Algebraic vs physical N=6 3-algebras

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Abstract

In our previous paper we classified linearly compact algebraic simple $N = 6$ 3-algebras. In the present paper we classify their "physical" counterparts, which actually appear in the $N = 6$ supersymmetric 3-dimensional Chern-Simons theories.

0 Introduction

In a series of papers on N-supersymmetric 3-dimensional Chern-Simons gauge theories various types of 3-algebras have naturally appeared ([10], [3], [4], [1], [13], [15], [2], . . .).

Recall that a 3-algebra is a vector space A over $\mathbb{C}$ with a 3-bracket $A^{\otimes 3} \to A$, $a \otimes b \otimes c \mapsto [a, b, c]$. If this bracket is linear in all arguments, the 3-algebra A is called *algebraic*. If this 3-bracket is linear in the first and the third argument, but anti-linear in the second argument (i.e. $[\lambda a, b, c] = [a, b, \lambda c] = \lambda[a, b, c]$, but $[a, \lambda b, c] = \bar{\lambda}[a, b, c]$, where $\bar{\lambda}$ is the complex conjugate of $\lambda \in \mathbb{C}$), then A is called a *physical* 3-algebra.

If C is an anti-linear involution of the vector space A (i.e. $C^2 = 1$ and $C(\lambda a) = \bar{\lambda}C(a)$), then an algebraic 3-algebra A with the 3-bracket $[a, b, c]$ is converted to a physical 3-algebra with the 3-bracket

$$[a, b, c]_{ph,C} = [a, C(b), c], \tag{0.1}$$

which we denote by $A_{ph,C}$, and viceversa.

An algebraic or physical 3-algebra A is called $N = 6$ 3-algebra if it satisfies the following two axioms, cf. [3] and [7] ($a, b, c, x, y, z \in A$):

$$[a, b, c] = -[c, b, a] \quad \text{(anti-commutativity)}$$

$$[a, b, [x, y, z]] = [[a, b, x], y, z] - [x, [b, a, y], z] + [x, y, [a, b, z]] \quad \text{(fundamental identity)}.$$

In our previous paper [7] we classified all linearly compact simple $N = 6$ algebraic 3-algebras. It is easy to see that if C is an anti-linear involution of the 3-bracket $[a, b, c]$ of an algebraic $N = 6$ 3-algebra, then $[a, b, c]_{ph,C}$, defined by (0.1), is a 3-bracket of a physical $N = 6$ 3-algebra. However not all physical $N = 6$ 3-algebras are obtained from the algebraic ones in this way, and we call them *trivially related* if they are. Indeed, it turned out that the theory of physical $N = 6$ 3-algebras is

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richer than that of algebraic ones. On the other hand, the method used in the classification of the former is similar to that of the latter.

Recall that the space $M_{m,n}(\mathbb{C})$ of all $m \times n$ matrices over $\mathbb{C}$ carries a structure of an algebraic $N = 6$ 3-algebra, given by

$$[a, b, c] = ab^t c - cb^t a, \tag{0.2}$$

where $a \mapsto a^t$ is the transpose, and this 3-algebra is simple. It is denoted by $A^3(m, n; t)$ in [7].

On the other hand, the space $M_{m,n}(\mathbb{C})$ carries the following structure of physical $N = 6$ 3-algebra, given by

$$[a, b, c] = aC_{p,q}(b)^t c - cC_{p,q}(b)^t a, \tag{0.3}$$

where $C_{p,q}$ is the anti-linear involution of the space $M_{m,n}(\mathbb{C})$, defined by

$$C_{p,q}(a) = S_q^m \bar{a} S_p^n, \text{ where } S_q^m = diag(I_q, -I_{m-q}),\ 0 \le q \le m,\ 0 \le p \le n$$

(and I_q stands for the $q \times q$ identity matrix). This is a simple physical $N = 6$ 3-algebra, denoted by $A^3(m, n; t)_{ph,C_{p,q}}$, which is, by definition, trivially related to the algebraic $N = 6$ 3-algebra $A^3(m, n; t)$; for $p = m$, $q = n$ it was introduced in [4].

The simplest example of a simple physical $N = 6$ 3-algebra, which is not trivially related to any algebraic one, is obtained by endowing the space $M_{n,n}(\mathbb{C})$ with the 3-bracket

$$[a, b, c] = \pm i(a\bar{b}c - c\bar{b}a) \tag{0.4}$$

(different signs give non-isomorphic 3-algebras). It is denoted by $A^3(n)_\pm$.

An interesting phenomen here is that physical $N = 6$ 3-algebras, trivially related to non-isomorphic algebraic $N = 6$ 3-algebras, may be isomorphic (see Remark 1.5).

The first main result of the paper is the classification of finite-dimensional simple physical $N = 6$ 3-algebras, given by the first part of Theorem 1.17. The answer consists of four series: the two series of physical 3-algebras $A^3(m, n; t)_{ph,C_{p,q}}$ and $A^3(n)_\pm$ of type A, mentioned above, and two more series of type C. One of the series of type C, denoted by $C^3(2n)_{ph,\pm C_p}$, where

$$C_{n-p}(u) = \bar{u}\, diag(S_p^n, S_p^n) \ \ (0 \le p \le n),$$

introduced for $p = n$ in [4], is trivially related to the algebraic 3-algebra $C^3(2n)$, introduced in [7], and the second series of type C, denoted by $C^3(2n, iS_n^{2n}, \pm i)$, introduced in Example 1.8 of the paper, is not trivially related to any algebraic $N = 6$ 3-algebra.

The second result of the paper is the classification of infinite-dimensional linearly compact simple physical $N = 6$ 3-algebras, given by the second part of Theorem 1.17. The third part of this theorem describes which of the examples are not trivially related to any algebraic $N = 6$ 3-algebra. The most interesting of them is the family of physical $N = 6$ 3-algebras $W_\beta^3(\varphi)_\pm$, depending on a complex parameter β of modulus 1, $\beta \ne \pm 1$, which is not trivially related to any algebraic $N = 6$ 3-algebra.

As in [7], in order to achieve the classification, we establish in Theorem 2.5 a bijection between the isomorphism classes of physical $N = 6$ 3-algebras A with zero center and the isomorphism classes of pairs $(\mathfrak{g}, \sigma)$, where $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is a $\mathbb{Z}$-graded Lie superalgebra with a consistent short grading, satisfying certain additional properties (which in case of simple A mean that $\mathfrak{g}$ is simple) and σ is an anti-linear graded conjugation of $\mathfrak{g}$. The only difference with the algebraic case, considered in [7], is that σ in [7] is a linear graded conjugation (graded conjugation means that $\sigma(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ and $\sigma^2|_{\mathfrak{g}_k} = (-1)^k$).

Thus, the classification in question reduces to the classification of anti-linear graded conjugations of simple linearly compact Lie superalgebras with consistent short $\mathbb{Z}$-gradings (such $\mathbb{Z}$-graded Lie superalgebras have been classified in [7]). This is technically the most difficult part of the paper.

Note a relation of $N = 6$ 3-algebras to Jordan triple systems (called also Jordan 3-algebras), introduced by Jacobson [11], as subspaces of an associative algebra, closed under the 3-bracket $[a, b, c] = abc + cba$, and axiomatized in [12]. Doing the same in an associative superalgebra we arrive at the following:

Definition 0.1 *A Jordan* 3*-superalgebra is a vector superspace endowed with an algebraic* 3*-bracket* $[\cdot, \cdot, \cdot]$ *satisfying the following axioms:*

(a) $\quad [a, b, c] = (-1)^{p(a)p(b)+p(a)p(c)+p(b)p(c)}[c, b, a]$

(b) $\quad [a, b, [x, y, z]] = [[a, b, x], y, z] - (-1)^{p(x)(p(a)+p(b))+p(a)p(b)+p(a)}[x, [b, a, y], z]+$

$$(-1)^{(p(x)+p(y))(p(a)+p(b))}[x, y, [a, b, z]]$$

Obviously, a purely even Jordan 3-superalgebra is a Jordan triple system, and a purely odd Jordan 3-superalgebra becomes an $N = 6$ 3-algebra after reversing the parity. We are planning to classify simple linearly compact Jordan 3-superalgebras in a subsequent paper [8].

We are grateful to the referee of our paper [7] who suggested us to study physical $N = 6$ 3-algebras.

1 Examples of physical N = 6 3-algebras

First, as has been mentioned in the Introduction, we have the following:

Remark 1.1 Let $(A, [\cdot, \cdot, \cdot])$ be an algebraic $N = 6$ 3-algebra over $\mathbb{C}$ and let C be an anti-linear involution of the 3-algebra A. Then it is straightforward to check that A with the 3-bracket $[a, b, c]_{ph,C} = [a, C(b), c]$ is a physical $N = 6$ 3-algebra. We shall denote it by $A_{ph,C}$.

Note also the following:

Remark 1.2 If $(A, [\cdot, \cdot, \cdot])$ is an algebraic $N = 6$ 3-algebra, then for every $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $(A, [\cdot, \cdot, \cdot]_\lambda)$ where $[a, b, c]_\lambda = \lambda[a, b, c]$, is an algebraic $N = 6$ 3-algebra isomorphic to $(A, [\cdot, \cdot, \cdot])$, the isomorphism being the map $f_\alpha : A \to A$, $a \mapsto \alpha a$, for $\alpha \in \mathbb{C}$ such that $\alpha^2 = \lambda^{-1}$.

Differently from the algebraic case, if $(A, [\cdot, \cdot, \cdot])$ is a physical $N = 6$ 3-algebra and $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $(A, [\cdot, \cdot, \cdot]_\lambda)$ is a physical $N = 6$ 3-algebra if and only if $\lambda \in \mathbb{R}$. Besides, the map f_α is an isomorphism of physical $N = 6$ 3-algebras if and only if $\lambda \in \mathbb{R}^{>0}$.

Now we introduce all the examples that appear in our classifications of algebraic and physical $N = 6$ 3-algebras. The fact that they indeed satisfy the fundamental identity either can be checked directly or follows automatically from Theorem 2.4.

Example 1.3 We denote by $A^3(m,n;t)$ the algebra $M_{m,n}(\mathbb{C})$ with 3-bracket:

$$[a,b,c] = ab^t c - cb^t a.$$

This is an algebraic $N = 6$ 3-algebra [7, Example 1.2]. The complex conjugation C_0 of matrices,

$$C_0 : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$$

$$(a_{ij}) \mapsto (\bar{a}_{ij}),$$

is an anti-linear involution of the 3-algebra $A^3(m,n;t)$, hence, by Remark 1.1, $M_{m,n}(\mathbb{C})$ with 3-bracket:

$$[a,b,c] = a\bar{b}^t c - c\bar{b}^t a$$

is a physical $N = 6$ 3-algebra that we shall denote by $A^3(m,n;t)_{ph,C_0}$.

More generally, let $^* : M_{m,n}(\mathbb{C}) \to M_{n,m}(\mathbb{C})$ be an anti-linear involutive map satisfying the following property:

$$(ab^*c)^* = c^*ba^*, \quad a,b,c \in M_{m,n}(\mathbb{C}). \tag{1.1}$$

Then $M_{m,n}(\mathbb{C})$ with the 3-bracket

$$[a,b,c] = ab^*c - cb^*a \tag{1.2}$$

is a physical $N = 6$ 3-algebra.

For example, let

$$S_p^n = diag(I_p, -I_{n-p}) \in GL_n(\mathbb{C}), \;\; 0 \le p \le n,$$

and define the map

$$\varphi_{p,q} : M_{m,n}(\mathbb{C}) \to M_{n,m}(\mathbb{C}), \quad u \longmapsto S_p^n \bar{u}^t S_q^m. \tag{1.3}$$

Then $\varphi_{p,q}$ satisfies property (1.1), hence $M_{m,n}(\mathbb{C})$ with the corresponding 3-bracket (1.2) is a physical $N = 6$ 3-algebra. Again, we can obtain this 3-algebra from the algebraic $N = 6$ 3-algebra $A^3(m,n;t)$, as explained in Remark 1.1, using the anti-linear involution

$$C_{p,q} : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C}), \quad u \mapsto S_q^m \bar{u} S_p^n.$$

We shall therefore denote the physical $N = 6$ 3-algebra $M_{m,n}(\mathbb{C})$ with the 3-bracket: $[a,b,c] = a\varphi_{p,q}(b)c - c\varphi_{p,q}(b)a$ by $A^3(m,n;t)_{ph,C_{p,q}}$. Note that $C_0 = C_{0,0} = C_{n,m}$ and $-C_0 = C_{n,0} = C_{0,m}$.

More generally, if $A \in GL_n(\mathbb{C})$ and $B \in GL_m(\mathbb{C})$ are such that $A = \lambda \bar{A}^t$, $B = \lambda \bar{B}^t$, for some $\lambda \in \mathbb{C}$, then the map $\varphi_{A,B} : M_{m,n}(\mathbb{C}) \to M_{n,m}(\mathbb{C})$ defined by: $\varphi_{A,B}(b) = A\bar{b}^t B^{-1}$, satisfies property (1.1), hence $M_{m,n}(\mathbb{C})$ with the corresponding 3-bracket (1.2) is a physical $N = 6$ 3-algebra. Note that $\varphi_{p,q} = \varphi_{S_p^n, S_q^m}$.

Lemma 1.4 *Let* $^* : M_{m,n}(\mathbb{C}) \to M_{n,m}(\mathbb{C})$ *be defined by:* $b^* = A\bar{b}^t B^{-1}$ *for some matrices* $A \in GL_n(\mathbb{C})$ *and* $B \in GL_m(\mathbb{C})$ *such that* $A = \lambda \bar{A}^t$, $B = \lambda \bar{B}^t$, *with* $\lambda \in \mathbb{C}$, $|\lambda| = 1$. *Then the physical* $N = 6$ *3-algebra with the corresponding bracket (1.2) is isomorphic to* $A^3(m,n;t)_{ph,C_{p,q}}$, p *and* q *being the numbers of positive eigenvalues of the hermitian matrices* $\lambda^{-1/2}A$ *and* $\lambda^{-1/2}B$, *respectively.*

Proof. It is immediate to check that, under the hypotheses of the lemma, the matrices $A' = \lambda^{-1/2}A$ and $B' = \lambda^{-1/2}B$ are hermitian. It follows that there exist unique matrices $h \in GL_n(\mathbb{C})$ and $k \in GL_m(\mathbb{C})$ such that $A' = hS_p^n\bar{h}^t$ and $B' = kS_q^m\bar{k}^t$, where p (resp. q) is the number of positive eigenvelues of A' (resp. B'). Now consider the map $f : M_{m,n}(\mathbb{C}) \to M_{m,n}(\mathbb{C})$ defined by $f(u) = kuh^{-1}$. For $a, b, c \in M_{m,n}(\mathbb{C})$, let $[a, b, c]_{p,q} = a\varphi_{p,q}(b)c - c\varphi_{p,q}(b)a$ where $\varphi_{p,q}$ is map (1.3), and $[a, b, c]_* = ab^*c - cb^*a$. Then we have:

$$f([a, b, c]_{p,q}) = f(a\varphi_{p,q}(b)c - c\varphi_{p,q}(b)a) = ka\varphi_{p,q}(b)ch^{-1} - kc\varphi_{p,q}(b)ah^{-1} = (kah^{-1})(hS_p^n\bar{b}^tS_q^mk^{-1})$$

$$(kch^{-1}) - (kch^{-1})(hS_p^n\bar{b}^tS_q^mk^{-1})(kah^{-1}) = f(a)(A(\bar{h}^t)^{-1}\bar{b}^t\bar{k}^tB^{-1})f(c) - f(c)(A(\bar{h}^t)^{-1}\bar{b}^t\bar{k}^tB^{-1})f(a)$$

$$= f(a)f(b)^*f(c) - f(c)f(b)^*f(a) = [f(a), f(b), f(c)]_*,$$

and this shows that the 3-brackets $[\, , \,]_{p,q}$ and $[\, , \,]_*$ are isomorphic. □

Remark 1.5 Recall [7] that for $m = 2h$ and $n = 2k$, the following algebraic $N = 6$ 3-bracket is defined on $M_{m,n}(\mathbb{C})$:

$$[a, b, c] = ab^{st}c - cb^{st}a,$$

where the map $st : M_{m,n}(\mathbb{C}) \to M_{n,m}(\mathbb{C})$, is defined by: $a \mapsto a^{st} := J_{2k}a^tJ_{2h}^{-1}$, with $J_{2k} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$. In [7] we denoted $M_{m,n}(\mathbb{C})$ with this 3-bracket by $A^3(m, n; st)$. By Remark 1.1, we can associate to $A^3(m, n; st)$ the physical $N = 6$ 3-algebra $A^3(m, n; st)_{ph,C_0}$ by defining on $M_{m,n}(\mathbb{C})$ the following 3-bracket:

$$[a, b, c] = a\bar{b}^{st}c - c\bar{b}^{st}a.$$

Since $\bar{J}_{2k}^t = -J_{2k}$, by Lemma 1.4, $A^3(2h, 2k; st)_{ph,C_0}$ is isomorphic to $A^3(2h, 2k; t)_{ph,C_{k,h}}$.

Example 1.6 Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ and let $^\dagger : \mathcal{A} \to \mathcal{A}$ be an anti-linear map such that:

$$(ab^\dagger c)^\dagger = a^\dagger bc^\dagger, \quad a, b, c \in \mathcal{A}. \tag{1.4}$$

Then $\mathcal{A}$ with the 3-bracket

$$[a, b, c] = i(ab^\dagger c - cb^\dagger a) \tag{1.5}$$

is an $N = 6$ physical 3-algebra. For example, if $\mathcal{A} = M_{n,n}(\mathbb{C})$ and † is the complex conjugation of matrices, then the corresponding bracket (1.5) defines on $\mathcal{A}$ a physical $N = 6$ 3-algebra structure that we shall denote by $A^3(n)_+$. Similarly, we shall denote by $A^3(n)_-$ the physical $N = 6$ 3-algebra $\mathcal{A}$ with the negative of the 3-bracket (1.5).

More generally, for $A \in GL_n(\mathbb{C})$ consider the following anti-linear map

$$\psi_A : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C}), \quad \psi_A(u) = A\bar{u}\bar{A}.$$

Then ψ_A satisfies property (1.4), hence $M_{n,n}(\mathbb{C})$ with the corresponding bracket (1.5) is a physical $N = 6$ 3-algebra.

Lemma 1.7 *For every $A \in GL_n(\mathbb{C})$, $M_{n,n}(\mathbb{C})$ with the 3-bracket (1.5) associated to the map ψ_A is isomorphic to $A^3(n)_+$.*

Proof. Let us denote by $[\cdot,\cdot,\cdot]_A$ bracket (1.5) corresponding to the map ψ_A, $A \in GL_n(\mathbb{C})$, i.e., $[a,b,c]_A = i(a\psi_A(b)c - c\psi_A(b)a)$ and let $[\cdot,\cdot,\cdot]$ be the 3-bracket in $A^3(n)_+$, i.e., $[a,b,c] = i(a\bar{b}c - c\bar{b}a)$. Now let us write $A = hk$ for some $h,k \in GL_n(\mathbb{C})$ and consider the map $f : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C})$ defined by $f(u) = \bar{k}uh$. Then we have: $f([a,b,c]_A) = i\bar{k}(a\psi_A(b)c - c\psi_A(b)a)h = i\bar{k}(aA\bar{b}\bar{A}c - cA\bar{b}\bar{A}a)h = i((\bar{k}ah)h^{-1}A\bar{b}\bar{A}(\bar{k})^{-1}(\bar{k}ch) - (\bar{k}ch)h^{-1}A\bar{b}\bar{A}(\bar{k})^{-1}(\bar{k}ah)) = i(f(a)k\bar{b}\bar{h}f(c) - f(c)k\bar{b}\bar{h}f(a)) = i(f(a)\overline{f(b)}f(c) - f(c)\overline{f(b)}f(a)) = [f(a),f(b),f(c)]$, which proves that the 3-brackets $[\cdot,\cdot,\cdot]_A$ and $[\cdot,\cdot,\cdot]$ are isomorphic. □

Example 1.8 Let us consider the map $\psi : M_{1,2n}(\mathbb{C}) \to M_{2n,1}(\mathbb{C})$, defined by: $\psi(X\ Y) = (Y\ -X)^t$, for $X, Y \in M_{1,n}$. Then $M_{1,2n}$ with the 3-bracket

$$[a,b,c] = -a\bar{b}^t c + c\bar{b}^t a - c\psi(a)(\psi(\bar{b}))^t \tag{1.6}$$

is a physical $N = 6$ 3-algebra, which we denote by $C^3(2n)_{ph,C_0}$, since it can be obtained from [7, Example 1.4] using Remark 1.1 with C_0 equal to the complex conjugation. Indeed $C^3(2n)$ denotes the algebra $M_{1,2n}(\mathbb{C})$ with 3-bracket

$$[a,b,c] = -ab^t c + cb^t a - c\psi(a)(\psi(b))^t$$

and the complex conjugation of matrices is an anti-linear involution of $C^3(2n)$ with respect to this bracket, since $\overline{\psi(x)} = \psi(\bar{x})$.

More generally, let $H \in M_{2n,2n}(\mathbb{C})$ be a symplectic matrix for the bilinear form with the matrix J_{2n}. Then $M_{1,2n}$ with 3-bracket

$$[a,b,c]_{H,\alpha} = -\alpha a H\overline{b}^t c + \alpha c H\overline{b}^t a - \alpha^{-1} c\psi(a)(\psi(\overline{b}H^t))^t, \tag{1.7}$$

where either $\alpha \in \mathbb{R}$ and H is hermitian, or $\alpha \in i\mathbb{R}$ and H is anti-hermitian (i.e., $H = -\bar{H}^t$), is a physical $N = 6$ 3-algebra which we denote by $C^3(2n, H; \alpha)$.

Let $S_p^n \in GL_n(\mathbb{C})$ be the matrix defined in Example 1.3. We set $H_p^{2n} = diag(S_p^n, S_p^n)$.

Lemma 1.9 *Let $H \in M_{2n,2n}(\mathbb{C})$ be a symplectic matrix (for the bilinear form with the matrix J_{2n}). Then*

- *(a) If H is hermitian, then there exists a symplectic matrix V such that $H = VH_p^{2n}\overline{V}^t$ where $2p$ is the number of positive eigenvalues of H.*
- *(b) If H is anti-hermitian, then there exists a symplectic matrix V such that $H = iVS_n^{2n}\overline{V}^t$.*

Proof. (a) follows immediately from [9, Proposition 3]. In order to prove (b) one can use the same argument as in [9, Proposition 3]. Namely, suppose that H is symplectic and anti-hermitian and set $K = iH$. Then K is hermitian and satisfies relation $K^t J_{2n} K = -J$. Let v be an eigenvector of K: $Kv = \lambda v$. Then $\lambda \in \mathbb{R}$ since K is hermitian. Besides, $K^t J_{2n} K v = -Jv$, i.e., $K^t(J_{2n}v) = -1/\lambda J_{2n} v$. By conjugating both sides of this equality we get: $\bar{K}^t(J_{2n}\bar{v}) = -1/\lambda J_{2n}\bar{v}$, i.e., since K is hermitian, $K(J_{2n}\bar{v}) = -1/\lambda J_{2n}\bar{v}$. Therefore if v is an eigenvector of K corresponding to the eigenvalue λ, then $J_{2n}\bar{v}$ is an eigenvector of K corresponding to the eigenvalue $-1/\lambda$. Notice that, since $\lambda \in \mathbb{R}$, $\lambda \neq -1/\lambda$. It follows that one can construct an orthonormal basis $\{v_1, \dots, v_n, J_{2n}\bar{v}_1, \dots, J_{2n}\bar{v}_n\}$ of $\mathbb{C}^{2n}$ consisting of eigenvectors of K. Therefore if we set $W = (v_1 \dots v_n - J_{2n}\bar{v}_1 \cdots - J_{2n}\bar{v}_n)$, W is a unitary matrix such that $K = W\Lambda\overline{W}^t$, where $\Lambda = diag(\Lambda_1, -\Lambda_1^{-1})$ is a diagonal matrix. One can check that W is symplectic. Then statement (b) follows using $H = -iK$. □

Lemma 1.10 *Consider the* 3*-bracket defined by (1.7) on* $M_{1,2n}$ *where* H *is a symplectic matrix.*

(a) If H *is hermitian and* $\alpha \in \mathbb{R}^{>0}$ *(resp.* $\alpha \in \mathbb{R}^{<0}$*) then the 3-algebra* $C^3(2n, H; \alpha)$ *is isomorphic to* $C^3(2n, H_p^{2n}; 1)$ *(resp.* $C^3(2n, H_p^{2n}; -1)$*), where* $2p$ *is the number of positive eigenvalues of* H*.*

(b) If H *is anti-hermitian and* $\alpha \in i\mathbb{R}^{>0}$ *(resp.* $\alpha \in i\mathbb{R}^{<0}$*) then* $C^3(2n, H; \alpha)$ *is isomorphic to* $C^3(2n, iS_n^{2n}; i)$ *(resp.* $C^3(2n, iS_n^{2n}; -i)$*).*

Proof. It is convenient to identify $M_{1,2n}(\mathbb{C})$ with the set of matrices of the form $\begin{pmatrix} 0 & 0 \\ \ell & m \end{pmatrix} \in M_{2,2n}$, with $\ell, m \in M_{1,n}$, and $M_{2n,1}(\mathbb{C})$ with the set of matrices of the form $\begin{pmatrix} n & 0 \\ p & 0 \end{pmatrix} \in M_{2n,2}$ with $n, p \in M_{n,1}$. Under these identifications, for $Z \in M_{1,2n}(\mathbb{C})$, $\psi(Z) = J_{2n} Z^t J_2^{-1}$. Let us first assume that H is symplectic and hermitian. Then, by Lemma 1.9 we can write $H = y H_p^{2n} \bar{y}^t$ for some matrix $y \in Sp_{2n}(\mathbb{C})$. Besides, assume that $\alpha \in \mathbb{R}^{>0}$. Then, for $h = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in GL_2(\mathbb{R})$, we can write $h = x\bar{x}$ for some diagonal matrix $x \in GL_2(\mathbb{C})$. Consider the map $\varphi : M_{1,2n}(\mathbb{C}) \to M_{1,2n}(\mathbb{C})$ defined by: $\varphi(u) = xuy^{-1}$. We have: $\varphi([a,b,c]_{H_p^{2n},1}) = x[a,b,c]_{H_p^{2n},1} y^{-1} = -xaH_p^{2n}\bar{b}^t cy^{-1} + xcH_p^{2n}\bar{b}^t ay^{-1} - xc\psi(a)(\psi(\bar{b}H_p^{2n}))^t y^{-1} = -(xay^{-1})yH_p^{2n}\bar{b}^t x^{-1}(xcy^{-1}) + (xcy^{-1})yH_p^{2n}\bar{b}^t x^{-1}(xay^{-1}) - (xcy^{-1})y\psi(a)(\psi(\bar{b}H_p^{2n}))^t y^{-1} = -\varphi(a)y\, H_p^{2n}\bar{b}^t x^{-1}\varphi(c) + \varphi(c)yH_p^{2n}\bar{b}^t x^{-1}\varphi(a) - \varphi(c)y\psi(a)(\psi(\bar{b}H_p^{2n}))^t y^{-1} = -\varphi(a)H\overline{\varphi(b)}^t h^{-1}\ \varphi(c) + \varphi(c)H\overline{\varphi(b)}^t h^{-1}\ \varphi(a) - \varphi(c)\psi(\varphi(a))h(\psi(\overline{\varphi(b)}H^t))^t = [\varphi(a), \varphi(b), \varphi(c)]_{H,\alpha}$ since $y\psi(a)(\psi(\bar{b}H_p^{2n}))^t y^{-1} = \psi(\varphi(a))h(\psi(\overline{\varphi(b)}H^t))^t$. Indeed we have: $\psi(\varphi(a))h(\psi(\overline{\varphi(b)}H^t))^t = J_{2n}(xay^{-1})^t J_2^{-1} h(J_{2n}H\overline{\varphi(b)}^t J_2^{-1})^t = -J_{2n}\ (y^{-1})^t a^t x^t h^{-1}\bar{x}\overline{by^{-1}}H^t J_{2n} = -J_{2n}(y^t)^{-1}a^t\bar{b}H_p^{2n}y^t J_{2n} = -yJ_{2n}a^t\bar{b}H_p^{2n}J_{2n}^t y^{-1} = y\psi(a)(\psi(\bar{b}H_p^{2n}))^t y^{-1}$. Statement (a) with $\alpha \in \mathbb{R}^{>0}$ then follows from Remark 1.2. Statement (a) with $\alpha \in \mathbb{R}^{<0}$ can be proven using the same argument with $h = -x\bar{x}$.

One proves statement (b) using the same arguments and the decomposition of an anti-hermitian symplectic matrix given in Lemma 1.9(b). Indeed, if H is anti-hermitian, then by Lemma 1.9(b), $H = iyS_n^{2n}\bar{y}^t$ for some symplectic matrix y. If $\alpha \in i\mathbb{R}^{>0}$, i.e., $\alpha = i\beta$ for some $\beta \in \mathbb{R}^{>0}$, we set $h = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$. Then we can write $h = x\bar{x}$ for some diagonal matrix $x \in GL_2(\mathbb{C})$. Let $\varphi : M_{1,2n}(\mathbb{C}) \to M_{1,2n}(\mathbb{C})$ be defined by $\varphi(u) = xuy^{-1}$. Then, by arguing as above, one can show that $\varphi([a,b,c]_{iS_n^{2n},i}) = [\varphi(a), \varphi(b), \varphi(c)]_{H,\alpha}$.

A similar argument proves (b) if $\alpha \in i\mathbb{R}^{<0}$. □

Remark 1.11 The $N = 6$ physical 3-algebra $C^3(2n, H_p^{2n}, 1)$ can be constructed as explained in Remark 1.1, using the following anti-linear involution C_{n-p} of the algebraic 3-algebra $C^3(2n)$:

$$C_{n-p}(u) = \bar{u}H_p^{2n},\ u \in M_{1,2n}(\mathbb{C}).$$

Therefore, we have $C^3(2n, H_p^{2n}, 1) = C^3(2n)_{ph,C_{n-p}}$.

Example 1.12 Consider the generalised Poisson algebra $P(m, 0)$ in the (even) indeterminates $p_1, \dots, p_k, q_1, \dots, q_k$ (resp. $p_1, \dots, p_k, q_1, \dots, q_k, t$) if $m = 2k$ (resp. $m = 2k+1$) endowed with the bracket:

$$\{f, g\} = (2 - E)(f)\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2 - E)(g) + \sum_{i=1}^{k}(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}), \tag{1.8}$$

where $E = \sum_{i=1}^{k}(p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i})$ (the first two terms in (1.8) vanish if m is even) and the derivation $D = 2\frac{\partial}{\partial t}$ (which is 0 if m is even). Let $\sigma_\varphi : f(p_i, q_i) \mapsto -f(\varphi(p_i), \varphi(q_i))$ (resp. $\sigma_\varphi : f(t, p_i, q_i) \mapsto -f(\varphi(t), \varphi(p_i), \varphi(q_i))$, where φ is an involutive linear change of variables (i.e. $\varphi^2 = 1$), multiplying by -1 the 1-form $\sum_i(p_i dq_i - q_i dp_i)$ if m is even (resp. $dt + \sum_i(p_i dq_i - q_i dp_i)$ if m is odd). Then the 3-bracket

$$[f, g, h] = \{f, \sigma_\varphi(g)\}h + \{f, h\}\sigma_\varphi(g) + f\{\sigma_\varphi(g), h\} + D(f)\sigma_\varphi(g)h - f\sigma_\varphi(g)D(h), \tag{1.9}$$

defines on $P(m, 0)$ an algebraic $N = 6$ 3-algebra structure that we denote by $P^3(m, \varphi)$ [7, Example 1.6].

If in (1.9) we replace σ_φ with the map $\bar{\sigma}_\varphi$ defined by: $\bar{\sigma}_\varphi : f(p_i, q_i) \mapsto -\bar{f}(\varphi(p_i), \varphi(q_i))$ (resp. $\bar{\sigma}_\varphi : f(t, p_i, q_i) \mapsto -\bar{f}(\varphi(t), \varphi(p_i), \varphi(q_i))$, we get a physical $N = 6$ 3-algebra that we denote by $P^3(m, \varphi; ^-)_+$. Similarly, we shall denote by $P^3(m, \varphi; ^-)_-$ the physical $N = 6$ 3-algebra with bracket (1.9) where σ_φ is replaced with the map $-\bar{\sigma}_\varphi$.

Note that $\overline{\sigma_\varphi(g)} = \sigma_\varphi(\bar{g})$ if and only if φ is a real change of variables, hence in this case the complex conjugation $C_0 : P^3(m, \varphi) \to P^3(m, \varphi)$, $C(f) = \bar{f}$, is an anti-linear involution of the algebraic $N = 6$ 3-algebra $P^3(m, \varphi)$ and, by Remark 1.1, $P^3(m, \varphi; ^-)_\pm = P^3(m, \varphi)_{ph, \pm C_0}$.

Example 1.13 Let $A = \mathbb{C}[[x]]^{\langle 1\rangle} \oplus \mathbb{C}[[x]]^{\langle 2\rangle}$ be the direct sum of two copies of the algebra $\mathbb{C}[[x]]$; for $f \in \mathbb{C}[[x]]$, denote by $f^{\langle i\rangle}$ the corresponding element in $\mathbb{C}[[x]]^{\langle i\rangle}$. Let $D = d/dx$, and let $a = (a_{ij}) \in M_{2,2}(\mathbb{C})$.

The following 3-bracket defines on A an algebraic $N = 6$ 3-algebra structure for every $a \in SL_2(\mathbb{C})$ such that either $a^2 = 1$ (and in this case $\varphi = 1$) or $a^2 = -1$ (and in this case $\varphi = -1$): for $i, j = 1$ or 2,

$$[f^{\langle i\rangle}, g^{\langle i\rangle}, h^{\langle i\rangle}] = (-1)^i a_{ij}((fD(h) - D(f)h)g(\varphi(x)))^{\langle i\rangle} \quad \text{for} \quad j \neq i;$$

$$[f^{\langle i\rangle}, g^{\langle j\rangle}, h^{\langle i\rangle}] = (-1)^i a_{jj}((fD(h) - D(f)h)g(\varphi(x)))^{\langle i\rangle} \quad \text{for} \quad j \neq i;$$

$$[f^{\langle 1\rangle}, g^{\langle j\rangle}, h^{\langle 2\rangle}] = a_{j1}((fD(g(\varphi(x))) - D(f)g(\varphi(x)))h)^{\langle 1\rangle}) + a_{j2}(f(hD(g(\varphi(x))) - D(h)g(\varphi(x))))^{\langle 2\rangle}.$$

This bracket is extended to A by skew-symmetry in the first and third entries. We denote this 3-algebra by $SW^3(a)$ [7, Example 1.7].

Similarly, for every $a \in SL_2(\mathbb{C})$ such that either $\bar{a}a = 1$ (and in this case $\lambda = \pm 1$) or $\bar{a}a = -1$ (and in this case $\lambda = \pm i$): we can define a physical $N = 6$ 3-algebra structure by setting

$$[f^{\langle i\rangle}, g^{\langle i\rangle}, h^{\langle i\rangle}] = \lambda(\exp(-t))(-1)^i \overline{a_{ij}}((fD(h) - D(f)h)\overline{g(\varphi_t(x))})^{\langle i\rangle} \quad \text{for} \quad j \neq i;$$

$$[f^{\langle i\rangle}, g^{\langle j\rangle}, h^{\langle i\rangle}] = \lambda(\exp(-t))(-1)^i \overline{a_{jj}}((fD(h) - D(f)h)\overline{(g(\varphi_t(x))})^{\langle i\rangle} \quad \text{for} \quad j \neq i;$$

$$[f^{\langle 1\rangle}, g^{\langle j\rangle}, h^{\langle 2\rangle}] = \lambda(\exp(-t))(\overline{a_{j1}}((f\overline{D(g(\varphi_t(x)))} - D(f)\overline{g(\varphi_t(x))})h)^{\langle 1\rangle})$$
$$+\overline{a_{j2}}(f(h\overline{D(g(\varphi_t(x)))} - D(h)\overline{g(\varphi_t(x))}))^{\langle 2\rangle}),$$

where $t \in i\mathbb{R}$ and $\varphi_t(x) = \exp(2t)x$, and extending it to A by skew-symmetry in the first and third entries. We shall denote this physical $N = 6$ 3-algebra by $SW^3(a; \varphi_t)_\pm$, depending on λ being equal to ± 1 (resp. $\pm i$) if $\bar{a}a = 1$ (resp. $\bar{a}a = -1$).

Note that if C_0 denotes the complex conjugation of $\mathbb{C}[[x]]$, i.e., for $f \in \mathbb{C}[[x]]$, $f = \sum_{i\in\mathbb{Z}} \alpha_i x^i$, $C_0(f) = \sum_{i\in\mathbb{Z}} \overline{\alpha}_i x^i$, then C_0 is an anti-linear involution of $SW^3(a)$ if and only if $a \in SL_2(\mathbb{R})$. Hence, by Remark 1.1, if $a \in SL_2(\mathbb{R})$ and $t = k\pi i$, $k \in \mathbb{Z}$, then $SW^3(a; \varphi_t)_\pm = SW^3(a)_{ph, \pm C_0}$.

Example 1.14 Let $A = \mathbb{C}[[x_1, x_2]]$ and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2$. Consider the following 3-bracket:

$$[f, g, h] = \det \begin{pmatrix} f & \varphi(g) & h \\ D_1(f) & D_1(\varphi(g)) & D_1(h) \\ D_2(f) & D_2(\varphi(g)) & D_2(h) \end{pmatrix}, \tag{1.10}$$

where φ is an automorphism of the associative algebra A. If φ is a linear change of variables with determinant equal to 1, and $\varphi^2 = 1$, then A with 3-bracket (1.10) is an algebraic $N = 6$ 3-algebra, which we denote by $W^3(\varphi)$ [7, Example 1.8]. Likewise, consider the following 3-bracket on A:

$$[f, g, h] = \det \begin{pmatrix} f & \overline{\varphi(g)} & h \\ D_1(f) & D_1(\overline{\varphi(g)}) & D_1(h) \\ D_2(f) & D_2(\overline{\varphi(g)}) & D_2(h) \end{pmatrix} \tag{1.11}$$

where φ is an automorphism of the algebra A. If φ is a linear change of variables such that $\bar{\varphi}\varphi = 1$ then A with the bracket (1.11) is a physical $N = 6$ 3-algebra that we shall denote by $W^3(\varphi, ^{-})_+$. Likewise we shall denote by $W^3(\varphi, ^{-})_-$ the 3-algebra A with the negative of the 3-bracket (1.11).

Note that $\overline{\varphi(g)} = \varphi(\bar{g})$ if and only if φ is a real change of variables, hence in this case the complex conjugation of A is an anti-linear involution of the algebraic $N = 6$ 3-algebra $W^3(\varphi)$, and, by Remark 1.1, $W^3(\varphi, ^{-})_\pm = W^3(\varphi)_{ph, \pm C_0}$.

Example 1.15 Let $A = \mathbb{C}[[x_1, x_2]]$ and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2$. Consider the following 3-bracket:

$$[f, g, h] = \det \begin{pmatrix} (2 - E)(f) & (2\beta - E)(\overline{\varphi(g)}) & (2 - E)(h) \\ D_1(f) & D_1(\overline{\varphi(g)}) & D_1(h) \\ D_2(f) & D_2(\overline{\varphi(g)}) & D_2(h) \end{pmatrix} \tag{1.12}$$

where $\beta \in \mathbb{C}$, $E = \sum_{i=1}^{2} x_i \frac{\partial}{\partial x_i}$ and φ is a linear change of indeterminates. If $\varphi \in GL_2$ is such that $\bar{\varphi}\varphi = I_2$, $\beta \notin \mathbb{R}$ and $|\beta| = 1$, then A with 3-bracket (1.12) is a physical $N = 6$ 3-algebra that we shall denote by $W^3_\beta(\varphi)_+$. Likewise, we shall denote by $W^3_\beta(\varphi)_-$ the 3-algebra A with the negative of the 3-bracket (1.12).

Example 1.16 Let $A = \mathbb{C}[[x_1, x_2, x_3]]$ and $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, 3$. Consider the following 3-bracket on A:

$$[f, g, h] = \det \begin{pmatrix} D_1(f) & D_1(\varphi(g)) & D_1(h) \\ D_2(f) & D_2(\varphi(g)) & D_2(h) \\ D_3(f) & D_3(\varphi(g)) & D_3(h) \end{pmatrix} \tag{1.13}$$

where φ is an automorphism of the algebra A. If φ is a linear change of variables with determinant equal to 1, and $\varphi^2 = 1$, then A with the bracket (1.13) is an algebraic $N = 6$ 3-algebra and $\mathbb{C}1$ is an ideal of this 3-algebra [7, Example 1.9]. We denote by $S^3(\varphi)$ the quotient 3-algebra $A/\mathbb{C}1$.

Likewise, consider the following 3-bracket on A:

$$[f, g, h] = \alpha \det \begin{pmatrix} D_1(f) & D_1(\overline{\varphi(g)}) & D_1(h) \\ D_2(f) & D_2(\overline{\varphi(g)}) & D_2(h) \\ D_3(f) & D_3(\overline{\varphi(g)}) & D_3(h) \end{pmatrix} \tag{1.14}$$

where $\alpha \in \mathbb{C}$ and φ is an automorphism of the associative algebra A. If φ is a linear change of variables with determinant equal to 1 (resp. -1), $\alpha = \pm 1$ (resp. $\alpha = \pm i$) and $\bar{\varphi}\varphi = 1$, then A with the bracket (1.14) is a physical $N = 6$ 3-algebra and $\mathbb{C}1$ is an ideal of this 3-algebra. We shall denote by $S^3(\varphi, ^-)_\pm$ the quotient 3-algebra $A/\mathbb{C}1$.

Note that $\overline{\varphi(g)} = \varphi(\bar{g})$ if and only if φ is a real change of variables, hence in this case the complex conjugation of A is an anti-linear involution of the algebraic $N = 6$ 3-algebra $S^3(\varphi)$, and, by Remark 1.1, $S^3(\varphi, ^-)_\pm = S^3(\varphi)_{ph,\pm C_0}$.

The main result of the paper is the following theorem.

Theorem 1.17 *The following is a complete list, up to isomorphisms, of simple linearly compact physical $N = 6$ 3-algebras:*

- *finite-dimensional:*
  - *(i)* $A^3(m, n; t)_{ph,C_{p,q}}$ *(*$0 \le p \le m$, $0 \le q \le n$*);*
  - *(ii)* $A^3(n)_\pm$*;*
  - *(iii)* $C^3(2n)_{ph,\pm C_p}$ *(*$1 \le p \le n$*);*
  - *(iv)* $C^3(2n, iS_n^{2n}, \pm i)$*.*
- *infinite-dimensional:*
  - *(a)* $P^3(m, \varphi; ^-)_\pm$ *(*$m \ge 1$*);*
  - *(b)* $SW^3(a, \varphi_t)_\pm$*;*
  - *(c)* $W^3(\varphi, ^-)_\pm$*;*
  - *(d)* $S^3(\varphi, ^-)_\pm$*;*
  - *(e)* $W^3_\beta(\varphi)_\pm$*.*

Among these a complete list of physical $N = 6$ 3-algebras which are trivially related to algebraic $N = 6$ 3-algebras over $\mathbb{C}$, is the following:

- *finite-dimensional:*
  - *(i)* $A^3(m, n; t)_{ph,C_{p,q}}$ *(*$0 \le p \le m$, $0 \le q \le n$*);*
  - *(ii)* $C^3(2n)_{ph,\pm C_p}$ *(*$1 \le p \le n$*).*
- *infinite-dimensional:*
  - *(a′)* $P^3(m, \varphi; ^-)_\pm = P^3(m, \varphi)_{ph,\pm C_0}$ *(*$m \ge 1$*) where φ is a real change of variables;*
  - *(b′)* $SW^3(a, \varphi_t)_\pm = SW^3(a)_{ph,\pm C_0}$ *where $a \in SL_2(\mathbb{R})$ and $t = k\pi i$, $k \in \mathbb{Z}$;*
  - *(c′)* $W^3(\varphi, ^-)_\pm = W^3(\varphi)_{ph,\pm C_0}$ *where φ is a real change of variables*
  - *(d′)* $S^3(\varphi, ^-)_\pm = S^3(\varphi)_{ph,\pm C_0}$ *where φ is a real change of variables.*

Proof. Theorem 2.5 from Section 2 reduces the classification in question to that of the pairs (L, σ), where $L = L_{-1} \oplus L_0 \oplus L_1$ is a simple linearly compact Lie superalgebra with a consistent $\mathbb{Z}$-grading satisfying properties (i) and (ii) of Theorem 2.5, and σ is an anti-linear graded conjugation of L. A complete list of possible such Lie superalgebras $L = L_{-1} \oplus L_0 \oplus L_1$ is given by Proposition 3.1 from Section 3. Finally, a complete list of anti-linear graded conjugations of these L is given, in the finite-dimensional case by Proposition 3.7 from Section 3, and in the infinite-dimensional case by Proposition 3.12 from Section 3.

By Theorem 2.5(b), the physical $N = 6$ 3-algebra is identified with ΠL_{-1}, on which the 3-bracket is given by the formula $[a, b, c] = [[a, \sigma(b)], c]$. This formula, applied to the $\mathbb{Z}$-graded finite-dimensional Lie superalgebras L with graded conjugations, described by Proposition 3.7 $(a), (b)$ and (c) produces the 3-algebras $A^3(m, n; t)_{ph,C_{p,q}}$, $A^3(n)_\pm$ and $C^3(2n)_{ph,\pm C_p}$, $C^3(2n, iS_n^{2n}; \pm i)$. The same formula, applied to the $\mathbb{Z}$-graded infinite-dimensional Lie superalgebras L with graded conjugations, described by Proposition 3.12 $(a), (b), (c), (d), (e)$ and (f) produces the 3-algebras $P^3(2k, \varphi; ^-)_\pm$ $(m \geq 1)$, $P^3(2k+1, \varphi; ^-)_\pm$, $SW^3(a, \varphi_t)_\pm$, $W^3(\varphi, ^-)_\pm$, $S^3(\varphi, ^-)_\pm$, $W^3_\beta(\varphi)_\pm$.

The fact that all of them are indeed phyisical $N = 6$ 3-algebras follows automatically from Theorem 2.5(b). $\square$

Remark 1.18 As we already noticed at the end of Example 1.3, we have $A^3(m, n; t)_{ph,C_{0,0}} = A^3(m, n; t)_{ph,C_{n,m}}$ and $A^3(m, n; t)_{ph,C_{n,0}} = A^3(m, n; t)_{ph,C_{0,m}}$.

2 Palmkvist's construction

This section is based on the ideas of [14].

Definition 2.1 *An element b of an $N = 6$ 3-algebra $\mathfrak{g}$ is called central if $[a, b, c] = 0$ for all a and c in $\mathfrak{g}$; the subspace consisting of central elements is called the center of $\mathfrak{g}$.*

Remark 2.2 It is immediate to see from the axioms that the center is an ideal of $\mathfrak{g}$. In particular the center of a simple $N = 6$ 3-algebra is zero (by definition the one-dimensional 3-algebra with zero 3-bracket is not simple).

Definition 2.3 *Let $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ be a Lie superalgebra over $\mathbb{C}$ with a consistent $\mathbb{Z}$-grading. A graded conjugation (resp. anti-linear graded conjugation) of $\mathfrak{g}$ is a linear (resp. anti-linear) Lie superalgebra automorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that*

1. $\varphi(\mathfrak{g}_j) = \mathfrak{g}_{-j}$
2. $\varphi^2(x) = (-1)^k x$ *for* $x \in \mathfrak{g}_k$.

Theorem 2.4 *Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a consistently $\mathbb{Z}$-graded Lie superalgebra with an anti-linear graded conjugation φ. Then the 3-bracket*

$$[u, v, w] := [[u, \varphi(v)], w] \qquad (2.1)$$

defines on $\Pi \mathfrak{g}_{-1}$ a physical $N = 6$ 3-algebra structure (here and further Π stands for the parity reversal).

Proof. Bracket (2.1) is obviously linear in the first and third argument and anti-linear in the second one. Since the grading of $\mathfrak{g}$ is consistent, $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are completely odd and $\mathfrak{g}_0$ is even. For $u, v, w \in \mathfrak{g}_{-1}$ we thus have: $[u, v, w] := [[u, \varphi(v)], w] = [u, [\varphi(v), w]] = -[[\varphi(v), w], u] = -[[w, \varphi(v)], u] = -[w, v, u]$, which proves anti-commutativity.

Besides, for $u, v, x, y, z \in \mathfrak{g}_{-1}$ we have: $[u, v, [x, y, z]] - [x, y, [u, v, z]] = [[u, \varphi(v)], [[x, \varphi(y)], z]] - [[x, \varphi(y)], [[u, \varphi(v)], z]] = [[u, \varphi(v)], [x, \varphi(y)]], z] + [[x, \varphi(y)], [[u, \varphi(v)], z]] - [[x, \varphi(y)], [[u, \varphi(v)], z]] = [[u, \varphi(v)], [x, \varphi(y)]], z] = [[[u, \varphi(v)], x], \varphi(y)], z] + [[[x, [[u, \varphi(v)], \varphi(y)]], z] = [[[u, \varphi(v)], x], \varphi(y)], z] - [[[x, \varphi([[\varphi(u), v], y])], z] = [[[u, v, x], y, z] - [x, [v, u, y], z]$ □

We shall now associate to a physical $N = 6$ 3-algebra T with zero center a $\mathbb{Z}$-graded Lie superalgebra $LieT = Lie_{-1}T \oplus Lie_0 T \oplus Lie_1 T$, as follows. For $x, y \in T$, denote by $L_{x,y}$ the endomorphism of T defined by $L_{x,y}(z) = [x, y, z]$. Besides, for $x \in T$, denote by φ_x the map in $Hom(\Pi T \otimes \Pi T, \Pi T)$ defined by $\varphi_x(y, z) = -[y, x, z]$.

We let $Lie_{-1}T = \Pi T$, $Lie_0 T = \langle L_{x,y} \mid x, y \in T\rangle$, $Lie_1 T = \langle \varphi_x \mid x \in T\rangle$, and let $LieT = Lie_{-1}T \oplus Lie_0 T \oplus Lie_1 T$. Define the map $\sigma : LieT \longrightarrow LieT$ by setting $(x, y, z \in T)$:

$$z \mapsto -\varphi_z, \quad \varphi_z \mapsto z, \;\; L_{x,y} \mapsto -L_{y,x},$$

and extending it on $LieT$ by anti-linearity. The map σ is well defined since the center of T is zero. Indeed, by definition, $\varphi_z(x, y) = -[x, z, y]$, hence $\varphi_z = 0$ only for $z = 0$. Besides, $L_{a,b} = 0$ for some $a, b \in T$ implies, by the fundamental identity, that $[b, a, y]$ is a central element for any y, hence $L_{b,a} = 0$. Besides, the following relations hold: $\varphi_{\alpha z} = \overline{\alpha}\varphi_z$ and $\alpha L_{x,y} = L_{\alpha x,y} = L_{x,\overline{\alpha}y}$, which are consistent with the anti-linearity of σ.

Theorem 2.5 *(a) $LieT$ is a $\mathbb{Z}$-graded Lie superalgebra with a short consistent grading, satisfying the following two properties:*

(i) any non-zero $\mathbb{Z}$-graded ideal of $LieT$ has a non-zero intersection with both $Lie_{-1}T$ and $Lie_1 T$;

(ii) $[Lie_{-1}T, Lie_1 T] = Lie_0 T$.

(b) σ is an anti-linear graded conjugation of the $\mathbb{Z}$-graded Lie superalgebra $LieT$ and the 3-bracket on T is recovered from the bracket on $LieT$ by the formula $[x, y, z] = [[x, \sigma(y)], z]$.

(c) The correspondence $T \longrightarrow (LieT, \sigma)$ is bijective between the isomorphism classes of physical $N = 6$ 3-algebras with zero center and the isomorphism classes of the pairs $(LieT, \sigma)$, where $LieT$ is a $\mathbb{Z}$-graded Lie superalgebra with a short consistent grading, satisfying properties (i) and (ii), and σ is an anti-linear graded conjugation of $LieT$.

(d) A 3-algebra T is simple (resp. finite-dimensional or linearly compact) if and only if $LieT$ is.

Proof. For $x, y, z \in T$, we have: $[\varphi_x, z] = -L_{z,x}$, and

$$[L_{x,y}, L_{x',y'}] = L_{[x,y,x'],y'} - L_{x',[y,x,y']}. \tag{2.2}$$

Note that $[L_{x,y}, \varphi_z] = -\varphi_{[y,x,z]}$. Finally, $[[\varphi_x, \varphi_y], z] = [\varphi_x, [\varphi_y, z]] + [\varphi_y, [\varphi_x, z]] = -[\varphi_x, L_{z,y}] - [\varphi_y, L_{z,x}] = [L_{z,y}, \varphi_x] + [L_{z,x}, \varphi_y] = -\varphi_{[y,z,x]} - \varphi_{[x,z,y]} = 0$. Hence $[\varphi_x, \varphi_y] = 0$.

It follows that $LieT$ is indeed a $\mathbb{Z}$-graded Lie superalgebra, satisfying (ii) and such that any non-zero ideal has a non-zero intersection with $Lie_{-1}T$. It is straightforward to check (b), hence any non-zero ideal of $LieT$ has a non-zero intersection with Lie_1T, which completes the proof of (a).

(c) follows from Theorem 2.4 where the inverse of the correspondence $T \longrightarrow (LieT, \sigma)$ is given. Finally, since the simplicity of T, by definition, means that all operators $L_{x,y}$ have no common non-trivial invariant subspace in T, it follows that T is a simple 3-algebra if and only if Lie_0T acts irreducibly on $Lie_{-1}T$. Hence, by the properties (i) and (ii) of $LieT$, T is simple if and only if $LieT$ is simple. The rest of (d) is clear as well. □

3 Classification of anti-linear graded conjugations

In this section we shall classify all anti-linear graded conjugations σ of all $\mathbb{Z}$-graded simple linearly compact Lie superalgebras $\mathfrak{g}$ with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

In [7, §3] we classified all short gradings of all simple linearly compact Lie superalgebras and deduced the following lists of simple finite-dimensional $\mathbb{Z}$-graded Lie superalgebras with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have the same dimension [7, Remark 3.2] and of simple infinite-dimensional $\mathbb{Z}$-graded Lie superalgebras with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have the same growth and size [7, Remark 3.4]:

Proposition 3.1 *(a) A complete list of simple finite-dimensional $\mathbb{Z}$-graded Lie superalgebras with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have the same dimension, is, up to isomorphism, as follows:*

- *$psl(m,n)$ with $m,n \geq 1$, $m+n \geq 2$, with the grading $f(\epsilon_1) = \cdots = f(\epsilon_m) = 1$, $f(\delta_1) = \cdots = f(\delta_n) = 0$;*
- *$osp(2,2n)$, $n \geq 1$, with the grading $f(\delta_i) = 0$ for all i, $f(\epsilon_1) = 1$.*

(b) A complete list of simple infinite-dimensional $\mathbb{Z}$-graded Lie superalgebras with a short consistent grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ have the same growth and size, is, up to isomorphism, as follows:

- *$S(1,2)$ with the grading of type $(0|1,1)$;*
- *$H(2k,2)$ with the grading of type $(0,\dots,0|1,-1)$;*
- *$K(2k+1,2)$ with the grading of type $(0,\dots,0|1,-1)$;*
- *$SHO(3,3)$ with the grading of type $(0,0,0|1,1,1)$;*
- *$SKO(2,3;\beta)$ with the grading of type $(0,0|1,1,1)$.*

In Examples 3.2 and 3.4 the $\mathbb{Z}$-grading of $\mathfrak{g}$ is given by Proposition 3.1(a).

Example 3.2 Let $\mathfrak{g} = psl(n,n)$. Then the maps $\tau_\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{d} & \pm i\bar{c} \\ \mp i\bar{b} & \bar{a} \end{pmatrix}$ are anti-linear graded conjugations of $\mathfrak{g}$.

Remark 3.3 Let σ be a graded conjugation of $\mathfrak{g}$ and let C be an anti-linear involution of $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ preserving the grading and such that $\sigma \circ C = C \circ \sigma$. Then $\tilde{\sigma} := C \circ \sigma$ is an anti-linear graded conjugation of $\mathfrak{g}$.

Example 3.4 The following examples of anti-linear graded conjugations can be deduced from [7, Propositions 4.3, 4.8], using Remark 3.3:

(a) $\mathfrak{g} = psl(m,n)$: $\tilde{\sigma}_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\bar{a}^t & \bar{c}^t \\ -\bar{b}^t & -\bar{d}^t \end{pmatrix}$.

(b) $\mathfrak{g} = osp(2,2n)$: $\tilde{\sigma}_1$.

Remark 3.5 Let φ_1 and φ_2 be two anti-linear graded conjugations of a consistently $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $f = \varphi_2 \varphi_1^{-1}$. Then f is a grading preserving linear automorphism of $\mathfrak{g}$ such that

$$(f \circ \varphi_1)^2(x_j) = (-1)^j x_j \quad \text{for } x_j \in \mathfrak{g}_j. \tag{3.1}$$

It follows that if φ_1 is an anti-linear graded conjugation of $\mathfrak{g}$, then any other such conjugation is of the form $f \circ \varphi_1$ for some grading preserving linear automorphism f of $\mathfrak{g}$ satisfying (3.1).

Definition 3.6 *Two (linear or anti-linear) graded conjugations σ_1 and σ_2 of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ are called equivalent if $\sigma_2 = \varphi \sigma_1 \varphi^{-1}$, where φ is a grading preserving automorphism of $\mathfrak{g}$.*

Proposition 3.7 *The following is a complete list, up to equivalence, of anti-linear graded conjugations of all simple finite-dimensional Lie superalgebras with the $\mathbb{Z}$-gradings from Proposition 3.1(a):*

(a) $\mathfrak{g} = psl(m,n)$: $Ad\, diag(S_p^m, S_q^n) \circ \tilde{\sigma}_1$ for $p = 0, \dots, m$ and $q = 0, \dots, n$, where S_p^m and S_q^n are defined in Example 1.3.

(b) $\mathfrak{g} = psl(n,n)$: $\sigma = \tau_{\pm}$;

(c) $\mathfrak{g} = osp(2,2n)$: $Ad\, diag(\pm A, H) \circ \tilde{\sigma}_1$ where either $A = I_2$ and $H = H_p^{2n}$ for some $p = 0, \dots, n$, or $A = diag(i, -i)$ and $H = iS_n^{2n}$.

Proof. The Lie superalgebra $sl(m,n)$ has a short consistent grading such that $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$ consists of matrices of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$, where $tr\alpha = tr\delta$, $\mathfrak{g}_{-1}$ is the set of matrices of the form $\begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$, and $\mathfrak{g}_1$ is the set of matrices of the form $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$.

For $m \neq n$ every automorphism of $\mathfrak{g} = sl(m,n)$ is either of the form $Ad\ diag(A,B)$ for some matrices $A \in GL_m(\mathbb{C})$, $B \in GL_n(\mathbb{C})$, or of the form $Ad\ diag(A,B) \circ \sigma_1$ [16], where, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl(m,n)$, $\sigma_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^t & c^t \\ -b^t & -d^t \end{pmatrix}$. Note that $Ad\ diag(A,B) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} AaA^{-1} & AbB^{-1} \\ BcA^{-1} & BdB^{-1} \end{pmatrix}$, hence $Ad\ diag(A,B) \circ \sigma_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -Aa^tA^{-1} & Ac^tB^{-1} \\ -Bb^tA^{-1} & -Bd^tB^{-1} \end{pmatrix}$. Therefore only automorphisms of the form $Ad\ diag(A,B)$ preserve the short consistent grading of $\mathfrak{g}$. By Remark 3.5 and Example 3.4, every anti-linear graded conjugation of $\mathfrak{g}$ is of the form $\varphi = Ad\ diag(A,B) \circ \tilde{\sigma}_1$. Condition (3.1) then implies $A = \lambda \bar{A}^t$ and $B = \lambda \bar{B}^t$, for some $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. Indeed we have: $\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -A\bar{a}^tA^{-1} & A\bar{c}^tB^{-1} \\ -B\bar{b}^tA^{-1} & -B\bar{d}^tB^{-1} \end{pmatrix}$, hence $\varphi^2 \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} =$

$\begin{pmatrix} A(\bar{A}^{-1})^t a(\bar{A})^t A^{-1} & 0 \\ 0 & B(\bar{B}^{-1})^t d(\bar{B})^t B^{-1} \end{pmatrix}$ from which we get $A = \lambda \bar{A}^t$ and $B = \mu \bar{B}^t$ for some $\lambda, \mu \in \mathbb{C}$. Besides, $\varphi^2 \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -A(\bar{A}^{-1})^t b \bar{B}^t B^{-1} \\ 0 & 0 \end{pmatrix}$, hence $\lambda = \sigma$. Finally, taking the conjugate transpose of both sides of the equality $A = \lambda \bar{A}^t$, we get $\bar{A}^t = \bar{\lambda} A = \bar{\lambda}\lambda \bar{A}^t$, i.e., $|\lambda| = 1$. Statement (a) then follows from Lemma 1.4.

If $m = n$, in addition to the automorphisms described above, $psl(n,n)$ has automorphisms of the form $Ad\ diag(A,B) \circ \Pi$, $Ad\ diag(A,B) \circ \Pi \circ \sigma_1$ and $Ad\ diag(A,B) \circ \sigma_1 \circ \Pi$, where $A, B \in GL_n(\mathbb{C})$, and Π is defined as follows: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl(n,n)$, $\Pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ [16]. Automorphisms of the form $Ad\ diag(A,B) \circ \Pi$ do not preserve the short consistent grading of $\mathfrak{g}$, hence, by Remark 3.5 and Example 3.4, every anti-linear graded conjugation of $\mathfrak{g}$ which is not one of those considered above, is either of the form $\varphi_1 = Ad\ diag(A,B) \circ \Pi \circ \sigma_1 \circ \tilde{\sigma}_1$ or of the form $\varphi_2 = Ad\ diag(A,B) \circ \sigma_1 \circ \Pi \circ \tilde{\sigma}_1$. We have: $\varphi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} A\bar{d}A^{-1} & -A\bar{c}B^{-1} \\ -B\bar{b}A^{-1} & B\bar{a}B^{-1} \end{pmatrix}$, hence $\varphi_1^2 \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} A\bar{B}a\bar{B}^{-1}A^{-1} & 0 \\ 0 & B\bar{A}d\bar{A}^{-1}B^{-1} \end{pmatrix}$. It follows that if φ_1 is an anti-linear graded conjugation of $\mathfrak{g}$, then $A\bar{B} = \lambda I_n$ and $B\bar{A} = \sigma I_n$ for some $\lambda, \sigma \in \mathbb{C}$. Besides, $\varphi_1^2 \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A\bar{B}b\bar{A}^{-1}B^{-1} \\ 0 & 0 \end{pmatrix}$, from which it follows that $\sigma^{-1}\lambda = -1$, i.e., $\sigma = -\lambda$. We have $A = \lambda(\bar{B})^{-1} = \lambda(\bar{\sigma})^{-1}A$, therefore $\bar{\sigma} = \lambda = -\bar{\lambda}$, i.e., $\lambda \in i\mathbb{R}$. By Lemma 1.7 φ_1 is thus equivalent either to the anti-linear graded conjugation τ_+ or to the anti-linear graded conjugation τ_-, depending on $\lambda \in R^{>0}$ or $\lambda \in R^{<0}$.

Similarly, we have: $\varphi_2 \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} A\bar{d}A^{-1} & A\bar{c}B^{-1} \\ B\bar{b}A^{-1} & B\bar{a}B^{-1} \end{pmatrix}$. Arguing as for the map φ_1 one gets statement (b).

The Lie superalgebra $osp(2,2n)$ has a short consistent grading such that $\mathfrak{g}_{\bar{0}}$ consists of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, where $a = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, $\alpha \in \mathbb{C}$, and d lies in the Lie algebra $sp(2n)$, defined by J_{2n}, $\mathfrak{g}_{-1}$ is the set of matrices of the form $\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \begin{matrix} 0 & 0 \\ \beta & \gamma \end{matrix} \\ \hline \begin{matrix} \gamma^t \\ -\beta^t \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & 0 \end{array}\right)$, and $\mathfrak{g}_1$ is the set of matrices of the form $\left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \begin{matrix} \beta & \gamma \\ 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} \gamma^t \\ -\beta^t \end{matrix} & 0 \end{array}\right)$, with $\beta, \gamma \in M_{1,n}$. Every automorphism of $osp(2,2n)$ is either of the form $Ad\ diag(A,B)$ for some matrices $A = diag(\alpha, \alpha^{-1})$, $\alpha \in \mathbb{C}^\times$, $B \in Sp_{2n}(\mathbb{C})$, or of the form $Ad\ diag(A,B) \circ \sigma_1$ [16]. One can easily check that only automorphism of the form $Ad\ diag(A,B)$ preserve the short consistent grading of $osp(2,2n)$, hence, by Remark 3.5 and Example 3.4, every anti-linear graded conjugation of $osp(2,2n)$ is of the form $\varphi = Ad\ diag(A,B) \circ \tilde{\sigma}_1$ for some matrices $A = diag(\alpha, \alpha^{-1})$, $\alpha \in \mathbb{C}^\times$, $B \in Sp_{2n}(\mathbb{C})$.

Since A and a are diagonal matrices, we have: $\varphi^2 \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & B(\bar{B}^{-1})^t d \bar{B}^t B^{-1} \end{pmatrix}$. Therefore if φ is an anti-linear graded conjugation of $\mathfrak{g}$, then $B(\bar{B}^{-1})^t = \lambda I_{2n} \in Sp_{2n}$, hence $B = \lambda \bar{B}^t$ with $\lambda = \pm 1$, i.e., either B is symplectic hermitian or it is symplectic anti-hermitian. Besides, $\varphi^2 \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \\ \beta & \gamma \end{pmatrix} \\ \begin{pmatrix} \gamma^t & 0 \\ -\beta^t & 0 \end{pmatrix} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda^{-1} A(\bar{A}^t)^{-1} \begin{pmatrix} 0 & 0 \\ \beta & \gamma \end{pmatrix} \\ -\lambda \begin{pmatrix} \gamma^t & 0 \\ -\beta^t & 0 \end{pmatrix} \bar{A}^t A^{-1} & 0 \end{pmatrix}$. Since $A(\bar{A}^t)^{-1} = diag(\alpha\bar{\alpha}^{-1}, \alpha^{-1}\bar{\alpha})$, if φ is an anti-linear graded conjugation of $\mathfrak{g}$, either $\lambda = 1$ and $\alpha = \bar{\alpha}$, i.e. B is hermitian and $\alpha \in \mathbb{R}$, or $\lambda = -1$ and $\alpha = -\bar{\alpha}$, i.e. B is anti-hermitian and $\alpha \in i\mathbb{R}$. Statement (c) then follows from Lemma 1.10. □

Remark 3.8 If $\mathfrak{g}$ is a simple infinite-dimensional linearly compact Lie superalgebra, then $Aut\ \mathfrak{g}$ contains a maximal reductive subgroup which is explicitely described in [6, Theorem 4.2]. We shall denote this subgroup by G. For any anti-linear graded conjugation σ of $\mathfrak{g}$, let S be the subgroup of the group of linear and anti-linear automorphisms of $\mathfrak{g}$ generated by σ. Then $\mathcal{G} := SAut\ \mathfrak{g} = Aut\ \mathfrak{g} \cup \sigma Aut\ \mathfrak{g}$ is a real algebraic group. If σ normalizes G, i.e., $\sigma^{-1} G \sigma = G$, then SG is a subgroup of $\mathcal{G}$ and it is maximal reductive. Therefore any maximal reductive subgroup of $\mathcal{G}$ is conjugate into SG, in particular any finite order element of $\mathcal{G}$ is conjugate to an element of SG. If, moreover, $\sigma G \sigma = G$, then $SG = G \cup \sigma G$, hence any anti-linear graded conjugation of $\mathfrak{g}$ is conjugate to one in σG.

Example 3.9 The grading of type $(0, \dots, 0|1, -1)$ of $\mathfrak{g} = H(2k, 2)$ (resp. $K(2k+1, 2)$) is short. Let $A = \mathbb{C}[[p_1, \dots, p_k, q_1, \dots, q_k]]$ (resp. $A = \mathbb{C}[[t, p_1, \dots, p_k, q_1, \dots, q_k]]$). We have:

$\mathfrak{g}_{-1} = \langle \xi_2 \rangle \otimes A$,
$\mathfrak{g}_0 = (\langle 1, \xi_1\xi_2 \rangle \otimes A)/\mathbb{C}1$ (resp. $\langle 1, \xi_1\xi_2 \rangle \otimes A$),
$\mathfrak{g}_1 = \langle \xi_1 \rangle \otimes A$.

For $f \in A$, $f = \sum_{\substack{(a_0, \dots, a_k) \in \mathbb{Z}_+^{k+1} \\ (b_1, \dots, b_k) \in \mathbb{Z}_+^k}} \alpha_{a_0, \dots, b_k} t^{a_0} p_1^{a_1} \dots p_k^{a_k} q_1^{b_1} \dots q_k^{b_k}$, $\alpha_{a_0, \dots, b_k} \in \mathbb{C}$, we denote by $\bar{f}$ the element $\bar{f} = \sum_{\substack{(a_0, \dots, a_k) \in \mathbb{Z}_+^k \\ (b_1, \dots, b_k) \in \mathbb{Z}_+^k}} \overline{\alpha_{a_0, \dots, b_k}} t^{a_0} p_1^{a_1} \dots p_k^{a_k} q_1^{b_1} \dots q_k^{b_k} \in A$ (here $a_0 = 0$ if $A = \mathbb{C}[[p_1, \dots, p_k, q_1, \dots, q_k]]$). Then, for every linear involutive change φ of the even variables, multiplying by -1 the 1-form $\sum_{i=1}^k (p_i dq_i - q_i dp_i)$ (resp. $dt + \sum_{i=1}^k (p_i dq_i - q_i dp_i)$), the following maps $\Sigma_\varphi^\pm$ are anti-linear graded conjugations of $\mathfrak{g}$:

$$
\begin{array}{ll}
f(p_i, q_i) \mapsto -\bar{f}(\varphi(p_i), \varphi(q_i)) & (\text{resp. } f(t, p_i, q_i) \mapsto -\bar{f}(\varphi(t), \varphi(p_i), \varphi(q_i)) \\
f(p_i, q_i)\xi_1\xi_2 \mapsto -\bar{f}(\varphi(p_i), \varphi(q_i))\xi_1\xi_2 & (\text{resp. } f(t, p_i, q_i)\xi_1\xi_2 \mapsto -\bar{f}(\varphi(t), \varphi(p_i), \varphi(q_i))\xi_1\xi_2) \\
f(p_i, q_i)\xi_1 \mapsto \pm\bar{f}(\varphi(p_i), \varphi(q_i))\xi_2 & (\text{resp. } f(t, p_i, q_i)\xi_1 \mapsto \pm\bar{f}(\varphi(t), \varphi(p_i), \varphi(q_i))\xi_2) \\
f(p_i, q_i)\xi_2 \mapsto \mp\bar{f}(\varphi(p_i), \varphi(q_i))\xi_1 & (\text{resp. } f(t, p_i, q_i)\xi_2 \mapsto \mp\bar{f}(\varphi(t), \varphi(p_i), \varphi(q_i))\xi_1).
\end{array}
\tag{3.2}
$$

Example 3.10 Let $\mathfrak{g} = S(1, 2)$, $SHO(3, 3)$, or $SKO(2, 3; 1)$. Then the algebra of outer derivations of $\mathfrak{g}$ contains $sl_2 = \langle e, h, f \rangle$, with $e = \xi_1\xi_2 \frac{\partial}{\partial x}$ and $h = \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}$ if $\mathfrak{g} = S(1, 2)$, $e = \xi_1\xi_3 \frac{\partial}{\partial x_2} - \xi_2\xi_3 \frac{\partial}{\partial x_1} - \xi_1\xi_2 \frac{\partial}{\partial x_3}$ and $h = \sum_{i=1}^3 \xi_i \frac{\partial}{\partial \xi_i}$ if $\mathfrak{g} = SHO(3, 3)$, $e = \xi_1\xi_2\tau$ and $h = 1/2(\tau - x_1\xi_1 - x_2\xi_2)$ if $\mathfrak{g} = SKO(2, 3; 1)$. Let us denote by G_{out} the subgroup of $Aut\ \mathfrak{g}$ generated by $\exp(ad(e))$, $\exp(ad(f))$ and $\exp(ad(h))$. We recall that $G_{out} \subset G$, where G is the subgroup of $Aut\ \mathfrak{g}$ introduced in Remark 3.8 [6, Remark 2.2, Theorem 4.2]. We shall denote by U_- the one parameter group of automorphisms

$\exp(ad(tf))$, and by G_{inn} the subgroup of G consisting of inner automorphisms. Finally, H will denote the subgroup of $Aut\ \mathfrak{g}$ consisting of invertible changes of variables multiplying the volume form (resp. the even supersymplectic form) by a constant if $\mathfrak{g} = S(1,2)$ (resp. $\mathfrak{g} = SHO(3,3)$), or the odd supercontact form by a function if $\mathfrak{g} = SKO(2,3;1)$ (see [6, Theorem 4.5]).

The gradings of type $(0|1,1)$, $(0,0,0|1,1,1)$ and $(0,0|1,1,1)$ of $\mathfrak{g} = S(1,2)$, $SHO(3,3)$ and $SKO(2,3;1)$, respectively, are short, and the subspaces $\mathfrak{g}_i$'s are as follows:

$\mathfrak{g} = S(1,2)$:

$\mathfrak{g}_{-1} = \langle \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2} \rangle \otimes \mathbb{F}[[x]]$

$\mathfrak{g}_0 = \{f \in \langle \frac{\partial}{\partial x}, \xi_i \frac{\partial}{\partial \xi_j} | i,j = 1,2 \rangle \otimes \mathbb{F}[[x]], div(f) = 0\}$

$\mathfrak{g}_1 = \{f \in \langle \xi_i \frac{\partial}{\partial x}, \xi_1\xi_2 \frac{\partial}{\partial \xi_i} | i = 1,2 \rangle \otimes \mathbb{F}[[x]], div(f) = 0\}$.

$\mathfrak{g} = SHO(3,3)$:

$\mathfrak{g}_{-1} = \mathbb{C}[[x_1,x_2,x_3]]/\mathbb{C}1$

$\mathfrak{g}_0 = \{f \in \langle \xi_1, \xi_2, \xi_3 \rangle \otimes \mathbb{C}[[x_1,x_2,x_3]] | \Delta(f) = 0\}$

$\mathfrak{g}_1 = \{f \in \langle \xi_i\xi_j, i,j = 1,2,3 \rangle \otimes \mathbb{C}[[x_1,x_2,x_3]] | \Delta(f) = 0\}$.

$\mathfrak{g} = SKO(2,3;1)$:

$\mathfrak{g}_{-1} = \mathbb{C}[[x_1,x_2]]$

$\mathfrak{g}_0 = \{f \in \langle \xi_1, \xi_2, \tau \rangle \otimes \mathbb{C}[[x_1,x_2]] | div_1(f) = 0\}$

$\mathfrak{g}_1 = \{f \in \langle \tau\xi_i, \xi_1\xi_2 \mid i = 1,2 \rangle \otimes \mathbb{C}[[x_1,x_2]] | div_1(f) = 0\}$.

In all these cases the map $s = \exp(ad(e))\exp(ad(-f))\exp(ad(e))$ is a graded conjugation of $\mathfrak{g}$: for $z \in \mathfrak{g}_{-1}$, $s(z) = [e,z]$; for $z \in \mathfrak{g}_1$, $s(z) = -[f,z]$, for $z \in \mathfrak{g}_0$, $s(z) = z$. Note that each of the above gradings can be extended to $Der\ \mathfrak{g} = \mathfrak{g} \rtimes \mathfrak{a}$, with $\mathfrak{a} \supset sl_2$, so that e has degree 2, h has degree 0, and f has degree -2.

We denote by C_0 the standard complex conjugation of $W(m,n)$ defined as follows: for $X = \sum_{i=1}^m P_i(x,\xi)\frac{\partial}{\partial x_i} + \sum_{j=1}^n Q_j(x,\xi)\frac{\partial}{\partial \xi_j} \in W(m,n)$, $C_0(X) = \sum_{i=1}^m \overline{P_i(x,\xi)}\frac{\partial}{\partial x_i} + \sum_{j=1}^n \overline{Q_j(x,\xi)}\frac{\partial}{\partial \xi_j}$. Note that C_0 maps $S(1,2)$ (resp. $SHO(3,3)$, $SKO(2,3;1)$) into itself. Moreover, $C_0 \circ s = s \circ C_0$, hence, by Remark 3.3, $C_0 \circ s$ is an anti-linear graded conjugation of $S(1,2)$ (resp. $SHO(3,3)$, $SKO(2,3;1)$).

Lemma 3.11 *The Lie superalgebra $SKO(2,3;\beta)$ admits an anti-linear graded conjugation if and only if $|\beta| = 1$, $\beta \neq \pm 1$.*

Proof. Consider the Lie superalgebra $\mathfrak{g} = SKO(2,3;\beta)$ with its short consistent grading of type $(0,0|1,1,1)$. Then, by [6, Theorem 4.2], there is no (linear) automorphism of $\mathfrak{g}$ exchanging $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$. Note that the restriction to $SKO(2,3;\beta)$ of the standard complex conjugation C_0 defined at the end of Example 3.10, defines an anti-linear Lie superalgebra isomorphism $C_\beta : SKO(2,3;\beta) \to SKO(2,3;\overline{\beta})$ such that $C_\beta \circ C_{\bar{\beta}} = C_{\bar{\beta}} \circ C_\beta = id$. It follows that if $\beta \in \mathbb{R}$ and σ is an anti-linear graded conjugation of $SKO(2,3;\beta)$, then $C_\beta \circ \sigma$ is a (linear) automorphism of $SKO(2,3;\beta)$ exchanging g_1 and g_{-1}. Since such an automorphism does not exist, we conclude that if $\beta \in \mathbb{R}$, then $SKO(2,3;\beta)$ has no anti-linear graded conjugations.

By [5, Remark 4.15], if $\beta \neq \pm 1$, then $SKO(2,3;\beta)_{\bar{0}} \cong W(2,0)$ and $SKO(2,3;\beta)_{\bar{1}} \cong \Omega^0(2)^{-\frac{1}{\beta+1}} \oplus \Omega^0(2)^{-\frac{\beta}{\beta+1}}$. Recall that a vector field $X \in W(2,0)$ acts on $\Omega^0(2)^\lambda$ as follows: $X.f = X(f) + \lambda div(X) f$. If σ is an anti-linear graded conjugation of $\mathfrak{g}$, then $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are anti-isomorphic $\mathfrak{g}_0$-modules, hence $-\frac{1}{\beta+1} = -\frac{\beta}{\beta+1}$, i.e., $|\beta| = 1$.

Now suppose $|\beta| = 1$, $\beta \neq \pm 1$. Let v (resp. w) be the element in $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_1$) corresponding to 1, and let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be defined as follows: for $X \in \mathfrak{g}_{\bar{0}}$, $\sigma(X) = \overline{X}$; for $f \in \mathbb{C}[[x, y]]$, $\sigma(fv) = \bar{f}w$, $\sigma(fw) = -\bar{f}v$. Then σ is an anti-linear graded conjugation of $SKO(2, 3; \beta)$. Indeed, one easily checks that $\sigma([X, fv]) = [\sigma(X), \bar{f}w]$. □

Proposition 3.12 *The following is a complete list, up to equivalence, of anti-linear graded conjugations of all simple infinite-dimensional linearly compact Lie superalgebras* $\mathfrak{g}$ *with the* $\mathbb{Z}$*-gradings from Proposition 3.1(b):*

a) $\mathfrak{g} = H(2k, 2)$*:* $\Sigma_\varphi^\pm$ *are the anti-linear automorphisms of* $\mathfrak{g}$ *defined by (3.2).*

b) $\mathfrak{g} = K(2k + 1, 2)$*:* $\Sigma_\varphi^\pm$ *are the anti-linear automorphisms of* $\mathfrak{g}$ *defined by (3.2).*

c) $\mathfrak{g} = S(1, 2)$*:* $\tilde{\sigma} = C_0 \circ s \circ \exp(ad(\alpha h)) \circ \exp(ad(th')) \circ \varphi$ *where* C_0 *is the standard complex conjugation, s is the graded conjugation of* $\mathfrak{g}$ *introduced in Example 3.10,* $t \in i\mathbb{R}$, $h' = 2x\frac{\partial}{\partial x} + \xi_1\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial \xi_2}$, φ *is an element of the* SL_2*-subgroup of* G_{inn} *generated by* $exp(ad(\xi_1\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial \xi_2}))$, $exp(ad(\xi_1\frac{\partial}{\partial \xi_2}))$ *and* $exp(ad(\xi_2\frac{\partial}{\partial \xi_1}))$ *such that* $\overline{\varphi}\varphi = I_2$ *(resp.* $\overline{\varphi}\varphi = -I_2$*), and* $\exp(\alpha) = \pm 1$ *(resp.* $\exp(\alpha) = \pm i$*).*

d) $\mathfrak{g} = SHO(3, 3)$*:* $\tilde{\sigma} = C_0 \circ s \circ \exp(ad(\alpha h)) \circ \varphi$ *with* φ *such that* $\overline{\varphi}\varphi = 1$ *(resp.* $\overline{\varphi}\varphi = -1$*), lying in the subgroup of* $Aut\mathfrak{g}$ *generated by* G_{inn} *and* $\exp(ad(\Phi))$, $\Phi = \sum_{i=1}^n (-x_i\frac{\partial}{\partial x_i} + \xi_i\frac{\partial}{\partial \xi_i})$, $\exp(\alpha) = \pm 1$ *(resp.* $\exp(\alpha) = \pm i$*).*

e) $\mathfrak{g} = SKO(2, 3; 1)$*:* $\tilde{\sigma} = C_0 \circ s \circ \exp(ad(\alpha h)) \circ \varphi$*, where* C_0 *is the standard complex conjugation, s is the graded conjugation of* $\mathfrak{g}$ *introduced in Example 3.10,* $\exp(\alpha) = \pm 1$ *and* φ *is an element in* G_{inn} *such that* $\overline{\varphi}\varphi = I_2$*.*

f) $\mathfrak{g} = SKO(2, 3; \beta)$, $|\beta| = 1$, $\beta \neq \pm 1$*:* $\tilde{\sigma} = \sigma \circ \exp(ad(\alpha h)) \circ \varphi$*, where* σ *is defined as in Lemma 3.11,* $h = \frac{\tau - x_1\xi_1 - x_2\xi_2}{2}$*,* φ *is an element in* G_{inn} *such that* $\overline{\varphi}\varphi = I_2$ *and* $\exp(\alpha) = \pm 1$*.*

Proof. Let $\mathfrak{g} = H(2k, 2)$ with the grading of type $(0, \dots, 0|1, -1)$ (see Example 3.9), let Σ_φ be the graded conjugation of $\mathfrak{g}$ defined in Example 3.9, and let σ be an anti-linear graded conjugation of $\mathfrak{g}$. By [6, Theorem 4.2], the group $G = \mathbb{C}^\times(Sp_{2k} \times O_2)$ consists of linear changes of variables preserving the symplectic form up to multiplication by a non-zero scalar [6, Theorem 4.2], hence the map Σ_φ satisfies both conditions $\Sigma_\varphi^{-1} G \Sigma_\varphi = G$ and $\Sigma_\varphi G \Sigma_\varphi = G$. Then, by Remark 3.5 and Remark 3.8, we may assume, up to conjugation, that $\sigma = F \circ \Sigma_\varphi$ where F is a grading preserving automorphism of $\mathfrak{g}$ such that $(F \circ \Sigma_\varphi)^2(x_j) = (-1)^j x_j$ for $x_j \in \mathfrak{g}_j$, and $F \in G$.

Since F preserves the grading and lies in G, we have: $F(\xi_1) = a\xi_1$ and $F(\xi_2) = b\xi_2$, for some $a, b \in \mathbb{C}^\times$. Besides, the condition $(F \circ \Sigma_\varphi)^2(\xi_i) = -\xi_i$ implies $b = (\bar{a})^{-1}$.

It follows that, for $f \in \mathbb{C}[[p_i, q_i]]$, $F(f) = F([\xi_1, f\xi_2]) = [a\xi_1, F(f\xi_2)] = [a\xi_1, \tilde{f}\xi_2] = a\tilde{f}$ for some $\tilde{f} \in \mathbb{C}[[p_i, q_i]]$, i.e., $F(f\xi_2) = a^{-1}F(f)\xi_2$. Likewise, $F(f\xi_1) = \bar{a}F(f)\xi_1$ and $F(f\xi_1\xi_2) = F(f)\xi_1\xi_2$. Note that $F([\xi_1\xi_2, \xi_1] = F(\xi_1) = a\xi_1$ and $F([\xi_1\xi_2, \xi_1] = [F(\xi_1\xi_2), a\xi_1] = a^2\bar{a}^{-1}\xi_1$, hence $a \in \mathbb{R}$.

It follows that $F \circ \Sigma_\varphi$ is defined as follows:

$$
\begin{aligned}
&f(p_i, q_i) \mapsto -\bar{f}(F \circ \varphi(p_i), F \circ \varphi(q_i)) \\
&f(p_i, q_i)\xi_1\xi_2 \mapsto -\bar{f}(F \circ \varphi(p_i), F \circ \varphi(q_i))\xi_1\xi_2 \\
&f(p_i, q_i)\xi_1 \mapsto a\bar{f}(F \circ \varphi(p_i), F \circ \varphi(q_i))\xi_2 \\
&f(p_i, q_i)\xi_2 \mapsto -a^{-1}\bar{f}(F \circ \varphi(p_i), F \circ \varphi(q_i))\xi_1
\end{aligned}
\tag{3.3}
$$

for some linear change of even variables φ, such that $(F\circ\varphi)^2=1$. Besides, since $F(\xi_1)=a\xi_1$ and $F(\xi_2)=a^{-1}\xi_2$, F preserves the odd part $d\xi_1 d\xi_2$ of the symplectic form, hence it preserves the symplectic form. Finally, by Remark 1.2 we may assume $a=\pm1$. This concludes the proof of a). Similar arguments prove b).

Let $\mathfrak{g}=S(1,2)$. We already noticed in Example 3.10 that $\tilde{s}=C_0\circ s$ is an anti-linear graded conjugation of $\mathfrak{g}$. Here G_{inn} is generated by $exp(ad(h'))$, $exp(ad(\xi_1\frac{\partial}{\partial\xi_1}-\xi_2\frac{\partial}{\partial\xi_2}))$, $exp(ad(\xi_1\frac{\partial}{\partial\xi_2}))$ and $exp(ad(\xi_2\frac{\partial}{\partial\xi_1}))$ and $G=\mathbb{C}^\times SO_4$ is generated by G_{inn} and the automorphisms $\exp(ad(e))$, $\exp(ad(f))$ and $\exp(ad(h))$ (see Example 3.10 and [6, Theorem 4.2]). Therefore $\tilde{s}$ satisfies both relations $\tilde{s}^{-1}G\tilde{s}=G$ and $\tilde{s}G\tilde{s}=G$, i.e, it satisfies the hypotheses of Remark 3.8, hence every anti-linear graded conjugation of $\mathfrak{g}$ is conjugate to one in $\tilde{s}G$. By Remark 3.5 we therefore aim to classify all automorphisms $\psi\in G$ preserving the grading of $\mathfrak{g}$ of type $(0|1,1)$, such that $(\tilde{s}\circ\psi)^2(x_j)=(-1)^j x_j$ for $x_j\in\mathfrak{g}_j$.

By [6, Remark 4.6], if ψ is an automorphism of $\mathfrak{g}$ lying in G, then either $\psi\in U_-H\cap G$ or $\psi\in U_-sH\cap G$. Note that $U_-H\cap G=U_-(H\cap G)$ and $U_-sH\cap G=U_-s(H\cap G)$, since $U_-\subset G$ and $s\in G$ [6, Theorem 4.2]. Here $H\cap G$ is the subgroup of $Aut\,\mathfrak{g}$ generated by $\exp(ad(e))$, $\exp(ad(h))$ and G_{inn}. Note that $G_{inn}\subset\exp(ad(\mathfrak{g}_0))$.

Let $\psi\in U_-(H\cap G)$. Then $\psi=\exp(ad(tf))\psi_0$ for some $t\in\mathbb{C}$ and some $\psi_0\in H\cap G$. For $x\in\mathfrak{g}_1$, we have: $\psi(x)=\exp(ad(tf))(\psi_0(x))=\psi_0(x)+t[f,\psi_0(x)]$, since $\psi_0(x)\in\mathfrak{g}_1$. Since ψ preserves the grading, $t=0$, i.e., $\psi=\psi_0\in H\cap G$. We may hence write $\psi=\varphi_0\varphi_1$ for some $\varphi_1\in G_{inn}$ and some φ_0 lying in the subgroup generated by $\exp(ad(h))$ and $\exp(ad(e))$, and assume $\varphi_0=\exp(ad(\beta e))\exp(ad(\alpha h))$ for some $\alpha,\beta\in\mathbb{C}$. For $z\in\mathfrak{g}_{-1}$ we have: $\psi(z)=\varphi_0(\varphi_1(z))=exp(-\alpha)\exp(ad(\beta e))(\varphi_1(z))=\exp(-\alpha)(\varphi_1(z)+\beta[e,\varphi_1(z)])$, since $\varphi_1(z)\in\mathfrak{g}_{-1}$. Since ψ preserves the grading, we have $\beta=0$, i.e., $\psi=\exp(ad(\alpha h))\varphi_1$, hence

$$\tilde{\sigma}=\tilde{s}\circ\psi=C_0\circ s\circ\exp(ad(\alpha h))\circ\varphi_1.$$

We have:

$$\varphi_1\circ C_0=C_0\circ\overline{\varphi_1},$$

where, for $\varphi=\prod_{i=1}^k\exp(ad(z_i))$, $\overline{\varphi_1}=\prod_{i=1}^k\exp(ad(C_0(z_i)))$, and, for $\alpha\in\mathbb{C}$,

$$\exp(ad(\alpha h))\circ C_0=C_0\circ\exp(ad(\bar{\alpha}h)),\quad \exp(ad(\alpha h))\circ s=s\circ\exp(ad(-\alpha h)).$$

It follows that $(\tilde{\sigma})^2=(\tilde{s})^2\circ\exp(ad(\alpha-\bar{\alpha})h)\circ\overline{\varphi_1}\varphi_1$, hence

$$\exp(ad(\alpha-\bar{\alpha})h)\circ\overline{\varphi_1}\varphi_1=id. \tag{3.4}$$

In particular, $\exp(ad(\alpha-\bar{\alpha})h)\circ\overline{\varphi_1}\varphi_1(\frac{\partial}{\partial x})=\frac{\partial}{\partial x}$. Since φ_1 lies in G_{inn}, we may assume that $\varphi_1=\exp(ad(th'))\varphi_2=\varphi_2\exp(ad(th'))$ for some φ_2 in the SL_2-subgroup of G_{inn} generated by $exp(ad(\xi_1\frac{\partial}{\partial\xi_1}-\xi_2\frac{\partial}{\partial\xi_2}))$, $exp(ad(\xi_1\frac{\partial}{\partial\xi_2}))$ and $exp(ad(\xi_2\frac{\partial}{\partial\xi_1}))$, therefore $\overline{\varphi_1}\varphi_1=\exp(ad(t+\bar{t})h')\overline{\varphi_2}\varphi_2$. Since $\overline{\varphi_2}\varphi_2(\frac{\partial}{\partial x})=\frac{\partial}{\partial x}$, and $\exp(ad(\alpha-\bar{\alpha})h)(\frac{\partial}{\partial x})=\frac{\partial}{\partial x}$ we have: $\exp(ad(t+\bar{t})h')(\frac{\partial}{\partial x})=\exp(-2(t+\bar{t}))(\frac{\partial}{\partial x})$, hence $t\in i\mathbb{R}$.

If we restrict condition (3.4) to the sl_2-subalgebra of $\mathfrak{g}_0$ generated by $\xi_1\frac{\partial}{\partial\xi_2}$ and $\xi_2\frac{\partial}{\partial\xi_1}$, we find that $\overline{\varphi_2}\varphi_2$ lies in $\langle I_2\rangle\cap SL_2$, i.e., $\overline{\varphi_2}\varphi_2=\pm I_2$. Then, if we restrict condition (3.4) to the subspace $S=\langle\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2}\rangle\subset\mathfrak{g}_{-1}$, we get:

$$\exp(ad(\alpha-\bar{\alpha})h)\exp(ad(t+\bar{t})h')\overline{\varphi_2}\varphi_2|_S=id,$$

i.e., either $\overline{\varphi_2}\varphi_2 = I_2$ and $\exp(-(\alpha-\bar{\alpha})) = 1$, or $\overline{\varphi_2}\varphi_2 = -I_2$ and $\exp(-(\alpha-\bar{\alpha})) = -1$. Using Remark 1.2, we get statement (c).

Finally note that, since ψ preserves the grading, it cannot lie in $U_-s(H\cap G)$. Indeed, suppose that $\psi \in U_-s(H\cap G)$. Then we may assume that $\psi = \exp(ad(tf))\circ s\circ \exp(ad(\beta e))\exp(ad(\alpha h))\varphi_1$, for some $\alpha,\beta,t \in \mathbb{C}$ and some $\varphi_1 \in G_{inn}$. Suppose that $x \in \mathfrak{g}_1$. Then $\varphi_1(x) \in \mathfrak{g}_1$, since $G_{inn} \subset \exp(ad(\mathfrak{g}_0))$, hence $\exp(ad(\beta e))(\varphi_1(x)) = \varphi_1(x)$. It follows that $\psi(x) = \exp(\alpha)\exp(ad(tf))(s(\varphi_1(x)))$ $= \exp(\alpha)s(\varphi_1(x))$, since $s|_{\mathfrak{g}_1} : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$. Therefore $\psi|_{\mathfrak{g}_1} : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$, a contradiction.

Let $\mathfrak{g} = SHO(3,3)$ with the grading of type $(0,0,0|1,1,1)$. We already noticed in Example 3.10 that $\tilde{s} = C_0\circ s$ is an anti-linear graded conjugation of $\mathfrak{g}$. Here G_{inn} is generated by automorphisms $exp(ad(x_i\xi_j))$ with $i \neq j$, $\exp(ad(e))$, $\exp(ad(f))$, $\exp(ad(h))$ and $\exp(ad(\Phi))$ (see Example 3.10 and [6, Theorem 4.2]). Therefore $\tilde{s}$ satisfies both relations $\tilde{s}^{-1}G\tilde{s} = G$ and $\tilde{s}G\tilde{s} = G$, i.e, it satisfies the hypotheses of Remark 3.8, hence every anti-linear graded conjugation of $\mathfrak{g}$ is conjugate to one in $\tilde{s}G$. By Remark 3.5 we therefore need to classify all automorphisms $\psi \in G$ preserving the grading of $\mathfrak{g}$ of type $(0,0,0|1,1,1)$, such that $(\tilde{s}\circ\psi)^2(x_j) = (-1)^j x_j$ for $x_j \in \mathfrak{g}_j$.

By [6, Remark 4.6], if ψ is an automorphism of $\mathfrak{g}$ lying in G, then either $\psi \in U_-H\cap G$ or $\psi \in U_-sH\cap G$. Like in the case of $S(1,2)$, we have: $U_-H\cap G = U_-(H\cap G)$ and $U_-sH\cap G = U_-s(H\cap G)$, since $U_- \subset G$ and $s \in G$ [6, Theorem 4.2]. Here $H\cap G$ is the subgroup of $Aut\ \mathfrak{g}$ generated by $\exp(ad(e))$, $\exp(ad(h))$, $\exp(ad(\Phi))$ and G_{inn}. Note that $G_{inn} \subset \exp(ad(\mathfrak{g}_0))$.

Let $\psi \in U_-(H\cap G)$. Then $\psi = \exp(ad(tf))\psi_0$ for some $t \in \mathbb{C}$ and some $\psi_0 \in H\cap G$. For $x \in \mathfrak{g}_1$, we have: $\psi(x) = \exp(ad(tf))(\psi_0(x)) = \psi_0(x) + t[f,\psi_0(x)]$, since $\psi_0(x) \in \mathfrak{g}_1$. Since ψ preserves the grading, $t = 0$, i.e., $\psi = \psi_0 \in H\cap G$. We may hence write $\psi = \exp(ad(\gamma e))\varphi$ for some $\gamma \in \mathbb{C}$ and some φ in the subgroup of $Aut\mathfrak{g}$ generated by $\exp(ad(h))$, $\exp(ad(\Phi))$ and G_{inn}. For $z \in \mathfrak{g}_{-1}$ we have: $\psi(z) = \varphi(z) + \gamma[e,\varphi(z)]$, since $\varphi(z) \in \mathfrak{g}_{-1}$. Since ψ preserves the grading, we have $\gamma = 0$, i.e., $\psi = \exp(ad(\beta\Phi))\exp(ad(\alpha h))\varphi_0$ for some $\alpha,\beta \in \mathbb{C}$ and $\varphi_0 \in G_{inn}$, hence every anti-linear graded conjugation $\tilde{\sigma}$ of $\mathfrak{g}$ is of the form:

$$\tilde{\sigma} = \tilde{s}\circ\psi = C_0\circ s\circ\exp(ad(\beta\Phi))\circ\exp(ad(\alpha h))\circ\varphi_0.$$

We have:

$$\varphi_0\circ C_0 = C_0\circ\overline{\varphi_0},$$

where, for $\varphi = \prod_{i=1}^k \exp(ad(z_i))$, $\overline{\varphi} = \prod_{i=1}^k \exp(ad(C_0(z_i)))$. Besides, for $\alpha,\beta \in \mathbb{C}$,

$$\exp(ad(\alpha h))\circ C_0 = C_0\circ\exp(ad(\bar{\alpha}h)), \quad \exp(ad(\alpha h))\circ s = s\circ\exp(ad(-\alpha h));$$

$\exp(ad(\beta\Phi))\circ C_0 = C_0\circ\exp(ad(\bar{\beta}h))$, $\exp(ad(\beta\Phi))\circ s|_{\mathfrak{g}_k} = exp(-3k\beta)s\circ\exp(ad(\beta\Phi))|_{\mathfrak{g}_k}$ $(k=-1,0,1)$.

It follows that $(\tilde{\sigma})^2|_{\mathfrak{g}_k} = exp(-3k\bar{\beta})(\tilde{s})^2\circ\exp(ad(\beta+\bar{\beta})\Phi)\circ\exp(ad(\alpha-\bar{\alpha})h)\circ\overline{\varphi_0}\varphi_0|_{\mathfrak{g}_k}$, hence, since φ_0, $\overline{\varphi_0}$, $\exp(ad(h))$ and $\exp(ad(\Phi))$ preserve the subspace $\mathfrak{g}_k$ for every k,

$$exp(-3k\bar{\beta})\exp(ad(\beta+\bar{\beta})\Phi)\circ\exp(ad(\alpha-\bar{\alpha})h)\circ\overline{\varphi_0}\varphi_0|_{\mathfrak{g}_k} = id\ (k=-1,0,1). \tag{3.5}$$

Let us restrict condition (3.5) to the subalgebra S of $\mathfrak{g}_0$ generated by the elements $x_i\xi_j$ with $1 \leq i \neq j \leq 2$: $S \cong sl_3$. Then G_{inn} act on S via the Adjoint action and $\exp(ad(h))|_S = id_S = \exp(ad(\Phi))|_S$. Hence (3.5) becomes simply $\overline{\varphi_0}\varphi_0|_S = id$, i.e., $\overline{\varphi_0}\varphi_0 = \lambda I_3$ for $\lambda \in \mathbb{C}$ such that $\lambda^3 = 1$.

Now let us restrict condition (3.5) to the subspace W of $\mathfrak{g}_0$ spanned by the elements ξ_i, for $i = 1,2,3$. Then G_{inn} acts on W via the standard action and $\exp(ad(h))|_W = id_W$, $\exp(ad(\beta\Phi))|_W =$

$\exp(\beta)id_W$. Hence (3.5) becomes: $\lambda\exp(\beta+\bar{\beta})=1$. Since $\lambda^3=1$, it follows that $\lambda=1$ and $\beta\in i\mathbb{R}$. Therefore (3.5) reduces to the following relation:

$$exp(-3k\bar{\beta})\exp(ad(\alpha-\bar{\alpha})h)\circ\overline{\varphi_0}\varphi_0|_{\mathfrak{g}_k}=id\ (k=-1,0,1). \tag{3.6}$$

By Remark 1.2, we can assume $\alpha\in\{i\pi/2, i\pi, i3\pi/2, 2i\pi\}$, i.e., $\exp(\bar{\alpha}-\alpha)=\pm1$. It follows that if we restrict condition (3.6) to the subspace $V=\langle x_1,x_2,x_3\rangle\subset\mathfrak{g}_{-1}$, we find that either $\exp(3\bar{\beta})=1$ if $\exp(\bar{\alpha}-\alpha)=1$, or $\exp(3\bar{\beta})=-1$ if $\exp(\bar{\alpha}-\alpha)=-1$. Hence $\varphi:=\exp(ad(\beta\Phi))\circ\varphi_0$ satisfies statement d).

Finally note that, since ψ preserves the grading, it cannot lie in $U_{-}s(H\cap G)$. Indeed, suppose that $\psi\in U_{-}s(H\cap G)$. Then we may assume that $\psi=\exp(ad(tf))\circ s\circ\varphi$, for some $t\in\mathbb{C}$ and some $\varphi\in H\cap G$. Suppose that $x\in\mathfrak{g}_1$. Then $\varphi(x)\in\mathfrak{g}_1$, hence $s(\varphi(x))\in\mathfrak{g}_{-1}$. It follows that $\psi(x)=\exp(ad(tf))(s(\varphi(x)))\in\mathfrak{g}_{-1}$. Therefore $\psi|_{\mathfrak{g}_1}:\mathfrak{g}_1\to\mathfrak{g}_{-1}$, a contradiction.

The argument for $\mathfrak{g}=SKO(2,3;1)$ and $\mathfrak{g}=SKO(2,3;\beta)$ with $\beta\neq1$ is the same as for $\mathfrak{g}=S(1,2)$. □

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