STABILITY OF STATIONARY EQUIVARIANT WAVE MAPS FROM THE HYPERBOLIC PLANE

ANDREW LAWRIE, SUNG-JIN OH, AND SOHRAB SHAHSHAHANI

Abstract. In this paper we initiate the study of equivariant wave maps from $2d$ hyperbolic space, $\mathbb{H}^2$, into rotationally symmetric surfaces. This problem exhibits markedly different phenomena than its Euclidean counterpart due to the exponential volume growth of concentric geodesic spheres on the domain.

In particular, when the target is $\mathbb{S}^2$, we find a family of equivariant harmonic maps $\mathbb{H}^2 \to \mathbb{S}^2$, indexed by a parameter that measures how far the image of each harmonic map wraps around the sphere. These maps have energies taking all values between zero and the energy of the unique co-rotational Euclidean harmonic map, $Q_{\rm euc}$, from $\mathbb{R}^2$ to $\mathbb{S}^2$, given by stereographic projection. We prove that the harmonic maps are asymptotically stable for values of the parameter smaller than a threshold that is large enough to allow for maps that wrap more than halfway around the sphere. Indeed, we prove Strichartz estimates for the operator obtained by linearizing around such a harmonic map. However, for harmonic maps with energies approaching the Euclidean energy of $Q_{\rm euc}$, asymptotic stability via a perturbative argument based on Strichartz estimates is precluded by the existence of gap eigenvalues in the spectrum of the linearized operator.

When the target is $\mathbb{H}^2$, we find a continuous family of asymptotically stable equivariant harmonic maps $\mathbb{H}^2 \to \mathbb{H}^2$ with arbitrarily small and arbitrarily large energies. This stands in sharp contrast to the corresponding problem on Euclidean space, where all finite energy solutions scatter to zero as time tends to infinity.

1. Introduction

In recent years there has been increased interest in the study of dispersive equations on curved spaces. Here we begin the investigation of a simple model problem, namely, equivariant wave maps from $2d$ hyperbolic space, $\mathbb{H}^2$, into rotationally symmetric surfaces M. In local coordinates the equation is given by

$$\psi_{tt} - \psi_{rr} - \coth r\, \psi_r + \frac{g(\psi)g'(\psi)}{\sinh^2 r} = 0, \tag{1.1}$$

where (ψ, θ) are geodesic polar coordinates on the target surface M, and g determines the metric, $ds^2 = d\psi^2 + g^2(\psi)d\theta^2$.

An intriguing feature of this problem is that there is an abundance of finite energy stationary solutions. This fact stems from two well-known geometric facts: conformal invariance of $2d$ harmonic maps (which are time independent solutions to the problem) and conformal equivalence between $\mathbb{H}^2$ and the unit disk $\mathbb{D}^2$. Another

Support of the National Science Foundation from grants DMS-1302782 and NSF 1045119 for the first and third authors, respectively, is gratefully acknowledged. The second author is a Miller Research Fellow, and acknowledges support from the Miller Institute. The authors thank Joachim Krieger, Wilhelm Schlag and Daniel Tataru for many helpful discussions, and also Ji Hyun Bak for help with numerical experiments at the initial stage of investigation.

notable feature of this problem is the lack of scaling symmetry. This feature, which is in stark contrast to the Euclidean case and akin to the exterior wave map problem considered by Kenig, Schlag and the first author [14, 16], rules out an *a priori* obstruction to asymptotic stability of stationary solutions. However, while the curved background eliminates any natural scaling invariance for the problem, the model still exhibits features of an energy critical equation. Indeed, solutions with highly localized initial data do not see the global geometry of the domain and thus can be well approximated by solutions to the corresponding scale invariant energy critical Euclidean equation $\mathbb{R}^{1+2} \to M$. We believe that such properties make (1.1) an interesting model for investigating stability of stationary solutions, and more ambitiously, asymptotic resolution for large general finite energy solutions into solitons (*soliton resolution*) in the case of negatively curved targets such as $\mathbb{H}^2$, and characterization of blow-up solutions in the case of positively curved targets such as $\mathbb{S}^2$.

In this paper we establish asymptotic stability of various time independent solutions (i.e., harmonic maps) to the equivariant wave map equation (1.1) on $\mathbb{R} \times \mathbb{H}^2$. More specifically, we consider two targets, namely the two sphere $\mathbb{S}^2$ and the hyperbolic plane $\mathbb{H}^2$. In each case we classify all finite energy equivariant harmonic maps, which exist in abundance in contrast to the Euclidean case. We then study the stability of each harmonic map by analyzing spectral properties of the linearized operator.

We begin with our results when the target is $\mathbb{H}^2$, as these are easier to describe. In this case, we show that the spectrum of the linearized operator about each harmonic map consists purely of the absolutely continuous part with no eigenvalue or resonance at the edge. This spectral information allows us to prove Strichartz estimates for the linearized operator, from which asymptotic stability of the harmonic map follows by a Picard iteration argument.

The picture changes drastically in the case of the $\mathbb{S}^2$ target. Let Q_λ be the family of finite energy equivariant harmonic maps from $\mathbb{H}^2$ to $\mathbb{S}^2$. These maps can be parametrized by $\lambda \in [0, \infty)$ in such a way that as $\lambda \to 0+$ the image of Q_λ contracts to the north pole, and as $\lambda \to \infty$ the image of Q_λ covers the whole of $\mathbb{S}^2$ except for the south pole. For Q_λ with small λ, we prove that the spectrum of the linearized operator is absolutely continuous as in the case of $\mathbb{H}^2$. This allows us to prove Strichartz estimates in this case and therefore asymptotic stability holds by a perturbative argument. This scenario applies, in particular, to the harmonic map covering the northern hemisphere of $\mathbb{S}^2$. On the other hand, for Q_λ with large λ, we show that there exists a unique simple *gap eigenvalue* μ_λ^2 in $(0, \frac{1}{4})$ ($\frac{1}{4}$ is the edge of the a.c. spectrum). Moreover, we demonstrate that μ_λ^2 migrates toward zero as $\lambda \to \infty$. While this phenomenon precludes the possibility of scattering to Q_λ by a linear mechanism, it nevertheless suggests an interesting picture concerning nonlinear stability of Q_λ and the rate of scattering; we refer the reader to Remark 3.

1.1. **Additional context for the problem.** Although the authors are not aware of any previous investigations into the model at hand, there has been substantial activity of late regarding dispersive equations on $\mathbb{R} \times \mathbb{H}^d$, and it is partially in this context in which this problem can be viewed. Perhaps the most relevant recent works are the proofs of Strichartz estimates for the free wave equation on $\mathbb{R} \times \mathbb{H}^d$ together with global small data theory for semi-linear equations with power-type nonlinearities in [18, 19, 3], see also the many references therein. There has also been substantial activity in this direction for the Schrödinger equation on $\mathbb{R} \times \mathbb{H}^d$ and we refer the reader to [4, 5, 6, 13, 12, 2] as well as the references therein for more details. For treatments of the semi-linear elliptic problem, see [9, 17]. As we are investigating the asymptotic stability of certain stationary solutions to (1.1), we are forced to confront the linearized operator, which amounts to a radial free evolution operator on hyperbolic space plus a potential term. The dispersive estimates for the free evolution from [3] thus make up an essential ingredient in the proof.

1.2. **Setup.** To explain the main results in more detail, we now give a more precise account of our setup. Consider polar coordinates on the hyperboloid model of $\mathbb{H}^2$:

$$[0,\infty) \times S^1 \ni (r,\omega) \mapsto (\sinh r \sin\omega, \sinh r \cos\omega, \cosh r) \in \mathbb{R}^{2+1}.$$

Denote this map by $\Psi : [0,\infty) \times S^1 \to (\mathbb{R}^{2+1}, \mathbf{m})$, where $\mathbf{m}$ is the Minkowski metric on $\mathbb{R}^{2+1}$. The hyperbolic metric $\mathbf{h}$ in these coordinates is given by the pull-back of the Minkowski metric on $\mathbb{R}^{2+1}$ by the map Ψ, i.e., $\mathbf{h} = \Psi^* \mathbf{m}$. We have

$$(\mathbf{h}_{jk}) = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 r \end{pmatrix}.$$

The volume element is $\sqrt{|\mathbf{h}(r,\omega)|} = \sinh r$, and hence for $f : \mathbb{H}^2 \to \mathbb{R}$ we have

$$\int_{\mathbb{H}^2} f(x)\, d\mathrm{Vol}_{\mathbf{h}} = \int_0^{2\pi} \int_0^\infty f(\Psi(r,\omega)) \sinh r\, dr\, d\omega.$$

For radial functions, $f : \mathbb{H}^2 \to \mathbb{R}$ we abuse notation and write $f(x) = f(r)$ and

$$\int_{\mathbb{H}^2} f(x)\, d\mathrm{Vol}_{\mathbf{h}} = 2\pi \int_0^\infty f(r) \sinh r\, dr.$$

We will focus attention on two rotationally symmetric target manifolds, namely, $M = \mathbb{S}^2$ and $M = \mathbb{H}^2$. We begin with the positively curved case, $\mathbb{S}^2$.

1.3. **Equivariant wave maps:** $\mathbb{R} \times \mathbb{H}^2 \to \mathbb{S}^2$**.** In this section we consider wave maps $U : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{S}^2$. As both the domain and the target are rotationally symmetric we can consider a restricted class of maps, U, satisfying the equivariance $U \circ \rho = \rho \circ U$, for all rotations $\rho \in SO(2)$. In fact we consider the special subclass of such maps known as 1-equivariant, or co-rotational, which corresponds to equivariant maps which in local coordinates take the form

$$U(t,r,\omega) = (\psi(t,r),\omega) \hookrightarrow (\sin\psi \sin\omega, \sin\psi \cos\omega, \cos\psi),$$

where ψ is the azimuth angle measured from the north pole of the sphere and the metric on $\mathbb{S}^2$ is given by $ds^2 = d\psi^2 + \sin^2\psi\, d\omega^2$ (for a more general class of equivariant maps one can consider an ansatz of the form $U(t,r,\omega) = (\psi(t,r), \omega + \chi(t,r))$). In this formulation, 1-equivariant wave maps are formal critical points of the Lagrangian

$$\mathcal{L}(U) = \frac{1}{2} \int_{\mathbb{R}} \int_0^\infty \left(-\psi_t^2(t,r) + \psi_r^2(t,r) + \frac{\sin^2\psi(t,r)}{\sinh^2 r} \right) \sinh r\, dr\, dt.$$

The Euler-Lagrange equations reduce to an equation for the azimuth angle ψ and we are led to the Cauchy problem:

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \coth r\, \psi_r + \frac{\sin(2\psi)}{2\sinh^2 r} = 0, \\ &\vec{\psi}(0) = (\psi_0, \psi_1). \end{aligned} \tag{1.2}$$

We will often use the notation $\vec{\psi}(t)$ to denote the pair $\vec{\psi}(t,r) := (\psi(t,r), \psi_t(t,r))$. The conserved energy is given by

$$\mathcal{E}(\vec{\psi})(t) = \frac{1}{2}\int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sin^2\psi}{\sinh^2 r}\right)\sinh r\,dr = \text{constant}. \tag{1.3}$$

Note that for initial data $\vec{\psi}(0) = (\psi_0, \psi_1)$ to have finite energy we need $\psi_0(0) = k\pi$ for some $k \in \mathbb{Z}$. For the solution to depend continuously on the initial data this integer k must be preserved by the evolution. Here we consider the case $k = 0$, corresponding to maps that send $r = 0$ (the vertex of the hyperboloid) to the north pole of $\mathbb{S}^2$, as the other cases are similar.

The behavior of finite energy data at $r = \infty$ is more flexible. One can check that $\psi_0(r)$ has a well defined limit as $r \to \infty$, but that this limit can be any finite number, i.e., $\mathcal{E}(\psi_0, \psi_1) < \infty$ implies there exists $\alpha \in \mathbb{R}$ so that $\lim_{r\to\infty}\psi_0(r) = \alpha$. This stands in sharp contrast to the corresponding problem for wave maps $\mathbb{R}^{1+2} \to \mathbb{S}^2$ where the endpoint can only be an integer multiple of π giving such maps a fixed topological degree. Here, the fact that any finite endpoint is allowed can be attributed to the rapid decay of $\sinh^{-1} r$ as $r \to \infty$ in the last term in the integrand of (1.3), and is ultimately responsible for the existence of the family of harmonic maps mentioned in the abstract.

In this paper we will consider initial data with endpoints $\psi_0(\infty) = \alpha$ for $\alpha \in [0, \pi)$, which means that we will only consider those ψ_0 which do not reach the south pole. This leads us to define the energy classes

$$\mathcal{E}_\lambda := \{(\psi_0, \psi_1) \mid \mathcal{E}(\psi_0, \psi_1) < \infty,\ \psi_0(0) = 0,\ \psi_0(\infty) = 2\arctan(\lambda)\}. \tag{1.4}$$

for $\lambda \in [0, \infty)$. The reason for this restriction to $\alpha \in [0, \pi)$ is given by the presence of a $1-$parameter family, Q_λ, of finite energy harmonic maps with such endpoints, i.e., solutions to

$$\begin{aligned} &Q_{rr} + \coth r\, Q_r = \frac{\sin 2Q}{2\sinh^2 r}, \\ &Q(0) = 0, \quad \lim_{r\to\infty} Q(r) = 2\arctan(\lambda). \end{aligned} \tag{1.5}$$

For every $\lambda \in [0, \infty)$ there is a unique finite energy solution Q_λ to (1.5), given by

$$Q_\lambda(r) = 2\arctan(\lambda\tanh(r/2)). \tag{1.6}$$

Moreover, $(Q_\lambda, 0)$ has energy

$$\mathcal{E}(Q_\lambda, 0) = 1 - \cos(Q_\lambda(\infty)) = 2\frac{\lambda^2}{\lambda^2 + 1}$$

which is minimal in $\mathcal{E}_\lambda$ – in other words for each angle $\alpha \in [0, \pi)$, there exists a map connecting 0 to α which uses the minimum possible amount of energy and this map is, in fact, the *harmonic map* Q_λ with $\lambda = \tan(\alpha/2)$. For endpoints $\alpha \geq \pi$ there are *no* finite energy harmonic maps. We provide a more detailed description of the Q_λ with proofs of the preceding statements in Section 2.1.

The existence of the Q_λ stands in stark contrast to the corresponding Euclidean problem, equivariant wave maps $\mathbb{R}^{1+2} \to \mathbb{S}^2$, which reduce to the following equation for the azimuth angle ψ:

$$\psi_{tt} - \psi_{rr} - \frac{1}{r}\psi_r + \frac{\sin 2\psi}{2r^2} = 0. \tag{1.7}$$

In fact, the unique (up to scaling) Euclidean equivariant harmonic map is given by $Q_{\mathrm{euc}}(r) = 2\arctan(r)$ which connects the north pole to the south pole of the sphere. Q_{euc} is the unique, nontrivial, finite energy solution to

$$Q_{rr} + \frac{1}{r}Q_r = \frac{\sin 2Q}{2r^2}, \quad Q(0) = 0. \tag{1.8}$$

We remark that Q_{euc} minimizes the Euclidean energy

$$\mathcal{E}_{\mathrm{euc}}(\psi_0, \psi_1) = \frac{1}{2}\int_0^\infty \left[(\partial_r \psi_0)^2 + \psi_1^2 + \frac{\sin^2 \psi_0}{r^2}\right] r\, dr \tag{1.9}$$

amongst all degree one maps, i.e., those which satisfy $\psi_0(0) = 0, \psi_0(\infty) = \pi$ and by direct computation one sees that $\mathcal{E}_{\mathrm{euc}}(Q_{\mathrm{euc}}, 0) = 2$. We note that for the hyperbolic harmonic maps Q_λ we have

$$\begin{aligned} &\mathcal{E}(Q_\lambda, 0) \to \mathcal{E}_{\mathrm{euc}}(Q_{\mathrm{euc}}, 0) \quad \text{as} \quad \lambda \to \infty, \\ &\mathcal{E}(Q_\lambda, 0) \to 0 \quad \text{as} \quad \lambda \to 0. \end{aligned}$$

1.3.1. *Asymptotic stability of* Q_λ. It is well known that Q_{euc} is *unstable* with respect to the Euclidean equivariant wave map flow and in fact, leads to finite time blow-up, see [10, 15].

A natural question to ask is whether $(Q_\lambda, 0)$ is asymptotically stable for fixed $\lambda \in [0, \infty)$ under the wave map evolution, (1.2), in $\mathcal{E}_\lambda$. It is this question that we address here. The natural space in which to consider solutions to (1.2) is the energy space

$$\|(\psi_0, \psi_1)\|^2_{\mathcal{H}_0} := \int_0^\infty \left[(\partial_r \psi_0)^2(r) + \psi_1^2(r) + \frac{\psi_0^2(r)}{\sinh^2 r}\right] \sinh r\, dr. \tag{1.10}$$

Indeed, we endow $\mathcal{E}_\lambda$ with the "norm"

$$\|(\psi_0, \psi_1)\|_{\mathcal{E}_\lambda} := \|(\psi_0, \psi_1) - (Q_\lambda, 0)\|_{\mathcal{H}_0}. \tag{1.11}$$

The first result is an affirmative answer to the above question for a range of $\lambda \in [0, \lambda_0)$ for some $\lambda_0 \geq \sqrt{15/8}$.

Theorem 1.1. *There exists* $\lambda_0 \geq \sqrt{15/8}$ *so that for every* $0 \leq \lambda < \lambda_0$*, the harmonic map* Q_λ *is asymptotically stable in the space* $\mathcal{E}_\lambda$*. In particular, there exists a* $\delta_0 > 0$ *such that for every* $(\psi_0, \psi_1) \in \mathcal{E}_\lambda$ *with*

$$\|(\psi_0, \psi_1) - (Q_\lambda, 0)\|_{\mathcal{H}_0} < \delta_0$$

there exists a unique global solution $\vec{\psi}(t) \in \mathcal{E}_\lambda$ *to* (1.2)*. Moreover,* $\vec{\psi}(t)$ *scatters to* $(Q_\lambda, 0)$ *as* $t \to \pm\infty$*.*

Remark 1. The phrase $\vec{\psi}(t)$ *scatters to* $(Q_\lambda, 0)$ *as* $t \to \pm\infty$ means that there exist solutions $\vec{\varphi}_L^{\pm}(t)$ to the linearized equation

$$\varphi_{tt} - \varphi_{rr} - \coth r\, \varphi_r + \frac{1}{\sinh^2 r}\varphi = 0, \tag{1.12}$$

so that

$$\|\vec{\psi}(t) - (Q_\lambda, 0) - \vec{\varphi}_L^{\pm}(t)\|_{\mathcal{H}_0} \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{1.13}$$

Remark 2. We note that the number $\sqrt{15/8}$ appears for a technical reason that will be further explained in Section 3. In short, it is the largest value for λ for which we have a simple proof that the linearized operator about Q_λ has no discrete spectrum. The number $2\arctan\sqrt{15/8}$ is slightly less than $3\pi/5$ which means that our stability result holds for maps which wrap more that halfway around the sphere.

The proof of Theorem 1.1 reduces to Strichartz estimates for the linearized operator after first passing to a radial wave equation on $\mathbb{R}\times\mathbb{H}^4$. The reason we can pass to waves on $\mathbb{H}^4$ comes from the fact that the nonlinearity in (1.2) contains a repulsive potential term:

$$\frac{\sin 2\psi}{2\sinh^2 r} = \frac{1}{\sinh^2 r}\psi + \frac{\sin 2\psi - 2\psi}{2\sinh^2 r}.$$

This indicates that the linear part of (1.2) has more dispersion than a free wave on $\mathbb{R}\times\mathbb{H}^2$. In fact, after linearizing about $(Q_\lambda, 0)$ we prove that for λ as in Theorem 1.1, the linear part has the same dispersion as a free wave on $\mathbb{R}\times\mathbb{H}^4$. This can be seen by making the following change of variables. For a solution $\vec{\psi}(t)\in\mathcal{E}_\lambda$ define $u(t)$ by

$$\sinh r\, u(t,r) := \psi(t,r) - Q_\lambda(r). \tag{1.14}$$

We obtain the following equation for $\vec{u}(t)$,

$$\begin{aligned} &u_{tt} - u_{rr} - 3\coth r\, u_r - 2u + V_\lambda(r)u = \mathcal{N}_{\mathbb{S}^2}(r,u)\\ &\vec{u}(0) = (u_0, u_1)\end{aligned} \tag{1.15}$$

where the *attractive* potential V_λ and the nonlinearity $\mathcal{N}_{\mathbb{S}^2}$ are given by

$$V_\lambda(r) := \frac{\cos 2Q_\lambda - 1}{\sinh^2 r} \le 0 \tag{1.16}$$

$$\mathcal{N}_{\mathbb{S}^2}(r,u) := \frac{\sin 2Q_\lambda}{\sinh^3 r}\sin^2(2\sinh r\, u) + \cos 2Q_\lambda \frac{2\sinh r\, u - \sin(2\sinh r\, u)}{2\sinh^3 r} \tag{1.17}$$

The underlying linear equation under consideration is then given by

$$v_{tt} - \Delta_{\mathbb{H}^4} v - 2v + V_\lambda v = 0 \tag{1.18}$$

for radially symmetric functions v. In Section 4 we prove Strichartz estimates in Proposition 4.2 for (1.18) with $\lambda\in[0,\lambda_0)$ using the spectral transformation, or the *distorted Fourier transform*, for the self-adjoint Schrödinger operators

$$\begin{aligned} H_0 &:= -\partial_{rr} - 3\coth r\,\partial_r - 2,\\ H_{V_\lambda} &:= -\partial_{rr} - 3\coth r\,\partial_r - 2 + V_\lambda,\end{aligned} \tag{1.19}$$

following roughly the argument in [16], which was based on techniques from [22], see also [24, 25]. The spectrum $\sigma(H_{V_\lambda})$ plays a central role in determining the dispersive properties of the wave equation (1.18). It is well known that the spectrum of the Laplacian on $\mathbb{H}^4$ is given by $\sigma(\Delta_{\mathbb{H}^4}) = [9/4,\infty)$ where here $9/4 = \left(\frac{d-1}{2}\right)^2$ for $d=4$, and thus we have $\sigma(H_0) = [1/4,\infty)$ for the shifted operator $H_0 = -\Delta_{\mathbb{H}^4} - 2$. The key to our analysis is the existence of $\lambda_0\in(0,\infty)$ (in fact we can prove that $\lambda_0 \ge \sqrt{15/8}$) so that for all $0\le\lambda<\lambda_0$, the perturbed operator H_{V_λ} has purely absolutely continuous spectrum equal to $[1/4,\infty)$. In particular, H_{V_λ} has no negative spectrum, no eigenvalues in the gap $[0,1/4)$, and the threshold $1/4$ is neither an eigenvalue nor a resonance.

However, as λ becomes large, which means that the harmonic map Q_λ wraps further around the sphere, we observe a change in the spectrum of H_{V_λ} which results in a breakdown in the dispersive behavior of solutions to the linear equation (1.18). In particular, as $\lambda \to \infty$ we establish the existence of a simple gap eigenvalue $\mu_\lambda^2 \in (0, 1/4)$. Moreover we show that as $\lambda \to \infty$ the eigenvalue μ_λ^2 migrates to 0. In particular, we prove the following result.

Theorem 1.2. *There exists $\Lambda_0 > 0$ so that for all $\lambda > \Lambda_0$, the Schrödinger operator H_{V_λ} has a unique, simple eigenvalue, μ_λ^2, in the spectral gap $(0, 1/4)$. That is, there exists a solution $\varphi_\lambda \in L^2(\mathbb{H}^4)$ to*

$$H_{V_\lambda}\varphi_\lambda = \mu_\lambda^2 \varphi_\lambda. \tag{1.20}$$

where $\mu_\lambda^2 \in (0, 1/4)$. Moreover, we have

$$\mu_\lambda^2 \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{1.21}$$

Finally, if we define

$$\begin{aligned} &\lambda_{\mathrm{sup}} := \sup\{\lambda \mid H_{V_{\tilde\lambda}} \text{ has no e-vals and no threshold resonance } \forall\, \tilde\lambda < \lambda\} \\ &\Lambda_{\mathrm{inf}} := \inf\{\lambda \mid H_{V_{\tilde\lambda}} \text{ has a gap e-val } \mu_{\tilde\lambda}^2 \in (0, 1/4)\ \forall\, \tilde\lambda > \lambda\} \end{aligned} \tag{1.22}$$

Then both $H_{\lambda_{\mathrm{sup}}}$ and $H_{\Lambda_{\mathrm{inf}}}$ have threshold resonances.

Remark 3. One immediate consequence of the presence of the gap eigenvalues for large λ is that we can no longer prove a stability result as in Theorem 1.1, by a perturbative argument based on the dispersive properties of the underlying linear equation, i.e., Strichartz estimates. On the other hand, a Struwe-type bubbling argument [29] suggests that any solution $\vec\psi(t)$ to (1.2) that blows up in finite time must bubble off a Euclidean harmonic map Q_{euc} and thus must have enough energy to wrap completely around the sphere. This gives some evidence towards a conjecture that in fact every Q_λ is stable – as small perturbations of Q_λ will not have enough energy to bubble off a Q_{euc} – but for large λ, the stability manifests via a completely nonlinear mechanism, possibly as in the work of Soffer, Weinstein [27].

Remark 4. At this point we do not know the precise location of $\lambda_0 = \lambda_{\mathrm{sup}}$ in Theorem 1.1, or of $\Lambda_0 = \Lambda_{\mathrm{inf}}$ in Theorem 1.2 or whether these two values are equal. Indeed, the existence of gap eigenvalues for large λ is demonstrated by a contradiction argument and thus does not reveal a precise geometric reason for the breakdown in *linear stability* described in Remark 3. On the other hand, this asymptotic-in-λ failure of linear stability is natural in view of the bubbling mentioned in Remark 3 and the explicit blow-up constructions for the corresponding Euclidean problem from [15, 23, 21]. Indeed the Euclidean blow-up constructions rely on energy concentration schemes which see only the local geometry of space, which suggests similar behavior is possible for the hyperbolic problem at hand, as long as the solution has enough energy to bubble off a Q_{euc}.

Remark 5. The existence of gap eigenvalues is a rather surprising feature of this model, as this contrasts greatly with the corresponding Euclidean wave maps problem. Key to the proof of Theorem 1.2 is the fact that after a renormalization, the Schrödinger operator H_{V_λ} formally approaches (as $\lambda \to \infty$) the operator $H_{V_{\mathrm{euc}}}$ obtained by linearizing (1.7) about Q_{euc}. Assuming, for contradiction, the nonexistence of a gap eigenvalue, this formal approximation can be made precise on a region that increases in size as λ increases. This fact allows us to treat the hyperbolic spectral picture as a perturbation of its Euclidean counterpart. We can then pair the existence of a threshold resonance for the Euclidean problem together with the existence of the spectral gap in the hyperbolic problem to force a contradiction. We refer the reader to Sections 3.2–3.5 for details.

1.4. **Equivariant wave maps:** $\mathbb{R} \times \mathbb{H}^2 \to \mathbb{H}^2$. We next consider wave maps $U : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{H}^2$, again restricting attention to co-rotational maps, meaning maps U, which in coordinates take the form

$$U(t, r, \omega) = (\psi(t, r), \omega) \hookrightarrow (\sinh \psi \sin \omega, \sinh \psi \cos \omega, \cosh \psi) \in \mathbb{R}^{2+1},$$

where the metric on the target $\mathbb{H}^2$ is given by $ds^2 = d\psi^2 + \sinh^2 \psi \, d\omega^2$. In this formulation, 1-equivariant wave maps are formal critical points of the Largrangian

$$\mathcal{L}(U) = \frac{1}{2} \int_{\mathbb{R}} \int_0^\infty \left(-\psi_t^2(t, r) + \psi_r^2(t, r) + \frac{\sinh^2 \psi(t, r)}{\sinh^2 r} \right) \sinh r \, dr \, dt.$$

The Euler-Lagrange equations reduce to an equation for ψ and we are led to the Cauchy problem:

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \coth r \, \psi_r + \frac{\sinh(2\psi)}{2 \sinh^2 r} = 0, \\ &\vec{\psi}(0) = (\psi_0, \psi_1). \end{aligned} \tag{1.23}$$

The conserved energy is given by

$$\mathscr{E}(\vec{\psi})(t) = \frac{1}{2} \int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{\sinh^2 \psi}{\sinh^2 r} \right) \sinh r \, dr = \text{constant}. \tag{1.24}$$

Note that for initial data $\vec{\psi}(0) = (\psi_0, \psi_1)$ to have finite energy we need $\psi_0(0) = 0$, which means that a finite energy map must fix the vertex of the hyperboloid. The behavior of ψ at $r = \infty$ is again more flexible than the corresponding Euclidean equation for wave maps $\mathbb{R}^{1+2} \to \mathbb{H}^2$ due to the rapid decay of $\sinh^{-1} r$ as $r \to \infty$. We note that for any finite energy data (ψ_0, ψ_1) the limit $\lim_{r \to \infty} \psi_0(r) = \alpha$ exists but can take any value $\alpha \in [0, \infty)$. We thus again define the energy classes

$$\mathscr{E}_\lambda := \{(\psi_0, \psi_1) \mid \mathscr{E}(\vec{\psi}) < \infty, \ \psi_0(0) = 0, \ \psi_0(\infty) = 2\text{arctanh}\,(\lambda)\} \tag{1.25}$$

with $\lambda \in [0, 1)$. We will demonstrate the presence of a family of nontrivial harmonic maps taking all energies ranging from 0 to infinity. This is a surprising and distinctive feature of this model in light of the fact that no such maps exist in the corresponding Euclidean problem. In this context a harmonic map is a solution P to the equation

$$\begin{aligned} &P_{rr} + \coth r \, P_r = \frac{\sinh 2P}{2 \sinh^2 r}, \\ &P(0) = 0, \quad \lim_{r \to \infty} P(r) = 2\text{arctanh}\,(\lambda). \end{aligned} \tag{1.26}$$

For every $\lambda \in [0, 1)$ there is a unique finite energy solution P_λ to (1.26) given by

$$P_\lambda(r) := 2\text{arctanh}\,(\lambda \tanh(r/2)) \tag{1.27}$$

In Section 2.2 we show that P_λ has energy

$$\mathscr{E}(P_\lambda, 0) = 2 \frac{\lambda^2}{1 - \lambda^2}$$

which minimizes the energy in $\mathscr{E}_\lambda$. Note that $\mathscr{E}(P_\lambda, 0) \to \infty$ as $\lambda \to 1^-$ and $\mathscr{E}(P_\lambda, 0) \to 0$ as $\lambda \to 0$.

We recall the well known fact that for the Euclidean case of wave maps from $\mathbb{R}^{1+2} \to \mathbb{H}^2$ there are *no finite energy nontrivial harmonic maps* due to the negative curvature of the target $\mathbb{H}^2$.

1.4.1. *Asymptotic stability of P_λ.* We now turn to the question of the asymptotic stability of P_λ in $\mathscr{E}_\lambda$ for fixed $\lambda \in [0, 1)$. We prove the following result.

Theorem 1.3. *For every $\lambda \in [0, 1)$ the harmonic map P_λ is asymptotically stable in $\mathscr{E}_\lambda$. In particular, for each $\lambda \in [0, 1)$ there exists a $\delta_0 > 0$ so that for every $(\psi_0, \psi_1) \in \mathscr{E}_\lambda$ with*

$$\|(\psi_0, \psi_1) - (P_\lambda, 0)\|_{\mathcal{H}_0} < \delta_0$$

there exists a unique, global solution $\vec{\psi}(t) \in \mathscr{E}_\lambda$ to (1.23). Moreover, $\vec{\psi}(t)$ scatters to $(P_\lambda, 0)$ as $t \to \pm\infty$.

The proof of Theorem 1.3 follows the same outline as in the previous subsection. In particular we establish Strichartz estimates for the operator obtained by linearizing about P_λ and then passing to an equation on $\mathbb{R} \times \mathbb{H}^4$. For a solution $\vec{\psi}(t) \in \mathscr{E}_\lambda$ to (1.23) we define $\vec{u}(t)$ by

$$\sinh r\, u(t, r) := \psi(t, r) - P_\lambda(r). \tag{1.28}$$

Then $\vec{u}(t)$ solves

$$\begin{aligned} &u_{tt} - u_{rr} - 3\coth r\, u_r - 2u + U_\lambda(r)u = \mathcal{N}_{\mathbb{H}^2}(r, u) \\ &\vec{u}(0) = (u_0, u_1) \end{aligned} \tag{1.29}$$

where the *repulsive* potential U_λ and the nonlinearity $\mathcal{N}_{\mathbb{H}^2}$ are given by

$$U_\lambda(r) := \frac{\cosh 2P_\lambda - 1}{\sinh^2 r} \geq 0 \tag{1.30}$$

$$\mathcal{N}_{\mathbb{H}^2}(r, u) := -\frac{\sinh 2P_\lambda}{\sinh^3 r}\sinh^2(2\sinh r\, u) + \cosh 2P_\lambda \frac{2\sinh r\, u - \sinh(2\sinh r\, u)}{2\sinh^3 r} \tag{1.31}$$

The underlying linear equation under consideration is then given by

$$v_{tt} - \Delta_{\mathbb{H}^4} v - 2v + U_\lambda v = 0 \tag{1.32}$$

for radially symmetric functions v. In Section 4 we prove Strichartz estimates in Proposition 4.2 for (1.32) using the spectral transformation, or the *distorted Fourier transform*, for the self-adjoint Schrödinger operators

$$\begin{aligned} H_0 &:= -\partial_{rr} - 3\coth r\, \partial_r - 2, \\ H_{U_\lambda} &:= -\partial_{rr} - 3\coth r\, \partial_r - 2 + U_\lambda, \end{aligned} \tag{1.33}$$

again following roughly the argument in [16, 22, 24, 25]. The key point here is that the repulsive potential U_λ rules out the possibility of discrete spectrum for $\sigma(H_{U_\lambda})$, and thus H_{U_λ} has essential spectrum $[1/4, \infty)$, with no negative spectrum, no gap eigenvalues, and no eigenvalue or resonance at the threshold $1/4$.

1.5. **Brief outline of the paper.** In Section 2 we establish the various facts about the harmonic maps Q_λ and P_λ defined above. We also give more details concerning the passage to equations on $\mathbb{R} \times \mathbb{H}^4$ outlined above. In particular, we show that the small data Cauchy problems, the $2d$ linearized problem in $\mathcal{H}_0$ and the $4d$ problem in $H^1 \times L^2(\mathbb{H}^4)$ are equivalent.

In Section 3 we study the spectrum of the linearized operator H_{V_λ}, which corresponds to $\mathbb{S}^2$ valued maps. We begin by showing that $\sigma(H_{V_\lambda})$ has no discrete spectrum for $\lambda < \sqrt{15/8}$. Beginning from Section 3.2, we then present the proof of Theorem 1.2.

In Section 4 we prove Strichartz estimates – Proposition 4.2– for the linearized equations (1.18) and (1.32). In the former case, we need to restrict to values of λ as in Theorem 1.1. In the latter case, the Strichartz estimates hold for all $\lambda \in [0, 1)$ since the potential U_λ is repulsive.

Finally, in Section 5 we prove Theorem 1.1 and Theorem 1.3 by the usual contraction mapping argument based on the Strichartz estimates proved in Proposition 4.2.

2. Preliminaries

In this section we establish the existence and uniqueness of the harmonic maps Q_λ and P_λ described in the introduction. We give simple geometric descriptions of these maps and prove several properties that we will need in the ensuing arguments. We also prove some additional preliminary facts including an equivalence between the $2d$ and $4d$ Cauchy problems described in the introduction.

We begin with the case of harmonic maps into $\mathbb{S}^2$.

2.1. **Harmonic maps into $\mathbb{S}^2$.** Here we prove various facts about the harmonic maps Q_λ. For convenience we collect these facts into a proposition.

Proposition 2.1. *For every $0 \le \alpha < \pi$ there exists a unique, finite energy stationary solution to* (1.2)*, i.e., a harmonic map,* $(Q_\lambda, 0) \in \mathcal{E}_\lambda$ *which solves* (1.5)*, where*

$$\begin{aligned} &Q_\lambda(r) = 2\arctan(\lambda \tanh(r/2)) \\ &\lambda \in [0, \infty),\ \alpha = \alpha(\lambda) = 2\arctan(\lambda) = \lim_{r \to \infty} Q_\lambda(r). \end{aligned} \tag{2.1}$$

Moreover, $(Q_\lambda, 0) \in \mathcal{E}_\lambda$ *has energy*

$$\mathcal{E}(Q_\lambda, 0) = 2\frac{\lambda^2}{1+\lambda^2}, \tag{2.2}$$

which is minimal in $\mathcal{E}_\lambda$. Finally, the Q_λ with $\lambda \in [0, \infty)$ are the only finite energy stationary solutions to (1.2)*.*

Proof. We are seeking to classify all stationary finite energy solutions to (1.2). Recall from the introduction that any finite energy harmonic map Q must have $Q(0) = 0$ and $Q(\infty) = \alpha \in [0, \infty)$. Thus we would like to find all solutions Q to

$$\begin{aligned} &Q_{rr} + \coth r\, Q_r = \frac{\sin 2Q}{2\sinh^2 r}, \\ &Q(0) = 0,\ \lim_{r \to \infty} Q(r) = \alpha \in [0, \infty). \end{aligned} \tag{2.3}$$

One can check directly that Q_λ, as defined in (2.1), satisfies (2.3) with $\alpha = 2\arctan(\lambda)$. One can also directly compute the energy to verify (2.2).

To prove the remaining statements in Proposition 2.1 we begin by giving a simple geometric interpretation of Q_λ. Recall that stereographic projection of $\mathbb{H}^2$ onto the Poincaré disc, $\mathbb{D}$, viewed as a subset of $\mathbb{R}^2$, is given by the map

$$(\sinh r\ \cos\omega, \sinh r\ \sin\omega, \cosh r) \mapsto (\tanh(r/2)\cos\omega, \tanh(r/2)\ \sin\omega).$$

Next, we rescale the disc by $\lambda \in [0,\infty)$ via

$$(\tanh(r/2)\cos\omega, \tanh(r/2)\ \sin\omega) \mapsto (\lambda\tanh(r/2)\cos\omega, \lambda\tanh(r/2)\ \sin\omega).$$

Finally, recall that the inverse of stereographic projection, $\mathbb{R}^2 \to \mathbb{S}^2 - \{\text{south pole}\}$ is given by

$$(\rho\cos\omega, \rho\sin\omega) \mapsto (\sin(2\arctan\rho)\ \cos\omega, \sin(2\arctan\rho)\ \sin\omega, \cos(2\arctan\rho)).$$

Then as solutions to (1.2) or (1.5) are expressed in terms of the azimuth angle on $\mathbb{S}^2$, we see that Q_λ is simply the composition of the above three maps.

This geometric interpretation motivates the following change of variables in (2.3). Setting

$$s := \log(\tanh(r/2)),\ \ \varphi(s) := Q(r), \tag{2.4}$$

we see that (2.3) reduces to the following equation for φ:

$$\begin{aligned} &\varphi'' = \frac{1}{2}\sin 2\varphi, \\ &\varphi(-\infty) = 0,\ \ \varphi(0) = \alpha. \end{aligned} \tag{2.5}$$

which is an autonomous ode and none other than the equation for the pendulum. Multiplying the first line in (2.5) by φ' and integrating from s_1 to s_2 yields the energy identity

$$\varphi_s^2(s_2) - \varphi_s^2(s_1) = \sin^2(\varphi(s_2)) - \sin^2(\varphi(s_1)) \tag{2.6}$$

A standard analysis of the phase portrait in (φ,φ') coordinates together with (2.6) mandates the condition that any nontrivial solution satisfies $0 < \varphi(0) < \pi$. In particular, we note that a solution with $\varphi(-\infty) = 0$ corresponds to the unstable manifold at $(0,0)$, (which connects to the stable manifold at $(\pi, 0)$ as $s \to +\infty$). Using (2.6) one sees that if there existed a nontrivial trajectory emanating from $(0,0)$ at $s = -\infty$ and such that $\varphi(s_0) = \pi$ for some $s_0 \in \mathbb{R}$, then we would have $\varphi_s(s_0) = 0$. But then $(\varphi,\varphi_s)(s_0) = (\pi, 0)$ and therefore φ must be a trivial solution, which contradicts our assumption.

One can also see this by noting that the unique positive solution (up to translation in s) is given by $\varphi(s) = 2\arctan(e^s)$. In particular, we have shown that for each $\alpha \in [0,\pi)$ there is a unique solution to (2.3). For $\alpha \geq \pi$ there are no solutions.

It remains to show that $(Q_\lambda, 0)$ minimizes the energy in $\mathcal{E}_\lambda$. This follows as a direct consequence of the following "Bogomol'nyi factorization": Let $\vec{\psi}(t) = (\psi(t), \psi_t(t)) \in \mathcal{E}_\lambda$. Then we have

$$\begin{aligned} \mathcal{E}(\vec{\psi}) &= \frac{1}{2}\int_0^\infty \psi_t^2 \sinh r\,dr + \frac{1}{2}\int_0^\infty \left(\psi_r - \frac{\sin\psi}{\sinh r}\right)^2 \sinh r\,dr + \int_0^\infty \sin\psi\psi_r\,dr \\ &= \frac{1}{2}\int_0^\infty \psi_t^2 \sinh r\,dr + \frac{1}{2}\int_0^\infty \left(\psi_r - \frac{\sin\psi}{\sinh r}\right)^2 \sinh r\,dr + \cos\psi(t,0) - \cos\psi(t,\infty) \\ &= \frac{1}{2}\int_0^\infty \psi_t^2 \sinh r\,dr + \frac{1}{2}\int_0^\infty \left(\psi_r - \frac{\sin\psi}{\sinh r}\right)^2 \sinh r\,dr + 1 - \cos(2\arctan(\lambda)) \end{aligned}$$

For the solution $\vec{\psi}(t) = (Q_\lambda, 0)$ the first two integrals–which we note are always non-negative–vanish identically, which proves that $(Q_\lambda, 0)$ uniquely minimizes the energy in $\mathcal{E}_\lambda$. Finally, a simple calculation yields

$$\mathcal{E}(Q_\lambda, 0) = 1 - \cos(2\arctan(\lambda)) = 2\frac{\lambda^2}{1+\lambda^2}$$

and this completes the proof. □

2.2. **Harmonic maps into $\mathbb{H}^2$.** Here we prove the analogous result for P_λ while providing a simple geometric interpretation.

Proposition 2.2. *For every $\beta \in [0, \infty)$ there exists a unique, finite energy stationary solution to* (1.23)*, i.e., a harmonic map, $(P_\lambda, 0) \in \mathscr{E}_\lambda$ which solves* (1.26)*, where*

$$\begin{aligned} &P_\lambda(r) = 2\,\mathrm{arctanh}\,(\lambda \tanh(r/2)) \\ &\lambda \in [0,1), \ \beta = \beta(\lambda) = 2\mathrm{arctanh}\,(\lambda) = \lim_{r\to\infty} P_\lambda(r). \end{aligned} \tag{2.7}$$

Moreover, $(P_\lambda, 0) \in \mathscr{E}_\lambda$ has energy

$$\mathscr{E}(P_\lambda, 0) = 2\frac{\lambda^2}{1-\lambda^2}, \tag{2.8}$$

which is minimal in $\mathscr{E}_\lambda$. Finally, the P_λ with $\lambda \in [0,1)$ are the only finite energy stationary solutions to (1.23)*.*

Proof. We would like to classify solutions to

$$\begin{aligned} &P_{rr} + \coth r\, P_r = \frac{\sinh 2P}{2\sinh^2 r}, \\ &P(0) = 0, \quad \lim_{r\to\infty} P(r) = \beta. \end{aligned} \tag{2.9}$$

for $\beta \in [0, \infty)$. One can check directly that P_λ as defined in (2.7) solves (2.9) with $\beta = 2\mathrm{arctanh}\,(\lambda)$, and the energy of $(P_\lambda, 0)$ satisfies (2.8).

For a geometric interpretation of the P_λ we recall again the stereographic of $\mathbb{H}^2$ onto the Poincaré disc, $\mathbb{D}$, which is a conformal isomorphism and is given, in coordinates by

$$(\sinh r\, \cos\omega, \sinh r\, \sin\omega, \cosh r) \mapsto (\tanh(r/2)\cos\omega, \tanh(r/2)\, \sin\omega).$$

Next, we perform the map $z \mapsto \lambda z$, $z \in \mathbb{D}$, which is *finite energy harmonic map* from $\mathbb{D} \to \mathbb{D}$ for $\lambda \in [0,1)$. In coordinates this is given by

$$(\tanh(r/2)\cos\omega, \tanh(r/2)\, \sin\omega) \mapsto (\lambda\tanh(r/2)\cos\omega, \lambda\tanh(r/2)\, \sin\omega).$$

Finally, recall that the inverse of stereographic projection, $\mathbb{D} \to \mathbb{H}^2$ given by

$$(\rho\cos\omega, \rho\sin\omega) \mapsto (\sinh(2\mathrm{arctanh}\,\rho)\, \cos\omega, \sinh(2\mathrm{arctanh}\,\rho)\, \sin\omega, \cosh(2\mathrm{arctanh}\,\rho)).$$

It is clear that P_λ is a composition of these three maps.

As is the case for maps to the sphere, we can also view (2.9) as an autonomous equation, with the change of variables

$$s = \log(\tanh(r/2)), \ \phi(s) = P(r).$$

Then (2.9) reduces to

$$\phi'' = \frac{1}{2}\sinh 2\phi$$
$$\phi(-\infty) = 0, \ \phi(0) = \beta$$

from which the existence and uniqueness of the P_λ is also apparent.

Finally, to show that P_λ minimizes the energy in $\mathscr{E}_\lambda$ we again preform the "Bogomol'nyi factorization": Let $\vec{\psi}(t) = (\psi(t), \psi_t(t)) \in \mathscr{E}_\lambda$. Then we have

$$\mathcal{E}(\vec{\psi}) = \frac{1}{2}\int_0^\infty \psi_t^2 \sinh r\, dr + \frac{1}{2}\int_0^\infty \left(\psi_r - \frac{\sinh\psi}{\sinh r}\right)^2 \sinh r\, dr + \int_0^\infty \sinh\psi\psi_r\, dr$$
$$= \frac{1}{2}\int_0^\infty \psi_t^2 \sinh r\, dr + \frac{1}{2}\int_0^\infty \left(\psi_r - \frac{\sinh\psi}{\sinh r}\right)^2 \sinh r\, dr + \cosh(2\text{arctanh}\,(\lambda)) - 1$$

For the solution $\vec{\psi}(t) = (P_\lambda, 0)$ the first two integrals–which are always non-negative–vanish identically, which proves that $(P_\lambda, 0)$ uniquely minimizes the energy in $\mathscr{E}_\lambda$. Finally, a simple calculation yields

$$\mathscr{E}(P_\lambda, 0) = \cosh(2\arctan(\lambda)) - 1 = 2\frac{\lambda^2}{1-\lambda^2}$$

and this completes the proof. □

2.3. **Reduction to equations on $\mathbb{R}\times\mathbb{H}^4$.** Next, we provide more details related to the $4d$ reductions to the Cauchy problems (1.15) and (1.29) outlined in the introduction.

First, we prove an estimate that gives an L^∞ bound on solutions to (1.2) and to (1.23) in terms of their energy. As the proof is the same in both cases we shorten the exposition by considering solutions to (1.1), namely

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \coth r\,\psi_r + \frac{g(\psi)g'(\psi)}{\sinh^2 r} = 0, \\ &E(\vec{\psi}) := \frac{1}{2}\int_0^\infty \left(\psi_t^2 + \psi_r^2 + \frac{g^2(\psi)}{\sinh^2 r}\right)\sinh r\, dr. \end{aligned} \tag{2.10}$$

where in the cases under consideration we have $g(\psi) = \sin\psi$, $E = \mathcal{E}$ for maps into $\mathbb{S}^2$, and $g(\psi) = \sinh\psi$, $E = \mathscr{E}$ for maps into $\mathbb{H}^2$.

Lemma 2.3. *Let $\vec{\psi}(t)$ be a finite energy solution to* (2.10) *defined on the interval $t \in I$ with $\psi(t,0) = 0$ for every $t \in I$. Then there exists a function C with $C(\rho) \to 0$ as $\rho \to 0$ so that*

$$\sup_{t\in I} \|\psi(t)\|_{L^\infty} \le C(E(\vec{\psi})). \tag{2.11}$$

Proof. Following, e.g., [26, Chapter 8] we define the function

$$G(\psi) = \int_0^\psi |g(\rho)|\, d\rho,$$

and we note that $G(0) = 0$, G is increasing, and $G(\psi) \to \infty$ as $\psi \to \infty$. For any fixed $t \in I$ we have

$$\begin{aligned} |G(\psi(t,r))| = |G(\psi(t,r)) - G(\psi(t,0))| &= \left| \int_{\psi(t,0)}^{\psi(t,r)} |g(\rho)| \, d\rho \right| \\ &= \int_0^r |g(\psi(t,r))| \, |\psi_r(t,r)| \, dr \le E(\vec{\psi}) \end{aligned} \tag{2.12}$$

Then (2.11) follows from (2.12) and the fact that G is increasing. □

Next, we establish an equivalence of the Cauchy problems (1.2) with (1.15) as well as (1.23) with (1.29) by proving an isomorphism between the spaces $\mathcal{H}_0$ and $H^1 \times L^2(\mathbb{H}^4)$, where $\mathcal{H}_0$ is defined as in (1.10) and where for radially symmetric $u, v : \mathbb{H}^4 \to \mathbb{R}$ we set

$$\|(u,v)\|_{H^1 \times L^2(\mathbb{H}^4)}^2 := \int_0^\infty \big(u_r^2(r) + v^2(r)\big) \sinh^3 r \, dr.$$

We use the notation H^1 for the above as opposed to $\dot{H}^1$ due to the embedding $\dot{H}^1(\mathbb{H}^d) \hookrightarrow L^2(\mathbb{H}^d)$ for $d \ge 2$. We prove the following simple lemma.

Lemma 2.4. *Let $(\psi, \phi) \in \mathcal{H}_0(\mathbb{H}^2)$ with $\psi(0) = 0$, $\psi(\infty) = 0$. Then if we define (u, v) by*

$$(\psi(r), \phi(r)) = (\sinh r u(r), \sinh r v(r))$$

we have

$$\|(\psi,\phi)\|_{\mathcal{H}_0}^2 \le \|(u,v)\|_{H^1 \times L^2(\mathbb{H}^4)}^2 \le 9 \|(\psi,\phi)\|_{\mathcal{H}_0}^2. \tag{2.13}$$

Remark 6. We note that Lemma 2.4 implies that in order to prove Theorem 1.1 and Theorem 1.3 it suffices to consider the corresponding results for the Cauchy problems (1.15), respectively (1.29), with initial data $\vec{u}(0) = (u_0, u_1) \in H^1 \times L^2(\mathbb{H}^4)$.

Proof. Since $\|\phi\|_{L^2(\mathbb{H}^2)}^2 = \|v\|_{L^2(\mathbb{H}^4)}^2$ it suffices to just consider u and $\psi = \sinh r u$. Integration by parts yields the following identity:

$$\int_0^\infty \left(\psi_r^2 + \frac{\psi^2}{\sinh^2 r} \right) \sinh r \, dr = \int_0^\infty u_r^2 \sinh^3 r \, dr - 2 \int_0^\infty u^2 \sinh^3 r \, dr, \tag{2.14}$$

which implies that

$$\int_0^\infty \left(\psi_r^2 + \frac{\psi^2}{\sinh^2 r} \right) \sinh r \, dr \le \int_0^\infty u_r^2 \sinh^3 r \, dr,$$

giving the left-hand inequality in (2.13). On the other hand,

$$\begin{aligned} \int_0^\infty \psi^2 \sinh r \, dr \le \int_0^\infty \psi^2 \cosh r \, dr &= -2 \int_0^\infty \psi \psi_r \sinh r \, dr \\ &\le 2 \left(\int_0^\infty \psi_r^2 \sinh r \, dr \right)^{\frac{1}{2}} \left(\int_0^\infty \psi^2 \sinh r \, dr \right)^{\frac{1}{2}}, \end{aligned}$$

which means that

$$\int_0^\infty u^2 \sinh^3 r \, dr = \int_0^\infty \psi^2 \sinh r \, dr \le 4 \int_0^\infty \psi_r^2 \sinh r \, dr.$$

Combining the above with (2.14) yields the right-hand-side of (2.13). □

3. The linearized operator H_{V_λ}: Analysis of the Spectrum

This section gives a detailed analysis of the spectrum of the Schrödinger operator H_{V_λ} defined in (1.19), which is self-adjoint on the domain $\mathcal{D} := H^2(\mathbb{H}^4)$, restricted to radial functions. In Section 3.1, we establish a positive result, Proposition 3.2, for a range of λ, namely $0 \le \lambda < \sqrt{15/8}$. We prove that for λ in this range the spectrum of H_{V_λ} coincides with that of the unperturbed operator $H_0 := -\Delta_{\mathbb{H}^4} - 2$. Next, we show that this breaks down for large λ. In particular, in the rest of this section, we prove that for λ large there is a unique simple eigenvalue μ_λ^2 in the spectral gap $(0, \frac{1}{4})$ and $\mu_\lambda^2 \to 0$ as $\lambda \to \infty$. This is the content of Theorem 1.2.

First we pass to the half-line by conjugating by $\sinh^{\frac{3}{2}} r$. Indeed, the map

$$L^2(\mathbb{H}^4) \ni \varphi \mapsto \sinh^{\frac{3}{2}} r\, \varphi =: \phi \in L^2(0,\infty) \tag{3.1}$$

is an isomorphism of $L^2(\mathbb{H}^4)$, restricted to radial functions, onto $L^2([0,\infty))$. If we define $\mathcal{L}_0, \mathcal{L}_{V_\lambda}$ by

$$\begin{aligned} \mathcal{L}_0 &:= -\partial_{rr} + \frac{1}{4} + \frac{3}{4\sinh^2 r}, \\ \mathcal{L}_{V_\lambda} &:= -\partial_{rr} + \frac{1}{4} + \frac{3}{4\sinh^2 r} + V_\lambda(r), \end{aligned} \tag{3.2}$$

we have

$$\begin{aligned} (H_0\varphi)(r) &= \sinh^{-\frac{3}{2}} r (\mathcal{L}_0 \phi)(r), \\ (H_{V_\lambda}\varphi)(r) &= \sinh^{-\frac{3}{2}} r (\mathcal{L}_{V_\lambda} \phi)(r). \end{aligned} \tag{3.3}$$

Hence it suffices to work with $\mathcal{L}_0$ and with $\mathcal{L}_{V_\lambda}$ on the half-line. We recall that V_λ is an *attractive* potential and is given by

$$V_\lambda(r) = \frac{\cos 2Q_\lambda - 1}{\sinh^2 r} \le 0. \tag{3.4}$$

Some elementary computations using the definition of Q_λ give us the explicit representation

$$V_\lambda(r) = \frac{-8\lambda^2}{[(1+\lambda^2)\cosh r + (1-\lambda^2)]^2}. \tag{3.5}$$

Below, we collect a few useful facts about V_λ.

Lemma 3.1. *The following statements hold for V_λ.*

(1) We have

$$V_\lambda' = \frac{16\lambda^2(1+\lambda^2)\sinh r}{[(1+\lambda)^2\cosh r + (1-\lambda^2)]^3}.$$

(2) The potential V_λ is attractive. *More precisely, V_λ is always non-decreasing on $[0,\infty)$ and*

$$V_\lambda(0) = -2\lambda^2, \quad \lim_{r\to\infty} V_\lambda(r) = 0.$$

(3) For $\lambda = 0$, $V_0 = 0$. For $\lambda = 1$,

$$V_1 = -\frac{2}{\cosh^2 r}.$$

(4) For $0 \le \lambda \le 1$, we have

$$V_\lambda \ge V_1.$$

Proof. Statements (1)–(3) are trivial. To see why (4) holds, note that for $0 \le \lambda \le 1$,

$$-V_\lambda \le \frac{8\lambda^2}{(1+\lambda^2)^2 \cosh^2 r} \le -V_1.$$

where we used $4\lambda^2 \le (1+\lambda^2)^2$. □

3.1. **Spectrum of H_{V_λ} for small λ.** We note that the spectrum for the self-adjoint operator $\mathcal{L}_0$ is purely absolutely continuous and is given by $\sigma(\mathcal{L}_0) = [1/4, \infty)$, and in particular there is no negative spectrum, no eigenvalue in the gap $[0, 1/4)$, and the threshold $1/4$ is neither an eigenvalue nor a resonance. The following result shows that in the case $0 < \lambda < \sqrt{15/8}$, the same can be said of the spectrum $\sigma(\mathcal{L}_{V_\lambda})$.

Proposition 3.2. *Let $0 \le \lambda < \sqrt{15/8}$. Then the spectrum for the self-adjoint operator $\mathcal{L}_{V_\lambda}$ is purely absolutely continuous and given by*

$$\sigma(\mathcal{L}_{V_\lambda}) = [1/4, \infty). \tag{3.6}$$

In particular, there is no negative spectrum, there are no eigenvalues in the gap $[0, 1/4)$, and the threshold $\frac{1}{4}$ is neither an eigenvalue nor a resonance.

Before we prove Proposition 3.2 we observe a few preliminary facts concerning solutions to

$$\mathcal{L}_{V_\lambda}\phi = \mu^2\phi, \quad \text{for} \;\; \mu^2 \in \mathbb{R}, \; \mu \in \mathbb{C}. \tag{3.7}$$

Lemma 3.3. *Let $\mu \in \mathbb{C}$, $\mu^2 \in \mathbb{R}$ and suppose that ϕ_μ is a solution to* (3.7) *such that $\phi_\mu \in L^2([0, c))$ for some $c > 0$. Then, there exists $a \in \mathbb{R}$ so that*

$$\phi_\mu(r) = a\, r^{\frac{3}{2}} + o(r^{\frac{3}{2}}) \quad as \quad r \to 0. \tag{3.8}$$

Proof. This follows from the fact that the operator $\mathcal{L}_0 - 1/4$ is well approximated near $r = 0$ by the singular operator

$$L_0 := -\partial_{rr} + \frac{3}{4r^2}.$$

L_0 is in the limit point case at $r = 0$ and a fundamental system for $L_0 f = 0$ is given by $\{r^{\frac{3}{2}}, r^{-\frac{1}{2}}\}$. It follows that a solution ϕ_μ as in Lemma 3.7 can be written in terms of these two solutions via the variation of parameters formula which converges for small r. The $L^2([0, c))$ requirement then guarantees that the coefficient in front of $r^{-\frac{1}{2}}$ must be 0 and the leading order behavior is given by $r^{\frac{3}{2}}$. □

Lemma 3.4. *Suppose ϕ_0 is a solution to* (3.7) *with $\mu^2 = \frac{1}{4}$. Then there exist constants $a, b \in \mathbb{R}$ so that*

$$\phi_0(r) = a + b\, r + O(re^{-2r}) \quad as \quad r \to \infty. \tag{3.9}$$

Proof. This follows from the fact that we can find constants $C_\lambda, C > 0$ so that for r large we have $V_\lambda(r) \le C_\lambda e^{-2r}$ and $\frac{3}{4\sinh^2 r} \le Ce^{-2r}$. Thus the operator

$$L_\infty := -\partial_{rr}$$

is a good approximation of $\mathcal{L}_{V_\lambda} - 1/4$ near $r = \infty$. A fundamental system for $L_\infty f = 0$ is given by $\{1, r\}$. The variation of parameters formula then yields the conclusions of Lemma 3.4. □

Definition 1. Given the conclusions of Lemma 3.7 and Lemma 3.4 we can give a precise definition of what we mean by threshold resonance. We say that that ϕ_0 is a *threshold resonance* for $\mathcal{L}_{V_\lambda}$ if ϕ_0 is not in $L^2(0,\infty)$ and it is a bounded solution to

$$\mathcal{L}_{V_\lambda}\phi_0 = \frac{1}{4}\phi_0.$$

In particular we can find non-zero $a, b \in \mathbb{R}$ so that

$$\begin{aligned}\phi_0(r) &= ar^{\frac{3}{2}} + o(r^{\frac{3}{2}}) \quad \text{as} \quad r \to 0,\\ \phi_0(r) &= b + O(re^{-2r}) \quad \text{as} \quad r \to \infty.\end{aligned} \tag{3.10}$$

We can now prove Proposition 3.2.

Proof of Proposition 3.2. Let $\mu \in \mathbb{C}$ with $\mu^2 \leq \frac{1}{4}$. Suppose that ϕ_μ is a solution to

$$\mathcal{L}_{V_\lambda}\phi_\mu = \mu^2\phi_\mu \tag{3.11}$$

If $\mu^2 \leq 1/4$ is an eigenvalue, we can assume that it is the smallest eigenvalue, and by a variational principle, we can further assume that corresponding eigenfunction $\phi_\mu \in L^2$ is unique, (i.e., μ^2 is simple) and strictly positive. If $\mu^2 = \frac{1}{4}$ and is not an eigenvalue, we assume that ϕ_μ is a threshold resonance. In either case, we know by Lemma 1.2 that $\phi_\mu(r) = O(r^{\frac{3}{2}})$ as $r \to 0$. If ϕ_μ is an eigenvalue, then $\phi_\mu(r) \to 0$ as $r \to \infty$. If $\phi_\mu(r)$ is a threshold resonance, we know by Definition 1 that $\phi_\mu(r) \to b > 0$ as $r \to \infty$.

Now, define the operator

$$\mathcal{K} := -\partial_{rr} + \frac{3}{4\sinh^2 r}. \tag{3.12}$$

Observe that the function

$$f(r) = \tanh^{\frac{3}{2}} r, \tag{3.13}$$

solves

$$\mathcal{K}f = \frac{15}{4\cosh^2 r}f. \tag{3.14}$$

Then, for any $R > 0$ we can integrate by parts, using (3.11) and (3.14) to obtain

$$\begin{aligned}\left(\mu^2 - \frac{1}{4}\right)\int_0^R \tanh^{\frac{3}{2}} r\, \phi_\mu(r)\, dr &= -\phi_\mu'(R)\tanh^{\frac{3}{2}} R + \frac{3}{2}\phi_\mu(R)\frac{\tanh^{\frac{1}{2}} R}{\cosh^2 R}\\ &\quad + \int_0^R \left(\frac{15}{4\cosh^2 r} + V_\lambda(r)\right)\tanh^{\frac{3}{2}} r\, \phi_\mu(r)\, dr\end{aligned} \tag{3.15}$$

Since $\mu^2 \leq \frac{1}{4}$ and $\frac{3}{2}\phi_\mu(R)\frac{\tanh^{\frac{1}{2}} R}{\cosh^2 R} \geq 0$ for all $R > 0$, we can deduce that

$$\int_0^R \left(\frac{15}{4\cosh^2 r} + V_\lambda(r)\right)\tanh^{\frac{3}{2}} r\, \phi_\mu(r)\, dr \leq \phi_\mu'(R)\tanh^{\frac{3}{2}} R \tag{3.16}$$

for all $R > 0$. For $0 < \lambda < \sqrt{15/8}$, we can plug in the definition (3.5) to see that for such a λ fixed, we have

$$\frac{15}{4\cosh^2 r} + V_\lambda(r) > 0, \quad \forall r \in [0,\infty). \tag{3.17}$$

This means that the left-hand-side of (3.16) is strictly positive and increasing in R and hence we can find $\delta > 0$ so that

$$0 < \delta \leq \phi_\mu'(R) \tanh^{\frac{3}{2}} R \tag{3.18}$$

for all $R > 0$. However, we know that $\phi_\mu'(R) \tanh^{\frac{3}{2}} R \to 0$ as $R \to \infty$, which means that (3.18) gives a contradiction for R large enough. This completes the proof of Proposition 3.2. □

3.2. **Spectrum of H_{V_λ} for large λ: Beginning of the proof of Theorem 1.2.** The rest of this section is devoted to the proof of Theorem 1.2 which asserts, in particular, existence of a unique simple gap eigenvalue $\mu_\lambda^2 \in (0, 1/4)$ of $\mathcal{L}_{V_\lambda}$ for large λ and migration of μ_λ^2 to 0 as λ tends to ∞. We remind the reader that $\mathcal{L}_{V_\lambda}$ is L^2-equivalent to H_{V_λ}.

In this subsection, we begin our proof of Theorem 1.2 by establishing some elementary facts concerning the spectrum of $\mathcal{L}_{V_\lambda}$ for all $\lambda \in [0, \infty)$. In particular, we prove that if an eigenvalue exists, then it must occur in the spectral gap $(0, 1/4)$. We also show that $\mathcal{L}_{V_\lambda}$ has a threshold resonance when $\lambda = \lambda_{\mathrm{sup}}$ or $\lambda = \Lambda_{\mathrm{inf}}$, where λ_{sup} and Λ_{inf} were defined in Theorem 1.2. Next, we briefly explain the idea of renormalization, which is key to the rest of our proof of Theorem 1.2. At the end of this subsection, we give an outline of the structure of the proof of Theorem 1.2.

To begin, we state and prove some general facts about the spectrum of $\mathcal{L}_{V_\lambda}$.

Proposition 3.5. *The following statements concerning $\mathcal{L}_{V_\lambda}$ hold.*

(i) For every $\lambda \geq 0$, the spectrum of $\mathcal{L}_{V_\lambda}$ does not contain any non-positive reals, i.e.,

$$\sigma(\mathcal{L}_{V_\lambda}) \cap (-\infty, 0] = \emptyset.$$

(ii) There does not exist any eigenvalue in $[\frac{1}{4}, \infty)$.

As a consequence of this proposition, any eigenvalue of the operator $\mathcal{L}_{V_\lambda}$ must occur in the *spectral gap* $(0, 1/4)$.

In our proof of the first statement of Proposition 3.5, we make use of the positive solution $\zeta_0^{(\lambda)}$ to the equation

$$\mathcal{L}_{V_\lambda} \zeta_0^{(\lambda)} = 0, \tag{3.19}$$

which is obtained by differentiating Q_λ with respect to λ and conjugating by $\sinh^{1/2} r$. It can be computed explicitly to be

$$\zeta_0^{(\lambda)}(r) = \frac{\tanh(r/2)}{1 + \lambda^2 \tanh^2(r/2)} \sinh^{1/2} r. \tag{3.20}$$

The explicit solution $\zeta_0^{(\lambda)}$ (more precisely, its conjugate $\zeta_\infty^{(\lambda)}$) will make another entrance in our proof of migration of the gap eigenvalue in Section 3.5.

Proof of Proposition 3.5. The existence of the solution ζ_0^λ rules out the possibility of an eigenvlaue at $\mu = 0$. Therefore, to prove the first statement, it suffices to rule out eigenvalues in $(-\infty, 0)$. Suppose that such an eigenvalue exists. Then, as in the proof of Proposition 3.2, there exists $\mu \in \mathbb{C}$ with $\mu^2 \leq 0$ and an L^2 solution ϕ_μ to (3.11) which is strictly positive. Proceeding as in (3.15) with $\zeta_0^{(\lambda)}$ in place of $\tanh^{3/2} r$, for any $R > 0$ we obtain

$$\mu^2 \int_0^R \zeta_0^{(\lambda)}(r) \phi_\mu(r) \, \mathrm{d}r = -\phi_\mu'(R) \zeta_0^{(\lambda)}(R) + \phi_\mu(R) (\zeta_0^{(\lambda)})'(R).$$

Arguing as in Proposition 3.2, we see that the left-hand side is strictly negative and decreasing in R. On the other hand, the right-hand side is non-negative for sufficiently large R, which is a contradiction.

The second statement follows from the fact that if $\mu^2 \geq 1/4$ then there does not exist any non-zero solution to $\mathcal{L}_{V_\lambda}\phi = \mu^2\phi$ in $L^2([1,\infty))$. To prove this fact, observe that $\mathcal{L}_{V_\lambda} - \mu^2$ is well-approximated by $-\partial_{rr} - (\mu^2 - 1/4)$, near $r = \infty$. Moreover, note that a fundamental system for $-\partial_{rr} f - (\mu^2 - 1/4)f = 0$ is $\{e^{\pm i\sqrt{\mu^2-1/4}\,r}\}$ when $\mu^2 > 1/4$ and $\{1, r\}$ when $\mu^2 = 1/4$, all of which do not decay as $r \to \infty$. □

Next we show, roughly speaking, that the transition from a λ-regime with no eigenvalue and threshold resonance to a λ-regime with a gap eigenvalue must be accompanied by a threshold resonance.

Proposition 3.6. *As in Theorem 1.2, define*

$$\lambda_{\sup} = \sup\{\lambda \mid \mathcal{L}_{V_{\tilde\lambda}} \textit{ has no e-vals and no threshold resonance } \forall\, \tilde\lambda < \lambda\}$$
$$\Lambda_{\inf} = \inf\{\lambda \mid \mathcal{L}_{V_{\tilde\lambda}} \textit{ has a gap e-val } \mu^2_{\tilde\lambda} \in (0, 1/4)\ \forall\, \tilde\lambda > \lambda\}$$

Then both $\mathcal{L}_{V_{\lambda_{\sup}}}$ and $\mathcal{L}_{V_{\Lambda_{\inf}}}$ have threshold resonances.

In view of the result that we will prove Section 3.3 (existence of a gap eigenvalue), we have $\lambda_{\sup} \leq \Lambda_{\inf} < \infty$.

Proof. To prove this proposition, we study the solution ϕ_0^λ to $\mathcal{L}_{V_\lambda}\phi_0^\lambda = (1/4)\phi_0^\lambda$ such that $\phi_0^\lambda(r) = r^{3/2} + o(r^{3/2})$. By Sturm's oscillation theory and Proposition 3.5 (which rules out negative spectrum), existence of an eigenvalue in $(0, 1/4)$ is equivalent to existence of a zero (i.e., a sign change) of ϕ_0^λ. As a consequence, we have the following alternative characterization of $\lambda_{\sup}$ and $\Lambda_{\inf}$ in terms of ϕ_0^λ:

$$\lambda_{\sup} = \sup\{\lambda \mid \phi_0^{\tilde\lambda} \text{ does not change sign and is not a threshold resonance} \forall\, \tilde\lambda < \lambda\}$$
$$\Lambda_{\inf} = \inf\{\lambda \mid \phi_0^{\tilde\lambda} \text{ changes sign } \forall\, \tilde\lambda > \lambda\}$$

Observe that A) "changing sign" and B) "not changing sign and not being a threshold resonance" are open conditions in λ for ϕ_0^λ. Indeed, that A) is an open condition is an easy consequence of pointwise continuity of $\phi_0^\lambda(r)$ in λ, r. That B) is an open condition follows from the fact that the coefficient $b = b(\lambda)$ from Lemma 3.4 is continuous in λ. By the above characterization of $\lambda_{\sup}$ and $\Lambda_{\inf}$, the only remaining possibility is that ϕ_0^λ is a threshold resonance for $\lambda = \lambda_{\sup}$ or $\lambda = \Lambda_{\inf}$, which proves the proposition. □

We now explain the idea of renormalization, which will play an important role in our arguments in the rest of this section. For each $\lambda > 0$, we define the rescaled Schrödinger operator $\widetilde{\mathcal{L}}_\lambda$ by

$$\widetilde{\mathcal{L}}_\lambda := -\partial_{\rho\rho} + \frac{3}{4}\frac{1}{\lambda^2 \sinh^2(\rho/\lambda)} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2}V_\lambda(\rho/\lambda). \tag{3.21}$$

We will refer to $\rho = \lambda r$ as the *renormalized coordinate*. The operator $\widetilde{\mathcal{L}}_\lambda$ is related to $\mathcal{L}_{V_\lambda}$ as follows: Given a function $\phi(r)$ on $(0, \infty)$, define $\widetilde{\phi}(\rho) := \phi(\rho/\lambda)$. Then

$$\widetilde{\mathcal{L}}_\lambda \widetilde{\phi} = \frac{1}{\lambda^2}(\mathcal{L}_{V_\lambda}\phi)(\cdot/\lambda) = \frac{1}{\lambda^2}\widetilde{(\mathcal{L}_{V_\lambda}\phi)}.$$

A simple but important observation is that in the limit $\lambda \to \infty$, $\widetilde{\mathcal{L}}_\lambda$ formally tends to the operator

$$\begin{aligned} \mathcal{L}_{\mathrm{euc}}\varphi &:= -\varphi_{\rho\rho} + \frac{3}{4}\frac{1}{\rho^2}\varphi + V_{\mathrm{euc}}(\rho)\varphi, \\ V_{\mathrm{euc}}(\rho) &:= -\frac{2}{(1+(\rho/2)^2)^2}. \end{aligned} \tag{3.22}$$

The equation $\mathcal{L}_{\mathrm{euc}}\varphi = 0$ possesses an explicit solution

$$\varphi_0(\rho) := \frac{\rho^{\frac{3}{2}}}{1+(\rho/2)^2}. \tag{3.23}$$

The Schrödinger operator $\mathcal{L}_{\mathrm{euc}}$ arises in linearizing the co-rotational wave maps equation $\mathbb{R}^{2+1} \to \mathbb{S}^2$ around the ground state harmonic map Q_{euc}. The explicit solution φ_0 is obtained from the scaling invariance of the problem, and is a resonance at zero of $\mathcal{L}_{\mathrm{euc}}$. See [15] for more details.

The idea of renormalization is to exploit the formal resemblance of $\widetilde{\mathcal{L}}_\lambda$ and $\mathcal{L}_{\mathrm{euc}}$ by (essentially) working with a fundamental system for $\mathcal{L}_{\mathrm{euc}}\varphi = 0$ consisting of the explicit solution φ_0 and its conjugate. More precisely, given a solution ψ to $\mathcal{L}_{V_\lambda}\psi = \overline{\mu}^2\psi$, we define its *renormalization* $g(\rho)$ by making a change of variable $\rho = \lambda r$ and dividing by $\varphi_0(\rho)$, i.e.,

$$g(\rho) := \frac{\psi(\rho/\lambda)}{\varphi_0(\rho)}. \tag{3.24}$$

Then $g(\rho)$ obeys the equation

$$(g'\varphi_0^2)' = \varphi_0^2 W_{\lambda,\overline{\mu}}\, g$$

where

$$W_{\lambda,\overline{\mu}}(\rho) := \frac{3}{4}\frac{1}{\lambda^2\sinh^2(\rho/\lambda)} - \frac{3}{4}\frac{1}{\rho^2} + \frac{1}{4\lambda^2} - \frac{\overline{\mu}^2}{\lambda^2} + \frac{1}{\lambda^2}V_\lambda(\rho/\lambda) - V_{\mathrm{euc}}(\rho).$$

We call $W_{\lambda,\overline{\mu}}$ the *renormalized potential*. By the same computation that shows $\widetilde{\mathcal{L}}_\lambda \to \mathcal{L}_{\mathrm{euc}}$, it follows that $W_{\lambda,\overline{\mu}}(\rho) \to 0$ for each $\rho > 0$ as $\lambda \to \infty$. This simple fact already suggests that we have a good control on $g(\rho)$ for $0 \leq \rho \lesssim 1$. This will be one of the main ideas of our proof of uniqueness of the gap eigenvalue.

Remarkably, the renormalization technique can be also made effective in a ρ-interval of the form $0 \leq \rho \lesssim \lambda$, which can be made arbitrarily long, provided that an appropriate *a priori* estimate for g holds. This observation is crucial in our proofs of existence and migration of the gap eigenvalue below, where we obtain an appropriate *a priori* estimate from a contradiction hypothesis.

We conclude this subsection with an outline of the structure of the proof of Theorem 1.2. In Section 3.3, we establish existence of an eigenvalue of $\mathcal{L}_{V_\lambda}$ in $(0, 1/4)$ for sufficiently large λ. Then in Section 3.4, we show that if an eigenvalue exists in $(0, 1/4)$, then it must be simple and unique if λ is sufficiently large. We also rule out threshold resonance for large λ. Finally, in Section 3.5, we show that the gap eigenvalue μ_λ^2 tends to 0 as $\lambda \to \infty$. Combined with Propositions 3.5 and 3.6, Theorem 1.2 then follows.

3.3. **Existence of gap eigenvalues for large λ.** The goal of this subsection is to prove existence of an eigenvalue of $\mathcal{L}_{V_\lambda}$ in the spectral gap $(0,1/4)$ for sufficiently large λ. The main result of this subsection is the following proposition:

Proposition 3.7. *There exists $\Lambda_0 > 0$ with the following property: Let $\lambda \geq \Lambda_0$ and let ϕ_0 be the solution to*

$$\mathcal{L}_{V_\lambda}\phi_0 = \frac{1}{4}\phi_0 \tag{3.25}$$

that satisfies $\phi_0(r) = r^{3/2} + o(r^{3/2})$ as $r \to 0$. Then $\phi_0(r)$ changes sign on $[0,\infty)$ at least once.

Existence of an eigenvalue $\mu^2 < \frac{1}{4}$ is then a direct consequence of Sturm's oscillation theory. By Proposition 3.5, it follows furthermore that $\mu^2 \in (0,1/4)$.

Henceforth, we will work with the rescaled solution $\psi_0^\lambda(\rho) := \lambda^{3/2}\phi_0(\rho/\lambda)$ with $\lambda > 1$, in anticipation of the application of the renormalization technique. Then ψ_0^λ solves the equation

$$\widetilde{\mathcal{L}}_\lambda \psi_0^\lambda = \frac{1}{4\lambda^2}\psi_0^\lambda. \tag{3.26}$$

Moreover, $\psi_0^\lambda(\rho) = \rho^{3/2} + o(\rho^{3/2})$ as $\rho \to 0$.

Proposition 3.7 will be a consequence of the following two lemmas.

Lemma 3.8. *For every $\lambda > 1$, there exists a unique solution ψ_∞^λ to (3.26) so that*

$$\begin{aligned} \psi_\infty^\lambda(\rho) &= 1 + O(e^{-2\rho/\lambda}) \quad \text{as} \quad \rho \to \infty \\ (\psi_\infty^\lambda)'(\rho) &= O(e^{-2\rho/\lambda}) \quad \text{as} \quad \rho \to \infty \end{aligned} \tag{3.27}$$

Moreover, for sufficiently large λ, $\psi_\infty^\lambda(\rho)$ satisfies

$$\psi_\infty^\lambda(\rho) > 0 \quad \forall \rho \in [\lambda,\infty) \tag{3.28}$$

Lemma 3.9. *For sufficiently large λ, the following statement holds: Let ψ_0^λ be the solution to (3.26) that satisfies*

$$\psi_0^\lambda(\rho) = \rho^{3/2} + o(\rho^{3/2}) \quad \text{as} \quad \rho \to 0. \tag{3.29}$$

Then, either ψ_0^λ changes sign on $[0,\lambda]$, or

$$\frac{(\psi_0^\lambda)'(\lambda)}{\psi_0^\lambda(\lambda)} < \frac{(\psi_\infty^\lambda)'(\lambda)}{\psi_\infty^\lambda(\lambda)} \tag{3.30}$$

where ψ_∞^λ is the function given by Lemma 3.8.

Before proving Lemma 3.8 and Lemma 3.9, we assume their conclusions and establish Proposition 3.7.

Proof of Propositon 3.7. Let $\psi_0 := \psi_0^\lambda$ be as in Lemma 3.9, and let $\psi_\infty := \psi_\infty^\lambda$ be as in Lemma 3.8.

Assume, for the sake of contradiction that ψ_0 does not change signs on $[0,\infty)$. Without loss of generality, we assume that $\psi_0(\rho) \geq 0$ for all $r \in [0,\infty)$.

Since there is no first order term in $\widetilde{\mathcal{L}}_\lambda$, the Wronskian of ψ_0 and ψ_∞ is constant. Hence we are free to evaluate it at any point $\rho \in [0,\infty)$. Since we are assuming that ψ_0 does not change its sign, (3.30) gives

$$W[\psi_0,\psi_\infty](\lambda) = \psi_0(\lambda)\psi_\infty'(\lambda) - \psi_0'(\lambda)\psi_\infty(\lambda) > 0. \tag{3.31}$$

Due to the rapid decay as $\rho \to \infty$ of

$$\frac{3}{4}\frac{1}{\lambda^2 \sinh^2(\rho/\lambda)} + \frac{1}{\lambda^2} V_\lambda(\rho/\lambda)$$

we can view any solution of (3.26) as a perturbation of a solution to the free equation

$$-\partial_{\rho\rho}\psi = 0 \tag{3.32}$$

at $\rho = \infty$. Since a fundamental system for (3.32) is given by $\{1, \rho\}$, we can use a variation of parameters argument to find $a, b \in \mathbb{R}$ such that

$$\begin{aligned} &\psi_0(\rho) = a + b\rho + o(1) \quad \text{as} \quad \rho \to \infty, \\ &\psi_0'(\rho) \to b \quad \text{as} \quad \rho \to \infty. \end{aligned} \tag{3.33}$$

Combining the above with (3.27) we can deduce that

$$\lim_{\rho \to \infty} W[\psi_0, \psi_\infty](\rho) = -b. \tag{3.34}$$

By (3.31) we can conclude that $b < 0$. But this means that $\psi_0(\rho) < 0$ for large ρ, which contradicts our assumption that $\psi_0(\rho) > 0$ for all $\rho \in [0, \infty)$. □

Lemma 3.8 follows from a standard Volterra iteration argument combined with an easy observation concerning the sign of $(3/4\sinh^2 r) + V_\lambda(r)$.

Proof of Lemma 3.8. Note that if ϕ_∞ is a solution of $\mathcal{L}_{V_\lambda}\phi_\infty = (1/4)\phi_\infty$, then $\psi^\lambda_\infty(\cdot) := \phi_\infty(\cdot/\lambda)$ is a solution of $\widetilde{\mathcal{L}}_\lambda \psi^\lambda_\infty = (1/4\lambda^2)\psi^\lambda_\infty$. Hence it suffices to prove the lemma for ϕ_∞, where the positivity statement is now $\phi_\infty(r) > 0$ for $r \geq 1$.

Existence and uniqueness of ϕ_∞ can be proved by a standard iteration argument applied to the Volterra equation

$$\phi_\infty(r) = 1 + \int_r^\infty (s - r) N(s) \phi_\infty(s) ds, \tag{3.35}$$

where

$$N(r) := \frac{3}{4\sinh^2 r} + V_\lambda(r).$$

From the definition (3.5) of $V_\lambda(r)$, it is not difficult to see that if λ is sufficiently large, then $N(r) \geq 0$ for $r \geq 1$ (for a proof of a stronger statement, see Lemma 3.13). Now it follows from a simple continuity argument that $\phi_\infty(r) \geq 1$ for $r \geq 1$, which proves the desired positivity statement. □

We have thus reduced matters to proving Lemma 3.9. In the proof we will make use of the following estimates for the renormalized potential $W_{\lambda,1/2}$.

Lemma 3.10. *Let $W(\rho, \lambda) := W_{\lambda,\frac{1}{2}}(\rho)$, i.e.,*

$$W(\rho, \lambda) = \frac{3}{4}\frac{1}{\lambda^2 \sinh^2(\rho/\lambda)} + \frac{1}{\lambda^2} V_\lambda(\rho/\lambda) - \frac{3}{4\rho^2} - V_{euc}(\rho).$$

Then there exists Λ_1 such that the following hold.

(i) $\left|\lambda^2 W(\rho, \lambda)\right|$ is uniformly bounded for $\lambda > \Lambda_1$ and $\rho \leq \lambda$. Moreover we can find $\rho_1 < \Lambda_1$ such that if $\lambda > \Lambda_1$ then $\lambda^2 W(\rho, \lambda) < -b$ in the region $\rho > \rho_1$, where $b > 0$ is a constant independent of ρ and λ.

(ii) $W(\rho, \lambda) < 0$ for $\rho \geq \lambda \geq \Lambda_1$.

Proof of Lemma 3.10. It is convenient to introduce the notation $\beta = \frac{1}{\lambda}$. Then the region $\rho \le \lambda$ corresponds to $\rho\beta \le 1$. With this in mind we first note that

$$\frac{3}{4\lambda^2 \sinh^2(\rho/\lambda)} - \frac{3}{4\rho^2} = \frac{3}{4}\left(\frac{(\rho\beta)^2 - \sinh^2(\rho\beta)}{(\rho\beta)^2\sinh^2(\rho\beta)}\right)\beta^2 \le -\frac{\beta^2}{10} \tag{3.36}$$

in the region $\rho\beta \le 1$. Next we write $\lambda^{-2}V_\lambda(\rho/\lambda) = \tilde{V}(\rho,\beta)$ where

$$\tilde{V}(\rho,\beta) := \frac{-8}{\left(\frac{\cosh(\rho\beta)-1}{\beta^2} + \cosh(\rho\beta) + 1\right)^2}$$

and note that $\lim_{\beta\to 0}\tilde{V}(\rho,\beta) = V_{\mathrm{euc}}(\rho)$, where V_{euc} was defined in (3.22). This, of course, is just a restatement of the fact that in the limit $\lambda \to \infty$ the hyperbolic potential approaches its Euclidean counterpart. It follows that

$$V_{\mathrm{euc}}(\rho) - \frac{V_\lambda(\rho/\lambda)}{\lambda^2} = V_{\mathrm{euc}}(\rho) - \tilde{V}(\rho,\beta) = -\int_0^\beta \partial_\beta \tilde{V}(\rho,\tau)d\tau. \tag{3.37}$$

Now

$$\partial_\beta \tilde{V}(\rho,\beta) = \frac{16\left(\left(\frac{(\rho\beta)\sinh(\rho\beta)-2(\cosh(\rho\beta)-1)}{(\rho\beta)^3}\right)\rho^3 + \rho\sinh(\rho\beta)\right)}{\left(\left(\frac{\cosh(\rho\beta)-1}{(\rho\beta)^2}\right)\rho^2 + \cosh(\rho\beta)+1\right)^3}. \tag{3.38}$$

But in the region $\rho\beta \le 1$

$$\left|\frac{(\rho\beta)\sinh(\rho\beta) - 2(\cosh(\rho\beta)-1)}{(\rho\beta)^3}\right| + |\sinh(\rho\beta)| \lesssim \rho\beta,$$
$$\left(\frac{\cosh(\rho\beta)-1}{(\rho\beta)^2}\right) \gtrsim 1,$$

and therefore

$$|\partial_\beta \tilde{V}(\rho,\beta)| \lesssim (1+\rho^2)^{-3}(1+\rho^4)\beta \lesssim \frac{\beta}{(1+\rho^2)}.$$

Inserting this into (3.37) we get

$$\left|V_{\mathrm{euc}}(\rho) - \frac{V_\lambda(\rho/\lambda)}{\lambda^2}\right| \lesssim \frac{\beta^2}{(1+\rho^2)}. \tag{3.39}$$

The conclusions of the first part of the lemma now follow from an inspection of (3.36) and (3.39).

The second part of the lemma concerns the region $\rho\beta \ge 1$. We first write

$$\frac{3}{4\lambda^2\sinh^2(\rho/\lambda)} - \frac{3}{4\rho^2} = \frac{3}{4\rho^2}\left(\frac{(\rho\beta)^2 - \sinh^2(\rho\beta)}{\sinh^2(\rho\beta)}\right) \le -\frac{c}{\rho^2}, \tag{3.40}$$

for some positive constant c and for $\rho\beta \ge 1$. Next note that $x\sinh(x) > 2(\cosh(x) - 1)$ for all real x, and therefore by (3.38) if $\rho\beta \ge 1$ and λ is sufficiently large,

$$|\partial_\beta \tilde{V}(\rho,\beta)| \lesssim \frac{\left(\frac{\sinh(\rho\beta)}{(\rho\beta)^2}\right)\rho^3 + \rho\sinh(\rho\beta)}{\left(\frac{\cosh(\rho\beta)-1}{(\rho\beta)^2}\right)^3\rho^6 + (\cosh(\rho\beta)+1)^3} \lesssim \frac{1}{\rho^3}.$$

The second part of the lemma now follows from combining this estimate with (3.40) and (3.37). □

Proof of Lemma 3.9. For simplicity, we will write $\psi_0 := \psi_0^\lambda$, $\psi_\infty := \psi_\infty^\lambda$ and $W(\rho,\lambda) := W_{\lambda,1/2}(\rho)$. We divide the proof into two steps.

Step 1: The first step in the proof consists of establishing the following claim, which compares ψ_∞ with the renormalized Euclidean resonance, φ_0 at $\rho = \lambda$.

Claim 3.11. *For sufficiently large λ, we have*

$$\frac{\varphi_0'(\lambda)}{\varphi_0(\lambda)} < \frac{\psi_\infty'(\lambda)}{\psi_\infty(\lambda)} \tag{3.41}$$

Proof. The proof follows from another comparison argument. Using that ψ_∞ solves (3.26) and φ_0 solves (3.22) we have

$$\begin{aligned}
\psi_\infty'(\lambda)\varphi_0(\lambda) - \psi_\infty(\lambda)\varphi_0'(\lambda) &= \int_\lambda^\infty \frac{d}{d\rho}\left(\psi_\infty(\rho)\varphi_0'(\rho) - \psi_\infty'(\rho)\varphi_0(\rho)\right)\, d\rho \\
&= \int_\lambda^\infty \psi_\infty(\rho)\varphi_0''(\rho) - \psi_\infty''(\rho)\varphi_0(\rho)\, d\rho \\
&= \int_\lambda^\infty \left[\frac{3}{4}\left(\frac{1}{\rho^2} - \frac{1}{\lambda^2\sinh^2(\rho/\lambda)}\right) - V_{\mathrm{euc}}(\rho) + \frac{1}{\lambda^2}V_\lambda(\rho/\lambda)\right]\psi_\infty(\rho)\varphi_0(\rho)\, d\rho \\
&= -\int_\lambda^\infty W(\rho,\lambda)\psi_\infty(\rho)\varphi_0(\rho)\, d\rho.
\end{aligned}$$

Therefore,

$$\frac{\psi_\infty'(\lambda)}{\psi_\infty(\lambda)} - \frac{\varphi_0'(\lambda)}{\varphi_0(\lambda)} = \frac{-1}{\psi_\infty(\lambda)\varphi_0(\lambda)}\int_\lambda^\infty W(\rho,\lambda)\psi_\infty(\rho)\varphi_0(\rho)\, d\rho$$

Note that by Lemma 3.8 we have that $\psi_\infty(\rho) > 0$ for all $\rho \in [\lambda, \infty)$. Also, $\varphi_0(\rho) \geq 0$ for all $\rho \geq 0$. Therefore, the claim follows from the second part of Lemma 3.10 □

Step 2: In this second step we prove the following claim.

Claim 3.12. *For sufficiently large λ, we have*

$$\frac{\psi_0'(\lambda)}{\psi_0(\lambda)} < \frac{\varphi_0'(\lambda)}{\varphi_0(\lambda)}. \tag{3.42}$$

Proof. We apply the renormalization technique introduced in Section 3.2. For simplicity of notation we write $W(\rho) = W_{\lambda,1/2}(\rho)$. We define g by the relation $\psi_0(\rho) = g(\rho)\varphi_0(\rho)$. Since by assumption ψ_0 does not change sign in $[0, \lambda]$ and since φ_0 is positive there, we must have $g(\rho) > 0$ for $\rho \in [0, \lambda]$. It follows that (3.42) is equivalent to

$$g'(\lambda) < 0. \tag{3.43}$$

Notice that g satisfies the equation

$$\left(g'\varphi_0^2\right)' = \varphi_0^2 W g.$$

Moreover, by our normalization $\psi_0(\rho) = \rho^{3/2} + o(\rho^{3/2})$, it follows that $(g, g')(0) = (1, 0)$. Therefore

$$g'(\rho) = \frac{1}{\varphi_0^2(\rho)}\int_0^\rho \varphi_0^2(\sigma)W(\sigma)g(\sigma)\, d\sigma, \tag{3.44}$$

$$g(\rho) = 1 + \int_0^\rho\int_0^\tau \frac{\varphi_0^2(\sigma)}{\varphi_0^2(\tau)}W(\sigma)g(\sigma)\, d\sigma d\tau. \tag{3.45}$$

Assume by contradiction that (3.43) does not hold, or in other words

$$g'(\lambda) \geq 0. \tag{3.46}$$

Take λ large enough so that Lemma 3.10 applies. Then in view of the representation (3.44) and the fact that $W(\rho)$ is negative for $\rho_1 \leq \rho \leq \lambda$ we must have $g'(\rho) \geq 0$ for $\rho \in [\rho_1, \lambda]$.

To derive the desired contradiction we begin by showing that g is bounded away from zero. According to the observation above, g can decrease only on the interval $[0, \rho_1]$. From Lemma 3.10, we see that $\sup_{\sigma\in[0,\rho_1]} |W(\sigma)| \to 0$ as $\lambda \to \infty$. Recalling the definition of φ_0 in (3.23), it follows (by carrying out an explicit integration) that

$$\int_0^{\rho_1} \Big(\int_\sigma^{\rho_1} \frac{1}{\varphi_0^2(\tau)}\, \mathrm{d}\tau \Big) \varphi_0^2(\sigma)\, |W(\sigma)|\, \mathrm{d}\sigma \lesssim_{\rho_1} \sup_{\sigma\in[0,\rho_1]} |W(\sigma)| \to 0$$

as $\lambda \to \infty$. By a Volterra-type iteration argument, we conclude that

$$\sup_{\rho\in[0,\rho_1]} |g(\rho) - 1| = o(1) \quad \text{as} \quad \lambda \to \infty.$$

Taking λ larger if necessary we can guarantee that $g(\rho) \geq 1/2$ for $\rho \leq \rho_1$. Since $g'(\rho) \geq 0$ for $\rho \in [\rho_1, \lambda]$ we have a global bound $g(\rho) \geq 1/2$ on $\rho \leq \lambda$. It follows that

$$\begin{aligned} \int_{\rho_1}^{\lambda} \varphi_0^2(\sigma) W(\sigma) g(\sigma)\, d\sigma &\leq -\frac{b}{2\lambda^2} \int_{\rho_1}^{\lambda} \varphi_0^2(\sigma) d\sigma \\ &\leq -\frac{C_1 b \log \lambda}{\lambda^2} \end{aligned}$$

for some universal constant C_1 independent of λ. On the other hand

$$\int_0^{\rho_1} \varphi_0^2(\sigma) |W(\sigma)| g(\sigma)\, d\sigma \leq \frac{C_2}{\lambda^2}$$

for another universal constant C_2 also independent of λ. Inserting the last two estimates into the representation (3.44) we conclude that if λ is sufficiently large, $g'(\lambda) < 0$ contradicting (3.46). ☐

Lemma 3.9 now follows from combining the conclusions of Steps 1 and 2. ☐

3.4. **Uniqueness of gap eigenvalues for H_{V_λ} for large λ.** Our next goal is to prove that for large λ, the eigenvalue found in the previous section is simple and unique, and moreover that $\mathcal{L}_{V_\lambda}$ does not have a threshold resonance at $1/4$. This will be accomplished by showing that eigenfunctions in the spectral gap and threshold resonances cannot change sign.

As before we need to treat the case of large and small r separately. We begin with the following technical lemma.

Lemma 3.13. *For λ sufficiently large, there is a constant C independent of λ such that for $r \geq \frac{C}{\lambda}$*

$$\frac{3}{4\sinh^2 r} + V_\lambda(r) \geq 0.$$

Proof. Note that since

$$|V_\lambda(r)| \leq \frac{8}{\lambda^2(\cosh r - 1)^2}$$

it suffices to show that for $r \geq C\lambda$

$$\frac{3}{4\sinh^2 r} - \frac{8}{\lambda^2(\cosh r - 1)^2} \geq 0.$$

Writing $\sinh^2 r = \cosh^2 r - 1$, we see that this is equivalent to

$$\Big(\frac{3}{4} - \frac{8}{\lambda^2}\Big)\cosh r \geq \frac{3}{4} + \frac{8}{\lambda^2}.$$

Assume that λ is large enough so that $3/4 - 8/\lambda^2 > 1/2$. Then from the elementary fact that $\cosh r \geq 1 + (1/2)r^2$, the preceding inequality holds for $r \geq C/\lambda$ with an absolute constant $C > 0$. □

We can now carry out the analysis for large r.

Lemma 3.14. *Let ϕ be either an eigenfunction (i.e., a nonzero L^2 solution) of $\mathcal{L}_{V_\lambda}\phi = \mu^2\phi$ with $\mu^2 \in (0, \frac{1}{4})$ or a threshold resonance at $\mu^2 = 1/4$. Let C be as in Lemma 3.13. Then for sufficiently large λ, ϕ cannot change sign in the region $r \in (\frac{C}{\lambda}, \infty)$.*

Proof. We begin with the case of an eigenvalue $\mu^2 \in (0, \frac{1}{4})$. Define $m > 0$ by $m^2 = \frac{1}{4} - \mu^2$. The idea is to compare ϕ with $f = e^{-mr}$, which up to scaling is the unique nonzero L^2 solution of $\partial_{rr} f = m^2 f$. As usual, after suitable renormalization we may assume that $\phi(r) = e^{-mr} + o(e^{-mr})$ as $r \to \infty$. Defining

$$W(r) := W[\phi, f](r) = \phi(r)f'(r) - \phi'(r)f(r),$$

we have

$$W'(r) = -\Big(\frac{3}{4\sinh^2 r} + V_\lambda(r)\Big)\phi(r)f(r).$$

Therefore, in view of Lemma 3.13, we see that $W'(r) \leq 0$ for $r \geq C/\lambda$ and so long as ϕ is positive (note that $f > 0$ everywhere). Now let R denote the largest zero of ϕ and for contradiction assume $R \geq C/\lambda$. Then $W'(r) < 0$ and $\phi \sim e^{-mr}$ as $r \to \infty$ imply that $W(R) \geq 0$. This means that

$$\lim_{r \to R^+} \frac{f'(r)}{f(r)} \geq \lim_{r \to R^+} \frac{\phi'(r)}{\phi(r)} = \infty,$$

and therefore we must have $f(R) = 0$ which is impossible.

In the case of a threshold resonance, we compare ϕ with $g = 1$, which up to scaling is the unique nonzero bounded solution of $\partial_{rr} g = 0$. We omit the details, which are very similar to the previous case. □

Our task now is to show that ϕ as in Lemma 3.14 does not change sign in the interval $r \leq C/\lambda$. For this purpose we use the technique of renormalization.

Lemma 3.15. *Let ϕ be either an eigenfunction (i.e., a nonzero L^2 solution) of $\mathcal{L}_{V_\lambda}\phi = \mu^2\phi$ with $\mu^2 \in (0, \frac{1}{4})$ or a threshold resonance at $\mu^2 = 1/4$. Let C be as in Lemma 3.13. Then for sufficiently large λ, ϕ cannot change sign in the region $r \in (0, \frac{C}{\lambda}]$.*

Proof. We work with the rescaled operator $\widetilde{\mathcal{L}}_\lambda$. It suffices to show that if $\psi(\rho)$ is a solution of $\widetilde{\mathcal{L}}_\lambda\psi = (\mu^2/\lambda^2)\psi$ in $L^2((0, C])$, then $\psi(\rho)$ does not change sign in the region $0 \leq \rho \leq C$. Arguing as in Lemma 3.3, we see that any $L^2((0, C])$ solution of $\widetilde{\mathcal{L}}_\lambda\psi = (\mu^2/\lambda^2)\psi$, after suitable normalization, has the behavior $\psi(\rho) = \rho^{3/2} + o(\rho^{3/2})$ and $\psi'(\rho) = (3/2)\rho^{1/2} + o(\rho^{1/2})$ as $\rho \to 0$.

Define $g(\rho) := \psi(\rho)/\varphi_0(\rho)$, where φ_0 is the Euclidean resonance defined in (3.23). Since φ_0 is always positive, we need to show that g is bounded away from zero in the region $0 \leq r \leq C$. Recall from Section 3.2 that g satisfies the equation

$$(g'\varphi_0^2)' = \varphi_0^2 W_{\lambda,\mu} g.$$

Note furthermore that we have $(g, g')(0) = (1, 0)$, thanks to our normalization of ψ. Therefore, we have the integral formula

$$g(\rho) = 1 + \int_0^\rho \int_0^\tau \frac{\varphi_0^2(\sigma)}{\varphi_0^2(\tau)} W_{\lambda,\mu}(\sigma) g(\sigma)\, d\sigma d\tau. \tag{3.47}$$

By a Volterra-type iteration argument as in the proof of Lemma 3.9 (see the proof of Claim 3.12), we see that

$$\sup_{\rho \in [0,C]} |g(\rho) - 1| = o(1) \quad \text{as} \quad \lambda \to \infty,$$

from which the lemma follows. □

Proposition 3.16. *If λ is sufficiently large, then $\mathcal{L}_{V_\lambda}$ has a unique simple eigenvalue in $(0, \frac{1}{4})$, with no threshold resonance at $1/4$.*

Proof. Existence was seen in the previous subsection, so it suffices to establish uniqueness and simpleness. Let ϕ be an eigenfunction corresponding to any eigenvalue $\mu^2 \in (0, \frac{1}{4})$, or a threshold resonance at $\mu^2 = \frac{1}{4}$. Combining Lemmas 3.14 and 3.15, it follows that ϕ cannot change sign on $(0, \infty)$. It follows by the variational principle that there is no eigenvalue below μ^2, and when $\mu^2 \in (0, \frac{1}{4})$ is an eigenvalue it is simple. The proposition follows. □

3.5. **Migration of the gap eigenvalue.** In this subsection, we conclude the proof of Theorem 1.2 by demonstrating that the gap eigenvalue μ_λ^2 approaches 0 as $\lambda \to \infty$. By Sturm's oscillation theory, Proposition 3.5 and the uniqueness of the gap eigenvalue, it suffices to establish the following proposition:

Proposition 3.17. *Let $\overline{\mu}^2 \in (0, 1/4]$. Then for λ sufficiently large (depending on $\overline{\mu}^2$), the solution ϕ_0 to the ODE*

$$\begin{cases} \mathcal{L}_{V_\lambda} \phi_0 = \overline{\mu}^2 \phi_0 \\ \phi_0 = r^{3/2} + o(r^{3/2}) \text{ as } r \to 0 \end{cases}$$

must change sign.

We will prove Proposition 3.17 by a contradiction argument, which is similar in spirit to the proof of Proposition 3.7. The key additional idea is to use $\zeta_\infty^{(\lambda)}$, which is the solution to the problem

$$\begin{cases} \mathcal{L}_{V_\lambda} \zeta_\infty^{(\lambda)} = 0, \\ \zeta_\infty^{(\lambda)} \sim c_\lambda e^{-r/2} \text{ as } r \to \infty, \end{cases} \tag{3.48}$$

for an appropriate $c_\lambda > 0$. An interesting feature of $\zeta_\infty^{(\lambda)}$ is that it is a conjugate solution to the explicit solution $\zeta_0^{(\lambda)}$ used in the proof of Proposition 3.5. By standard ODE theory, it follows that $\zeta_\infty^{(\lambda)}$ can also be explicitly determined.

We now briefly explain why $\zeta_\infty^{(\lambda)}$ is useful for proving Proposition 3.17. By a comparison argument, the contradiction hypothesis $\phi_0 > 0$ leads to a lower bound for ϕ_0 in terms of $\zeta_\infty^{(\lambda)}$, i.e.,

$$\phi_0(r) \geq \frac{\phi_0(r_0)}{\zeta_\infty(r_0)} \zeta_\infty(r) \qquad \text{for } 0 < r_0 \leq r.$$

(For details, see the proof of (3.56) below.) Thanks to the explicit expression for $\zeta_\infty^{(\lambda)}$, we are able to derive from this inequality a uniform lower bound for ϕ_0 in an r-interval of length $\simeq 1$. In the renormalized coordinate, this lower bound holds on a ρ-interval of length $\simeq \lambda$, which can be made arbitrarily large. This gives enough 'time' for the renormalized potential (which is *negative* since $\overline{\mu}^2 > 0$) to force a sign change of ϕ_0, which is a contradiction.

Remark 7. In fact, our proof of Proposition 3.17 does not depend on Proposition 3.7, and therefore furnishes an alternative proof of existence of a gap eigenvalue. We have nevertheless elected to include both proofs in this paper, since the proof of Proposition 3.7 presented in Section 3.3 requires a weaker hypothesis (in particular, there is no need for the knowledge of the explicit solution $\zeta_\infty^{(\lambda)}$) and therefore might be of independent interest.

Some lemmas needed for proving Proposition 3.17 are in order. The first lemma consists of an upper and lower bound on the explicit solution $\zeta_\infty^{(\lambda)}$.

Lemma 3.18. *There exist $\epsilon_1 > 0$ such that*

$$\zeta_\infty^{(\lambda)}(r) \simeq \lambda^{1/2}\Big(\lambda^2 + \frac{1}{\lambda^2 r^2}\Big)\frac{(\lambda r)^{3/2}}{1+\lambda^2 r^2} \qquad \text{for } 0 \leq r \leq \epsilon_1,\ \lambda > 0, \tag{3.49}$$

where the implicit constants are independent of r and λ.

Proof. We begin by computing $\zeta_\infty^{(\lambda)}$ explicitly. For simplicity, we will omit writing the superscript (λ).

Since ζ_0 and ζ_∞ solve the same equation (with no first order term), their Wronskian is constant. We choose c_λ in the definition of ζ_∞ so that

$$W[\zeta_0, \zeta_\infty] = \zeta_0 \zeta_\infty' - \zeta_0' \zeta_\infty = -1$$

Dividing by ζ_0^2, we have

$$\Big(\frac{\zeta_\infty}{\zeta_0}\Big)' = -\frac{1}{\zeta_0^2}.$$

Because of the vanishing condition as $r \to \infty$, it follows that

$$\begin{aligned}\zeta_\infty(r) =& \Big(\int_r^\infty \frac{1}{\zeta_0^2(s)} \, \mathrm{d}s\Big)\zeta_0(r) \\ =& \Big(\int_r^\infty \frac{1}{\zeta_0^2(s)} \, \mathrm{d}s\Big)\frac{\tanh(r/2)}{1+\lambda^2\tanh^2(r/2)} \sinh^{1/2} r.\end{aligned}$$

We now compute the s-integral. Using the identity

$$\sinh s = 2\cosh(s/2)\sinh(s/2) = \frac{2\tanh(s/2)}{1-\tanh^2(s/2)},$$

we see that

$$\frac{1}{\zeta_0^2(s)} = \frac{(1+\lambda^2\tanh^2(s/2))^2(1-\tanh^2(s/2))}{2\tanh^3(r/2)}$$

Therefore, making a change of variables $u = \tanh(s/2)$, we have

$$\begin{aligned}
&\int_r^\infty \frac{1}{\zeta_0^2(s)}\,\mathrm{d}s \\
&= \int_{\tanh(r/2)}^1 \frac{(1+\lambda^2 u^2)^2}{u^3}\,\mathrm{d}u \qquad (3.50)\\
&= -\frac{1}{2}(1-\tanh^{-2}(r/2)) - 2\lambda^2 \log\tanh(r/2) + \frac{\lambda^4}{2}(1-\tanh^2(r/2)).
\end{aligned}$$

To prove (3.49), it suffices to establish

$$\frac{\tanh(r/2)}{1+\lambda^2\tanh^2(r/2)}\sinh^{1/2} r \simeq \frac{r^{3/2}}{1+\lambda^2 r^2}, \tag{3.51}$$

$$\int_r^\infty \frac{1}{\zeta_0^2(s)}\mathrm{d}s \simeq \lambda^2\Big(\lambda^2 + \frac{1}{\lambda^2 r^2}\Big), \tag{3.52}$$

for $0 \le r \le \epsilon_1$, where $\epsilon_1 > 0$ is to be chosen below.

Estimate (3.51) is an immediate consequence of the easier estimates

$$\tanh(r/2) \simeq r, \quad \sinh r \simeq r, \quad 1+\lambda\tanh(r/2) \simeq 1+\lambda r.$$

which holds when $\epsilon_1 > 0$ is sufficiently small. On the other hand, (3.52) is easy to prove (taking $\epsilon_1 > 0$ smaller if necessary) by directly estimating the integral (3.50), whose integrand is positive. We leave the details to the reader. ☐

In the following lemma, we collect useful facts for estimating the renormalized potential $W_{\lambda,\overline{\mu}}$.

Lemma 3.19. *For $0 \le \rho \le \lambda$ we have*

$$\Big|\frac{3}{4}\frac{1}{\lambda^2\sinh^2(\rho/\lambda)} - \frac{3}{4}\frac{1}{\rho^2} + \frac{1}{4\lambda^2} - \frac{\overline{\mu}^2}{\lambda^2}\Big| \lesssim \frac{1}{\lambda^2}, \tag{3.53}$$

$$\Big|\frac{1}{\lambda^2}V_\lambda(\rho/\lambda) - V_{euc}(\rho)\Big| \lesssim \frac{1}{\lambda^2}\frac{1}{(1+\rho^2)}. \tag{3.54}$$

Moreover, there exists $\epsilon_2 = \epsilon_2(\overline{\mu}^2) > 0$, which are independent of ρ and λ, such that for $0 \le \rho \le \epsilon_2\lambda$ we have

$$\frac{3}{4}\frac{1}{\lambda^2\sinh^2(\rho/\lambda)} - \frac{3}{4}\frac{1}{\rho^2} + \frac{1}{4\lambda^2} - \frac{\overline{\mu}^2}{\lambda^2} \le -\frac{\overline{\mu}^2}{2\lambda^2}. \tag{3.55}$$

Proof. Estimate (3.53) follows from (3.36) and estimate (3.54) is exactly (3.39) in the proof of Lemma 3.10. To prove (3.55), we begin by observing that the Taylor expansion of $r^2/\sinh^2 r$ at $r = 0$ is given by

$$\frac{r^2}{\sinh^2 r} = 1 - \frac{1}{3}r^2 + \frac{1}{2}\int_0^r (r-r')^2 E(r')\,\mathrm{d}r'$$

where

$$E(r) = \frac{\mathrm{d}^3}{\mathrm{d}r^3}\Big(\frac{r^2}{\sinh^2 r}\Big).$$

Note that $E(r)$ obviously enjoys the bound $\sup_{r\in[0,1]}|E(r)| \le C$ for some absolute constant $C > 0$. Therefore,

$$\Big|\frac{r^2}{\sinh^2 r} - 1 + \frac{1}{3}r^2\Big| \le Cr^3$$

for $0 \leq r \leq 1$. Making a change of variable $r = \rho/\lambda$ and restricting to $0 \leq \rho \leq \epsilon_2 \lambda$, it follows that

$$\Big|\frac{3}{4}\frac{1}{\lambda^2 \sinh^2(\rho/\lambda)} - \frac{3}{4}\frac{1}{\rho^2} + \frac{1}{4\lambda^2}\Big| \leq \epsilon_2 \frac{C}{\lambda^2}.$$

Choosing $\epsilon_2 > 0$ sufficiently small compared to $\overline{\mu}^2 > 0$, (3.55) follows. □

We are now ready to prove Proposition 3.17.

Proof of Proposition 3.17. In what follows, we will omit the superscript (λ) in $\zeta_\infty^{(\lambda)}$ for simplicity. For the sake of contradiction, suppose that ϕ_0 does not change sign, i.e., ϕ_0 is positive.

Step 1: We claim that

$$W[\phi_0, \zeta_\infty] = \phi_0 (\zeta_\infty)' - \phi_0' \zeta_\infty \leq 0. \tag{3.56}$$

Indeed, suppose the contrary. Then at some $R > 0$, we must have

$$W[\phi_0, \zeta_\infty](R) > 0.$$

We introduce an auxiliary function ζ which solves the equation

$$\begin{cases} \mathcal{L}_{V_\lambda} \zeta =0, \\ (\zeta, \zeta')(R) =(\phi_0, \phi_0')(R). \end{cases}$$

Since ζ and ζ_∞ solve the same equation, their Wronskian is constant. Therefore,

$$W[\zeta, \zeta_\infty](r) = W[\zeta, \zeta_\infty](R) = W[\phi_0, \zeta_\infty](R) > 0$$

for all $r \geq 0$. It follows that ζ must tend to $-\infty$ as $r \to \infty$, thus changing sign. Then a comparison argument between ϕ_0 and ζ as in Propositions 3.2 and 3.5, using crucially the fact that $\overline{\mu}^2 > 0$, shows that ϕ_0 must also change sign, which is a contradiction.

Step 2: As discussed earlier, the benefit of (3.56) is that it gives a lower bound on ϕ_0 in terms of ζ_∞ on an arbitrarily long interval in the renormalized coordinate. Indeed, by (3.56), we have

$$\frac{\mathrm{d}}{\mathrm{d}r} \log \phi_0(r) \geq \frac{\mathrm{d}}{\mathrm{d}r} \log \zeta_\infty(r). \tag{3.57}$$

Thus, for any $r \geq r_0 > 0$,

$$\phi_0(r) \geq \frac{\phi_0(r_0)}{\zeta_\infty(r_0)} \zeta_\infty(r). \tag{3.58}$$

To translate this lower bound to the renormalized picture, we make the change of variable $\rho = \lambda r$ (thus $\rho_0 = \lambda r_0$) and define

$$\begin{aligned} \widetilde{\zeta_\infty}(\rho) :=& \lambda^{-1/2} \zeta_\infty(\rho/\lambda), \\ g(\rho) :=& \frac{\lambda^{3/2} \phi_0(\rho/\lambda)}{\varphi_0(\rho)}, \end{aligned}$$

where we remind the reader that

$$\varphi_0(\rho) = \frac{\rho^{3/2}}{1 + (\rho/2)^2}.$$

Since $\varphi_0 > 0$, our contradiction hypothesis $\phi_0 > 0$ is equivalent to $g > 0$. Moreover, (3.58) translates to

$$g(\rho) \geq g(\rho_0)\Big(\frac{\widetilde{\zeta_\infty}(\rho_0)}{\varphi_0(\rho_0)}\Big)^{-1}\Big(\frac{\widetilde{\zeta_\infty}(\rho)}{\varphi_0(\rho)}\Big). \tag{3.59}$$

Applying Lemma 3.18 and plugging in the definition of φ_0, we see that

$$g(\rho) \geq C_{\rho_0} g(\rho_0), \qquad \text{for } \rho_0 \leq \rho \leq \epsilon_1\lambda, \quad \lambda \geq 1, \tag{3.60}$$

where $C > 0$ is independent of ρ and λ.

Step 3: Note that g satisfies $(g, g')(0) = (1, 0)$ and

$$\big(g'\varphi_0^2\big)' = \varphi_0^2 W_{\lambda,\overline{\mu}} g \tag{3.61}$$

where

$$W_{\lambda,\overline{\mu}}(\rho) = \frac{3}{4}\frac{1}{\lambda^2\sinh^2(\rho/\lambda)} - \frac{3}{4}\frac{1}{\rho^2} + \frac{1}{4\lambda^2} - \frac{\overline{\mu}^2}{\lambda^2} + \frac{1}{\lambda^2}V_\lambda(\rho/\lambda) - V_{\text{euc}}(\rho).$$

At this point, we fix a large enough $\rho_0 > 0$ so that we have

$$W_{\lambda,\overline{\mu}}(\rho) \leq -\frac{\overline{\mu}^2}{4\lambda^2} \qquad \text{for } \rho_0 \leq \rho \leq \epsilon_2\lambda. \tag{3.62}$$

Indeed, (3.62) follows by combining (3.54) and (3.55). We now claim that the following bounds hold for $g(\rho)$ and $g'(\rho)$: For $0 \leq \rho \leq \rho_0$ and $\lambda \geq \rho_0$, we have

$$|g(\rho) - 1| \leq \frac{C_{\rho_0}}{\lambda^2}, \tag{3.63}$$

$$|g'(\rho)| \leq \frac{C_{\rho_0}}{\lambda^2}. \tag{3.64}$$

These bounds are proved in a similar fashion to the proof of uniqueness of μ_λ^2, cf. Lemma 3.15. From (3.61) it follows that

$$\begin{aligned} g'(\rho) &= \frac{1}{\varphi_0^2(\rho)}\int_0^\rho \varphi_0^2(\sigma)W_{\lambda,\overline{\mu}}(\sigma)g(\sigma)\,\mathrm{d}\sigma, \\ g(\rho) &= 1 + \int_0^\rho\int_0^\tau \frac{\varphi_0^2(\sigma)}{\varphi_0^2(\tau)}W_{\lambda,\overline{\mu}}(\sigma)g(\sigma)\,\mathrm{d}\sigma\mathrm{d}\tau. \end{aligned}$$

Using (3.53) and (3.54) to estimate $W_{\lambda,\overline{\mu}}$, substituting φ_0 by its explicit definition and estimating the resulting integral, it follows that

$$\int_0^{\rho_0} \varphi_0^2(\sigma)\,|W_{\lambda,\overline{\mu}}(\sigma)|\,\mathrm{d}\sigma \lesssim_{\rho_0} \frac{1}{\lambda^2},$$

$$\int_0^{\rho_0}\Big(\int_\sigma^{\rho_0}\frac{1}{\varphi_0^2(\tau)}\,\mathrm{d}\tau\Big)\varphi_0^2(\sigma)\,|W_{\lambda,\overline{\mu}}(\sigma)|\,\mathrm{d}\sigma \lesssim_{\rho_0} \frac{1}{\lambda^2}.$$

Then by a Volterra-type iteration, (3.63) follows. Moreover, (3.64) is an immediate consequence of plugging in (3.63) to the formula for $g'(\rho)$.

Step 4: We now derive a contradiction. Our starting point is the identity

$$g(\rho) = g(\rho_0) + g'(\rho_0)\int_{\rho_0}^\rho \frac{\varphi_0^2(\rho_0)}{\varphi_0^2(\tau)}\,\mathrm{d}\tau + \int_{\rho_0}^\rho\int_{\rho_0}^\tau \frac{\varphi_0^2(\sigma)}{\varphi_0^2(\tau)}W_{\lambda,\overline{\mu}}(\sigma)g(\sigma)\,\mathrm{d}\sigma\mathrm{d}\tau,$$

for $\rho \geq \rho_0$, which is obtained by integrating (3.61) twice.

Fix $\epsilon > 0$ so that $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Taking $\lambda \geq \rho_0/\epsilon$, let $\rho = \epsilon\lambda$ in the preceding identity. Then since $\epsilon \leq \epsilon_2$, we can apply (3.55) and conclude the following one-sided inequality:

$$g(\epsilon\lambda) \leq g(\rho_0)\Big(1 - \frac{C\overline{\mu}^2}{\lambda^2}\int_{\rho_0}^{\epsilon\lambda}\int_{\rho_0}^{\tau}\frac{\varphi_0^2(\sigma)}{\varphi_0^2(\tau)}\,\mathrm{d}\sigma\mathrm{d}\tau\Big) + |g'(\rho_0)|\int_{\rho_0}^{\epsilon\lambda}\frac{\varphi_0^2(\rho_0)}{\varphi_0^2(\tau)}\,\mathrm{d}\tau. \tag{3.65}$$

Recalling the definition of φ_0, we easily compute

$$\begin{aligned}\int_{\rho_0}^{\epsilon\lambda}\frac{1}{\varphi_0^2(\tau)}\,\mathrm{d}\tau \leq& C_{\rho_0}\epsilon^2\lambda^2,\\ -\frac{1}{\lambda^2}\int_{\rho_0}^{\epsilon\lambda}\int_{\rho_0}^{\tau}\frac{\varphi_0^2(\sigma)}{\varphi_0^2(\tau)}\,\mathrm{d}\sigma\mathrm{d}\tau \leq& -C_{\rho_0}\epsilon^2\log(2+\epsilon\lambda).\end{aligned}$$

Therefore, we obtain

$$g(\epsilon\lambda) \leq g(\rho_0)\big(1 - C_{\rho_0,\epsilon}\,\overline{\mu}^2\log(2+\epsilon\lambda)\big) + C_{\rho_0}\epsilon^2(\lambda^2\,|g'(\rho_0)|).$$

We now recall the bounds (3.63) and (3.64) for $g(\rho_0)$ and $\lambda^2\,|g'(\rho_0)|$. Thanks to the term $\log(2+\epsilon\lambda)$, we then see that the right-hand side is negative when λ is sufficiently large. It follows that $\phi_0(\epsilon) = g(\epsilon\lambda)\varphi(\epsilon\lambda) < 0$, which contradicts our hypothesis that $\phi_0 > 0$. □

4. Strichartz estimates for the linearized operators

The goal of this section is to prove Strichartz estimates for a radial shifted [1] linear wave equation in $\mathbb{R}\times\mathbb{H}^4$, perturbed by a radial potential V, i.e.,

$$\begin{aligned}&u_{tt} - u_{rr} - 3\coth r\, u_r - 2u + Vu = F,\\ &\vec{u}(0) = (u_0, u_1).\end{aligned} \tag{4.1}$$

We will make several assumptions about V, which are consistent with the potentials V_λ, U_λ, where V_λ is as in Proposition 3.2, and U_λ is as in (1.30). First define

$$H_V := -\partial_{rr} - 3\coth r\,\partial_r - 2 + V(r).$$

We will work under the assumptions that

(*A*) V is real-valued, smooth, radial, and bounded on $\mathbb{H}^4$, and H_V is self-adjoint on the domain $\mathcal{D} = H^2(\mathbb{H}^4)$. Moreover $V(r) \leq Ce^{-2r}$ as $r\to\infty$.

(*B*) The operator H_V defined above has purely absolutely continuous spectrum

$$\sigma(H_V) = [1/4, \infty)\,.$$

In particular, H_V has no negative spectrum and no eigenvalues in the gap $[0, 1/4)$. Moreover, the threshold energy $\frac{1}{4}$ is neither an eigenvalue nor a resonance.

Remark 8. We note that by Proposition 3.2, H_{V_λ} satisfies (*A*) and (*B*) above for $0 \leq \lambda < \sqrt{15/8}$. On the other hand, H_{U_λ} satisfies (*A*) and (*B*) for all $\lambda \in [0, 1)$, because $U_\lambda \geq 0$ is a repulsive potential. In fact, we have

$$U_\lambda(r) := \frac{\cosh 2P_\lambda - 1}{\sinh^2 r} = \frac{8\lambda^2}{[\cosh r + 1 - \lambda^2(\cosh r - 1)]^2} \geq 0. \tag{4.2}$$

[1] In [3] and most of the related literature "the shifted wave equation" refers to the equation $(\Box_{\mathbf{g}} - \frac{9}{4})u = F$ because the spectrum of the Laplacian $\Delta_{\mathbf{g}}$ on $\mathbb{H}^4$ is $[9/4, \infty)$. Nevertheless, we have preferred to use the term "a shifted wave equation" here as well since there is little risk for confusion.

Strichartz estimates for the free equation, that is with $V \equiv 0$, were proved by Anker-Pierfelice, [3], and we briefly recall their set-up and main result. The corresponding free shifted linear wave equation on $\mathbb{R} \times \mathbb{H}^4$ is given by

$$\begin{aligned} (\Box_{\mathbf{g}} - 2)v &:= v_{tt} - \Delta_{\mathbf{g}} v - 2v = F, \\ \vec{v}(0) &= (v_0, v_1). \end{aligned} \tag{4.3}$$

A triple (p, q, σ) is called hyperbolic-admissible if

$$p, q > 2, \quad \frac{1}{p} + \frac{3}{2q} \le \frac{3}{4}, \quad \frac{1}{p} + \frac{4}{q} = 2 - \sigma. \tag{4.4}$$

Proposition 4.1. [3, Corollary 5.3] *Suppose $\vec{v}(t)$ is a solution to* (4.3) *with initial data $\vec{v}(0) = (v_0, v_1)$ and let $0 \in I \subset \mathbb{R}$ be any time interval. Let (p, q, σ) and (a, b, γ) be any two hyperbolic-admissible triples. Then we have the estimates*

$$\|v\|_{L^p(I; W^{1-\sigma, q}(\mathbb{H}^4))} + \|\partial_t v\|_{L^p(I; W^{-\sigma, q}(\mathbb{H}^4))} \lesssim \|\vec{v}(0)\|_{H^1 \times L^2(\mathbb{H}^4)} + \|F\|_{L^{a'}(I; W^{\gamma, b'}(\mathbb{H}^4))}. \tag{4.5}$$

We will use Proposition 4.1 together with a perturbative argument to establish the corresponding estimates for (4.1). In particular we prove the following result.

Proposition 4.2. *Suppose $\vec{u}(t)$ is a solution to* (4.1) *with initial data $\vec{u}(0) = (u_0, u_1)$ and with V satisfying assumptions (A) and (B) above. Let $0 \in I \subset \mathbb{R}$ be any time interval. Let (p, q, σ) and (a, b, γ) be any two hyperbolic-admissible triples. Then we have the estimates*

$$\|u\|_{L^p(I; W^{1-\sigma, q}(\mathbb{H}^4))} + \|\partial_t u\|_{L^p(I; W^{-\sigma, q}(\mathbb{H}^4))} \lesssim \|\vec{u}(0)\|_{H^1 \times L^2(\mathbb{H}^4)} + \|F\|_{L^{a'}(I; W^{\gamma, b'}(\mathbb{H}^4))}. \tag{4.6}$$

Proof of Proposition 4.2. The proof roughly follows the approach in [16, Section 5], which in turn is a variant of an argument in [22]. Note that by the standard TT^* argument and Minkowski's inequality it suffices to consider the case $F = 0$.

The argument hinges on certain estimates related to the distorted Fourier transforms relative to the self-adjoint operators $H_0 := -\Delta_{\mathbf{g}} - 2$ and $H_V = -\Delta_{\mathbf{g}} - 2 + V$ on the domain $\mathcal{D} := H^2(\mathbb{H}^4)$, restricted to radial functions. First though, we can reduce the proof of Proposition (4.2) to a pair of local energy estimates, in particular (4.11) and (4.12) below.

Indeed, define the operator

$$A := \sqrt{-\Delta_{\mathbf{g}} - 2},$$

and note that

$$\|Af\|_{L^2(\mathbb{H}^4)} \simeq \|f\|_{H^1(\mathbb{H}^4)}. \tag{4.7}$$

For any real valued $\vec{u} = (u_0, u_1) \in H^1 \times L^2(\mathbb{H}^4)$ we set

$$w := Au_0 + iu_1$$

Then (4.7) implies that $\|w\|_{L^2} \simeq \|\vec{u}\|_{H^1 \times L^2}$. Moreover, $\vec{u}(t)$ solves (4.1) if and only if

$$\begin{aligned} i\partial_t w &= Aw + Vu, \\ w(0) &= Au_0 + iu_1. \end{aligned} \tag{4.8}$$

The Duhamel formula then gives us

$$w(t) = e^{-itA} w(0) - i \int_0^t e^{-i(t-s)A} V u(s) \, ds$$

and by (4.7) we need to show that

$$\|Pw\|_X \le C\|w(0)\|_{L^2}$$

where $P := A^{-1}\,\mathrm{Re}$, and $X := L^p_t W^{1-\sigma,q}_x$ is any Strichartz norm, i.e., (p,q,σ) is hyperbolic admissible. By Proposition 4.1, we have

$$\|Pe^{-itA}w(0)\|_X \le C\|w(0)\|_{L^2}.$$

By the Christ-Kislelev lemma, see [8, 28], it then suffices to show

$$\left\| P\int_{-\infty}^{\infty} e^{-i(t-s)A} Vu(s)\,ds \right\|_X \lesssim \|w(0)\|_{L^2} \simeq \|\vec{u}(0)\|_{H^1\times L^2}. \tag{4.9}$$

To prove (4.9) we factor the potential $V(r) = V_1(r)V_2(r)$ so that each factor decays like e^{-r} as $r\to\infty$. Then,

$$\begin{aligned} \left\| P\int_{-\infty}^{\infty} e^{-i(t-s)A} Vu(s)\,ds \right\|_X &\le \|K\|_{L^2_{t,x}\to X}\|V_2u\|_{L^2_{t,x}} \\ (Kf)(t) := P\int_{-\infty}^{\infty} e^{-i(t-s)A}V_1 f(s)\,ds & \end{aligned} \tag{4.10}$$

Next, note that for each $f\in L^2_{t,x}(\mathbb{R}\times\mathbb{H}^4)$ we have

$$\|Kf\|_X \le \|Pe^{-itA}\|_{L^2_x\to X}\left\| \int_{-\infty}^{\infty} e^{isA}V_1 f(s)\,ds\right\|_{L^2_x}$$

The first factor on the right-hand-side above is bounded by a fixed constant by Proposition 4.1. We claim that the second factor is bounded by $C\|f\|_{L^2_{t,x}}$. By duality, this is equivalent to the localized energy bound:

$$\|V_1e^{-itA}\varphi\|_{L^2_{t,x}} \le C\|\varphi\|_{L^2_x}, \quad \forall\varphi\in L^2(\mathbb{H}^4) \tag{4.11}$$

Therefore, by (4.10) we have reduced the proof of Proposition 4.2 to proving (4.11) in addition to a local energy estimate for the perturbed evolution, namely

$$\|V_2u\|_{L^2_{t,x}} \le C\|\vec{u}(0)\|_{H^1\times L^2} \simeq C\|w(0)\|_{L^2} \tag{4.12}$$

To prove (4.11) and (4.12) we will need to develop some machinery. First we pass to an equation on the half-line by conjugating by $\sinh^{\frac32} r$. Indeed, the map

$$L^2(\mathbb{H}^4) \ni \varphi \mapsto \sinh^{\frac32} r\,\varphi =: \phi \in L^2(0,\infty) \tag{4.13}$$

is an isomorphism of $L^2(\mathbb{H}^4)$, restricted to radial functions, onto $L^2([0,\infty))$. If we define $\mathcal{L}_0, \mathcal{L}_V$ by

$$\begin{aligned} \mathcal{L}_0 &:= -\partial_{rr} + \frac14 + \frac{3}{4\sinh^2 r}, \\ \mathcal{L}_V &:= -\partial_{rr} + \frac14 + \frac{3}{4\sinh^2 r} + V(r), \end{aligned} \tag{4.14}$$

we have

$$\begin{aligned} (H_0\varphi)(r) &= \sinh^{-\frac32} r(\mathcal{L}_0\phi)(r), \\ (H_V\varphi)(r) &= \sinh^{-\frac32} r(\mathcal{L}_V\phi)(r). \end{aligned} \tag{4.15}$$

We claim that we can associate with $\mathcal{L}_0$ and $\mathcal{L}_V$ distorted Fourier bases, $\phi_0(r;\xi)$, respectively $\phi(r;\xi)$, that satisfy

$$\mathcal{L}_0\phi_0(r;\xi) = (\frac{1}{4}+\xi^2)\phi_0(r;\xi), \quad \phi_0(r;\xi) \in L^2([0,b]),\ \forall\ 0<b<\infty \tag{4.16}$$

$$\mathcal{L}_V\phi(r;\xi) = (\frac{1}{4}+\xi^2)\phi(r;\xi), \quad \phi(r;\xi) \in L^2([0,b]),\ \forall\ 0<b<\infty. \tag{4.17}$$

To prove (4.11) we will need the following additional information about $\phi_0(r;\xi)$. For all $g \in L^2([0,\infty))$, set

$$\tilde{g}(\xi) := \int_0^\infty \phi_0(r;\xi) g(r)\, dr. \tag{4.18}$$

We can find a positive measure $\rho_0(d\xi) = \omega_0(\xi)\, d\xi$ so that

$$g(r) = \int_0^\infty \phi_0(r;\xi)\, \tilde{g}(\xi)\, \rho_0(d\xi), \tag{4.19}$$

$$\|g\|^2_{L^2(0,\infty)} = \int_0^\infty |\tilde{g}(\xi)|^2\ \rho_0(d\xi), \tag{4.20}$$

$$\sup_{r>0,\xi>0} \frac{|\phi_0(r;\xi)|^2}{(r+1)^2}\frac{\langle\xi\rangle}{\xi}\omega_0(\xi) \le C < \infty, \tag{4.21}$$

where we are using the notation $\langle\xi\rangle := \sqrt{\xi^2+\frac{1}{4}}$. We remark here that the distorted Fourier basis (4.16) is explicit and is obtained by setting $\phi_0(r,\xi) := \sinh^{\frac{3}{2}} r\Phi_\xi(r)$ where the Φ_ξ are the hyperbolic spherical functions in $\mathbb{H}^4$. The spectral measure $\rho_0(d\xi) := |c(\xi)|^{-2}\, d\xi$ where $c(\xi)$ is the Harish-Chandra c-function. Then (4.19) and (4.20) follow from the corresponding facts about the Helgason-Fourier transform on $L^2(\mathbb{H}^4)$. The estimates (4.21) will follow from well known estimates for Φ_ξ and $c(\xi)$ and we will sketch the proof in Lemma 4.3 below.

With (4.19)–(4.21) we can easily prove (4.11). First, note that after passing to the half-line formulation via conjugation by $\sinh^{\frac{3}{2}} r$, (4.11) is equivalent to proving

$$\|V_1 e^{-it\sqrt{\mathcal{L}_0}} g\|_{L^2_{t,x}(\mathbb{R}\times(0,\infty))} \lesssim \|g\|_{L^2(0,\infty)}, \tag{4.22}$$

for $g \in L^2(0,\infty)$. As $\mathcal{L}_0$ is given by multiplication by $\langle\xi\rangle$ on the Fourier side, the above, using (4.19), reduces to showing that

$$\int_{-\infty}^\infty \left\| V_1 \int_0^\infty e^{-it\langle\xi\rangle}\phi_0(r;\xi)\tilde{g}(\xi)\rho_0(d\xi)\right\|^2_{L^2(0,\infty)} dt \lesssim \|g\|^2_{L^2(0,\infty)}. \tag{4.23}$$

Expanding and carrying out the t-integration, the left-hand-side becomes

$$\begin{aligned}&\int_0^\infty V_1^2(r)\int_0^\infty\int_0^\infty \phi_0(r;\xi)\phi_0(r;\mu)\tilde{g}(\xi)\overline{\tilde{g}(\mu)}\delta(\langle\xi\rangle-\langle\mu\rangle)\omega_0(\xi)d\xi\,\omega_0(\mu)d\mu\,dr\\ &= \int_0^\infty V_1^2(r)\int_0^\infty \phi_0^2(r;\xi)\,|\tilde{g}(\xi)|^2\ \omega_0^2(\xi)\frac{\langle\xi\rangle}{\xi}\,d\xi\,dr\end{aligned} \tag{4.24}$$

Using the estimates (4.21), the exponential decay of $V_1(r)$, and (4.20), we can bound the above as follows:

$$\int_0^\infty V_1^2(r) \int_0^\infty \phi_0^2(r;\xi) \left|\tilde{g}(\xi)\right|^2 \omega_0^2(\xi) \frac{\langle\xi\rangle}{\xi} \, d\xi \, dr \lesssim$$
$$\int_0^\infty V_1(r)(r+1)^2 \, dr \int_0^\infty \left|\tilde{g}(\xi)\right|^2 \omega_0(\xi) \, d\xi \lesssim \int_0^\infty \left|\tilde{g}(\xi)\right|^2 \omega_0(\xi) \, d\xi = \|g\|_{L^2(0,\infty)},$$

which proves (4.23) and hence (4.11).

The key point here is that we can establish the analogs of (4.18)–(4.21) for the perturbed operator $\mathcal{L}_V$ and $\phi(r;\xi)$ as in (4.17). In particular, for $f \in L^2(0,\infty)$ we set

$$\hat{f}(\xi) := \int_0^\infty \phi(r;\xi) f(r) \, dr. \tag{4.25}$$

We can find a positive measure $\rho(d\xi) = \omega(\xi)\, d\xi$ so that

$$f(r) = \int_0^\infty \phi(r;\xi) \, \hat{f}(\xi) \, \rho(d\xi), \tag{4.26}$$

$$\|f\|_{L^2(0,\infty)}^2 = \int_0^\infty \left|\hat{f}(\xi)\right|^2 \rho(d\xi), \tag{4.27}$$

$$\|\sqrt{\mathcal{L}_V} f\|_{L^2(0,\infty)} = \int_0^\infty \langle\xi\rangle^2 \left|\hat{f}(\xi)\right|^2 \rho(d\xi), \tag{4.28}$$

$$\sup_{r>0,\xi>0} \frac{\left|\phi(r;\xi)\right|^2}{(r+1)^3} \frac{\langle\xi\rangle}{\xi} \omega(\xi) \leq C < \infty. \tag{4.29}$$

The proof of (4.29) is where the spectral information for H_V plays a crucial role. In particular, the fact that the spectrum of H_V is supported on $(1/4,\infty)$ and that the threshold energy $1/4$ is not a resonance allows us to establish the same rate of decay for $\omega(\xi)$ as for the free spectral measure $\omega_0(\xi)$ as $\xi \to 0+$. We postpone the proof of (4.17), (4.25)–(4.29) to Lemma 4.4 below, and first use these estimates to prove (4.12).

In light of (4.17), (4.26), and (4.7), conjugating by $\sinh^{\frac{3}{2}} r$ reduces the local energy estimate (4.12) to showing that

$$\int_{-\infty}^\infty \int_0^\infty \left| V_2(r) \int_0^\infty \phi(r;\xi) \left(\cos(t\langle\xi\rangle) \hat{f}(\xi) + \frac{\sin(t\langle\xi\rangle)}{\langle\xi\rangle} \hat{g}(\xi) \right) \rho_0(d\xi) \right|^2 dt$$
$$\lesssim \|(\sqrt{\mathcal{L}_V} f, g)\|_{L^2\times L^2(0,\infty)}^2. \tag{4.30}$$

For simplicity, we first consider the case $g = 0$ above. After expanding and carrying out the t-integration on the left-hand-side as before, one obtains

$$\int_0^\infty V_2^2(r) \int_0^\infty \phi^2(r;\xi) \left|\hat{f}(\xi)\right|^2 \omega^2(\xi) \frac{\langle\xi\rangle}{\xi} \, d\xi \, dr \lesssim$$
$$\lesssim \sup_{r>0,\xi>0} \left[\frac{\left|\phi(r;\xi)\right|^2}{(r+1)^3} \frac{1}{\xi\langle\xi\rangle} \omega(\xi) \right] \int_0^\infty V_2^2(r)(r+1)^3 \, dr \int_0^\infty \langle\xi\rangle^2 \left|\hat{f}(\xi)\right|^2 \omega(\xi) \, d\xi$$
$$\lesssim \int_0^\infty \langle\xi\rangle^2 \left|\hat{f}(\xi)\right|^2 \omega(\xi) \, d\xi = \|\sqrt{\mathcal{L}_V} f\|_{L^2(0,\infty)}$$

The case $f = 0$ is similar. This proves (4.30) and hence (4.12). This finishes the proof of Proposition 4.2 pending the proofs of the technical statements regarding the distorted Fourier transform in Lemma 4.3 and Lemma 4.4 below. $\square$

4.1. **Distorted Fourier Transform for $\mathcal{L}_0$ and $\mathcal{L}_V$.** In this subsection we establish the technical statements regarding the distorted Fourier transforms for $\mathcal{L}_0$ and $\mathcal{L}_V$.

We begin with the free case, $\mathcal{L}_0$, which will follow from well known estimates for the Helgason-Fourier transform in $\mathbb{H}^4$ restricted to radial functions.

Lemma 4.3. *The half-line operator $\mathcal{L}_0$ admits a distorted Fourier basis satisfying* (4.16)*,* (4.18)*–*(4.21)*,*

Proof. As we remarked earlier, the distorted Fourier basis associated to $\mathcal{L}_0$ is explicit and is given by

$$\begin{aligned}\phi_0(r;\xi) &:= \sinh^{\frac{3}{2}} r\, \Phi_\xi(r)\\ &= C \sinh^{\frac{3}{2}} r \int_0^\pi (\cosh r - \sinh r \cos\theta)^{-i\xi - \frac{3}{2}} \sin^2\theta\, d\theta, \quad (4.31)\end{aligned}$$

where here $\Phi_\xi(r)$ are the elementary spherical functions and serve as the Helgason-Fourier basis for $\Delta_{\mathbb{H}^4}$. Indeed, the elementary spherical functions $\Phi_\xi(r)$ in $\mathbb{H}^4$ satisfy

$$H_0 \Phi_\xi(r) = (\xi^2 + 1/4)\Phi_\xi(r).$$

For radial functions $G \in L^2(\mathbb{H}^4)$ we can define the Helgason-Fourier transform by

$$\tilde{G}(\xi) = \int_0^\infty \Phi_\xi(r) G(r) \sinh^3 r\, dr.$$

The associated inversion formula is

$$G(r) = C \int_0^\infty \Phi_\xi(r)\, \tilde{G}(\xi)\, |c(\xi)|^{-2}\, d\xi, \quad (4.32)$$

where C is a normalizing constant and $c(\xi)$ is the Harish-Chandra c-function

$$c(\xi) = \frac{4\Gamma(i\xi)}{\pi^{1/2}\Gamma(\frac{3}{2} + i\xi)}.$$

The Plancherel theorem also holds. In particular, the Helgason-Fourier transform extends to an isometry of radial functions $L^2_{\mathrm{rad}}(\mathbb{H}^4) \to L^2(\mathbb{R}_+, |c(\xi)|^{-2}\, d\xi)$ with

$$\int_0^\infty f_1(r)\overline{f_2(r)} \sinh^3 r\, dr = C \int_0^\infty \tilde{f}_1(\xi)\overline{\tilde{f}_2(\xi)}\ |c(\xi)|^{-2}\, d\xi. \quad (4.33)$$

For the estimates, it follows from the definition that $|c(\xi)|^{-2}$ satisfies the bound

$$|c(\xi)|^{-2} \lesssim |\xi|^2 (1 + |\xi|). \quad (4.34)$$

For the spherical functions $\Phi_\xi(r)$ we separate two cases and we refer the reader to [7, 13, 3] for more details. We will follow the notation of [13]. For $r \geq r_0 > 0$ with $r_0 < 1$ fixed, we can write

$$\Phi_\xi(r) = e^{-\frac{3}{2}r}(e^{ir\xi} c(\xi) m_1(r,\xi) + e^{-ir\xi} c(-\xi) m_1(r,-\xi)) \quad (4.35)$$

where the function $|m_1(r,\xi)|$ is uniformly bounded in r and ξ. For small r, say $r \le 1$, we can write

$$\Phi_\xi(r) = e^{ir\xi} m_2(r,\xi) + e^{-ir\xi} m_2(r,-\xi) \tag{4.36}$$

where $m_2(r,\xi)$ satisfies

$$|m_2(r,\xi)| \lesssim (1 + r\,|\xi|)^{-\frac{3}{2}} \tag{4.37}$$

Finally, we recall the uniform estimate

$$\sup_{\xi \ge 0} |\Phi_\xi(r)| \le \Phi_0(r) \lesssim e^{-\frac{3}{2}r}(1+r) \tag{4.38}$$

which can be proved directly from (4.31).

We can now transfer these results to $\phi_0(r;\xi)$ and define the distorted Fourier transform relative to $\mathcal{L}_0$. First note that since

$$\sinh^{\frac{3}{2}} r (H_0 \Phi_\xi)(r) = (\mathcal{L}_0 \phi_0(\cdot;\xi))(r),$$

we have

$$\mathcal{L}_0 \phi_0(r;\xi) = (\xi^2 + 1/4)\phi_0(r;\xi)$$

For $g \in L^2(0,\infty)$ we can then define

$$\tilde{g}(\xi) = \int_0^\infty \phi_0(r;\xi) g(r)\, dr$$

It then follows from (4.32) and the isometry

$$L^2_{\mathrm{rad}}(\mathbb{H}^4) \ni G \mapsto \sinh^{\frac{3}{2}} r\, G =: g \in L^2(0,\infty)$$

that the following inversion formula holds

$$g(r) = C \int_0^\infty \phi_0(r;\xi)\tilde{g}(\xi)\, |c(\xi)|^{-2}\, d\xi$$

Hence we can define the spectral measure $\rho_0(d\xi) = \omega_0(\xi)\, d\xi := C\, |c(\xi)|^{-2}\, d\xi$, which proves (4.19). Plancherel's theorem (4.20) follows directly from (4.33).

The estimates (4.21) now follow easily as well. Recall that

$$\phi_0(r;\xi) = \sinh^{3/2} r \Phi_\xi(r).$$

For $r \ge r_0 > 0$, with $r_0 < 1$ fixed, (4.35) implies

$$|\phi_0(r;\xi)|^2 \lesssim |c(\xi)|^2, \quad \text{for} \quad r \ge r_0 > 0. \tag{4.39}$$

For $r \le 1$, (4.36) and (4.37) translate to

$$|\phi_0(r,\xi)|^2 \lesssim r^3 (1 + r\,|\xi|)^{-3} \quad \text{for} \quad r \le 1 \tag{4.40}$$

We also note that (4.38) allows us to deduce that

$$\sup_{\xi \ge 0} |\phi_0(r,\xi)| \lesssim 1 + r \tag{4.41}$$

Therefore, using (4.39), (4.40), and (4.34) for $\xi \ge \xi_0 > 0$, we have

$$\frac{|\phi_0(r;\xi)|^2}{(r+1)^2} \frac{\langle \xi \rangle}{\xi} |c(\xi)|^{-2} \lesssim \begin{cases} \frac{1}{(1+r)^2} \le C & \text{if} \quad r \ge r_0 > 0 \\ \frac{r^3}{(1+r)^2} \frac{|\xi|^3}{(1+r|\xi|)^3} \le C & \text{if} \quad r \le 1 \end{cases}$$

For $\xi \le \xi_0 \le 1$ the uniform estimate (4.41) as well as (4.34) imply

$$\frac{|\phi_0(r;\xi)|^2}{(r+1)^2}\frac{\langle\xi\rangle}{\xi}|c(\xi)|^{-2} \lesssim \frac{|\phi_0(r;\xi)|^2}{(r+1)^2}\langle\xi\rangle^2|\xi| \lesssim |\xi| \lesssim 1$$

This completes the proof of Lemma 4.3. □

We now extend these results to $\mathcal{L}_V$ via a perturbation argument. The key point that allows the analysis to go through is that the threshold $\frac{1}{4}$ is neither a resonance nor an eigenvalue for $\mathcal{L}_V$. The Weyl-Titchmarsh theory for half-line operators with a singular potential used below is standard and can be found in [11, Section 3].

Lemma 4.4. *The half-line operator $\mathcal{L}_V$ admits a distorted Fourier basis satisfying* (4.17)*, as well as* (4.25)*–* (4.29)*.*

Proof. First, we remark that $\mathcal{L}_V$ is in the limit-point case at both $r=0$ and $r=\infty$. Moreover, $\mathcal{L}_V$ admits an entire Weyl-Titchmarsh solution $\phi(r;z)$ of

$$\mathcal{L}_V\phi(r;z) = z^2\phi(r;z), \quad z\in\mathbb{C},\ r\in(0,\infty), \tag{4.42}$$

which satisfies

(a) For all $r\in(0,\infty)$, $\phi(r;\cdot)$ is entire.
(b) $\phi(r;z)$ is real-valued for $z\in\mathbb{R}$.
(c) $\phi(\cdot,z)\in L^2([0,b])$ for all $b\in(0,\infty)$ and $z\in\mathbb{C}$.

The above follows from [11, Lemma 3.12]. In addition, by [11, Lemma 3.3] we can find $\theta(r;z)$ so that

$$\mathcal{L}_V\theta(r;z) = z^2\theta(r;z) \quad z\in\mathbb{C},\ r\in(0,\infty),$$

where $\theta(r;z)$ is real-valued for $z\in\mathbb{R}$, entire in $z\in\mathbb{C}$ and so that

$$W(\theta(\cdot,z),\phi(\cdot;z)) = 1, \quad z\in\mathbb{C} \tag{4.43}$$

We also introduce another solution, namely the *Weyl-Titchmarsh solution* at infinity, $\psi(r;z)$, which, since $\mathcal{L}_V$ is in the limit-point case at $r=\infty$, is uniquely characterized (up to constant multiples) by

$$\begin{aligned}&\mathcal{L}_V\psi(r;z) = z^2\psi(r;z)\\ &\psi(r;z)\in L^2[a,\infty) \quad \forall z\in\mathbb{C} \text{ with } \mathrm{Im}\,(z)>0 \ \text{ and } \ \forall a>0\end{aligned} \tag{4.44}$$

Since $\{\phi(\cdot,z),\theta(\cdot,z)\}$ form a fundamental system we can find a function $m(z)$ so that

$$\psi(r;z) = \theta(r;z) + m(z)\phi(r;z). \tag{4.45}$$

This function $m(z)$ is analytic in $\mathrm{Im}\,(z)>0$ and is referred to as the *Weyl-Titchmarsh function.* We note that $m(z)$ determines the spectral measure. Indeed,

$$\rho(d\xi) = 2\xi\mathrm{Im}\,m(\xi+i0)\,d\xi \tag{4.46}$$

If for $f\in L^2(0,\infty)$ we define the Distorted Fourier transform

$$\hat{f}(\xi) := \int_0^\infty \phi(r;\xi)\,f(r)\,dr$$

then (4.26)–(4.28) hold.

We also record some useful relations. Note that for $\xi\in(0,\infty)$ we have

$$2i\mathrm{Im}\,m(\xi) = W(\overline{\psi(\cdot;\xi)},\psi(\cdot;\xi)) \tag{4.47}$$

and

$$\phi(r;\xi) = \frac{\operatorname{Im}\psi(r;\xi)}{\operatorname{Im} m(\xi)} \tag{4.48}$$

Our goal now, is to approximate $\phi(r;\xi)$ and $\omega(\xi) := 2\xi \operatorname{Im} m(\xi + i0)$ well enough to prove the estimate (4.29). We will accomplish this in two steps. The first step addresses the case $0 < \xi \ll 1$ and the second provides estimates for $\xi \gg 1$.

Step 1: We begin by considering the case of small ξ, i.e., $0 < \xi \ll 1$. We will construct $\phi(r;\xi)$ in this case by perturbing around a fundamental system at energy 1/4. In particular, we first establish:

Claim 4.5. *There exists a fundamental system, $\{\phi(r), \theta(r)\}$ to $\mathcal{L}_V f = \frac{1}{4} f$ with $W(\phi, \theta) = 1$. Moreover we have the asymptotic behavior*

$$\begin{aligned} \phi(r) &= r^{\frac{3}{2}} + o(r^{\frac{3}{2}}) \quad \textit{as} \quad r \to 0, \\ \theta(r) &= -\frac{1}{2} r^{-\frac{1}{2}} + o(r^{-\frac{1}{2}}) \quad \textit{as} \quad r \to 0, \end{aligned} \tag{4.49}$$

as well as

$$\begin{aligned} \phi(r) &= a_1 + b_1 r + o(1) \quad \textit{as} \quad r \to \infty, \ \textit{with } b_1 \neq 0, \\ \theta(r) &= a_2 + b_2 r + o(1) \quad \textit{as} \quad r \to \infty, \end{aligned} \tag{4.50}$$

where $a_1 b_2 - a_2 b_1 = 1$. The fact that $\mathcal{L}_V$ has no point spectrum allows us to ensure that $b_1 \neq 0$.

To prove the claim, we begin by constructing solutions with the desired behavior at $r = 0$. The behavior at $r = \infty$, in particular the fact that $b_1 \neq 0$, will then follow from the fact that $\frac{1}{4}$ is neither an eigenvalue, nor a resonance for $\mathcal{L}_V$.

Since $V(r)$ is smooth, we can write

$$V(r) = V(0) + O(r) \quad \text{as} \quad r \to 0$$

Similarly, we have

$$\frac{3}{4\sinh^2 r} - \frac{3}{4r^2} = -\frac{1}{4} + O(r^2) \quad \text{as} \quad r \to 0$$

For small r we can thus view solutions to $\mathcal{L}_V$ as a perturbations of solutions to

$$\mathcal{L}_E f = (1/4 - V(0)) f \tag{4.51}$$

where $\mathcal{L}_E := -\partial_{rr} + \frac{3}{4r^2}$. A fundamental system for (4.51) is given by

$$\begin{aligned} f_0(r) &= c_0 r^{\frac{1}{2}} J_1(ar) = r^{\frac{3}{2}} + o(r^{\frac{3}{2}}) \quad \text{as} \quad r \to 0 \\ f_1(r) &= c_1 r^{\frac{1}{2}} Y_1(ar) = -\frac{1}{2} r^{-\frac{1}{2}} + o(r^{-\frac{1}{2}}) \quad \text{as} \quad r \to 0 \\ a &:= \sqrt{\frac{1}{4} - V(0)} \end{aligned} \tag{4.52}$$

where J_1, Y_1 are the order one Bessel functions and c_0, c_1 are chosen to obtain the desired behavior at $r = 0$ on the right above. Indeed, if we set

$$\mathcal{N}_0(r) := -\frac{3}{4\sinh^2 r} + \frac{3}{4r^2} - \frac{1}{4} - V(r) + V(0) = O(r) \quad \text{as} \quad r \to 0 \tag{4.53}$$

then using the variation of constants formula we can write,

$$\begin{aligned}\phi(r) &= f_0(r) + \int_0^r G(s,r)\mathcal{N}_0(s)\phi(s)\,ds\\ \theta(r) &= f_1(r) + \int_0^r G(s,r)\mathcal{N}_0(s)\theta(s)\,ds\end{aligned}\tag{4.54}$$

where the Green's function $G(r,s)$ is given by

$$G(r,s) = -f_0(s)f_1(r) + f_0(r)f_1(s)$$

The Volterra iterations in (4.54) converge for r small enough and yield the desired behavior (4.49). Since both $V(r)$ and $\frac{3}{4\sinh^2 r}$ decay exponentially as $r \to \infty$, the behavior (4.50) is the only possible one. The fact that $b_1 \neq 0$ is the important consequence of the fact that $\mathcal{L}_V$ is non-resonant at $\frac{1}{4}$. This completes the proof of Claim 4.5.

We can now construct $\phi(r;\xi)$ for small ξ and $r \lesssim \xi^{-1/3}$. In particular we claim that there exists $\varepsilon > 0$ and $\xi_0 > 0$ such that for all $\xi \le \xi_0$ we have the estimates

$$\begin{aligned}\phi(r;\xi) &= \phi(r) + O(r\xi), \quad \text{for} \quad 0 \le r \le \varepsilon\xi^{-\frac{1}{3}},\\ \phi'(r;\xi) &= \phi'(r) + O(\xi) \quad \text{for} \quad 0 \le r \le \varepsilon\xi^{-\frac{1}{3}}\end{aligned}\tag{4.55}$$

Moreover we can require the fixed normalization

$$\phi(r;\xi) = \phi(r) + o(r^{\frac{3}{2}}) \quad \text{as} \quad r \to 0. \tag{4.56}$$

To prove (4.55) we first define a function $u_0(r;\xi)$ by

$$u_0(r;\xi) = \phi(r) - \xi^2\int_0^r \theta(r)\phi(s)u_0(s;\xi)\,ds - \xi^2\int_r^{\varepsilon\xi^{-1/3}} \phi(r)\theta(s)u_0(s;\xi)\,ds \tag{4.57}$$

We will then obtain $\phi(r;\xi)$ from $u_0(r;\xi)$ by a renormalization so as to obtain the precise behavior (4.56) in addition to (4.55).

Although (4.57) is not a Volterra integral equation, it can still be solved by a contraction argument. To see this, define $u_1(r;\xi)$ by

$$u_0(r;\xi) = \phi(r) + r\xi u_1(r;\xi) \tag{4.58}$$

Rewriting (4.57) in terms of u_1 we can define a linear map $T_{\varepsilon,\xi}$ by

$$\begin{aligned}u_1(r;\xi) &= -r^{-1}\xi\theta(r)\int_0^r \phi(s)[\phi(s) + s\xi u_1(s;\xi)]\,ds\\ &\quad - r^{-1}\xi\phi(r)\int_r^{\varepsilon\xi^{-\frac{1}{3}}} \theta(s)[\phi(s) + s\xi u_1(s;\xi)]\,ds\\ &=: T_{\varepsilon,\xi}u_1(r;\xi)\end{aligned}\tag{4.59}$$

One can then check that there exists $\xi_0 > 0$ and $\varepsilon > 0$ fixed so that for all $0 \le \xi \le \xi_0$, the map $T_{\varepsilon,\xi}$ is a contraction in a ball of fixed size in the space $C([0,\varepsilon\xi^{-\frac{1}{3}}])$. Hence, there is a unique solution $u_1(r;\xi)$ to (4.59) satisfying

$$|u_1(r;\xi)| \le C, \quad \forall\, 0 \le r \le \varepsilon\xi^{-1/3}$$

and all $0 < \xi \le \xi_0$. By plugging these estimates for u_1 into (4.58) we establish that

$$u_0(r;\xi) = \phi(r) + O(r\xi), \quad \text{for} \quad 0 \le r \le \varepsilon\xi^{-\frac{1}{3}},$$

Next, from (4.57) we see that

$$u_0(r,\xi) = r^{\frac{3}{2}}(1 + O(\varepsilon^3\xi + \xi^2)) + o(r^{\frac{3}{2}}) \quad \text{as} \quad r \to 0 \tag{4.60}$$

Finally, we obtain $\phi(r;\xi)$ from $u_0(r,\xi)$ by multiplication by a constant $C = C(\varepsilon,\xi)$ determined by (4.60) to ensure that (4.56) holds. A similar argument can be made for the derivative $\phi'(r;\xi)$ to prove the second line in (4.55).

To estimate $\phi(r,\xi)$ in the regime $r \geq \varepsilon\xi^{-1/3}$ and to find the spectral measure $\omega(\xi)$, we define the function $\tilde{\psi}(r;\xi)$ by

$$\begin{aligned} \tilde{\psi}(r;\xi) &= e^{ir\xi} - \int_r^\infty \frac{\sin\xi(r-s)}{\xi}\mathcal{N}_2(s)\tilde{\psi}(s;\xi)\,ds \\ \mathcal{N}_2(r) &:= -\frac{3}{4\sinh^2 r} - V(r) \end{aligned} \tag{4.61}$$

Because of the exponential decay of $\mathcal{N}_2(r)$ as $r \to \infty$ the Volterra iteration converges for r large enough and $\tilde{\psi}(r;\xi)$ solves

$$\mathcal{L}_V\tilde{\psi}(r;\xi) = (1/4 + \xi^2)\tilde{\psi}(r;\xi)$$

In addition, we can deduce that

$$\begin{aligned} \tilde{\psi}(r;\xi) &= e^{ir\xi} + O(re^{-2r}) \quad \text{as} \quad r \to \infty \\ \tilde{\psi}'(r;\xi) &= i\xi e^{ir\xi} + O(re^{-2r}) \quad \text{as} \quad r \to \infty \end{aligned} \tag{4.62}$$

This implies that

$$W(\tilde{\psi}(\cdot;\xi), \overline{\tilde{\psi}(\cdot;\xi)}) = W(e^{ir\xi}, e^{-ir\xi}) = -2i\xi \tag{4.63}$$

Moreover, since the Weyl-Titchmarsh solution $\psi(r;z)$ is uniquely characterized up to constant multiples by $\psi(\cdot;z) \in L^2([c,\infty))$ for all $\mathrm{Im}\,(z) > 0$ and $c > 0$ we can find a smooth function $a(\xi)$ so that

$$a(\xi)\tilde{\psi}(r;\xi) = \psi(r;\xi) \tag{4.64}$$

Observe that in light of (4.43) and (4.45), $a(\xi)$ is given by

$$a(\xi) = \frac{1}{W(\tilde{\psi}(\cdot;\xi), \phi(\cdot;\xi))} \tag{4.65}$$

By (4.47) and (4.63), estimates for the function $a(\xi)$ can then be used to estimate the m-function. Indeed we have the relation

$$2i\mathrm{Im}\,m(\xi) = W(\overline{\psi(\cdot,\xi)}, \psi(\cdot;\xi)) = |a(\xi)|^2\, W(\overline{\tilde{\psi}(\cdot;\xi)}, \tilde{\psi}(\cdot;\xi)) = 2i\xi\,|a(\xi)|^2 \tag{4.66}$$

We therefore look to bound the right hand side of (4.65). We will show that $W(\tilde{\psi}(\cdot;\xi), \phi(\cdot;\xi))$ is bounded away from 0 for small ξ. To see this we evaluate the Wronskian at $r = \xi^{-1/6}$, which is large enough for small ξ so that we can accurately approximate $\tilde{\psi}(r;\xi)$, but also within the range in which we have good control of $\phi(r;\xi)$ by (4.55). The spectral information on $\mathcal{L}_V$ enters crucially at this point as evaluating the Wronksian at $r = \xi^{-1/6}$ yields

$$\left| W(\tilde{\psi}(\cdot;\xi), \phi(\cdot;\xi))|_{r=\xi^{-1/6}} \right| \geq c\,|b_1| > 0 \tag{4.67}$$

where we have used (4.55) and (4.62) as well as (4.50) above. Hence $|a(\xi)| \simeq 1$ and we can conclude that

$$\mathrm{Im}\,m(\xi) \simeq |\xi| \quad \text{as} \quad \xi \to 0 \tag{4.68}$$

For the spectral measure $\omega(\xi)d\xi$ we then have

$$\omega(\xi) \simeq |\xi|^2 \quad \text{as} \quad \xi \to 0 \tag{4.69}$$

Finally, we would like to estimate $\phi(r;\xi)$ for small ξ and $r \geq \varepsilon\xi^{-1/3}$. Since the functions $\tilde{\psi}(r;\xi)$ and $\overline{\tilde{\psi}(r;\xi)}$ form a fundamental system and since $\phi(r;\xi) \in \mathbb{R}$ for $\xi \in \mathbb{R}$, we can find $b(\xi)$ so that

$$\phi(r;\xi) = b(\xi)\tilde{\psi}(r;\xi) + \overline{b(\xi)}\overline{\tilde{\psi}(r;\xi)} \tag{4.70}$$

We then have

$$b(\xi) = \frac{W(\phi(\cdot;\xi), \overline{\tilde{\psi}(\cdot;\xi)})}{W(\tilde{\psi}(\cdot;\xi), \overline{\tilde{\psi}(\cdot;\xi)})} \tag{4.71}$$

We use (4.63) for the denominator whereas we evaluate the numerate again at $r = \xi^{-1/6}$ as above to prove that

$$|b(\xi)| = O(\xi^{-1}) \quad \text{as} \quad \xi \to 0$$

This gives us the estimate

$$\sup_{r \geq \varepsilon\xi^{-1/3}} \xi\,|\phi(r;\xi)| = O(1) \quad \text{as} \quad \xi \to 0 \tag{4.72}$$

We can now prove (4.29) for small $\xi \to 0$. Using (4.69) we can deduce that

$$\frac{|\phi(r;\xi)|^2}{(r+1)^3}\frac{\langle\xi\rangle}{\xi}\omega(\xi) \lesssim \frac{|\phi(r;\xi)|^2}{(r+1)^3}|\xi| \quad \forall \xi \leq \xi_0 \tag{4.73}$$

For $r \leq \varepsilon\xi^{-1/3}$ we can use the estimate (4.55) together with the conclusions of Claim 4.5 to see that

$$|\phi(r;\xi)| \lesssim \min(1, r), \quad \forall\, r \leq \varepsilon\xi^{-1/3}$$

Hence the right hand side of (4.73) is bounded by a constant in the regime $r \leq \varepsilon\xi^{-1/3}$. If $r \geq \varepsilon\xi^{-1/3}$ then we use (4.72) to conclude that

$$\frac{|\phi(r;\xi)|^2}{(r+1)^3}|\xi| \lesssim \frac{1}{r^3\xi} \lesssim 1, \quad \forall\, r \geq \varepsilon\xi^{-1/3}$$

This finishes the proof of (4.29) for $\xi \ll 1$.

Step 2: We now consider the case $\xi \geq \Xi_0$ for Ξ_0 large. This is somewhat easier than the small ξ case as we can perturb around explicit solutions to a well understood Euclidean problem. Define the half-line Schrödinger operator

$$\mathcal{L}_E f := -f_{rr} + \frac{3}{4r^2}f$$

Then $\mathcal{L}_E f = z^2 f$ are Bessel equations and we define the Weyl-Titchmarsh solution at $r = 0$, by

$$\phi_E(r;z) := 2z^{-1}r^{\frac{1}{2}}J_1(zr). \tag{4.74}$$

Together with

$$\theta_E(r,z) := \frac{\pi}{4}zr^{\frac{1}{2}}[-Y_1(zr) + \pi^{-1}\log(z^2)J_1(zr)], \tag{4.75}$$

we have a fundamental system with $W(\theta_E(\cdot;z),\phi_E(\cdot;z)) = 1$ for all $z \in \mathbb{C}$, see [11, Section 4] for more discussion of this system. The Weyl-Titchmarsh solution at $r = \infty$ is given by

$$\psi_E(r;z) := zr^{\frac{1}{2}} iH_1^{(1)}(zr) = \theta_E(r;z) + m_E(z)\phi_E(r;z), \tag{4.76}$$

where above J_1, Y_1 are the order one Bessel functions and $H_1^{(1)}$ is the order one Hankel function. The Weyl-Titchmarsh m-function is also explicit and is given by

$$m_E(z) = \frac{\pi}{4} z^2 [i - \pi^{-1}\log(z^2)]. \tag{4.77}$$

We now rewrite $\mathcal{L}_V f = (\frac{1}{4} + \xi^2) f$ as

$$-f_{rr} + \frac{3}{4r^2} f - \xi^2 f = \frac{3}{4}\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right) f - V(r) f.$$

The variation of parameters formula then gives the Volterra integral equation for the outgoing solutions

$$\tilde{\psi}_+(r;\xi) = \psi_E(r;\xi) + \int_r^\infty G_E(r,s,\xi)\mathcal{N}_3(s)\tilde{\psi}_+(s;\xi)\, ds, \tag{4.78}$$

where

$$G_E(r,s,\xi) := \frac{\psi_E(r;\xi)\overline{\psi_E(s;\xi)} - \psi_E(s;\xi)\overline{\psi_E(r;\xi)}}{W(\psi_E(\cdot;\xi), \overline{\psi_E(\cdot;\xi)})},$$

and

$$\mathcal{N}_3(r) := \frac{3}{4}\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right) - V(r).$$

The Wronskian is given by

$$W(\overline{\psi_E(\cdot;\xi)}, \psi_E(\cdot;\xi)) = 2i \operatorname{Im} m_E(\xi) = \frac{\pi}{2} i \xi^2. \tag{4.79}$$

Let $\varepsilon > 0$ be a small number to be determined below. We will solve (4.78) on the region $r \in [\frac{1}{2}\epsilon\xi^{-1}, \infty)$. Combining (4.79) with (4.76), we note that the Green's function G_E takes the form

$$G_E(r,s,\xi) = \frac{2}{i\pi} r^{1/2} s^{1/2} 2 \operatorname{Im}\left(H_1^{(1)}(r\xi)\overline{H_1^{(1)}(s\xi)}\right) \tag{4.80}$$

Using the standard asymptotic expansions and estimates for the Hankel function, $H_1^{(1)}(x)$ with $x > 0$ large (see for example [1, 20]), we can deduce the bound

$$|G_E(r,s,\xi)| \lesssim_\varepsilon |\xi|^{-1}, \quad \forall\ \frac{\varepsilon}{2\xi} \le r, s \tag{4.81}$$

Moreover, we have the estimate

$$\mathcal{N}_3(r) \lesssim \frac{1}{(1+r)^2}, \quad \forall\, r > 0 \tag{4.82}$$

Thus, if we set $K(r,s,\xi) := G_E(r,s,\xi)\mathcal{N}_3(s)$ we have the estimate

$$|K(r,s,\xi)| \lesssim_\varepsilon \xi^{-1} \langle s \rangle^{-2}, \quad \forall\ \frac{\varepsilon}{2\xi} \le r, s$$

and thus

$$\int_{\frac{\varepsilon}{2\xi}}^\infty \sup_{r > \frac{\varepsilon}{2\xi}} |K(r,s,\xi)|\, ds \lesssim_\varepsilon \xi^{-1}$$

This means that the Volterra integral (4.78) has a unique solution $\tilde{\psi}_+(r;\xi)$ defined for all $r \geq \frac{1}{2}\varepsilon\xi^{-1}$ and satisfying

$$\tilde{\psi}_+(r;\xi) = \psi_E(r;\xi)(1 + O(\xi^{-1}\langle r\rangle^{-1})), \quad \forall\, r \geq \frac{\varepsilon}{2\xi} \tag{4.83}$$

We can then deduce that

$$W(\overline{\tilde{\psi}_+(\cdot;\xi)}, \tilde{\psi}_+(\cdot;\xi)) = W(\overline{\psi_E(\cdot;\xi)}, \psi_E(\cdot;\xi)) = \frac{\pi i}{2}\xi^2 \tag{4.84}$$

Since the Weyl-Titchmarsh solution at $r = \infty$,

$$\psi(r;\xi) := \theta(r;\xi) + m(\xi)\phi(r;\xi) \tag{4.85}$$

is uniquely determined up to constant multiples by (4.44), we can find a smooth function $d(\xi)$ such that

$$\psi(r;\xi) = d(\xi)\tilde{\psi}_+(r;\xi) \tag{4.86}$$

Using (4.85) we see that

$$d(\xi) = \frac{1}{W(\tilde{\psi}_+(\cdot;\xi), \phi(\cdot;\xi))} \tag{4.87}$$

We claim that

$$|d(\xi)| \simeq 1 \quad \text{as} \quad \xi \to \infty \tag{4.88}$$

To prove (4.88), we first note that for $\varepsilon > 0$ small enough, we can find constants c_1, C_2, which are independent of ξ, so that

$$\begin{aligned} c_1 r^{\frac{3}{2}} \leq \phi(r;\xi) \leq C_2 r^{\frac{3}{2}}, \quad \forall r \leq 2\varepsilon\xi^{-1} \\ c_1 r^{\frac{1}{2}} \leq \phi'(r;\xi) \leq C_2 r^{\frac{1}{2}}, \quad \forall r \leq 2\varepsilon\xi^{-1} \end{aligned} \tag{4.89}$$

By rewriting the equation for $\phi(r;\xi)$ as

$$\begin{aligned} & -\phi'' + \frac{3}{4r^2}\phi = \mathcal{N}_4(r,\xi)\phi \\ & \mathcal{N}_4(r) = \xi^2 + \frac{3}{4}\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right) + V(r) \end{aligned}$$

we see that (4.89) follows directly from an iteration argument on the interval $[0, 2\varepsilon\xi^{-1}]$ for

$$\phi(r;\xi) = r^{\frac{3}{2}} - \frac{1}{2}\int_0^r (r^{3/2}s^{-1/2} - r^{-1/2}s^{3/2})\mathcal{N}_4(s,\xi)\phi(s,\xi)\, ds$$

as long as $\varepsilon > 0$ is chosen small enough. Now (4.88) follows by combining (4.83) and (4.89) to evaluate the right-hand-side of (4.87) at the point $r = \varepsilon\xi^{-1}$, and then taking $\xi \to \infty$. Here we remark that after closing the Volterra iteration for $\tilde{\psi}_+(r,\xi)$ in (4.78), the control of $\tilde{\psi}'_+(r;\xi)$ near the point $r = \varepsilon\xi^{-1}$ needed for (4.88) follows by essentially differentiating (4.78), plugging in the bound (4.83) for $\tilde{\psi}_+(r;\xi)$ and using the asymptotics of the derivative of the Hankel function, $H_1^{(1)}$, which can be found for example in Abramowitz-Stegun [1, p. 364, 9.2.13].

By (4.47) and (4.63), estimate (4.88) for the function $d(\xi)$ can then be used to estimate the m-function. Indeed we have the relation

$$\begin{aligned} 2i\mathrm{Im}\, m(\xi) &= W(\overline{\psi(\cdot,\xi)}, \psi(\cdot;\xi)) = |d(\xi)|^2 W(\overline{\tilde{\psi}_+(\cdot;\xi)}, \tilde{\psi}_+(\cdot;\xi)) \\ &= \frac{\pi i}{2}\xi^2 + O(1) \quad \text{as} \quad \xi \to \infty \end{aligned} \tag{4.90}$$

We therefore have proved that

$$\mathrm{Im}\, m(\xi) \simeq |\xi|^2 \quad \text{as} \quad \xi \to \infty. \tag{4.91}$$

For the spectral measure $\rho(d\xi) = \omega(\xi)d\xi$ we then have

$$\omega(\xi) \simeq |\xi|^3 \quad \text{as} \quad \xi \to \infty \tag{4.92}$$

Finally we estimate $\phi(r;\xi)$ using the relation (4.48).

$$\begin{aligned} \phi(r;\xi) &= \frac{\mathrm{Im}\,\psi(r;\xi)}{\mathrm{Im}\, m(\xi)} \simeq \xi^{-2}\mathrm{Im}\,[\psi_E(r;\xi)(1 + O(\xi^{-1}\langle r\rangle^{-1}))] \\ &= \xi^{-1}(r^{1/2}J_1(r\xi)(1 + O(\langle r\rangle^{-1})) \end{aligned} \tag{4.93}$$

Using asymptotic formulas for $J_1(z)$ we obtain the estimate

$$\xi^{\frac{3}{2}}|\phi(r;\xi)| \lesssim 1 \quad \text{as} \quad \xi \to \infty \tag{4.94}$$

for all $r > 0$. Together with (4.92) this establishes (4.29) for all $\xi \geq \Xi_0$ for some $\Xi_0 > 0$ large. This completes the proof of Lemma 4.4. □

5. Proofs of Theorem 1.1 and Theorem 1.3

In this final section we prove the two asymptotic stability results, Theorem 1.1 and Theorem 1.3. We remark that due to Lemma 2.4 it will suffice to work in the $4d$ setting and consider the Cauchy problems (1.15) for the case of maps into $\mathbb{S}^2$ and (1.29) for the case of maps into $\mathbb{H}^2$.

As we will be working with initial data $\vec{u}(0) = (u_0, u_1)$ that is small in the energy space $H^1 \times L^2(\mathbb{H}^4)$, we will also require only rough estimates on the size of the nonlinearities $\mathcal{N}_{\mathbb{S}^2}$ and $\mathcal{N}_{\mathbb{H}^2}$. In particular, we will make use of the following simple lemmas.

Lemma 5.1. *Let $\vec{\psi} \in \mathcal{E}_\lambda$ and define u by $\sinh r\, u(r) = \psi(r) - Q_\lambda(r)$. Let $\mathcal{N}_{\mathbb{S}^2}(r,u)$ be defined as in* (1.17)*, i.e.,*

$$\begin{aligned} \mathcal{N}_{\mathbb{S}^2}(r,u) &:= \frac{\sin 2Q_\lambda}{\sinh^3 r}\sin^2(2\sinh r\, u) + \cos 2Q_\lambda \frac{2\sinh r\, u - \sin(2\sinh r\, u)}{2\sinh^3 r} \\ &=: \mathcal{F}_{\mathbb{S}^2}(r,u) + \mathcal{G}_{\mathbb{S}^2}(r,u). \end{aligned}$$

Then, there exist constants $C_1 = C(\lambda) > 0$ and $C_2 > 0$ so that

$$\begin{aligned} |\mathcal{F}_{\mathbb{S}^2}(r,u)| &\leq C_1 \langle \sinh r\rangle^{-1} |u|^2 \\ |\mathcal{G}_{\mathbb{S}^2}(r,u| &\leq C_2 |u|^3 \end{aligned} \tag{5.1}$$

Proof. To prove the first estimate in (5.1) we consider two regions, $r \leq 1$ and $r \geq 1$. When $r \geq 1$ we use the estimate,

$$|\mathcal{F}_{\mathbb{S}^2}(r,u)| \leq C \sinh^{-1} r\, |u|^2$$

Next, recall that $Q_\lambda(r) = 2\arctan(\lambda\tanh(r/2))$. Then

$$\begin{aligned}\sin 2Q_\lambda &= 4\tan(Q_\lambda/2)\cos^2(Q_\lambda/2)\cos(Q_\lambda)\\ &= 4\lambda\tanh(r/2)\cos^2(Q_\lambda/2)\cos(Q_\lambda)\end{aligned}$$

Therefore, for $r \le 1$ we can conclude that

$$|\mathcal{F}_{\mathbb{S}^2}(r,u)| \le C\lambda\tanh(r/2)\sinh^{-1} r\,|u|^2 \le C(\lambda)\,|u|^2$$

which proves the first estimate in (5.1). The second estimate in (5.1) is clear. □

Lemma 5.2. *Let $(\psi, 0) \in \mathscr{E}_\lambda$ and define u by $\sinh r\, u(r) = \psi(r) - P_\lambda(r)$. Suppose that*

$$\|(\psi,0) - (P_\lambda,0)\|_{\mathcal{H}_0} \le A \tag{5.2}$$

Let $\mathcal{N}_{\mathbb{H}^2}(r,u)$ be defined as in (1.31), i.e.,

$$\begin{aligned}\mathcal{N}_{\mathbb{H}^2}(r,u) &:= -\frac{\sinh 2P_\lambda}{\sinh^3 r}\sinh^2(2\sinh r\, u) + \cosh 2P_\lambda \frac{2\sinh r\, u - \sinh(2\sinh r\, u)}{2\sinh^3 r}\\ &=: \mathcal{F}_{\mathbb{H}^2}(r,u) + \mathcal{G}_{\mathbb{H}^2}(r,u).\end{aligned}$$

Then, there exists a constant $K = K(\lambda, A) > 0$ so that

$$\begin{aligned}|\mathcal{F}_{\mathbb{H}^2}(r,u)| &\le K\,\langle\sinh r\rangle^{-1}\,|u|^2\,,\\ |\mathcal{G}_{\mathbb{H}^2}(r,u)| &\le K\,|u|^3\,.\end{aligned} \tag{5.3}$$

Proof. First we note that the estimate (5.2) gives us an a priori estimate on the size of ψ. Indeed, since $\psi(0) - P_\lambda(0) = 0$ we have

$$\begin{aligned}(\psi(\rho) - P_\lambda(\rho))^2 &= \int_0^\rho \partial_r(\psi(r) - P_\lambda(r))^2\,dr\\ &= 2\int_0^\rho (\psi_r(r) - \partial_r P_\lambda(r))(\psi(r) - P_\lambda(r))\,dr\\ &\le \|(\psi,0) - (P_\lambda,0)\|^2_{\mathcal{H}_0}\end{aligned}$$

This means that

$$\sup_{r\in[0,\infty)} |\sinh r u(r)| = \sup_{r\in[0,\infty)} |\psi(r) - P_\lambda(r)| \le A. \tag{5.4}$$

Next, note that

$$\begin{aligned}\sinh 4\vartheta &= 2\sinh 2\vartheta\cosh 2\vartheta\\ &= 4\sinh\vartheta\cosh\vartheta(\cosh^2\vartheta + \sinh^2\vartheta)\\ &= 4\frac{\tanh\vartheta(1+\tanh^2\vartheta)}{(1-\tanh^2\vartheta)^2}.\end{aligned}$$

Plugging in $\vartheta = \operatorname{arctanh}(\lambda\tanh(r/2))$, we see that

$$\begin{aligned}\sinh(2P_\lambda(r)) &= 4\left(\frac{\lambda\tanh(r/2)(1+\lambda^2\tanh^2(r/2)}{(1-\lambda^2\tanh^2(r/2))^2})\right)\\ &\le 4\lambda\tanh(r/2)\left(\frac{1+\lambda^2}{1-\lambda^2}\right)\\ &\le C(\lambda)\lambda\tanh(r/2)\end{aligned} \tag{5.5}$$

In light of (5.4) and (5.5) we can find $K = K(\lambda, A)$ so that

$$|\mathcal{F}_{\mathbb{H}^2}(r,u)| = \left|\frac{\sinh 2P_\lambda}{\sinh^3 r}\sinh^2(2\sinh r\, u)\right| \le K(\lambda, A)\frac{\tanh(r/2)}{\sinh^3 r}\sinh^2 r\,|u|^2$$

and this proves the first estimate in (5.3). To prove the second estimate we define

$$Z(\rho) := 4\frac{\rho - \sinh(\rho)}{\rho^3}$$

Then,

$$\mathcal{G}_{\mathbb{H}^2}(r,u) = \cosh 2P_\lambda Z(\sinh ru)u^3$$

By (5.4), and the fact that $\cosh 2P_\lambda \le \cosh(4\operatorname{arctanh}\lambda)$, we can find $K = K(\lambda, A)$ so that $|\cosh 2P_\lambda Z(\sinh r\, u(r))| \le K(\lambda, A)$, which completes the proof of the second estimate in (5.3). □

In light, of Lemma 5.1 and Lemma 5.2 we can handle the small data theory for (1.15) and (1.29) simultaneously. Indeed, consider the more general Cauchy problem

$$\begin{aligned} &u_{tt} - u_{rr} - 3\coth r\, u_r - 2u + Vu = \mathcal{F}(r,u) + G(r,u) \\ &\vec{u}(0) = (u_0, u_1) \in H^1 \times L^2(\mathbb{H}^4) \end{aligned} \tag{5.6}$$

where $V = V(r)$ is a radial potential satisfying assumptions (A) and (B) as defined in the beginning of Section 4 so that Proposition 4.2 applies. We assume that for $\vec{u}$ and $A > 0$ such that

$$\|\vec{u}\|_{H^1\times L^2} \le A, \tag{5.7}$$

the nonlinearities $\mathcal{F}, \mathcal{G}$ satisfy

$$\begin{aligned} &|\mathcal{F}(r,u)| \lesssim_A \langle \sinh r\rangle^{-1} |u|^2, \\ &|\mathcal{G}(r,u)| \lesssim_A |u|^3. \end{aligned} \tag{5.8}$$

We can now formulate the local well-posedness theory for (5.6). For a time interval $0 \in I \subset \mathbb{R}$, define the norms $S(I)$ and $N(I)$ by

$$\begin{aligned} &\|u\|_{S(I)} := \|u\|_{L^3_t(I;L^6(\mathbb{H}^4))} \\ &\|F\|_{N(I)} := \|F\|_{L^1_t(I;L^2(\mathbb{H}^4)) + L^{\frac{3}{2}}_t(I;L^{\frac{12}{7}}(\mathbb{H}^4))} \end{aligned} \tag{5.9}$$

Proposition 5.3. *Let $\vec{u}(0) = (u_0, u_1) \in H^1 \times L^2(\mathbb{H}^4)$ be radial. Then there is a unique solution $\vec{u}(t) \in H^1 \times L^2(\mathbb{H}^4)$ to (5.6) defined on a maximal interval of existence $0 \in I_{\max}(\vec{u}) = (-T_-, T_+)$, and for any compact interval $J \subset I_{\max}$ we have*

$$\|u\|_{S(J)} + \|\vec{u}\|_{L^\infty_t(J;H^1\times L^2)} < \infty.$$

Moreover, a globally defined solution $\vec{u}(t)$ to (5.6) for $t \in [0,\infty)$ scatters as $t \to \infty$ to a free shifted wave, i.e., a solution $\vec{u}_L(t) \in H^1 \times L^2(\mathbb{H}^4)$ of

$$v_{tt} - v_{rr} - 3\coth r\, v_r - 2v = 0, \tag{5.10}$$

if and only if

$$\|u\|_{S([0,\infty))} + \|\vec{u}\|_{L^\infty_t([0,\infty);H^1\times L^2)} < \infty.$$

Here scattering as $t \to \infty$ means that

$$\|\vec{u}(t) - \vec{u}_L(t)\|_{H^1\times L^2(\mathbb{H}^4)} \to 0 \quad \text{as} \quad t \to \infty. \tag{5.11}$$

In particular, there exists a constant $\delta > 0$ so that

$$\|\vec{u}(0)\|_{H^1\times L^2} < \delta \Rightarrow \|u\|_{S(\mathbb{R})} + \|\vec{u}\|_{L^\infty_t(\mathbb{R};H^1\times L^2)} \lesssim \|\vec{u}(0)\|_{H^1\times L^2} \lesssim \delta \tag{5.12}$$

and hence $\vec{u}(t)$ scatters to free waves as $t \to \pm\infty$.

Remark 9. In our applications of this proposition to the equivariant wave map equation, we note that an *a priori* $L^\infty_t(H^1\times L^2)$ bound holds thanks to conservation of energy. In particular, the criterion for scattering as $t\to\infty$ is simply

$$\|u\|_{S([0,\infty))} < \infty,$$

where u is defined as in (1.14) or (1.28) depending on the target M and the parameter λ.

Proof. The proof of Proposition 5.3 follows from the usual contraction mapping argument based on the Strichartz estimates in Proposition 4.2 and we give a brief sketch as the details are standard. Indeed, suppose that the bootstrap assumption

$$\|\vec{u}\|_{L^\infty_t(I;H^1\times L^2)} \le A \tag{5.13}$$

holds for some $A > 0$. Then applying Proposition 4.2 to any time interval I we have

$$\begin{aligned}
&\|u\|_{S(I)} + \|\vec{u}(t)\|_{L^\infty(I;H^1\times L^2)}\\
&\quad\lesssim \|\vec{u}(0)\|_{H^1\times L^2} + \|\mathcal{F}(\cdot,u)+\mathcal{G}(\cdot,u)\|_{N(I)}\\
&\quad\lesssim \|\vec{u}(0)\|_{H^1\times L^2} + C_A(\|\langle\sinh r\rangle^{-1}u^2\|_{L^{\frac{3}{2}}_t(I;L^{\frac{12}{7}}_x)} + \|u^3\|_{L^1_t(I;L^2_x)})\\
&\quad\lesssim \|\vec{u}(0)\|_{H^1\times L^2} + C_A(\|\langle\sinh r\rangle^{-1}\|_{L^\infty_t L^4_x}\|u^2\|_{L^{\frac{3}{2}}_t L^3_x} + \|u\|^3_{S(I)})\\
&\quad\lesssim \|\vec{u}(0)\|_{H^1\times L^2} + C_A(\|u\|^2_{S(I)} + \|u\|^3_{S(I)}).
\end{aligned}$$

By the usual continuity argument, (expanding I), this implies the a priori estimate (5.12) for small data. The scattering is also standard and based on a global Strichartz estimate. Indeed, if we denote by $S_{-2+V}(t)$ the propagator of the free shifted wave equation on $\mathbb{H}^4$ perturbed by the radial potential $V(r)$, i.e., the propagator for (4.1) with $F=0$, we seek initial data $\vec{v}_L(0)\in H^1\times L^2$ so that

$$\vec{u}(t) = S_{-2+V}(t)\vec{v}_L(0) + o_{H^1\times L^2}(1) \quad\text{as}\quad t\to\infty$$

In view of the Duhamel representation for $\vec{u}(t)$ and using the group property and unitarity of S_{-2+V} this is tantamount to

$$\vec{v}_L(0) = \vec{u}(0) + \int_0^\infty S_{-2+V}(-s)(0,(\mathcal{F}+\mathcal{G})(\cdot,u)(s))\,ds$$

The integral on the right-hand side above is absolutely convergent in $H^1\times L^2$ as long as $\|u\|_{S([0,\infty))} + \|\vec{u}\|_{L^\infty([0,\infty);H^1\times L^2)} < \infty$. That the finiteness of these norms of u is a necessary condition is due the fact that free shifted waves satisfy it, whence by the small data theory (applied to large times) it carries over to any nonlinear wave that scatters. Now that we have found the linear wave $\vec{v}_L(t) = S_{-2+V}(t)\vec{v}_L(0)$ which approaches $\vec{u}(t)$ in the energy space we can easily pass to a solution to (5.10) with the same property. Indeed we can define the scattering data

$$\vec{u}_L(0) = \vec{v}_L(0) + \int_0^\infty S_{-2}(-s)(0,Vv_L)(s)\,ds \tag{5.14}$$

where S_{-2} is the free propagator for (5.10). The fact that (5.14) is in $H^1 \times L^2$ follows from (4.9) with the space $X = L^\infty_t H^1_x$. □

Finally, we remark that in light of the reductions to the $4d$ equations, (1.15) and (1.29), as well as the estimates in Lemma 5.1 and Lemma 5.2 we have also completed the proofs of Theorem 1.1 and Theorem 1.3.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] J.-P. Anker and V. Pierfelice. Nonlinear Schrödinger equation on real hyperbolic spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1853–1869, 2009.

[3] J.-P. Anker and V. Pierfelice. Wave and Klein-Gordon equations on hyperbolic spaces. *Preprint*, 2011.

[4] V. Banica. The nonlinear Schrödinger equation on hyperbolic space. *Comm. Partial Differential Equations*, 32(10-12):1643–1677, 2007.

[5] V. Banica, R. Carles, and T. Duyckaerts. On scattering for NLS: from Euclidean to hyperbolic space. *Discrete Contin. Dyn. Syst.*, 24(4):1113–1127, 2009.

[6] V. Banica, R. Carles, and G. Staffilani. Scattering theory for radial nonlinear Schrödinger equations on hyperbolic space. *Geom. Funct. Anal.*, 18(2):367–399, 2008.

[7] W. O. Bray. Aspects of harmonic analysis on real hyperbolic space. In *Fourier analysis (Orono, ME, 1992)*, volume 157 of *Lecture Notes in Pure and Appl. Math.*, pages 77–102. Dekker, New York, 1994.

[8] M. Christ and A. Kiselev. Maximal functions associated to filtrations. *J. Funct. Anal.*, 179(2):409–425, 2001.

[9] H. Christianson and J. L. Marzuola. Existence and stability of solitons for the nonlinear Schrödinger equation on hyperbolic space. *Nonlinearity*, 23(1):89–106, 2010.

[10] R. Côte. Instability of nonconstant harmonic maps for the $(1+2)$-dimensional equivariant wave map system. *Int. Math. Res. Not.*, (57):3525–3549, 2005.

[11] F. Gesztesy and M. Zinchenko. On spectral theory for Schrödinger operators with strongly singular potentials. *Math. Nachr.*, 279(9-10):1041–1082, 2006.

[12] A. D. Ionescu, B. Pausader, and G. Staffilani. On the global well-posedness of energy-critical Schrödinger equations in curved spaces. *Analysis and PDE*, 5(4):705–746, 2012.

[13] A. D. Ionescu and G. Staffilani. Semilinear Schrödinger flows on hyperbolic spaces: scattering H^1. *Math. Ann.*, 345(1):133–158, 2009.

[14] C. E. Kenig, A. Lawrie, and W. Schlag. Relaxation of wave maps exterior to a ball to harmonic maps for all data. *Geom. Funct. Anal.*, 24(2):610–647, 2014.

[15] J. Krieger, W. Schlag, and D. Tataru. Renormalization and blow up for charge one equivariant wave critical wave maps. *Invent. Math.*, 171(3):543–615, 2008.

[16] A. Lawrie and W. Schlag. Scattering for wave maps exterior to a ball. *Advances in Mathematics*, 232(1):57–97, 2013.

[17] G. Mancini and K. Sandeep. On a semilinear elliptic equation in $\mathbb{H}^n$. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 7(4):635–671, 2008.

[18] J. Metcalfe and M. Taylor. Nonlinear waves on 3D hyperbolic space. *Trans. Amer. Math. Soc.*, 363(7):3489–3529, 2011.

[19] J. Metcalfe and M. Taylor. Dispersive wave estimates on 3D hyperbolic space. *Proc. Amer. Math. Soc.*, 140(11):3861–3866, 2012.

[20] F. W. J. Olver. *Asymptotics and special functions*. AKP Classics. A K Peters, Ltd., Wellesley, MA, 1997. Reprint of the 1974 original [Academic Press, New York; MR0435697 (55 #8655)].

[21] P. Raphael and I. Rodnianski. Stable blow up dynamics for the critical corotational wave map and equivariant Yang-Mills problems. *Publi. I.H.E.S., in press*, 2009.

[22] I. Rodnianski and W. Schlag. Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. *Invent. Math.*, 155(3):451–513, 2004.

[23] I. Rodnianski and J. Sterbenz. On the formation of singularities in the critical $O(3)$ σ-model. *Ann. of Math.*, 172:187–242, 2010.

[24] W. Schlag, A. Soffer, and W. Staubach. Decay for the wave and Schrödinger evolutions on manifolds with conical ends. I. *Trans. Amer. Math. Soc.*, 362(1):19–52, 2010.

[25] W. Schlag, A. Soffer, and W. Staubach. Decay for the wave and Schrödinger evolutions on manifolds with conical ends. II. *Trans. Amer. Math. Soc.*, 362(1):289–318, 2010.

[26] J. Shatah and M. Struwe. *Geometric wave equations*. Courant Lecture notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York. American Mathematical Society, Providence RI, 1998.

[27] A. Soffer and M. I. Weinstein. Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations. *Invent. Math.*, 136(1):9–74, 1999.

[28] C. D. Sogge. *Lectures on non-linear wave equations*. International Press, Boston, MA, second edition, 2008.

[29] M. Struwe. Equivariant wave maps in two space dimensions. *Comm. Pure Appl. Math.*, 56(7):815–823, 2003.

Andrew Lawrie, Sung-Jin Oh

Department of Mathematics, The University of California, Berkeley
970 Evans Hall #3840, Berkeley, CA 94720, U.S.A.
alawrie@math.berkeley.edu, sjoh@math.berkeley.edu

Sohrab Shahshahani

Department of Mathematics, The University of Michigan
2074 East Hall, 530 Church Street Ann Arbor, MI 48109-1043, U.S.A.
shahshah@umich.edu