GOPAKUMAR-VAFA INVARIANTS VIA VANISHING CYCLES

DAVESH MAULIK AND YUKINOBU TODA

Abstract. In this paper, we propose an ansatz for defining Gopakumar-Vafa invariants of Calabi-Yau threefolds, using perverse sheaves of vanishing cycles. Our proposal is a modification of a recent approach of Kiem-Li, which is itself based on earlier ideas of Hosono-Saito-Takahashi. We conjecture that these invariants are equivalent to other curve-counting theories such as Gromov-Witten theory and Pandharipande-Thomas theory.

Our main theorem is that, for local surfaces, our invariants agree with PT invariants for irreducible one-cycles. We also give a counter-example to the Kiem-Li conjectures, where our invariants match the predicted answer. Finally, we give examples where our invariant matches the expected answer in cases where the cycle is non-reduced, non-planar, or non-primitive.

Contents

1. Introduction

1.1. **Background.** Let X be a smooth projective Calabi-Yau threefold over $\mathbb{C}$. For $g \geq 0$ and $\beta \in H_2(X, \mathbb{Z})$, the corresponding *Gromov-Witten invariant*

$$\mathrm{GW}_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}$$

enumerates stable maps $f\colon C \to X$ from connected, nodal curves C of arithmetic genus g such that $f_*[C] = \beta$. In general, these invariants are given by an infinite sequence of rational numbers; nevertheless, for fixed β, they are expected to be controlled by a finite collection of integer invariants. Indeed, based on the string duality between type IIA and M theory, Gopakumar-Vafa [GV] conjectured the existence of integer-valued invariants

$$n_{g,\beta} \in \mathbb{Z},\ g \geq 0,\ \beta \in H_2(X, \mathbb{Z}) \tag{1.1}$$

which vanish for sufficiently large g and which determine the Gromov-Witten series by the identity

$$\sum_{\beta>0, g\geq 0} \mathrm{GW}_{g,\beta} \lambda^{2g-2} t^{\beta} = \sum_{\beta>0, g\geq 0, k\geq 1} \frac{n_{g,\beta}}{k} \left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2} t^{k\beta}. \tag{1.2}$$

The invariants (1.1) are called *Gopakumar-Vafa (GV for short) invariants.*

In order to define these invariants directly[1], the original approach of Gopakumar-Vafa was to use the $sl_2 \times sl_2$-action on the cohomology of a certain moduli space of D-branes, which should be given by a moduli space of one-dimensional sheaves. The goal of this paper is to propose an ansatz for making this mathematically precise. As we will review shortly, there have been earlier efforts in this direction, most notably by Hosono-Saito-Takahashi [HST01] and Kiem-Li [KL]. Our approach is a modification of the recent Kiem-Li proposal via perverse sheaves of vanishing cycles, where we use the perverse filtration for the Hilbert-Chow map instead of the action of $sl_2 \times sl_2$. We then show that our definition of GV invariants matches with stable pair invariants introduced by Pandharipande-Thomas [PT09] in several cases, in particular for irreducible one-cycles on local surfaces.

1.2. **Proposed definition.** Let $\mathrm{Sh}_{\beta}(X)$ denote the moduli space of one-dimensional stable sheaves E on X satisfying[2]

$$[E] = \beta \in H_2(X, \mathbb{Z}),\ \chi(E) = 1.$$

Let $\mathrm{Chow}_{\beta}(X)$ denote the Chow variety parameterizing effective one-cycles on X with homology class β. There is a Hilbert-Chow map

$$\pi \colon \mathrm{Sh}_{\beta}^{\mathrm{red}}(X) \to \mathrm{Chow}_{\beta}(X) \tag{1.3}$$

sending a stable sheaf to its fundamental one-cycle. In [BBD$^+$, KL], a certain perverse sheaf $\phi_{\mathcal{S}h}$ on $\mathrm{Sh}_{\beta}(X)$ is constructed whose Euler characteristic recovers the usual invariants of *Donaldson-Thomas (DT for short) theory* [Tho00]. Roughly speaking, the moduli space $\mathrm{Sh}_{\beta}(X)$ is locally written as a critical locus of some function on a smooth scheme, and $\phi_{\mathcal{S}h}$ is obtained by gluing together the locally-defined sheaves of vanishing cycles. An important subtlety in this construction is that the gluing is not uniquely determined and depends on a choice of *orientation data*, that is a square root of the virtual canonical line bundle on $\mathrm{Sh}_{\beta}(X)$.

We propose the following definition of GV invariants:

[1] One can take equation (1.2) as an indirect definition of GV invariants, in which case integrality and vanishing become conjectures. A symplectic approach to proving the integrality is pursued in [IP].

[2] The case for other choices of $\chi(E)$ will be considered in Subsection 3.3.

Definition 1.1. (Definition 3.7) *We define the GV invariants $n_{g,\beta}$ by the identity*

$$\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_* \phi_{\mathcal{S}h}))y^i = \sum_{g\geq 0} n_{g,\beta}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}. \tag{1.4}$$

Here ${}^p\mathcal{H}^i(-)$ is the i-th cohomology functor with respect to the perverse t-structure. By the self-duality of $\phi_{\mathcal{S}h}$ and the Verdier duality, the LHS of (1.4) is uniquely written as the form of the RHS. The GV invariants in Definition 1.1 are obviously integers, which vanish for sufficiently large g.

As currently formulated our GV invariants in (1.4) depend on a choice of an orientation, and a canonical choice is not known. We impose an additional restriction that the orientation data is trivial along the fibers of (1.3). These are denoted *Calabi-Yau* orientations, and we conjecture that such choices always exist. As we will see, the invariants $n_{g,\beta}$ defined by (1.4) are independent of the choice of CY orientation data.

Our definition of GV invariants in Definition 1.1 is related to the character formula of the $sl_2 \times sl_2$-actions in earlier approaches [HST01, KL], see Subsection 1.5. As far as we know, the observation that the character formula can be reformulated via perverse cohomology goes back to work of Chuang-Diaconescu-Pan [CDP14] on the Hitchin fibration and the survey article of Pandharipande-Thomas [PT14] in their discussion of [HST01].

One advantage of this reformulation is that it naturally lifts to a definition of *local GV invariants*, i.e. we can define a constructible function (see Definition 3.9)

$$n^{\mathrm{loc}}_{g,-} \colon \mathrm{Chow}_\beta(X) \to \mathbb{Z} \tag{1.5}$$

whose integral over the Chow variety gives $n_{g,\beta}$. This makes it possible to compare our GV invariants with stable pair invariants for a fixed one-cycle.

1.3. **PT/GV correspondence for irreducible cycles.** Given X as before, a *stable pair* on X consists of a pair

$$(F, s), \ \ s \colon \mathcal{O}_X \to F$$

such that F is a pure one-dimensional sheaf on X, and s is a morphism whose cokernel is at most zero-dimensional. Virtual invariants for stable pair spaces were introduced by Pandharipande-Thomas (PT for short) in [PT09]. In [PT10], *local PT invariants* are defined by taking the constructible function

$$P^{\mathrm{loc}}_{n,-} \colon \mathrm{Chow}_\beta(X) \to \mathbb{Z} \tag{1.6}$$

obtained by integrating the Behrend function [Beh09] over the locus of the moduli space of stable pairs with fixed fundamental cycle. The usual *PT invariant* $P_{n,\beta} \in \mathbb{Z}$ is determined from the local invariants by integrating over the Chow variety; when X is projective, this agrees with the original definition by virtual classes [Beh09].

For any threefold, there is a conjectural equivalence [MNOP06, PT09] between GW and PT invariants, which is proved in many cases [MNOP06, BP08, PP]. In particular, formula (1.2) implies a conjectural equivalence between PT invariants and our GV invariants. Furthermore, since both PT invariants and our GV invariants can be refined to local invariants (1.5), (1.6), we can conjecture a local PT/GV correspondence (see Figure 1). For an irreducible one-cycle, it is given as follows:

Conjecture 1.2. *For an irreducible one-cycle $\gamma \in \mathrm{Chow}_\beta(X)$, we have the identity*

$$\sum_{n\in\mathbb{Z}} P^{\mathrm{loc}}_{n,\gamma} q^n = \sum_{g\geq 0} n^{\mathrm{loc}}_{g,\gamma} (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2g-2}.$$

We will formulate the conjecture for general one-cycles in Conjecture 3.14; it requires contributions coming from effective summands of the cycle γ.

The first main result of this paper is to prove Conjecture 1.2 for local surfaces.

Theorem 1.3. (Theorem 5.15) *Let S be a smooth projective surface with $H^1(\mathcal{O}_S) = 0$, and*

$$p\colon X = \mathrm{Tot}(K_S) \to S$$

the non-compact CY 3-fold. Then Conjecture 1.2 is true for any one-cycle γ on X such that $p_*\gamma$ is irreducible.

We also prove Conjecture 1.2 for smooth one-cycles on an arbitrary Calabi-Yau threefold X.

Theorem 1.4. (Theorem 6.4) *For a smooth curve $C \subset X$, Conjecture 1.2 is true for the one cycle $\gamma = [C]$.*

The situations in Theorem 1.3, 1.4 include many cases where both $\mathrm{Sh}_\beta(X)$ and $\mathrm{Chow}_\beta(X)$ are singular. In particular, in these cases, the sheaf $\phi_{\mathcal{S}h}$ will typically not be pure and the Behrend function will not be constant. The main idea to prove the above theorems is to use the vanishing cycles functor to reduce to the generalized Macdonald formula for versal deformations of locally planar curves, proved in [MY14, MS13]. The basic outline of this reduction is explained in Section 4, which we recommend for readers who just want to see the essential concept without the (many) technical details.

1.4. **Examples for non-reduced cycles.** One limitation of the theorems above is they only apply in cases where the one-cycle is integral. In Section 8, we produce an infinite family of examples where the local PT/GV correspondence holds for one-cycles that are non-reduced or non-planar.

The idea for the construction is to study how our invariants behave for certain 3-fold flops and combine this analysis with Theorem 1.3, 1.4. We will show the following result:

Theorem 1.5. (Corollary 8.9) *Let $\phi\colon X \dashrightarrow X^\dagger$ be a flop between CY 3-folds and $C \subset X$ an irreducible curve which is not contained in $\mathrm{Ex}(\phi)$. Suppose that Conjecture 1.2 holds for $\gamma = [C]$. Then the local PT/GV correspondence holds for $\phi_*\gamma$.*

In typical examples, the flopped cycle $\phi_*\gamma$ can be arranged to be non-reduced and non-planar. In such cases, the correspondence requires the more general formulation of Conjecture 3.14. In the examples obtained above, the contributions from effective summands of the reducible one-cycle

$$
\begin{tikzcd}
& \boxed{\text{GW invariants}} \arrow[dl, leftrightarrow, "\text{GW/PT conjecture}"'] \arrow[dr, leftrightarrow, "\text{GV conjecture}"] & \\
\boxed{\text{PT invariants}} \arrow[rr, leftrightarrow] & & \boxed{\text{GV invariants}} \\
\boxed{\text{local PT invariants}} \arrow[u, rightsquigarrow, "\text{integration}"] \arrow[rr, leftrightarrow] & & \boxed{\text{local GV invariants}} \arrow[u, rightsquigarrow, "\text{integration}"']
\end{tikzcd}
$$

FIGURE 1. GW/PT/GV correspondence

will typically be nonzero. Furthermore, we can iterate the above theorem to obtain more complicated one-cycles where our conjecture holds.

In Section 9, we also give examples of the local PT/GV correspondence where the one-cycle is non-primitive. One interesting family of such examples is due to Chuang-Diaconescu-Pan [CDP14], when X is the total space of $K_C \oplus \mathcal{O}_C$ over a curve C of genus g. In this case, the conjecture for $\beta = r[C]$ is a consequence of the $P = W$ conjecture for Higgs bundles of rank r on the curve C; since the $P = W$ conjecture is proven in rank 2, this gives an infinite family of non-primitive examples.

1.5. **Previous history.** A mathematical approach for GV invariants was first proposed by Hosono-Saito-Takahashi (HST for short) [HST01]. They considered the intersection complex $\mathrm{IC}(\mathrm{Sh}_\beta(X))$ and applied the decomposition theorem [BBD82] with respect to the map (1.3) to define the action of $sl_2 \times sl_2$ on the intersection cohomology $IH^*(\mathrm{Sh}_\beta(X))$. They rearrange this as

$$IH^*(\mathrm{Sh}_\beta(X)) = \bigoplus_{g \geq 0} I_g \otimes R_g$$

where I_g is the cohomology of a g-dimensional complex torus with its natural left sl_2-action, and R_g is a certain (virtual) right sl_2-representation. The HST definition is given by the Euler characteristic of R_g. Since the IC-sheaf is not sensitive to the virtual structure of $\mathrm{Sh}_\beta(X)$, one can find examples where this approach does not match the expected answer, even in genus 0.

In genus 0, Katz [Kat08] proposed as a definition the virtual integral

$$n_{0,\beta} = \int_{[\mathrm{Sh}_\beta(X)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Since this can be written as a Behrend-weighted Euler characteristic, one can use a wall-crossing argument to prove this is compatible with the GV invariants defined via stable pairs spaces.

More recently, Kiem-Li proposed a combination of these two approaches, using the perverse sheaf $\phi_{\mathcal{S}h}$ mentioned earlier. One ambiguity in their proposal is they do not specify how to choose an orientation on $\mathrm{Sh}_\beta(X)$ to define $\phi_{\mathcal{S}h}$, and different choices lead to different prescriptions. Once chosen, they consider a natural lift of $\phi_{\mathcal{S}h}$ to the category of mixed Hodge modules. In order to apply the decomposition theorem, they require a pure Hodge module, so take the associated graded $\mathrm{gr}^{\bullet}_{W}(\phi_{\mathcal{S}h})$ with respect to the weight filtration, and apply the formalism of [HST01] described earlier. In genus 0, independent of orientation, this recovers Katz's definition.

1.6. **Counter-example to Kiem-Li conjecture.** We will give an example that the earlier definitions [HST01, KL] are not deformation-invariant and, in particular, do not match the invariants as calculated via PT theory. Let S be an Enriques surface and E an elliptic curve. Let $\sigma\colon \widetilde{S} \to S$ be its K3 cover. We take a CY 3-fold

$$X = (\widetilde{S} \times E)/\langle \tau \rangle.$$

Here τ is an involution of $\widetilde{S} \times E$ which acts on $\widetilde{S}$ by the covering involution of σ and acts on E by $x \mapsto -x$. An Enriques surface S always admits an elliptic fibration $S \to \mathbb{P}^1$, with a double fiber $2C$. We take a curve class β in X by

$$\beta = ([C], 0) \in H_2(S, \mathbb{Z}) \oplus \mathbb{Z}[E] = H_2(X, \mathbb{Z}).$$

Proposition 1.6. (Proposition 7.6) *Suppose that C is of type I_n for $n \geq 2$, i.e. C is a circle of $\mathbb{P}^1$ with n-irreducible components. Then the HST, KL, our definitions, and the expected answers (from GW or PT theory) are given in the following table:*

	HST	KL	ours	expected
$n_{0,\beta}$	$-8n$	0	0	0
$n_{1,\beta}$	$4n$	$4n$	4	4
$n_{\geq 2,\beta}$	0	0	0	0

The result of Proposition 1.6 is a summary of the computations in Subsection 7.4. Since all Enriques surfaces are deformation equivalent, the resulting invariants $n_{g,\beta}$ should be independent of the type of C. The result of $g = 1$ for KL definition is true for any choice of orientation data, which does not match with the predicted answer. Therefore Proposition 1.6 gives a counter-example to the Kiem-Li conjecture [KL, Conjecture 7.4].

One interesting feature of this example is that our invariants are preserved under deformations despite the fact that the Chow variety itself jumps in dimension. Although our invariant in this example is deformation-invariant, we do not see a mechanism for this in general families of CY threefolds, due to our poor understanding of the Chow variety. For this reason, our definition of GV invariants still may not be the final one and a better understanding of the Hilbert-Chow map and deformation invariance will be needed.

One can also ask what happens if we study the motivic vanishing cycles associated to $\mathrm{Sh}_{\beta}(X)$ in the sense of Bussi-Joyce-Meinhardt [BJM], and define the motivic GV invariants as in [Tod08]. It turns out that the motivic GV invariants are different from Kiem-Li's invariants due to some rearrangement of weights, but in any case they also do not give a correct answer. An example already occurs in the case of the nodal rational curve in Section 5.11, where the genus one motivic invariant becomes zero.

1.7. **Outline of the paper.** In Section 2, we review Joyce's notion of d-critical structures and introduce GV type invariants for d-critical schemes. In Section 3, we define GV invariants on Calabi-Yau 3-folds and formulate conjectures relating them with PT and GW invariants. In Section 4, we explain the idea proving PT/GV correspondence using versal deformations of curves. In Section 5, we prove Theorem 1.3. In Section 6, we prove Theorem 1.4. In Section 7, we prove Proposition 1.6. In Section 8, we produce examples for non-reduced cycles using 3-fold flops. In Section 9, we discuss some examples for non-primitive cycles. In Appendix A, we discuss Calabi-Yau orientation data for d-critical schemes.

1.8. **Acknowledgments.** We are grateful to Tomoyuki Abe, Jim Bryan, Duiliu-Emanuel Diaconescu, Igor Dolgachev, Jesse Kass, Young-Hoon Kiem, Jun Li, Georg Oberdieck, Rahul Pandharipande, Christian Schnell and Richard Thomas for many discussions and valuable comments. We are also grateful to the referees for many suggestions and comments. This article was written while Y. T. was visiting Massachusetts Institute of Technology from 2015 October to 2016 September by the JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers. D. M. is supported by NSF grants DMS-1645082 and DMS-1564458. Y. T. is supported by World Premier International Research Center Initiative (WPI initiative), MEXT, Japan, and Grant-in Aid for Scientific Research grant (No. 26287002) from MEXT, Japan.

1.9. **Notation and convention.** In this paper, all varieties and schemes are defined over $\mathbb{C}$. For a scheme M, we will only consider constructible sheaves with $\mathbb{Q}$-coefficients. We denote by $\mathrm{Perv}(M)$ the category of perverse sheaves on M, which is the heart of a t-structure on the derived category of constructible sheaves on M (see [BBD82]). Let $\iota\colon M^{\mathrm{red}} \hookrightarrow M$ be the reduced part of M. Since ι is a homeomorphism, we always identity $\mathrm{Perv}(M)$ with $\mathrm{Perv}(M^{\mathrm{red}})$ in a natural way.

For a bounded complex E of constructible sheaves on M, we denote by ${}^p\mathcal{H}^i(E)$ the i-th cohomology with respect to the perverse t-structure, and $\chi(E)$ is the the Euler characteristic of $\mathbf{R}\Gamma(M, E)$. For a constructible function ν on a scheme M, the weighted Euler characteristic is denoted by

$$\int_M \nu \, de := \sum_{m \in \mathbb{Z}} m \cdot e(\nu^{-1}(m)).$$

Here $e(-)$ is the topological Euler characteristic. We will use the fact that, for a complex E of constructible sheaves on a finite type scheme M, we have

$$\chi(E) = \int_M \nu_E \, de$$

where ν_E is the constructible function given by $p \mapsto \chi(E|_p)$.

2. GV type invariants for d-critical schemes

In this section, we review Joyce's work on (algebraic) d-critical structures and define GV-type invariants in this setting[3]. We then introduce the notion of Calabi-Yau d-critical structures, which allows us to fix the ambiguity due to choice of orientation.

2.1. d-critical schemes.

We recall the notion of d-critical structures introduced by Joyce [Joy15]. For any complex scheme M, Joyce [Joy15] shows that there exists a canonical sheaf of $\mathbb{C}$-vector spaces $\mathcal{S}_M$ on M satisfying the following property: for any Zariski open subset $R \subset M$ and a closed embedding $i \colon R \hookrightarrow V$ into a smooth scheme V, there is an exact sequence

$$0 \longrightarrow \mathcal{S}_M|_R \longrightarrow \mathcal{O}_V/I^2 \xrightarrow{d_{\mathrm{DR}}} \Omega_V/I \cdot \Omega_V. \tag{2.1}$$

Here $I \subset \mathcal{O}_V$ is the ideal sheaf which defines R and d_{DR} is the de-Rham differential. Moreover there is a natural decomposition

$$\mathcal{S}_M = \mathcal{S}_M^0 \oplus \mathbb{C}_M$$

where $\mathbb{C}_M$ is the constant sheaf on M. The sheaf $\mathcal{S}_M^0$ restricted to R is the kernel of the composition

$$\mathcal{S}_M|_R \hookrightarrow \mathcal{O}_V/I^2 \twoheadrightarrow \mathcal{O}_{R^{\mathrm{red}}}.$$

For example, suppose that $f \colon V \to \mathbb{A}^1$ is a regular function such that

$$R = \{df = 0\}, \ f|_{R^{\mathrm{red}}} = 0. \tag{2.2}$$

Then $f + I^2$ is an element of $\Gamma(R, \mathcal{S}_M^0|_R)$.

Definition 2.1. ([Joy15]) *A pair (M, s) for a complex scheme M and $s \in \Gamma(M, \mathcal{S}_M^0)$ is called a d-critical scheme if for any $x \in M$, there is an open neighborhood $x \in R \subset M$, a closed embedding $i \colon R \hookrightarrow V$ into a smooth scheme V, a regular function $f \colon V \to \mathbb{A}^1$ satisfying (2.2) such that $s|_V = f + I^2$ holds. In this case, the data*

$$\xi = (R, V, f, i) \tag{2.3}$$

is called a d-critical chart. The section s is called a d-critical structure of M.

Roughly speaking, a d-critical scheme (M, s) is locally written as a critical locus of some function f on a smooth scheme, and the section s remembers the function f. Given a d-critical scheme (M, s), there exists a line bundle $K_{M,s}$ on M^{red}, called the *virtual canonical line bundle*, such that for any d-critical chart (2.3) there is a natural isomorphism

$$K_{M,s}|_{R^{\mathrm{red}}} \stackrel{\cong}{\to} K_V^{\otimes 2}|_{R^{\mathrm{red}}}. \tag{2.4}$$

[3]In [Joy15], Joyce also introduces an analytic version of d-critical structures; this is equivalent to the notion of virtual critical structures in [KL]. Although we work with algebraic d-critical structures, the arguments of this section also apply for analytic d-critical structures.

Definition 2.2. ([Joy15]) *An orientation of a d-critical scheme* (M, s) *is a choice of a square root line bundle* $K_{M,s}^{1/2}$ *for* $K_{M,s}$ *on* M^{red} *and an isomorphism*

$$(K_{M,s}^{1/2})^{\otimes 2} \xrightarrow{\cong} K_{M,s}. \tag{2.5}$$

A d-critical scheme with an orientation is called an oriented d-critical scheme.

2.2. **Sheaves of vanishing cycles.** Let $f\colon V \to \mathbb{A}^1$ be a regular function on a smooth scheme V, and set $R = \{df = 0\}$. Suppose that $f|_{R^{\mathrm{red}}} = 0$ and set $V_0 = f^{-1}(0)$. We have the associated vanishing cycle functor (see [Dim04, Theorem 5.2.21])

$$\phi_f\colon \mathrm{Perv}(V) \to \mathrm{Perv}(V_0).$$

Let $\mathrm{IC}(V) \in \mathrm{Perv}(V)$ be the intersection complex on V, which coincides with $\mathbb{Q}_V[\dim V]$ since V is smooth. We have the perverse sheaf of vanishing cycles supported on $R^{\mathrm{red}} \subset V_0$

$$\phi_f(\mathrm{IC}(V)) \in \mathrm{Perv}(R) \subset \mathrm{Perv}(V_0). \tag{2.6}$$

Let (M, s) be a d-critical scheme. For a d-critical chart (R, V, f, i) as in (2.3), we have the sheaf of vanishing cycles (2.6) on R. In [BBD+] it is proved that if (M, s) is oriented, then the sheaves of vanishing cycles (2.6) glue to give a global perverse sheaf on M. Let

$$(K_{M,s}^{1/2}|_{R^{\mathrm{red}}})^{\otimes 2} \cong K_V^{\otimes 2}|_{R^{\mathrm{red}}} \tag{2.7}$$

be the isomorphism given by the composition of (2.4) and (2.5). Then there is a $\mathbb{Z}/2\mathbb{Z}$-principal bundle

$$\tau_R\colon \widetilde{R}^{\mathrm{red}} \to R^{\mathrm{red}}$$

which parametrizes local square roots of the isomorphism (2.7). We have the decomposition

$$\tau_{R*}\mathbb{Q}_{\widetilde{R}^{\mathrm{red}}} = \mathbb{Q}_{R^{\mathrm{red}}} \oplus \mathcal{L}_\xi$$

for a rank one local system $\mathcal{L}_\xi$ on R^{red}. The following result is proved in [BBD+] (also see [KL] for the same result in the framework of virtual critical structures):

Theorem 2.3. ([BBD+, Theorem 6.9]) *For an oriented d-critical scheme* $\mathcal{M} = (M, s, K_{M,s}^{1/2})$, *there exists a natural perverse sheaf* $\phi_{\mathcal{M}}$ *on* M *such that for any d-critical chart (2.3) there is a natural isomorphism*

$$\phi_{\mathcal{M}}|_R \xrightarrow{\cong} \phi_f(\mathrm{IC}(V)) \otimes \mathcal{L}_\xi. \tag{2.8}$$

Moreover there exists a natural isomorphism $\mathbb{D}_M(\phi_{\mathcal{M}}) \cong \phi_{\mathcal{M}}$, *where* $\mathbb{D}_M$ *is the Verdier dualizing functor.*

2.3. **GV type invariants.** For an oriented d-critical scheme $\mathcal{M} = (M, s, K_{M,s}^{1/2})$, let

$$\pi\colon M^{\mathrm{red}} \to T$$

be a projective morphism to a finite type complex scheme T. We will use the perverse sheaf $\phi_{\mathcal{M}}$ in Theorem 2.3 through the following lemma:

Lemma 2.4. *There exist unique* $\mathrm{GV}_{g,\mathcal{M}/T} \in \mathbb{Z}$ *for* $g \in \mathbb{Z}_{\geq 0}$*, and* $\mathrm{GV}_{g,\mathcal{M}/T} = 0$ *for* $g \gg 0$*, such that*

$$\sum_{i \in \mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi_{\mathcal{M}}))y^i = \sum_{g \geq 0} \mathrm{GV}_{g,\mathcal{M}/T}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}. \tag{2.9}$$

Proof. By the isomorphism $\mathbb{D}_M(\phi_{\mathcal{M}}) \cong \phi_{\mathcal{M}}$ and the Verdier duality, we have the isomorphism

$$\mathbb{D}_T(\mathbf{R}\pi_*\phi_{\mathcal{M}}) \cong \mathbf{R}\pi_*\phi_{\mathcal{M}}.$$

Since $\mathbb{D}_T$ preserves the perverse t-structure, it follows that

$$\mathbb{D}_T({}^p\mathcal{H}^{-i}(\mathbf{R}\pi_*\phi_{\mathcal{M}})) \cong {}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi_{\mathcal{M}}).$$

Therefore the LHS of (2.9) is a polynomial of $y^{\pm 1}$ which is invariant under $y \mapsto y^{-1}$. By the induction of the degree of y, it is easy to see that any polynomial of $y^{\pm 1}$ invariant under $y \mapsto y^{-1}$ is uniquely written as the form of the RHS of (2.9). □

We also have the following local version of Lemma 2.4, whose proof is identical to Lemma 2.4.

Lemma 2.5. *In the situation of Lemma 2.4, for* $t \in T$ *there exist unique* $\mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,t} \in \mathbb{Z}$ *for* $g \in \mathbb{Z}_{\geq 0}$*, and* $\mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,t} = 0$ *for* $g \gg 0$*, such that*

$$\sum_{i \in \mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi_{\mathcal{M}})|_t)y^i = \sum_{g \geq 0} \mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,t}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}.$$

For each $g \in \mathbb{Z}_{\geq 0}$, we have the constructible function on T

$$\mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,-} \colon T \to \mathbb{Z}, \ t \mapsto \mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,t}.$$

Then the integer $\mathrm{GV}_{g,\mathcal{M}/T}$ in (2.9) is written as

$$\mathrm{GV}_{g,\mathcal{M}/T} = \int_T \mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,-} \ de. \tag{2.10}$$

In genus zero, our GV type invariants are described in terms of Behrend's constructible function. For the perverse sheaf $\phi_{\mathcal{M}}$ in Theorem 2.3, the *Behrend constructible function* [Beh09] on M is defined by

$$\nu_M \colon M \to \mathbb{Z}, \ p \mapsto \chi(\phi_{\mathcal{M}}|_p). \tag{2.11}$$

The constructible function ν_M is independent of the choice of orientation data. Indeed, it is proved in [Beh09] that ν_M only depends on the scheme structure of M.

Lemma 2.6. *We have the identities*

$$\mathrm{GV}_{0,\mathcal{M}/T} = \int_M \nu_M \ de, \ \mathrm{GV}^{\mathrm{loc}}_{0,\mathcal{M}/T,t} = \int_{\pi^{-1}(t)} \nu_M \ de.$$

In particular, $\mathrm{GV}_{0,\mathcal{M}/T}$*,* $\mathrm{GV}^{\mathrm{loc}}_{0,\mathcal{M}/T,t}$ *are independent of the choice of orientation.*

Proof. By substituting $y = -1$ to (2.9), we obtain $\mathrm{GV}_{0,\mathcal{M}/T} = \chi(\phi_{\mathcal{M}})$. Therefore the first identity holds. The second identity also holds in the similar way. □

2.4. **Strictly Calabi-Yau conditions.** In this subsection, we introduce a certain CY condition for d-critical schemes. While not needed for our definitions, it is convenient for proofs later on.

Definition 2.7. *(i) A d-critical scheme (M, s) is called strictly CY if there is a global d-critical chart*

$$(M, J, f, i),\ i\colon M \hookrightarrow J,\ f\colon J \to \mathbb{A}^1 \tag{2.12}$$

of (M, s) such that $K_J = \mathcal{O}_J$. Here i is a closed embedding and f is a regular function. A d-critical chart (2.12) is called a CY d-critical chart.

(ii) A d-critical scheme (M, s) with a projective morphism $\pi\colon M^{\mathrm{red}} \to T$ is called strictly CY at $t \in T$ if there is an open neighborhood $t \in U \subset T$ such that (M_U, s_U) is strictly CY. Here $M_U := \iota(\pi^{-1}(U))$, $\iota\colon M^{\mathrm{red}} \hookrightarrow M$ is the natural closed embedding, and $s_U = s|_{M_U}$.

Remark 2.8. *If a d-critical scheme (M, s) is strictly CY, the line bundle $i^*K_J = \mathcal{O}_M$ together with the identity map* $\mathrm{id}\colon \mathcal{O}_M^{\otimes 2} \to \mathcal{O}_M$ *gives a CY orientation (see Definition A.2) of (M, s).*

In the situation of Definition 2.7 (ii), let us take a CY d-critical chart (M_U, J, f, i). Moreover, let us make the additional assumption that the ring $H^0(\mathcal{O}_J)$ is finitely generated. The function f factors through the affinization

$$f\colon J \xrightarrow{\pi_J} T := \operatorname{Spec} H^0(\mathcal{O}_J) \xrightarrow{g} \mathbb{A}^1.$$

Suppose that $U \subset T$ is affine. The Stein factorization of $\pi\colon M_U^{\mathrm{red}} \to U$ is given by

$$\pi\colon M_U^{\mathrm{red}} \xrightarrow{\pi_1} \overline{U} := \operatorname{Spec} H^0(\mathcal{O}_{M_U^{\mathrm{red}}}) \xrightarrow{\pi_2} U.$$

By the property of the affinization, we have the commutative diagram

$$\begin{tikzcd} M_U^{\mathrm{red}} \arrow[r, hook, "\iota"] \arrow[d, "\pi_1"'] & M_U \arrow[r, hook, "i"] & J \arrow[d, "\pi_J"'] \arrow[dr, "f"] & \\ \overline{U} \arrow[rr, "i'"] & & T \arrow[r, "g"] & \mathbb{A}^1. \end{tikzcd} \tag{2.13}$$

In the proof of Theorem 1.3, 1.4, we will show the strictly CY conditions for moduli of one dimensional sheaves and use the diagrams as above.

Remark 2.9. *Let $\mathcal{M}_U = (M_U, s_U, i^*K_J)$ be the oriented d-critical scheme given by the CY d-critical chart (M_U, J, i, f), and $\phi_{\mathcal{M}_U}$ the perverse sheaf given in Theorem 2.3. Although $\phi_{\mathcal{M}_U}$ is not necessary pure, one can use the BBD decomposition theorem* [BBD82] *for $\mathbf{R}\pi_{J*}\,\mathrm{IC}(J)$, the compatibility of ϕ_g with proper push forwards* [Dim04, Proposition 4.2.11], *to show the decomposition*

$$\mathbf{R}\pi_*\phi_{\mathcal{M}_U} \cong \bigoplus_{j \in \mathbb{Z}} {}^p\mathcal{H}^j(\mathbf{R}\pi_*\phi_{\mathcal{M}_U})[-j]. \tag{2.14}$$

By (2.14), one can interpret $\mathrm{GV}^{\mathrm{loc}}_{g,(M,s)/T,t}$ in terms of the character of sl_2-action on the RHS of (2.14). As this fact will not be used in this paper, we omit the details.

3. GV invariants on Calabi-Yau 3-folds

Let X be a smooth quasi-projective CY 3-fold, i.e. there is an isomorphism

$$\mathcal{O}_X \stackrel{\cong}{\to} K_X. \tag{3.1}$$

Below, we fix the isomorphism (3.1) In this section, we define GV invariants on X and formulate the conjectures relating them with PT and GW invariants.

3.1. Definition of GV invariants.

Let

$$\mathrm{Coh}_{\le 1}(X) \subset \mathrm{Coh}(X)$$

be the subcategory consisting of sheaves whose supports are compact and have dimensions less than or equal to one. For an ample divisor ω on X and $0 \neq E \in \mathrm{Coh}_{\le 1}(X)$, the ω-slope $\mu_\omega(E)$ is defined by

$$\mu_\omega(E) := \frac{\chi(E)}{\omega \cdot [E]} \in \mathbb{Q} \cup \{\infty\}.$$

Definition 3.1. *An object $E \in \mathrm{Coh}_{\le 1}(X)$ is called ω-(semi)stable if for any subobject $0 \neq E' \subsetneq E$, we have the inequality $\mu_\omega(E') < (\le) \mu_\omega(E)$.*

Remark 3.2. *For $E \in \mathrm{Coh}_{\le 1}(X)$ with $\chi(E) = 1$, its ω-stability is equivalent to that $\chi(E') \le 0$ for any subsheaf $0 \subsetneq E' \subsetneq E$. In particular, it is independent of a choice of h.*

For $\beta \in H_2(X, \mathbb{Z})$, we denote by

$$\mathrm{Sh}_\beta(X) \tag{3.2}$$

the moduli space of ω-stable $E \in \mathrm{Coh}_{\le 1}(X)$ with $[E] = \beta$, $\chi(E) = 1$. Note that $\mathrm{Sh}_\beta(X)$ is independent of h by Remark 3.2. Also the condition $\chi(E) = 1$ implies that $\mathrm{Sh}_\beta(X)$ is fine, i.e. there is no strictly ω-semistable $E \in \mathrm{Coh}_{\le 1}(X)$ with $[E] = \beta$, $\chi(E) = 1$, and there is a universal sheaf

$$\mathcal{E} \in \mathrm{Coh}(X \times \mathrm{Sh}_\beta(X)).$$

Then it is known that $\mathrm{Sh}_\beta(X)$ admits a canonical d-critical structure:

Theorem 3.3. ([BBBBJ15]) *A choice of the trivialization (3.1) gives a canonical d-critical structure s_{Sh} on $\mathrm{Sh}_\beta(X)$ such that its virtual canonical bundle is given by*

$$K_{\mathrm{Sh}}^{\mathrm{vir}} := \det \Big(\mathbf{R} p_{\mathrm{Sh}*} \mathbf{R}\mathcal{H}om_{X \times \mathrm{Sh}_\beta(X)}(\mathcal{E}, \mathcal{E}) \Big) |_{\mathrm{Sh}_\beta^{\mathrm{red}}(X)}. \tag{3.3}$$

Here $p_{\mathrm{Sh}} \colon X \times \mathrm{Sh}_\beta(X) \to \mathrm{Sh}_\beta(X)$ is the projection.

Remark 3.4. *The canonical d-critical structure in Theorem 3.3 is induced from derived deformation theory. Let $\widehat{\mathrm{Sh}}_\beta(X)$ be the derived moduli space of h-stable sheaves E with $[E] = \beta$, $\chi(E) = 1$. If X is projective, by* [PTVV13], *a choice of (3.1) gives a canonical (-1)-shifted symplectic structure on $\widehat{\mathrm{Sh}}_\beta(X)$, which is written as a Darboux form by* [BBBBJ15], *and induce the canonical d-critical structure on its truncation $\mathrm{Sh}_\beta(X) \subset \widehat{\mathrm{Sh}}_\beta(X)$ in Theorem 3.3.*

Remark 3.5. *Since we only assume that X is quasi-projective, we need a generalization of the result in* [PTVV13] *to the quasi-projective case. If X is written as $(u \neq 0)$ for a smooth projective variety Y and $0 \neq u \in H^0(-K_Y)$, Bussi* [Bus, Theorem 5.2] *shows that $\widehat{\mathrm{Sh}}_\beta(X)$ has a canonical (-1)-shifted symplectic structure. In general, this follows from the work of Preygel* [Pre, Theorem 3.0.6] *combined with the original argument of* [PTVV13].

Let

$$\pi\colon \mathrm{Sh}_\beta^{\mathrm{red}}(X) \to \mathrm{Chow}_\beta(X) \tag{3.4}$$

be the Hilbert-Chow morphism. Here $\mathrm{Chow}_\beta(X)$ is the Chow variety which parametrizes compactly supported effective one-cycles on X with homology class β (see [Kol96]), and the map (3.4) sends a one dimensional sheaf E to the associated fundamental cycle of E. Because $\mathrm{Sh}_\beta(X)$ is a fine moduli space, the morphism (3.4) is a projective morphism.

Remark 3.6. *Here we use the classical definition of the Chow variety, which is a reduced scheme and denoted as* $\mathrm{Chow}'(X)$ *in* [Kol96]. *The existence of the map (3.4) follows from, for example, by the argument of* [Ryd, Corollary 7.15].

The d-critical scheme in Theorem 3.3 always admits a (non-canonical) orientation [NO]. Let us take one of them and consider an oriented d-critical scheme

$$\mathcal{S}h_\beta(X) = (\mathrm{Sh}_\beta(X), s_{\mathrm{Sh}}, (K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2}).$$

In the following definition, we take a special type of orientation data $(K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2}$, called *CY orientation data* (see Definition A.2). We expect that such an orientation data always exist (or at least locally on $\mathrm{Chow}_\beta(X)$, see Conjecture A.7).

Definition 3.7. *We define the invariants $n_{g,\beta} \in \mathbb{Z}$ by (see Lemma 2.4)*

$$n_{g,\beta} := \mathrm{GV}_{g, \mathcal{S}h_\beta(X)/\,\mathrm{Chow}_\beta(X)}\,.$$

The invariant $n_{g,\beta}$ is called the genus g Gopakumar-Vafa invariant with curve class β.

Remark 3.8. *Once we take a CY orientation data, the resulting invariant $n_{g,\beta}$ is independent of an orientation data as long as it is CY (see Lemma A.5).*

The local version is defined in the similar way:

Definition 3.9. *We define the invariants $n_{g,\gamma}^{\mathrm{loc}} \in \mathbb{Z}$ for $g \in \mathbb{Z}_{\geq 0}$ by (see Lemma 2.5)*

$$n_{g,\gamma}^{\mathrm{loc}} := \mathrm{GV}^{\mathrm{loc}}_{g, \mathcal{S}h_\beta(X)/\,\mathrm{Chow}_\beta(X), \gamma}\,.$$

The invariant $n_{g,\gamma}^{\mathrm{loc}}$ is called the local genus g Gopakumar-Vafa invariant at γ.

Remark 3.10. *Similarly to the global case, the local GV invariant $n_{g,\gamma}^{\mathrm{loc}}$ is defined using a CY orientation data. However for the local case, we only need such an orientation data locally on* $\mathrm{Chow}_\beta(X)$ *near γ (i.e. only need* *to assume that* $\mathrm{Sh}_\beta(X)$ *is CY at* γ *in Definition A.2 (iii)). If the latter condition holds for any* $\gamma \in \mathrm{Chow}_\beta(X)$*, (i.e. (3.4) is a CY fibration in Definition A.2 (iv)), we can define the global GV invariants by the identity (see Remark A.6)*

$$n_{g,\beta} = \int_{\mathrm{Chow}_\beta(X)} n^{\mathrm{loc}}_{g,-} \, de.$$

If $\mathrm{Sh}_\beta(X)$ *has a global CY orientation data, the above definition coincides with the one in Definition 3.7.*

If X is projective, our genus zero GV invariant agrees with Katz's definition [Kat08]:

Lemma 3.11. *If* X *is projective, we have the identity*

$$n_{0,\beta} = \int_{[\mathrm{Sh}_\beta(X)]^{\mathrm{vir}}} 1.$$

Proof. The assumption implies that $\mathrm{Sh}_\beta(X)$ is also projective and the RHS makes sense. Then the lemma follows from Lemma 2.6 together with the result of Behrend [Beh09] that the degree of the virtual class of $\mathrm{Sh}_\beta(X)$ coincides with the weighted Euler number of its Behrend constructible function. □

3.2. **Conjectures on the relation to PT/GW invariants.** Let X be a smooth quasi-projective CY 3-fold as before. By definition, a *stable pair* introduced by Pandharipande-Thomas [PT09] consists of a pair

$$(F, s), \ F \in \mathrm{Coh}_{\le 1}(X), \ s \colon \mathcal{O}_X \to F$$

where F is pure one-dimensional and s is surjective in dimension one. For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, let

$$P_n(X, \beta)$$

be the moduli space of stable pairs (F, s) on X with $[F] = \beta$ and $\chi(F) = n$. By [BBBBJ15], the moduli space of stable pairs $P_n(X, \beta)$ admits a canonical d-critical structure s_P whose virtual canonical line bundle K_P^{vir} is given similarly to (3.3) for universal stable pairs. By [NO], it always has a (non-canonical) orientation $(K_P^{\mathrm{vir}})^{1/2}$. An oriented d-critical scheme

$$\mathcal{P}_n(X, \beta) = (P_n(X, \beta), s_P, (K_P^{\mathrm{vir}})^{1/2})$$

determines the sheaf of vanishing cycles $\phi_{\mathcal{P}}$ on $P_n(X, \beta)$ by Theorem 2.3. Similarly to (2.11), let

$$\nu_P \colon P_n(X, \beta) \to \mathbb{Z}, \ p \mapsto \chi(\phi_{\mathcal{P}}|_p)$$

be the Behrend constructible function on $P_n(X, \beta)$. The PT invariant is defined by

$$P_{n,\beta} := \int_{P_n(X,\beta)} \nu_P \, de.$$

If X is projective, it coincides with the integration of the zero-dimensional virtual class of $P_n(X, \beta)$ [Beh09].

Remark 3.12. *We don't need the d-critical structure on $P_n(X,\beta)$ to define $P_{n,\beta}$, as we only need the Behrend constructible function ν_P to define it. The d-critical structure on $P_n(X,\beta)$ and the associated vanishing cycle sheaf will be used later, e.g. in Subsection 4.2.*

By [PT09, Lemma 3.1], there exist unique integers

$$n^P_{g,\beta} \in \mathbb{Z},\ \beta > 0,\ g \in \mathbb{Z}$$

such that the logarithm of the generating series of stable pairs is written as

$$\log\left(1 + \sum_{\beta>0, n\in\mathbb{Z}} P_{n,\beta}(-q)^n t^\beta\right) = \sum_{\beta>0} \sum_{g\in\mathbb{Z}, k\geq 1} \frac{n^P_{g,\beta}}{k}(-1)^{g-1}(q^{\frac{k}{2}} - q^{-\frac{k}{2}})^{2g-2} t^{k\beta}. \tag{3.5}$$

Here $\beta > 0$ means that β is a homology class of an effective one cycle on X.

Conjecture 3.13. *Under the situation of Definition 3.7, we have the identity*

$$n^P_{g,\beta} = n_{g,\beta},\ \beta > 0,\ g \in \mathbb{Z}. \tag{3.6}$$

The local PT invariants are also defined in a similar way. For $\gamma \in \mathrm{Chow}_\beta(X)$, let $P^{\mathrm{loc}}_n(X,\gamma) \subset P_n(X,\beta)$ be the closed subset corresponding to stable pairs (F,s) such that the fundamental cycle of F coincides with γ. We define

$$P^{\mathrm{loc}}_{n,\gamma} := \int_{P^{\mathrm{loc}}_n(X,\gamma)} \nu_P\ de.$$

We have the constructible function

$$P^{\mathrm{loc}}_{n,-} \colon \mathrm{Chow}_\beta(X) \to \mathbb{Z},\ \gamma \mapsto P^{\mathrm{loc}}_{n,\gamma}.$$

Similarly to (3.5), there exist locally constructible functions

$$n^{P,\mathrm{loc}}_{g,-} \colon \mathrm{Chow}(X) := \coprod_{\beta>0} \mathrm{Chow}_\beta(X) \to \mathbb{Z},\ g \in \mathbb{Z}$$

such that we have the identity

$$\log\left(1 + \sum_{n\in\mathbb{Z}} P^{\mathrm{loc}}_{n,-} q^n\right) = \sum_{g\in\mathbb{Z}, k\geq 1} (k)_* \frac{n^{P,\mathrm{loc}}_{g,-}}{k}(-1)^{g-1}((-q)^{\frac{k}{2}} - (-q)^{-\frac{k}{2}})^{2g-2}. \tag{3.7}$$

The above identity is interpreted as the identity of $\mathbb{Q}(\!(q)\!)$-valued functions on $\mathrm{Chow}(X)$. Here for functions n_1, n_2 on $\mathrm{Chow}(X)$, the product is defined by

$$n_1 \cdot n_2 := (+)_*(n_1 \boxtimes n_2) \tag{3.8}$$

where $+$ is an obvious addition map of one cycles

$$+\colon \mathrm{Chow}(X) \times \mathrm{Chow}(X) \to \mathrm{Chow}(X).$$

The logarithm of the LHS of (3.7) is taken with respect to the product (3.8). Also $(k)_*$ in the RHS of (3.7) is the push-forward of the map $\gamma \mapsto k\gamma$ on $\mathrm{Chow}(X)$.

Conjecture 3.14. *Under the situation of Definition 3.9, we have the identity*

$$n^{P,\mathrm{loc}}_{g,\gamma} = n^{\mathrm{loc}}_{g,\gamma}, \ \gamma \in \mathrm{Chow}_\beta(X), \ g \in \mathbb{Z}. \tag{3.9}$$

Remark 3.15. *By integrating over the Chow variety, it is easy to see that Conjecture 3.14 implies Conjecture 3.13.*

Remark 3.16. *For an irreducible one cycle γ, Conjecture 3.14 is equivalent to Conjecture 1.2.*

Remark 3.17. *The identity (3.6) in particular implies $n^P_{g,\beta} = 0$ for $g < 0$, which is the strong rationality conjecture in* [PT09, Conjecture 3.14]*. By* [Tod12, Theorem 6.4]*, the above vanishing is equivalent to the multiple cover conjecture of generalized DT invariants of one dimensional semistable sheaves (see* [Tod12, Conjecture 6.3]*), and in this case the identity (3.6) holds for $g = 0$. Similarly in the local case, the vanishing $n^{P,\mathrm{loc}}_{g,\gamma} = 0$ for $g < 0$ is equivalent to the multiple cover conjecture of local generalized DT invariants (see* [Tod14, Conjecture 4.13]*) and in this case (3.9) holds for $g = 0$.*

If X is projective, the comparison with Gromov-Witten invariants is also formulated in a similar way. Let

$$\mathrm{GW}_{g,\beta} \in \mathbb{Q}$$

be the genus g Gromov-Witten invariant on X with curve class β. Then there exist (a priori rational numbers)

$$n^{GW}_{g,\beta} \in \mathbb{Q}, \ g \in \mathbb{Z}_{\geq 0}, \ \beta > 0$$

satisfying the identity:

$$\sum_{\beta>0, g\geq 0} \mathrm{GW}_{g,\beta} \lambda^{2g-2} t^\beta = \sum_{\beta>0, g\geq 0, k\geq 1} \frac{n^{GW}_{g,\beta}}{k} \left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2} t^{k\beta}.$$

Conjecture 3.18. *Suppose that X is a smooth projective CY 3-fold. Then under the situation of Definition 3.7, we have the identity*

$$n^{GW}_{g,\beta} = n_{g,\beta}, \ \beta > 0, \ g \in \mathbb{Z}_{\geq 0}.$$

Remark 3.19. *Suppose that we know either $n^P_{g,\beta} = 0$ for $g < 0$ or $n^{GW}_{g,\beta} = 0$ for $g \gg 0$. Then the conjectural GW/DT(PT) correspondence* [MNOP06, PT09] *implies $n^P_{g,\beta} = n^{GW}_{g,\beta}$.*

3.3. Independence of Euler characteristic.

For $k \in \mathbb{Z}$, let

$$\mathrm{Sh}_{\beta,k}(X)$$

be the moduli space of ω-stable $E \in \mathrm{Coh}_{\leq 1}(X)$ satisfying $[E] = \beta$, $\chi(E) = k$. We note that, for $k \neq 1$, the moduli space $\mathrm{Sh}_{\beta,k}(X)$ may depend on a choice of ω. Similarly to the $k = 1$ case, we have the Hilbert-Chow map

$$\pi_k \colon \mathrm{Sh}^{\mathrm{red}}_{\beta,k}(X) \to \mathrm{Chow}_\beta(X).$$

In Subsection 3.1, we used the moduli space for $k = 1$ and the map $\pi = \pi_1$ to define GV invariants. For other value of k, assuming that $\mathrm{Sh}_{\beta,k}(X)$ is fine and π_k is a CY fibration (see Definition A.2), we expect that we can also use $\mathrm{Sh}_{\beta,k}(X)$ to define GV invariants. Namely by replacing $\mathrm{Sh}_{\beta}(X)$ by $\mathrm{Sh}_{\beta,k}(X)$ in Definition 3.7 and Definition 3.9, we have the GV type invariants

$$n_{g,\beta,k,\omega} \in \mathbb{Z}, \quad n^{\mathrm{loc}}_{g,\gamma,k,\omega} \in \mathbb{Z} \tag{3.10}$$

for $\beta > 0$ and $\gamma \in \mathrm{Chow}_{\beta}(X)$ respectively. A priori, the invariants (3.10) may also depend on ω.

Conjecture 3.20. *Assuming that* $\mathrm{Sh}_{\beta,k}(X)$ *is fine, we have the following:*

(i) π_k *is a CY fibration. In particular, the invariants (3.10) are defined.*
(ii) *The invariants (3.10) are independent of* ω*.*
(iii) *The invariants (3.10) are independent of* k*.*

In genus zero, Conjecture 3.20 (ii) is known to be true by a wall-crossing argument (see [JS12, Theorem 6.6]), and Conjecture 3.20 (iii) is a special case of the multiple cover conjecture for generalized DT invariants (see [Tod14, Conjecture 4.13]). By another wall-crossing argument, Conjecture 3.20 (iii) for a primitive one cycle γ is proved when $g = 0$ (see [Tod, Lemma 2.12]), and we expect that a similar argument may be applied for $g > 0$. For an irreducible one cycle γ, Conjecture 3.20 (iii) should follow along with the same argument of [PT10, Proposition 2.1] by tensoring a local line bundle with degree one on the support of γ.

4. PT/GV correspondence for locally planar curves

In this section, we explain an approach for proving Conjecture 3.14 for integral planar curves. More precisely, we show that the existence of strictly CY d-critical chart (see Definition 2.7) together with the results of [MY14, MS13] implies the conjecture.

4.1. GV formula for locally versal deformations.

Let C be an integral projective curve with at worst planar singularities. Let g be the arithmetic genus of C, and $\{c_1, \ldots, c_k\}$ the singular set of C. Let

$$\pi_T \colon \mathcal{C} \to T \tag{4.1}$$

be a flat family of curves with smooth base T such that $C = \pi_T^{-1}(0)$ for $0 \in T$. We recall the notion of locally versal family:

Definition 4.1. *A family (4.1) is called locally versal at* $0 \in T$ *if the natural map*

$$\mathrm{Tan}_0(T) \to \prod_{i=1}^{k} \mathrm{Tan}_0(\mathrm{Def}(C, c_i)) \tag{4.2}$$

is surjective. Here $\mathrm{Def}(C, c_i)$ *is the miniversal deformation space of the singularity* $c_i \in C$*, and* $\mathrm{Tan}_0(*)$ *is the tangent space at* 0*. If a family (4.1) is locally versal at any* $t \in T$*, it is called a locally versal family.*

We will use the following lemma:

Lemma 4.2. *For any projective flat family of curves $\pi_H \colon \mathcal{C}_H \to H$ with at worst planar singularities, there is a locally versal family of curves $\mathcal{C} \to T$ and a closed immersion $H \hookrightarrow T$ such that $\mathcal{C}_H = \mathcal{C} \times_T H$.*

Proof. Since π_H is projective, there is a closed embedding $\mathcal{C}_H \subset \mathbb{P}^n \times H$ as H-schemes. Since π_H is flat, it induces the morphism of schemes $h \colon H \to \mathrm{Hilb}(\mathbb{P}^n)$, where $\mathrm{Hilb}(\mathbb{P}^n)$ is the Hilbert scheme of one dimensional subschemes in $\mathbb{P}^n$. Let

$$\pi_{\mathrm{Hilb}} \colon \mathcal{C}_{\mathrm{Hilb}} \to \mathrm{Hilb}(\mathbb{P}^n) \tag{4.3}$$

be the universal curve. We check that π_{Hilb} is locally versal at any point in the image of h. For $t \in H$, let us write $C = \pi_H^{-1}(t) \subset \mathbb{P}^n$. Let $I \subset \mathcal{O}_{\mathbb{P}^n}$ be the ideal sheaf of C. By the exact sequence

$$0 \to I/I^2 \to \Omega_{\mathbb{P}^n}|_C \to \Omega_C \to 0$$

we have the exact sequence

$$\begin{aligned} \mathrm{Hom}(I/I^2, \mathcal{O}_C) \to \mathrm{Ext}^1_C(\Omega_C, \mathcal{O}_C) \to H^1(C, T_{\mathbb{P}^n}|_C) \\ \to \mathrm{Ext}^1_C(I/I^2, \mathcal{O}_C) \to \mathrm{Ext}^2_C(\Omega_C, \mathcal{O}_C) \to 0. \end{aligned}$$

By replacing the embedding $\mathcal{C}_H \subset \mathbb{P}^n \times H$ if necessary, we may assume that $H^1(C, \mathcal{O}_C(1)) = 0$ holds for any choice of $t \in H$. Then the vanishing $H^1(C, T_{\mathbb{P}^n}|_C) = 0$ also holds. Since the singularities of C are planar, we also have $\mathrm{Ext}^2_C(\Omega_C, \mathcal{O}_C) = 0$. By the above exact sequence, we have $\mathrm{Ext}^1_C(I/I^2, \mathcal{O}_C) = 0$, therefore $\mathrm{Hilb}(\mathbb{P}^n)$ is smooth at $h(t)$. Moreover we have the surjections

$$\mathrm{Hom}(I/I^2, \mathcal{O}_C) \twoheadrightarrow \mathrm{Ext}^1_C(\Omega_C, \mathcal{O}_C) \twoheadrightarrow H^0(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)).$$

The above map is identified with the map (4.2) for the family (4.3), therefore π_{Hilb} is locally versal at t.

Let $h(H) \subset U \subset \mathrm{Hilb}(\mathbb{P}^n)$ be an open subset on which the corresponding one dimensional subscheme have at worst planar singularities, and π_{Hilb} is locally versal. We set $T = H \times U$, and $\mathcal{C} = H \times \pi_{\mathrm{Hilb}}^{-1}(U)$. Then $(\mathrm{id} \times \pi_{\mathrm{Hilb}}) \colon \mathcal{C} \to T$ is a locally versal family. By taking the embedding $(\mathrm{id}, h) \colon H \hookrightarrow T$, the lemma follows. $\square$

For a locally versal family (4.1), let

$$\pi^{[n]} \colon \mathcal{C}^{[n]} \to T, \ \pi_J \colon \overline{J} \to T$$

be the π_T-relative Hilbert scheme of n-points, π_T-relative rank one torsion free sheaves with Euler characteristic one, respectively. The following is the main result of [MY14, MS13].

Theorem 4.3. ([MY14, MS13]) *Both of $\mathcal{C}^{[n]}$ and $\overline{J}$ are smooth for any $n \geq 1$. After replacing T by its étale cover if necessary, we have the identity in $K(\mathrm{Perv}(T))((q))$:*

$$\sum_{n \geq 0} \mathbf{R}\pi^{[n]}_* \, \mathrm{IC}(\mathcal{C}^{[n]}) q^{n+1-g} = \frac{q}{(1+q)^2} \sum_{i \in \mathbb{Z}} {}^p\mathcal{H}^i(\mathbf{R}\pi_{J*} \, \mathrm{IC}(\overline{J})) q^i. \tag{4.4}$$

Let $g\colon T \to \mathbb{A}^1$ be a regular function. We define $f^{[n]}$ and f_J by the commutative diagram

$$\begin{tikzcd}
\overline{J} \arrow[dr, "\pi_J"'] \arrow[drr, "f_J"] & & \\
& T \arrow[r, "g"] & \mathbb{A}^1 \\
\mathcal{C}^{[n]} \arrow[ur, "\pi^{[n]}"] \arrow[urr, "f^{[n]}"'] & &
\end{tikzcd}$$

Let us consider the associated vanishing cycle sheaves:

$$\begin{aligned}
\phi^{[n]} &:= \phi_{f^{[n]}}(\mathrm{IC}(\mathcal{C}^{[n]})) \in \mathrm{Perv}(\mathcal{C}^{[n]}), \\
\phi_J &:= \phi_{f_J}(\mathrm{IC}(\overline{J})) \in \mathrm{Perv}(\overline{J}).
\end{aligned} \tag{4.5}$$

Applying Theorem 4.3, we have the following lemma:

Lemma 4.4. *After replacing T by its étale cover if necessary, we have the identity in $K(\mathrm{Perv}(T))(\!(q)\!)$:*

$$\sum_{n\geq 0} \mathbf{R}\pi_*^{[n]}\phi^{[n]}q^{n+1-g} = \frac{q}{(1+q)^2}\sum_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(\mathbf{R}\pi_{J*}\phi_J)q^i. \tag{4.6}$$

Proof. By the compatibility of vanishing cycles with proper push forwards (see [Dim04, Proposition 4.2.11]), we have

$$\phi_g(\mathbf{R}\pi_*^{[n]}\,\mathrm{IC}(\mathcal{C}^{[n]})) = \mathbf{R}\pi_*^{[n]}\phi^{[n]},\ \ \phi_g(\mathbf{R}\pi_{J*}\,\mathrm{IC}(\overline{J})) = \mathbf{R}\pi_{J*}\phi_J.$$

Since ϕ_g preserves the perverse t-structure, it commutes with the perverse cohomology functor ${}^p\mathcal{H}^i(-)$. Therefore we have

$$\phi_g({}^p\mathcal{H}^i(\mathbf{R}\pi_{J*}\,\mathrm{IC}(\overline{J}))) = {}^p\mathcal{H}^i(\mathbf{R}\pi_{J*}\phi_J).$$

Therefore the lemma follows by applying the vanishing cycle functor ϕ_g to both sides of (4.4). $\square$

4.2. **GV formula for Calabi-Yau 3-folds.** Let X be a smooth quasi-projective CY 3-fold, and $C \subset X$ an integral projective curve with at worst planar singularities with arithmetic genus g. For $\beta = [C] \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, let

$$(P = P_{n+1-g}(X, \beta), s_P),\ (\mathrm{Sh} = \mathrm{Sh}_\beta(X), s_{\mathrm{Sh}})$$

be the d-critical schemes considered in Section 3.2 and Section 3.1 respectively. We have the Hilbert-Chow morphisms

$$\begin{tikzcd}
P^{\mathrm{red}} \arrow[dr, "\pi_P"'] & & \mathrm{Sh}^{\mathrm{red}} \arrow[dl, "\pi_{\mathrm{Sh}}"] \\
& \mathrm{Chow}_\beta(X). &
\end{tikzcd}$$

For an open neighborhood $[C] \in U \subset \mathrm{Chow}_\beta(X)$, let $\mathrm{Sh}_U \subset \mathrm{Sh}$, $P_U \subset P$ be the open subschemes whose underlying spaces are $\pi_{\mathrm{Sh}}^{-1}(U)$, $\pi_P^{-1}(U)$

respectively. Suppose that there exist a locally versal deformation $\pi_T \colon \mathcal{C} \to T$ of C and closed embeddings

$$i_T \colon U \hookrightarrow T, \ i_P \colon P_U \hookrightarrow \mathcal{C}^{[n]}, \ i_{\mathrm{Sh}} \colon \mathrm{Sh}_U \hookrightarrow \overline{J}$$

such that we have the commutative diagrams

$$\begin{tikzcd}
\mathrm{Sh}_U^{\mathrm{red}} \arrow[r, hook, "\iota"] \arrow[d, "\pi_{\mathrm{Sh}}"'] & \mathrm{Sh}_U \arrow[r, hook, "i_{\mathrm{Sh}}"] & \overline{J} \arrow[d, "\pi_J"'] \arrow[dr, "f_J"] & \\
U \arrow[rr, hook, "i_T"] & & T \arrow[r, "g"] & \mathbb{A}^1,
\end{tikzcd}
\quad
\begin{tikzcd}
P_U^{\mathrm{red}} \arrow[r, hook, "\iota"] \arrow[d, "\pi_P"'] & P_U \arrow[r, hook, "i_P"] & \mathcal{C}^{[n]} \arrow[d, "\pi^{[n]}"'] \arrow[dr, "f^{[n]}"] & \\
U \arrow[rr, hook, "i_T"] & & T \arrow[r, "g"] & \mathbb{A}^1
\end{tikzcd}$$

giving d-critical charts

$$(\mathrm{Sh}_U, \overline{J}, f_J, i_{\mathrm{Sh}}), \ (P_U, \mathcal{C}^{[n]}, f^{[n]}, i_P)$$

of $(\mathrm{Sh}_U, s_{\mathrm{Sh}}|_{\mathrm{Sh}_U})$, $(P_U, s_P|_{P_U})$ respectively. Since the π_J-relative canonical line bundle of $\overline{J}$ is trivial (see [MRV, Theorem A]), $(\mathrm{Sh}, s_{\mathrm{Sh}})$ has an orientation data on $\mathrm{Sh}_U^{\mathrm{red}}$ which is trivial as a line bundle, i.e. CY orientation in Definition A.2. Indeed the above diagram for Sh is a strictly CY d-critical chart (see Definition 2.7 and the diagram (2.13)). Therefore the local GV invariant $n_{g,\gamma}^{\mathrm{loc}} \in \mathbb{Z}$ is defined as in Definition 3.9, using a CY orientation data.

Proposition 4.5. *Suppose that the above assumption holds for all $n \geq 0$. Then Conjecture 3.14 holds for $\gamma = [C] \in \mathrm{Chow}_\beta(X)$.*

Proof. We set $\phi^{[n]}, \phi_J$ as in (4.5). By the definition of $P_{n,\gamma}^{\mathrm{loc}}$, we have

$$\begin{aligned}
\sum_{n \geq 0} \chi(\mathbf{R}\pi_*^{[n]} \phi^{[n]}|_\gamma) q^{n+1-g} &= \sum_{n \geq 0} \chi(\phi^{[n]}|_{P_n^{\mathrm{loc}}(X,\gamma)}) q^{n+1-g} \\
&= \sum_{n \in \mathbb{Z}} P_{n,\gamma}^{\mathrm{loc}} q^n.
\end{aligned}$$

Also by the definition of $n_{g,\gamma}^{\mathrm{loc}}$, we have

$$\sum_{i \in \mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_{J*}\phi_J)|_\gamma) q^i = \sum_{g \geq 0} n_{g,\gamma}^{\mathrm{loc}} (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2g}.$$

The proposition follows by taking the Euler characteristics of (4.6) at $i_T(\gamma) \in T$. $\square$

5. GV formula for local surfaces

Let S be a smooth projective surface with $H^1(\mathcal{O}_S) = 0$. In this section, we prove Theorem 1.3 for the non-compact CY 3-fold X

$$p \colon X := \mathrm{Tot}(K_S) \to S. \tag{5.1}$$

This will require a lengthy discussion on the natural obstruction theory for sheaves on S and the d-critical structure for sheaves on X. Once these technical details are in place, we will use them to apply the general approach of the last section.

5.1. **Overview of the proof.** Here we give an outline of the proof of Theorem 1.3. Let Sh_X, P_X be the moduli spaces of one dimensional stable sheaves, stable pairs on X respectively. Our strategy is to find $\mathrm{Chow}(X)$-local d-critical charts of Sh_X, P_X via versal deformations of irreducible curves with at worst planar singularities as in the last section. We construct such d-critical charts by relating these moduli spaces with similar moduli spaces Sh_S, P_S on the *surface* S.

Namely we have the natural projections

$$p_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_S,\ p_*\colon P_X \to P_S.$$

The fiber of the morphism $p_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_S$ at each point $[F] \in \mathrm{Sh}_S$ is the vector space

$$\mathrm{Hom}(F, F \otimes K_S) = \mathrm{Ext}^2_S(F, F)^\vee.$$

If Sh_S is smooth so that $\mathrm{Ext}^2_S(F, F)$ is constant for $[F] \in \mathrm{Sh}_S$, then Sh_X is just a vector bundle on Sh_S. Indeed as $\mathrm{Ext}^2_S(F, F)$ is the obstruction space for the deformations of the sheaf F inside S, the vector bundle $\mathrm{Sh}_X \to \mathrm{Sh}_S$ is nothing but the dual of the obstruction bundle on Sh_S. In general, Sh_X is the *dual obstruction cone* [JT] over Sh_S, determined by a perfect obstruction theory of Sh_S. The property of dual obstruction cones tells us how to write Sh_X as a critical locus. A similar argument also applies to $p_*\colon P_X \to P_S$.

Let H be the Chow variety on S (which is smooth by our assumption $H^1(\mathcal{O}_S) = 0$) with HC maps

$$P_S \to H \leftarrow \mathrm{Sh}_S. \tag{5.2}$$

For each $[C] \in H$, we will take an embedding $H \subset T$ (locally on C), where T is the base of a locally versal deformation $\mathcal{C} \to T$ of C (see Lemma 4.2). Then we will see that the diagram

$$\mathcal{C}^{[n]} \to T \leftarrow \overline{J} \tag{5.3}$$

in the last section restricts to the diagram (5.2) on the closed subscheme $H \subset T$.

As both of H, T are smooth, locally near $[C] \in H$ we can realize H as a zero section of a regular section s of a vector bundle E. Then we can form the following diagram

$$\begin{tikzcd}
\mathcal{C}^{[n]} \times_T E^\vee \arrow[r] \arrow[dr, "f^{[n]}"'] & E^\vee \arrow[d, "g"] & \overline{J} \times_T E^\vee \arrow[l] \arrow[dl, "f_J"] \\
& \mathbb{A}^1 &
\end{tikzcd} \tag{5.4}$$

Here g is the function on $E^\vee$ defined by

$$g(a, e) = (s(a), e),\ a \in T, e \in E^\vee|_a.$$

We will show that, locally near $[C] \in H$ we have smooth surjections

$$P_X \twoheadrightarrow \{df^{[n]} = 0\},\ \mathrm{Sh}_X \twoheadrightarrow \{df_J = 0\},\ \mathrm{Chow}(X) \twoheadrightarrow Q \subset E^\vee \tag{5.5}$$

for some closed subset $Q \subset E^\vee$, compatible with the maps in (5.4) and HC maps on P_X, Sh_X. The fibers of the maps in (5.5) are $H^1(\mathcal{O}_C(C))^\vee$, so they are isomorphisms if $H^1(\mathcal{O}_C(C))=0$. Then the same argument of Proposition 4.5 shows the desired result.

The existence of smooth surjections (5.5) is the most technical part in the proof. For this purpose, we will carefully compare the obstruction theories on Sh_S, P_S induced by their deformation theories with those induced by the embedding of (5.2) into (5.3), and investigate the Chow variety on X.

5.2. **Chow variety on local surfaces.** Let us take an algebraic curve class $\beta \in H_2(S, \mathbb{Z})$. By the assumption $H^1(\mathcal{O}_S)=0$, there is a unique $L_\beta \in \mathrm{Pic}(S)$ such that $c_1(L_\beta)=\beta$. The Chow variety on S is just the linear system

$$\mathrm{Chow}_\beta(S) = |L_\beta|.$$

In this case, $\mathrm{Chow}_\beta(S)$ is also identified with the Hilbert scheme of pure one-dimensional subschemes in S with homology class β. We define the open subscheme

$$H_\beta \subset \mathrm{Chow}_\beta(S)$$

consisting of irreducible one-cycles. Note that any curve in H_β has arithmetic genus g given by

$$g = 1 + \frac{1}{2}\beta(K_S+\beta). \tag{5.6}$$

Let $\mathcal{C}_\beta$ be the universal curve

$$\pi_H \colon \mathcal{C}_\beta \subset S \times H_\beta \to H_\beta.$$

By [KT14, Appendix], there is a perfect obstruction theory on H_β

$$\mathcal{U}_H^\bullet := (\mathbf{R}\pi_{H*}\mathcal{O}_{\mathcal{C}_\beta}(\mathcal{C}_\beta))^\vee \to \mathbb{L}_{H_\beta}.$$

As H_β is smooth, we have the distinguished triangle

$$R^1\pi_{H*}\mathcal{O}_{\mathcal{C}_\beta}(\mathcal{C}_\beta)^\vee[1] \to \mathcal{U}_H^\bullet \to \mathbb{L}_{H_\beta}.$$

For the non-compact CY 3-fold (5.1), we have the push-forward map

$$p_* \colon \mathrm{Chow}_\beta(X) \to \mathrm{Chow}_\beta(S). \tag{5.7}$$

Lemma 5.1. *For $[C] \in H_\beta$, the set of closed points of $(p_*)^{-1}([C])$ is identified with $H^0(\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S)$. Here $\nu \colon \widetilde{C} \to C$ is the normalization of C.*

Proof. Let $C' \subset X$ be an irreducible curve with $p(C')=C$. By taking the diagonal map $C' \hookrightarrow X \times_S X = \mathrm{Tot}(p^*K_S)$, we obtain the section of $H^0(p^*K_S|_{C'}) = H^0(p_*\mathcal{O}_{C'} \otimes K_S)$. Since the normalization ν factors through $p|_{C'}$

$$\nu \colon \widetilde{C} \to C' \stackrel{p|_{C'}}{\to} C$$

we obtain the section of $\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S$. Conversely a section of $\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S$ gives a morphism $\widetilde{C} \to \mathrm{Tot}(\nu^*K_S)$. By composing it with the projection $\mathrm{Tot}(\nu^*K_S) \to X$, we obtain the morphism $\widetilde{C} \to X$ whose image gives a point of $(p_*)^{-1}([C])$. $\square$

5.3. **Moduli spaces of one-dimensional stable sheaves on surfaces.** Let $\mathrm{Sh}_\beta(S)$ be the moduli space of one-dimensional stable sheaves F on S satisfying $[F]=\beta$ and $\chi(F)=1$. We set

$$\mathrm{Sh}_S \subset \mathrm{Sh}_\beta(S)$$

to be the open subscheme consisting of sheaves F whose fundamental cycles are irreducible. Then we have the Hilbert-Chow morphism

$$\rho_{\mathrm{Sh}} \colon \mathrm{Sh}_S \to H_\beta \tag{5.8}$$

sending F to its fundamental cycle.

Remark 5.2. *Since S is a surface the morphism (5.8) is defined without taking the reduced part of Sh_S. Indeed let $\mathcal{F} \in \mathrm{Coh}(S \times T)$ be a flat family of objects in Sh_S. Then there is an exact sequence*

$$0 \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to \mathcal{F} \to 0$$

for vector bundles $\mathcal{F}^i$. By taking the determinants, we obtain the global section of $\det(\mathcal{F}^0) \otimes \det(\mathcal{F}^{-1})^{-1} = L_\beta \boxtimes \mathcal{O}_T$, giving the T-valued point of H_β.

By Lemma 4.2, there is a locally versal deformation $\pi_T \colon \mathcal{C}_T \to T$ and a closed embedding $i \colon H_\beta \hookrightarrow T$ such that we have the Cartesian square

$$\begin{tikzcd} \mathcal{C}_\beta \arrow[r, hook] \arrow[d, "\pi_H"'] \arrow[dr, phantom, "\square"] & \mathcal{C}_T \arrow[d, "\pi_T"] \\ H_\beta \arrow[r, hook, "i"'] & T. \end{tikzcd} \tag{5.9}$$

Let

$$\pi_J \colon \overline{J} \to T$$

be the π_T-relative moduli space of rank one torsion free sheaves with Euler characteristic one, which is non-singular by Theorem 4.3. We have the Cartesian square

$$\begin{tikzcd} \mathrm{Sh}_S \arrow[r, hook] \arrow[d, "\rho_{\mathrm{Sh}}"'] \arrow[dr, phantom, "\square"] & \overline{J} \arrow[d, "\pi_J"] \\ H_\beta \arrow[r, hook, "i"'] & T. \end{tikzcd} \tag{5.10}$$

Since π_J is flat (see [MRV, Theorem C (ii)]), the above diagram in particular implies that Sh_S has at worst locally complete intersection singularities.

We next discuss obstruction theories on Sh_S. Let $\mathcal{C}'_\beta$ be the universal curve over Sh_S

$$\mathcal{C}'_\beta := \mathcal{C}_\beta \times_{H_\beta} \mathrm{Sh}_S \subset S \times \mathrm{Sh}_S$$

and $\mathbb{F}$ the universal sheaf

$$\mathbb{F} \in \mathrm{Coh}(\mathcal{C}'_\beta) \subset \mathrm{Coh}(S \times \mathrm{Sh}_S).$$

By a standard deformation theory of sheaves, we have the perfect obstruction theory on Sh_S

$$\mathcal{U}_{\mathrm{Sh}}^{\bullet} := (\tau_{\geq 1}\mathbf{R}\mathcal{H}om_{p_{\mathrm{Sh}}}(\mathbb{F}, \mathbb{F}))^{\vee}[-1] \to \mathbb{L}_{\mathrm{Sh}_S} \tag{5.11}$$

where $p_{\mathrm{Sh}} \colon \mathrm{Sh}_S \times S \to \mathrm{Sh}_S$ is the projection.

Proposition 5.3. *There is a natural distinguished triangle*

$$\rho_{\mathrm{Sh}}^{*} R^1\pi_{H*}\mathcal{O}_{\mathcal{C}_\beta}(\mathcal{C}_\beta)^{\vee}[1] \to \mathcal{U}_{\mathrm{Sh}}^{\bullet} \to \mathbb{L}_{\mathrm{Sh}_S}. \tag{5.12}$$

Proof. Let us consider the natural morphism in $D^b(S \times \mathrm{Sh}_S)$

$$\mathcal{O}_{\mathcal{C}'_\beta} \otimes p_S^{*}K_S \to \mathbf{R}\mathcal{H}om_{S\times \mathrm{Sh}_S}(\mathbb{F}, \mathbb{F}) \otimes p_S^{*}K_S.$$

By pushing forward to Sh_S and using the Grothendieck duality, we obtain the morphism in $D^b(\mathrm{Sh}_S)$

$$\mathbf{R}\mathcal{H}om_{p_{\mathrm{Sh}}}(\mathbb{F}, \mathbb{F})[1] \to \mathbf{R}\mathcal{H}om_{p_{\mathrm{Sh}}}(\mathcal{O}_{\mathcal{C}'_\beta}, \mathcal{O}_{\mathrm{Sh}_S \times S})[1]. \tag{5.13}$$

Note that the RHS is identified with

$$\mathbf{R}p_{\mathrm{Sh}*}\mathcal{O}_{\mathcal{C}'_\beta}(\mathcal{C}'_\beta) = \rho_{\mathrm{Sh}}^{*}\mathbf{R}\pi_{H*}\mathcal{O}_{\mathcal{C}_\beta}(\mathcal{C}_\beta).$$

By taking the truncations $\tau_{\geq 1}$ of (5.13) and dualizing, we obtain the morphism

$$\rho_{\mathrm{Sh}}^{*}\mathcal{U}_H^{\bullet} \to \mathcal{U}_{\mathrm{Sh}}^{\bullet}.$$

Let $\mathcal{G}_{\mathrm{Sh}}^{\bullet}$ be the cone of the above morphism. Then there is a morphism $\mathcal{G}_{\mathrm{Sh}}^{\bullet} \to \mathbb{L}_{\mathrm{Sh}_S/H_\beta}$ which fits into the morphism of distinguished triangles:

$$\begin{array}{ccccc} \rho_{\mathrm{Sh}}^{*}\mathcal{U}_H^{\bullet} & \longrightarrow & \mathcal{U}_{\mathrm{Sh}}^{\bullet} & \longrightarrow & \mathcal{G}_{\mathrm{Sh}}^{\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ \rho_{\mathrm{Sh}}^{*}\mathbb{L}_{H_\beta} & \longrightarrow & \mathbb{L}_{\mathrm{Sh}_S} & \longrightarrow & \mathbb{L}_{\mathrm{Sh}_S/H_\beta}. \end{array} \tag{5.14}$$

We claim that the morphism $\mathcal{G}_{\mathrm{Sh}}^{\bullet} \to \mathbb{L}_{\mathrm{Sh}_S/H_\beta}$ is a quasi-isomorphism. First we check that $\mathcal{G}_{\mathrm{Sh}}^{\bullet}$ is concentrated on $[-1, 0]$. For a sheaf F giving a closed point $p \in \mathrm{Sh}_S$, let $C \subset S$ be the support of F. By the top distinguished triangle in (5.14), we have the exact sequence

$$0 \to \mathcal{H}^{-2}(\mathcal{G}_{\mathrm{Sh}}^{\bullet}|_p) \to H^1(\mathcal{O}_C(C))^{\vee} \to \mathrm{Ext}_S^1(F, F)^{\vee}.$$

By the Serre duality, the right morphism is identified with the natural morphism

$$H^0(K_S|_C) \to \mathrm{Hom}(F, F \otimes K_S)$$

which is clearly injective as F is a torsion free sheaf on C. Therefore $\mathcal{H}^{-2}(\mathcal{G}_{\mathrm{Sh}}^{\bullet}|_p) = 0$ and $\mathcal{G}_{\mathrm{Sh}}^{\bullet}$ is concentrated on $[-1, 0]$. Also note that the left and the middle morphisms in the diagram (5.14) satisfy that $\mathcal{H}^0(*)$ are isomorphisms and $\mathcal{H}^{-1}(*)$ are surjective. Then an easy diagram chasing shows that $\mathcal{G}_{\mathrm{Sh}}^{\bullet} \to \mathbb{L}_{\mathrm{Sh}_S/H_\beta}$ is a ρ_{Sh}-relative perfect obstruction theory. Its virtual dimension is

$$1 - \chi(F, F) - \chi(\mathcal{O}_C(C)) = g$$

by the Riemann-Roch theorem, where g is the arithmetic genus of C given by (5.6).

On the other hand, since T and $\overline{J}$ in the diagram (5.10) are smooth, it follows that

$$\mathbb{L}_{\mathrm{Sh}_S/H_\beta} = (\pi_J^*\Omega_T \to \Omega_{\overline{J}})|_{\mathrm{Sh}_S}.$$

Therefore $\mathrm{id}\colon \mathbb{L}_{\mathrm{Sh}_S/H_\beta} \to \mathbb{L}_{\mathrm{Sh}_S/H_\beta}$ is also a ρ_{Sh}-relative perfect obstruction theory, whose virtual dimension is also g as π_J has relative dimension g. Therefore $\mathcal{G}^\bullet_{\mathrm{Sh}} \to \mathbb{L}_{\mathrm{Sh}_S/H_\beta}$ must be a quasi-isomorphism.

By taking the cones of the diagram (5.14), we obtain the distinguished triangle (5.12). $\square$

5.4. **Dual obstruction cone.** In general, let M be a complex scheme with a perfect obstruction theory $\mathcal{U}^\bullet \to \mathbb{L}_M$. The *dual obstruction cone* is a cone over M defined by

$$\mathrm{Obs}^*(\mathcal{U}^\bullet) := \mathrm{Spec}_{\mathcal{O}_M}\left(\bigoplus_{i\geq 0}\mathrm{Sym}^i(\mathcal{H}^1(\mathcal{U}^{\bullet\vee}))\right). \tag{5.15}$$

By [JT], the dual obstruction cone (5.15) is locally written as a critical locus of some function on a smooth scheme. Suppose that M is cut out by a section s of a vector bundle $E \to A$ on a smooth scheme A, and the obstruction theory $\mathcal{U}^\bullet$ is given by

$$\begin{tikzcd} \mathcal{U}^\bullet = (E^\vee|_M \arrow[r, "ds"] \arrow[d, "s"'] & \Omega_A|_M) \arrow[d, equal] \\ \mathbb{L}_M = (I/I^2 \arrow[r] & \Omega_A|_M). \end{tikzcd} \tag{5.16}$$

Here $I \subset \mathcal{O}_A$ is the ideal sheaf of M in A. Then the dual obstruction cone (5.15) is written as the critical locus of the function

$$f\colon \mathrm{Tot}(E^\vee) \to \mathbb{A}^1,\ f(a,e) = (s(a), e) \tag{5.17}$$

where $a \in A$ and $e \in E^\vee|_a$ (see [JT, Section 2] for details). Since any perfect obstruction theory is locally of the form (5.16), the dual obstruction cone (5.15) is locally written as a critical locus.

Suppose that M has at worst locally complete intersection singularities. Then $\mathrm{id}\colon \mathbb{L}_M \to \mathbb{L}_M$ is a perfect obstruction theory, and $\mathrm{Obs}^*(\mathbb{L}_M)$ carries a natural d-critical structure which locally is given by the above construction. Indeed, the dual obstruction cone is the underlying scheme of the (-1)-shifted cotangent derived scheme (see [PTVV13, Definition 1.20])

$$\Omega^\bullet_M[-1] \to M$$

and the d-critical structure is induced by the natural (-1)-shifted symplectic structure.

In the above situation, suppose that $\mathcal{U}^\bullet \to \mathbb{L}_M$ is a perfect obstruction theory. Then we have the distinguished triangle

$$\mathcal{V}[1] \to \mathcal{U}^\bullet \to \mathbb{L}_M$$

for a vector bundle $\mathcal{V}$ on M. By taking the dual and the long exact sequence of cohomology, we have the exact sequence

$$0 \to \mathcal{H}^1(\mathbb{L}_M^\vee) \to \mathcal{H}^1(\mathcal{U}^{\bullet\vee}) \to \mathcal{V}^\vee \to 0.$$

Therefore we have the smooth surjection

$$\mathrm{Obs}^*(\mathcal{U}^\bullet) \twoheadrightarrow \mathrm{Obs}^*(\mathbb{L}_M).$$

Assume the perfect obstruction theory on M has a local model as given at the beginning of this section. One can then show that the pullback of the canonical d-critical structure on $\mathrm{Obs}^*(\mathbb{L}_M)$ agrees with the d-critical structure

$$(\mathrm{Obs}^*(\mathcal{U}^\bullet), s_{\mathcal{U}}), \ s_{\mathcal{U}} \in \Gamma(\mathcal{S}^0_{\mathrm{Obs}^*(\mathcal{U}^\bullet)}). \tag{5.18}$$

Indeed, since the underlying scheme structure on M is lci, one can show the dg-scheme structure is locally split on M, which implies the claim. Locally, the d-critical structure can be defined by critical charts, as described in equation (5.17).

5.5. **Moduli spaces of stable sheaves on local surfaces.** We apply the above dual obstruction cone construction to the perfect obstruction theory $\mathcal{U}^\bullet_{\mathrm{Sh}} \to \mathbb{L}_{\mathrm{Sh}_S}$ given in (5.11), and compare it with the moduli space of one-dimensional stable sheaves on $X = \mathrm{Tot}(K_S)$. Let $\mathrm{Sh}_\beta(X)$ be the moduli space of compactly supported one-dimensional stable sheaves E on X with $[p_*E] = \beta$ and $\chi(E) = 1$. Let

$$\mathrm{Sh}_X \subset \mathrm{Sh}_\beta(X)$$

be the open subset corresponding to sheaves E such that the fundamental cycle of p_*E is irreducible.

Lemma 5.4. *We have the following commutative diagram:*

$$\begin{tikzcd} \mathrm{Sh}_X \arrow[rr, "\cong"] \arrow[dr, "p_*"'] & & \mathrm{Obs}^*(\mathcal{U}^\bullet_{\mathrm{Sh}}) \arrow[dl] \\ & \mathrm{Sh}_S. & \end{tikzcd} \tag{5.19}$$

Here the top morphism is a canonical isomorphism, p_* is induced by the projection $p\colon X \to S$ and the right morphism is the natural morphism defined from the cone structure.

Proof. For a closed point $[E] \in \mathrm{Sh}_X$, the sheaf p_*E on S is stable as it is pure and has irreducible support. Therefore we have the morphism $p_*\colon \mathrm{Sh}_X \to \mathrm{Sh}_S$. The fiber of this morphism at $[F] \in \mathrm{Sh}_S$ is given by the $\mathcal{O}_X$-module structures on F, that is $\mathrm{Hom}(F, F \otimes K_S)$. On the other hand, the fiber of the right morphism of (5.19) at $[F] \in \mathrm{Sh}_S$ is given by $\mathrm{Ext}^2_S(F,F)^\vee$ which is isomorphic to $\mathrm{Hom}(F, F \otimes K_S)$. Therefore the fibers of left and right morphisms in (5.19) are canonically identified. It is straightforward to generalize this argument for flat families of sheaves in Sh_S, which shows the existence of a canonical isomorphism in the diagram (5.19). $\square$

By the diagram (5.10), Sh_S has at worst locally complete intersection singularities. Therefore the construction in (5.18) yields the d-critical scheme

$$(\mathrm{Sh}_X, s_{\mathrm{Sh}}), \ s_{\mathrm{Sh}} \in \Gamma(\mathcal{S}^0_{\mathrm{Sh}_X}). \tag{5.20}$$

Let us take the dual and the long exact sequence of cohomologies of (5.12). Then we obtain the exact sequence of sheaves

$$0 \to \mathcal{H}^1(\mathbb{L}^\vee_{\mathrm{Sh}_S}) \to \mathcal{H}^1(\mathcal{U}^{\bullet\vee}_{\mathrm{Sh}}) \to \rho^*_{\mathrm{Sh}} R^1\pi_{H*}\mathcal{O}_{\mathcal{C}_\beta}(\mathcal{C}_\beta) \to 0. \tag{5.21}$$

We define O_H to be the total space of the vector bundle

$$O_H := R^1\pi_{H*}\mathcal{O}_{\mathcal{C}_\beta}(\mathcal{C}_\beta)^\vee \to H_\beta \tag{5.22}$$

on H_β. By the exact sequence (5.21) and Lemma 5.4, the H_β-group scheme O_H acts on Sh_X fiberwise over H_β without fixed points. The quotient space is

$$\eta_{\mathrm{Sh}} \colon \mathrm{Sh}_X \twoheadrightarrow \mathrm{Sh}_X/O_H = \mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_S}). \tag{5.23}$$

By the Riemann-Roch computation, the relative dimension d of η_{Sh} is given by

$$d = \dim H_\beta - \frac{1}{2}\beta^2 + \frac{1}{2}K_S \cdot \beta. \tag{5.24}$$

By the construction of (5.20), the d-critical structure s_{Sh} is pulled back from the smooth surjection η_{Sh}.

Remark 5.5. *One subtlety in this discussion is the compatibility of the d-critical structure s_{Sh} in (5.20) with the d-critical structure $s^{\mathrm{der}}_{\mathrm{Sh}}$ induced by the derived deformation theory as in Remark 3.4. In fact, we expect a stronger matching of (-1)-shifted symplectic structures is known to experts; however, since a reference is unavailable, we sketch a proof of the weaker statement as follows.*

First, both s_{Sh} and $s^{\mathrm{der}}_{\mathrm{Sh}}$ are homogeneous with weight 1 with respect to the natural action of $\mathbb{C}^*$ on Sh_X. Using this, it suffices to show their formal completions agree at sheaves $[E] \in \mathrm{Sh}_X$ which are pushed forward from S, i.e. of the form $E = i_*F$ for a sheaf F on S and the inclusion $i \colon S \hookrightarrow X$ by the zero section.

At any such point, the d-critical structure $s^{\mathrm{der}}_{\mathrm{Sh}}$ is determined formal-locally by the cyclic pairing on the L_∞-algebra $L_{X,E} = \mathbf{R}\mathrm{Hom}_X(E,E)$ induced by Serre duality. In this case, since $[E]$ is $\mathbb{C}^*$-fixed, $L_{X,E}$ inherits an extra grading compatible with the higher operations and the cyclic structure; using this grading, we can identify

$$L_{X,E} = L_{S,F} \oplus L^\vee_{S,F}[1]$$

where $L_{S,F} = \mathbf{R}\mathrm{Hom}_S(F,F)$ and the right-hand side has a cyclic dgla structure from the coadjoint action and the natural pairing. If we use this decomposition to compute $s^{\mathrm{der}}_{\mathrm{Sh}}$ and compare with equation (5.17), we see that $s_{\mathrm{Sh}} = s^{\mathrm{der}}_{\mathrm{Sh}}$.

5.6. **Hilbert-Chow map on Sh_X.** We set

$$\widehat{H}_\beta := p_*^{-1}(H_\beta) \subset \mathrm{Chow}_\beta(X)$$

where p_* is the morphism (5.7). We have the Hilbert-Chow map $\pi_{\mathrm{Sh}} \colon \mathrm{Sh}_X^{\mathrm{red}} \to \widehat{H}_\beta$ for X and the commutative diagram

$$\begin{tikzcd} \mathrm{Sh}_X^{\mathrm{red}} \arrow[r, "p_*"] \arrow[d, "\pi_{\mathrm{Sh}}"'] \arrow[dr] & \mathrm{Sh}_S \arrow[d, "\rho_{\mathrm{Sh}}"] \\ \widehat{H}_\beta \arrow[r, "p_*"'] & H_\beta. \end{tikzcd} \tag{5.25}$$

Below we fix a point

$$c = [C] \in H_\beta \tag{5.26}$$

for an irreducible curve $C \subset S$. Let $\mathrm{Sh}_{X,c}^{\mathrm{red}}$ be the reduced fiber at c of the morphism $\mathrm{Sh}_X^{\mathrm{red}} \to H_\beta$ in the diagram (5.25). Let $\nu \colon \widetilde{C} \to C$ be the normalization. By Lemma 5.1, we obtain the morphism

$$\pi_{\mathrm{Sh},c} \colon \mathrm{Sh}_{X,c}^{\mathrm{red}} \to H^0(\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S). \tag{5.27}$$

Note that the fiber of (5.22) at c is

$$H^1(\mathcal{O}_C(C))^\vee = H^0(K_S|_C).$$

The O_H-action on Sh_X induces the $H^0(K_S|_C)$-action on $\mathrm{Sh}_{X,c}^{\mathrm{red}}$. On the other hand, $H^0(K_S|_C)$ is contained in $H^0(\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S)$, hence acts on it by the translation.

Lemma 5.6. *The morphism (5.27) is equivariant with respect to the $H^0(K_S|_C)$-action.*

Proof. A closed point of $\mathrm{Sh}_{X,c}^{\mathrm{red}}$ consists of a pair (F, s), where $F \in \mathrm{Coh}(S)$ gives a closed point of Sh_S with fundamental cycle c, and s is a morphism $s \colon F \to F \otimes K_S$ (see the proof of Lemma 5.4). Let $\widetilde{F} = \nu^*F/T$ where $T \subset \nu^*F$ is the torsion part. Since $\widetilde{C}$ is smooth, $\widetilde{F}$ is a line bundle on $\widetilde{C}$ and we have the embedding

$$\mathcal{E}nd(F) \subset \nu_*\mathcal{E}nd(\widetilde{F}) = \nu_*\mathcal{O}_{\widetilde{C}}. \tag{5.28}$$

Combined with the inclusion $\mathcal{O}_C \subset \mathcal{E}nd(F)$, we have

$$H^0(K_S|_C) \subset \mathrm{Hom}(F, F \otimes K_S) \subset H^0(\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S). \tag{5.29}$$

The $H^0(K_S|_C)$-action on $\mathrm{Sh}_{X,c}^{\mathrm{red}}$ is given by the translation with respect to the first embedding of (5.29). Also the morphism (5.27) is given by the second embedding of (5.29). Therefore (5.27) is $H^0(K_S|_C)$-equivariant. $\square$

For the normalization $\nu \colon \widetilde{C} \to C$, we set Q_c to be

$$Q_c := \nu_*\mathcal{O}_{\widetilde{C}}/\mathcal{O}_C.$$

Lemma 5.7. *We have*

$$H^0(\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S)/H^0(K_S|_C) = \overline{Q}_c := \mathrm{Ker}\left(Q_c \xrightarrow{u} \Omega_{H_\beta}|_c\right)$$

for some morphism u.

Proof. By the exact sequence
$$0 \to K_S|_C \to \nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S \to Q_c \to 0$$
we obtain the long exact sequence
$$0 \to H^0(K_S|_C) \to H^0(\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S) \to Q_c \to H^1(K_S|_C).$$
Then the lemma follows from the identification
$$H^1(K_S|_C) = H^0(\mathcal{O}_C(C))^\vee = \Omega_{H_\beta}|_c.$$
□

Let $\mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_S})_c^{\mathrm{red}}$ be the reduced fiber of the composition at $c = [C] \in H_\beta$:
$$\mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_S})^{\mathrm{red}} \to \mathrm{Sh}_S \stackrel{\rho_{\mathrm{Sh}}}{\to} H_\beta$$
where the first morphism is the projection. By Lemma 5.6 and Lemma 5.7, we obtain the Cartesian square

$$\begin{tikzcd} \mathrm{Sh}_{X,c}^{\mathrm{red}} \arrow[r] \arrow[d, "\pi_{\mathrm{Sh},c}"'] \arrow[dr, phantom, "\square"] & \mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_S})_c^{\mathrm{red}} \arrow[d] \\ H^0(\nu_*\mathcal{O}_{\widetilde{C}} \otimes K_S) \arrow[r] & \overline{Q}_c \end{tikzcd} \tag{5.30}$$

such that horizontal morphisms are smooth morphisms with fiber $H^0(K_S|_C)$.

5.7. **CY condition for** Sh_X**.** We consider the embedding $i\colon H_\beta \hookrightarrow T$ in the diagram (5.9). We take an open neighborhood U in T of the point (5.26), and a bundle E on it with a section s
$$c \in U \subset T,\ E \to U,\ s \in \Gamma(U, E) \tag{5.31}$$
such that s is a regular section which cuts out H_β, i.e.
$$H_U := H_\beta \cap U = (s = 0) \subset U.$$
By taking the fiber products of the diagrams (5.10), (5.25) with U over T, we obtain the diagrams

$$\begin{tikzcd} \mathrm{Sh}_{S,U} \arrow[r, hook] \arrow[d, "\rho_{\mathrm{Sh}}"'] \arrow[dr, phantom, "\square"] & \overline{J}_U \arrow[d, "\pi_J"] \\ H_U \arrow[r, hook, "i"'] & U, \end{tikzcd} \quad \begin{tikzcd} \mathrm{Sh}_{X,U}^{\mathrm{red}} \arrow[r, "p_*"] \arrow[d, "\pi_{\mathrm{Sh}}"'] \arrow[dr] & \mathrm{Sh}_{S,U} \arrow[d, "\rho_{\mathrm{Sh}}"] \\ \widehat{H}_U \arrow[r, "p_*"'] & H_U. \end{tikzcd} \tag{5.32}$$

Since s is a regular section of E, we have the isomorphism
$$(\pi_J^* E^\vee|_{\mathrm{Sh}_{S,U}} \stackrel{d(\pi_J^* s)}{\longrightarrow} \Omega_{\overline{J}_U}|_{\mathrm{Sh}_{S,U}}) \stackrel{\cong}{\to} \mathbb{L}_{\mathrm{Sh}_{S,U}}.$$
By the argument in Subsection 5.4, we have the commutative diagram

$$\begin{tikzcd} \mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_{S,U}}) \arrow[r, "\cong"] & \{df_J = 0\} \arrow[r, hook] & \overline{J}_U \times_U E^\vee \arrow[d, "\pi_J"'] \arrow[dr, "f_J"] & \\ & & E^\vee \arrow[r, "g"'] & \mathbb{A}^1 \end{tikzcd} \tag{5.33}$$

giving a d-critical chart of $\mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_{S,U}})$. Here g is the function defined as in (5.17), i.e.

$$g(a,e) = (s(a), e), \ a \in U, \ e \in E_a^\vee := E^\vee|_a.$$

By shrinking U if necessary, the canonical line bundle of $\overline{J}_U$ is trivial (see [MRV, Theorem A]). Then the above d-critical structure is strictly CY and (5.33) is a CY d-critical chart (see Definition 2.7).

Lemma 5.8. *Let $\overline{J}_C$ be the fiber of $\pi_J \colon \overline{J} \to T$ at the point (5.26), and E_c the fiber of $E \to U$ at $c \in H_U \subset U$. There is a closed embedding $\overline{Q}_c \subset E_c^\vee$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_S})_c^{\mathrm{red}} \arrow[r, hook] \arrow[d] & \overline{J}_C \times E_c^\vee \arrow[d] \\ \overline{Q}_c \arrow[r, hook] & E_c^\vee. \end{tikzcd} \tag{5.34}$$

Here the top morphism is induced by the embedding (5.33), and the left morphism is given in (5.30).

Proof. By the description (5.23), $\mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_S})_c^{\mathrm{red}}$ is a cone over $\overline{J}_C$ whose fiber at a closed point $[F] \in \overline{J}_C$ is the quotient of $\mathrm{Hom}(F, F \otimes K_S)$ by the action of $H^0(K_S|_C)$. Similarly to Lemma 5.7, we see that

$$\mathrm{Hom}(F, F \otimes K_S)/H^0(K_S|_C) = \mathrm{Ker}\left(\mathcal{E}nd(F)/\mathcal{O}_C \to \Omega_{H_\beta}|_c\right) \tag{5.35}$$

which is a closed subspace of $E_c^\vee$ by the diagram (5.33). We have the embedding (5.28), such that the equality holds if $F = \nu_* L$ for some line bundle L on $\widetilde{C}$. Applying (5.35) for such F, we obtain the embedding $\overline{Q}_c \subset E_c^\vee$ which makes the diagram (5.34) commutative. $\square$

We write $O_{H,U} = O_H \times_{H_\beta} H_U$, where O_H is given by (5.22). By the above lemma, we have the commutative diagram

$$\begin{tikzcd} \mathrm{Sh}_{X,U}^{\mathrm{red}} \arrow[r, "\eta_{\mathrm{Sh}}"] \arrow[d, "\pi_{\mathrm{Sh}}"'] \arrow[dr, phantom, "\square"] & \mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_{S,U}})^{\mathrm{red}} \arrow[r, hook] \arrow[d] & \overline{J}_U \times_U E^\vee \arrow[d, "\pi_J"] \\ \widehat{H}_U \arrow[r, "\eta"] & \widehat{H}_U/O_{H,U} \arrow[r] & E^\vee. \end{tikzcd} \tag{5.36}$$

where the left horizontal morphisms are quotient maps with respect to the free $O_{H,U}$-actions and the right horizontal morphisms are bijections onto their images. Combined with the diagram (5.33), we obtain the following:

Corollary 5.9. *The d-critical scheme $(\mathrm{Sh}_X, s_{\mathrm{Sh}})$ in (5.20) together with the morphism $\mathrm{Sh}_X^{\mathrm{red}} \to \widehat{H}_\beta$ in (5.25) is a CY fibration in the sense of Definition A.2.*

5.8. **GV invariants on local surfaces.** We keep the notation in the previous subsection. Let

$$\phi_{\mathcal{S}h} \in \mathrm{Perv}(\mathrm{Sh}_{X,U})$$

be the perverse sheaf of vanishing cycles on $\mathrm{Sh}_{X,U}$ determined by its canonical d-critical structure with a CY orientation data (see Corollary 5.9). Let

$$\phi_J := \phi_{f_J}(\mathrm{IC}(\overline{J}_U \times_U E^\vee)) \in \mathrm{Perv}(\mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_{S,U}}))$$

be the perverse sheaf of vanishing cycles associated to the CY d-critical chart (5.33).

Lemma 5.10. *We can choose a CY orientation data on* $\mathrm{Sh}_{X,U}$ *such that the following identity holds:*

$$\phi_{\mathcal{S}h} = \eta_{\mathrm{Sh}}^* \phi_J[d] \in \mathrm{Perv}(\mathrm{Sh}_{X,U}).$$

Here η_{Sh} *is the quotient map (5.23), and* d *is the relative dimension of* η_{Sh} *given by (5.24).*

Proof. The lemma follows since the d-critical structure and the virtual canonical line bundle on $\mathrm{Sh}_{X,U}$ are pulled back from $\mathrm{Obs}^*(\mathbb{L}_{\mathrm{Sh}_{S,U}})$. □

Let us take a one-cycle γ on X (see the diagram (5.25))

$$\gamma \in p_*^{-1}(c) \subset \widehat{H}_U \subset \widehat{H}_\beta \subset \mathrm{Chow}_\beta(X) \tag{5.37}$$

where $c = [C] \in H_\beta$ is the point (5.26). By Definition 3.9, the local GV invariant $n_{g,\gamma}^{\mathrm{loc}} \in \mathbb{Z}$ is given by

$$\sum_{i \in \mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_{\mathrm{Sh}*}\phi_{\mathcal{S}h})|_\gamma) y^i = \sum_{g \geq 0} n_{g,\gamma}^{\mathrm{loc}} (y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}.$$

The following lemma obviously follows from the diagram (5.36) together with Lemma 5.10.

Lemma 5.11. *Let* $\overline{\gamma} \in E^\vee$ *be the image of* γ *under the map* $\widehat{H}_U \to E^\vee$ *in the diagram (5.36). Then we have the following identity (see the diagram (5.33)):*

$$\sum_{i \in \mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_{J*}\phi_J)|_{\overline{\gamma}}) y^i = (-1)^d \sum_{g \geq 0} n_{g,\gamma}^{\mathrm{loc}} (y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}.$$

5.9. **Stable pairs on local surfaces.** Let $P_n(S, \beta)$ be the moduli space of stable pairs (F, s) on S with $[F] = \beta$, $\chi(F) = n$. For $n \geq 0$, we set

$$P_S \subset P_{n+1-g}(S, \beta)$$

to be the open subscheme of stable pairs (F, s) such that the fundamental cycle of F is irreducible. Here g is the arithmetic genus of curves in H_β given by (5.6). Then we have the Hilbert-Chow map $\rho_P \colon P_S \to H_\beta$, and the Cartesian square by [PT10, Appendix]

$$\begin{tikzcd} P_S \arrow[r, hook] \arrow[d, "\rho_P"'] \arrow[dr, phantom, "\square"] & \mathcal{C}_T^{[n]} \arrow[d, "\pi^{[n]}"] \\ H_\beta \arrow[r, hook, "i"'] & T. \end{tikzcd} \tag{5.38}$$

Since T is a base of a locally versal family, the relative Hilbert scheme $\mathcal{C}_T^{[n]}$ is non-singular by Theorem 4.3. As $\pi^{[n]}$ is flat (see [MY92, Lemma 2.6]), the diagram (5.38) implies that P_S has only complete intersection singularities. Let

$$\mathbb{I}_S := (\mathcal{O}_{S\times P_S} \to \mathbb{F}) \in D^b(S \times P_S)$$

be a universal pair on $S \times P_S$. By [KT14], we have the perfect obstruction theory

$$\mathcal{U}_P^\bullet := \mathbf{R}\mathcal{H}om_{p_P}(\mathbb{I}_S, \mathbb{F})^\vee \to \mathbb{L}_{P_S}.$$

Here $p_P\colon S \times P_S \to P_S$ is the projection.

Lemma 5.12. *There is a natural distinguished triangle*

$$\rho_P^* R^1\pi_{H*}\mathcal{O}_{\mathcal{C}_\beta}(\mathcal{C}_\beta)^\vee[1] \to \mathcal{U}_P^\bullet \to \mathbb{L}_{P_S}. \tag{5.39}$$

Proof. The proof is similar to Proposition 5.3, so we just give an outline. By [KT14, Appendix], there is a natural morphism $\rho_P^*\mathcal{U}_H^\bullet \to \mathcal{U}_P^\bullet$ such that its cone $\mathcal{G}_P^\bullet$ fits into a distinguished triangle of perfect obstruction theories:

$$\begin{array}{ccccc} \rho_P^*\mathcal{U}_H^\bullet & \longrightarrow & \mathcal{U}_P^\bullet & \longrightarrow & \mathcal{G}_P^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ \rho_P^*\mathbb{L}_{H_\beta} & \longrightarrow & \mathbb{L}_{P_S} & \longrightarrow & \mathbb{L}_{P_S/H_\beta}. \end{array} \tag{5.40}$$

Similarly to the proof of Proposition 5.3, the morphism $\mathcal{G}_P^\bullet \to \mathbb{L}_{P_S/H_\beta}$ is shown to be a quasi-isomorphism. Therefore taking the cones of (5.40) gives the desired distinguished triangle (5.39). □

Let $P_n(X,\beta)$ be the moduli space of stable pairs (E,s) on $X = \mathrm{Tot}(K_S)$ such that $[p_*E] = \beta$ and $\chi(E) = n$. Let

$$P_X \subset P_{n+1-g}(X,\beta)$$

be the open subscheme of stable pairs (E,s) such that the fundamental cycle of p_*E is irreducible. We first observe the following lemma, which is an analogue of Lemma 5.4.

Lemma 5.13. *We have the following commutative diagram:*

$$\begin{tikzcd} P_X \arrow[rr, "\cong"] \arrow[dr, "p_*"'] & & \mathrm{Obs}^*(\mathcal{U}_P^\bullet) \arrow[dl] \\ & P_S. & \end{tikzcd} \tag{5.41}$$

Here the top morphism is a canonical isomorphism, p_* is induced by the projection $p\colon X \to S$ and the right morphism is the natural morphism defined from the cone structure.

Proof. For a stable pair (E,s) giving a closed point of P_X, we have the morphism $p_*s\colon \mathcal{O}_S \to p_*E$ by the adjunction. Note that p_*E is a pure one dimensional sheaf on S. Since the fundamental cycle of p_*E is irreducible and p_*s is non-zero, the pair (p_*E, p_*s) is a stable pair on S. This argument can be generalized to families of stable pairs, so we obtain the morphism $p_*\colon P_X \to P_S$. For a closed point (F,s) in Sh_S, the fiber of $p_*\colon P_X \to P_S$ at (F,s) consists of $\mathcal{O}_X$-module structures of F, i.e. $\mathrm{Hom}(F, F\otimes K_S)$. Indeed if an $\mathcal{O}_X$-module structure of F is given, the morphism $s\colon \mathcal{O}_S \to F$ is extended to an $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X \to F$ by the adjunction. On the other hand, the fiber of the right morphism in (5.41) at (F, s) consists of $\mathrm{Ext}^1_S(I, F)^\vee$, where $I = (\mathcal{O}_S \stackrel{s}{\to} F)$ is the two term complex. We have the exact sequence

$$\mathrm{Ext}^1_S(F, F) \to H^1(F) \to \mathrm{Ext}^1_S(I, F) \to \mathrm{Ext}^2_S(F, F) \to 0.$$

In [PT10, Proposition C.2], the morphism $\mathrm{Ext}^1_S(F, F) \to H^1(F)$ is shown to be surjective. Hence we have the isomorphism $\mathrm{Ext}^1_S(I, F) \xrightarrow{\cong} \mathrm{Ext}^2_S(F, F)$, and the fiber of the right morphism in (5.41) is identified with $\mathrm{Ext}^2_S(F, F)^\vee \cong \mathrm{Hom}(F, F \otimes K_S)$. Then similarly to Lemma 5.4, we obtain the desired isomorphism in the diagram (5.41). $\square$

Similarly to (5.25), we have the commutative diagram

$$\begin{tikzcd} P_X^{\mathrm{red}} \arrow[r, "p_*"] \arrow[d, "\pi_P"'] \arrow[dr] & P_S \arrow[d, "\rho_P"] \\ \widehat{H}_\beta \arrow[r, "p_*"'] & H_\beta \end{tikzcd}$$

where the vertical morphisms are Hilbert-Chow maps. Let us take $\gamma \in \widehat{H}_\beta$ as in (5.37). The local stable pair invariant is given by

$$P^{\mathrm{loc}}_{n+1-g,\gamma} = \int_{\pi_P^{-1}(\gamma)} \nu_P \ de$$

where ν_P is the Behrend function on P_X.

We describe the local stable pair invariant in terms of data (5.31). By the diagram (5.38), we have the isomorphism

$$(\pi^{[n]*} E^\vee|_{P_S} \stackrel{ds}{\to} \Omega_{\mathcal{C}_T^{[n]}}|_{P_S}) \xrightarrow{\cong} \mathbb{L}_{P_S}.$$

Similarly to (5.33), we have the commutative diagram

$$\begin{tikzcd} \mathrm{Obs}^*(\mathbb{L}_{P_S}) \xrightarrow{\cong} \{df^{[n]} = 0\} \arrow[r, hook] & \mathcal{C}_T^{[n]} \times_T E^\vee \arrow[d, "\pi^{[n]}"'] \arrow[dr, "f^{[n]}"] & \\ & E^\vee \arrow[r, "g"'] & \mathbb{A}^1. \end{tikzcd}$$

Let $\phi^{[n]}$ be the associated perverse sheaf of vanishing cycles

$$\phi^{[n]} := \phi_{f^{[n]}}(\mathrm{IC}(\mathcal{C}_T^{[n]} \times_T E^\vee)) \in \mathrm{Perv}(\mathrm{Obs}^*(\mathbb{L}_{P_S})).$$

Lemma 5.14. *By taking d and $\overline{\gamma} \in E^\vee$ as in Lemma 5.11, we have the identity:*

$$P^{\mathrm{loc}}_{n+1-g,\gamma} = (-1)^d \chi(\mathbf{R}\pi^{[n]}_* \phi^{[n]}|_{\overline{\gamma}}). \tag{5.42}$$

Proof. By taking the dual and the long exact sequence of cohomologies of (5.39), we obtain the exact sequence of sheaves

$$0 \to \mathcal{H}^{-1}(\mathbb{L}^\vee_{P_S}) \to \mathcal{H}^1(\mathcal{U}_P^{\bullet\vee}) \to \rho_P^* R^1\pi_{H*}\mathcal{O}_{\mathcal{C}_\beta}(\mathcal{C}_\beta) \to 0.$$

By Lemma 5.13, the total space of the vector bundle (5.22) acts on P_X fiberwise over H_β without fixed points. The quotient space is

$$P_X/O_H = \mathrm{Obs}^*(\mathbb{L}_{P_S}).$$

The canonical d-critical structure on the RHS as in Subsection 5.4 is pulled-back to the d-critical structure on P_X (also see Remark 5.5). Similarly to (5.36), we have the commutative diagram

$$\begin{tikzcd} P^{\mathrm{red}}_{X,U} \arrow[r, "\eta_P"] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\Box"] & \mathrm{Obs}^*(\mathbb{L}_{P_{S,U}})^{\mathrm{red}} \arrow[r, hook] \arrow[d] & \mathcal{C}_U^{[n]} \times_U E^\vee \arrow[d, "\pi^{[n]}"] \\ \widehat{H}_U \arrow[r, "\eta"] & \widehat{H}_U/O_{H,U} \arrow[r] & E^\vee \end{tikzcd} \tag{5.43}$$

giving a d-critical chart of $\mathrm{Obs}^*(\mathbb{L}_{P_{S,U}})$. Here $-_U$ refers to the pull-back by the open immersion (5.31), and the right horizontal morphisms are bijections onto their images. Let ϕ_P be the perverse sheaf on $P_{X,U}$ given by

$$\phi_P := \eta_P^* \phi^{[n]}[d] \in \mathrm{Perv}(P_{X,U}).$$

By the definition of the Behrend function, we have

$$\int_{\pi_P^{-1}(\gamma)} \nu_P \ de = \chi(\mathbf{R}\pi_{P*}\phi_P|_\gamma).$$

Therefore the identity (5.42) follows from the commutative diagram (5.43). □

5.10. **Proof of Theorem 1.3.** By combining the arguments so far, Theorem 1.3 now immediately follows:

Theorem 5.15. *Suppose that $C \subset S$ be an irreducible curve. Then for $\gamma \in \mathrm{Chow}_\beta(X)$ with $p_*\gamma = [C]$, we have the identity*

$$\sum_{n\in\mathbb{Z}} P^{\mathrm{loc}}_{n,\gamma} q^n = \sum_{g \ge 0} n^{\mathrm{loc}}_{g,\gamma} (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2g-2}.$$

Proof. By Lemma 5.11 and Lemma 5.14, the result follows from the argument of Proposition 4.5. □

By taking the integration over $\mathrm{Chow}_\beta(S)$, we also obtain the following:

Corollary 5.16. *Conjecture 3.13 holds for $X = \mathrm{Tot}(K_S)$ with irreducible curve class $\beta \in H_2(X, \mathbb{Z}) = H_2(S, \mathbb{Z})$.*

5.11. **Examples from rigid singular rational curves.** The result of Theorem 5.15 includes many examples where the moduli space Sh_X is singular. We describe what the above argument looks like in some examples arising from rigid singular rational curves on surfaces.

Let S be a smooth projective surface with $H^1(\mathcal{O}_S) = 0$ and

$$C \subset S$$

an irreducible curve whose normal bundle $\mathcal{O}_C(C)$ is a non-trivial degree zero line bundle on C. We assume C is either a nodal rational curve with one node, or a cuspidal rational curve. For example, we can construct such an example as follows: we first embed $C \subset \mathbb{P}^2$ as a cubic curve and blow-up $\mathbb{P}^2$

at 9-points on C in a general position. Then the resulting blow-up $S \to \mathbb{P}^2$ and the strict transform of C to S give such an example.

Let $\beta \in H_2(S, \mathbb{Z})$ denote the homology class of C. Since $\mathcal{O}_C(C)$ is non-trivial, we have

$$H^0(\mathcal{O}_C(C)) = H^1(\mathcal{O}_C(C)) = 0. \tag{5.44}$$

In particular, the Chow variety $\mathrm{Chow}_\beta(S)$ on S is one point. Moreover we have the isomorphism

$$C \xrightarrow{\cong} \mathrm{Sh}_S, \ x \mapsto I_x^\vee.$$

Here $I_x \subset \mathcal{O}_C$ is the ideal sheaf of x in C and $I_x^\vee$ is its (non-derived) dual on C, viewed as a sheaf on S.

Let $\nu \colon \mathbb{P}^1 \to C$ be the normalization. Since $K_S|_C = \mathcal{O}_C(-C)$ is degree zero, we have

$$H^0(\nu^* K_S) = H^0(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C},$$

where this section does not descend to C. By Lemma 5.1, we see that $\mathrm{Chow}_\beta(X) = \mathbb{A}^1$, where $X = \mathrm{Tot}(K_S)$. The origin $0 \in \mathbb{A}^1$ corresponds to $C \subset S \subset X$, and $0 \neq a \in \mathbb{A}^1$ corresponds to the embedding

$$i_a \colon \mathbb{P}^1 \hookrightarrow X$$

whose projection to S is C. The moduli space Sh_X is (set theoretically) identified with the union of $\mathrm{Sh}_S = C$ and $\mathbb{A}^1$. Here $a \in \mathbb{A}^1 \setminus \{0\}$ corresponds to the sheaf $i_{a*}\mathcal{O}_{\mathbb{P}^1}$, and $\mathbb{A}^1$ intersects with C at the singular point $p \in C$. We have the Hilbert-Chow map

$$\pi_{\mathrm{Sh}} \colon \mathrm{Sh}_X^{\mathrm{red}} = C \cup \mathbb{A}^1 \to \mathrm{Chow}_\beta(X) = \mathbb{A}^1$$

which sends C to the origin and restricts to id on $\mathbb{A}^1$. Below we describe Sh_X as critical locus, and compute the local GV invariants

$$n_{g,-}^{\mathrm{loc}} \colon \mathrm{Chow}_\beta(X) = \mathbb{A}^1 \to \mathbb{Z}. \tag{5.45}$$

Let $\pi_T \colon \mathcal{C} \to T$ be a versal deformation of C, with a point $0 \in T$ such that $\pi_T^{-1}(0) = C$. We first treat the case that $p \in C$ is a nodal singularity. In this case, we can take T to be a sufficiently small open neighborhood of $0 \in \mathbb{A}^1$, and

$$\pi_T \colon \mathcal{C} = \{zy^2 = x^3 + zx^2 + tz^3\} \subset \mathbb{P}^2 \times T \to T$$

where $[x : y : z]$ is the homogeneous coordinate of $\mathbb{P}^2$, t is the coordinate of T and the right arrow is the projection. The generic fiber of π_T is a smooth elliptic curve, and the π_T-relative compactified Jacobian is isomorphic to $\mathcal{C}$ itself. Since Sh_S is cut out by $t = 0$ in $\mathcal{C}$, by the diagram (5.33) and the vanishing (5.44), we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Sh}_X \xrightarrow{\cong} \{df = 0\} \arrow[r, hook] & \mathcal{C} \times \mathbb{A}^1 \arrow[d, "\pi_T \times \mathrm{id}"'] \arrow[dr, "f"] & \\
& T \times \mathbb{A}^1 \arrow[r, "g"] & \mathbb{A}^1.
\end{tikzcd}$$

Here g is defined by

$$g(t, u) = tu.$$

The moduli space Sh_X is singular at the node $p \in C$. We have the following affine open neighborhood of $p \in \mathrm{Sh}_X$:

$$\operatorname{Spec} \mathbb{C}[x, y, u]/(y^2 - x^3 - x^2, yu, (3x^2 + 2x)u).$$

This is the critical locus of the following function

$$f : \mathbb{A}^3 \to \mathbb{A},\ (x, y, u) \mapsto u(y^2 - x^3 - x^2).$$

After taking the completion at 0 and coordinate change, the singularity at p is simplified as

$$\widehat{\mathcal{O}}_{\mathrm{Sh}_X, p} \cong \mathbb{C}[[x, y, u]]/(xy, yu, ux).$$

This is the critical locus of the super potential $xyu \in \mathbb{C}[[x, y, u]]$.

As for the local GV invariants (5.45), the result is as follows:

$$n^{\mathrm{loc}}_{0,-} \equiv -1,\ n^{\mathrm{loc}}_{1,-} = \delta_0,\ n^{\mathrm{loc}}_{\geq 2,-} \equiv 0. \tag{5.46}$$

Here $\delta_0(t) = 1$ for $t = 0$ and $\delta_0(t) = 0$ for $t \neq 0$. Indeed we have the perverse decomposition

$$\mathbf{R}(\pi_T \times \mathrm{id})_* \operatorname{IC}(\mathcal{C} \times \mathbb{A}^1) = \operatorname{IC}(T \times \mathbb{A}^1)[1] \oplus V \oplus \operatorname{IC}(T \times \mathbb{A}^1)[-1]$$

where $V = R^1(\pi_T \times \mathrm{id})_* \mathbb{Q}[2]$ is a constructible sheaf on $T \times \mathbb{A}^1$. Applying ϕ_g, we obtain

$$\begin{aligned} \mathbf{R}\pi_{\mathrm{Sh}*}\phi_f &= \phi_g(\mathbf{R}(\pi_T \times \mathrm{id})_* \operatorname{IC}(\mathcal{C} \times \mathbb{A}^1)) \\ &= \mathbb{Q}_0[1] \oplus \phi_g(V) \oplus \mathbb{Q}_0[-1]. \end{aligned}$$

The above perverse decomposition immediately implies (5.46) for $n^{\mathrm{loc}}_{\geq 1,-}$. The computation of $n^{\mathrm{loc}}_{0,-}$ easily follows by the computation of the Behrend function. Note that we don't have to compute $\phi_g(V)$ in the above computation.

We next treat the case that $p \in C$ is a cusp. In this case, we can take T to be a sufficiently small open neighborhood of $0 \in \mathbb{A}^2$, and

$$\pi_T : \mathcal{C} = \{zy^2 = x^3 + t_1 xz + t_2 z^3\} \subset \mathbb{P}^2 \times T \to T.$$

Here (t_1, t_2) is the coordinate of T. Similarly to the nodal case, we have the commutative diagram

$$\begin{tikzcd} \mathrm{Sh}_X \xrightarrow{\cong} \{df = 0\} \arrow[r, hook] & \mathcal{C} \times \mathbb{A}^2 \arrow[d, "\pi_T \times \mathrm{id}"'] \arrow[dr, "f"] & \\ & T \times \mathbb{A}^2 \arrow[r, "g"] & \mathbb{A}^1. \end{tikzcd}$$

Here g is defined by

$$g(t_1, t_2, u_1, u_2) = t_1 u_1 + t_2 u_2.$$

We have the following affine open neighborhood at $p \in \mathrm{Sh}_X$:

$$\operatorname{Spec} \mathbb{C}[x, y, u]/(x^2 u, yu, y^2 - x^3).$$

In this case, $\mathbb{A}^1$ is a double line in Sh_X. The above affine open neighborhood is the critical locus of the following function:

$$f : \mathbb{A}^5 \to \mathbb{A},\ (x, y, t_1, u_1, u_2) \mapsto u_1 t_1 + u_2(y^2 - x^3 - t_1 x).$$

This is simplified as the critical locus of $u(y^2 - x^3) \in \mathbb{C}[x, y, z]$. Similarly to the nodal case, the local GV invariants (5.45), are computed as follows:

$$n_{0,-}^{\mathrm{loc}} \equiv -2,\ n_{1,-}^{\mathrm{loc}} = \delta_0,\ n_{\geq 2,-}^{\mathrm{loc}} \equiv 0.$$

6. Smooth curve case

Let X be a smooth quasi-projective CY 3-fold and $C \subset X$ a smooth projective curve with homology class β and genus g. Note that we no longer are assuming that X is a local surface.

In this section, we apply Proposition 4.5 to prove Conjecture 3.14 at the point

$$\gamma = [C] \in \mathrm{Chow}_\beta(X).$$

6.1. **Moduli space at a smooth one-cycle.**

We set

$$\mathrm{Sh} = \mathrm{Sh}_\beta(X),\ \mathcal{E} \in \mathrm{Coh}(X \times \mathrm{Sh})$$

where $\mathcal{E}$ is a universal sheaf. We will use the following lemma:

Lemma 6.1. *Let* $\mathrm{Sh}' \subset \mathrm{Sh}$ *be the subset consisting of sheaves of the form* j_*L *for a smooth curve* $j\colon Z \hookrightarrow X$ *and* $L \in \mathrm{Pic}(Z)$. *Let* $\mathcal{Z} \subset X \times \mathrm{Sh}$ *be the closed subscheme defined by the following ideal sheaf* $\mathcal{I}_{\mathcal{Z}}$

$$\mathcal{I}_{\mathcal{Z}} := \mathrm{Ker}\left(\mathcal{O}_{X\times \mathrm{Sh}} \to \mathcal{E}nd_{\mathcal{O}_{X\times\mathrm{Sh}}}(\mathcal{E})\right). \tag{6.1}$$

Then $\mathcal{Z}$ *is flat over* Sh *at any point in* Sh'.

Proof. Let us take a point $y \in \mathrm{Sh}'$ corresponding to j_*L for a smooth curve $j\colon Z \hookrightarrow X$ and $L \in \mathrm{Pic}(Z)$. We also take a point $x \in X$ and set

$$z = (x, y) \in X \times \mathrm{Sh}'.$$

We need to show that $\mathcal{O}_{\mathcal{Z},z}$ is a flat $\mathcal{O}_{\mathrm{Sh},y}$-module. Since $\mathcal{E}$ is a universal sheaf, we have an isomorphism $\mathcal{E}|_{X\times\{y\}} \cong j_*L$. Since $(j_*L)_x$ is generated by one element as an $\mathcal{O}_{X,x}$-module, for a sufficiently ample line bundle $\mathcal{L}$ on X there is a morphism

$$s\colon \mathcal{O}_{X\times\mathrm{Sh}} \to \mathcal{E} \otimes p_X^*\mathcal{L}$$

which is surjective at z. Here $p_X\colon X \times \mathrm{Sh} \to X$ is the projection. Therefore the morphism s induces an isomorphism $\mathcal{E}nd_{\mathcal{O}_{X\times\mathrm{Sh},z}}(\mathcal{E}_z) \cong \mathcal{E}_z$ as $\mathcal{O}_{X\times\mathrm{Sh},z}$-modules. Since we have the factorization

$$s_z\colon \mathcal{O}_{X\times\mathrm{Sh},z} \twoheadrightarrow \mathcal{O}_{\mathcal{Z},z} \hookrightarrow \mathcal{E}nd_{\mathcal{O}_{X\times\mathrm{Sh},z}}(\mathcal{E}_z) \cong \mathcal{E}_z$$

and the above composition is surjective, we see that $\mathcal{O}_{\mathcal{Z},z} \cong \mathcal{E}_z$ as $\mathcal{O}_{X\times\mathrm{Sh},z}$-modules. Since $\mathcal{E}$ is flat over Sh, it follows that $\mathcal{O}_{\mathcal{Z},z}$ is a flat $\mathcal{O}_{\mathrm{Sh},y}$-module. □

Let $\mathrm{Hilb}(X)$ be the Hilbert scheme of compactly supported closed subschemes in X. We take the open subscheme

$$[C] \in H \subset \mathrm{Hilb}(X)$$

consisting of smooth subschemes $Z \subset X$ with $\dim Z = 1$, homology class β and arithmetic genus g. Let

$$\begin{tikzcd} \mathcal{C}_H \arrow[r, hook] \arrow[d, "\pi_H"'] & X \times H \arrow[dl, "p_H"] \\ H & \end{tikzcd} \tag{6.2}$$

be the universal curve. We denote by $J_H \to H$ the π_H-relative moduli space of line bundles with Euler characteristic one, which is a smooth abelian fibration with relative dimension g. We take the universal line bundle

$$\mathcal{L} \in \mathrm{Pic}(\mathcal{C}_H \times_H J_H).$$

Let $i \colon \mathcal{C}_H \times_H J_H \hookrightarrow X \times J_H$ be the closed embedding induced by the diagram (6.2). The object

$$i_*\mathcal{L} \in \mathrm{Coh}(X \times J_H) \tag{6.3}$$

is a J_H-flat family of one dimensional stable sheaves on X. The object (6.3) determines the morphism of schemes

$$h \colon J_H \to \mathrm{Sh}.$$

Lemma 6.2. *The subset* $\mathrm{Sh}' \subset \mathrm{Sh}$ *in Lemma 6.1 is open, and the morphism* h *gives an isomorphism of schemes* $h \colon J_H \xrightarrow{\cong} \mathrm{Sh}'$.

Proof. By Lemma 6.1, the subset $\mathrm{Sh}' \subset \mathrm{Sh}$ coincides with the set of points $y \in \mathrm{Sh}$ such that $\mathcal{Z}$ is flat over Sh at y and $\mathcal{Z}_y := \mathcal{Z}|_{X \times y}$ is smooth. Since the latter conditions are open conditions, the subset $\mathrm{Sh}' \subset \mathrm{Sh}$ is open.

By Lemma 6.1, the subscheme

$$\mathcal{Z}' := \mathcal{Z}|_{X \times \mathrm{Sh}'} \subset X \times \mathrm{Sh}'$$

is flat over Sh'. Hence it defines the morphism

$$\pi_{\mathrm{Sh}} \colon \mathrm{Sh}' \to H \tag{6.4}$$

such that $\mathcal{Z}' = \mathcal{C}_H \times_H \mathrm{Sh}'$. Then by the definition of $\mathcal{Z}$ in (6.1), we have

$$\mathcal{E}|_{X \times \mathrm{Sh}'} \in \mathrm{Coh}(\mathcal{Z}') = \mathrm{Coh}(\mathcal{C}_H \times_H \mathrm{Sh}').$$

The above object is a Sh'-flat family of line bundles on the fibers of $\mathcal{C}_H \to H$, hence determines the morphism $h' \colon \mathrm{Sh}' \to J_H$. The morphism h' gives an inverse of h, hence h is an isomorphism. $\square$

6.2. **CY condition for** Sh**.**

Here we prove the CY condition for Sh:

Proposition 6.3. *The d-critical scheme*

$$(\mathrm{Sh} = \mathrm{Sh}_\beta(X), s_{\mathrm{Sh}})$$

in Theorem 3.3 is strictly CY at the point $\gamma = [C] \in \mathrm{Chow}_\beta(X)$ *for a smooth projective curve* $C \subset X$ *(see Definition 2.7).*

Proof. For the universal curve $\pi_H \colon \mathcal{C}_H \to H$ in (6.2), by Lemma 4.2 we can take a locally versal family and a closed embedding

$$\pi_T \colon \mathcal{C} \to T, \ j \colon H \hookrightarrow T$$

such that $\mathcal{C}_H = \mathcal{C} \times_T H$. Let $\pi_J \colon J \to T$ be the π_T-relative moduli space of line bundles with Euler characteristic one. By Lemma 6.2, we have the Cartesian squares:

$$\begin{tikzcd} \mathrm{Sh}^{\mathrm{red}} \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sh}'^{\mathrm{red}} \arrow[l, hook'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sh}' \arrow[r, hook, "i"] \arrow[d, "\pi_{\mathrm{Sh}}"'] \arrow[dr, phantom, "\square"] & J \arrow[d, "\pi_J"] \\ \mathrm{Chow}_\beta(X) & H^{\mathrm{red}} \arrow[l, hook'] \arrow[r] & H \arrow[r, hook, "j"] & T. \end{tikzcd} \tag{6.5}$$

Here π_{Sh} is given in (6.4), the left bottom arrow is an injection induced by the cycle $\mathcal{C}_H \times_H H^{\mathrm{red}}$, and the left top arrow is an open immersion. By the diagram (6.5), it is enough to show the following: there is an open neighborhood $\gamma = [C] \in U \subset T$ and a regular function $g \colon U \to \mathbb{A}^1$ such that by setting the commutative diagram

$$\begin{tikzcd} \mathrm{Sh}'_U \arrow[r, hook, "i"] \arrow[d, "\pi_{\mathrm{Sh}}"'] \arrow[dr, phantom, "\square"] & J_U \arrow[d, "\pi_J"] \arrow[dr, "f_J"] & \\ H_U \arrow[r, hook, "j"] & U \arrow[r, "g"] & \mathbb{A}^1 \end{tikzcd}$$

where $-_U$ refers to the pull-back by $U \subset T$, the data

$$(\mathrm{Sh}'_U, J_U, f_J, i) \tag{6.6}$$

is a d-critical chart of $(\mathrm{Sh}'_U, s_{\mathrm{Sh}}|_{\mathrm{Sh}'_U})$.

Let $I \subset \mathcal{O}_T$ be the ideal sheaf which defines the subscheme $H \subset T$. Since π_J is smooth, the ideal sheaf $\pi_J^* I \subset \mathcal{O}_J$ defines the subscheme $\mathrm{Sh}' \subset J$. By the property of the sheaves $\mathcal{S}_{\mathrm{Sh}'}$ and $\mathcal{S}_H$ (see (2.1)), we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{S}_H) \arrow[r] \arrow[d, "\pi_{\mathrm{Sh}}^*"] & H^0(\mathcal{O}_T/I^2) \arrow[r] \arrow[d, "\pi_J^*"] & H^0(\Omega_T/I \cdot \Omega_T) \arrow[d, "\pi_J^*"] \\ 0 \arrow[r] & H^0(\mathcal{S}_{\mathrm{Sh}'}) \arrow[r] & H^0(\mathcal{O}_J/\pi_J^* I^2) \arrow[r] & H^0(\Omega_J/\pi_J^* I \cdot \Omega_J). \end{tikzcd} \tag{6.7}$$

Here the horizontal arrows are exact sequences of vector spaces. By the derived base change, we have

$$\mathbf{R}\pi_{J*}\mathcal{O}_J \overset{\mathbf{L}}{\otimes} \mathcal{O}_T/I^2 \cong \mathbf{R}\pi_{J*}(\mathcal{O}_J/\pi_J^* I^2).$$

Since each $R^i\pi_{J*}\mathcal{O}_J$ is locally free, it follows that $\pi_{J*}(\mathcal{O}_J/\pi_J^* I^2) \cong \mathcal{O}_T/I^2$, and the middle vertical arrow of (6.7) is an isomorphism. Since we have the exact sequence of locally free sheaves

$$0 \to \pi_J^*\Omega_T \to \Omega_J \to \Omega_{J/T} \to 0$$

we have the injection

$$\pi_J^*\Omega_T/\pi_J^* I \cdot \pi_J^*\Omega_T \hookrightarrow \Omega_J/\pi_J^* I \cdot \Omega_J.$$

Therefore the right vertical arrow of (6.7) is an injection, hence the left vertical arrow of (6.7) is an isomorphism. It follows that there exists $s_H \in H^0(\mathcal{S}_H^0)$ such that the identity $s_{\mathrm{Sh}}|_{\mathrm{Sh}'} = \pi_{\mathrm{Sh}}^* s_H$ holds.

Since π_{Sh} is a smooth surjective morphism, by [Joy15, Proposition 2.8], the section s_H is a d-critical structure of H. By [Joy15, Proposition 2.7], we can find an open neighborhood $\gamma \in U \subset T$ and a regular function $g\colon U \to \mathbb{A}^1$ such that (H_U, U, g, j) is a d-critical chart of $(H_U, s_H|_{H_U})$. Since π_J is a smooth morphism and $s_{\mathrm{Sh}}|_{\mathrm{Sh}'} = \pi_{\mathrm{Sh}}^* s_H$, the data (6.6) obviously gives a desired d-critical chart. □

6.3. **Proof of Theorem 1.4.** The following is the main result of this section, which proves Theorem 1.4:

Theorem 6.4. *For a smooth projective curve $C \subset X$ with genus g, Conjecture 3.14 is true for $\gamma = [C] \in \mathrm{Chow}_\beta(X)$. In this case, we have*

$$n_{h,\gamma}^{\mathrm{loc}} = \begin{cases} (-1)^g \nu_{\mathrm{Sh}}(\mathcal{O}_C), & h = g, \\ 0, & h \neq g. \end{cases} \tag{6.8}$$

Here ν_{Sh} is the Behrend function on $\mathrm{Sh} = \mathrm{Sh}_\beta(X)$.

Proof. By the proof of Proposition 6.3, the moduli space Sh satisfies the assumption in Proposition 4.5 at γ. As for the moduli space of stable pairs let

$$P' \subset P = P_{n+1-g}(X, \beta)$$

be the subset consisting of stable pairs (F, s) such that $F = j_* L$ for a smooth subscheme $j\colon Z \hookrightarrow X$ and $L \in \mathrm{Pic}(Z)$. Then the same argument of Lemma 6.1 shows that P' is an open subset of P, and isomorphic to the relative Hilbert scheme of n-points of $\pi_H\colon \mathcal{C}_H \to H$. Then similarly to Lemma 6.3, one can show show that P also satisfies the assumption in Proposition 4.5 at γ. Therefore applying Proposition 4.5, Conjecture 3.14 is true in this case. The formula (6.8) for $n_{h,[C]}^{\mathrm{loc}}$ follows from the GV formula for stable pairs given in [PT10, Proposition 3.6]. □

7. Comparison with the former definitions

In this section, we review the previous definitions of GV invariants [HST01, KL] using $sl_2 \times sl_2$-actions, and compare them with our definition. We also give an example that the previous definitions do not match with the predicted answer.

7.1. $sl_2 \times sl_2$**-action.** Let X be a smooth projective CY 3-fold. As in Subsection 3.1, we take the moduli space of one-dimensional stable sheaves on X with its canonical d-critical structure

$$(\mathrm{Sh}_\beta(X), s_{\mathrm{Sh}}) \tag{7.1}$$

and the Hilbert-Chow map

$$\pi\colon \mathrm{Sh}_\beta^{\mathrm{red}}(X) \to \mathrm{Chow}_\beta(X). \tag{7.2}$$

Let $\mathrm{MHM}(\mathrm{Sh}_\beta(X))$ be the abelian category of polarized mixed Hodge modules on $\mathrm{Sh}_\beta(X)$, whose basics we refer to [Sai, Sch]. We take a perverse sheaf ϕ on $\mathrm{Sh}_\beta(X)$ which underlies a polarized mixed Hodge module, i.e. there is an object $\phi^H \in \mathrm{MHM}(\mathrm{Sh}_\beta(X))$ such that $\mathrm{rat}(\phi^H) = \phi$ for the forgetful functor

$$\mathrm{rat}\colon \mathrm{MHM}(\mathrm{Sh}_\beta(X)) \to \mathrm{Perv}(\mathrm{Sh}_\beta(X)).$$

We say that ϕ is *pure* if ϕ^H is a pure Hodge module. If ϕ is pure, then we have the BBD decomposition theorem [BBD82]

$$\mathbf{R}\pi_*\phi \cong \bigoplus_{i\in\mathbb{Z}} {}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi)[-i].$$

Then the hypercohomology of ϕ decomposes into

$$H^*(\mathrm{Sh}_\beta(X),\phi) = \bigoplus_{i,j} H^{i,j},\ H^{i,j} := H^j(\mathrm{Chow}_\beta(X), {}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi)).$$

Let ω_L be a π-ample divisor on $\mathrm{Sh}_\beta^{\mathrm{red}}(X)$ and ω_R an ample divisor on $\mathrm{Chow}_\beta(X)$. Since each ${}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi)$ is also pure (see [Sch, Theorem 16.1]), we have the Hard-Lefschetz isomorphisms

$$\omega_L^i\colon H^{-i,j} \xrightarrow{\cong} H^{i,j},\ \omega_R^j\colon H^{i,-j} \xrightarrow{\cong} H^{i,j}.$$

The above isomorphisms define the $sl_2 \times sl_2$-action on $H^*(\mathrm{Sh}_\beta(X),\phi)$. The multiplication by ω_L defines the left sl_2-action, and the multiplication by ω_R defines the right sl_2-action.

Let I_g be the sl_2-representation given by

$$I_g = IH^*(A,\mathbb{Q})$$

where A is a g-dimensional abelian variety with its sl_2-action given by the Hard-Lefschetz theorem. For $2j \in \mathbb{Z}$, let (j) be the unique irreducible sl_2-representation with dimension $2j+1$. The sl_2-representation I_g is written as

$$I_g = \left(\left(\frac{1}{2}\right) \oplus 2(0)\right)^{\otimes g}.$$

By the Clebsch-Gordan rule, one can write

$$H^*(\mathrm{Sh}_\beta(X),\phi) = \bigoplus_{g\geq 0} (I_g)_L \otimes (R_g)_R \tag{7.3}$$

for some virtual right sl_2-representation R_g. Here $-_L$, $-_R$ refer to left, right sl_2-representations respectively. We can write R_g as

$$R_g = \sum_{2j\in\mathbb{Z}} R_{g,j} \otimes (j),\ R_{g,j} \in K(\mathrm{Vect}(\mathbb{Q})).$$

Following the previous works [GV, HST01], we define

$$n_{g,\beta}(\phi) := \sum_{2j\in\mathbb{Z}} (-1)^{2j}(2j+1)\cdot \dim R_{g,j}. \tag{7.4}$$

The invariants (7.4) are characterized by the character formula, as in Lemma 2.4:

Lemma 7.1. *We have the identity*

$$\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi))y^i = \sum_{g\geq 0} n_{g,\beta}(\phi)(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}. \tag{7.5}$$

Proof. By taking the characters of the identity (7.3), we have the following identity in $K(\mathrm{Vect}(\mathbb{Q}))[x^{\pm 1}, y^{\pm 1}]$:

$$\sum_{i,j\in\mathbb{Z}} H^{i,j}x^j y^i = \sum_{g\geq 0, 2j\in\mathbb{Z}} R_{g,j}(x^{-2j} + x^{-2j+2} + \cdots + x^{2j})(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}.$$

By substituting $x=-1$, we obtain the identity (7.5). □

7.2. **HST and KL definitions.** Both of HST [HST01] and KL [KL] definitions are given by (7.4) for some pure perverse sheaf ϕ. Let us take the normalization and the intersection complex

$$\nu\colon \widetilde{\mathrm{Sh}}_\beta(X) \to \mathrm{Sh}_\beta(X),\ \phi=\nu_* \mathrm{IC}(\widetilde{\mathrm{Sh}}_\beta(X)).$$

The HST definition is given by

$$n_{g,\beta}^{\mathrm{HST}} := n_{g,\beta}(\phi=\nu_* \mathrm{IC}(\widetilde{\mathrm{Sh}}_\beta(X))).$$

The KL definition uses sheaves of vanishing cycles. Let us choose an orientation $(K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2}$ for the d-critical scheme (7.1), and set

$$\mathcal{S}h=(\mathrm{Sh}_\beta(X), s_{\mathrm{Sh}}, (K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2}).$$

Let $\phi_{\mathcal{S}h}$ be the gluing of local sheaves of vanishing cycles as in Theorem 2.3. By [BBD$^+$, Theorem 6.9], the perverse sheaf $\phi_{\mathcal{S}h}$ underlies a polarized mixed Hodge module, but it is not pure in general. Let

$$\mathrm{gr}_W^\bullet(\phi_{\mathcal{S}h}) \in \mathrm{Perv}(\mathrm{Sh}_\beta(X))$$

be the associated graded sheaf with respect to the weight filtration in $\mathrm{MHM}(\mathrm{Sh}_\beta(X))$, which is now a pure perverse sheaf. The KL definition is given by

$$n_{g,\beta}^{\mathrm{KL}} := n_{g,\beta}(\phi=\mathrm{gr}_W^\bullet(\phi_{\mathcal{S}h})). \tag{7.6}$$

Remark 7.2. *By Lemma 7.1, the HST and KL definitions are also given by substituting*

$$\phi=\nu_* \mathrm{IC}(\widetilde{\mathrm{Sh}}_\beta(X)),\ \phi=\mathrm{gr}_W^\bullet(\phi_{\mathcal{S}h})$$

to the formula (7.5) respectively. On the other hand, the character formula (7.5) makes sense even if ϕ is not pure. As in Lemma 2.4, if ϕ is self-dual $\mathbb{D}(\phi)=\phi$, the invariant $n_{g,\beta}(\phi)$ is uniquely determined by the identity (7.5). This is our point of view defining GV invariants in Definition 1.1.

Remark 7.3. *In* [KL]*, Kiem-Li used the semi-normalization of (7.2) as the definition of HC map, following the convention of the Chow variety in* [Kol96]*. Since taking the semi-normalization is a homeomorphism, this step does not affect the definition of the GV invariants.*

7.3. **Dependence on orientation data of KL definition.** In Kiem-Li's paper [KL], they did not specify how to choose an orientation data. Indeed as the following example shows, the KL invariant (7.6) depends on a choice of an orientation data.

Let E be an elliptic curve, which is embedded into a CY 3-fold

$$i\colon E \hookrightarrow X$$

whose normal bundle is written as

$$N_{E/X}=L\oplus L^{-1},\ L\in \mathrm{Pic}^0(E)\setminus\{\mathcal{O}_E\}.$$

Then E is rigid inside X. Let us take the homology class

$$\beta=[E]\in H_2(X,\mathbb{Z}).$$

Suppose that E is the unique curve in X with homology class β, i.e. the Chow variety is a point

$$\mathrm{Chow}_\beta(X) = \{[E]\}.$$

Then we have the isomorphism (see [HST01, Proposition 4.4])

$$E \xrightarrow{\cong} \mathrm{Sh}_\beta(X),\ x \mapsto i_*\mathcal{O}_E(x).$$

Since E is smooth, we have $K_{\mathrm{Sh}}^{\mathrm{vir}} = \mathcal{O}_E$. Then an orientation $(K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2}$ of $(\mathrm{Sh}_\beta(X), s_{\mathrm{Sh}})$ is a 2-torsion element of $\mathrm{Pic}^0(E)$. For the oriented d-critical scheme $\mathcal{S}h = (\mathrm{Sh}_\beta(X), s_{\mathrm{Sh}}, (K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2})$, we have

$$\phi_{\mathcal{S}h} = \mathcal{L}[1]$$

for a rank one local system $\mathcal{L}$ on E such that $\mathcal{L}^{\otimes 2} \cong \mathbb{Q}_E$. The local system $\mathcal{L}$ is trivial if and only if $(K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2} = \mathcal{O}_E$. Since $H^*(E, \mathcal{L}) = 0$ if $\mathcal{L}$ is non-trivial, we obtain $n_{g,\beta}^{\mathrm{KL}} = 0$ for $g \neq 1$ and

$$n_{1,\beta}^{\mathrm{KL}} = \begin{cases} 1, & (K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2} = \mathcal{O}_E, \\ 0, & (K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2} \neq \mathcal{O}_E. \end{cases}$$

The expected answer is $n_{1,\beta} = 1$, so should choose an orientation data to be CY, i.e. $(K_{\mathrm{Sh}}^{\mathrm{vir}})^{1/2} = \mathcal{O}_E$.

Remark 7.4. *Suppose that* $\mathrm{Sh}_\beta(X)$ *is non-singular and the usual canonical line bundle* K_{Sh} *on* $\mathrm{Sh}_\beta(X)$ *is pulled back from the Hilbert-Chow map (7.2). Then as* $K_{\mathrm{Sh}}^{\mathrm{vir}} = K_{\mathrm{Sh}}^{\otimes 2}$*, we can take the oriented d-critical scheme*

$$\mathcal{S}h = (\mathrm{Sh}_\beta(X), s_{\mathrm{Sh}} = 0, K_{\mathrm{Sh}})$$

which is a CY fibration over $\mathrm{Chow}_\beta(X)$*. In this case, we have* $\phi_{\mathcal{S}h} = \mathrm{IC}(\mathrm{Sh}_\beta(X))$ *and all the definitions agree:*

$$n_{g,\beta}^{\mathrm{HST}} = n_{g,\beta}^{\mathrm{KL}} = n_{g,\beta}.$$

Even if we choose a CY orientation data, our definition may not agree with KL definition. In general, there is a *weight spectral sequence*

$$E_1^{i,j} = {}^p\mathcal{H}^{i+j}(\mathbf{R}\pi_*\mathrm{gr}_W^{-i}(\phi_{\mathcal{S}h})) \Rightarrow {}^p\mathcal{H}^{i+j}(\mathbf{R}\pi_*\phi_{\mathcal{S}h}) \tag{7.7}$$

which always degenerates at E_2 by considering weights (see [Sch, Section 17] for the similar spectral sequence for nearby cycles). It is easy to see that, for a choice of CY orientation data, our definition agrees with KL definition if the spectral sequence (7.7) degenerates at E_1. In the next subsection, we will see an example where (7.7) does not degenerate, which also gives a counter-example to the conjectures of Kiem-Li.

7.4. **Counter-example to Kiem-Li conjecture.** In this subsection, we prove Proposition 1.6. Let S be an Enriques surface. It always admits an elliptic fibration

$$h\colon S \to \mathbb{P}^1. \tag{7.8}$$

Let $\sigma\colon \widetilde{S} \to S$ be a K3 cover and E an elliptic curve. We set

$$X = (\widetilde{S} \times E)/\langle \tau \rangle.$$

Here τ is an involution on $\widetilde{S} \times E$ which acts on $\widetilde{S}$ as a covering involution of σ, and acts on E by $x \mapsto -x$. The 3-fold X is a smooth projective CY 3-fold, and its GW invariants were studied in [MP08].

We first describe the geometry of X via fibrations over $\mathbb{P}^1$ and S. The projections from $\widetilde{S} \times E$ onto each factors induce the fibrations

$$p \colon X \to E/\langle \tau \rangle = \mathbb{P}^1, \ \widehat{p} \colon X \to S. \tag{7.9}$$

We denote by

$$e_1, e_2, e_3, e_4 \in E, \ q_1, q_2, q_3, q_4 \in \mathbb{P}^1$$

the 2-torsion points of E and their images under the quotient map $E \to \mathbb{P}^1$ respectively. The fiber of p at $x \in \mathbb{P}^1$ is $\widetilde{S}$ for $x \neq q_i$, and the fiber at q_i is a double fiber $2S$. The map $\widehat{p}$ is a smooth fibration with fiber E.

Remark 7.5. *The CY 3-fold X is closely related to the non-compact CY 3-fold*

$$X' = \mathrm{Tot}(K_S).$$

Indeed X' is given by the quotient of $\widetilde{S} \times \mathbb{A}^1$ by τ, where τ acts on $\mathbb{A}^1$ by $x \mapsto -x$. Similarly to (7.9), we have the fibration

$$p' \colon X' \to \mathbb{A}^1/\tau = \mathbb{A}^1.$$

Then there exist analytic open neighborhoods $q_i \in U_i \subset \mathbb{P}^1$, $0 \in U \subset \mathbb{A}^1$ such that $p^{-1}(U_i)$ is isomorphic to $p'^{-1}(U)$.

Let $2C$ be a double fiber the elliptic fibration (7.8), and set $X_C := \widehat{p}^{-1}(C)$, $\widetilde{C} := \sigma^{-1}(C) \subset \widetilde{S}$. Note that we have

$$X_C = (\widetilde{C} \times E)/\langle \tau \rangle.$$

Here if $\widetilde{C}$ is smooth, τ acts on $\widetilde{C} \times E$ by $(x, y) \mapsto (x + t, -y)$ where t is a two torsion point in $\widetilde{C}$. We will use the following diagram

$$\begin{tikzcd} X_C \arrow[r, "p|_{X_C}"] \arrow[d, "\widehat{p}|_{X_C}"'] & \mathbb{P}^1 \\ C. \end{tikzcd} \tag{7.10}$$

Let C_i be the reduced fiber of $p|_{X_C}$ at q_i for $1 \le i \le 4$, giving four sections of $\widehat{p}|_{X_C}$. Note that we have

$$C_i = C \subset 2S = p^{-1}(q_i). \tag{7.11}$$

Also we have

$$p|_{X_C}^{-1}(x) = \widetilde{C} \subset p^{-1}(x) = \widetilde{S}, \ x \neq q_i.$$

Using the Künneth formula, we see that

$$H_2(X, \mathbb{Z}) = H_2(S, \mathbb{Z}) \oplus \mathbb{Z}[E] \tag{7.12}$$

where $[E]$ is the fiber class of the projection $\widehat{p} \colon X \to S$. Let β be the homology class given by

$$\beta = ([C], 0) \in H_2(X, \mathbb{Z})$$

under the decomposition (7.12). By (7.11), each C_i has homology class β. The computations in this subsection are summarized below (which is also stated in Proposition 1.6):

Proposition 7.6. *Suppose that C is of type I_n for $n \geq 2$, i.e. C is a circle of $\mathbb{P}^1$ with n-irreducible components. Then the HST, KL, our definitions, and the expected answers (from GW or PT theory) are given in the following table:*

	HST	KL	ours	expected
$n_{0,\beta}$	$-8n$	0	0	0
$n_{1,\beta}$	$4n$	$4n$	4	4
$n_{\geq 2,\beta}$	0	0	0	0

As before, we consider the moduli space $\mathrm{Sh}_\beta(X)$ and the Hilbert-Chow map

$$\pi\colon \mathrm{Sh}_\beta^{\mathrm{red}}(X) \to \mathrm{Chow}_\beta(X). \tag{7.13}$$

We have the following lemma on the Chow variety $\mathrm{Chow}_\beta(X)$:

Lemma 7.7. *A closed point of $\mathrm{Chow}_\beta(X)$ corresponds to a one cycle γ on X_C in the diagram (7.10) satisfying $\widehat{p}_*\gamma = C$ and $p_*\gamma = 0$.*

Proof. For a one cycle $\gamma \in \mathrm{Chow}_\beta(X)$, the cycle $\widehat{p}_*\gamma$ is a one cycle on S with homology class $[C]$. Then we have $\widehat{p}_*\gamma = C$ as C is the unique effective one cycle on S with homology class $[C]$. In particular, γ is supported on X_C. The last statement $p_*\gamma = 0$ is obvious from $p_*\beta = 0$. $\square$

We first assume that C is of type I_0, i.e. C is a smooth elliptic curve. In this case, we have the following:

Lemma 7.8. *If C is of type I_0, then (7.13) is*

$$\begin{aligned} \mathrm{Sh}_\beta^{\mathrm{red}}(X) &\to \mathrm{Chow}_\beta(X) \\ \amalg_{i=1}^4 C_i &\to \{q_1, q_2, q_3, q_4\} \\ x \in C_i &\mapsto q_i. \end{aligned}$$

In particular, (7.13) is a CY fibration with a CY orientation data.

Proof. Let γ be a one cycle on X_C satisfying the conditions in Lemma 7.7. Since C is irreducible, the cycle γ must be irreducible as well. Then $p(\gamma) \in \mathbb{P}^1$ is a one point, so γ is supported on either on C_i for $1 \leq i \leq 4$ or $\widetilde{C} \subset p^{-1}(x)$ for $x \neq q_i$. Since $\widehat{p}_*\widetilde{C} = \sigma_*\widetilde{C} = 2C$, the latter possibility is excluded. It follows that

$$\mathrm{Chow}_\beta(X) = \{[C_1], [C_2], [C_3], [C_4]\}.$$

Let $\mathrm{Pic}^1(C_i) \cong C_i$ be the moduli space of line bundles on C_i of degree 1. We have the natural closed embedding

$$\amalg_{i=1}^4 \mathrm{Pic}^1(C_i) \hookrightarrow \mathrm{Sh}_\beta(X) \tag{7.14}$$

which is bijective on closed points as C_i is smooth. On the other hand, we have the exact sequence

$$0 \to H^1(\mathcal{O}_{C_i}) \to \mathrm{Ext}_X^1(\mathcal{O}_{C_i}, \mathcal{O}_{C_i}) \to H^0(N_{C_i/X}).$$

Since $N_{C_i/X}$ is a rank two vector bundle given by an extension of non-trivial 2-torsion line bundles, we have $H^0(N_{C_i/X}) = 0$ and the second arrow of the above sequence is an isomorphism. This shows that (7.14) induces isomorphisms on tangent spaces. As $\mathrm{Pic}^1(C_i)$ is smooth, the embedding (7.14) is an isomorphism. □

By the above lemma, we can define $n_{g,\beta} \in \mathbb{Z}$ as in Definition 3.7. Together with the argument in Subsection 7.3, we have the following:

Corollary 7.9. *If C is of type I_0, we have the identity:*

$$n_{g,\beta} = \begin{cases} 4, & g = 1, \\ 0, & g \neq 1. \end{cases} \tag{7.15}$$

The same identity holds for $n^{\mathrm{KL}}_{g,\beta}$ if and only if we take a CY orientation data.

Remark 7.10. *As in the discussion of Subsection 7.9, the answer (7.15) matches with the expected one (see* [HST01, Section 4]*).*

We next consider the case that C is of type I_n for $n \geq 2$, i.e. C is a nodal curve of a circle of n smooth rational curves. Such an Enriques surface exists when $2 \leq n \leq 9$ by [CD89, Theorem 5.7.5]. We denote the irreducible components of C as

$$C = C^{(1)} \cup \cdots \cup C^{(n)}, \ C^{(j)} = \mathbb{P}^1.$$

The nodal points are denoted as

$$p^{(j)} = C^{(j)} \cap C^{(j+1)}, \ j \in \mathbb{Z}/n\mathbb{Z}.$$

In this case, the Chow variety is described as follows:

Lemma 7.11. *If C is of type I_n for $n \geq 2$, we have*

$$\mathrm{Chow}_\beta(X) = E^{\times n}. \tag{7.16}$$

For $1 \leq i \leq n$, let $\Gamma_i \subset E^{\times n}$ be the closed subvariety defined by

$$\Gamma_1 = \{(x_1, \dots, x_n) \in E^{\times n} : x_1 = \cdots = x_n\},$$
$$\Gamma_i = \{(x_1, \dots, x_n) \in E^{\times n} : x_i = \cdots = x_n = -x_1 = \cdots = -x_{i-1}\}, \ i \geq 2.$$

Then the image of (7.13) is identified with

$$\mathrm{Im}\,\pi = \bigcup_{i=1}^n \Gamma_i \subset E^{\times n}.$$

The cycle $[C_i] \in \mathrm{Chow}_\beta(X)$ corresponds to the point $y_i = (e_i, \dots, e_i)$.

Proof. By Lemma 7.7, giving a point of $\mathrm{Chow}_\beta(X)$ is equivalent to giving a one cycle γ on X_C written as

$$\gamma = \gamma_1 + \cdots + \gamma_n, \ \widehat{p}_* \gamma_j = C^{(j)}.$$

Since $\mathbb{P}^1$ is simply connected, the fibration $\widehat{p}$ is trivial over each irreducible component $C^{(j)} \subset C$. It follows that a choice of γ_j is equivalent to a choice of a section of the trivial bundle $E \times C^{(j)} \to C^{(j)}$. As there is no non-constant morphism $\mathbb{P}^1 \to E$, such a section is determined by a point in E. Therefore the set of choices of γ is identified with $E^{\times n}$, and (7.16) holds.

For a stable sheaf $[F] \in \mathrm{Sh}_\beta(X)$, its support must be connected. Suppose that a one cycle $\gamma \in \mathrm{Chow}_\beta(X)$ corresponds to a point $(x_1, \dots, x_n) \in E^{\times n}$. Since the monodromy of $\widehat{p}|_{X_C}$ around the generator of $\pi_1(C) = \mathbb{Z}$ is given by $\tau \colon E \to E$ sending x to $-x$, the cycle γ is connected if and only if $(x_1, \dots, x_n) \in \Gamma_i$ for some i. Conversely if γ is connected, then its underlying curve is either C or a partial normalization of C at one of $p^{(j)}$. In each case, there exist line bundles on it giving closed points of $\mathrm{Sh}_\beta(X)$, so $\gamma \in \mathrm{Im}\,\pi$ holds. The case $\gamma = [C_i]$ is only possible when $x_1 = x_2 = \cdots = x_n = -x_1$, so it corresponds to y_i. $\square$

For $1 \le i \le 4$ and $1 \le j \le n$, we denote by $p_i^{(j)}$ the nodal point in C_i corresponding to $p^{(j)} \in C$. We have the following lemma describing (7.13) (see Figure 2 for $n = 2$ case):

Lemma 7.12. *For $y \in \cup_{i=1}^n \Gamma_i$, we have (set theoretically)*

$$\pi^{-1}(y) = \begin{cases} C_i, & y = y_i, \\ \text{one point}, & y \neq y_i. \end{cases}$$

Moreover the moduli space $\mathrm{Sh}_\beta(X)$ is non-singular except points $p_i^{(j)}$ in $\pi^{-1}(y_i) = C_i$. At $p_i^{(j)}$, the singularity of $\mathrm{Sh}_\beta(X)$ is analytically isomorphic to the critical locus of

$$f \colon \mathbb{A}^3 \to \mathbb{A}, \ (x, y, z) \mapsto xyz \tag{7.17}$$

at the origin $0 \in \mathbb{A}^3$.

Proof. It is well-known that the moduli of rank one stable sheaves on C with Euler characteristic one is isomorphic to C itself, by the map $x \mapsto I_x^\vee$. Therefore $\pi^{-1}(y_i) = C_i$ follows. For $y \neq y_i$, $\pi^{-1}(y)$ consists of rank one stable sheaves L on a partial normalization $C' \to C$ at one of $p^{(j)}$ with $\chi(L) = 1$. So it consists of a one point $\{\mathcal{O}_{C'}\}$. For a point in $\mathrm{Sh}_\beta(X)$ except nodal points in $\pi^{-1}(y_i)$, the corresponding sheaf is a line bundle on the underlying curve. Therefore the same argument of Lemma 7.8 shows that $\mathrm{Sh}_\beta(X)$ is smooth except points $p_i^{(j)}$.

By Remark 7.5, we can also describe the singularities of $\mathrm{Sh}_\beta(X)$ at $p_i^{(j)}$ by the same argument as in Subsection 5.11. Namely let $\pi_T \colon \mathcal{C} \to T$ be a versal deformation of C, with $0 \in T$ such that $C = \pi_T^{-1}(C)$. We can take T to be a sufficiently small open neighborhood of $0 \in \mathbb{A}^n$. We define $B \subset \mathcal{C} \times \mathbb{A}^n$ by the commutative diagram

$$\begin{tikzcd} B = \{df' = 0\} \arrow[r, hook] \arrow[dr, "\pi'"'] & \mathcal{C} \times \mathbb{A}^n \arrow[d, "\pi_T \times \mathrm{id}"'] \arrow[dr, "f'"] & \\ & T \times \mathbb{A}^n \arrow[r, "g'"] & \mathbb{A}^1. \end{tikzcd} \tag{7.18}$$

Here g' is defined by

$$g'(t_1, \dots, t_n, u_1, \dots, u_n) = t_1 u_1 + \cdots + t_n u_n.$$

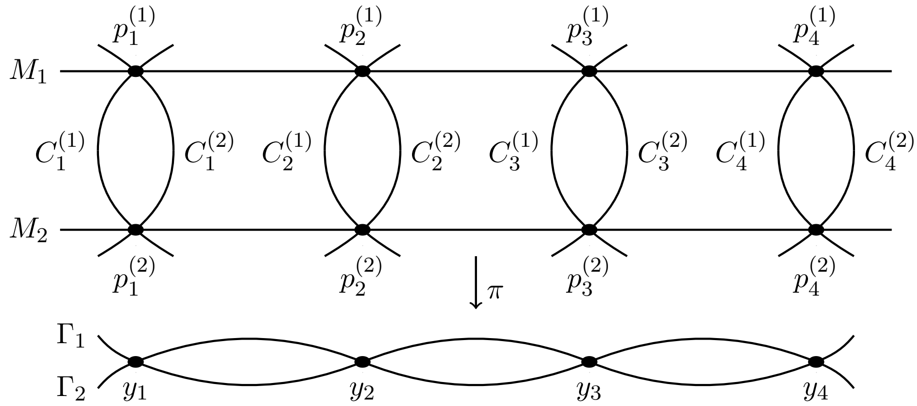

FIGURE 2. Picture of HC map for $n = 2$

Then there exist analytic open neighborhoods $y_i \in V_i \subset \mathrm{Chow}_\beta(X)$, $0 \in V \subset T \times \mathbb{A}^n$ such that $\pi^{-1}(V_i)$ is isomorphic to $\pi^{'-1}(V)$[4].

It is easy to see that B is only singular at $(p^{(j)}, 0) \in C \times \{0\} \subset \mathcal{C} \times T$ for $1 \le j \le n$, and near these points f' is analytically isomorphic to

$$f' \colon (x, t_1, \dots, t_n, u_1, \dots, u_n) \mapsto xt_1u_1 + t_2u_2 + \cdots + t_nu_n \tag{7.19}$$

at the origin. The critical locus of the above function is isomorphic to the critical locus of (7.17). $\square$

Lemma 7.13. *In the above situation, the morphism (7.13) is a CY fibration, and one can take a CY orientation data of* $\mathrm{Sh}_\beta(X)$.

Proof. The lemma follows since the canonical line bundle of $\mathcal{C} \times \mathbb{A}^n$ in the diagram (7.18) is trivial. $\square$

Now we compute the KL definition for type I_n case with $n \ge 2$, and see that it does not match with the predicted answer. Let us take a (not necessary CY) orientation data on $\mathrm{Sh}_\beta(X)$, and

$$\phi_{\mathcal{S}h} \in \mathrm{Perv}(\mathrm{Sh}_\beta(X))$$

the perverse sheaf as in Theorem 2.3. Let

$$M_1, \dots, M_n \subset \mathrm{Sh}_\beta(X) \tag{7.20}$$

be the irreducible components of $\mathrm{Sh}_\beta(X)$ which are mapped to Γ_i by the map (7.13). We denote by

$$C_i^{(1)}, \dots, C_i^{(n)} \subset \pi^{-1}(y_i) = C_i \subset \mathrm{Sh}_\beta(X) \tag{7.21}$$

the irreducible components of C_i. By Lemma 7.12, the subvarieties (7.20), (7.21) form the set of irreducible components of $\mathrm{Sh}_\beta(X)$. We also denote by $\mathrm{Sh}^{\mathrm{sm}} \subset \mathrm{Sh}_\beta(X)$ the smooth locus of $\mathrm{Sh}_\beta(X)$, whose complement is the points $p_i^{(j)}$ by Lemma 7.12. We first compute the associated graded sheaf of the weight filtration of $\phi_{\mathcal{S}h}$.

[4]Indeed the image of π' lies in $0 \times \mathbb{A}^n$ so one can instead take an open neighborhood $0 \in V \subset \mathbb{A}^n$

Lemma 7.14. *There exist rank one local systems* $\mathcal{L}_k$ *on* $M_k \cap \mathrm{Sh}^{\mathrm{sm}}$ *for* $1 \le k \le n$ *such that* $\mathrm{gr}^i_W(\phi_{\mathcal{S}h})$ *is written as*

$$\mathrm{gr}^0_W(\phi_{\mathcal{S}h}) = \bigoplus_{i=1}^{n} \mathrm{IC}(\mathcal{L}_i) \oplus \bigoplus_{i=1}^{4}\bigoplus_{j=1}^{n} \mathrm{IC}(C_i^{(j)}), \tag{7.22}$$

$$\mathrm{gr}^{\pm 1}_W(\phi_{\mathcal{S}h}) = \bigoplus_{i=1}^{4}\bigoplus_{j=1}^{n} \mathbb{Q}_{p_i^{(j)}}, \ \mathrm{gr}^k_W(\phi_{\mathcal{S}h}) = 0, \ |k| \ge 2.$$

Proof. Note that $\phi_{\mathcal{S}h}|_{\mathrm{Sh}^{\mathrm{sm}}}$ is written as $\mathcal{L}[1]$ for a rank one local system $\mathcal{L}$ on $\mathrm{Sh}^{\mathrm{sm}}$, which is of weight zero by our weight convention as in [BBD+]. Note that any perverse sheaf on $\mathrm{Sh}_\beta(X)$ which underlines a pure Hodge module is a direct sum of intersection complexes of some local systems on dense open subsets of closed irreducible subvarieties in $\mathrm{Sh}_\beta(X)$. Therefore, $\mathrm{gr}^0_W(\phi_{\mathcal{S}h})$ is written as

$$\mathrm{gr}^0_W(\phi_{\mathcal{S}h}) = \bigoplus_{i=1}^{n} \mathrm{IC}(\mathcal{L}|_{M_i \cap \mathrm{Sh}^{\mathrm{sm}}}) \oplus \bigoplus_{i=1}^{4}\bigoplus_{j=1}^{n} \mathrm{IC}(\mathcal{L}|_{C_i^{(j)} \cap \mathrm{Sh}^{\mathrm{sm}}}) \oplus Q.$$

Here Q and $\mathrm{gr}^k_W(\phi_{\mathcal{S}h})$ for $k \ne 0$ are supported on points $p_i^{(j)}$.

Let us first take a CY orientation data of $\mathrm{Sh}_\beta(X)$ and show that $\mathcal{L}|_{C_i^{(j)} \cap \mathrm{Sh}^{\mathrm{sm}}}$ is a trivial local system. In the diagram (7.18), we set

$$\phi_B = \phi_{f'}(\mathrm{IC}(\mathcal{C} \times \mathbb{A}^n)) \in \mathrm{Perv}(B).$$

By the description of f' in (7.19) and the Thom-Sebastiani theorem, ϕ_B is locally near $(p^{(j)}, 0)$ calculated as the vanishing cycle sheaf of (7.17). Let N be the critical locus of (7.17) and N_i for $1 \le i \le 3$ the irreducible components of N. For $\phi_N = \phi_f(\mathrm{IC}(\mathbb{A}^3))$, its weight filtration is easily computed (for example see the last part of Section 6 in [Efi])

$$\mathrm{gr}^0_W(\phi_N) = \bigoplus_{i=1}^{3} \mathrm{IC}(N_i), \ \mathrm{gr}^{\pm 1}_W(\phi_N) = \mathbb{Q}_0, \ \mathrm{gr}^i_W(\phi_N) = 0, \ |i| \ge 2. \tag{7.23}$$

The above local calculation implies that the monodromy of $\mathcal{L}|_{C_i^{(j)} \cap \mathrm{Sh}^{\mathrm{sm}}}$ around $p_i^{(j)}$ is trivial. Since $C_i^{(j)} \cap \mathrm{Sh}^{\mathrm{sm}} = \mathbb{C}^*$, the monodromy around $p_i^{(j)}$ determines the local system on it. Therefore $\mathcal{L}|_{C_i^{(j)} \cap \mathrm{Sh}^{\mathrm{sm}}}$ is trivial. The computation (7.23) also shows that $Q = 0$ and gives the result for $\mathrm{gr}^k_W(\phi_{\mathcal{S}h})$ with $k \ne 0$. Therefore we obtain the identities (7.22) for a CY orientation data of $\mathrm{Sh}_\beta(X)$.

Next let us take an orientation data of $\mathrm{Sh}_\beta(X)$ which is not necessary CY, and denote by $\phi_{\mathcal{S}h'}$ the resulting perverse sheaf on $\mathrm{Sh}_\beta(X)$. Then by Theorem 2.3, $\phi_{\mathcal{S}h'} = \phi_{\mathcal{S}h} \otimes \mathcal{L}'$ for some rank one local system $\mathcal{L}'$ on $\mathrm{Sh}_\beta(X)$. Since $\otimes \mathcal{L}'$ preserves the purity, and each $\mathcal{L}'|_{C_i^{(j)}}$ is trivial as $C_i^{(j)} = \mathbb{P}^1$ is simply connected, the associated graded sheaf $\mathrm{gr}^\bullet_W(\phi_{\mathcal{S}h'})$ is still of the form (7.22). $\square$

Lemma 7.15. *Suppose that* C *is of type* I_n *for* $n \ge 2$. *Then for any choice of orientation data of* $\mathrm{Sh}_\beta(X)$, *we have* $n^{\mathrm{KL}}_{1,\beta} = 4n$.

Proof. By Lemma 7.14, we have

$$\sum_{i\in\mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_* \operatorname{gr}_W^{\bullet}(\phi_{\mathcal{S}h})))y^i = 4n(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^2 + (\text{constant}).$$

Then the lemma follows by Remark 7.2. □

Since an Enriques surface with an I_n type double fiber can be deformed to a one with an I_0 type double fiber, by Lemma 7.8, 7.15, we have the following:

Corollary 7.16. *The Kiem-Li invariant $n_{g,\beta}^{\mathrm{KL}}$ is not deformation invariant.*

Now we show that our invariants agree with the result in Corollary 7.9. So we can observe that our invariants are deformation invariant in this example despite the fact that the Chow variety jumps in dimension. Below we fix a CY orientation data of $\mathrm{Sh}_\beta(X)$.

Lemma 7.17. *Suppose that C is of type I_n for $n \geq 2$. Then we have $n_{1,\beta} = 4$.*

Proof. Let us consider the spectral sequence (7.7). The differential $E_1^{0,-1} \to E_1^{1,-1}$ is induced by taking the ${}^p\mathcal{H}^0(\mathbf{R}\pi_*(-))$ of the canonical morphism

$$\operatorname{gr}_W^0(\phi_{\mathcal{S}h})[-1] \to \operatorname{gr}_W^{-1}(\phi_{\mathcal{S}h}). \tag{7.24}$$

Let ν_i be the normalization of C_i

$$\nu_i \colon \widetilde{C}_i := \coprod_{j=1}^{n} C_i^{(j)} \to C_i.$$

By Lemma 7.14, we have

$$E_1^{0,-1} = \bigoplus_{i=1}^{4} H^0(\widetilde{C}_i, \mathbb{Q}) \otimes \mathbb{Q}_{y_i}, \ E_1^{1,-1} = \bigoplus_{i=1}^{4}\bigoplus_{j=1}^{n} H^0(p_i^{(j)}, \mathbb{Q}_{p_i^{(j)}}) \otimes \mathbb{Q}_{y_i}.$$

The differential $E_1^{0,-1} \to E_1^{1,-1}$ at y_i is induced by taking the global sections of the exact sequences

$$0 \to \mathbb{Q}_{C_i} \to \nu_{i*}\mathbb{Q}_{\widetilde{C}_i} \to \bigoplus_{j=1}^{n} \mathbb{Q}_{p_i^{(j)}} \to 0.$$

Indeed this follows from the local calculation of the function $(x, y, z) \mapsto xyz$ as in (7.23), where we can easily see that the canonical morphism

$$\bigoplus_{i=1}^{3} \mathbb{Q}_{N_i} = \operatorname{gr}_W^0(\phi_N)[-1] \to \operatorname{gr}_W^{-1}(\phi_N) = \mathbb{Q}_0$$

is the surjection of sheaves which is non-zero on each component N_i.

Therefore $E_1^{0,-1} \to E_1^{1,-1}$ is rank $n-1$ at y_i, and the E_2 term is

$$E_2^{0,-1} = \bigoplus_{i=1}^{4} \mathbb{Q}_{y_i}.$$

Since $E_2^{0,-1}$ is the only term which contributes to ${}^p\mathcal{H}^{-1}(\mathbf{R}\pi_*\phi_{\mathcal{S}h})$, and the spectral sequence does not contribute to ${}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi_{\mathcal{S}h})$ for $|i| \geq 2$, we have

$$\sum_{i \in \mathbb{Z}} \chi({}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi_{\mathcal{S}h}))y^i = 4(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^2 + (\text{constant}).$$

Therefore the lemma follows. $\square$

Remark 7.18. *In type I_n case with $n \geq 2$, the genus zero invariant $n_{0,\beta}$ can be directly checked to be zero as follows. As a singularity of $\mathrm{Sh}_\beta(X)$ is the critical locus of (7.17), the Behrend function on $\mathrm{Sh}_\beta(X)$ is constant -1, and $n_{0,\beta} = -e(\mathrm{Sh}_\beta(X))$. By Lemma 7.12, we have*

$$e(\mathrm{Sh}_\beta(X)) = 4 \cdot e(C) + n \cdot (e(E) - 4) = 0$$

as expected.

Remark 7.19. *Contrary to Lemma 7.15, our GV invariant in Lemma 7.17 gives a different answer if we take a non-CY orientation data, as it affects the map $E_1^{0,-1} \to E_1^{1,-1}$ in the proof of Lemma 7.17.*

The HST invariants are similarly computed, which we leave the readers for details. The computations in this subsection are summarized in the table in Proposition 7.6.

8. Non-reduced examples from 3-fold flops

In this section, using the results of the previous sections and derived equivalences under 3-fold flops, we give some examples where Conjecture 3.14 holds for non-reduced, non-planar one cycles.

8.1. **3-fold flops.** Let X, $X^\dagger$ be smooth quasi-projective CY 3-folds which are connected by a flop

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X^\dagger \arrow[dl, "f^\dagger"] \\ & Y. & \end{tikzcd} \tag{8.1}$$

This means that $f, f^\dagger$ are birational morphisms which are isomorphic in codimension one with relative Picard number one, Y has only Gorenstein singularities, and ϕ is a non-isomorphic birational map. The exceptional loci of $f, f^\dagger$ are chains of smooth rational curves. By the result of Bridgeland [Bri02], there is an equivalence of derived categories

$$\Phi \colon D^b(\mathrm{Coh}(X)) \xrightarrow{\sim} D^b(\mathrm{Coh}(X^\dagger)) \tag{8.2}$$

given by the Fourier-Mukai transform whose kernel is $\mathcal{O}_{X \times_Y X^\dagger}$. The above equivalence restricts to the equivalence of triangulated subcategories (see [Tod08, Proposition 5.2])

$$\Phi \colon D^b(\mathrm{Coh}_{\leq 1}(X)) \xrightarrow{\sim} D^b(\mathrm{Coh}_{\leq 1}(X^\dagger)) \tag{8.3}$$

The equivalence (8.3) preserves the hearts of perverse t-structures. Namely there exist hearts of bounded t-structures defined in [Bri02, Section 3]

(8.4)

$$\begin{aligned} &{}^p\mathrm{Per}_{\le 1}(X/Y) \\ &:= \left\{ E \in D^b(\mathrm{Coh}_{\le 1}(X)) \colon \begin{array}{c} \mathbf{R}f_*E \in \mathrm{Coh}(Y) \\ \mathrm{Hom}^{<-p}(E, \mathcal{C}_X) = \mathrm{Hom}^{<p}(\mathcal{C}_X, E) = 0 \end{array} \right\} \end{aligned}$$

where $\mathcal{C}_X$ is defined by

$$\mathcal{C}_X := \{F \in \mathrm{Coh}(X) : \mathbf{R}f_*F = 0\}.$$

Below we always take $p \in \{-1, 0\}$. The equivalence (8.3) restricts to the equivalence

$$\Phi \colon {}^0\mathrm{Per}_{\le 1}(X/Y) \xrightarrow{\sim} {}^{-1}\mathrm{Per}_{\le 1}(X^\dagger/Y). \tag{8.5}$$

We can describe the hearts (8.4) in terms of tilting. Let $\mathrm{Coh}(X/Y)$ be the subcategory of $\mathrm{Coh}_{\le 1}(X)$ consisting of sheaves supported on the exceptional locus of f. For an ample divisor ω on X, we set

$$\begin{aligned} {}^0\mathcal{F} &:= \langle E \in \mathrm{Coh}(X/Y) : E \text{ is } \omega\text{-semistable with } \mu_\omega(E) < 0 \rangle \\ {}^{-1}\mathcal{F} &:= \langle E \in \mathrm{Coh}(X/Y) : E \text{ is } \omega\text{-semistable with } \mu_\omega(E) \le 0 \rangle. \end{aligned}$$

Here $\langle * \rangle$ is the smallest extension closed subcategory which contains $*$. Let ${}^p\mathcal{T}$ be the orthogonal complement of ${}^p\mathcal{F}$

$${}^p\mathcal{T} := \{E \in \mathrm{Coh}_{\le 1}(X) \colon \mathrm{Hom}(E, {}^p\mathcal{F}) = 0\}.$$

Lemma 8.1. *We have the identity*

$${}^p\mathrm{Per}(X/Y) = \langle {}^p\mathcal{F}[1], {}^p\mathcal{T} \rangle.$$

Proof. See [Tod15, Lemma 2.5]. □

Remark 8.2. *Below, we will use the fact that the equivalence Φ commutes with $\mathbf{R}f_*$ and $\mathbf{R}f_*^\dagger$ (see* [Bri02]*). By the definition of ${}^p\mathrm{Per}_{\le 1}(X/Y)$, we have*

$$\mathcal{C}_X[-p] = \{E \in {}^p\mathrm{Per}_{\le 1}(X/Y) : \mathbf{R}f_*E = 0\}.$$

In particular, Φ restricts to the equivalence of $\mathcal{C}_X$ and $\mathcal{C}_{X^\dagger}[1]$.

8.2. **Isomorphism of moduli spaces.** Let us consider the moduli space of one dimensional stable sheaves $\mathrm{Sh}_\beta(X)$ as in (3.2). We have the following lemma:

Lemma 8.3. *For $[E] \in \mathrm{Sh}_\beta(X)$, we have*

$$E \in {}^p\mathcal{T} = {}^p\mathrm{Per}_{\le 1}(X/Y) \cap \mathrm{Coh}_{\le 1}(X). \tag{8.6}$$

Proof. By the h-stability of E, we have $\mathrm{Hom}(E, {}^p\mathcal{F}) = 0$, hence $E \in {}^p\mathcal{T}$ follows. The right identity of (8.6) is due to Lemma 8.1. □

We define the open subscheme

$$\mathrm{Sh}_\beta^\circ(X) \subset \mathrm{Sh}_\beta(X)$$

to be consisting of sheaves E whose supports are irreducible and not contained in $\mathrm{Ex}(f)$. For the birational map ϕ in (8.1), let

$$\phi_* \colon H_2(X, \mathbb{Z}) \xrightarrow{\cong} H_2(X^\dagger, \mathbb{Z})$$

be the induced map on homology groups.

Lemma 8.4. *For $[E] \in \mathrm{Sh}^{\circ}_{\beta}(X)$, we have $[\Phi(E)] \in \mathrm{Sh}_{\phi_*\beta}(X^{\dagger})$.*

Proof. We have $\Phi(E) \in {}^{-1}\mathrm{Per}_{\le 1}(X^{\dagger}/Y)$ by Lemma 8.3 and the equivalence (8.5). Let us set $A = \mathcal{H}^{-1}(\Phi(E))$ and $B = \mathcal{H}^{0}(\Phi(E))$. We have the exact sequence

$$0 \to A[1] \to \Phi(E) \to B \to 0$$

in ${}^{-1}\mathrm{Per}_{\le 1}(X^{\dagger}/Y)$. By applying $\mathbf{R}f^{\dagger}_{*}$ and noting Remark 8.2, we obtain the exact sequence of sheaves

$$0 \to \mathbf{R}f^{\dagger}_{*}(A[1]) \to f_{*}E \to \mathbf{R}f^{\dagger}_{*}B \to 0.$$

Since f_*E is a pure one dimensional sheaf on Y, we have $\mathbf{R}f^{\dagger}_{*}(A[1]) = 0$, i.e. $A \in \mathcal{C}_{X^{\dagger}}$. Suppose that $A \neq 0$. Then $\Phi^{-1}(A[1])$ is a non-zero object in $\mathcal{C}_X$ (see Remark 8.2), which admits a non-zero morphism to E. This contradicts to the assumption that the support of E does not contain irreducible components in $\mathrm{Ex}(f)$. Hence $A = 0$, and $\Phi(E) \in \mathrm{Coh}_{\le 1}(X^{\dagger})$ follows.

It remains to show $\Phi(E)$ is a stable sheaf. Let

$$0 \to P \to \Phi(E) \to Q \to 0$$

be an exact sequence in $\mathrm{Coh}_{\le 1}(X^{\dagger})$ with $P, Q \neq 0$. By pushing forward to Y, we obtain the exact sequence of sheaves

$$0 \to f^{\dagger}_{*}P \to f_{*}E \to f^{\dagger}_{*}Q \to R^1 f^{\dagger}_{*}P.$$

Since $\mathbf{R}f_*E = f_*E$ is a stable sheaf with Euler characteristic one, we have $\chi(f^{\dagger}_{*}P) \le 1$. Since $R^1 f^{\dagger}_{*}P$ is a zero dimensional sheaf, we have

$$\chi(P) = \chi(f^{\dagger}_{*}P) - \chi(R^1 f^{\dagger}_{*}P) \le 1.$$

In order to conclude the stability of $\Phi(E)$, we need to exclude the case of $\chi(P) = 1$. In this case, $\mathbf{R}f^{\dagger}_{*}Q = 0$, i.e. $Q \in \mathcal{C}_{X^{\dagger}}$. Since $\Phi^{-1}(Q) \in \mathcal{C}_X[-1]$, and $\mathrm{Hom}(E, \mathcal{C}_X[-1]) = 0$, we obtain the contradiction. □

We define the open subscheme

$$\mathrm{Sh}^{\circ}_{\phi_*\beta}(X^{\dagger}) \subset \mathrm{Sh}_{\phi_*\beta}(X^{\dagger})$$

to be consisting of sheaves E such that the support of $\Phi^{-1}(E)$ is irreducible and not contained in $\mathrm{Ex}(f)$.

Proposition 8.5. *The equivalence Φ induces the isomorphism*

$$\Phi_* \colon \mathrm{Sh}_{\beta}(X) \xrightarrow{\cong} \mathrm{Sh}_{\phi_*\beta}(X^{\dagger}). \tag{8.7}$$

Proof. By the definition of $\mathrm{Sh}^{\circ}_{\phi_*\beta}(X^{\dagger})$ and Lemma 8.4, we have the well-defined morphism (8.7). Let us take an object $[E] \in \mathrm{Sh}^{\circ}_{\phi_*\beta}(X^{\dagger})$. It is enough to show that $\Phi^{-1}(E)$ is an object in $\mathrm{Sh}^{\circ}_{\beta}(X)$. We first note that $\mathbf{R}f^{\dagger}_{*}E = f^{\dagger}_{*}E$ is a pure sheaf on Y. Indeed otherwise, there is $y \in Y$ and a non-zero morphism $\mathcal{O}_y \to f^{\dagger}_{*}E$. By the adjunction, we have the non-zero morphism $f^{\dagger *}\mathcal{O}_y \to E$. Since $f^{\dagger *}\mathcal{O}_y$ is a stable sheaf (see the proof of [Kat08, Lemma 3.2]) with Euler characteristic one supported on $\mathrm{Ex}(f^{\dagger})$, this contradicts to the stability of E. Therefore $f^{\dagger}_{*}E$ is a pure sheaf.

Next we show that $\Phi^{-1}(E)$ is a coherent sheaf. By Lemma 8.3, we have $\Phi^{-1}(E) \in {}^{0}\mathrm{Per}_{\le 1}(X/Y)$. We set $A = \mathcal{H}^{-1}(\Phi^{-1}(E))$ and $B = \mathcal{H}^{0}(\Phi^{-1}(E))$. We have the exact sequence in ${}^{0}\mathrm{Per}_{\le 1}(X/Y)$

$$0 \to A[1] \to \Phi^{-1}(E) \to B \to 0. \tag{8.8}$$

Suppose that $A \neq 0$. Then $\mathbf{R}f_{*}(A[1])$ is a subsheaf of $f^{\dagger}_{*}E$ which is at most zero dimensional. By the purity of $f^{\dagger}_{*}E$, it follows that $\mathbf{R}f_{*}(A[1]) = 0$, i.e. $A \in \mathcal{C}_X$. This implies $A \in {}^{0}\mathrm{Per}_{\le 1}(X/Y)$, which contradicts to that (8.8) is an exact sequence in ${}^{0}\mathrm{Per}_{\le 1}(X/Y)$. Hence $A = 0$ and $\Phi^{-1}(E) \in \mathrm{Coh}_{\le 1}(X)$ follows.

It remains to show that $\Phi^{-1}(E)$ is a stable sheaf, or equivalently a pure sheaf as the support of $\Phi^{-1}(E)$ is irreducible. If otherwise, there is $x \in X$ and a non-zero morphism $\mathcal{O}_x \to \Phi^{-1}(E)$. Let P be its image in ${}^{0}\mathrm{Per}_{\le 1}(X/Y)$. Then $\mathbf{R}f_{*}P$ is a subsheaf of $f^{\dagger}_{*}E$ which is at most zero dimensional, hence $\mathbf{R}f_{*}P = 0$ by the purity of $f^{\dagger}_{*}E$. Then $P \in \mathcal{C}_X$, and $\Phi(P) \in \mathcal{C}_{X^{\dagger}}[1]$ is a subobject of E in ${}^{-1}\mathrm{Per}_{\le 1}(X^{\dagger}/Y)$. Since $\mathrm{Hom}(\mathcal{C}_{X^{\dagger}}[1], E) = 0$ as E is a sheaf, this is a contradiction. $\square$

Remark 8.6. *The equivalence of derived categories (8.2) induces the isomorphism*

$$H^0(X, K_X) \xrightarrow{\cong} H^0(X^{\dagger}, K_{X^{\dagger}}) \tag{8.9}$$

by taking the induced isomorphism on Hochschild homologies. Therefore the trivialization (3.1) induces the trivialization $\mathcal{O}_{X^{\dagger}} \xrightarrow{\cong} K_{X^{\dagger}}$, hence a d-critical stricture on $\mathrm{Sh}_{\phi_{*}\beta}(X^{\dagger})$ by Theorem 3.3.

The isomorphism (8.7) can be proved to preserve the d-critical structures of both sides. Indeed the Fourier-Mukai equivalence (8.2) lifts to a dg quasi-functor between the enhancements of both sides of (8.2), therefore the derived moduli schemes $\widehat{\mathrm{Sh}}_{\beta}(X)$ and $\widehat{\mathrm{Sh}}_{\phi_{*}\beta}(X^{\dagger})$ in Remark 3.4 are equivalent. On the other hand by the announced work [BD, Theorem 1.2]*, the (-1)-shifted symplectic structures on the above derived schemes are canonically constructed from the CY dg enhancements (see* [BD, Theorem 1.2]*), where the CY structures are given by non-zero elements in (8.9) by* [BD, Lemma 5.11]*. By taking the truncations of $\widehat{\mathrm{Sh}}_{\beta}(X)$ and $\widehat{\mathrm{Sh}}_{\phi_{*}\beta}(X^{\dagger})$, we have the matching of d-critical structures of both sides of (8.7).*

8.3. **Comparison of GV invariants.** We define the open subset

$$U_{\beta} \subset \mathrm{Chow}_{\beta}(X)$$

to be consisting of irreducible one cycles which do not contain irreducible components of $\mathrm{Ex}(f)$. For $\gamma \in U_{\beta}$, by taking pull-back and push-forward along with the morphisms

$$X \leftarrow X \times_Y X^{\dagger} \to X^{\dagger}$$

we have the map

$$\phi_{*} \colon U_{\beta} \to \mathrm{Chow}_{\phi_{*}\beta}(X^{\dagger}).$$

The above map is obviously injective. Let $U_{\phi_*\beta}$ be the image of the above map. We have the commutative diagram

$$\begin{tikzcd} \mathrm{Sh}^{\mathrm{red}}_{\beta}(X) \arrow[r, "\Phi_*"] \arrow[d] & \mathrm{Sh}^{\mathrm{red}}_{\phi_*\beta}(X^{\dagger}) \arrow[d] \\ U_{\beta} \arrow[r, "\phi_*"'] & U_{\phi_*\beta}. \end{tikzcd}$$

Here the vertical arrows are Hilbert-Chow maps. Then from Proposition 8.5, we obtain the following:

Corollary 8.7. *For $\gamma \in U_{\beta}$, the d-critical scheme $(\mathrm{Sh}_{\beta}(X), s_{\mathrm{Sh}})$ is CY at γ if and only if $(\mathrm{Sh}_{\phi_*\beta}(X^{\dagger}), s_{\mathrm{Sh}})$ is CY at $\phi_*\gamma$. In this case, we have the identity $n^{\mathrm{loc}}_{g,\gamma} = n^{\mathrm{loc}}_{g,\phi_*\gamma}$.*

As for stable pairs, let us set

$$\mathrm{PT}(X) := 1 + \sum_{n \in \mathbb{Z}, \beta > 0} P_{n,\beta} q^n t^{\beta},$$
$$\mathrm{PT}(X/Y) := 1 + \sum_{n \in \mathbb{Z}, f_*\beta = 0} P_{n,\beta} q^n t^{\beta}.$$

The following result is proved using wall-crossing formulas in the derived category:

Theorem 8.8. ([Tod13, Cal16]) *We have the following identity:*

$$\phi_* \frac{\mathrm{PT}(X)}{\mathrm{PT}(X/Y)} = \frac{\mathrm{PT}(X^{\dagger})}{\mathrm{PT}(X^{\dagger}/Y)}. \tag{8.10}$$

Here ϕ_* is the variable change $t^{\beta} \mapsto t^{\phi_*\beta}$.

By taking the logarithm of both sides of (8.10) and comparing the coefficient at t^{β}, we have

$$n^{P}_{g,\beta} = n^{P}_{g,\phi_*\beta}, \ \beta \in H_2(X, \mathbb{Z}), \ f_*\beta > 0.$$

The arguments of [Tod13, Cal16] also apply to the local version, which give

$$n^{P,\mathrm{loc}}_{g,\gamma} = n^{P,\mathrm{loc}}_{g,\phi_*\gamma}, \ \gamma \in \mathrm{Chow}_{\beta}(X), \ f_*\gamma > 0.$$

By combining with Corollary 8.7, we obtain the following:

Corollary 8.9. *Under the situation of Corollary 8.7, for $\gamma \in U_{\beta}$ we have $n^{P,\mathrm{loc}}_{g,\gamma} = n^{\mathrm{loc}}_{g,\gamma}$ if and only if $n^{P,\mathrm{loc}}_{g,\phi_*\gamma} = n^{\mathrm{loc}}_{g,\phi_*\gamma}$.*

8.4. **Examples via flops.** For $\gamma \in U_{\beta}$, the one cycle $\phi_*\gamma$ is not a reduced cycle if it intersects with $\mathrm{Ex}(f)$ with multiplicity bigger than or equal to two. So if we know $n^{P,\mathrm{loc}}_{g,\gamma} = n^{\mathrm{loc}}_{g,\gamma}$, by Corollary 8.9 we obtain examples of non-reduced one cycles γ' where $n^{P,\mathrm{loc}}_{g,\gamma'} = n^{\mathrm{loc}}_{g,\gamma'}$ holds. We give two examples where such an argument applies.

First, the following corollary obviously follows from Theorem 6.4 and Corollary 8.9:

Corollary 8.10. *Let $\phi\colon X \dashrightarrow X^{\dagger}$ be a flop as in (8.1), and $C \subset X$ a smooth curve which is not contained in (but may intersect with) the exceptional locus of ϕ. Then Conjecture 3.14 holds for the one cycle $\phi_*[C]$ on $X^{\dagger}$.*

We state the next example. Let S be a smooth projective surface with $H^1(\mathcal{O}_S) = 0$, and take a blow-up

$$h\colon S^{\dagger} \to S$$

at a point $p \in S$. Then there exist smooth projective 3-folds X, $X^{\dagger}$ and a flop diagram (8.1) satisfying the following conditions (see [Tod15, Lemma 4.2])

- Both of the exceptional locus $Z = \mathrm{Ex}(f)$, $Z^{\dagger} = \mathrm{Ex}(f^{\dagger})$ are irreducible $(-1,-1)$-curves.
- There are closed embeddings

$$i\colon S \hookrightarrow X, \quad i^{\dagger}\colon S^{\dagger} \hookrightarrow X^{\dagger} \tag{8.11}$$

  such that $S \cap Z$ consists of one point, the strict transform of S in $X^{\dagger}$ coincides with $S^{\dagger}$, and $Z^{\dagger} \subset S^{\dagger}$ coincides with the exceptional locus of $h\colon S^{\dagger} \to S$.
- There are open neighborhoods $S \subset X_0$, $S^{\dagger} \subset X_0^{\dagger}$ and isomorphisms

$$X_0 \cong \mathrm{Tot}(K_S), \ X_0^{\dagger} \cong \mathrm{Tot}(K_{S^{\dagger}}) \tag{8.12}$$

  such that the embeddings (8.11) are identified with the zero sections.

Below we regard one cycles on S, $S^{\dagger}$ as one cycles on X, $X^{\dagger}$ by isomorphisms (8.12) and zero sections. Applying the result of Theorem 5.15 and the argument of Corollary 8.7, we have the following:

Corollary 8.11. *(i) For any irreducible curve $C \subset S$, Conjecture 3.14 holds for the one cycle $\phi_*C = h^*C$ on $X^{\dagger}$.*

(ii) For any irreducible curve $C^{\dagger} \subset S^{\dagger}$, Conjecture 3.14 holds for the one cycle $\phi_*^{-1}C^{\dagger}$.

Remark 8.12. *Although X, $X^{\dagger}$ may not be CY, they are CY at $X_0 \cup C$ and $X_0^{\dagger}$, so the statements of Corollary 8.11 make sense.*

Remark 8.13. *In Corollary 8.11 (i), suppose that the multiplicity of C at p is m. Then $\phi_*C = \overline{C} + mZ$, where $\overline{C} \subset S^{\dagger}$ is the strict transform of C. In particular, ϕ_*C is not reduced, but it is planar.*

Remark 8.14. *In Corollary 8.11 (ii), suppose that the curves $C^{\dagger}$, $Z^{\dagger}$ in $S^{\dagger}$ intersect with multiplicity m. Then $\phi_*^{-1}C^{\dagger} = h(C) + mZ$. In this case, the cycle $\phi_*^{-1}C^{\dagger}$ is not reduced, not planar.*

9. Non-primitive examples

We give some examples where Conjecture 3.14 holds for non-primitive one cycles and $g \geq 1$.

9.1. **Super rigid elliptic curves.** Let C be an elliptic curve and

$$X = \mathrm{Tot}(L \oplus L^{-1})$$

for a for generic line bundle L on C with degree zero. Note that X is a non-compact CY 3-fold, with unique projective curve $C \subset X$ given by the zero section. For $\beta = m[C]$, the Chow variety $\mathrm{Chow}_\beta(X)$ consists of one point $m[C]$, and the moduli space $\mathrm{Sh}_{m[C]}(X)$ is isomorphic to C itself (see [HST01, Proposition 4.4]). Therefore we have

$$n_{1,m[C]} = 0, \ n_{g,m[C]} = 0, \ g \neq 1.$$

As C is smooth, our GV invariants agree with the invariants defined in [HST01]. In this case, Conjecture 3.18 is checked in [HST01] using the result of [Pan99]. Combined with DT/GW correspondence for local curves [BP08], Conjecture 3.14 holds for the one cycle $\gamma = m[C]$.

9.2. **Elliptic fibrations.** Let X be a smooth projective CY 3-fold with an elliptic fibration

$$\pi \colon X \to S$$

such that every scheme theoretic fiber is an integral curve. Let $F \in H_2(X, \mathbb{Z})$ be a fiber class of π, and set $\beta = n[F]$. Then $\mathrm{Chow}_\beta(X) = \mathrm{Sym}^n(S)$ and we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[d, "\pi"'] & \mathrm{Sh}_\beta(X) \arrow[d] \\ S \arrow[r, hook] & \mathrm{Sym}^n(S). \end{tikzcd} \tag{9.1}$$

Here the right arrow is the Hilbert-Chow map, and the bottom arrow is the diagonal map. We have the decomposition

$$\mathbf{R}\pi_* \mathrm{IC}(X) = \mathrm{IC}(S)[1] \oplus V \oplus \mathrm{IC}(S)[-1]$$

where $V = R^1\pi_*\mathbb{Q}_X[2]$ is a perverse sheaf on S. For $s \in S$, let X_s be the fiber of π at s which is either an elliptic curve, rational curve with one node or a cusp. In any case, we have

$$R^1\pi_*\mathbb{Q}_X|_s = H^1(X_s, \mathbb{Q}) = \mathbb{Q}^{2-e(X_s)}.$$

Then an easy calculation shows

$$\chi(\mathrm{IC}(S))y^{-1} + \chi(V) + \chi(\mathrm{IC}(S))y = -e(X) + e(S)(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^2.$$

By the diagram (9.1), we obtain

$$n_{0,\beta} = -e(X), \ n_{1,\beta} = e(S), \ n_{g,\beta} = 0, \ g \geq 2.$$

The invariants $n^P_{g,\beta}$ from stable pairs are computed in [Tod12, Theorem 6.9] via wall-crossing method, and completely agree with the above $n_{g,\beta}$. The argument here is easily applied to the local version, proving Conjecture 3.14 for the one cycle $\gamma = n[X_s]$ for some $s \in S$.

9.3. **Hitchin moduli spaces.** Let C be a smooth projective curve and

$$X = \mathrm{Tot}(\mathcal{O}_C \oplus K_C)$$

a non-compact CY 3-fold. Let $C \subset X$ be the zero section, and take the curve class $\beta = r[C]$. Then the moduli space $\mathrm{Sh}_\beta(X)$ is isomorphic to the product of $\mathbb{A}^1$ with Hitchin moduli space of rank r and Euler characteristic one stable Higgs bundles on C. In this case, by the work of Chuang-Diaconescu-Pan [CDP14], Conjecture 3.13 (or rather its refined version) is reduced to the following conjectures:

- Cataldo-Hausel-Migliorini's $P = W$ conjecture [dCHM12]: it claims the perverse filtration on the Hitchin moduli space coincides with the weight filtration of the character variety under their natural diffeomorphism. Here the character variety is the moduli space of representations of $\pi_1(C)$.
- Hausel-Rodriguez-Villegas's conjecture [HRV08]: it describes the generating series of mixed Poincaré polynomials of character varieties in terms of explicit sum of rational functions associated to Young diagrams.

The first conjecture is proven in [dCHM12] when $r = 2$; for character varieties, the weight polynomial (ignoring cohomological degree) is calculated in [HRV08]. By combining these results, Conjecture 3.13 holds for $\beta = 2[C]$. By applying the fiberwise $(\mathbb{C}^*)^{\times 2}$-action on X and localizing, we also obtain Conjecture 3.14 for the one cycle $\gamma = 2[C]$.

Appendix A. Calabi-Yau orientation data

In this Appendix, we discuss the role of orientation data on our definition of GV type invariants in Section 2.3. In general, they depend on a choice of an orientation data. Our idea to solve this issue is that we impose an additional condition (called CY condition) of an orientation data, and show that the resulting GV type invariants are independent of an orientation data as long as it satisfies the CY condition.

Roughly speaking, a CY condition of an orientation data is that it is trivial along the fibers of the map $\pi \colon M^{\mathrm{red}} \to T$ as a line bundle. This is a quite strong restriction, and such an orientation data does not always exist for arbitrary d-critical schemes. However for the moduli space of one dimensional sheaves and its HC map, we expect that such an orientation data exists. In other words, we expect that the HC map for the moduli space of one dimensional sheaves is a kind of Calabi-Yau fibration in a somewhat virtual sense.

A.1. **Dependence on orientation data.** Let us consider the situation in Section 2.3, i.e. $\mathcal{M} = (M, s, K^{1/2}_{M,s})$ be an oriented d-critical scheme and $\pi \colon M^{\mathrm{red}} \to T$ be a projective morphism for a finite type complex scheme T. For $g \geq 0$ and $t \in T$, let

$$\mathrm{GV}_{g,\mathcal{M}/T} \in \mathbb{Z}, \ \mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,t} \in \mathbb{Z} \tag{A.1}$$

be the GV type invariants given in Lemma 2.4, Lemma 2.5 respectively. For $g = 0$, they do not depend on a choice of an orientation data $K^{1/2}_{M,s}$ (see

Lemma 2.6). For $g \geq 1$, they may depend on a choice of an orientation data. However we have the following lemma:

Lemma A.1. *Let $K'^{1/2}_{M,s}$ be another orientation of (M, s). Suppose that for a Stein factorization*

$$\pi \colon M^{\mathrm{red}} \stackrel{\pi_1}{\to} \overline{T} \stackrel{\pi_2}{\to} T \tag{A.2}$$

i.e. $\overline{T} = \mathrm{Spec}_T(\pi_* \mathcal{O}_{M^{\mathrm{red}}})$, there exists a line bundle $L_{\overline{T}}$ on $\overline{T}$ such that

$$K'^{1/2}_{M,s} \cong K^{1/2}_{M,s} \otimes \pi_1^* L_{\overline{T}}$$

as line bundles. Then for $\mathcal{M}' = (M, s, K'^{1/2}_{M,s})$ and $t \in T$, we have

$$\mathrm{GV}_{g,\mathcal{M}/T} = \mathrm{GV}_{g,\mathcal{M}'/T}, \ \mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,t} = \mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}'/T,t}.$$

Proof. The isomorphism (2.5) and the similar isomorphism for $K'^{1/2}_{M,s}$ gives an isomorphism $s \colon \pi_1^* L_{\overline{T}}^{\otimes 2} \stackrel{\cong}{\to} \mathcal{O}_{M^{\mathrm{red}}}$. Since π_1 satisfies $\pi_{1*}\mathcal{O}_{M^{\mathrm{red}}} = \mathcal{O}_{\overline{T}}$, the isomorphism s is pulled back from an isomorphism $s' \colon L_{\overline{T}}^{\otimes 2} \stackrel{\cong}{\to} \mathcal{O}_{\overline{T}}$ via π_1^*. Let $\tau_T \colon \widetilde{T} \to \overline{T}$ be the $\mathbb{Z}/2\mathbb{Z}$-principal bundle which parametrizes local square roots of s', and $\mathcal{L}_{\overline{T}}$ the rank one local system on $\overline{T}$ given by $\tau_{T*}\mathbb{Q}_{\widetilde{T}} = \mathbb{Q}_{\overline{T}} \oplus \mathcal{L}_{\overline{T}}$. Let $\phi_{\mathcal{M}'}$ be the vanishing cycle sheaf on M in Theorem 2.3 defined from the oriented d-critical scheme $\mathcal{M}'$. Then the property (2.8) shows that $\phi_{\mathcal{M}'} = \phi_{\mathcal{M}} \otimes \pi_1^* \mathcal{L}_{\overline{T}}$. Also since π_2 is finite, $\mathbf{R}\pi_{2*} = \pi_{2*}$ takes perverse sheaves on $\overline{T}$ to perverse sheaves on T. Therefore we obtain the isomorphisms

$${}^p\mathcal{H}^i(\mathbf{R}\pi_* \phi_{\mathcal{M}'}) \cong \pi_{2*}\left({}^p\mathcal{H}^i(\mathbf{R}\pi_{1*}\phi_{\mathcal{M}}) \otimes \mathcal{L}_{\overline{T}}\right).$$

The lemma follows from the above isomorphisms. □

A.2. **Calabi-Yau d-critical schemes.** We introduce the following notion of CY d-critical schemes:

Definition A.2. *(i) A d-critical scheme (M, s) is called Calabi-Yau (CY for short) if $K_{M,s} \cong \mathcal{O}_{M^{\mathrm{red}}}$.*

(ii) An oriented d-critical scheme $(M, s, K^{1/2}_{M,s})$ is called CY if $K^{1/2}_{M,s} \cong \mathcal{O}_{M^{\mathrm{red}}}$.

(iii) A d-critical scheme (M, s) with a projective morphism $\pi \colon M^{\mathrm{red}} \to T$ is called CY at $t \in T$ if there is an open neighborhood $t \in U \subset T$ such that, by setting $M^{\mathrm{red}}_U := \pi^{-1}(U)$, the d-critical scheme

$$(M_U, s_U), \ M_U := \iota(M^{\mathrm{red}}_U), \ s_U := s|_{M_U} \tag{A.3}$$

is CY. Here $\iota \colon M^{\mathrm{red}} \hookrightarrow M$ is the closed immersion.

(iv) A d-critical scheme (M, s) with a projective morphism $\pi \colon M^{\mathrm{red}} \to T$ is called a CY fibration over T if and only if it is CY at all of $t \in T$.

The following lemma is obvious.

Lemma A.3. *For a d-critical scheme (M, s) with a projective morphism $\pi \colon M^{\mathrm{red}} \to T$, it is a CY fibration over T if and only if there is a line bundle L on $\overline{T}$ for the Stein factorization (A.2) such that $K_{M,s} \cong \pi_1^* L$*

We introduce the following notion of CY orientation data:

Definition A.4. *For a d-critical scheme (M, s) with a projective morphism $\pi\colon M^{\mathrm{red}} \to T$, suppose that it is a CY fibration over T. A CY orientation data of (M, s) is an orientation data $K_{M,s}^{1/2}$ satisfying $K_{M,s}^{1/2} \cong \pi_1^* L^{1/2}$ for a line bundle $L^{1/2}$ on $\overline{T}$. Here $M^{\mathrm{red}} \stackrel{\pi_1}{\to} \overline{T} \to T$ is the Stein factorization.*

By Lemma A.1, we immediately have the following lemma:

Lemma A.5. *The GV type invariants (A.1) are independent of a choice of an orientation data as long as it is CY orientation data.*

Remark A.6. *If $\pi\colon M^{\mathrm{red}} \to T$ is a CY fibration, it is not a priori true that it always has a CY orientation data as in Definition A.4. However of course such an orientation data always exists locally on T. Therefore using a local CY orientation data, we can define the local GV type invariant $\mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,t} \in \mathbb{Z}$. Then following the relation (2.10), we can define the global GV type invariant by the integration*

$$\mathrm{GV}_{g,\mathcal{M}/T} := \int_T \mathrm{GV}^{\mathrm{loc}}_{g,\mathcal{M}/T,t} \ de.$$

A.3. **Conjecture on Calabi-Yau conditions.** We keep the situation and notation from the previous subsection. We conjecture that the GV invariants are always well-defined:

Conjecture A.7. *The d-critical scheme $(\mathrm{Sh}_\beta(X), s)$ in Theorem 3.3 is a CY fibration over $\mathrm{Chow}_\beta(X)$ so that the local/global GV invariants $n^{\mathrm{loc}}_{g,\gamma}$, $n_{g,\beta}$ are defined by Definition 3.9, Remark 3.10 respectively.*

We have the following evidence of the above conjecture:

Proposition A.8. *Let T be a normal quasi-projective variety and $\mathcal{F} \in \mathrm{Coh}(X \times T)$ be a T-flat family of one dimensional sheaves on X such that the fundamental cycle $[\mathcal{F}_t] \in \mathrm{Chow}(X)$ for $t \in T$ is constant. Then we have*

$$\det\left(\mathbf{R}p_{T*}\mathbf{R}\mathcal{H}om_{X\times T}(\mathcal{F}, \mathcal{F})\right) \cong \mathcal{O}_T.$$

Proof. For a smooth quasi-projective variety Y, let

$$K^{\geq i}(Y) \subset K(Y)$$

be the subgroup generated by sheaves whose supports have codimensions bigger than or equal to i. By a classical result of Grothendieck (see [Gil05, Theorem 3.10]), the smoothness of Y implies that the tensor product on the K-theory restricts to the map

$$\otimes\colon K^{\geq i}(Y) \times K^{\geq j}(Y) \to K^{\geq i+j}(Y). \tag{A.4}$$

For a normal variety T, any line bundle on it is determined by its smooth part, so we may assume that T is smooth. Let $C \subset X$ be a subscheme whose fundamental cycle coincides with $[\mathcal{F}_t]$. By the property (A.4), we have

$$[\mathcal{O}_C] \otimes [\mathcal{O}_C] \in K^{\geq 4}(X) = 0. \tag{A.5}$$

Also by the assumption, we have $[\mathcal{F}], [\mathcal{F}]^\vee \in K^{\geq 2}(X \times T)$ and

$$[\mathcal{F}], [\mathcal{F}]^\vee \in [\mathcal{O}_{C\times T}] + K^{\geq 3}(X \times T). \tag{A.6}$$

By (A.4), (A.5) and (A.6), we have

$$[\mathbf{R}\mathcal{H}om_{X\times T}(\mathcal{F},\mathcal{F})] = [\mathcal{F}]^{\vee} \otimes [\mathcal{F}] \in K^{\geq 5}(X\times T).$$

Since X is three-dimensional, it follows that

$$[\mathbf{R}p_{T*}\mathbf{R}\mathcal{H}om_{X\times T}(\mathcal{F},\mathcal{F})] \in K^{\geq 2}(T).$$

By taking the determinant, we obtain the proposition. □

Remark A.9. *The above proposition in particular implies that the virtual canonical line bundle of* $\mathrm{Sh}_{\beta}(X)$ *is numerically trivial on any fiber of the HC map*

$$\pi\colon\ \mathrm{Sh}_{\beta}^{\mathrm{red}}(X) \to \mathrm{Chow}_{\beta}(X)$$

and trivial on any fiber of π *with at worst normal singularities. These are necessary conditions for Conjecture A.7. When* $\mathrm{Chow}_{\beta}(X)$ *is a one point, the above proposition implies Conjecture A.7 if* $\mathrm{Sh}_{\beta}^{\mathrm{red}}(X)$ *is normal. On the other hand if* $\mathrm{Sh}_{\beta}^{\mathrm{red}}(X)$ *is not normal, the argument of the above proposition does not imply Conjecture A.7.*

Remark A.10. *We may ask a question whether* $(\mathrm{Sh}_{\beta}(X), s)$ *is strictly CY at any point in* $\mathrm{Chow}_{\beta}(X)$ *(see Definition 2.7), which is stronger than Conjecture A.7. In Section 5, we more or less proved such a statement for the local surface case. As we have no other evidence, we just leave it as a question (rather than a conjecture) in this paper.*

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Davesh Maulik
Massachusetts Institute of Technology, Departement of Mathematics, 77 Massachusetts Avenue Cambridge, MA 02139, US.
E-mail address: maulik@mit.edu

Yukinobu Toda
Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan.
E-mail address: yukinobu.toda@ipmu.jp