A FAMILY OF MAPS WITH MANY SMALL FIBERS

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Abstract. The waist inequality states that for a continuous map from S^n to $\mathbb{R}^q$, not all fibers can have small $(n-q)$-dimensional volume. We construct maps for which most fibers have small $(n-q)$-dimensional volume and all fibers have bounded $(n-q)$-dimensional volume.

Let $n, q \in \mathbb{N}$ with $n > q \geq 1$, and let $f : S^n \to \mathbb{R}^q$ be a continuous map. Let $\widehat{p} : \mathbb{R}^{n+1} \to \mathbb{R}^q$ be a surjective linear map, and let $p = \widehat{p}|_{S^n}$. The waist inequality states that the largest fiber of f is at least as large as the largest fiber of p:

$$\sup_{y\in\mathbb{R}^q} \operatorname{Vol}_{n-q} f^{-1}(y) \geq \sup_{y\in\mathbb{R}^q} \operatorname{Vol}_{n-q} p^{-1}(y).$$

See [1], [3], [4], and [6] for proofs of the waist inequality, or [5] for a survey. In the case $q = 1$, the waist inequality is a consequence of the isoperimetric inequality on S^n. The isoperimetric inequality can also be used to prove that the portion of S^n covered by small fibers of f is not very big; that is, for all ε, we have

$$\operatorname{Vol}_n \ f^{-1}\{y : \operatorname{Vol}_{n-q} f^{-1}(y) < \varepsilon\} \leq \operatorname{Vol}_n \ p^{-1}\{y : \operatorname{Vol}_{n-q} p^{-1}(y) < \varepsilon\}.$$

The theorem presented in this paper describes how the same statement does not hold in the case $q > 1$. We have also included an appendix with a more precise statement of the waist inequality and the isoperimetric inequality.

Theorem 1. *For every $n, q \in \mathbb{N}$ with $n > q > 1$, and for every $\varepsilon > 0$, there is a continuous map $f : S^n \to \mathbb{R}^q$ such that all but ε of the n-dimensional volume of S^n is covered by fibers that have $(n-q)$-dimensional volume at most ε. Moreover, we may require that every fiber of f has $(n-q)$-dimensional volume bounded by $C_{n,q}$, a constant not depending on ε.*

In what follows, $I^n = [0,1]^n$ denotes the n-dimensional unit cube, and ∂I^n denotes its boundary. A *tree* refers to the topological space corresponding to a graph-theoretic tree: topologically, a tree is a finite 1-dimensional simplicial complex that is contractible.

The bulk of the construction comes from the following lemma, in which we construct a preliminary "tree map" $t_{n,r,\delta}$ from I^n to a tree. Later, to construct f we will change the domain from I^n to S^n by gluing several tree maps together, and we will change the range from the tree to $\mathbb{R}^q$ by composing with a map from a thickened tree to $\mathbb{R}^q$. In the tree map $t_{n,r,\delta}$, the parameter r corresponds to the depth of the tree. As r increases, the typical fiber of the map becomes smaller. The parameter δ corresponds to the total volume of the larger fibers.

Lemma 1. *For every $n, r \in \mathbb{N}$, there is a rooted tree $T_{n,r}$ such that for every $\delta > 0$ there is a continuous map $t_{n,r,\delta} : I^n \to T_{n,r}$ with the following properties:*

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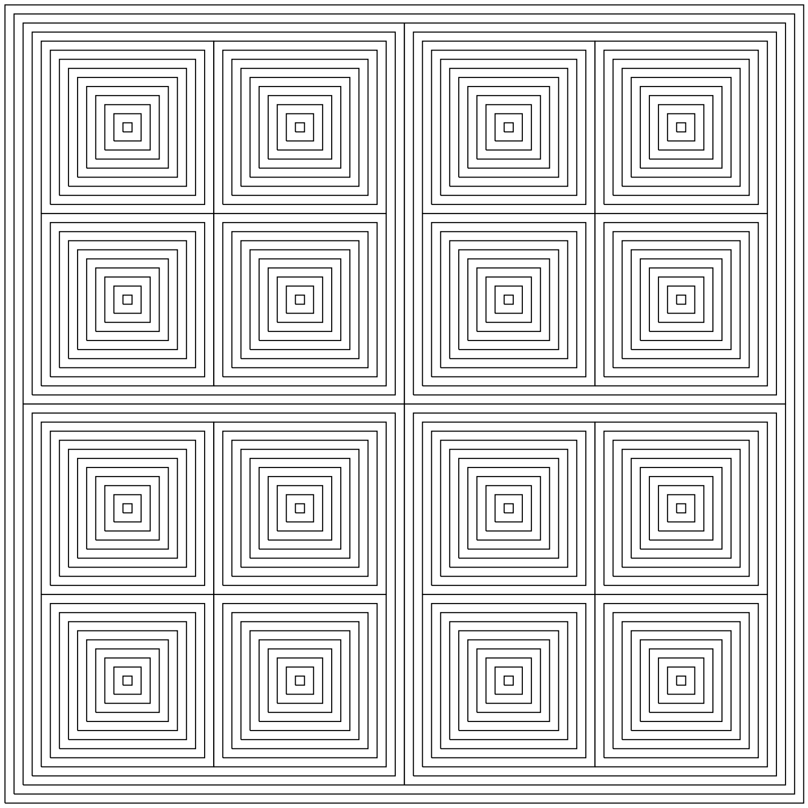

FIGURE 1. Every fiber of $t_{2,2,\delta}$ has length at most 6, and most fibers have length at most 1.

(1) *Every fiber of $t_{n,r,\delta}$ is either a single point, the boundary of an n-dimensional cube of side length at most 1, or the $(n-1)$-skeleton of a $2 \times 2 \times \cdots \times 2$ array of n-dimensional cubes each of side length at most $\frac{1}{2}$.*
(2) *All but δ of the volume of I^n is covered by fibers of $t_{n,r,\delta}$ that are boundaries of n-dimensional cubes of side length at most 2^{-r}.*
(3) *$t_{n,r,\delta}(\partial I^n)$ is a single point, the root of $T_{n,r}$.*
(4) *Each vertex has at most 2^n daughter vertices.*

Proof. We construct the tree and tree map recursively in r. For $r = 0$, the tree $T_{n,0}$ is a single edge which we may identify with the interval $[0, \frac{1}{2}]$, with 0 being the root. For any δ, we set $t_{n,0,\delta}(x) = \mathrm{dist}(x, \partial I^n)$ for all $x \in I^n$.

Now let $r > 0$. To construct $T_{n,r}$, we take the disjoint union of one copy of $[0, 1]$ and 2^n copies of $T_{n,r-1}$, and identify the root of every copy of $T_{n,r-1}$ with $1 \in [0, 1]$. The root of $T_{n,r}$ is $0 \in [0, 1]$. We define $t_{n,r,\delta}$ piecewise as follows. For some small choice of $\delta_1 > 0$, we define $t_{n,r,\delta}$ on the closed δ_1-neighborhood of ∂I^n to $[0, 1] \subset T_{n,r}$ by

$$t_{n,r,\delta}(x) = \frac{1}{\delta_1}\mathrm{dist}(x, \partial I^n).$$

Then, translating the coordinate hyperplanes to pass through the center of I^n we divide the remainder of the cube into a $2 \times 2 \times \cdots \times 2$ array of cubes $Q_1, \ldots, Q_{2^n}$ each of side length slightly less than $\frac{1}{2}$. For each $j = 1, \ldots 2^n$, let $\lambda_j : Q_j \to I^n$ be the map that scales Q_j up to unit size, and let $i_j : T_{n,r-1} \to T_{n,r}$ be the inclusion of the jth copy of $T_{n,r-1}$ into $T_{n,r}$. Then for some small choice of $\delta_2 > 0$, we put

$$t_{n,r,\delta}|_{Q_j} = i_j \circ t_{n,r-1,\delta_2} \circ \lambda_j.$$

Properties 1, 3, and 4 are easily satisfied by the construction. To ensure property 2, we need to choose δ_1 and δ_2. The volume of I^n that is covered by large fibers—fibers not equal to the boundary of a cube of side length at most 2^{-r}—is at most $\delta_1 \cdot 2n + 2^n \cdot \delta_2 \cdot 2^{-n}$, because the area of ∂I^n is $2n$ and because the portion of each Q_j that is covered by large fibers has volume at most $\delta_2 \cdot \mathrm{Vol}(Q_j) < \delta_2 \cdot 2^{-n}$. Thus we may choose $\delta_1 = \frac{\delta}{4n}$ and $\delta_2 = \frac{\delta}{2}$. $\square$

Proof of Theorem 1. We may replace S^n by ∂I^{n+1} by composing with the (bi-Lipschitz) homeomorphism $\psi : S^n \to \partial I^{n+1}$ given by lining up the centers of S^n and ∂I^{n+1} in $\mathbb{R}^{n+1}$ and projecting radially. We start by constructing a tree T and a tree map $t : \partial I^{n+1} \to T$. For some large choice of r, let T be the tree obtained by identifying the roots of $2(n+1)$ copies of $T_{n,r}$, one for each n-dimensional face of ∂I^{n+1}. For some small choice of δ, define t on each n-dimensional face of ∂I^{n+1} to be the composition of $t_{n,r,\delta}$ with the inclusion of the corresponding $T_{n,r}$ into T.

The fibers of t have dimension $n-1$. In order to cut the fibers down to dimension $n-q$, we next construct a projection map $p : \partial I^{n+1} \to \mathbb{R}^{q-1}$ such that the fibers of p intersect the fibers of t transversely. The fibers of t have codimension 2 in $\mathbb{R}^{n+1}$ and are aligned with the standard coordinates, so we achieve transversality by using other linear coordinates to construct p. We choose $q-1$ linearly independent vectors $v_1, \ldots, v_{q-1} \in \mathbb{R}^{n+1}$ such that for every two standard basis vectors $e_i, e_j \in \mathbb{R}^{n+1}$ the spaces $\mathrm{span}\{e_i, e_j\}^\perp$ and $\mathrm{span}\{v_1, \ldots, v_{q-1}\}^\perp$ intersect transversely; equivalently, the set $e_i, e_j, v_1, \ldots, v_{q-1}$ is linearly independent. For $k = 1, \ldots, q-1$, define the kth component of p to be the dot product of the input with v_k. Then the fibers of $t \times p : \partial I^{n+1} \to T \times \mathbb{R}^{q-1}$ are codimension $q-1$ transverse linear cross-sections of the $(n-1)$-dimensional fibers of t, and have $(n-q)$-dimensional volume bounded by some constant depending on n and q.

There exists M large enough that $p(\partial I^{n+1})$ is contained in the $(q-1)$-dimensional ball $B(M)$ of radius M. We define a map $\phi : T \times B(M) \to \mathbb{R}^q$ such that the number of points in each fiber of ϕ is at most the maximum degree of T, which is $2^n + 1$. Then we define $f = \phi \circ (t \times p)$. The fibers of f, like the fibers of $t \times p$, have $(n-q)$-dimensional volume bounded by a constant $C_{n,q}$.

The map ϕ is constructed as follows. Let $\phi|_{T \times \{0\}}$ be an embedding of T into $\mathbb{R}^q$ in which the edges map to straight line segments and each daughter vertex has x_1-coordinate greater than that of its parent. Let d be the minimum distance between disjoint edges of $\phi(T \times \{0\})$. Then for every $p \in T$ and $x \in B(M)$, we set

$$\phi(p, x) = \phi(p, 0) + \frac{d}{4}\left(0, \frac{x}{M}\right),$$

where $\left(0, \frac{x}{M}\right)$ denotes the point in $\mathbb{R}^q$ constructed by adding onto $\frac{x}{M} \in \mathbb{R}^{q-1}$ a first coordinate of 0. If $\phi(p, x) = \phi(p', x')$, then $\phi(p, 0)$ and $\phi(p', 0)$ are at most $\frac{d}{2}$ apart, so p and p' lie on two incident edges of T; also, $\phi(p, 0)$ and $\phi(p', 0)$ have the same

x_1-coordinate, so these two edges are between two daughters and a common parent, rather than a daughter, a parent, and a grandparent.

To finish the proof, we show that δ and r may be chosen such that all but ε of the n-dimensional volume of ∂I^{n+1} is covered by fibers with $(n-q)$-dimensional volume at most ε. The maximum number of daughter vertices of any vertex of T is 2^n, and most of ∂I^{n+1} is covered by fibers of f that are unions of at most 2^n codimension $q-1$ transverse linear cross-sections of boundaries of n-dimensional cubes of side length at most 2^{-r}. We choose r large enough that every codimension $q-1$ transverse linear cross-section of $2^{-r}\partial I^n$ has $(n-q)$-dimensional volume at most $\frac{\varepsilon}{2^n}$. The volume of the portion of ∂I^{n+1} covered by larger fibers is at most $2(n+1)\cdot\delta$, so we choose $\delta < \frac{\varepsilon}{2(n+1)}$. □

Appendix: The waist inequality and the isoperimetric inequality

In order to be precise about the waist inequality, we need a notion of $(n-q)$-dimensional volume of arbitrary closed subsets in S^n. Gromov's version of the waist inequality is stated in terms of the Lebesgue measures Vol_n of the ε-neighborhoods $f^{-1}(y)_\varepsilon$ of the fibers $f^{-1}(y)$ of a continuous map f.

Theorem 2 (Waist inequality, [4])**.** *Let $f : S^n \to \mathbb{R}^q$ be a continuous map. Then there exists a point $y \in \mathbb{R}^q$ such that for all $\varepsilon > 0$, we have*

$$\mathrm{Vol}_n(f^{-1}(y)_\varepsilon) \geq \mathrm{Vol}_n(S^{n-q}_\varepsilon),$$

where $S^{n-q} \subset S^n$ denotes an equatorial $(n-q)$-sphere.

The paper [6] gives a detailed exposition of the proof of the waist inequality and fills in some small gaps in the original argument. For convenience we introduce a notation for comparing the ε-neighborhoods of two sets: given $E, F \subseteq S^n$, we say that E is ***larger in neighborhood*** than F, denoted $E \geq_{\mathrm{nbd}} F$, if for all $\varepsilon > 0$ we have

$$\mathrm{Vol}_n(E_\varepsilon) \geq \mathrm{Vol}_n(F_\varepsilon).$$

Then the waist inequality states that for some $y \in \mathbb{R}^q$ we have $f^{-1}(y) \geq_{\mathrm{nbd}} S^{n-q}$.

In the case $q = 1$, we would like to say that the waist inequality is a consequence of the isoperimetric inequality. The classical isoperimetric inequality applies only to regions with smooth boundary, so we need the following version, which is stated and proved in [2] and attributed to [7]:

Theorem 3 (Isoperimetric inequality)**.** *Let $A \subseteq S^n$ be a closed set and $B \subseteq S^n$ be a closed ball with* $\mathrm{Vol}_n(B) = \mathrm{Vol}_n(A)$*. Then we have*

$$A \geq_{\mathrm{nbd}} B.$$

In the introduction we claimed that in the case $q = 1$, the isoperimetric inequality could be used to prove, in addition to the waist inequality, another statement about the volume of S^n covered by small fibers. Here we formulate the statement more precisely and prove it. The proof implies the waist inequality for $q = 1$.

Theorem 4. *Let $f : S^n \to \mathbb{R}$ be a continuous map, and $p : S^n \to \mathbb{R}$ be the restriction to S^n of a surjective linear map $\widehat{p} : \mathbb{R}^{n+1} \to \mathbb{R}$. Then for all $y \in p(S^n)$, we have*

$$\mathrm{Vol}_n\{x \in S^n : f^{-1}(f(x)) \geq_{\mathrm{nbd}} p^{-1}(y)\} \geq \mathrm{Vol}_n\{x \in S^n : p^{-1}(p(x)) \geq_{\mathrm{nbd}} p^{-1}(y)\}.$$

The proof of this theorem is based on the following lemma:

Lemma 2. *Let $X, Y \subset S^n$ be closed sets with $X \cup Y = S^n$. Let $B^X, B^Y \subset S^n$ be closed balls such that their two centers are antipodal in S^n and $\mathrm{Vol}_n(B^X) = \mathrm{Vol}_n(X)$ and $\mathrm{Vol}_n(B^Y) = \mathrm{Vol}_n(Y)$. Then we have*

$$X \cap Y \geq_{\mathrm{nbd}} B^X \cap B^Y.$$

Proof. First we claim that $(X \cap Y)_\varepsilon$ is the disjoint union of $X_\varepsilon \setminus X$, $Y_\varepsilon \setminus Y$, and $X \cap Y$. It is clear that $(X \cap Y)_\varepsilon$ is the disjoint union of its intersections with $S^n \setminus X$, $S^n \setminus Y$, and $X \cap Y$. Thus it suffices to show that

$$(X \cap Y)_\varepsilon \cap (S^n \setminus X) = X_\varepsilon \setminus X.$$

Because $(X \cap Y)_\varepsilon \subseteq X_\varepsilon$, we immediately have

$$(X \cap Y)_\varepsilon \cap (S^n \setminus X) \subseteq X_\varepsilon \setminus X.$$

For the reverse inclusion, let $y \in X_\varepsilon \setminus X$, and let $\gamma : [0, 1] \to S^n$ be a curve of length at most ε with $\gamma(0) = y$ and $\gamma(1) = x \in X$. Let $t \in [0, 1]$ be the greatest value with $\gamma(t) \in Y$. Then $\gamma(t) \in X \cap Y$, so $y \in (X \cap Y)_\varepsilon$.

Thus, applying the isoperimetric inequality and additivity of measure, we have

$$\begin{aligned}
\mathrm{Vol}_n((X \cap Y)_\varepsilon) &= \mathrm{Vol}_n(X_\varepsilon) - \mathrm{Vol}_n(X) + \mathrm{Vol}_n(Y_\varepsilon) - \mathrm{Vol}_n(Y) + \mathrm{Vol}_n(X \cap Y) \geq \\
&\geq \mathrm{Vol}_n(B^X_\varepsilon) - \mathrm{Vol}_n(B^X) + \mathrm{Vol}_n(B^Y_\varepsilon) - \mathrm{Vol}_n(B^Y) + \mathrm{Vol}_n(B^X \cap B^Y) = \\
&= \mathrm{Vol}_n((B^X \cap B^Y)_\varepsilon).
\end{aligned}$$

□

Proof of Theorem 4. Without loss of generality we assume $p(S^n) = [0, 1]$ and $y \leq \frac{1}{2}$. Then on the right-hand side of the desired inequality we have

$$\{x \in S^n : p^{-1}(p(x)) \geq_{\mathrm{nbd}} p^{-1}(y)\} = p^{-1}[y, 1 - y].$$

Define $\alpha, \beta \in \mathbb{R}$ as

$$\alpha = \sup\{t \in \mathbb{R} : \mathrm{Vol}_n f^{-1}(-\infty, t) \leq \mathrm{Vol}_n p^{-1}[0, y]\},$$

$$\beta = \inf\{t \in \mathbb{R} : \mathrm{Vol}_n f^{-1}(t, \infty) \leq \mathrm{Vol}_n p^{-1}[y, 1]\}.$$

For each $t \in [\alpha, \beta]$, apply the lemma with $X = f^{-1}(-\infty, t]$ and $Y = f^{-1}[t, \infty)$ to get $f^{-1}(t) \geq_{\mathrm{nbd}} p^{-1}[y_1, y_2]$ for some $y_1, y_2 \in [y, 1 - y]$. In particular, we have

$$f^{-1}(t) \geq_{\mathrm{nbd}} p^{-1}(y_1) \geq_{\mathrm{nbd}} p^{-1}(y).$$

Thus, we have

$$f^{-1}[\alpha, \beta] \subseteq \{x \in S^n : f^{-1}(f(x)) \geq_{\mathrm{nbd}} p^{-1}(y)\}.$$

Because $\mathrm{Vol}_n f^{-1}(-\infty, \alpha) \leq \mathrm{Vol}_n p^{-1}[0, y]$ and $\mathrm{Vol}_n f^{-1}(\beta, \infty) \leq \mathrm{Vol}_n p^{-1}[y, 1]$ we have

$$\mathrm{Vol}_n f^{-1}[\alpha, \beta] \geq \mathrm{Vol}_n p^{-1}[y, 1 - y].$$

□

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