BOUNDARY BEHAVIOUR OF WEIL–PETERSSON AND FIBER METRICS FOR RIEMANN MODULI SPACES

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Abstract. The Weil–Petersson and Takhtajan–Zograf metrics on the Riemann moduli spaces of complex structures for an n-fold punctured oriented surface of genus g, in the stable range $g + 2n > 2$, are shown here to have complete asymptotic expansions in terms of Fenchel–Nielsen coordinates at the exceptional divisors of the Knudsen–Deligne–Mumford compactification. This is accomplished by finding a full expansion for the hyperbolic metrics on the fibers of the universal curve as they approach the complete metrics on the nodal curves above the exceptional divisors and then using a push-forward theorem for conormal densities. This refines a two-term expansion due to Obitsu–Wolpert for the conformal factor relative to the model plumbing metric which in turn refined the bound obtained by Masur. A similar expansion for the Ricci metric is also obtained.

Contents

Introduction

The universal curve, $\mathcal{C}_g = \mathcal{M}_{g,1}$, of Riemann surfaces of genus $g \geq 2$ may be identified with the moduli space of pointed curves as a stack or as a smooth orbifold fibration $\psi : \mathcal{C}_g \longrightarrow \mathcal{M}_g$ over the moduli space (we distinguish notationally between these spaces since later they have different real resolutions). Deligne and Mumford [3] gave compactifications $\overline{\psi} : \overline{\mathcal{C}}_g = \overline{\mathcal{M}}_{g,1} \longrightarrow \overline{\mathcal{M}}_g$ in which nodal curves are added covering exceptional divisors corresponding to the pinching of geodesics on the Riemann surfaces to pairs of nodal points resulting in a surface, or surfaces, of lower genus but with the same arithmetic genus. The holomorphic map $\overline{\psi}$ then has Lefschetz singularities and is universal in this sense. Each fiber of $\overline{\mathcal{M}}_{g,1}$ carries a unique metric of finite area and curvature -1, complete outside the nodal points.

A resolution of the complex compactification is given below

$$\begin{tikzcd} \widehat{\mathcal{C}}_g \arrow[r, "\beta"] \arrow[d, "\hat{\psi}"'] & \overline{\mathcal{M}}_{g,1} \arrow[d, "\overline{\psi}"] \\ \widehat{\mathcal{M}}_g \arrow[r, "\beta"'] & \overline{\mathcal{M}}_g \end{tikzcd} \tag{1}$$

in the category of real manifolds with corners (or more correctly tied orbifolds), which resolves this fiber metric. In particular, the real fibration in (1) is a b-fibration in terms of which the fiber metric is conformal, with a log-smooth conformal factor, to a smooth metric on a rescaling of the fiber tangent bundle. The resolution (1) (in both domain and range) involves a transcendental step, introducing variables comparable to the length of the shrinking cycles, i.e. Fenchel-Nielsen coordinates.

The regularity properties of the fiber hyperbolic metrics have been widely studied, see [21, 23, 24, 25, 26, 27], and the results effectively applied, see [5, 6, 2]. The log-smoothness of the hyperbolic fiber metric up to the boundaries, produced by the real blow-up, corresponds to a refinement, to infinite order, of the 2-term expansion obtained by Obitsu and Wolpert [15] for the fiber metric relative to the 'plumbing metric' on the local model for nodal curves. The case of a single shrinking geodesic was considered in [13] and the log-smoothness of the constant curvature fiber metrics here is proved by an extension of the method used there (which in structure goes back to Obitsu and Wolpert loc. cit.) We further extend these results to the case of the universal curve over the moduli space, $\mathcal{M}_{g,n}$, of marked Riemann surfaces in the stable case that $2g+n>2$. The universal curve over $\mathcal{M}_{g,n}$ may be identified as $\mathcal{C}_{g,n}=\mathcal{M}_{g,n+1}$ but in which one, here by convention the last, variable is distinguished as the fiber variable in the holomorphic fibration $\mathcal{C}_{g,n}\longrightarrow\mathcal{M}_{g,n}$.

In the unpointed case, let $\mathcal{L}$ be the fiber tangent bundle of ψ, then the cotangent bundle of $\mathcal{M}_g$ is naturally identified with the bundle of holomorphic quadratic differentials, i.e. holomorphic sections of $\mathcal{L}^{-2}$, on the fibers of $\mathcal{C}_g=\mathcal{M}_{g,1}$

$$q:\Lambda^{1,0}\mathcal{M}_g\simeq Q\mathcal{M}_g. \tag{2}$$

Using this identification, the Weil–Petersson (co-)metric is defined by

$$G_{\mathrm{WP}}(\zeta_1,\zeta_2)=\int_{\mathrm{fib}}\frac{\zeta_1\overline{\zeta_2}}{\mu_H},\ \zeta_1,\ \zeta_2\in Q_m,\ m\in\mathcal{M}_g \tag{3}$$

where μ_H is the area form of the fiber hyperbolic metric and the integrand itself may be identified as a fiber 2-form.

More generally in the pointed case, the Knudsen-Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the n-pointed moduli space may again be considered as a smooth complex orbifold. The 'boundary' $\overline{\mathcal{M}}_{g,n}\setminus\mathcal{M}_{g,n}$ is a union of normally intersecting, and self-intersecting, divisors. Expanding on an idea of Robbin and Salamon [17] we extend (2) to the compactification, showing that the logarithmic cotangent bundle ${}^{\mathcal{D}}\Lambda^{1,0}\overline{\mathcal{M}}_{g,n}$, with local sections the sheaf of differentials which are logarithmic across the exceptional divisors, is naturally isomorphic to a corresponding holomorphic extension of the bundle of holomorphic quadratic differentials on the fibers of

$\overline{\mathcal{C}}_{g,n}$. More precisely, the projection

$$\overline{\psi} : \overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n} \tag{4}$$

is a Lefschetz map (as defined explicitly below) and hence has a well-defined, and surjective, differential between the logarithmic tangent bundles. The null bundle, $\widetilde{\mathcal{L}}$, is thus a holomorphic line bundle over $\overline{\mathcal{C}}_{g,n}$ (with sections being the holomorphic vector fields on the fibers which vanish at marked points and nodes – in the marked case nodes also arise from the collision between, or more precisely the separation of, the fixed divisors corresponding to the marked points). Then $Q\mathcal{M}_{g,n}$ extends as a holomorphic vector bundle $\widetilde{Q}\,\overline{\mathcal{M}}_{g,n}$ where the fiber of $\widetilde{Q}$ consists of the holomorphic sections of $\widetilde{\mathcal{L}}^{-2}$ which vanish at marked points and have consistent values at nodes. That is, if the fibers of $\overline{\psi}$ are 'resolved' into a disjoint union of marked Riemann surfaces, by separating the nodes, then elements of the fiber of $\widetilde{Q}$ may be interpreted as meromorphic quadratic differentials in the ordinary sense, with at most simple poles at marked points and double poles at nodes but where the double residues at the two points representing a node are the same. Notice that this is meaningful since the double residue, which is the leading coefficient of $f(z)\frac{dz^2}{z^2}$, is a well-defined complex number. Then q in (2) extends to a global holomorphic isomorphism

$$\widetilde{q} : {}^{\mathcal{D}}\Lambda^{1,0}\overline{\mathcal{M}}_{g,n} \simeq \widetilde{Q}\,\overline{\mathcal{M}}_{g,n}. \tag{5}$$

This allows (3) to be evaluated asymptotically near the divisors, allowing the full description of the singularities of the Weil–Petersson co-log-metric, i.e. on ${}^{\mathcal{D}}\Lambda^{1,0}\overline{\mathcal{M}}_{g,n}$.

One of the properties of the log-cotangent bundle is that ${}^{\mathcal{D}}\Lambda^{1,0}\overline{\mathcal{M}}_{g,n}$ has the cotangent bundle of the divisor (or the log-cotangent bundle in the case of intersecting divisors) as a subbundle over the divisor. These 'tangential elements' are identified by $\widetilde{q}$ with the quadratic differentials with at most simple poles at the corresponding (separated) nodal points. Thus the quadratic differentials corresponding to the log-normal directions are the most singular and these produce the singularities in G_{WP} as a co-log-metric. Moreover, the restriction to the log-cotangent bundle of the divisor gives the Weil–Petersson metric for the finite covering of the divisor as a product of pointed moduli spaces; this is already noted by Masur [10].

In the pointed case we again have a 'metric resolution' to real manifolds with corners extending (1) and again involving the introduction of logarithmic coordinates

$$\begin{array}{ccc} \widehat{\mathcal{C}}_{g,n} & \xrightarrow{\beta} & \overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \\ \Big\downarrow{\scriptstyle\hat{\psi}} & & \Big\downarrow{\scriptstyle\overline{\psi}} \\ \widehat{\mathcal{M}}_{g,n} & \xrightarrow{\beta} & \overline{\mathcal{M}}_{g,n}. \end{array} \tag{6}$$

As remarked above $\widehat{\mathcal{C}}_{g,n} = \widehat{\mathcal{M}}_{g,n+1}$ involves an extra step of resolution compared to $\widehat{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$, without which the map $\hat{\psi}$ is not defined. This extra blow-up is of the codimension-two variety of double points of $\overline{\psi}$ which therefore lifts to a collection of boundary hypersurfaces. As a result the boundary hypersurfaces of $\widehat{\mathcal{C}}_{g,n}$ fall into three distinct classes, the 'fixed' hypersurfaces corresponding to the marked points, the 'type I' boundary hypersurfaces corresponding to the resolved singular fibers and the 'type II' boundary hypersurfaces corresponding to the singular set of $\overline{\psi}$, so the image in the base of the type II hypersurfaces lies within that of the type I hypersurfaces.

The line bundle $\widetilde{\mathcal{L}}$ lifts to a complex line bundle, $\hat{\mathcal{L}}$, on $\widehat{\mathcal{C}}_{g,n}$ but this is not precisely the fiber b-tangent bundle

$$ {}^{\hat{\psi}}T\widehat{\mathcal{C}}_{g,n} \subset {}^{\mathrm{b}}T\widehat{\mathcal{C}}_{g,n} \tag{7}$$

of $\hat{\psi}$, although the latter is a well-defined smooth bundle. Namely $\widetilde{\mathcal{L}}$ is a rescaling of the fiber tangent bundle of $\hat{\psi}$ at the type II boundary hypersurfaces. Its smooth sections are precisely those of the form $\rho_{\mathrm{II}}^{-1}V$ where V is a smooth tangential vector field on $\widehat{\mathcal{C}}_{g,n}$ which is tangent to the circle fibration of the type II hypersurface and such that $\hat{\psi}_*(V) = 0$ and ρ_{II} is a collective defining function for the hypersurfaces of type II. We think of this as a 'cusp structure'. Extending the special case in [13]:

Theorem 1. *The complete metrics of constant curvature -1 on the fibers of $\mathcal{C}_{g,n}$ over $\mathcal{M}_{g,n}$ extend to be conformal to a smooth family of fiber Hermitian metrics on the line bundle $\hat{\mathcal{L}}$ over the fibers of $\widehat{\mathcal{C}}_{g,n} \longrightarrow \widehat{\mathcal{M}}_{g,n}$ with a positive definite conformal factor which is log-smooth.*

A log-smooth function on a manifold with corners is smooth in the interior and log-smooth at the boundary with an expansion with integer powers (as for the Taylor series of a smooth function) with logarithmic factors with powers growing at most linearly. A more detailed description of the asymptotic expansion is given in §5 and §6.

It then follows that the Weil–Petersson metric is also log-smooth on the resolved manifold $\widehat{\mathcal{M}}_{g,n}$, which corresponds to the introduction of logarithmic coordinates around the divisors. The passage to logarithmic variables means that the circle bundles corresponding to the fibrations of the boundary hypersurfaces of $\widehat{\mathcal{M}}_{g,n}$, over the divisors, extend off the boundary to infinite order. In §7 the push-forward theorem from [12] is applied to (3) to yield the second major result of this paper (see also the results of Mazzeo and Swoboda [11]). A holomorphic local defining function for an exceptional divisor

$$ z_j = \exp(-s_j^{-1} + i\theta_j) \tag{8}$$

induces a defining function s_j for the corresponding boundary hypersurface of $\widehat{\mathcal{M}}_{g,n}$ and also locally trivializes the normal circle bundle.

Theorem 2. *On the resolved space $\widehat{\mathcal{M}}_{g,n}$ the Weil–Petersson metric is θ-invariant to infinite order at each of the boundary hypersurfaces and near any point in a corner of codimension k, takes the form*

$$ g_{\mathrm{WP}} = \pi \sum_{j=1}^{k} s_j \left(\frac{ds_j^2}{s_j^2} + s_j^2 d\theta_j^2 \right) + g'_{\mathrm{WP}},\ g'_{\mathrm{WP}}(\partial_{\theta_j}, \cdot) = O(s_j^4), \tag{9}$$

in terms of (8) for local holomorphic defining functions for the divisors. Here g'_{WP} is a log-smooth hermitian tensor which restricts to the corner to be the lift of the Weil–Petersson metric on the k-fold intersection of divisors.

The leading order part of the metric in (9) is the same as the model metric given by Yamada [27]. Note that the local fiber differentials $d\theta_j$ may be replaced by connections forms, in fact it is natural to take (extensions of) the connections forms, α_j, fixed by holomorphy and the hermitian structures on the normal bundles. Then

$$g_{\mathrm{WP}} = \pi \sum_{j=1}^{k} s_j \left(\frac{ds_j^2}{s_j^2} + s_j^2 \alpha_j^2 \right) + g''_{\mathrm{WP}} \tag{10}$$

where g''_{WP} has the same restriction properties as g'_{WP}. It is also permissible to replace the s_j by $h_j = s_j + O(s_j^2)$ where the h_j are the length functions arising from the hermitian structures on the normal bundles, without changing the conclusion regarding the remainder term (in fact the proof of (9) passes through this change of variable).

Given the regularity results below for the lengths of the short closed geodesics, the same form occurs in Fenchel–Nielsen coordinates, i.e. with the s_j interpreted as the normalized lengths of the nearby shrinking geodesics for the non-fixed divisors. The tangential metrics come from the covering of the components of the k-fold intersection of divisors by a product of pointed moduli spaces. For $\mathcal{M}_g$ such an asymptotic expansion has been deduced, from [13], using related methods by Mazzeo and Swoboda [11].

As a corollary, in §11 we derive the formula for the length of the shrinking closed geodesic near the j-th non-fixed divisor in terms of the logarithmic coordinates $s_j = 1/\log(|z_j|^{-1})$ for which we use the abbreviation $\operatorname{ilog}|z_j|$.

Corollary 1. *The length of the shrinking geodesic is a log-smooth function of s_j, and has the form*

$$L_j(s_j) = 2\pi^2 s_j (1 + s_j e(s_j)) \tag{11}$$

where e is a log-smooth function up to the boundaries.

In the paper [20] of Wolf and Wolpert, it is claimed that $L_j(s_j)$ is real-analytic in s_j, which would preclude the appearance of logarithmic terms in $e(s_j)$ (as well as being a stronger analytic statement). However there is an error in the bound deduced from equation (2.2) in [20] which appears to invalidate the argument.

From the asymptotic expansion of the Weil–Petersson metric it follows that the Ricci curvature has a similar expansion, see (8.2). Trapani in [19] showed that the corresponding Ricci metric is complete. In §8 we show that this metric is locally equal, to leading order, to a product of cusp metrics near the intersection of divisors; this refines a result of Liu, Sun and Yau in [9].

The curvature tensor of the Weil–Petersson metric is also computed and specifically the decay rates of the sectional curvature along the normal and tangential directions near the divisors are given, see (9.6).

The curvature form of the fiber hyperbolic metric on the vertical tangent bundle of ψ over $\mathcal{M}_g$ was computed by Wolpert [22] who showed that it pushes forward to a multiple of the Kähler form of the Weil–Petersson metric. Reinterpreting this as a local index theorem, Takhtajan and Zograf in [18] extended the result to the pointed moduli space $\mathcal{M}_{g,n}$, finding extra 'boundary terms' in the push-forward as an additional Kähler form. This metric is given, as a cometric lifted using the Weil–Petersson metric, by a sum over the fixed divisors in $\mathcal{M}_{g,n+1}$:

$$G_{\mathrm{TZ}}(\zeta_1, \zeta_2) = \sum_j \int_{\mathrm{fib}} E_j \frac{\zeta_1 \overline{\zeta_2}}{\mu_H},\ \zeta_1,\ \zeta_2 \in Q_p,\ p \in \mathcal{M}_{g,n}. \tag{12}$$

Here E_j is a boundary forcing term, the solution of $(\Delta+2)E_j = 0$ which is in L^2 on the fibers including up to the marked points and nodes, except for a prescribed singularity (corresponding to the non-L^2 formal solution) at the point corresponding to the jth fixed divisor denoted as F_j; in [18] it is obtained as an Eisenstein series. Such a function is well-defined on any stable Riemann surface with cusps; it is strictly positive away from the cusps and, with these resolved to boundaries, E_j is smooth up to, and vanishes simply at each cusp boundary except the 'forcing boundary' where it has a singularity s^{-2} where the metric is locally $\frac{ds^2}{s^2} + s^2 d\theta^2$. The asymptotic behaviour of G_{TZ} is determined by the structure of the E_j. In [16] the Kähler potential and Chern forms of this metric were calculated.

Each boundary hypersurface of $\widehat{\mathcal{M}}_{g,n}$ corresponds to either two or three boundary hypersurfaces in the resolved universal curve $\widehat{\mathcal{C}}_{g,n}$, depending on whether the node to which this gives rise disconnects the Riemann surface or not. These have interior fibers which are one or two connected Riemann surfaces with two nodes and a cylindrical 'neck' joining the nodes. The behavior of E_j is slightly different in the two cases. If the Riemann surface remains connected without the neck, then E_j approaches the corresponding boundary forcing term for this Riemann surface and vanishes simply at the neck with coefficient being a bridging function discussed in the body of the paper. In case of separation into two Riemann surfaces again E_j approaches the 'local' E_j for the component that meets F_j, it vanishes simply on the neck and vanishes to fourth order at the second component with leading coefficient which is the product of a scattering (or L-function) constant and the singular boundary term E_* on this Riemann surface with pole at the node where it meets the neck. In all cases E_j is globally log-smooth once multiplied by the square of a defining function for F_j.

This description can be iterated to determine the precise leading term of E_j at a boundary surface of codimension k in $\widehat{\mathcal{M}}_{g,n}$. This corresponds to the intersection of k divisors in $\overline{\mathcal{M}}_{g,n}$ and the fiber is the initial Riemann surface subject to k degenerations, each either the shrinking of a geodesic or the 'bubbling off' of a sphere due to the collision of marked points (or marked points with nodes). In all cases the fixed divisor F_j meets one of the component surfaces and E_j approaches the corresponding boundary forcing term there. Any other component is connected to F_j through one or more paths, passing through a sequence of nodes and necks. Consider those paths which pass through the minimum number, σ, of necks. Each of these gives rise to part of the leading term of E_j at the Riemann surface in question; it vanishes to order 4σ there with coefficient the E_* for that Riemann surface with pole at the node through which the path entered, and another constant coefficient formed by a product of scattering (L-function) factors corresponding to the sequence of nodes through which the path passes; thus the leading term of E_j is in general a sum of such terms but all are positive. At the necks essentially the same conclusion holds except that the order of vanishing is $4\sigma+1$ with the coefficient a bridging function.

The asymptotic behavior of E_j at the boundary of $\widehat{\mathcal{M}}_{g,n}$ leads to a corresponding asymptotic expansion for the Takhtajan–Zograf metric, analogous to (9), but with combinatorial complications; this refines results of Obitsu, To and Weng [14]. At the interior of a boundary hypersurface of $\widehat{\mathcal{M}}_{g,n}$ the behavior is relatively simple if the corresponding divisor lifts from $\overline{\mathcal{M}}_{g,n}$. Namely the divisor itself has a local covering by either a moduli space $\overline{\mathcal{M}}_{g-1,n+2}$ if the node does not separate and otherwise by some product $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}$, $n_1+n_2=n$, $g_1+g_2=g-1$. Then

$$\begin{aligned} g_{\mathrm{TZ}} &= h(ds^2+s^4d\theta^2)+sh_1g_{\mathrm{TZ},g_1,n_1}+sh_2g_{\mathrm{TZ},g_2,n_2}+O(s^2),\ n_1,n_2>0 \\ g_{\mathrm{TZ}} &= h(ds^2+s^4d\theta^2)+sh_1g_{\mathrm{TZ},g_1,n}+s^4h_2g_{\mathrm{TZ},g_2,1}+O(s^2),\ n_1=n \end{aligned} \tag{13}$$

has log-smooth coefficients. Here, the coefficients h, h_1 and h_2 are $\theta-$invariant to all orders in s and positive but not constant. We do not completely explore the asymptotics of g_{TZ}, but it is bounded above by a multiple of the Weil–Petersson metric and always vanishes relative to it in normal directions to the boundary faces; the same is therefore true of the corresponding Kähler form.

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1. Lefschetz maps

We consider a complex manifold with normally intersecting and self-intersecting divisors, (C, G_*). Thus the $\{G_i\}$ are a finite collection of closed immersed connected complex hypersurfaces and near each point of C there are *admissible coordinates* (z_*, τ_*) where the z_l define the *local* divisors passing through the point as $\{z_l = 0\}$. On such a manifold there is a well-defined 'logarithmic' complex tangent bundle ${}^{\mathcal{D}}T^{1,0}C$ and corresponding cotangent bundle ${}^{\mathcal{D}}\Lambda^{1,0}C$ determined by the G_*. Namely the spaces of locally holomorphic sections of ${}^{\mathcal{D}}T^{1,0}C$ are the holomorphic vector fields which are tangent to all the local divisors. In admissible coordinates ${}^{\mathcal{D}}T^{1,0}C$ is spanned by the holomorphic vector fields $z_l\partial_{z_l}$ and ∂_{τ_k}. The complex dual of this bundle, ${}^{\mathcal{D}}\Lambda^{1,0}C$, is locally spanned in these coordinates by the dz_l/z_l and $d\tau_k$. The universal curve $\overline{\mathcal{C}}_{g,n}$ has such a structure (except for the orbifold points). And there is a natural map from $\overline{\mathcal{C}}_{g,n}$ to the compactified moduli space $\overline{\mathcal{M}}_{g,n}$ which will be discussed below.

We consider Lefschetz maps, which have singularities modelled on the 'plumbing variety'

$$\phi : \mathbb{C}^2 \ni (z,w) \longmapsto t = zw \in \mathbb{C}. \tag{1.1}$$

Definition 1. A *Lefschetz map* $\phi : C \longrightarrow M$ is a holomorphic map between complex manifolds (subsequently orbifolds) with the following properties

(1) The fiber dimension is one: $\dim_{\mathbb{C}} C = \dim_{\mathbb{C}} M + 1$.
(2) The fibers of ϕ are compact.
(3) Both domain and range carry normally intersecting (and self-intersecting) divisors which will be denoted (C, G_*, F_*) and (M, G'_*).
(4) The F_l, $l = 1, \ldots, n$ are 'fixed divisors', without self-intersections or intersections with the other F_* and such that, for each i,

$$\phi : F_i \longrightarrow M \text{ is a biholomorphism.} \tag{1.2}$$

(5) The other divisors, G_i, in C map onto the divisors in M :

$$\phi(G_i) = G'_i. \tag{1.3}$$

(6) The G_i have self-intersections, S_i, of codimension two (if they exist) and for different i, these are also disjoint in C; outside the S_i, ϕ has surjective differential.

(7) Near each double point $p \in S_i \subset G_i$ there are admissible coordinates z, w, z_r, τ_j in C and coordinates t, z'_r, τ'_j near the image of p in M such that

$$\phi^* t = zw,\ \phi^* \tau'_j = \tau_j,\ \phi^* z'_r = z_r, \tag{1.4}$$

where G_i is locally defined by $\{z=0\} \cup \{w=0\}$ and the z'_r define the local divisors through p other than G'_i which is defined by $t=0$.

It follows that each of the fibers is a nodal Riemann surface, with marked points from the intersections with the F_l. The Lefschetz map is said to be *stable* if each of the component Riemann surfaces (when the nodal points are separated) in the fibers are stable; i.e. for each component the genus g and the total number of nodes and punctures n satisfy $2g-2+n>0$.

Consider the invariance properties of the local form of the 'plumbing' model for a Lefschetz map as in (1.1) near the singular surfaces S_i :

$$\{(z,w) \in \mathbb{C}; |z|, |w| < \delta\} \ni (z,w) \longmapsto zw \in \mathbb{C}. \tag{1.5}$$

This normal form can be regained after separate holomorphic changes of coordinates in the disk $z=0$ and $w=0$.

Lemma 1. *Suppose that*

$$z \longmapsto z(1+zf(z)),\ w \longmapsto w(1+wg(w)) \tag{1.6}$$

are separate holomorphic coordinate changes fixing the origins, then there are holomorphic functions (germs near the origin) $a(w)$, $b(z)$ *and* $e(z,w)$ *such that*

$$\begin{aligned} Z &= z(1+zf(z)+wa(w)+zwe(z,w)),\\ W &= w(1+wg(w)+zb(z)+zwe(z,w)) \text{ satisfy } ZW=t. \end{aligned} \tag{1.7}$$

Proof. The identity we aim for is

$$zw(1+zf(z)+wa(w)+wze(z,w))(1+wg(w)+zb(z)+wze(z,w)) = zw. \tag{1.8}$$

Cancelling factors and expanding out this becomes

$$\begin{aligned} &wg(w)+zb(z)+zwe(z,w)+zf(z)+wa(w)+zwe(z,w)+zwf(z)g(w)\\ &+z^2b(z)f(z)+z^2wf(z)e(z,w)+w^2g(w)a(w)+zwa(w)b(z)+zw^2a(w)e(z,w)\\ &+w^2zg(w)e(z,w)+z^2wb(z)e(z,w)+z^2w^2e^2(z,w)=zwh(zw) \end{aligned} \tag{1.9}$$

Separating out the 'pure terms' it follows that (1.9) is a consequence of demanding

$$\begin{aligned} &w\left(g(w)+a(w)+wg(w)a(w)\right)=0 \Longleftrightarrow a(w)=-(1+wg(w))^{-1}g(w)\\ &z\left(b(z)+f(z)+zb(z)f(z)\right)=0 \Longleftrightarrow b(z)=-(1+zf(z))^{-1}f(z)\\ &zw\big(2e(z,w)+f(z)g(w)+zf(z)e(z,w)+a(w)b(z)+wa(w)e(z,w)\\ &\qquad +wg(w)e(z,w)+zb(z)e(z,w)+zwe^2(z,w)\big)\big)=0. \end{aligned} \tag{1.10}$$

By the implicit function theorem the last equation has a unique, and holomorphic, solution with $e(z,w)$ close to $-\frac{1}{2}f(z)g(w)$ for z, w small. $\square$

Holomorphic dependence on parameters follows from the same argument, so the result carries over to the case of a Lefschetz map with base of dimension greater than one, in the sense above, locally near each singular surface.

Corollary 2. *For a stable Lefschetz fibration there are holomorphic coordinates near each singular point in terms of which both the Lefschetz map and the family of hyperbolic metrics on the singular fibre are in normal form.*

Conversely:

Lemma 2. *Any local holomorphic coordinate transformation in z, w near a singular point which leaves the form of the family of hyperbolic metrics and the coordinate form of the Lefschetz map unchanged can only change the defining function for the divisor in the base from t to $ct(1 + th(t))$ where $|c| = 1$.*

Proof. If a coordinate transformation preserves the normal form for the Lefschetz map it must fix the singular point and hence the preimage of its image in the base. Thus it must map the surfaces $z = 0$ and $w = 0$ into themselves, or each other. The latter possibility can be ignored, since we may simply exchange z and w and preserve the form. Thus the normal form of the hyperbolic metric near the ends fixes the conformal structure near the end and hence the coordinates up to a constant factor of norm 1. The coordinate transformation on the total space must therefore be of the form $z \longmapsto e^{i\theta} z(1+zf(z)+wF(z,w))$, $w \longmapsto e^{i\theta'} w(1+wg(w)+zG(z,w))$ with θ and θ' real. Thus

$$t \longmapsto e^{i(\theta+\theta')} t(1 + O(z, w)) = ct(1 + th(t)) \Longrightarrow |c| = 1.$$

□

Proposition 1. *The divisors G'_i in the base of a stable Lefschetz map have natural hermitian structures on their normal bundles.*

Proof. The possible changes of local holomorphic defining function for a divisor G^*_i in the base, in terms of which the Lefschetz map takes normal form near the corresponding S_i, are limited to have differential of norm one on the divisor and hence induce a hermitian structure. □

As the name indicates, if one fiber of a Lefschetz map is stable then the map is stable in a neighborhood of that fiber. In the case of a singular fiber the arithmetic genus g_a is the sum of the number of pairs of nodal points and the genus from each of the components of the nodal surface with nodes separated and it follows that $g_a + 2n > 2$ is then constant.

Away from the self-intersections, S_i, of the G_i, ϕ is a holomorphic submersion mapping the divisors G_i onto the G'_i so the logarithmic differentials dz'_i/z'_i pull back to be dz_i/z_i. Near $p \in S_i$ this remains true for the divisors other than G'_i, locally defined by t which, by (1.4) satisfies

$$\phi^*(dt/t) = dz/z + dw/w. \tag{1.11}$$

So at these points, ϕ^* maps ${}^{\mathcal{D}}\Lambda^{1,0}_{\phi(p)}M$ injectively to ${}^{\mathcal{D}}\Lambda^{1,0}_{p}C$ and hence there is a well-defined dual log-differential which is still surjective

$$\phi_* : {}^{\mathcal{D}}T^{1,0}_p C \longrightarrow {}^{\mathcal{D}}T^{1,0}_{\phi(p)}M, \ \forall\ p \in C. \tag{1.12}$$

In this sense a Lefschetz map is a 'log fibration' (the complex analog of the b-fibrations considered in the real case below).

Lemma 3. *The null bundle of the logarithmic differential $L = L_\phi \subset {}^{\mathcal{D}}T^{1,0}C$ is a holomorphic subbundle which reduces to the fiber tangent bundle at regular points.*

Proof. This is immediate from the local form of ϕ required in the definition. Near regular points of the map. the fiber is smooth and L is spanned by a non-vanishing holomorphic vector field tangent to the fibers. Near singular points there are coordinates as in (1.4) and the log-differential has null space spanned by $z\partial_z - w\partial_w$. □

The fiber $\overline{\partial}$-operator on regular fibers is naturally a differential operator

$$\overline{\partial} : \mathcal{C}^\infty(C_{\mathrm{reg}}; L) \longrightarrow \mathcal{C}^\infty(C_{\mathrm{reg}}; L \otimes \overline{L}^{-1}).$$

Lemma 4. *The fiber $\overline{\partial}$-operator extends smoothly to a 'log-differential' operator*

$$(1.13) \qquad {}^{\mathcal{D}}\overline{\partial} : \mathcal{C}^\infty(C; L) \longrightarrow \{u \in \mathcal{C}^\infty(C; L \otimes \overline{L}^{-1}); u = 0 \text{ at } \bigcup_i S_i \cup \bigcup_l F_l\}.$$

Proof. This is clear away from the singular points $S_i \subset C$. Near each such point in local coordinates (1.4),

$$(1.14) \qquad {}^{\mathcal{D}}\overline{\partial}(a(z\partial_z - w\partial_w)) = \frac{1}{2}\left(\bar{z}\partial_{\bar{z}} - \bar{w}\partial_{\bar{w}} a\right)(z\partial_z - w\partial_w) \cdot \left(\frac{d\bar{z}}{\bar{z}} + \frac{d\bar{w}}{\bar{w}}\right)$$

where the two local components of the singular fiber are $z = 0$ and $w = 0$ and the coefficient vanishes at the S_i. □

The stability assumption on the fibers and an application of Riemann–Roch show that ${}^{\mathcal{D}}\overline{\partial}$ is injective on each fiber since the null space consists of holomorphic vector fields vanishing at the fixed divisors and, on the G_i at the S_i.

As already noted the essential property of the Knudsen-Deligne-Mumford Lefschetz map $\overline{\mathcal{C}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ is that it is universal for stable Lefschetz maps. Namely

Theorem 3 (Knudsen–Deligne–Mumford[3, 7])**.** *For any stable Lefschetz map, in the sense of Definition 1, there is a unique commuting square of holomorphic maps*

$$(1.15) \qquad \begin{tikzcd} C \arrow[r, "\chi^\#"] \arrow[d, "\phi"'] & \overline{\mathcal{C}}_{g,n} \arrow[d, "\overline{\psi}"] \\ M \arrow[r, "\chi"] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

where $\chi^\#$ is a fiber biholomorphism.

In [17], Robbin and Salamon give an infinitesimal criterion for such universality at the germ level. We proceed to review this result.

The operator ${}^{\mathcal{D}}\overline{\partial}$ in (1.13) is not surjective, in fact again by Riemann-Roch its image has complement of dimension $3g_a - 3 + n$. As noted above, a Lefschetz map has surjective log-differential. It follows that any smooth log vector field (i.e. tangent to the divisors) on the base is ϕ-related to such a smooth vector field on C. Consider the sheaf over M with sections over an open set $O \subset M$

$$(1.16) \quad \mathcal{E}(O) = \{V \in \mathcal{C}^\infty(\phi^{-1}(O); T^{1,0}C) : \exists\, v \in \mathcal{C}^\infty(O; T^{1,0}M); \overline{\partial}v = 0,\ \phi_{p*}V(p) = v(\phi(p))\}$$

consisting of the vector fields which are ϕ-related to a holomorphic vector field on O. As the null space of ϕ_*

$$\mathcal{C}^\infty(\phi^{-1}(O); L) \subset \mathcal{E}(O) \tag{1.17}$$

with the quotient being the holomorphic vector fields on O.

Lemma 5. *For a Lefschetz map the operator* ${}^{\mathcal{D}}\overline{\partial}$ *extends to* $\mathcal{E}$:

$$ {}^{\mathcal{D}}\overline{\partial} : \mathcal{E}(O) \longrightarrow \{u \in \mathcal{C}^\infty(\phi^{-1}(O); L \otimes \overline{L}^{-1}); u = 0 \ \mathit{at} \ \bigcup_i S_i \cup \bigcup_l F_l\}. \tag{1.18}$$

Proof. On a coordinate patch $U \subset \phi^{-1}(O) \subset C$, as in the definition of a Lefschetz map, an element $u \in \mathcal{E}(O)$ restricts to be of the form

$$u = v + w, \ \ w \in \mathcal{C}^\infty(U; L) \tag{1.19}$$

and defining ${}^{\mathcal{D}}\overline{\partial} u = {}^{\mathcal{D}}\overline{\partial} w$ is independent of choices. □

Now, the result alluded to above is

Proposition 2 (Robbin–Salamon [17])**.** *A Lefschetz map is universal (at the germ level) at a given fiber if and only if* ${}^{\mathcal{D}}\overline{\partial}$ *in* (1.18) *is an isomorphism.*

Proof. See [17]. A proof using techniques much closer to those used here can also be constructed. □

Note that Robbin and Salamon in [17] proceed to construct such germs of universal Lefschetz fibrations (as 'unfoldings' of the central fiber) and use these to (re-)construct the Knudsen-Deligne-Mumford compactification.

Proposition 3. *For each fiber of* ϕ*, sections of the bundle* $L \otimes \overline{L}^{-1}$ *which vanish at the* F_l *and* S_i *may be paired with 'quadratic differentials' to give a natural and non-degenerate complex pairing*

$$\begin{aligned} &\{u \in \mathcal{C}^\infty(\phi^{-1}(O); L \otimes \overline{L}^{-1})\big|_{\phi^{-1}(m)}; u = 0 \ \mathit{at} \ \bigcup_i S_i \ \mathit{and} \ \bigcup_l F_l\} \\ &\qquad \times \{q \in \mathcal{C}^\infty(\phi^{-1}(O); L^{-2})\big|_{\phi^{-1}(m)}; q = 0 \ \mathit{at} \ \bigcup_l F_l\} \\ &\qquad\qquad \ni (u, q) \longmapsto \int_{\mathrm{fib}} (u \cdot q) \in \mathbb{C}. \end{aligned} \tag{1.20}$$

Proof. At regular fibers, away from the F_l, the sections are of the form $u = u'\partial_z \cdot \overline{dz}$ and $q = q' dz^2$ with u' and q' smooth. The complex pairing of ∂_z and dz allows this to be interpreted as a local area form

$$u' q' \overline{dz} \cdot dz. \tag{1.21}$$

Near a fixed divisor with z and admissible fiber coordinate $u = u' z \partial_z \cdot d\bar{z}/\bar{z}$ and $q = q'(dz/z)^2$ where by hypothesis u' vanishes at $z = 0$ as does q'. Working in polar coordinates $z = re^{i\theta}$ the area form in (1.21) becomes

$$u' q' \frac{d\bar{z}}{\bar{z}} \cdot \frac{dz}{z} = u' q' \frac{dr}{r} d\theta. \tag{1.22}$$

Since both u' and q' vanish at $r = 0$ this is a multiple of the standard area form in polar coordinates $r dr d\theta$ and so the integral pairing extends across these divisors.

On the singular fibers the same discussion applies near the fixed divisors. Near a nodal point, in some S_i, there are Lefschetz coordinates z and w. By assumption $u = u'(z\partial_z - w\partial_w)\cdot(\frac{d\bar z}{\bar z} + \frac{d\bar w}{\bar w})$ with u' vanishing at $z = 0$ or $w = 0$ on the two intersecting parts of the fiber. Similarly

$$q = q'(\frac{dz}{z} + \frac{dw}{w})^2 \tag{1.23}$$

where the compatibility condition on q' is that it have a well-defined value at $z = w = 0$, approached continuously from both local parts of the fiber. Thus the local area form given by the pairing becomes

$$u'q'\frac{dr}{r}d\theta \tag{1.24}$$

which is bounded by $Cdrd\theta$ locally. The resulting pairing integral is finite, but more significantly the leading terms in the integral at $w = 0$ and $z = 0$ cancel.

That the resulting pairing is non-degenerate on the smooth sections is then clear. □

Although somewhat ad hoc in appearance this pairing actually corresponds to a natural distributional pairing on real resolved spaces.

Proposition 4. *In terms of the pairing* (1.20)*, the range of* ${}^{\mathcal{D}}\overline{\partial}$ *in* (1.13) *is naturally identified, over each fiber with the annihilator of*

$$\widetilde{Q}(m) = \{\zeta \in \mathcal{C}^\infty(\pi^{-1}(m); L^{-2}); \zeta = 0 \text{ at } \bigcup_l F_l \text{ and } \overline{\partial}\zeta = 0\}. \tag{1.25}$$

Proof. The range of ${}^{\mathcal{D}}\overline{\partial}$ in (1.14) is closed with finite-dimensional complement for each fiber since the domain differs from the standard domain on the disjoint union of the Riemann surfaces, into which each fiber decomposed, by a finite-dimensional space. The restriction of the pairing to a finite dimensional complement and to an appropriate finite-dimensional subspace of the space of quadratic differentials in (1.20) is therefore non-degenerate. On the other hand the space of holomorphic quadratic differentials, in this sense, pairs to zero with the range of ${}^{\mathcal{D}}\overline{\partial}$ and has the same dimension as the complementary space so is indeed the annihilator of the range. □

Note there is a certain inconsistency in notation regarding the extension of Q to the compactification by $\widetilde{Q}$, since following the convention of denoting extensions to the compactification by a 'bar' would conflict with the notation for complex conjugation.

That $\widetilde{Q}$ is a holomorphic bundle over M is again a consequence of stability, that its rank is constant. This follows from algebraic-geometric arguments or from the known holomorphy of the log cotangent bundle to $\overline{\mathcal{M}}_{g,n}$ and the pointwise identification of it with $\widetilde{Q}$.

Now the extended operator ${}^{\mathcal{D}}\overline{\partial}$ in (1.18) defines a bundle map

$${}^{\mathcal{D}}T^{1,0}M \longrightarrow (\widetilde{Q})',\ {}^{\mathcal{D}}\Lambda^{1,0}M \longrightarrow \widetilde{Q} \tag{1.26}$$

and Proposition 2 of Robbin-Salamon asserts that this is an isomorphism, precisely when the Lefschetz map is universal at a germ level. This is the identification $\widetilde{q}$ of (5) in the case of $\overline{\mathcal{M}}_{g,n}$.

2. Real and metric resolutions

Although regular, or at least minimally singular, in themselves (in particular 'flat'), it is necessary for analytic purposes to resolve the Lefschetz fibrations considered above. The construction here is a direct generalization of the constructions in [13]. Although it is not discussed here, there is a simpler (non-logarithmic) resolution which is appropriate for the analysis of the fiber ${}^{\mathcal{D}}\overline{\partial}$.

The first step in the resolution is to blow up, in the real sense, the divisors, in both domain and range. Since any intersections or self-intersections are normal, i.e. transversal, there is no ambiguity in terms of the order chosen in doing this. Since we insist, as a matter of definition or notation, that the boundary hypersurfaces of a manifold with corners be embedded, and this need not be the case here, the result may only be a *tied manifold* – a smooth manifold locally modelled on the products $[0,\infty)^k \times (-\infty,\infty)^{n-k}$ but with possibly non-embedded boundary hypersurfaces.

Definition 2. If (M, G_*) is a complex manifold with normally intersecting divisors, then the real resolution

$$\tilde{M} = [M; G_*]_{\mathbb{R}} \tag{2.1}$$

is a tied manifold (so with corners).

Locally the complex variables z_l defining the divisors lift to $r_l e^{i\theta_l}$ in the real blow-up so the boundaries carry circle fibrations. The logarithmic tangent and cotangent bundles lift to be canonically isomorphic to the corresponding b-tangent and cotangent bundles, which therefore carry induced complex structures; namely

$$z\partial_z \text{ lifts to } r\partial_r + i\partial_\theta. \tag{2.2}$$

This definition can be applied to both the domain and the range for the 'plumbing variety' model for a Lefschetz map (1.1). Real blow-up of the two divisors $z = 0$ and $w = 0$, introduces polar coordinates $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ and similarly blow up of $t = 0$ introduces $t = re^{i\theta}$ in the range space. The local Lefschetz maps then lifts to be a fibration in the angular variables with a simple b-fibration condition satisfied in the radial variables

$$\begin{gathered}\tilde{\phi}(r_1, r_2, \theta_1, \theta_2) = (r, \theta)\\ r = r_1 r_2,\ \theta = \theta_1 + \theta_2.\end{gathered} \tag{2.3}$$

Thus the lifted map is indeed a b-fibration. Globalizing this statement gives:

Proposition 5. *A Lefschetz map between complex manifolds with divisors lifts to a b-fibration*

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{\phi}"] \arrow[d, "\beta_{\mathbb{R}}"'] & \tilde{M} \arrow[d, "\beta_{\mathbb{R}}"] \\ C \arrow[r, "\phi"] & M \end{tikzcd} \tag{2.4}$$

which is 'simple' in the sense that it is the real analog of a Lefschetz map and each boundary defining function in the base lifts, locally, to be a product of at most two factors as in (2.3).

However, to resolve the metric we need to go further.

Definition 3. The metric resolution of a Lefschetz fibration is defined from the real resolution in Definition 2 by

(1) Logarithmic resolution of the boundary hypersurfaces in both domain and range, i.e. by introduction of the function $\operatorname{ilog}\rho = 1/\log(1/\rho)$ in place of each (local) boundary defining function ρ . This results in tied manifolds in domain and range mapping smoothly, and homeomorphically, to the resolutions in Definition 2; the resolved range space is denote $\widehat{M}$.
(2) Further radial blow-up, in the domain, of the preimages after the first step of the boundary faces of codimension-two in Definition 2.4 resulting in the tied manifold with corners manifolds $\widehat{C}$.

To illustrate the resolution, we give the example when there is one nodal intersection explicitly in terms of local coordinates. Locally the fibration is given by the following plumbing variety

$$P = \{(z,w) \in \mathbb{C}^2;\ \exists\, t \in \mathbb{C},\ zw = t,\ |z| \le \frac{3}{4},\ |w| \le \frac{3}{4},\ |t| \le \frac{1}{2}\},$$
$$P \xrightarrow{\phi} \mathbb{D}_{\frac{1}{2}} = \{t \in \mathbb{C}; |t| \le \frac{1}{2}\}.$$

The real resolution introduces polar coordinates

$$z = r_z e^{i\theta_z},\ w = r_w e^{i\theta_w},\ t = r_t e^{i\theta_t}.$$

Step (1) above, the logarithmic resolution, introduces variables $\operatorname{ilog}\rho = 1/\log(1/\rho)$ for the radial variables

$$s_z = \operatorname{ilog} r_z,\ s_w = \operatorname{ilog} r_w,\ s = \operatorname{ilog} r_t.$$

Step (2), the radial blow-up, resolves the singularity in $s = \frac{s_z s_w}{s_z + s_w}$ by introducing polar coordinates for (s_z, s_w) :

$$(s_z, s_w) = (RR_z, RR_w),\ R = \sqrt{s_z^2 + s_w^2}.$$

Figure 1 indicate the resolution of the domain space. The first step illustrates the blow-up of complex divisors, each becoming a boundary hypersurface (with circle bundle). The second 'trivial' step on the left corresponds to the replacement of the radial defining functions by their (doubly inverted) logarithms. The radial blow up amounts to the introduction of polar coordinates around the corners which correspond to the nodal surfaces.

In the case of a multi-Lefschetz fibration, the local coordinate description (with more parameters) is the same but in various factors. Performing the two operations (1) and (2) in the opposite order is by no means equivalent. Although we use the same notation for the resolutions in domain and range note that an extra step is involved in the domain, where the codimension two intersections of the divisors corresponding to the nodal surfaces are resolved to become boundary hypersurfaces. It is for this reason that we distinguish notationally between $\mathcal{C}_{g,n}$ and $\mathcal{M}_{g,n+1}$, which are the same space, but their real resolutions, $\widehat{\mathcal{C}}_{g,n}$ and $\widehat{\mathcal{M}}_{g,n+1}$, are different – because the former is viewed as the domain of a Lefschetz map while the latter is the range (of a different map); so it is the Lefschetz map itself which is resolved by this construction.

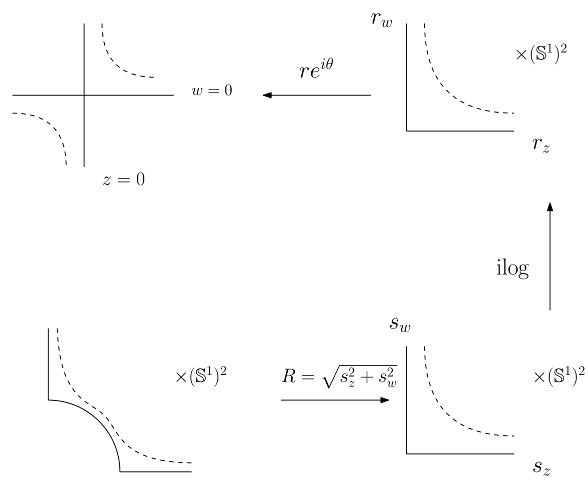

FIGURE 1. Local form of the metric resolution

Proposition 6. *After the 'metric resolution' of domain and range a Lefschetz map $\phi : C \longrightarrow M$ lifts to a b-fibration*

$$\widehat{\phi} : \widehat{C} \longrightarrow \widehat{M}. \tag{2.5}$$

Proof. As noted in Proposition 5, after the initial real resolution the lift of ϕ is smooth and a b-fibration. However, after the logarithmic step in Definition 3 regularity (and indeed existence) fails precisely in the new normal variables, the defining functions $\operatorname{ilog}\rho$. Tangential regularity is unaffected and there is no issue where the map is a fibration, but from the local form (2.3) away from the new boundaries,

$$\phi^* \operatorname{ilog} r = \frac{\operatorname{ilog} r_1 \operatorname{ilog} r_2}{\operatorname{ilog} r_1 + \operatorname{ilog} r_2} \tag{2.6}$$

and the right side is not smooth. However, the second blow-up introduces the radial variable $R = \operatorname{ilog} r_1 + \operatorname{ilog} r_2$ as defining function for the new hypersurface and makes the 'angular functions' $w_1 = \operatorname{ilog} r_1/R$ and $w_2 = \operatorname{ilog} r_2/R$ smooth local boundary defining functions, so then

$$\widehat{\phi}^* \operatorname{ilog} r = R w_1 w_2 \tag{2.7}$$

is indeed smooth and shows the resulting map to be a b-fibration. Although the lifted boundary defining function is the product of three boundary defining functions only two of these can vanish simultaneously. □

After the metric resolution, the domain space $\widehat{C}$ has three types of boundary hypersurfaces. The fixed hypersurfaces denoted F_*, which correspond to the marked points. These do not have self-intersections nor do they intersect among themselves. The second class of boundary hypersurfaces are the lifts (proper transforms) of the original Lefschetz divisors, we denote them $H_{\mathrm{I},*}$. The third class, $H_{\mathrm{II},*}$ arise from the final blow up of the codimension two surfaces formed by the singular points of the Lefschetz map. These last two classes fiber under $\widehat{\phi}$. The fibers of the $H_{I,*}$ consist of circle bundles over the component Riemann surfaces of singular fibers of ϕ with the marked points blown up in the base of the circle fibrations (forming the intersections with the F_*) and the nodal points similarly blown up (and separated) forming the intersections with the $H_{\mathrm{II},*}$. Thus these fibers are all circle bundles over Riemann surfaces with (resolved) cusp boundaries. The fibers of the $H_{\mathrm{II},*}$ cylinders each carry torus bundles, linking the boundary curves of the fibers of $H_{I,*}$ corresponding to the nodes.

The boundaries of the resolved base $\widehat{M}$ also carry circle bundles, over which the boundary faces above fiber. From Proposition 1, the normal bundle of any G'_i is trivial, hence there is a uniquely defined circle bundle. The faces $H_{\mathrm{II},*}$ fiber over the original intersection of the divisors in M as a torus bundle over a cylinder, with a circle subbundle corresponding to the diagonal action in the angular variables in (2.3).

Proposition 7. *The circle bundles, and torus bundles in the case of $H_{II} \subset \widehat{C}$, over the boundary hypersurfaces of $\widehat{M}$ and $\widehat{C}$ have well-defined extensions off the boundaries up to infinite order and the rotation-invariant defining functions are determined up to second order.*

Proof. A complex hypersurface in a complex manifold has a local defining function z which is well-defined up to a non-vanishing complex multiple so another defining function is

$$z' = \alpha z(1 + z\beta(z)) \tag{2.8}$$

where α is independent of z. Thus a branch of the logarithm satisfies

$$\log z' = \log z + \log \alpha + \log(1 + z\beta). \tag{2.9}$$

In terms of $\rho = \operatorname{ilog}|z|$ the last term vanishes to infinite order with ρ so the circle action, given by the imaginary part of (2.9) is defined up to infinite order. The real part of (2.9) shows that the new defining function for the hypersurface is

$$\rho' = \operatorname{ilog}|z'| = \frac{\operatorname{ilog}|z|}{1 - i\log|z|\log|\alpha| + \log|1 + z\beta|} = \rho + O(\rho^2).$$

Thus there is a determined class of defining functions differing only by second order terms. Under the further, radial, blow up the torus bundle and defining function

$$\rho_{\mathrm{II}} = \rho_1 + \rho_2 \tag{2.10}$$

have similarly well-defined extensions. □

Definition 4. The space of smooth 'rotationally flat' functions on $\widehat{C}$ which are invariant to infinite order under the circle and torus actions will be denoted $\mathcal{C}^\infty_\theta(\widehat{C}) \subset \mathcal{C}^\infty(\widehat{C})$.

There is a similar notion on $\widehat{M}$ and for metrics on $\widehat{C}$ and $\widehat{M}$ and other bundles to which the circle and torus actions naturally extend. The discussion above shows that the boundary defining functions are all rotationally flat,

$$\rho_F,\ \rho_{\mathrm{I}},\ \rho_{\mathrm{II}} \in \mathcal{C}^\infty_\theta(\widehat{C}). \tag{2.11}$$

Such a defining function, determined to leading order, is particularly relevant at the boundary hypersurfaces, $H_{\mathrm{II},*} \subset \widehat{C}$, corresponding to the blow-up of the variety of singular points of the Lefschetz fibration; we say it determines a cusp structure there and denote by $\rho_{\mathrm{II}} \in \mathcal{C}^\infty(\widehat{M})$ an admissible defining function in this sense. As a direct consequence of Proposition 6, the null bundle of the differential $\widehat{\phi}_*$ extends smoothly to a subbundle of ${}^{\mathrm{b}}T\widehat{M}$. The holomorphy of ϕ means this has a complex structure away from the boundaries but this does not extend to a complex structure on the null bundle, just corresponding to the fact that the null bundle after the first resolution does not lift to the null bundle of the differential after the second stage. Rather it lifts to a rescaled version of this, corresponding to the 'cusp' structure at $H_{\mathrm{II},*}$.

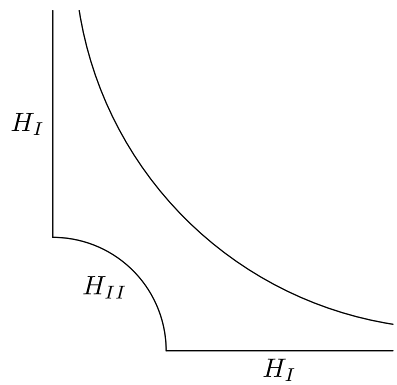

FIGURE 2. The metric resolution that introduces two boundary hypersurfaces H_{I} and H_{II}

Lemma 6. *The null bundle of the Lefschetz map lifts, with its complex structure, to the rescaled bundle which has global sections*

$$\mathcal{C}^\infty(\widehat{C};\hat{\mathcal{L}}) = \{v \in \mathcal{C}^\infty(\widehat{C};{}^{\mathrm{b}}T\widehat{M}); \widehat{\phi}_*(v) = 0 \text{ and } v\rho_{II} = O(\rho_{II}^2)\} \tag{2.12}$$

and is equivariant to infinite order with respect to the circle and torus actions.

Proof. The result is immediate away from $H_{\mathrm{II},*}$. In plumbing coordinates $z = r_1 e^{i\theta_1}$, $w = r_2 e^{i\theta_2}$ near the singular points the null space of the differential of ϕ lifts in terms of $\rho_i = \mathrm{ilog}\, r_i$ to be spanned by

$$\begin{gathered} v = r_1\partial_{r_1} - r_2\partial_{r_2} + i(\partial_{\theta_1} - \partial_{\theta_2}) = \rho_1^2\partial_{\rho_1} - \rho_2^2\partial_{\rho_2} + i(\partial_{\theta_1} - \partial_{\theta_2}), \\ v(\rho_1+\rho_2) = \rho_1^2 - \rho_2^2 = (\rho_1+\rho_2)^2\left(\frac{\rho_1}{\rho_1+\rho_2} - \frac{\rho_2}{\rho_1+\rho_2}\right). \end{gathered} \tag{2.13}$$

Since $\rho_{\mathrm{II}} = \rho_1 + \rho_2$ is a local admissible defining function for $H_{\mathrm{II},*} \subset \widehat{M}$ and the projective functions are smooth on $\widehat{M}$ this shows that the null bundle lifts to (2.12) with its complex structure remaining smooth. □

3. Log-smoothness of the fiber metrics

We proceed to outline the proof of Theorem 1 in the context of the metric resolution of a general smooth stable Lefschetz fibration, as defined above, $\widehat{C} \longrightarrow \widehat{M}$. Before the real resolution, the regular fibers of $C \longrightarrow M$ are Riemann surfaces, with punctures where they intersect the fixed divisors $F_i \subset C$ which are mapped surjectively to M. These fixed divisors cannot intersect, and in consequence the regular fibers are naturally pointed Riemann surfaces. Similarly, the singular fibers are nodal Riemann surfaces, with punctures at the intersections with the F_i, which are disjoint from the nodes. The assumed stability implies the stability of each of the component punctured Riemann surfaces, after each of the nodes has been separated into a pair of punctures.

Under these assumptions of stability on each fiber there is a unique Riemannian metric of curvature -1 and finite area which is complete outside the nodes and marked points; these are the hyperbolic fiber metrics. We extend the main result of [13], which corresponds to the formation of a single node, to the general case. The basic properties of log-smooth functions on a manifold with corners, in this case $\widehat{C}$, are recalled in the Appendix. In particular, rotational flatness, meaning that application of a generator of the circle action corresponding to a resolved divisor in $\widehat{M}$ lifted from the base yields an object vanishing to infinite order at the preimage of that resolved divisor, extends directly to the log-smooth case.

Theorem 4. *The hyperbolic fiber metrics for a stable Lefschetz map form a log-smooth rotationally flat family of hermitian metrics on the complex line bundle $\hat{\mathcal{L}}$ over $\widehat{C}$.*

Theorem 1, the universal case $\widehat{\mathcal{C}}_{g,n} \longrightarrow \widehat{\mathcal{M}}_{g,n}$, follows from the uniqueness of the fiber metrics by localizing in the base and passing to a finite cover to remove the orbifold points.

The proof is carried out in the subsequent sections concluding in Section 6 as follows:

(1) In Section 4 by slightly extending the 'grafting' construction of Obitsu and Wolpert we construct, in Proposition 8, a smooth family of Hermitian fiber metrics on $\hat{\mathcal{L}}$ which has constant curvature -1 near the nodal parts and the fixed divisors, is rotational invariant to infinite order at the preimage of each divisor in the base and reduces to the standard cusp metric on the singular fibers up to quadratic error.

(2) In Section 5 we examine in detail the properties of $(\Delta+2)^{-1}$ for this family of metrics since this is the operator appearing in the linearization of the curvature equation in the form

$$(\Delta_{g_{\mathrm{pl}}} + 2)u = h = -1 - R(g_{pl}). \tag{3.1}$$

First we show the uniform boundedness of this operator on appropriate spaces in which the parameters are incorporated. Next we show, inductively, the existence of rotationally invariant solutions in the sense of formal log-power series at the boundary faces lying in the preimages of the divisors. Combining these two results shows that $(\Delta+2)^{-1}$ acts on log-smooth rotationally invariant data.

(3) From the 'grafted' family of metrics from Proposition 8 below, the regularity of the constant curvature family is obtained in Section 6 by solving the equation for the conformal factor e^{2f}

$$\Delta_{g_{\mathrm{pl}}} f + e^{2f} + R(g_{pl}) = 0. \tag{3.2}$$

This is first shown to have a formal log-power series solution, by iteration of the corresponding result for the linearized equation, and then an application of the implicit function theorem shows that this is the expansion of the unique solution to (3.2) on the regular fibers.

In Section 7 this result is applied to yield a corresponding regularity statement for the Weil-Petersson metric.

4. The grafted metric

We carry through the construction of a family of metrics on $\hat{\mathcal{L}}$ near a given singular fiber of $\hat{\psi}$ as described in Step (1) above. As already noted, this is essentially the construction of Obitsu and Wolpert [15] . Initially we work locally in the base in the original holomorphic setting, for the universal case passing to a finite cover of the base to remove orbifold points, but then the metric can be averaged over the finite group action.

If the base fiber has k pairs of nodal points it lies in the intersection of k local divisors over which the family is holomorphic. The uniformization theorem, with smooth parameters, shows the existence of a smooth family of hyperbolic metrics for this restricted family. Away from the $2k$ nodal points the family forms a smooth fibration, so with smoothly varying complex structures and in a smooth local trivialization give a smooth family of Hermitian metrics.

Near each of the nodal points there are holomorphic coordinates in which the map is the product of a Lefschetz map and a projection; the k-fold divisor is a smooth submanifold of the zero set of the Lefschetz factor and the hyperbolic family over this submanifold has cusp singularities along an intersection of $k-1$ divisors contained in the intersecting pair of divisors defined by the local Lefschetz singularity. In this sense the complex structure on the fiber near the nodal points can be arranged to be locally constant. On each fiber there is a holomorphic coordinate near the cusp point in terms of which the metric take the standard cusp form. This fiber holomorphic coordinate can be extended smoothly to yield a smooth, but not holomorphic, complex defining function for the nodal surface. This allows the family of constant curvature metrics to be extended off the singular surface, near the nodal crossing, by the plumbing metric. This yields a local extension of the initial family of metrics to hyperbolic metrics which are smooth on $\hat{\mathcal{L}}$ near each of the nodal points.

Finally these $2k+1$ families – the plumbing metric near each of the $2k$ resolved nodes and a smooth family elsewhere – may be combined to give a smooth family of rotationally flat metrics on $\hat{\mathcal{L}}$.

Proposition 8. *Near each point in a k-fold intersection of divisors in $\widehat{M}$ there is a smooth family of Hermitian metrics on $\hat{\mathcal{L}}$ which restricts to the hyperbolic metrics over the intersection of divisors, is equal to the plumbing metric near the resolved nodes $H_{II,*}$ and to the standard metric near the fixed divisors, is rotationally flat and has curvature equal to -1 to second order at the divisors in $H_{I,*}$.*

Proof. The various families discussed above are Hermitian, for the same bundle and are rotationally flat. The different local components of $H_{\mathrm{II},*}$ do not intersect so the discussion reduces to the case of a single Lefschetz factor in $\widehat{C}$, patching the extension of the limiting metric to the grafting metric. Since they differ only by quadratic terms in ρ_{I} away from (but near) $H_{\mathrm{II},*}$ it is only necessary to use a rotationally flat partition of unity to combine the conformal factors. □

5. The linearized operator

The boundary of the metric resolution of the total space, $\widehat{C}$, has in general, three types of boundary hypersurfaces. There are the 'fixed' hypersurfaces, F_*, arising from the resolution of marked points, the collective hypersurfaces $H_{\mathrm{I},*}$ which are fibered by the resolved Riemann surface and the 'necks', $H_{\mathrm{II},*}$, coming from the resolution of the nodal surfaces. Defining functions for these hypersurfaces will be denoted $\rho_{\mathcal{F}}$, ρ_{I} and ρ_{II} with the same notation used for the restrictions of these functions to the other hypersurfaces, where they are again boundary defining functions. Note that the collective hypersurfaces $\mathcal{F}$ and H_{II} (each consisting of embedded boundary hypersurfaces intersecting to produce corners) do not meet each other, so each only meets the collective hypersurface H_{I}. It should be noted that, the defining function $\rho_{\mathrm{I},i}$ can be taken to be the preimage of a defining function for the corresponding resolved divisor in the base, except near the component, $H_{\mathrm{II},i}$, of H_{II} corresponding to the related node, where $\rho_{\mathrm{I},i}\rho_{\mathrm{II},i}$ is a multiple of the defining function from the base.

The grafted metric is equal to the plumbing metric near the intersection of the boundary faces H_{I} and H_{II} and so, in terms of the local coordinates introduced in Section 2, takes the form

$$g = \frac{\pi^2 s^2}{\sin^2(\frac{\pi s}{s_w})}\left(\frac{ds_w^2}{s_w^4} + d\theta_w^2\right) = \frac{\pi^2 s^2}{\sin^2(\frac{\pi}{1+\rho_z})}\left(\frac{d\rho_z^2}{s^2(1+\rho_z)^4} + d\theta_z^2\right). \tag{5.1}$$

Here $s = \operatorname{ilog}|t|$, $s_w = \operatorname{ilog}|w|$, $\rho_z = \operatorname{ilog}|z|/\operatorname{ilog}|w|$.

The Laplacian of the metric near the boundary of H_{I} is

$$\varDelta_{\mathrm{I}} = -\frac{\sin^2(\frac{\pi s}{s_w})}{(\frac{\pi s}{s_w})^2}\left((s_w\partial_{s_w})^2 + s_w\partial_{s_w} + \frac{1}{s_w^2}\partial_{\theta_w}\right), \tag{5.2}$$

where $s_w = \rho_{\mathrm{II}}$ is the boundary defining function of H_{II} and hence a variable restricted on H_{I}. Similarly, near H_{II} the Laplacian is

$$\varDelta_{\mathrm{II}} = -\frac{\sin^2(\frac{\pi}{1+\rho_z})}{(\frac{\pi}{1+\rho_z})^2}\left((1+\rho_z)^2\partial_{\rho_z}^2 + 2(1+\rho_z)\partial_{\rho_z} + \frac{\partial_\theta^2}{s^2(1+\rho_z)^2}\right) \tag{5.3}$$

with $\rho_z = \rho_{\mathrm{I}}$ is the boundary defining function of H_{I} and a variable on H_{II}.

The plumbing metric is rotationally invariant to infinite order near all boundary hypersurfaces. Let $\mathcal{C}^\infty_\theta(\widehat{C}) \subset \mathcal{C}^\infty(\widehat{C})$ denote the subspace annihilated to infinite order at each boundary hypersurface by the corresponding generator(s) of the circle bundle(s) D_θ, see Definition 4. So to each hypersurface in H_{I} and $\mathcal{F}$ there corresponds one generator and to each component of H_{II} there correspond two; note that as discussed in Proposition 7 these are all defined up to infinite order in view of the introduction of logarithmic variables.

As indicated in Step (2) in §3, we first study the linearized equation (3.1):

$$(\varDelta_{g_{\mathrm{pl}}} + 2)u = h = -1 - R(g_{pl}).$$

Since -2 is outside the spectrum of $\varDelta$ the resolvent $\|(\varDelta+2)^{-1}\| \le 1$ on the regular fibers of $\widehat{C}$ has a bound, in terms of the L^2 norm on the fibers, which is independent of the parameters. In [13, Proposition 3] the space $L^2_{\mathrm{b}}(\widehat{M}; L^2(dg))$ of L^2 functions, with respect to the b-volume form on the base, with values in the fiber L^2 spaces is identified with the total weighted L^2 space $\rho_{\mathrm{II}}^{-\frac{1}{2}} L^2_{\mathrm{b}}(\widehat{C})$. This statement is a tautology except near the nodal hypersurfaces defined by the ρ_{II}. Since these are disjoint, the argument for a single node in [13] carries over unchanged, as does the commutation argument giving higher regularity with respect to tangential differentiation in all variables. In consequence the uniform solvability of (3.1), in spaces including the parameters, follows from these same arguments.

Proposition 9. *For the Laplacian of the grafted metric*

$$(\Delta + 2)^{-1} : \rho_{II}^{-\frac{1}{2}} H^k_{\mathrm{b}}(\widehat{C}) \longrightarrow \rho_{II}^{-\frac{1}{2}} H^k_{\mathrm{b}}(\widehat{C}) \ \forall \ k \in \mathbb{N}. \tag{5.4}$$

In particular this result holds for $k = \infty$, where the spaces become the L^2 based conormal spaces, so conormality on $\widehat{C}$ up to the boundaries for the solution of $(\Delta + 2)u = h$ follow from conormality for the forcing term. Defining functions in the base commute with the operator so (vanishing) weights with respect to these commute with the inverse.

More refined regularity properties for the family $(\Delta+2)^{-1}$ follow from an iterative construction of formal series solutions locally near the preimage of a point in an intersection of divisors in $\widehat{M}$, and is based on the solution to two model problems which are described explicitly below and we note their basic solvability properties here.

The first model operator is $\Delta_{\mathrm{I}} + 2$, obtained from $\Delta + 2$ by restriction to the fibers of the hypersurface H_{I} above the interior of a particular corner in $\widehat{M}$, i.e. intersection of divisors. These fibers are Riemann surfaces with constant curvature cusp metrics. The main issue here is the appearance of logarithmic terms, so of order $\log\log 1/|t|$ with respect to the original complex parameters. Thus the solution we obtain is log-smooth rather than a true formal power series.

Lemma 7. *The L^2 inverse of $\Delta + 2$, for a cusp metric on a Riemann surface S, applied to functions rotationally-invariant at the ends, satisfies*

$$\begin{aligned} h = \sum_{0 \le j \le k} \rho_{II} (\log \rho_{II})^j h_j,\ h_j \in \mathcal{C}^\infty_\theta(S) \Longrightarrow& \\ (\Delta + 2)^{-1} h = \sum_{0 \le j \le k+1} \rho_{II} (\log \rho_{II})^j u_j,&\ u_j \in \mathcal{C}^\infty_\theta(S). \end{aligned} \tag{5.5}$$

Proof. The proof is the same as in [13, Lemma 4]. The Laplacian is essentially self-adjoint and non-negative, so $\Delta+2$ is invertible. Near the boundary the zero Fourier mode satisfies a reduced, ordinary differential, equation with regular singular points and having indicial roots 1 and -2 in terms of a defining function for the (resolved) cusps. Then the form of solution (5.5) follows. □

The second model operator is the ordinary differential operator arising from $\Delta+2$ conjugated by ρ_{II} and restricted to the 'necks' H_{II} and then projected onto the the zero Fourier mode. The fibers are now cylinders, projecting to interval, I. These operator have a regular singular points with two indicial roots, 0 and 3, where the first corresponds to the simple vanishing in (5.5).

Lemma 8. *For the (reduced) model operator $\Delta_{II} + 2$ the Dirichlet problem is uniquely solvable and for smooth boundary data and a smooth forcing term has solution*

$$\begin{aligned} &\text{If } v_\pm \in \mathbb{R},\ r = \sum_{0 \le j \le k} (\log \rho_I)^j r_j,\ r_j \in \mathcal{C}^\infty(I) \text{ then} \\ &(\Delta_{II} + 2) v = r,\ v\big|_{\partial I} = v_\pm \Longrightarrow\ v = \sum_{0 \le j \le k} (\log \rho_I)^j v'_j + \sum_{0 \le j \le k+1} (\log \rho_I)^j \rho_I^3 v''_j. \end{aligned} \tag{5.6}$$

Proof. This is the same proof as [13, Lemma 6]. Unique solvability follows by integration by parts and positivity. The initial source of logarithmic terms is the 'second' indicial root for $\Delta_{\mathrm{II}}+2$, even if $r=0$ but the boundary data differ this introduces a logarithmic term with coefficient vanishing to third order at the ends. Ultimately this is an effect of scattering on H_{I}. $\square$

In the inductive argument we begin by solving (5.5) on the highest codimension boundary face in the base, where it is uniform, and then extend the solution to a neighborhood of the preimage. Near the (fixed) cusps this can be done so that the extension remains in the null space of $\Delta+2$ since the operator is actually independent of the parameters. However along the boundary, H_{II}, corresponding to the resolved nodes this produces an error which is not rapidly vanishing. On iteration this requires us to solve the same equation, but now with $h \in \rho\,\mathcal{C}^\infty(S)$ and then the result is that the solution is a sum of two terms, one in $\rho\,\mathcal{C}^\infty(S)$ and a more singular term in the space $\rho\log\rho\,\mathcal{C}^\infty(S)$. Repeating this procedure results in higher powers of logarithms. For this reason we devote considerable effort below to controlling the growth rate of the powers of the logs of ρ_{I} and ρ_{II} in the formal power series solution.

In [13] the log-smooth expansion of solutions to (3.1) was shown in the simplest case where only one pair of nodes forms. Following that argument, with the minor modifications due to the presence of the other cusps, this generalizes directly to the case of a neighborhood of a point in a single hypersurface in the base. In fact we extend this a little further by considering a neighborhood of a point of (possibly) higher codimension in the base but with the coefficients of the forcing term vanishing to infinite order in all but one of the defining functions.

Without loss of generality suppose that s_1 is the distinguished defining function and denote the others collectively as s'. Similarly, let $\rho_{\mathrm{I}}=\rho_{\mathrm{I},1}$ and $\rho_{\mathrm{II}}=\rho_{\mathrm{II},1}$ be defining functions for $H_{\mathrm{I},1}$ and $H_{\mathrm{II},1}$ in $\widehat{C}$, always taken to reduce to s_1 outside a small neighborhood of the intersection $H_{\mathrm{I},1}\cap H_{\mathrm{II},1}$. Then, as in [13], to capture the special structure of the expansion, we introduced spaces of polynomials in $\log\rho_{\mathrm{I}}$ and $\log\rho_{\mathrm{II}}$ with coefficients now in $(s')^\infty\mathcal{C}^\infty_\theta(\widehat{C})$, the space of smooth functions in $\widehat{C}$ vanishing to infinite order at all the hypersurfaces defined by the s' and also rotationally invariant to infinite order under all the circle actions:

$$\mathcal{P}^k=\left\{u=\sum_{0\le l+p\le k}(\log\rho_{\mathrm{I}})^l(\log\rho_{\mathrm{II}})^p u_{l,p},\ u_{l,p}\in(s')^\infty\mathcal{C}^\infty_\theta(\widehat{C})\right\}$$

$$\mathcal{P}^{k,m}_{\mathrm{II}}=\left\{u=\sum_{0\le l+p\le k,\ p\le m}(\log\rho_{\mathrm{I}})^l(\log\rho_{\mathrm{II}})^p u_{l,p},\ u_{l,p}\in(s')^\infty\mathcal{C}^\infty_\theta(\widehat{C})\right\},\ m\le k.$$

The second collection give the filtration of the first spaces by the maximal order of powers of $\log\rho_{II}$:

Proposition 10 (Compare Proposition 6 of [13])**.** *For each* k

$$h\in\rho_{II}\mathcal{P}^k+\rho_I\rho_{II}\mathcal{P}^{k+1}\Longrightarrow\ \exists\ u\in\rho_{II}\mathcal{P}^{k+1}+\rho_I^2\rho_{II}\mathcal{P}^{k+2,k+1}_{II}\tag{5.7}$$

such that

$$(\Delta+2)u-h\in s\left(\rho_{II}\mathcal{P}^{k+1}+\rho_I\rho_{II}\mathcal{P}^{k+2}\right).\tag{5.8}$$

The form of the error allows for immediate iteration.

As noted above, this follows by slight extension of the arguments in [13], and also directly from the more general result below.

We proceed by induction over the local codimension m in the base. Thus the boundary hypersurfaces in $\widehat{M}$ have defining functions $s_i = s_{t,i}$, $i \in \{1, \dots, m\}$. We proceed by separating these boundary hypersurfaces into collections corresponding to choice of a subset $L \subset \{1, \dots, m\}$, $\#(L) = p$, $L = \{j_1, \dots, j_p\}$. First consider the space of polynomials determined by a given multidegree $\kappa = (\kappa_{j_1}, \dots, \kappa_{j_p}) \in \mathbb{N}_0^p$ with smooth coefficients independent of the appropriate angular variables at the boundary hypersurfaces faces and vanishing rapidly at the boundary faces on which $s_j = 0$ for $j \notin L$; this space will be denoted

$$s_{\complement L}^{\infty} \mathcal{C}_{\theta}^{\infty}(\widehat{C}),\ s_{\complement L} = \left(\prod_{j \notin L} s_j \right). \tag{5.9}$$

Then the space of polynomials is

$$\mathcal{P}_L^{\kappa} = \{ \sum_{\alpha_{\mathrm{I},j} + \alpha_{\mathrm{II},j} \le \kappa_j} a_\alpha \prod_{j \in L} (\log \rho_{\mathrm{I},j})^{\alpha_{\mathrm{I},j}} (\log \rho_{\mathrm{II},j})^{\alpha_{\mathrm{II},j}}, a_\alpha \in s_{\complement L}^{\infty} \mathcal{C}_{\theta}^{\infty}(\widehat{C}) \}.$$

Here $\alpha = (\alpha_{j_1}, \dots, \alpha_{j_p}) \in (\mathbb{N}_0^2)^p$ with $\alpha_j = (\alpha_{\mathrm{I},j}, \alpha_{\mathrm{II},j}) \in \mathbb{N}_0^2, j \in L$ are the indices for the powers of logarithms in the two boundary defining functions $\rho_{\mathrm{I},j}$ and $\rho_{\mathrm{II},j}$ near the hypersurface $H_{\mathrm{II},j}$ such that $s_j = \rho_{\mathrm{I},j} \rho_{\mathrm{II},j}$. This gives a well-defined space of functions near the union of the boundary faces determined by L. We also make use of subspaces which have relatively lower order in the second set of variables

$$\mathcal{P}_{L,\mathrm{II}}^{\kappa} = \{ \sum_{\substack{\alpha_{\mathrm{I},j} + \alpha_{\mathrm{II},j} \le \kappa_j \\ \alpha_{\mathrm{II},j} \le \kappa_j - 1}} a_\alpha \prod_{j \in L} (\log \rho_{\mathrm{I},j})^{\alpha_{\mathrm{I},j}} (\log \rho_{\mathrm{II},j})^{\alpha_{\mathrm{II},j}}; a_\alpha \in s_{\complement L}^{\infty} \mathcal{C}_{\theta}^{\infty}(\widehat{C}) \}$$

Let e_j be the multi-index $(0, ..., 0, 1, 0, ..., 0)$ where the j-th entry is 1, and set

$$\kappa + n e_j = (\kappa_1, \dots, \kappa_j + n, \dots, \kappa_p) \in \mathbb{N}_0^p.$$

Then consider the polynomials with an extra restriction on one index j that $\log \rho_{\mathrm{II},j}$ cannot reach the top degree:

$$\mathcal{P}_{L,\mathrm{II}}^{\kappa+2e_j} = \{ \sum_{\substack{\alpha_{\mathrm{I},i} + \alpha_{\mathrm{II},i} \le \kappa_i + 2\delta_{ij} e_j \\ \alpha_{\mathrm{II},j} \le \kappa_j}} a_\alpha \prod_{j \in L} (\log \rho_{\mathrm{I},j})^{\alpha_{\mathrm{I},j}} (\log \rho_{\mathrm{II},j})^{\alpha_{\mathrm{II},j}}; a_\alpha \in s_{\complement L}^{\infty} \mathcal{C}_{\theta}^{\infty}(\widehat{C}) \}.$$

Let $\phi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R})$ be a cut-off function with support near $x = 0$. Since $\rho_{\mathrm{II},i}$ never vanishes simultaneously with any other $\rho_{\mathrm{II},j}$, if we take ϕ with sufficiently small support $\phi(\rho_{\mathrm{II},i})$ is a function that is 1 near $\rho_{\mathrm{II},i} = 0$ and vanishes near $\rho_{\mathrm{II},j} = 0$ for $j \ne i$. We also consider a collective boundary defining function for the $H_{\mathrm{II},i}$

$$\rho_{\mathrm{II}}^L = \prod_{j \in L} \rho_{\mathrm{II},j}.$$

Proposition 11. *Suppose $L \subset \{1, 2, \dots, m\}$ and any $\kappa \in \mathbb{N}_0^p$ then for any*

$$h \in \mathcal{L}_L^{\kappa} = \rho_{II}^L \mathcal{P}_L^{\kappa} + \rho_{II}^L \sum_{j \in L} \rho_{I,j} \phi(\rho_{II,j}) \mathcal{P}_{L,II}^{\kappa}, \tag{5.10}$$

there exists

$$u \in \mathcal{U}_L^{\kappa} = \rho_{II}^L \sum_{i \in L} \mathcal{P}_{L,II}^{\kappa+e_i} + \rho_{II}^L \sum_{j \in L} \rho_{I,j}^2 \phi(\rho_{II,j}) \mathcal{P}_{L,II}^{\kappa+2e_j} \tag{5.11}$$

such that

$$(\Delta+2)u-h=\sum_{j\in L}s_jg_j,\ g_j\in\mathcal{L}_L^{\kappa+1}. \tag{5.12}$$

Proof. We first solve the equation on the most singular fiber $S=\{\rho_{\mathrm{I},j}=0,\forall j\in L\}$ which is a punctured Riemann surface with cusp ends (from both the fixed divisors and the nodes), where the function h has a finite expansion

$$h=\sum_{\alpha_j\le\kappa_j}a_\alpha\prod_{j\in L}(\log\rho_{\mathrm{I},j})^{\alpha_j} \tag{5.13}$$

with coefficients a_α which are polynomials in $(\log\rho_{\mathrm{II},j})$'s of multi-index orders at most $(\kappa_j-\alpha_j)$'s respectively. For fixed $(\alpha_{\mathrm{I},1},\dots,\alpha_{\mathrm{I},p})$ the equation

$$(\Delta+2)\sum_{\alpha_j\le\kappa_j}v_\alpha\prod_{j\in L}(\log\rho_{\mathrm{I},j})^{\alpha_j}=h+F, \tag{5.14}$$

where F is to vanish at S, induces equation on the coefficients restricted to S. These form an upper-triangular matrix of b-differential operators, with diagonal entries Δ_S+2, so can be solved iteratively, over decreasing $\sum_{i\in L}\alpha_{\mathrm{I},i}$, and at each level take the form

$$(\Delta_S+2)v_\alpha=g_\alpha=h_\alpha+\sum_{|\beta|>|\alpha|}P_\beta v_\beta, \tag{5.15}$$

where P_β is the operator that extracts $\prod(\log\rho_{\mathrm{I},i})^{\alpha_{\mathrm{I},i}}$ terms from v_β. It therefore follows inductively that $g_\alpha\in\mathcal{P}_L^{\kappa-\alpha}$ and that $v_\alpha\in\mathcal{P}_L^{\kappa-\alpha+1}$. It is important to recall here that $\rho_{\mathrm{II},i}$ and $\rho_{\mathrm{II},i'}$ cannot vanish simultaneously when $i\neq i'$. We proceed to choose an extension of v_α off S, but do this separately near $\rho_{\mathrm{II},i}=0$ for each i. Away from these hypersurfaces any smooth extension will suffice.

Consider the extension of the v_α off S near $\rho_{\mathrm{II},i}=0$. Since $\rho_{\mathrm{II},i'}>0$ for any $i'\neq i$, we may replace $\rho_{\mathrm{I},i'}$ by $s_{i'}$ in this region. The coefficients h_α change, but the form is preserved and the solutions v_α are similarly transformed and the resulting system is diagonal in i'. Replacing f by $\chi(s_i)f$ where χ is a smooth cutoff near zero introduces an error which is in $s_i\mathcal{L}_L^\kappa$ and so can be absorbed in the next iteration. Now we can extend the v_α from S to $\rho_{\mathrm{I},i}=0$ near $\rho_{\mathrm{II},i}=0$ as solutions of the same equations although the Laplacian now may depend on s_i as a parameter. The error term vanishes identically at $s_i=0$ near $\rho_{\mathrm{II},i}=0$. We may now proceed exactly as in [13] to extend the coefficients v_α along $H_{\mathrm{II},i}$ in such a way as to remove the leading coefficient in $\rho_{\mathrm{II},i}$. Since this is all now uniform in s_i, and the same is true for the removal of the second term in (5.10), which only involves the induced differential equation on $H_{\mathrm{II},i}$, this part of the solution is certainly of the form (5.11) and an error as in (5.12). A similar construction near $\rho_{\mathrm{II},i'}=0$ for all other i' completes the proof. $\square$

The results above allow the log-smoothness of the solution of the linearized equation to be deduced from appropriate log-smoothness of the data in terms of the spaces defined in (5.10) and (5.11) where we drop the suffix L when all boundary hypersurfaces are involved. Here we abbreviate the indices and write

$$s^\alpha=\prod_{i=1}^m s_i^{\alpha_i},\ \alpha=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}_0^m.$$

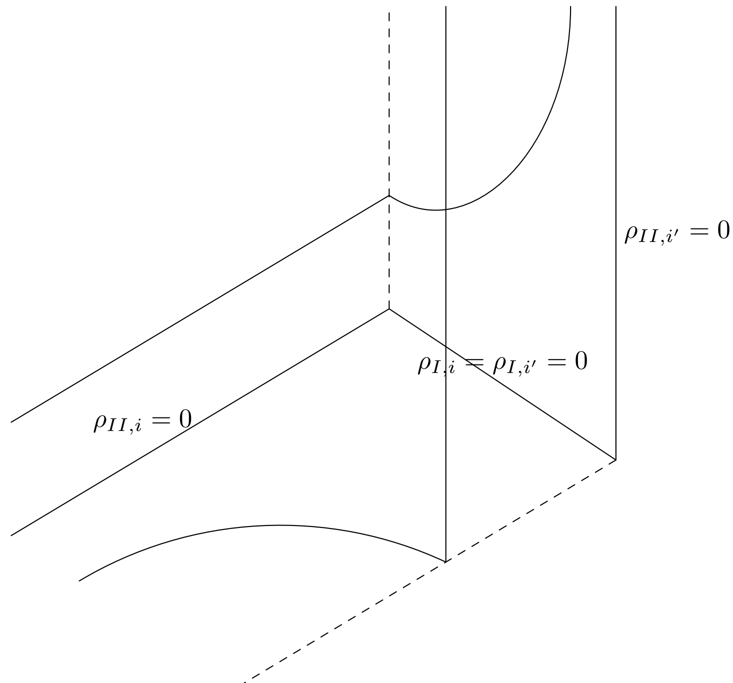

FIGURE 3. The induction procedure with two nodes: curves with two different degenerating directions approaching the nodal curve with two nodes

Proposition 12. *If $h \in \rho_{II}\mathcal{C}^\infty_{\log}(\widehat{C})$ has expansions at each of the of boundary faces of the form*

$$h \simeq \sum_{\alpha\in\mathbb{N}_0^m} f_\alpha s^\alpha,\ \ f_\alpha \in \mathcal{L}^\alpha \tag{5.16}$$

then $u = (\varDelta + 2)^{-1}h \in \mathcal{C}^\infty_{\log}(\widehat{C})$ has similar expansion

$$u \simeq \sum_\alpha u_\alpha s^\alpha,\ \ u_\alpha \in \mathcal{U}^\alpha. \tag{5.17}$$

Proof. We first recall (see also the Appendix) the structure of the proof of Borel's Lemma, summing formal power series, in this context. Suppose $x_i \geq 0$, $i = 1, \dots, \ell$ and y are respectively boundary defining functions and tangential coordinates near a boundary point of codimension ℓ on a manifold with corners and

$$\sum_{\substack{\alpha\in\mathbb{N}_0^\ell,\beta\in\mathbb{N}_0^\ell\\ \alpha_i<\beta_i}} f_{\alpha,\beta}(y)(\log x)^\alpha x^\beta \tag{5.18}$$

is a formal power series with smooth coefficients of fixed compact support. Then, choosing a cutoff function $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^\ell)$ which is identically equal to 1 near zero, the series

$$u = \sum_{\substack{\alpha\in\mathbb{N}_0^\ell,\beta\in\mathbb{N}_0^\ell\\ \alpha_i<\beta_i}} f_{\alpha,\beta}(y)(\log x)^\alpha x^\beta \chi(x/\epsilon_\alpha) \in \mathcal{C}^\infty_{\log} \tag{5.19}$$

converges provided the ϵ_α decrease rapidly enough. In fact for an appropriate choice, base on the norms of the coefficients, all the finite differences

$$u(x,y) - \sum_{|\alpha|\le N} f_{\alpha,\beta}(y)(\log x)^\alpha x^\beta \tag{5.20}$$

are conormal and vanish to order N at the corner. The same arguments apply when the coefficients are assumed to be on a manifold with corners but to vanish to infinite order at all boundaries.

The 'asymptotic sum' so obtained is independent of the choices up to an error in $\mathcal{C}^\infty_{\log}$ which vanishes to infinite order at the corner. Such an error can be decomposed, using a partition of unity in polar coordinates around (i.e. after blowing up) the corner, and such that each term is non-zero only near one of the ℓ faces of codimension $\ell-1$, i.e. $\{x_i = 0\}$. This allows for the iterative construction of a log-smooth function with expansions at each of the boundary hypersurfaces, provided these expansions are formally compatible in essentially the same sense as for Taylor series.

So, to prove that the solution $(\Delta+2)^{-1}h$ is log-smooth we use Proposition 11 repeatedly, first on the highest codimension boundary face in the base $\widehat{M}$ so with $\complement L = \emptyset$ and with a smooth forcing term h with support disjoint from H_{II} and the fixed hypersurfaces in $\widehat{C}$. The formal solution obtained from iteration can be summed to give

$$u_m = \sum_\alpha s^\alpha v_\alpha \chi(s/\epsilon_\alpha) \in \rho_{\mathrm{II}} \mathcal{C}^\infty_{\log}(\widehat{C}). \tag{5.21}$$

Note that although the preimage of $s=0$ in $\widehat{C}$ has points of codimension $m+1$ the series is constructed uniformly at the union of the $m+1$ faces of codimension m and the same asymptotic summation principles apply to (5.21). Moreover the error term

$$h_{m-1} = h - (\Delta+2)u_m \tag{5.22}$$

is log-smooth, has expansion with powers coming only from those in the errors in (5.12) at *all* boundary hypersurfaces and vanishes to infinite order at the preimage of $s=0$. Thus, using a polar partition of unity in s it may be divided into m pieces, each of which vanishes to infinite order with one of the s_i.

This allows Proposition 11 to be applied, now with $\#(L) = m-1$ and the general case to be proved by induction. □

6. The curvature equation

In the previous section we showed that the solution to the linearized equation (3.1) with a log-smooth forcing term is log-smooth. Now we iteratively apply Proposition 12 to the full curvature equation to arrive at the same conclusion for the solution to (3.2). Note that the log-smooth functions form a ring.

Proposition 13. *The curvature equation* (3.2) *has a solution in log-power series*

$$f \sim \sum_\alpha u_\alpha s^\alpha,\ u_\alpha \in \mathcal{U}^\alpha \tag{6.1}$$

where the $\mathcal{U}^\alpha$ *are defined by* (5.11).

Proof. To get the expansion for the nonlinear equation, we write it as

$$(\Delta+2)f=-R-1-(e^{2f}-1-2f). \tag{6.2}$$

Consider the formal solution of the linearized equation $(\Delta+2)f_1\sim -R-1$. from (5.17)

$$f_1=\sum_{\alpha\geq 2}s^\alpha f_{1,\alpha},\ \ f_{1,\alpha}\in\mathcal{U}^\alpha.$$

Now we solve the nonlinear equation (6.2) by taking

$$f=\sum_{i=1}^{\infty}f_i,\ \ f_i=\sum_{\alpha\geq 2i}s^\alpha f_{i,\alpha}, \tag{6.3}$$

which gives

$$\sum_{i=1}^{\infty}(\Delta+2)f_i=-R-1-\sum_{k=2}^{\infty}\frac{2^k}{k!}(\sum_{i=1}^{\infty}f_i)^k.$$

Now we require that for $j\geq 2$

$$(\Delta+2)f_j=-\sum_{k=2}^{\infty}\frac{2^k}{k!}\left[(\sum_{i=1}^{j-1}f_i)^k-(\sum_{i=1}^{j-2}f_i)^k\right]=f_{j-1}Q_j(f_1,\ldots,f_{j-1})$$

where Q_j is a polynomial with no constant term. Assuming inductively that

$$f_k=\sum_{\alpha\geq 2k}s^\alpha f_{k,\alpha},\ \ f_{k,\alpha}\in\mathcal{U}^\alpha \tag{6.4}$$

for $k\leq j-1$ as in (6.1), the right hand side is in $\sum_{\alpha\geq 2j}s^\alpha\mathcal{L}^\alpha$. Applying Proposition 12, we obtain the inductive conclusion that

$$f_j=\sum_{\alpha\geq 2j}s^\alpha f_{j,\alpha},\ \ f_{j,\alpha}\in\mathcal{U}^\alpha.$$

This shows the existence of a formal solution as in (6.1). □

Now the structure of the actual solution follows from an application of the Implicit Function Theorem.

Proof of Theorem 4. It suffices to show that there is a function $f\in\mathcal{C}^\infty_{\log}(\widehat{C})$ satisfying the curvature equation (3.2) since then $e^{2f}g_{pl}$ is the family of hyperbolic metrics.

Writing $f=f_0+\tilde{f}$, where f_0 is a sum of the formal solution obtained from Proposition 13 by Borel's lemma. Then $\tilde{f}$ should satisfy

$$(\Delta+2)\tilde{f}=-\left(2\tilde{f}(e^{2f_0}-1)+e^{2f_0}(e^{2\tilde{f}}-1-2\tilde{f})-g\right).$$

We can apply Proposition 9 to see the solvability of this equation in the space $s^NH^N_{\mathrm{b}}(\widehat{C})$ restricted to a region where at least one of the s_i is small. The uniqueness of the hyperbolic metric shows that the $\tilde{f}$ is in fact smooth and vanishes to infinite order with each of the s_i, essentially as in [13, Proposition 8]. □

7. The Weil–Petersson metric

The Weil–Petersson metric is most easily realized at $m \in \mathcal{M}_{g,n}$ in terms of the dual metric, an Hermitian metric on the logarithmic cotangent bundle, ${}^{\mathcal{D}}\Lambda^{(1,0)}_m$. The definition uses the identification (3). The bundle of quadratic logarithmic differentials on the resolution of the marked Riemann surface $Z = Z_m$ representing $m \in \mathcal{M}_{g,n}$ satisfies

$$({}^{\mathcal{D}}\Lambda^{(1,0)}Z)^2 \otimes (\overline{{}^{\mathcal{D}}\Lambda^{(1,0)}Z})^2 = ({}^{\mathcal{D}}\Lambda^{(1,1)}Z)^2.$$

The hyperbolic metric on Z, complete outside the marked points and fixed uniquely by the complex structure, has area form given in terms of a holomorphic coordinate vanishing at a marked point

$$\frac{dz d\bar{z}}{|z|^2(\log|z|)^2}. \tag{7.1}$$

By definition, the vector space QZ of holomorphic quadratic differentials on a punctured Riemann surface consists of those holomorphic sections of $({}^{\mathcal{D}}\Lambda^{(1,0)}Z)^2$ which vanish at the marked points. Dividing the product of one such form and the complex conjugate of another by the area form therefore gives a continuous section of ${}^{\mathcal{D}}\Lambda^{(1,1)}Z$ which is smooth away from the marked points near which it has a bound

$$\Big|\frac{q_1\bar{q}_2}{\mu_m}\Big| \le C(\log|z|)^2 dz d\bar{z},\ q_1,\ q_2 \in QZ, \tag{7.2}$$

and so is integrable. This integral gives the dual Weil–Petersson metric, as an Hermitian form, by push-forward under the Lefschetz map

$$\begin{gathered} {}^{\mathcal{D}}\Lambda^{(1,0)}_m\mathcal{M}_{g,n} = QZ_m \\ {}^{\mathcal{D}}\Lambda^{(1,0)}_m\mathcal{M}_{g,n} \otimes {}^{\mathcal{D}}\Lambda^{(0,1)}_m\mathcal{M}_{g,n} \ni (q_1, \overline{q_2}) \longmapsto G_{\mathrm{WP}}(\zeta_1,\zeta_2) = \phi_*\left(\frac{q_1\bar{q}_2}{\mu_m}\right). \end{gathered} \tag{7.3}$$

Standard proofs of the Riemann mapping theorem for marked Riemann surfaces show the smoothness (indeed real-analyticity) of μ_m on the moduli space and hence that G_{WP} is similarly smooth as a metric on $\mathcal{M}_{g,n}$.

We proceed to analyze the behavior of the Weil–Petersson (co-)metric near a boundary point $\bar{m} \in \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$, so in some intersection of the G_i', corresponding to the number of geodesics which have been pinched to nodes. The fiber of $\overline{\psi}$ above $\bar{m}$ is a nodal Riemann surfaces lying in one or more of the G_i. The null bundle $\widetilde{\mathcal{L}}_{\overline{\psi}}$, of the log differential of $\overline{\psi}$ again reduces to the log tangent bundle of the nodal surface, in effect separating nodes to punctures. However the corresponding space of holomorphic quadratic differentials, $\widetilde{Q}_{\bar{m}}$, which is naturally isomorphic to the logarithmic cotangent space ${}^{\mathcal{D}}\Lambda_{\bar{m}}\overline{\mathcal{M}}_{g,n}$ consists of those holomorphic sections of the square of the dual bundle $(\widetilde{\mathcal{L}}_{\overline{\psi}})^{-2}$ which vanish at the marked points but at nodal points, in one of the S_i, are simply required to take a consistent value, the same at the two punctures representing the node. Thus in place of (7.2), which again holds at the marked points, only the much weaker bound

$$\Big|\frac{q_1\bar{q}_2}{\mu_{\bar{m}}}\Big| \le C(\log|z|)^2\frac{dz d\bar{z}}{|z|^2},\ q_1,\ q_2 \in \widetilde{Q}_{\bar{m}} \tag{7.4}$$

generally holds near the nodes. This does not imply integrability so the co-metric is singular at $\bar{m}$.

The structure of the area form at the singular fibers has been analysed in detail, above, by passing to the real resolution $\hat{\psi} : \widehat{\mathcal{C}}_{g,n} \longrightarrow \widehat{\mathcal{M}}_{g,n}$; see Theorem 1. The compactified moduli space is resolved, to a tied manifold, with boundary hypersurfaces replacing the exceptional divisors with their logarithmic real blow up, as for the fixed divisors (which are in the universal curve). Thus each boundary hypersurface has a local defining function of the form $s_i = \operatorname{ilog}|z_i|$ and forms a circle bundle over the corresponding divisor given locally by $z_i = 0$.

Lemma 9. *The lift of the logarithmic tangent bundle ${}^{\mathcal{D}}\Lambda\overline{\mathcal{M}}_{g,n}$ to $\widehat{\mathcal{M}}_{g,n}$ is naturally identified with corresponding (iterated) cusp bundle with local spanning sections $\frac{ds_i}{s_i^2}$, $d\theta_i$ and dw_j at any boundary face.*

Proof. This is just the computation

$$\frac{dz_i}{z_i} = \frac{d|z_i|}{|z_i|} + id\theta = \frac{ds_i}{s_i^2} + id\theta,\ s_i = \operatorname{ilog}|z_i| = (\log|z_i|^{-1})^{-1}. \tag{7.5}$$

□

The preimage in $\widehat{\mathcal{M}}_{g,n}$ of $\bar{m} \in \overline{\mathcal{M}}_{g,n}$ lying in a k-fold intersection of the divisors, $G'_{i'}$, is a product of k circles through a boundary point of codimension k in $\widehat{\mathcal{M}}_{g,n}$. The preimage in $\widehat{\mathcal{C}}_{g,n}$ of the singular fiber $Z_{\bar{m}} \subset \overline{\mathcal{C}}_{g,n}$ above $\bar{m}$ is a product over this k-torus with a factor which is an 'articulated manifold' (the real analog of a nodal surface) in the sense that it is a union of compact surfaces with boundaries meeting only at (some) of their boundary faces $\widehat{Z} = Z_{\mathrm{I}} \cup Z_{\mathrm{II}}$. The component manifolds forming Z_{I} are resolutions of the Riemann surfaces (including 'bubbled off spheres') into which the original Riemann surface has decomposed under nodal degeneration. These can (and in the case of spheres must) have boundary faces formed by the fixed divisors, with collective boundary defining function $\rho_{\mathcal{F}}$; the other boundary faces arise from the resolution of the (separated) nodes which are in common with components in Z_{II}. The Z_{II} are all cylinders, joining the two circles forming the resolution of a node.

The hyperbolic fiber metric of the original Riemann surface induces the hyperbolic fiber metric on each of the components of Z_{I} (which are necessarily stable) which is therefore of the form (7.1) near its boundaries. The fiber metrics degenerate at the Z_{II} in an adiabatic fashion; namely the metric approaches the pull back of metric on the base of the circle fibration with the tangential part vanishing to second order.

For the resolved universal curve, $\hat{\psi} : \widehat{\mathcal{C}}_{g,n} \longrightarrow \widehat{\mathcal{M}}_{g,n}$, as shown above, is a real Lefschetz map in the sense that it is a b-fibration, so has surjective b-differential and the defining functions on the base each lift to be everywhere either locally a defining function or the product of two

$$\hat{\psi}^* s_i = \rho_{\mathrm{I},i}\rho_{\mathrm{II},i}. \tag{7.6}$$

The collective boundary hypersurfaces $H_{\mathrm{I}} = \bigcup_i \{\rho_{\mathrm{I},i} = 0\}$ and $H_{\mathrm{II}} = \bigcup_i \{\rho_{\mathrm{II},i} = 0\}$ are without self-intersections (because in the case of H_{I} these have been replaced by intersections with components of H_{II} through the blow up of the S_i) and are the unions of the parts of the fibers just described.

Lemma 10. *If q_1, q_2 are families of holomorphic logarithmic quadratic differentials on $\widehat{\mathcal{C}}_{g,n}$, near a boundary fiber, then if $\nu_{\hat{\psi}}$ is a positive section of the b-fiber density* *bundle for* $\hat{\psi}$,

$$\frac{q_1\bar{q}_2}{\mu_H} = a\rho_{II}^{-3}\rho_{\mathcal{F}}^{\infty}\nu_{\hat{\psi}},\ 0 < a \in \mathcal{C}^{\infty}_{\log}(\widehat{\mathcal{C}}_{g,n}) \tag{7.7}$$

Proof. This is a refinement of the computation above leading to (7.2), (7.4). Near the fixed hypersurfaces the metric has a cusp singularity so the fiber area form is

$$\mu_H = \alpha\frac{d\rho_{\mathcal{F}}d\theta_{\mathcal{F}}}{\rho_{\mathcal{F}}},\ 0 < \alpha \in \mathcal{C}^{\infty}_{\log}. \tag{7.8}$$

The product of the holomorphic quadratic differentials vanishes in terms of the holomorphic coordinates so

$$q_1\bar{q}_2 = e^{-2/\rho_{\mathcal{F}}}b(\frac{d\rho_{\mathcal{F}}d\theta_{\mathcal{F}}}{\rho_{\mathcal{F}}^2})^2,\ b \in \mathcal{C}^{\infty}. \tag{7.9}$$

The quotient is therefore rapidly decreasing at the fixed boundary hypersurfaces, giving the formal factor of $\rho_{\mathcal{F}}^{\infty}$ in (7.7).

It remains to analyse the behaviour at Z_{II}, including at the corresponding boundary faces of Z_{I}. These cover the nodal points at which the holomorphic quadratic differentials do not necessarily vanish. The structure of the area form is essentially the same as in (7.8) with ρ_{II} replacing $\rho_{\mathcal{F}}$ but extends along H_{II}; Lemma 9 (and the uniform analysis of the metric) shows that

$$\mu_H = \alpha\frac{d\rho_{\mathrm{II}}d\theta_{\mathrm{II}}}{\rho_{\mathrm{II}}},\ 0 < \alpha \in \mathcal{C}^{\infty}_{\log}. \tag{7.10}$$

The exponential factor in (7.9) is then missing, so

$$q_1\bar{q}_2 = b\rho_{\mathrm{II}}^{-4}(d\rho_{\mathrm{II}}d\theta_{\mathrm{II}}))^2,\ b \in \mathcal{C}^{\infty}. \tag{7.11}$$

and (7.7) follows. □

The holomorphic quadratic differentials which vanish at a nodal point (so in some S_i) correspond to the 'tangential' (logarithmic) differentials on $\overline{\mathcal{M}}_{g,n}$ at that point, those which are smooth up to the divisor (although possibly singular along it as logarithmic differentials). If either q_1 or q_2 lies in this subspace then at least one exponentially vanishing factor occurs and (7.7) is replaced by

$$\frac{q_1\bar{q}_2}{\mu_H} = a\rho_{\mathrm{II},i}^{\infty}\rho_{\mathcal{F}}^{\infty}\nu_{\hat{\psi}},\ 0 < a \in \mathcal{C}^{\infty}_{\log}(\widehat{\mathcal{C}}_{g,n}) \text{ near } H_{\mathrm{II},i}, \tag{7.12}$$

the corresponding component of Z_{II}. In this case the area form is locally integrable whereas in the 'normal' case, (7.7) it is singular at Z_{II}.

Proof of Theorem 2. As noted in (7.3), the Weil–Petersson metric, defined through the dual metric on ${}^{\mathcal{D}}\Lambda^{(1,0)}M$, is given by push forward under the Lefschetz map ψ. Lifting under the metric resolution, it follows that

$$G_{\mathrm{WP}}(\zeta_1,\zeta_2) = \hat{\psi}_*(\frac{q_1\bar{q}_2}{\mu_m}) \tag{7.13}$$

where the q_i are holomorphic quadratic differentials representing the ζ_i. Here we may think of the ζ_i as holomorphic sections of ${}^{\mathcal{D}}\Lambda^{(1,0)}\overline{\mathcal{M}}_{g,n}$ near some point $\bar{m}$ and the q_i as the corresponding sections of $\widetilde{Q}\overline{\mathcal{M}}_{g,n}$, hence holomorphic quadratic differentials near the fiber above $\bar{m}$. Since $\hat{\psi}$ is a b-fibration the push-forward theorem in [12] applies. In principal this is in the context of manifolds with corners, rather than the slightly more general case of tied manifolds with orbifold points as encountered here. However the fibers *are* globally manifolds with boundaries and the result is essentially local in the base. The same remark applies to the presence of orbifold points since these are also in the base directions, so one can always apply the result in [12] to an appropriate finite local cover and then take the quotient.

Proposition 14 (See [12]). *If $\widehat{\phi} : \widehat{C} \longrightarrow \widehat{M}$ is a b-fibration with compact fibers, between manifolds with corners, with multiplicity at most two, in the sense that each boundary defining function in the range is locally the product of at most two boundary defining functions in the domain and $\nu_{\widehat{\phi}}$ is a non-vanishing fibre b-density then*

$$\mathcal{C}^{\infty}_{\log}(\widehat{C}) \ni a \longmapsto \widehat{\phi}_*(a\nu_\phi) = g + \sum_{i'} a_{i'} \log \rho_{i'}, \;\; g, \;\; a_{i'} \in \mathcal{C}^{\infty}_{\log}(\widehat{M}). \tag{7.14}$$

If a vanishes to order j at the codimension two faces occurring as the common zero surface in (7.6) *for $\widehat{\phi}^* \rho_{i'}$ then the corresponding coefficient $a_{i'}$ vanishes to order j at $\rho_{i'} = 0$.*

In brief the logarithmic multiplicity in the generalized Taylor series at boundary faces in the range of such a push-forward is at most one degree higher, there is at most one more factor of $\log \rho_{i'}$, and this only arises from the Taylor series at the codimension two faces mapping onto a given boundary hypersurface. One can be more precise about the sense in which it is the diagonal terms in the Taylor series at the corners which contribute to the logarithmic coefficients.

Near a point in the local intersection of k exceptional divisors in the base, $\overline{\mathcal{M}}_{g,n}$, we may always choose a local coordinate basis of ${}^{\mathcal{D}}\Lambda^{(1,0)}$, $\zeta_{i'} = dz_{i'}/z_{i'}$, $i' = 1, \dots, k$, $\zeta'_j = dw_j$, over an open neighborhood $O \subset \overline{\mathcal{M}}_{g,n}$ so that the corresponding holomorphic quadratic differentials are $q_{i'}$ and q'_j, where each $q_{i'}$ vanishes at all $S_{l'}$ with $l' \neq i'$ and takes the value 1 at $S_{i'}$ and the q'_j are tangential in the sense that they vanish all local $S_{l'}$. Thus the q'_j are holomorphic quadratic differentials when the nodal points are separated and regarded as marked points on the resulting possibly non-connected Riemann surface.

Applying Proposition 14 to compute the coefficients of the metric via (7.3) using Lemma 10 and the subsequent remark we conclude that locally

$$\begin{gathered} G_{\mathrm{WP}}(\frac{dz_{i'}}{z_{i'}}, \frac{dz_{i'}}{z_{i'}}) \in \rho_{i'}^{-3}\mathcal{C}^{\infty}(\widehat{O}) + \log \rho_{i'} \mathcal{C}^{\infty}_{\log}(\widehat{O}), \\ G_{\mathrm{WP}}(\frac{dz_{i'}}{z_{i'}}, \frac{dz_{j'}}{z_{j'}}) \in \mathcal{C}^{\infty}_{\log}(\widehat{O}), \;\; i' \neq j' \\ G_{\mathrm{WP}}(dw_j, dw_k) \in \mathcal{C}^{\infty}_{\log}(\widehat{O}) \;\forall\; j, k \end{gathered} \tag{7.15}$$

where $\widehat{O}$ is the preimage of O in $\widehat{\mathcal{M}}_{g,n}$.

The singular coefficients in (7.15) arise only from the boundary hypersurface $H_{\mathrm{II},i'}$ resolving $S_{i'}$ in $\widehat{\mathcal{C}}_{g,n}$. Consider the leading terms in the length, with respect to the Weil–Petersson co-metric, of $dz_{i'}/z_{i'}$, dropping the index for notational simplicity. This is given by the push-forward formula. Since we have shown that they differ by quadratic terms at the divisors, it suffices to replace the Weil–Petersson metric by the grafting metric in the computation of the leading terms. The explicit computation of the integral depends on the fibration being in model Lefschetz form and the metric reducing to the plumbing metric near the nodal points. To accomplish this we choose z to be fiber holomorphic but only smooth in the base, i.e. arising from a non-holomorphic defining function for the corresponding G'_i.

Now the integral of the fiber area form which may be written out explicitly as

$$\int_{|t|<|z|<1} \mu_s \tag{7.16}$$

where μ_s is the quotient of $|dz|^4/|z|^4$ – the square of a quadratic differential with a double pole – and the area form of the plumbing metric. So,

$$\mu_s = \frac{|dz|^2}{|z|^2}(\log|z|)^2\left(\frac{\log|t|}{\pi\log|z|}\sin\frac{\pi\log|z|}{\log|t|}\right)^2. \tag{7.17}$$

So, setting $s = -1/\log|t|$, $r = -1/\log|z|$ we find

$$\begin{gathered}\frac{|dz|^2}{|z|^2} = \frac{drd\theta}{r^2},\\ \mu_s = \frac{drd\theta}{r^4}\left(\frac{r}{\pi s}\sin\frac{\pi s}{r}\right)^2.\end{gathered} \tag{7.18}$$

The integral then becomes

$$2\pi\int_s^1 \frac{dr}{\pi^2 s^2 r^2}\sin^2\frac{\pi s}{r}. \tag{7.19}$$

Changing variable to $\tau = s/r$, so $dr/r^2 = -d\tau/s$, gives

$$\frac{1}{\pi s^3}\int_s^1(1-\cos 2\pi\tau)d\tau = \frac{1}{\pi s^3}(1 - s + \frac{\sin 2\pi s}{2\pi}). \tag{7.20}$$

The higher order terms in the coefficients of these diagonal terms arise from either the difference of the conformal factor for the degenerating family of metrics, which may give a term in $\log\rho_{i'}\mathcal{C}^\infty_{\log}(\widehat{O})$, and the higher order terms in the quadratic differential which produce a term in $\mathcal{C}^\infty_{\log}(\widehat{O})$.

To obtain the form (7.5) of the metric we must change from the non-holomorphic defining functions for the divisors to holomorphic ones. As discussed in Proposition 7 this only changes the real boundary defining functions s_j by quadratic terms and the given decomposition of the metric is not altered by such changes. □

8. Ricci curvature and metric

The Ricci curvature of the Weil–Petersson metric itself defines a Kähler metric on the moduli space; the quasi-isometry class was found by Trapani [19] and the leading asymptotics at a divisor by Liu, Sun and Yau [9, 8]. Near the intersection of k divisors, written out in terms of the singular coordinate basis

$$\alpha_j = d\log z_i = d(-s_i^{-1} + i\theta_i), 1 \le i \le k;\ \alpha_j = dz_j, 3g-3+n \ge j > k \tag{8.1}$$

the full asymptotic expansion of the Weil–Petersson metric in turn yields a full description of the asymptotic behaviour of the Ricci metric at the exceptional divisors:

Theorem 5. *In terms of the coordinates s_i, θ_i and z_l near an intersection of exceptional divisors, the Ricci metric derived from the Weil–Petersson metric is*

θ_i-invariant to infinite order at $s_i = 0$, has log-smooth coefficients as an Hermitian form in $-ds_i^{-1} + id\theta_i$ and dz_j and in this sense has leading part

$$g_{Ri} = \frac{3}{4}\sum_{i=1}^{k}\left(\frac{ds_i^2}{s_i^2} + s_i^2 d\theta_i^2\right) + h \tag{8.2}$$

where h is log-smooth and restricts to the exceptional divisor to be the induced Ricci metric.

Proof. In terms of the coordinates in (8.1), the Weil–Petersson metric g_{WP} is as in (9). Computed in terms of the these complex differentials, the determinant of the metric takes the form

$$\begin{gathered}\det(g_{\mathrm{WP}}) = (\prod_{i=1}^{k}\pi\frac{1}{(\log|z_i|)^3|z_i|^2})\det(g_{\mathrm{WP},z})\left(1+\sum_{i=1}^{k}(\mathrm{ilog}\,|s_i|)f_i\right)\\ \log\det(g_{\mathrm{WP}}) = C + \sum_{i=1}^{k} -3i\log\log|z_i| + \log\det g_{\mathrm{WP},z} + \sum_i \mathrm{ilog}\,|s_i|\tilde{f}_i\end{gathered} \tag{8.3}$$

where the f_i and $\tilde{f}_i$ are log-smooth with respect to the z_i variable.

Since $-\log\det(g_{\mathrm{WP}})$ is a Kähler potential for the Ricci metric the leading normal part of the metric is

$$\frac{3}{4}\sum_i \frac{|dz_i|^2}{(\log|z_i|)^2|z_i|^2} \tag{8.4}$$

Changing variables back to $s_i = -1/\log|z_i|$ gives (8.2) with the constants matching Corollary 4.2 of [9]. □

Thus the Ricci metric of the Weil–Petersson metric is of the form of a 'multi-cusp' metric, as exemplified by the product of Riemann surfaces with cusps. It is shown in [9] that the Kähler-Einstein metric on the moduli space is quasi-isometric to the Ricci metric; it presumably has similar regularity although this has not been demonstrated. Since such metrics also appear in the setting of locally symmetric spaces it is very natural to enquire as to the structure of the continuous spectrum in these settings.

9. Sectional curvature

Written out in terms of the coordinate introduced by the real resolution as in (8.1), the Weil–Petersson metric is given by the Hermitian form

$$\begin{gathered} g_{\mathrm{WP}} = \sum g_{j\bar{l}}\alpha_j\overline{\alpha_l},\\ g_{j\bar{l}} = \begin{cases}\pi s_j^3(1+s_j^2\gamma_{j\bar{j}}) & j = l \le k\\ s_j^3 s_l^3\gamma_{j\bar{l}} & j \ne l \le k\\ s_j^3\gamma_{j\bar{l}} & j \le k,\ l > k\\ \gamma_{j\bar{l}} & j,l > k\end{cases}\end{gathered} \tag{9.1}$$

where the $\gamma_{j\bar{l}}$ are log-smooth, have θ_j derivatives vanishing to infinite order at $s_j = 0$ and for $j,l > k$, $g_{j\bar{l}}$ restricts to the corner $\cap_{j\le k}\{s_j = 0\}$ to give the induced Weil–Petersson metric. The structure of the metric comes directly from the formula (9) which shows that the co-metric with respect to the dual basis is log-smooth, with invertible tangential part, except for the diagonal components in the 'normal' directions which are of the form

$$g^{j\bar{j}} = \pi^{-1} s_j^{-3}(1 + s_j^2 \delta_j),\ 1 \le j \le k \tag{9.2}$$

with δ_j log-smooth. The Fenchel–Nielsen coordinates are geodesic coordinates [1], therefore restricting at $\{s_j = 0\}$ the cross terms $g_{j\bar{l}}$, $j \neq l$, vanish there. Note that this is different from the plumbing coordinates used in (8.1). This implies that the Kähler potential of the metric near the intersection of k divisors is given by a log-smooth function with θ_j derivatives vanishing to infinite order at $s_j = 0$ and with expansion

$$u(s_1, \dots, s_k, z_{k+1}, \dots, z_{3g-3+n}) = \psi(z, \bar{z}) + \sum_{i=1}^{k} 2\pi s_i + \sum_{i=1}^{k} s_i^2 \phi_i(s, z, \bar{z}), \tag{9.3}$$

where $\phi_i = \sum_{j \neq i} O(s_j^2)$. The Weil–Petersson metric, given by the Kähler form $g = \partial\overline{\partial} u$, is

$$g = \begin{pmatrix} \pi s_i^3 + \frac{3}{2} s_i^4 \phi_i + O(s_i^5) & O(s_i^3 s_j^3) & s_i^3 \phi_{i,\bar{z}} \\ O(s_i^3 s_j^3) & \pi s_j^3 + \frac{3}{2} s_j^4 \phi_j + O(s_j^5) & s_j^3 \phi_{j,\bar{z}} \\ s_i^3 \phi_{i,z} & s_j^3 \phi_{j,z} & \psi_{z\bar{z}} + O(\sum_i s_i^2) \end{pmatrix}, \quad 1 \le i \neq j \le k \tag{9.4}$$

with the dual metric

$$g^{-1} = \begin{pmatrix} \pi^{-1} s_i^{-3} + O(s_i^{-2}) & O(1) & O(1) \\ O(1) & \pi^{-1} s_j^{-3} + O(s_j^{-2}) & O(1) \\ O(1) & O(1) & \psi_{z\bar{z}}^{-1} + \sum O(s_i^2) \end{pmatrix}. \tag{9.5}$$

Proposition 15. *The leading order of the curvature tensors of the Weil–Petersson metric near the intersection of k divisors $\cap_{j=1}^k \{s_j = 0\}$ in the orthonormal basis $\{\frac{ds_j}{s_j^{\frac{1}{2}}} + s_j^{\frac{3}{2}} i d\theta_j,\ dz_l,\ 1 \le j \le k,\ l > k\}$ are given by the matrices below with entries $(q, p) \in \{i, j, *\} \times \{i, j, *\}$:*

$$\tilde{R}_{i\bar{i}q\bar{p}} = O \begin{pmatrix} s_i^{-1} & s_i^{\frac{1}{2}} & s_i^{\frac{1}{2}} \\ s_i^{\frac{1}{2}} & s_i^{\frac{1}{2}} s_j^{\frac{1}{2}} & s_i^{\frac{1}{2}} \\ s_i^{\frac{1}{2}} & s_i^{\frac{1}{2}} & s_i \end{pmatrix}$$

$$\begin{aligned} \tilde{R}_{i\bar{j}q\bar{p}} &= O \begin{pmatrix} s_i^{\frac{1}{2}} & s_i s_j & s_i \\ s_i s_j & s_j^{\frac{1}{2}} & s_j \\ s_i & s_j & s_i^{\frac{3}{2}} s_j^{\frac{3}{2}} \end{pmatrix}, \quad \tilde{R}_{i\bar{*}q\bar{p}} = O \begin{pmatrix} s_i^{\frac{1}{2}} & s_i & s_i^2 \\ s_i & s_j^{\frac{1}{2}} & s_i^{\frac{3}{2}} s_j^{\frac{3}{2}} \\ s_i^2 & s_i^{\frac{3}{2}} s_j^{\frac{3}{2}} & s_i^{\frac{3}{2}} \end{pmatrix} \\ \tilde{R}_{*\bar{i}q\bar{p}} &= O \begin{pmatrix} s_i^{\frac{1}{2}} & s_i & s_i^2 \\ s_i & s_j^{\frac{1}{2}} & s_i^{\frac{3}{2}} s_j^{\frac{3}{2}} \\ s_i^2 & s_i^{\frac{3}{2}} s_j^{\frac{3}{2}} & s_i^{\frac{3}{2}} \end{pmatrix}, \quad \tilde{R}_{*\bar{*}q\bar{p}} = O \begin{pmatrix} s_i & s_i^{\frac{3}{2}} s_j^{\frac{3}{2}} & s_i^{\frac{3}{2}} \\ s_i^{\frac{3}{2}} s_j^{\frac{3}{2}} & s_j & s_j^{\frac{3}{2}} \\ s_i^{\frac{3}{2}} & s_j^{\frac{3}{2}} & 1 \end{pmatrix} \end{aligned} \tag{9.6}$$

$$(q, p) \in \{i, j, *\} \times \{i, j, *\},\ 1 \le i, j \le k,\ k + 1 \le * \le 3g - 3 + n,$$

where more specifically the sectional curvature of the normal direction $\tilde{R}_{i\bar{i}i\bar{i}}$ is given by

$$\tilde{R}_{i\bar{i}i\bar{i}} = -\frac{3\pi}{4} s_i^{-1} + O(s_i). \tag{9.7}$$

Remark 1. The leading coefficient in (9.7) matches with the scalar curvature of the leading term in (9) which gives

$$R = -\frac{3\pi}{2} s_i^{-1}.$$

Proof. We show the computation of the case with one single divisor first $k = 1$, and the computation with multiple divisors are similar. Consider the Kähler potential $u(s, z, \bar{z})$ for the Weil–Petersson metric, which should be

$$u = \phi(z, \bar{z}) + 2\pi s + s^2 \psi(s, z, \bar{z}) \tag{9.8}$$

the metric $g_{i\bar{j}}$ is given by (here $\partial_\alpha = \frac{1}{2}(s^2\partial_s - i\partial_\theta)$ and similarly $\partial_{\bar{\alpha}} = \frac{1}{2}(s^2\partial_s + i\partial_\theta)$)

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u = \begin{pmatrix} u_{\alpha\bar{\alpha}} & u_{\alpha\bar{z}} \\ u_{\bar{\alpha}z} & u_{z\bar{z}} \end{pmatrix} = \begin{pmatrix} \pi s^3 + \frac{3}{2}s^4\psi + \frac{3}{2}s^5\psi_s + \frac{1}{4}s^6\psi_{ss} & s^3\psi_{\bar{z}} + \frac{1}{2}s^4\psi_{s\bar{z}} \\ s^3\psi_z + \frac{1}{2}s^4\psi_{sz} & \phi_{z\bar{z}} + s^2\psi_{z\bar{z}} \end{pmatrix}. \tag{9.9}$$

Note the metric itself, is of the form

$$\begin{pmatrix} s^3(\pi + a's) & s^3 b' \\ s^3\overline{b'} & h' \end{pmatrix} \tag{9.10}$$

where a', b' and h' are again log-smooth and h' is invertible.

The dual metric (always using the b-basis which becomes the cusp basis) is of the form

$$\begin{aligned} g^{-1} &= (\det g_{i\bar{j}})^{-1} \begin{pmatrix} u_{z\bar{z}} & -u_{\alpha\bar{z}} \\ -u_{\bar{\alpha}z} & u_{\alpha\bar{\alpha}} \end{pmatrix} \\ &= \begin{pmatrix} \pi^{-1}s^{-3} - \frac{3}{2}s^{-2}\pi^{-2}\psi & -\frac{1}{\pi}\phi_{z\bar{z}}^{-1}\psi_{\bar{z}} \\ -\frac{1}{2\pi}\phi_{z\bar{z}}^{-1}\psi_z & \phi_{z\bar{z}}^{-1} \end{pmatrix} + O\begin{pmatrix} s^{-1} & s \\ s & s^2 \end{pmatrix} \end{aligned} \tag{9.11}$$

From here we compute the curvature tensor:

$$\begin{aligned} R_{1\bar{1}q\bar{p}} &= -\begin{pmatrix} \frac{3\pi}{4}s^5 + O(s^6) & O(s^5) \\ O(s^5) & \frac{3}{2}s^4\psi_{z\bar{z}} + O(s^5) \end{pmatrix} \\ R_{*\bar{1}q\bar{p}} &= -\begin{pmatrix} \frac{3}{4}\psi_z s^5 + O(s^6) & O(s^5) \\ O(s^5) & O(s^3) \end{pmatrix} \\ R_{1\bar{*}q\bar{p}} &= -\begin{pmatrix} \frac{3}{4}\psi_{\bar{z}} s^5 + O(s^6) & O(s^5) \\ O(s^5) & O(s^3) \end{pmatrix} \\ R_{*\bar{*}q\bar{p}} &= -\begin{pmatrix} \frac{3}{2}\psi_{z\bar{z}} s^4 + O(s^5) & O(s^3) \\ O(s^3) & O(1) \end{pmatrix} \end{aligned} \tag{9.12}$$

In the orthonormal basis, we need to rescale the basis by changing from $\frac{ds}{s^2} + id\theta$ to unit length vector $\frac{ds}{s^{\frac{1}{2}}} + s^{\frac{3}{2}} id\theta$, so effectively multiplying each entry with a 'i' or '$\bar{i}$' by $s_i^{-\frac{3}{2}}$. Therefore,

$$\begin{aligned} \tilde{R}_{1\bar{1}q\bar{p}} &= \begin{pmatrix} -\frac{3\pi}{4}s^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O\begin{pmatrix} 1 & s^{\frac{1}{2}} \\ s^{\frac{1}{2}} & s \end{pmatrix} \\ \tilde{R}_{*\bar{1}q\bar{p}} = O\begin{pmatrix} s^{\frac{1}{2}} & s^2 \\ s^2 & s^{\frac{3}{2}} \end{pmatrix}&, \ \tilde{R}_{1\bar{*}q\bar{p}} = O\begin{pmatrix} s^{\frac{1}{2}} & s^2 \\ s^2 & s^{\frac{3}{2}} \end{pmatrix}, \ \tilde{R}_{*\bar{*}q\bar{p}} = O\begin{pmatrix} s & s^{\frac{3}{2}} \\ s^{\frac{3}{2}} & 1 \end{pmatrix}. \end{aligned} \tag{9.13}$$

□

10. Takhtajan-Zograf metrics

To analyse the behaviour of the Takhtajan-Zograf metric(s) at the divisors of $\overline{\mathcal{M}}_{g,n}$, $n \geq 1$, we need (in addition to everything above) to see what happens to the appropriate 'Eisenstein series' under degeneration.

If we have a punctured Riemann surface undergoing nodal degeneration with a fixed cusp B_i we need to understand the behaviour of the Eisenstein series, which can be realized as the solution to

$$(\varDelta+2)E_i(z)=0,\ E_i(z)=x^{-2}\chi+E_i',\ E_i'\in L^2 \tag{10.1}$$

where χ is a cut-off near B_i. So the behaviour of this follows from the analysis of $(\varDelta+2)^{-1}$ above.

Lemma 11. *On any connected compact Riemann surface with punctures, the Eisenstein series E_i determined by* (10.1) *for the puncture B_i is strictly positive.*

Proof. Maximum principle. □

For this Eisenstein series associated with any one marked point on a marked Riemann surface the limit at another marked point

$$L(i,p)=\lim \rho_p^{-1}E_i \tag{10.2}$$

is the value of some L-function.

Now consider the degeneracy of E_i at some k-fold (self-)intersection of divisors at the boundary of $\overline{\mathcal{M}}_{g,n}$. The nodal Riemann surface corresponds to at most k components of connected Riemann surfaces M_j of genus g_j. Each component has (possibly empty) finite set of n_j+k_j points, consisting of n_j labelled marked points and k_j distinct but unlabelled nodal points; each component is stable in the sense that $2g_j+n_j+k_j>2$ and there is an involution pairing the nodal points (collectively). At each boundary point, the b-cotangent bundle of $\mathcal{M}_{g,n}$ contains as a subspace the b-cotangent bundle of $\mathcal{M}_{g_j,n_j+k_j}$.

Now, for such a nodal Riemann surface in the boundary of the moduli space, the ith marked point appears in precisely one of the components corresponding to $j=m(i)$. Each of the other components is connected to this particular component by a chain of separating nodes. Let $s(m(i),j)$ be the number of these nodes and let $n(i,j)$ formally denote the last node which is in the jth component Riemann surface. We also let $L(m(i),j)$ be the product of the L-functions corresponding to the chain of intervening 'linking' nodes.

Proposition 16. *Each Eisenstein series E_i is log-smooth on $\widehat{\mathcal{C}}_{g,n}$ (the metric resolution of the universal curve) and at a k-fold corner is of the form*

$$E_i=\sum_j \rho_i^{4\sigma(m(i),j)}L(m(i),j)E_{n(i,j)}(\mathcal{M}_{g_j,n_j+k_j}). \tag{10.3}$$

For one of the 'fixed divisors' in $\overline{\mathcal{M}}_{g,n}$, $n>0$, the corresponding Takhtajan-Zograf metric is the push-forward in

$$(q_1,q_2)_{\mathrm{TZ}}=\phi_*(\frac{E_i^{-1}q_1\bar{q}_2}{\mu_H}) \tag{10.4}$$

and the total metric is the sum over i; see [14].

The extra factor is therefore at worst $O(x^2)$ in terms of the logarithmic coordinates, so does not affect the exponential decay from the vanishing (i.e. simple pole) of the quadratic differentials at the cusp face with which it is associated. In fact the final effect is that there is one extra order of singularity at the nodal face relative to the Weil–Petersson metric. The leading term can be extracted as before, except that there is an overall constant which is global in nature and the Takhtajan-Zograf metric will have the behaviour

$$g_{\mathrm{TZ}} = cds^2 + c's^4 d\theta^2 \tag{10.5}$$

in the normal direction to the divisor. According to Obitsu-To-Weng there is some degeneracy in the tangential directions, see [14].

11. Lengths of short geodesics

For the plumbing model the shortest geodesic occurs in the middle of the hyperbolic neck, that is, at $|z(\theta)| = \sqrt{|t|}$ in terms of the original complex coordinates. The length of this circle is $2\pi^2 s$, $s = \mathrm{ilog}\,|t|$. This provides an approximation to the degenerating geodesic for the global hyperbolic metric both in terms of the length and the position of the circle.

Proposition 17. *In terms of the local fiber coordinate $w = \mathrm{ilog}\,|z|/s$ near the front face of the metric resolution the short closed geodesic near a nodal point is of the form*

$$\gamma_s(\theta) = (w(s,\theta),\theta) = (2 + g(s) + g'(s,\theta),\theta) \tag{11.1}$$

where $g(s)$ is log-smooth with $g(0) = 0$ and $g'(s,\theta)$ is smooth and vanishes to infinite order as $s \downarrow 0$; it follows that its length is

$$L_\gamma(s) = 2\pi^2 s(1 + se(s)) \tag{11.2}$$

where e is log-smooth.

Proof. On the metric resolution, near a nodal point, the degenerating hyperbolic metric, takes the form

$$g = e^{2s^2 f} Z(w)^2 \left(\frac{dw^2}{w^2} + w^2 s^2 d\theta^2\right),\ Z(w) = \frac{\pi/w}{\sin(\pi/w)} \tag{11.3}$$

where $f = f(w,s,\theta))$ is log-smooth and $\partial_\theta f = O(s^\infty)$.

To show that the actual degenerating geodesic is close to the curve for the model, $\gamma(\theta) = (2,\theta)$, consider the length of families of curves of the form

$$\gamma(\theta,s) = (w(s,\theta),\theta) = (2 + h(s) + su(\theta,s),\theta), \tag{11.4}$$

where $h \in H^1([0,1))$ and $u \in H^1_0(\mathbb{S} \times [0,1))$ lies in the Hilbert subspace without constant term, so $\int u d\theta = 0$. Then the length of the family satisfies

$$\frac{L(\gamma)}{s} = \int e^{s^2 f(\gamma)} Z(w) E^{\frac{1}{2}} d\theta,\ E = \frac{(u')^2}{w^2} + w^2. \tag{11.5}$$

This is a C^2 function near zero in $[0,1)_s \times H^1([0,1)) \times H^1_0(\mathbb{S} \times [0,1))$ which for small s has a non-degenerate minimum at $(h,u) = 0$. Here the smoothness uses the fact $w > 0$ and $E > 0$, so its inverse and square-root are strictly positive functions in

L^∞. Indeed, the derivative with respect to (h, u) evaluated on the tangent vector (κ, υ) may be written in the form

$$\int \kappa\alpha + \upsilon'\beta$$

$$\alpha = e^{s^2 f} Z(w) E^{\frac{1}{2}} \left(s^2 f_w - \frac{1}{w}(1 - \frac{\pi}{w}\cot\frac{\pi}{w}) + E^{-1}(\frac{(u')^2}{w^3} + w) \right) \tag{11.6}$$

$$\beta = e^{s^2 f} Z(w) E^{-1/2} \frac{u'}{w^2} - s \int \alpha$$

where the last integral is the unique in θ without constant term. So using L^2 duality this may be identified as a map, rather than a linear form,

$$D_s : H^1([0,1) \times L^2_0(\mathbb{S}; H^1([0,1)) \ni (h, u) \longmapsto \\ (\alpha, \beta) \in H^1([0,1)) \times L^2_0(\mathbb{S}; H^1([0,1)) \tag{11.7}$$

where the 0 subscript indicates the absence of the constant mode. As such it is again C^2 and its derivative at $s = 0$ is invertible. From the Implicit Function Theorem, D_s has a unique 0 near 0 and from this the stated regularity of the curve giving the unique geodesic follows directly. The log-smoothness of the length can then be seen by evaluating the integral as a push-forward. $\square$

Appendix: Log-smooth functions

In this appendix we recall the definition, and some of the basic properties, of log-smooth conormal functions on a manifold with corners.

For a general compact manifold with corners the space of log-smooth functions, denoted by $\mathcal{C}^\infty_{\log}(M)$, is well defined in terms of iterated expansions at each of the boundary faces. Proceeding by induction one can suppose that $\mathcal{C}^\infty_{\log}(M)$ is well-defined for any manifold with faces of codimension at most k. Then on a manifold N with boundary faces of codimension up to $k+1$ a function $u \in \mathcal{C}^\infty(N \setminus \partial N)$ is in $\mathcal{C}^\infty_{\log}(N)$ if it has expansions at each boundary hypersurface $H = \{\rho = 0\}$ with defining function ρ and for any (one) choice of collar decomposition $\{\rho < \epsilon\} = H \times [0, \epsilon)$:

$$u \simeq \sum_{l=0}^{\infty} \sum_{j=0}^{l} u_{l,j} (\log \rho)^j \rho^l, \ u_{l,j} \in \mathcal{C}^\infty_{\log}(H) \tag{.8}$$

where H can have boundary faces only up to codimension k. The precise meaning of the expansion can be given in terms of conormal estimates. Namely if $\mathcal{A}(N)$ is the space of bounded conormal functions, defined by the stability condition

$$\mathrm{Diff}^m_{\mathrm{b}}(N) \cdot u \subset L^\infty(N) \ \forall \ m \tag{.9}$$

then the remainder terms in (.8) are required to satisfy

$$\phi(\rho) \left[u - \sum_{l=0}^{M} \sum_{j=0}^{l} u_{l,j} (\log \rho)^j \rho^l \right] \in \rho^{M+1} \mathcal{A}(N) \ \forall \ M \tag{.10}$$

where $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ has support in the collar and $\phi = 1$ near 0.

Such an expansion at a boundary face implies similar expansions near the corners contained in it. Indeed, again proceeding by induction over boundary codimension, the expansion of the coefficients in (.8) gives an expansion at any boundary face F of H of the form

$$u \simeq \sum_{\alpha,\beta\le\alpha}^{\infty} u_{\alpha,\beta}(\log\rho)^{\beta}\rho^{\alpha},\ u_{\alpha,\beta}\in\mathcal{C}^{\infty}_{\log}(F) \tag{.11}$$

where ρ now stands for the codim(F) defining functions of F, one of which is by assumption a defining function for N. Again the meaning of this expansion is that the difference of u and the terms with $|\alpha|\le L$ should lie in $R_F^{L-1}\mathcal{A}(M)$ where R_F is a radial defining function for F. The function u determines all the coefficients in the expansion at any boundary face and it follows that there are compatibility conditions across the higher codimension faces.

These compatibility conditions are between the expansions at different boundary faces but the expansion at any one boundary face is unrestricted. This can be seen by constructing appropriate elements of $\mathcal{C}^{\infty}_{\log}(M)$. The series in (.11) can be summed by choosing a cutoff $\chi\in\mathcal{C}^{\infty}_{\mathrm{c}}(\mathbb{R}^p)$, where p is the codimension and $\chi=1$ in a neighbourhood of the origin. Then, provided $\epsilon_l\downarrow 0$ converges sufficiently rapidly, depending on the coefficients $u_{\alpha,\beta}\in\mathcal{C}^{\infty}_{\log}(F)$,

$$u=\sum_{l}\sum_{|\alpha|=l,\beta\le\alpha}^{\infty} u_{\alpha,\beta}(\log\rho)^{\beta}\rho^{\alpha}\chi(\frac{\rho}{\epsilon_l})\in\mathcal{C}^{\infty}_{\log}(M) \tag{.12}$$

satisfies (.11). Moreover, if $u'\in\mathcal{C}^{\infty}_{\log}(M)$ has the same expansion at F then the difference can be decomposed near F as a sum over the boundary hypersurfaces containing F

$$u'-u=u''+\sum_{H\supset F} v_H,\ v_H\in\mathcal{C}^{\infty}_{\log}(M),\ \mathrm{supp}(u'')\cap F=\emptyset \tag{.13}$$

where each v_H has a trivial expansion at all faces of codimension two or higher which are not contained in H.

Since we construct functions below by iteration over such asymptotic sums it is useful to consider subspaces of $\mathcal{C}^{\infty}_{\log}(M)$ for which the expansions at a given collection of boundary faces are trivial. We use the notation $u\overset{F}{\equiv}0$ to indicate that the expansion at the boundary face F is trivial.

Lemma 12. *If $u\in\mathcal{C}^{\infty}_{\log}(M)$ and $u\overset{F}{\equiv}0$ at all boundary faces of codimension k then u may be decomposed into a sum over boundary faces $\{G\}$ of codimension $k-1$*

$$u=\sum_G u_G,\ u_G\in\mathcal{C}^{\infty}_{\log}(M),\ u_G\overset{F}{\equiv}0.$$

Proof. If ff is a boundary face of M under the blow up of F the space $\{u\in\mathcal{C}^{\infty}_{\log}(M); u\overset{F}{\equiv}0\}$ lifts isomorphically to $\{v\in\mathcal{C}^{\infty}_{\log}([M;F]); v\overset{\mathrm{ff}}{\equiv}0\}$. After the blow up of all boundary faces of codimension k, the lifts of the faces of codimension $k-1$ are disjoint. Thus, on the blown up space the lift of u can be divided into pieces each of which has support disjoint from one of the (lifted) boundary faces of codimension $k-1$ by use of a partition of unity. These pieces therefore are the lifts of a decomposition as desired. □

References

[1] Lars V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math. (2) **74** (1961), 171–191. MR 0204641

[2] K. Burns, H. Masur, and A. Wilkinson, *The Weil-Petersson geodesic flow is ergodic*, Ann. of Math. (2) **175** (2012), no. 2, 835–908. MR 2993753

[3] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109. MR 0262240

[4] William Fulton and Robert MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) **139** (1994), no. 1, 183–225. MR 1259368 (95j:14002)

[5] Jesse Gell-Redman and Jan Swoboda, *Spectral and hodge theory of 'Witt' incomplete cusp edge spaces*, arXiv preprint arXiv:1509.06359, 2015.

[6] Lizhen Ji, Rafe Mazzeo, Werner Müller, and András Vasy, *Spectral theory for the Weil-Petersson Laplacian on the Riemann moduli space*, Commentarii Mathematici Helvetici **89** (2014), no. 4, 867–894.

[7] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199. MR 702953

[8] Kefeng Liu, Xiaofeng Sun, and Shing-Tung Yau, *Canonical metrics on the moduli space of Riemann surfaces. I*, J. Differential Geom. **68** (2004), no. 3, 571–637. MR 2144543

[9] ———, *Canonical metrics on the moduli space of Riemann surfaces. II*, J. Differential Geom. **69** (2005), no. 1, 163–216. MR 2169586

[10] Howard Masur, *The extension of the Weil-Petersson metric to the boundary of Teichmüller space*, Duke Mathematical Journal **43** (1976), no. 3, 623–635.

[11] Rafe Mazzeo and Jan Swoboda, *Asymptotics of the Weil-Petersson metric*, Int. Math. Res. Not. IMRN (2017), no. 6, 1749–1786. MR 3658182

[12] Richard B. Melrose, *Calculus of conormal distributions on manifolds with corners*, Internat. Math. Res. Notices **1992**, no. 3, 51–61. MR 93i:58148

[13] Richard B. Melrose and Xuwen Zhu, *Resolution of the canonical fiber metrics for a Lefschetz fibration*, to appear in Journal of Differential Geometry.

[14] Kunio Obitsu, Wing-Keung To, and Lin Weng, *The asymptotic behavior of the Takhtajan-Zograf metric*, Comm. Math. Phys. **284** (2008), no. 1, 227–261. MR 2443304 (2009g:32026)

[15] Kunio Obitsu and Scott A. Wolpert, *Grafting hyperbolic metrics and Eisenstein series*, Math. Ann. **341** (2008), no. 3, 685–706. MR 2399166 (2009d:32011)

[16] Jinsung Park, Leon A. Takhtajan, and Lee-Peng Teo, *Potentials and Chern forms for Weil-Petersson and Takhtajan-Zograf metrics on moduli spaces*, Adv. Math. **305** (2017), 856–894. MR 3570149

[17] Joel W. Robbin and Dietmar A. Salamon, *A construction of the Deligne-Mumford orbifold*, J. Eur. Math. Soc. (JEMS) **8** (2006), no. 4, 611–699. MR 2262197 (2009d:32012)

[18] L. A. Takhtajan and P. G. Zograf, *A local index theorem for families of $\overline{\partial}$-operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces*, Comm. Math. Phys. **137** (1991), no. 2, 399–426. MR 1101693 (92g:58121)

[19] Stefano Trapani, *On the determinant of the bundle of meromorphic quadratic differentials on the Deligne-Mumford compactification of the moduli space of Riemann surfaces*, Math. Ann. **293** (1992), no. 4, 681–705. MR 1176026

[20] Michael Wolf and Scott A. Wolpert, *Real analytic structures on the moduli space of curves*, Amer. J. Math. **114** (1992), no. 5, 1079–1102. MR 1183532

[21] Scott A. Wolpert, *On the Weil-Petersson geometry of the moduli space of curves*, American Journal of Mathematics (1985), 969–997.

[22] ———, *Chern forms and the Riemann tensor for the moduli space of curves*, Invent. Math. **85** (1986), no. 1, 119–145. MR 842050 (87j:32070)

[23] ———, *The hyperbolic metric and the geometry of the universal curve*, J. Differential Geom. **31** (1990), no. 2, 417–472. MR 1037410 (91a:32030)

[24] ———, *Geometry of the Weil-Petersson completion of Teichmüller space*, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), Surv. Differ. Geom., VIII, Int. Press, Somerville, MA, 2003, pp. 357–393. MR 2039996 (2005h:32032)

[25] ———, *Infinitesimal deformations of nodal stable curves*, Adv. Math. **244** (2013), 413–440. MR 3077878

[26] ______, *Equiboundedness of the Weil-Petersson metric*, Trans. Amer. Math. Soc. **369** (2017), no. 8, 5871–5887. MR 3646782

[27] Sumio Yamada, *On the geometry of Weil-Petersson completion of Teichmüller spaces*, Math. Res. Lett. **11** (2004), no. 2-3, 327–344. MR 2067477

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