \overline{H} \wedge X & \xrightarrow{i_H \wedge 1} & S^{-1} \wedge Y \wedge \overline{H} \wedge X \end{array}$$

in which $X = \overline{BP}^{\wedge s}$ and $Y = \overline{H}^{\wedge t}$, and the cycles indicate the appropriate permutations of factors. The top inner square commutes, and it remains to check that the bottom square commutes up to the indicated sign.

For this, note that we can drop the terminal X. We can also move the Y to the right end and drop it; so we need to check that

$$\begin{array}{ccc} \overline{H} \wedge \overline{BP} & \xrightarrow{1 \wedge i_{BP}} & \overline{H} \wedge S^1 \\ \downarrow{\scriptstyle 1 \wedge \delta} \quad - & & \downarrow{\scriptstyle (12)} \\ \overline{H} \wedge \overline{H} & \xrightarrow{i_H \wedge 1} & S^1 \wedge \overline{H} \end{array} \tag{9.3.11}$$

commutes up to sign. To verify this, first map out to $S^1 \wedge S^1$:



Now $i_H \delta = i_{BP}$, and the switch map on $S^1 \wedge S^1$ multiplies by -1, so the outer diagram commutes up to sign. Thus the sum of the two maps across (9.3.11) lifts through a map $\overline{H} \wedge \overline{BP} \longrightarrow S^1 \wedge H$. But $H^1(\overline{H} \wedge \overline{BP}) = 0$.

This completes the proof of the first part of the theorem. To relate the Adams differential to the algebraic Novikov differential, recall from (9.3.8) and (9.3.9) that in the algebraic Novikov E_1 term

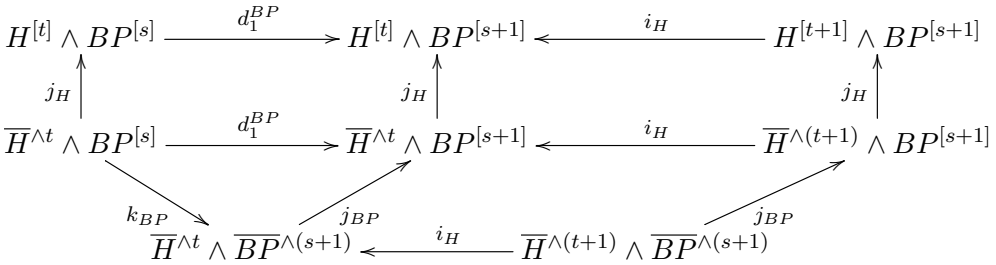
$$x \in F^s_{CE} E_2^{s+t,u+t}(\mathbb{S}; H) \text{ is represented the class of } j_{BP} z' \in \pi_{u+t}(\overline{H}^{[t]} \wedge \overline{BP}^{[s]}).$$

Similarly, from (9.3.10),

$$d_2^A x \in F^{s+1}_{CE} E_2^{s+t+2,u+t+1}(\mathbb{S}; H) \text{ is represented by } j_{BP} j_H y_0 \in \pi_{u+t+1}(H^{[t+1]} \wedge BP^{[s+1]}).$$

We need to see that these two classes are also related by d_1^{AN}.

The computation of the differential d_1^{AN} is organized by the top of the following diagram.

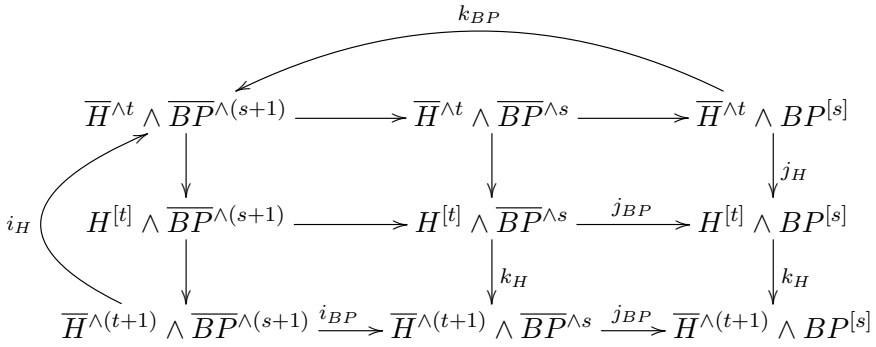


Let $a \in H^{t,u}(P; Q^s)$ be an element in the algebraic Novikov E_1 term. To compute $d_1^{AN} a$, find a representative $a' \in \pi_{u+t}(H^{[t]} \wedge BP^{[s]})$. By the collapse at E_2 of the Adams spectral sequence for $BP^{[s]}$, a' lifts to an element $y_1 \in \pi_{t+u}(\overline{H}^{\wedge t} \wedge BP^{[s]})$. Since a' is a cocycle, $d_1^{BP} y_1 = i_H b \in \pi_{u+t}(\overline{H}^{\wedge t} \wedge BP^{[s+1]})$ for some $b \in \pi_{u+t+1}(\overline{H}^{\wedge(t+1)} \wedge BP^{[s+1]})$. Then $j_H b$ is a representative for $d_1^{AN} a$.

With $a' = j_{BP} z'$, our earlier work suggests that we might choose $b = j_{BP} y_0$; then $j_H b = j_H j_{BP} y_0 = j_{BP} j_H y_0$ would be our representative for $d_1^{AN} a$, reaching the conclusion we want. For this to work, we need $j_{BP} y_0$ to satisfy the equation required of b; that is, $i_H j_{BP} y_0 = d_1^{BP} y_1 = j_{BP} k_{BP} y_1$. Since $i_H j_{BP} = j_{BP} i_H$, this will be guaranteed if

$$i_H y_0 = k_{BP} y_1 \in \pi_{u+t}(\overline{H}^{\wedge t} \wedge \overline{BP}^{\wedge(s+1)}) \,.$$

But this is precisely what is guaranteed to us by the application of May's lemma 9.3.2 to the following diagram.



This concludes the proof. □

9.4 Motivic conclusions.

This proof works in much greater generality. As in [19], the square of spectral sequences can be set up for any map of ring spectra $A \longrightarrow B$ and any spectrum X for which the B-Adams spectral sequence

$$E_2(A \wedge \overline{A}^{\wedge s} \wedge X; B) \implies \pi_*(A \wedge \overline{A}^{\wedge s} \wedge X)$$

converges and collapses at the E_2-page for all s. The proof holds whenever $B^1(\overline{B} \wedge \overline{A}) = 0$.

In particular, the proof works in the motivic context, at least over $\mathbb{C}$, with H replaced by the mod 2 motivic Eilenberg Mac Lane spectrum (also denoted H) and BP by the spectrum BPM considered by Hu, Kriz, and Ormsby [14]. They observe that as a comodule for A_{Mot}

$$H_*(BPM) = \mathbb{A}_{\text{Mot}} \square_E \mathbb{M}_2 \,,$$

where A_{Mot} is the motivic dual Steenrod algebra

$$A_{\text{Mot}} = V[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_n^2 = \tau \xi_{n+1}) \,,$$

$$|\tau_n| = (2^{n+1} - 1, 2^n - 1), \ |\xi_n| = (2^{n+1} - 2, 2^n - 1) \,,$$

and E is the exterior algebra over $\mathbb{M}_2$ generated by $\tau_0, \tau_1 \ldots$ By change of rings, then,

$$E_2(BPM; H) = \mathbb{F}_2[\tau, v_0, v_1, \ldots]$$

where v_i is the class of the primitive τ_i. The motivic Adams spectral sequence for BPM thus collapses.

We also need to check that $H^{1,0}(\overline{BPM} \wedge \overline{H}) = 0$. This follows from the fact that BPM and H are cellular ([14]), using the computations of $H_*(BPM)$ and $H_*(H)$ and the Künneth theorem [5].

We have

$$h_0 \in E_2^{1,2,1}(\mathbb{S}_{\text{Mot}}; H) \,, \quad v_n \in E_2^{1,2^{n+1}-1,2^n-1}(\mathbb{S}_{\text{Mot}}; H) \,,$$

and the differentials constructed in Proposition 6.3 produce the following differentials in the motivic Adams spectral sequence:

$$d_2 v_{n+1} \equiv v_n^2 h_0, \ n \geq 2$$

modulo terms of higher Cartan-Eilenberg filtration. Recalling that $\deg = t - 2w = 1$ we see $\deg(h_0) = 0$ and $\deg(v_n) = 1$. Thus $v_n^2 h_0$ has Chow degree 2 and Adams filtration 3 and any other such element must be of the form $v_i v_j h_0$. We find that

$$v_n^2 h_0 \in E_2^{3,2^{n+2},2^{n+1}-1}(\mathbb{S}_{\mathrm{Mot}}; H)$$

is the only element in its trigrading so, in fact, our calculation has no indeterminancy. This is our last theorem.

Theorem 9.4.1. *In the η-localized motivic Adams spectral sequence,*

$$d_2 v_{n+1} = v_n^2 h_0\,, \quad n \geq 2\,.$$

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