arXiv:1202.3680v2 [math.PR] 14 Feb 2014

Record-dependent measures on the symmetric groups

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25.09.13

Abstract

A probability measure P_n on the symmetric group $\mathfrak{S}_n$ is said to be record-dependent if $P_n(\sigma)$ depends only on the set of records of a permutation $\sigma \in \mathfrak{S}_n$. A sequence $P = (P_n)_{n \in \mathbb{N}}$ of consistent record-dependent measures determines a random order on $\mathbb{N}$. In this paper we describe the extreme elements of the convex set of such P. This problem turns out to be related to the study of asymptotic behavior of permutation-valued growth processes, to random extensions of partial orders, and to the measures on the Young-Fibonacci lattice.

1 Introduction

Let $\mathfrak{S}_n$ be the group of permutations of $[n] := \{1, \ldots, n\}$. A position $j \in [n]$ is called an *upper record position* or simply a *record* in a permutation $\sigma \in \mathfrak{S}_n$ if $\sigma(j) = \max_{i \in [j]} \sigma(i)$. Let $R(\sigma) \subset [n]$ be the set of records of σ. A probability measure P_n on $\mathfrak{S}_n$ is called *record-dependent* (RD) if P_n is conditionally uniform given the set of records, or, equivalently, if the probability mass function $P_n(\sigma)$ depends only on $R(\sigma)$.

A natural way to connect permutations of different sizes is suggested by viewing a generic permutation $\sigma \in \mathfrak{S}_n$ as a (total) order on $[n]$, in which i precedes j if i appears in a lower position, that is $\sigma^{-1}(i) < \sigma^{-1}(j)$. Restricting the order to the smaller set $[n-1]$ yields a projection $\pi_{n-1}^n : \mathfrak{S}_n \to \mathfrak{S}_{n-1}$, by

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which each permutation σ is mapped to a permutation which we call *coherent* with σ. Likewise, two probability measures, P_n on $\mathfrak{S}_n$ and P_{n-1} on $\mathfrak{S}_{n-1}$, are said to be coherent if the restriction sends P_n to P_{n-1}. It turns out that for coherent measures if P_n is RD, so is P_{n-1}.

In this paper we are interested in coherent sequences of RD-measures $P = (P_n)_{n \in \mathbb{N}}$. Each such sequence defines a Markovian permutation growth process, with backward transition probabilities like under the uniform distributions. On the other hand, a coherent sequence uniquely determines a probability measure, denoted by the same symbol P, on a '$n = \infty$' object, which is the space of orders on $\mathbb{N}$. The measure P will be called an RD-measure, meaning that the projection of P to each $[n]$ is RD. Our main results (Theorem 3.3 and Proposition 3.5) explicitly characterize the extreme elements of the convex set of such RD-measures P. The characterization problem belongs to the circle of de Finetti-type questions around sufficiency and stochastic symmetries [1, 14], and can be viewed in different contexts like the boundary problem for a branching scheme [16] or processes on causal posets [2, 3].

A straightforward example of an RD-measure on $\mathfrak{S}_n$ is

$$P_n(\sigma) = \frac{1}{Z}\, \theta^{|R(\sigma)|}, \quad \sigma \in \mathfrak{S}_n, \tag{1}$$

where $\theta \in (0, \infty)$ and $Z = \theta(\theta + 1) \cdots (\theta + n - 1)$. For $\theta = 1$ this is the uniform distribution. By the 'fundamental bijection', which identifies the one-row notation for writing a permutation with the cycle notation for another permutation, this measure is mapped to the well-known Ewens distribution with weights $\frac{1}{Z}\theta^{|\mathrm{cycles}(\sigma)|}$ [19, 7]. However, unless $\theta = 1$ the measures (1) are not coherent in the sense of the present paper. A qualitative difference appears if we look how the number of records $|R(\sigma)|$ grows with n. For (1) the order of growth is logarithmic, while for the measures studied here the right scale for $|R(\sigma)|$ is linear, with the case of the uniform distribution being the sole exception.

The case of the uniform distribution is special among the RD measures. The uniform distribution on $\mathfrak{S}_n$ is important in statistics, as it appears by ranking samples from continuous distributions. In this connection various quantities, such as record times, record values, interrecord times and others attracted lots of attention. We refer the reader to the review [4] for classical results on the theory of records.

In the literature there is a number of other permutation growth models build on principles similar to ours: the probability mass function depends, for each n, on a statistic S, while coherence of random permutations of different sizes is defined via a system of projections. When S is the cycle type of a

permutation, the model can be embedded into Kingman's theory of exchangeable partitions [16, 25]. When S is the set of descents, a coherent sequence of random permutations is associated with a random order on $\mathbb{N}$ which has the property of spreadability [9]. Also Pitman's partially exchangeable partitions [24] and random sequences of compositions of integers [17, 7] can be recast in terms of coherent random permutations. Furthermore, certain parametric deformations of the uniform distribution fit in the framework, with S being a numerical statistic like the number of cycles, descents, pikes, inversions, etc (see [10, 7] for examples and references).

Both the choice of a statistic S and the choice of a system of projections connecting symmetric groups affect properties of the permutation growth model. If we replace $|R(\sigma)|$ in (1) by the number of *lower* records, the measures will become coherent under our projections π_{n-1}^n. On the other hand, changing the system of projections one can achieve the coherence of distributions (1). The alternative projections were considered in [7] for permutations with distribution depending on statistics of both upper and lower records.

Comparing with the previous work, the main distinction of the present setting is in the structure of the set of extreme RD-measures P. In e.g. [7, 16, 17, 24] the extremes are described in terms of infinitely many continuous parameters, which are asymptotic frequencies (on a linear scale) of certain components of a growing permutation. In contrast to that, in our model the parametrization of the extreme RD-measures involves an integer sequence and a real number.

Somewhat unexpectedly, a parametrization of extremes similar to ours has appeared in the work of Goodman and Kerov [12]. They studied random growth processes on the Young-Fibonacci lattice, which is an important example of a differential poset (as introduced by Stanley [26] and Fomin [6]), and proved a result very much in line with our Theorem 3.3. Although we see some further similarities with the setting of [12], the connection remains obscure, and it would be very interesting to have more clarity in this point.

A prototypical instance of differential poset is Young's lattice. The study of coherent measures on Young's lattice, that is growth processes with values in Young diagrams, is a deeply explored subject related, in particular, to the theory of group representations, total positivity of matrices, and asymptotics of Schur symmetric functions, see [16, 27] and references therein. In particular, the identification of the extreme coherent measures is equivalent to the classification of the characters and finite factor representations of the infinite symmetric group $\mathfrak{S}_\infty$. Our results on the RD-measures can be also interpreted in this spirit, as the classification of traces on the AF–algebra associated with the branching scheme (Bratteli diagram) of permutations.

Finally, we mention that some of the extreme RD-measures can be viewed

in the context of order-invariant measures on fixed causal sets, as introduced recently by Brightwell and Luczak [3]. We add details to this aspect of our study in Section 7.

2   Record-dependent measures and orders on $\mathbb{N}$

We write permutations $\sigma \in \mathfrak{S}_n$ in the one-row notation as permutation words $\sigma(1) \ldots \sigma(n)$. Note that position 1 is the smallest and $\sigma^{-1}(n)$ is the largest element of the set of records $R(\sigma)$. For instance, permutation $\sigma = 2\,6\,5\,7\,1\,4 \in \mathfrak{S}_7$ has records $R(\sigma) = \{1, 2, 4\}$.

With $\sigma \in \mathfrak{S}_n$ one associates an order on $[n]$, in which letter i precedes j if i appears in a lower position, meaning that $\sigma^{-1}(i) < \sigma^{-1}(j)$. The restriction of the order to $[n-1]$ yields a projection $\pi_{n-1}^n : \mathfrak{S}_n \to \mathfrak{S}_{n-1}$, which amounts to removing letter n from the permutation word. Explicitly, for $i \in [n-1]$

$$\pi_{n-1}^n(\sigma)(i) = \begin{cases} \sigma(i), & \text{if } i < \sigma^{-1}(n), \\ \sigma(i+1), & \text{if } i \geq \sigma^{-1}(n). \end{cases}$$

For example, π_3^4 sends $3\,4\,1\,2$ to $3\,1\,2$. More generally, the iterated projection $\pi_m^n : \mathfrak{S}_n \to \mathfrak{S}_m$ is defined for $1 \leq m < n$ as the operation of deleting letters $m+1, \ldots, n$ from the permutation word $\sigma(1) \ldots \sigma(n)$.

Let $\mathcal{O}$ be the projective limit of the symmetric groups $\mathfrak{S}_n$ taken together with projections π_{n-1}^n. Thus, an element of $\mathcal{O}$ is a coherent sequence $(\sigma_n)_{n \in \mathbb{N}}$, which has $\sigma_n \in \mathfrak{S}_n$ and $\sigma_{n-1} = \pi_{n-1}^n(\sigma_n)$ for $n > 1$. Let π_n^∞ denote the coordinate map sending $(\sigma_1, \sigma_2, \ldots) \in \mathcal{O}$ to $\sigma_n \in \mathfrak{S}_n$. The coherence of permutations immediately implies the following statement.

Proposition 2.1. *$\mathcal{O}$ is in bijection with the set of total orders on $\mathbb{N}$. The projection π_n^∞ amounts to restricting the order from $\mathbb{N}$ to $[n]$.*

In what follows we identify elements of $\mathcal{O}$ with the orders corresponding to them. We endow each $\mathfrak{S}_n$ with the discrete topology, and endow $\mathcal{O}$ with the product topology of projective limit, which corresponds to the coordinate-wise convergence. In this topology $\mathcal{O}$ is a compact totally disconnected space.

Let $\mathcal{M}(\mathcal{O})$ be the space of Borel probability measures on $\mathcal{O}$. Each measure $P \in \mathcal{O}$ has marginal measures $P_n := \hat{\pi}_n^\infty(P)$ which satisfy the coherence condition $P_{n-1} = \hat{\pi}_{n-1}^n(P_n)$ for $n > 1$, where and henceforth $\hat{f}$ denotes the pushforward of measures under a mapping f. Conversely, by Kolmogorov's measure extension theorem each coherent sequence (P_n) determines a unique measure on $\mathcal{O}$.

Let $\mathcal{M}_R(\mathfrak{S}_n)$ be the set of RD-measures on $\mathfrak{S}_n$, as defined in the Introduction. We call $P \in \mathcal{M}(\mathcal{O})$ an RD-measure if the marginal measures satisfy $P_n \in \mathcal{M}_R(\mathfrak{S}_n)$ for every n. We denote $\mathcal{M}_R(\mathcal{O})$ the convex set of such RD-measures, and denote $\text{ext}\, \mathcal{M}_R(\mathcal{O})$ the set of extreme elements of $\mathcal{M}_R(\mathcal{O})$.

The permutation statistic $R(\sigma)$ is consistent with projections $\hat{\pi}^n_{n-1}$ in the following sense.

Lemma 2.2. *If P_n is an RD-measure on $\mathfrak{S}_n$ then its projection $P_{n-1} := \hat{\pi}^n_{n-1}(P_n)$ is an RD-measure on $\mathfrak{S}_{n-1}$.*

Proof. Recall that for $\sigma \in \mathfrak{S}_n$, the position $\sigma^{-1}(n)$ is the maximal element of $R(\sigma)$. Note that σ can be uniquely recovered from $\sigma^{-1}(n)$ and $\pi^n_{n-1}(\sigma)$. Thus, for $\tau \in \mathfrak{S}_{n-1}$ and $A \subset [n]$ the number k of permutations σ which satisfy $R(\sigma) = A$ and $\tau = \pi^n_{n-1}(\sigma)$ depends only on A and $B := R(\tau) \subset [n-1]$. Specifically, $k = 1$ if for some $j \in [n]$ the set A can be obtained by the following deletion-insertion operation: delete from B all elements greater than $j - 1$ then insert j in the remaining set. Otherwise $k = 0$. For instance, taking $n = 7$ and $B = \{1, 3, 5\}$ we have $k = 1$ for

$$A = \{1\}, \{1, 2\}, \{1, 3\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 3, 5, 6\}, \{1, 3, 5, 7\}.$$

The assertion follows since

$$P_{n-1}(\tau) = \sum_{A \subset [n]} \quad \sum_{\sigma: \pi^n_{n-1}(\sigma) = \tau, R(\sigma) = A} P_n(\sigma)$$

depends only on B. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

An alternative coordinatization of permutations is sometimes useful. For a permutation σ define r_i as the rank of $\sigma(i)$ among $\sigma(1), \ldots, \sigma(i)$. That is to say, $r_i = k$ if $\sigma(i)$ is the kth smallest element in $\{\sigma(1), \ldots, \sigma(i)\}$. We call the r_i's *ranks* (other terminology found in the literature is 'relative ranks' or 'initial ranks'). The correspondence $\sigma \mapsto (r_1, \ldots, r_n)$ is a bijection between $\mathfrak{S}_n$ and $[1] \times \cdots \times [n]$. Under the uniform distribution on $\mathfrak{S}_n$ the ranks are independent random variables, with r_i uniformly distributed on $[i]$.

Remark. The representation of permutations by rank sequences does not sit well with the projections π^n_{n-1}, which in terms of the ranks r_i are rather involved. In [7] other projections $(r_1, \ldots, r_n) \mapsto (r_1, \ldots, r_{n-1})$ were used to study measures (1) and their generalizations.

For $\rho \subset [n]$ satisfying $1 \in \rho$, let P^ρ be the *elementary measure*, which is the uniform distribution on the set of permutations $\{\sigma \in \mathfrak{S}_n : R(\sigma) = \rho\}$.

Note that P^ρ is a product measure in the rank coordinates: the r_i's are independent, r_i is uniformly distributed on $[i-1]$ for $i \notin \rho$, and $r_i = i$ almost surely for $i \in \rho$. The set of RD-measures $\mathcal{M}_R(\mathfrak{S}_n)$ is a simplex with 2^{n-1} extreme elements P^ρ.

The convex set $\mathcal{M}_R(\mathcal{O})$ is a projective limit of the finite-dimensional simplices $\mathcal{M}_R(\mathfrak{S}_n)$. By the general theory (see e.g. [11]) $\mathcal{M}_R(\mathcal{O})$ is a Choquet simplex, i.e. a convex compact set with the property of uniqueness of the representation of a generic point as a convex mixture of the elements of $\text{ext} \, \mathcal{M}_R(\mathcal{O})$. In view of this property it is important to determine the set of extreme RD-measures $\text{ext} \, \mathcal{M}_R(\mathcal{O})$.

3 Constructions of the extreme RD-measures

A natural concept of 'a uniformly distributed random order' on $\mathbb{N}$ is the probability measure $P^* \in \mathcal{M}_R(\mathcal{O})$ whose projection to $\mathfrak{S}_n$ is the uniform distribution for every n. A characteristic feature of P^* is exchangeability, that is the invariance under bijections of $\mathbb{N}$. The order can be neatly constructed in terms of a sequence (ξ_i) of independent random variables uniformly distributed on the unit interval, by letting i to precede j iff $\xi_i < \xi_j$. It is clear from this construction that the exchangeable order is almost surely dense and has neither maximal, nor minimal elements. Thus, $\mathbb{N}$ with this order is isomorphic (as an ordered space) to $(\mathbb{Q}, <)$ P^*-almost surely, as it follows from the classical characterization of dense orders due to Hausdorff (see [13, Section III.11]).

Under the uniform distribution on $\mathfrak{S}_n$ the number of records satisfies $|R(\sigma)|/\log n \to 1$ in probability, as is well known [4]. It will be clear from what follows that P^* is the only RD-measure with sublinear growth of $|R(\sigma)|$ as $n \to \infty$. We introduce next a family of random orders for which the number of records is asymptotically linear in n. The idea is to exploit the ranks as in the construction of extreme elements of $\mathcal{M}_R(\mathfrak{S}_n)$. To that end, we need some preliminaries.

We define an infinite permutation as a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and denote $\mathfrak{S}$ the set of such bijections. The symmetric group $\mathfrak{S}_n$ is naturally embedded in $\mathfrak{S}$ as the set of bijections that satisfy $\sigma(j) = j$ for $j > n$, and the infinite symmetric group is identified with $\mathfrak{S}_\infty = \cup_{n=1}^\infty \mathfrak{S}_n$. Each $\sigma \in \mathfrak{S}$ defines an order on $\mathbb{N}$ by the familiar rule: i precedes j iff $\sigma^{-1}(i) < \sigma^{-1}(j)$. This order is of the type of the ordered set $(\mathbb{N}, <)$, with the jth minimal element being $\sigma(j)$. By the virtue of this correspondence $\mathfrak{S}$ is embedded in $\mathcal{O}$. Note that the support of the exchangeable order is disjoint with $\mathfrak{S}$, i.e. $P^*(\mathfrak{S}) = 0$.

With $\sigma \in \mathfrak{S}$ we associate an infinite sequence of ranks $r_1, r_2, \ldots$, where

r_i is the rank of $\sigma(i)$ among $\{\sigma(1), \ldots, \sigma(i)\}$. More generally, for $i \leq j$ let $r_{i,j}$ be the rank of $\sigma(i)$ among $\{\sigma(1), \ldots, \sigma(j)\}$. The bivariate array $r_{i,j}$ is determined by the diagonal entries r_i by the virtue of the recursion

$$r_{i,i} = r_i, \quad r_{i,j+1} = r_{i,j} + 1(r_{j+1} \leq r_{i,j}), \tag{2}$$

where $1('C')$ is 1 when the condition $'C'$ is true and is 0 otherwise. Moreover, the sequence $r_{i,i}, r_{i,i+1}, \ldots$ is nondecreasing and eventually stabilizes at the value

$$\sigma(i) = \lim_{j \to \infty} r_{i,j}. \tag{3}$$

Thus, $\sigma \in \mathfrak{S}$ is uniquely determined by the sequence of ranks (r_i). This correspondence suggests a criterion to identify the sequences of ranks corresponding to infinite permutations.

Lemma 3.1. *A sequence $(r_i)_{i \in \mathbb{N}}$ with $r_i \in [i]$ defines a bijection $\sigma \in \mathfrak{S}$ iff for every i the nondecreasing sequence $r_{i,i}, r_{i,i+1}, \ldots$ defined recursively by (2) is bounded. In this case σ is given by (3).*

Now let $\alpha = (\alpha_k)$ be a strictly increasing sequence of positive integers. For notational convenience we assume that the sequence is infinite, but our considerations also apply to finite sequences with obvious modifications. Assume that

$$\sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty. \tag{4}$$

Let $P^{(\alpha,1)}$ be the product measure on $[1] \times [2] \times \cdots$ which makes the coordinates r_i independent and satisfying

(i) $r_i \equiv i$ for $i \notin \{\alpha_1 + 1, \alpha_2 + 1, \ldots\}$,

(ii) r_i is uniformly distributed on $[i - 1]$ for $i \in \{\alpha_1 + 1, \alpha_2 + 1, \ldots\}$.

Lemma 3.2. *A random sequence (r_i) with distribution $P^{(\alpha,1)}$ almost surely determines, via (2) and (3), a random element of $\mathfrak{S}$.*

Proof. Fix i and condition on the event $r_i = s$. If $i \notin \{\alpha_1, \alpha_2, \ldots\}$ then $r_{i,i+1} = s$. If $i \in \{\alpha_1, \alpha_2, \ldots\}$ then the expected value of $r_{i,i+1}$ is $s + s/i$. Iterating we see that the expected value of $r_{i,j}$ converges, as $j \to \infty$, to

$$s \prod_{k \,:\, \alpha_k \geq i} \left(1 + \frac{1}{\alpha_k}\right),$$

which is finite in view of (4). Therefore, Fatou's lemma implies that $r_{i,j}$ is bounded in j and Lemma 3.1 can be applied. $\square$

Using the correspondence between the rank sequences (r_i) and the infinite permutations we consider $P^{(\alpha,1)}$ as a measure on $\mathcal{O}$ supported by $\mathfrak{S}$. Similarly to $\mathfrak{S}_n$, for $\sigma \in \mathfrak{S}$ we define position j to be a record if $\sigma(j) = \max_{i \in [j]} \sigma(i)$. Then under $P^{(\alpha,1)}$ the records are positions not of the kind $\alpha_k + 1$. Note that if the sequence α is finite, then $P^{(\alpha,1)}$ is supported by $\mathfrak{S}_\infty$.

The dual algorithm There is a dual stochastic algorithm that produces $P^{(\alpha,1)}$ via the entries of the inverse infinite permutation $\sigma^{-1}(1), \sigma^{-1}(2), \cdots$. Note that the position of integer 1 belongs to $\{1, \alpha_1 + 1, \alpha_2 + 1, \dots\}$, and that in terms of ranks we have $\sigma^{-1}(1) = \max\{i : r_i = 1\}$. Hence, introducing a random variable ν_1 with distribution

$$\mathrm{Prob}\{\nu_1 = 0\} = \prod_{m=1}^{\infty} \left(1 - \frac{1}{\alpha_m}\right), \tag{5}$$

$$\mathrm{Prob}\{\nu_1 = k\} = \frac{1}{\alpha_k} \prod_{m=1}^{\infty} \left(1 - \frac{1}{\alpha_{k+m}}\right), \quad k = 1, 2, \dots, \tag{6}$$

the position of 1 can be defined as $\sigma^{-1}(1) = y_1$, where

$$y_1 = 1(\nu_1 = 0) + (\alpha_{\nu_1} + 1)1(\nu_1 \neq 0).$$

Given the value $\sigma^{-1}(1)$, define a new sequence α' by the following rules

(i) if $\sigma^{-1}(1) = 1$ and $\alpha_1 \geq 2$ then $\alpha'_k = \alpha_k - 1$ for $k \geq 1$,

(ii) if $\sigma^{-1}(1) = 1$ and $\alpha_1 = 1$ then $\alpha'_k = \alpha_{k+1} - 1$ for $k \geq 1$,

(iii) if $\sigma^{-1}(1) = \alpha_i + 1$ then $\alpha'_k = \alpha_k$ for $k < i$ and $\alpha'_k = \alpha_{k+1} - 1$ for $k \geq i$.

Let ν_2 be distributed as in the right-hand side of (5), (6) but with α' in place of α. Finally, let $\sigma^{-1}(2)$ be the y_2th element of $\mathbb{N} \setminus \{\sigma^{-1}(1)\}$ for $y_2 = 1(\nu_2 = 0) + (\alpha_{\nu_2} + 1)1(\nu_2 \neq 0)$. Then we iterate on $\mathbb{N} \setminus \{\sigma^{-1}(1), \sigma^{-1}(2)\}$ and so on.

In Section 6 we will give a more direct proof that all positions eventually get filled, hence the output of the dual algorithm is indeed a random permutation $\sigma^{-1} \in \mathfrak{S}$.

Finally, we construct a larger family of RD-measures by interpolating between P^* and $P^{(\alpha,1)}$'s. Fix α satisfying (4) and $0 < p \leq 1$. Split $\mathbb{N}$ in two infinite subsets N_1 and N_2 by assigning each integer independently to N_1 with probability p and to N_2 with probability $1 - p$. Using increasing bijections we can identify N_1 and N_2 with two copies of $\mathbb{N}$. We construct an order by requiring that every $i \in N_1$ precedes every $j \in N_2$ and (using the

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identifications with $\mathbb{N}$) by ordering N_1 according to $P^{(\alpha,1)}$ and ordering N_2 according to P^*. The distribution of the resulting order is denoted $P^{(\alpha,p)}$.

Let Ω denote the set comprised of a point $*$ and of pairs (α, p), where α is a strictly increasing sequence of positive integers satisfying (4), and $0 < p \leq 1$. The space Ω is a topological cone obtained by collapsing one face of the cylinder $\{\alpha\} \times [0, 1]$ in the point $*$. In this topology the convergence to (α, p) is component-wise, and the convergence to $*$ means that the p-component goes to 0. For a generic point ω of Ω (either $*$ or some (α, p)), P^ω will denote the corresponding measure.

Theorem 3.3. *Measures P^ω with $\omega \in \Omega$ comprise the set $\operatorname{ext}\mathcal{M}_R(\mathcal{O})$ of the extreme RD-measures. The topology on Ω agrees with the topology of weak convergence of measures on $\mathcal{O}$.*

Taken together with the uniqueness property of Choquet simplex, this result implies:

Corollary 3.4. *For every $P \in \mathcal{M}_R(\mathcal{O})$ there exists a unique probability measure μ on Ω, such that*

$$P = \int_\Omega P^\omega \mu(d\omega).$$

The next result is a law of large numbers for the extreme measures P^ω.

Proposition 3.5. *Let O be a random element of $\mathcal{O}$ distributed according to $P^{(\alpha,p)}$. Then $P^{(\alpha,p)}$-almost surely*

(i) *for every k the kth non-record position in $\pi_n^\infty(O)$ converges to $\alpha_k + 1$ as $n \to \infty$,*

(ii)
$$\liminf_{n \to \infty} \frac{(\pi_n^\infty(O))^{-1}(n)}{n} = p.$$

The proofs of Theorem 3.3 and Proposition 3.5 are postponed to later sections.

Remark 1. If α_k is not defined, then under convergence to $\alpha_k + 1$ we mean that if for large n the kth non-record position in $\pi_n^\infty(O)$ exists, then it converges to $+\infty$.

Remark 2. Under $P^{(\alpha,p)}$ the number of records is asymptotically linear in n, so that $|R(\pi_n^\infty(O))|/(np) \to 1$ in probability. Under P^* the position of n in $\pi_n^\infty(O)$ has uniform distribution on $[n]$, hence, relation (ii) holds with $p = 0$.

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4   The branching graph representation

In this section we recast the setting of RD-measures on permutations within a general formalism of central measures on branching graphs [16, 18].

The succession of permutations of different sizes and their record sets is representable in the form of an infinite graded graph $\mathcal{R}$. It is convenient to encode each admissible $\rho \subset [n]$ into a binary word $\rho(1)\ldots\rho(n)$ starting with $\rho(1) = 1$. For instance, $\{1, 3, 4\} \subset [5]$ becomes 10110. Let $\mathcal{R}_n$ denote the set of all 2^{n-1} such binary words of length n.

Consider a graded graph $\mathcal{R}$ with the set of vertices $\bigcup_{n=1}^{\infty} \mathcal{R}_n$, and with edges connecting vertices on neighboring levels according to the rule: two vertices $\rho = \rho(1)\ldots\rho(n) \in \mathcal{R}_n$ and $\tau = \tau(1)\ldots\tau(n+1) \in \mathcal{R}_{n+1}$ are connected by an edge, denoted $\rho \nearrow \tau$, if there exists $k \in [n+1]$ such that

(i) $\rho(i) = \tau(i)$ for $i < k$,

(ii) $\tau(k) = 1$,

(iii) $\tau(i) = 0$ for $i > k$.

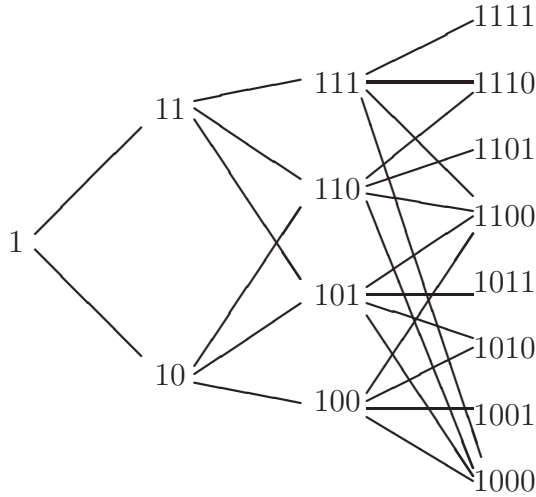The first four levels of $\mathcal{R}$ are shown in Figure 1.

Figure 1: The first four levels of graph $\mathcal{R}$.

A (standard) path in $\mathcal{R}$ is a sequence of vertices (ρ_i) such that $\rho_i \in \mathcal{R}_i$ and $\rho_i \nearrow \rho_{i+1}$. Let Γ be the set of infinite paths $\rho_1 \nearrow \rho_2 \nearrow \ldots$, and let Γ_n be the set of paths $\rho_1 \nearrow \ldots \nearrow \rho_n$ of length n. We view Γ as the projective

limit of finite sets Γ_n, and we equip Γ with the usual topology of projective limit of discrete spaces. Recall that $R(\sigma) \in \mathcal{R}_n$ for $\sigma \in \mathfrak{S}_n$.

Proposition 4.1. *The map*

$$\Phi_n : \sigma \rightarrow (R(\pi_j^n(\sigma)), \ j \in [n])$$

(where π_n^n is the identity map) is a bijection between $\mathfrak{S}_n$ and Γ_n. Similarly, the map

$$\Phi : O \rightarrow (R(\pi_n^\infty(O)), n \in \mathbb{N})$$

is a homeomorphism between $\mathcal{O}$ and Γ.

Proof. Let $\sigma_n = \pi_n^\infty(O)$. As in the proof of Lemma 2.2, $\sigma_n^{-1}(n)$ is uniquely determined by $R(\sigma_n)$ and $R(\sigma_{n-1})$. On the other hand, σ_n is uniquely determined by $\sigma_j^{-1}(j)$, $j \in [n]$, by the virtue of a correspondence analogous to the bijection between $\mathfrak{S}_n$ and the sequence of n ranks. $\square$

Identifying finite paths with permutations, and infinite paths with orders on $\mathbb{N}$ we use the same symbols as above for measures and projections. For instance, π_n^∞ denotes the projection $\Gamma \rightarrow \Gamma_n$ which cuts the tail of a path up to the first n terms.

A probability measure P_n on Γ_n is called *central* if the probability of a path $\rho_1 \nearrow \rho_2 \cdots \nearrow \rho_n$, depends only on ρ_n, i.e. all paths with fixed endpoint ρ_n are equiprobable. Similarly, a probability measure P on Γ is central if all its projections $P_n = \hat{\pi}_n^\infty(P)$ on Γ_n are central. Let $\mathcal{M}_C(\Gamma)$ denote the space of all central measures on Γ. We remark that each $P \in \mathcal{M}_C(\Gamma)$ is associated with a random walk which moves along the paths in $\mathcal{R}$ and has standard (not depending on P) backward transition probabilities determined by the condition of centrality, see [16] for more details.

Now Proposition 4.1 implies the following statement.

Proposition 4.2. *The map $\widehat{\Phi}$ is a an affine isomorphism of the convex sets $\mathcal{M}_R(\mathcal{O})$ and $\mathcal{M}_C(\Gamma)$.*

Let $\rho_n \in \mathcal{R}_n$. The elementary measure P^{ρ_n} is a unique central measure on Γ_n supported by the set of paths of length n with endpoint ρ_n. Under the bijection between $\mathfrak{S}_n$ and Γ_n, the elementary measure corresponds to the uniform distribution on the set of permutations $\sigma \in \mathfrak{S}_n$ with fixed records $R(\sigma) = \rho_n$. The next standard fact (see e.g. [22, Proposition 10.8] or [5, Theorem 1.1]) is the main technical tool to identify the set $\mathcal{M}_C(\Gamma)$.

Lemma 4.3. *Let P be an extreme point of the convex set $\mathcal{M}_C(\Gamma)$. Then for P-almost all paths $\rho_1 \nearrow \rho_2 \nearrow \dots$ in $\mathcal{R}$*

$$\lim_{n \to \infty} \hat{\pi}_k^n(P^{\rho_n})(A) = \hat{\pi}_k^\infty(P)(A), \tag{7}$$

for all $A \subset \Gamma_k$ and $k \in \mathbb{N}$.

The family of probability measures $P \in \mathcal{M}_C(\mathcal{R})$ representable as limits (7) of elementary measures along some paths (ρ_n) is called the *Martin boundary* of the graph $\mathcal{R}$. (We remark that sometimes the Martin boundary is defined as a larger set of limits along arbitrary sequences (ρ_n).) By Lemma 4.3 the set of extremes $\text{ext}\,\mathcal{M}_C(\Gamma)$ is a part of the Martin boundary. Convergence (7) is the same as the weak convergence of projections on every Γ_k. With this in mind we simply write $P^{\rho_n} \to P$. The boundary problem has a straightforward reformulation in terms of permutations.

5 The Martin boundary identification

Theorem 5.1. *Let $(\rho_n) \in \Gamma$ be a path such that the elementary RD-measures on $\mathfrak{S}_n$ weakly converge, i.e. $P^{\rho_n} \to P$. Then $P = P^\omega$ for some $\omega \in \Omega$. Thus, the Martin boundary of $\mathcal{R}$ can be identified with Ω.*

To prove the result we need a number of auxiliary propositions.

For $\rho \in \mathcal{R}_n$ with k zeros let $\ell_1(\rho) < \ell_2(\rho) < \cdots < \ell_k(\rho)$ be the positions of zeros listed in increasing order,

$$\{\ell_1(\rho), \dots, \ell_k(\rho)\} = \{i : \rho(i) = 0\},$$

and define

$$L(\rho) := \prod_{i=1}^k \left(1 - \frac{1}{\ell_i(\rho) - 1}\right).$$

The following algorithm produces a random permutation with distribution P^ρ. Let $m_1 > m_2 > \cdots > m_{n-k} = 1$ be the positions of 1's listed in decreasing order. Since $\sigma^{-1}(n)$ is the largest record we must have $\sigma(m_1) = n$. Next, $\sigma(m_1 + 1), \dots, \sigma(n)$ is an equiprobable sample without replacement from $\{1, \dots, n-1\} = \{1, \dots, n\} \setminus \{\sigma(m_1)\}$. Furthermore, $\sigma(m_2)$ is the maximal element of $\{1, \dots, n\} \setminus \{\sigma(m_1), \dots, \sigma(n)\}$, thus there is only one choice for $\sigma(m_2)$ after $\sigma(m_1), \dots, \sigma(n)$ have been determined. Now $\sigma(m_3 + 1), \dots, \sigma(m_2 - 1)$ is an equiprobable sample from $\{1, \dots, n\} \setminus \{\sigma(m_1), \dots, \sigma(n)\} \setminus \{\sigma(m_2)\}$. The process is continued until all positions are filled.

The above algorithm for sampling permutations from P^ρ readily implies the following.

Proposition 5.2. *Let $\rho \in \mathcal{R}_n$ and let σ be a random permutation from $\mathfrak{S}_n$ distributed according to P^ρ. The random variable $\sigma^{-1}(1)$ has the following distribution:*

$$
P^\rho(\sigma^{-1}(1) = h) = \begin{cases} \prod_{i=1}^{k}\left(1 - \frac{1}{\ell_i - 1}\right), & \text{if } h = 1, \\ \frac{1}{\ell_j - 1}\prod_{i=j+1}^{k}\left(1 - \frac{1}{\ell_i - 1}\right), & \text{if } h = \ell_j, \\ 0, & \text{otherwise.} \end{cases}
$$

Proposition 5.3. *Let $\rho \in \mathcal{R}_n$, and let σ be a random permutation from $\mathfrak{S}_n$ distributed according to P^ρ. For $2 \le t \le n$ the conditional distribution of $\sigma^{-1}(t)$ given $\sigma^{-1}(1) = s_1, \ldots, \sigma^{-1}(t-1) = s_{t-1}$ is*

$$
P^\rho(\sigma^{-1}(t) = h \mid \sigma^{-1}(1) = s_1, \ldots, \sigma^{-1}(t-1) = s_{t-1}) =
$$

$$
\begin{cases} \prod_{i=1}^{k'}\left(1 - \frac{1}{\ell_i' - 1 - w(\ell_i')}\right), & \text{if } h = \min(\{1,\ldots,n\}\setminus\{s_1,\ldots,s_{t-1}\}), \\ \frac{1}{\ell_j' - 1 - w(\ell_j')}\prod_{i=j+1}^{k'}\left(1 - \frac{1}{\ell_i' - 1 - w(\ell_j')}\right), & \text{if } h = \ell_j', \\ 0, & \text{otherwise,} \end{cases}
$$

where $\ell_1' < \cdots < \ell_{k'}'$ satisfy

$$
\{\ell_1', \ldots, \ell_{k'}'\} = \{\ell_1, \ldots, \ell_k\}\setminus\{s_1, \ldots, s_{t-1}\}
$$

and

$$
w(x) = |\{s_1, \ldots, s_{t-1}\} \cap \{1, \ldots, x-1\}|.
$$

Proposition 5.4. *If a path $(\rho_n) \in \Gamma$ satisfies $L(\rho_n) \to 0$, then $P^{\rho_n} \to P^*$.*

Proof. Observe that if for $\sigma \in \mathfrak{S}_n$ positions of $1, \ldots, m$ are not records, i.e. if $\{\sigma^{-1}(1), \ldots, \sigma^{-1}(m)\} \subset [n] \setminus R(\sigma)$, then the set of records remains unaltered when the positions of $1, \ldots, m$ are exchanged. Therefore, under the RD-measure P^{ρ_n} the permutation $\pi_m^n(\sigma)$ is uniformly distributed given $\{\sigma^{-1}(1), \ldots, \sigma^{-1}(m)\} \subset [n] \setminus R(\sigma)$. Finally, by Propositions 5.2 and 5.3 if $L(\rho_n) \to 0$, then

$$
P^{\rho_n}(\sigma^{-1}(m) \notin R(\sigma)) \to 1,
$$

hence $\hat{\pi}_m^n(P^{\rho_n})$ converges to the uniform distribution on $\mathfrak{S}_m$, for every m. $\qquad\square$

Proposition 5.5. *Let $(\rho_n) \in \Gamma$ be a path such that $P^{\rho_n} \to P$ and $P \ne P^*$. Then there exists a $0-1$ sequence $\rho_\infty = (\rho_\infty(1), \rho_\infty(2), \ldots)$ such that*

$$
\lim_{n\to\infty} \rho_n(i) = \rho_\infty(i)
$$

for every i.

Proof. Suppose that for some i the sequence $\rho_n(i)$ does not converge. Then for infinitely many n_j we have $\rho_{n_j}(i) = 0$ and $\rho_{n_j-1}(i) = 1$. Then, since $\rho_{n_j-1} \nearrow \rho_{n_j}$, we have

$$\rho_{n_j}(i) = \rho_{n_j}(i+1) = \cdots = \rho_{n_j}(n_j) = 0.$$

Therefore, $L(\rho_{n_j}) \to 0$ as $j \to \infty$ and Proposition 5.4 implies that $\widehat{\pi}_k^{n_j}(P^{\rho_{n_j}})$ converges to the uniform measure on $\mathfrak{S}_k$, so $P^{\rho_{n_j}} \to P^*$ which is a contradiction. $\qquad\square$

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. If $P^{\rho_n} \to P^*$ then there is nothing to prove. Otherwise by Propositions 5.4 and 5.5, passing if necessary to a subsequence, we have as $n \to \infty$

1. $L(\rho_n) \to p_1$ for some $0 < p_1 \le 1$,

2. $\lim_{n \to \infty} \rho_n(i) = \rho_\infty(i)$ for some $0-1$ sequence $\rho_\infty = (\rho_\infty(1), \rho_\infty(2), \dots)$.

   Let $\ell_1(\rho_\infty) < \ell_2(\rho_\infty) < \dots$ be positions of zeros in ρ_∞:

$$\{\ell_1(\rho_\infty), \ell_2(\rho_\infty), \dots\} = \{i : \rho_\infty(i) = 0\}.$$

Set $\alpha_i = \ell_i(\rho_\infty) - 1$ for all i such that $\ell_i(\rho_\infty)$ is defined. Observe that convergence of $L(\rho_n)$ entails that $\prod_i(1 - 1/\alpha_i)$ converges to some p_2 with $p_1 \le p_2 \le 1$. Now set $p = p_1/p_2$. We claim that

$$P_k = \widehat{\pi}_k^\infty(P^{(\alpha,p)}).$$

The claim is shown by comparing the description of $P^{(\alpha,p)}$ via the dual algorithm given in Section 3) with the description of the elementary measures P^ρ given in Propositions 5.2 and 5.3. $\qquad\square$

Corollary 5.6. *The measures P^ω, $\omega \in \Omega$, are record-dependent.*

Proof. Indeed, by Theorem 5.1 they are weak limits of record-dependent measures. $\qquad\square$

6 The laws of large numbers

In this section we exploit the algorithmic description of measures $P^{(\alpha,p)}$ to prove Proposition 3.5 and to finish the proof of Theorem 3.3.

First, suppose that $p = 1$ and fix a sequence α such that $\sum_{i=1}^\infty 1/\alpha_i < \infty$. Recall, that the dual algorithm for $P^{(\alpha,1)}$ constructs successively the entries $\sigma^{-1}(1), \sigma^{-1}(2), \dots$ of the inverse permutation $\sigma^{-1} : \mathbb{N} \to \mathbb{N}$.

Lemma 6.1. *For every $\varepsilon > 0$ there exist constants $C > 1$ and n_0 such that the estimate*

$$P^{(\alpha,1)}(\sigma^{-1}(k) > Cn \mid \sigma^{-1}(1) = s_1, \ldots, \sigma^{-1}(k-1) = s_{k-1}) < \varepsilon. \quad (8)$$

holds for $n > n_0$, $k \le n$ and arbitrary distinct $s_1, \ldots, s_{k-1}$.

Proof. For shorthand, we write Q for the conditional probability in (8). As follows from the description of the dual algorithm in Section 3,

$$P^{(\alpha,1)}(\sigma^{-1}(1) > Cn) = 1 - \prod_{i:\alpha_i>Cn}\left(1 - \frac{1}{\alpha_i}\right).$$

More generally, a similar formula holds for Q with α_i being replaced by other sequence β_i. Following a procedure in Section 3 to derive (β_i), we pass from (α_i) to a subsequence and then subtract from each term a nonnegative integer not exceeding k. Therefore,

$$Q \le 1 - \prod_{i:\,\alpha_i>Cn}\left(1 - \frac{1}{\alpha_i - k}\right) \le 1 - \prod_{i:\,\alpha_i>Cn}\left(1 - \frac{1}{\alpha_i - n}\right).$$

Since $\ln(1 + x) \ge 2x$ for $-1/2 \le x \le 0$, we have the following estimate

$$-\frac{1}{2}\ln\left(\prod_{i:\,\alpha_i>Cn}\left(1 - \frac{1}{\alpha_i - n}\right)\right) \le \sum_{i:\,\alpha_i>Cn}\frac{1}{\alpha_i - n}$$

$$= \sum_{i:\,\alpha_i>Cn}\frac{1}{\alpha_i} + \sum_{i:\,\alpha_i>Cn}\frac{n}{\alpha_i(\alpha_i - n)} \le \sum_{i:\,\alpha_i>Cn}\frac{1}{\alpha_i} + \sum_{j=Cn}^{\infty}\frac{n}{(j-n-1)(j-n)}$$

$$= \sum_{i:\,\alpha_i>Cn}\frac{1}{\alpha_i} + \frac{n}{Cn} = \sum_{i:\,\alpha_i>Cn}\frac{1}{\alpha_i} + \frac{1}{C}$$

Now choose small enough $\delta > 0$ to have $1 - e^{-\delta} < \varepsilon$. Let $C > \frac{1}{4\delta}$ and choose n_0 such that

$$\sum_{i:\,\alpha_i>Cn_0}\frac{1}{\alpha_i} < \delta/4$$

(this is possible, since $\sum 1/\alpha_i$ converges). Then for $n > n_0$ we obtain $Q < 1 - e^{-\delta} < \varepsilon$, as desired. $\square$

Proposition 6.2. *Let O be a random order with distribution $P^{(\alpha,1)}$ and let $\sigma_n = \pi_n^\infty(O)$ be the projection of O on $\mathfrak{S}_n$. Then $P^{(\alpha,1)}$-almost surely*

$$\frac{\sigma_n^{-1}(n)}{n} \to 1.$$

Proof. Choose $\varepsilon > 0$. Recall that a real-valued random variable X stochastically dominates another such variable Y if for any bounded non-decreasing function f the expected values satisfy $Ef(X) \geq Ef(Y)$. Observe that by Lemma 6.1 the random variable $X = |\{1 \leq i \leq n \mid \sigma^{-1}(i) \leq Cn\}|$ stochastically dominates a sum of $n - n_0$ independent Bernoulli random variables with the probability of 1 equal to $1 - \varepsilon$ (see Lemma 1.1 in and [20] and Lemma 1 in [23]). Now using a standard large deviations estimate for the sum of independent Bernoulli random variables (see e.g. [15], Chapter 27), we conclude that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$P^{(\alpha,1)}(|\{1 \leq i \leq n \mid \sigma^{-1}(i) \leq Cn\}| > (1 - 2\varepsilon)n) > 1 - \exp(-C_2 n) \quad (9)$$

for $n > n_1$.

Observe that the set $\{\sigma^{-1}(1), \ldots, \sigma^{-1}(n)\}$ is the union of an integer interval $\{1, \ldots, M\}$ and a subset of the set $\{\alpha_1 + 1, \alpha_2 + 1, \ldots\}$.

The convergence of the series $\sum 1/\alpha_i$ implies that

$$\frac{|\{i \mid \alpha_i \leq Cn\}|}{n} \to 0.$$

Therefore, (9) implies that

$$P^{(\alpha,1)}\left(\{1, \ldots, \lfloor(1 - 3\varepsilon)n\rfloor\} \subset \{\sigma^{-1}(1), \ldots, \sigma^{-1}(n)\}\right) > 1 - \exp(-C_2 n)$$

for $n > n_2$. But on the event $\{1, \ldots, \lfloor(1 - 3\varepsilon)n\rfloor\} \subset \{\sigma^{-1}(1), \ldots, \sigma^{-1}(n)\}$ we have $(\pi_{n+1}^\infty(O))^{-1}(n + 1) > (1 - 3\varepsilon)n$. Hence, for $n > n_2$ we have

$$P^{(\alpha,1)}\left(\frac{\sigma_{n+1}^{-1}(n + 1)}{n} > (1 - 3\varepsilon)\right) > 1 - \exp(-C_2 n). \quad (10)$$

Since $\sum_{n=n_2+1}^\infty \exp(-C_2 n) < \infty$, from (10) and the Borel-Cantelli lemma follows that for all but finitely many n we have

$$\frac{\sigma_{n+1}^{-1}(n + 1)}{n} > 1 - 3\varepsilon,$$

whence

$$\liminf_{n \to \infty} \frac{\sigma_n^{-1}(n)}{n} > 1 - 3\varepsilon$$

almost surely. To finish the proof it remains to observe that $\varepsilon > 0$ is arbitrary and $\sigma_n^{-1}(n) \leq n$ always holds. $\qquad\square$

Corollary 6.3. *The dual algorithm for $P^{(\alpha,1)}$ eventually fills every position, so that the output is indeed a bijection $\sigma : \mathbb{N} \to \mathbb{N}$.*

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Proof. Indeed, in the proof of Proposition 6.2 we have shown that for every k the probability of the event $\{1, \ldots, k\} \subset \{\sigma^{-1}(1), \ldots, \sigma^{-1}(n)\}$ tends to 1 as $n \to \infty$. $\qquad\square$

Now we seek for an analogue of Proposition 6.2 for more general P^ω.

Proposition 6.4. *Let O be a random order with distribution P^* and let $\sigma_n = \pi_n^\infty(O)$ be the projection of O on $\mathfrak{S}_n$. Then P^*-almost surely*

$$\liminf_{n \to \infty} \frac{\sigma_n^{-1}(n)}{n} \to 0.$$

Proof. Under P^* the permutation σ_{n-1} and the position $\sigma_n^{-1}(n)$ are independent, and the latter is uniformly distributed on $[n]$. Since $P^*(\sigma_n^{-1}(n) = 1) = 1/n$, the event $\{\sigma_n^{-1}(n) = 1\}$ almost surely occurs infinitely often as $n \to \infty$, and the statement becomes trivial. $\qquad\square$

The analogous statement for general $P^{(\alpha,p)}$ interpolates between Propositions 6.2 and 6.4.

Proposition 6.5. *Let $0 < p < 1$, then for the order O with distribution $P^{(\alpha,p)}$, almost surely*

$$\liminf_{n \to \infty} \frac{(\pi_n^\infty(O))^{-1}(n)}{n} \to p.$$

Proof. Let O_0 and O_α be two independent linear orders on $\mathbb{N}$, such that the distributions of O_0 and O_α are P^* and $P^{(\alpha,1)}$, respectively. Recall that the $P^{(\alpha,p)}$-distributed order O is constructed from O_0 and O_α by splitting $\mathbb{N}$ into two subsets N_1 and N_2 (with the aid of a coin landing heads up with probability p), setting O_0 on N_2 and O_α on N_1, and requiring that N_1 precedes N_2.

Projecting to $[n]$ yields $\sigma = \pi_n^\infty(O)$, constructed as follows. Let $M_1 = N_1 \cap [n]$ and $M_2 = N_2 \cap [n]$. The permutation $\sigma_1 = \pi_{|M_1|}^\infty(O_\alpha)$ uniquely defines a permutation $\bar{\sigma}_1$ of the set M_1 and $\sigma_2 = \pi_{|M_2|}^\infty(O_0)$ uniquely defines a permutation $\bar{\sigma}_2$ of the set M_2. Permutation σ is obtained by first writing $\bar{\sigma}_1$ and then writing $\bar{\sigma}_2$.

Let us analyze $\sigma^{-1}(n)$. Choose $\varepsilon > 0$. Almost surely for large enough n we have

1. $p - \varepsilon \le |M_1|/n \le p + \varepsilon$,

2. $1 - \varepsilon \le (\sigma_1)^{-1}(|M_1|)/|M_1| \le 1$.

The latter is just the statement of Proposition 6.2 and the former follows from the law of large numbers for Bernoulli trials. Now if $n \in M_1$, then $\sigma^{-1}(n) = \sigma_1^{-1}(|M_1|)$ and, thus,

$$\frac{\sigma^{-1}(n)}{n} \geq (p - \varepsilon)(1 - \varepsilon).$$

If $n \in M_1$, then

$$\sigma^{-1}(n) > |M_1| \geq (p - \varepsilon)n.$$

Since ε is arbitrary, we conclude that

$$\liminf_{n \to \infty} \frac{(\pi_n^\infty(O))^{-1}(n)}{n} \geq p.$$

Next, using Proposition 6.4 we conclude that almost surely there exists an increasing sequence n_m such that for $n = n_m$, $m = 1, 2, \ldots$ we have

1. $n \in M_2$,

2. $\sigma_2^{-1}(|M_2|) = 1$.

This implies that for large enough m,

$$\sigma^{-1}(n) = |M_1| + 1 \leq (p + \varepsilon)n + 1.$$

Therefore,

$$\liminf_{n \to \infty} \frac{(\pi_n^\infty(O))^{-1}(n)}{n} \leq (p + \varepsilon)$$

Since ε is arbitrary, we are done. $\qquad\square$

Proposition 6.6. *If $p > 0$, then under $P^{(\alpha,p)}$ the position of the ith non-record in $\pi_k^\infty(O)$ converges to $\alpha_i + 1$ as $k \to \infty$ almost surely.*

Proof. First, suppose that $p = 1$ and recall the algorithmic description of $P^{(\alpha,1)}$. The permutation $\pi_k^\infty(O)$ is read from the order of numbers $1, \ldots, k$ after the first k steps of the algorithm. Moreover, observe that if after k steps of the algorithm all positions $1, \ldots, \alpha_i + 1$ are filled, then $\alpha_i + 1$ is precisely the position of the ith non-record in $\pi_k^\infty(O)$. Therefore, our claim is implied by Corollary 6.3.

For the general p a bulk of integers is appended at the right end of the permutation, thus not affecting positions of the first few non-records. $\qquad\square$

Proof of Theorem 3.3. The set of extremes $\operatorname{ext} \mathcal{M}_R(\mathcal{O})$ is contained in the Martin boundary by Lemma 4.3. On the other hand, by Proposition 3.5 each measure $P^\omega, \omega \in \Omega$, satisfies a law of large numbers specific for this particular P^ω. It follows that the supports of P^ω's are disjoint, hence none of the measures can be represented as a nontrivial convex mixture over the Martin boundary. Thus every P^ω is extreme, so $\operatorname{ext} \mathcal{M}_R(\mathcal{O}) = \{P^\omega, \ \omega \in \Omega\}$. The coincidence of topologies immediately follows from the explicit description of measures P^ω given in Section 3. $\qquad\square$

7   Two connections

Order-invariant measures on causal sets   We describe now a connection of the record-dependent measures $P^{(\alpha,1)}$ to a recent work on random partial orders [2, 3].

A partial order $\lhd$ on $\mathbb{N}$ defines a *causal set* $(\mathbb{N}, \lhd)$ if every element is preceded by finitely many other elements. A *natural extension* of $\lhd$ is an order-preserving bijection $\sigma : \mathbb{N} \to \mathbb{N}$, i.e. $i \lhd j$ implies $\sigma^{-1}(i) < \sigma^{-1}(j)$. A *stem* is a finite collection of positions $j_1, \ldots, j_k$ such that there exists a natural extension with $\sigma^{-1}(1) = j_1, \ldots, \sigma^{-1}(k) = j_k$.

If $j_1, \ldots, j_k$ is a stem, every $D_\ell = \{j_1, \ldots, j_\ell\}$, $1 \le \ell \le k$, is a down-set (lower ideal). A stem can be identified with a chain of down-sets $D_1 \subset \cdots \subset D_k$, where $|D_\ell| = \ell$. It is not hard to see that an infinite chain of down-sets $D_1 \subset D_2 \subset \ldots$ (where $|D_\ell| = \ell$) with $\cup D_\ell = \mathbb{N}$ uniquely corresponds to a natural extension of $\lhd$.

Brightwell and Luczak [2, 3] defined an *order-invariant measure* as a probability measure P on the set of natural extensions of $\lhd$, such that

$$P(\sigma^{-1}(1) = j_1, \ldots, \sigma^{-1}(k) = j_k) = P(\sigma^{-1}(1) = \ell_1, \ldots, \sigma^{-1}(k) = \ell_k),$$

provided $\{j_1, \ldots, j_k\} = \{\ell_1, \ldots, \ell_k\}$. The condition means that the probability of a stem only depends on the corresponding down-set $D_k = \{j_1, \ldots, j_k\}$. It is possible to interpret order-invariant measures as central measures on the path space of a graded graph of down-sets.

Let (α_k) be a strictly increasing sequence of integers as in Section 3, and let (β_k) be the (infinite) sequence complimentary to $(\alpha_k + 1)$, so that $\{\alpha_1 + 1, \alpha_2 + 1, \ldots\} \cup \{\beta_1, \beta_2, \ldots\} = \mathbb{N}$. Consider a partial order $\lhd$ generated by the relations

$$\beta_1 \lhd \beta_2 \lhd \ldots, \qquad \alpha_i + 1 \lhd \max\{\beta_k : \beta_k \le \alpha_i\},$$

which mean that (β_k) is a chain, and each segment $\beta_k + 1, \beta_k + 2, \ldots, \beta_{k+1} - 1$ is an antichain covered by β_k.

Obviously from the definitions, $\sigma : \mathbb{N} \to \mathbb{N}$ is a natural extension of $\lhd$ if and only if $\{\beta_k\}$ is the set of records of σ.

Proposition 7.1. $P^{(\alpha,1)}$ *is a unique order-invariant measure for the causal set* $(\mathbb{N}, \lhd)$.

Sketch of the proof. Every finite down-set with elements arranged in increasing order is a sequence γ of the kind

$$1, 2, \ldots, \beta_k, \beta_k + 1, \beta_k + 2, \ldots, \beta_{k+1} - 1, \alpha_{i_1} + 1, \ldots, \alpha_{i_\ell} + 1,$$

where either of the segments $\beta_k + 1, \beta_k + 2, \ldots, \beta_{k+1} - 1$ or $\alpha_{i_1}, \ldots, \alpha_{i_\ell}$ can be empty. Call such γ admissible.

The conditions ensuring that the set of records is (β_k) and that γ is admissible impose constraints on permutation that can be expressed in terms of the r_i. We illustrate this with γ of the form

$$1, \ldots, a, b, c$$

where $a + 1 \in \{b_k\}$ and $b, c \in \{\alpha_k + 1\}$. The constraints on the ranks become $r_j = j$ for $j \notin \{\alpha_k + 1\}$, $r_j < j$ for $j \in \{\alpha_k + 1\}$ and, to guarantee the admissibility,

$$
\begin{aligned}
r_i &\geq a + 1 \quad \text{for} \quad a < i < b, \\
r_b &\leq a + 1, \\
r_i &\geq a + 2 \quad \text{for} \quad b < i < c, \\
r_c &= a + 2, \\
r_i &\geq a + 3 \quad \text{for} \quad i > c.
\end{aligned}
$$

Under $P^{(\alpha,1)}$ the r_i's are independent and each r_i is uniformly distributed on a suitable range. Therefore each possible stem associated with γ has the same probability, equal to the probability of admissible realization of $r_j, j \leq \alpha_{i_\ell}$. The order-invariance of the measure follows.

For a *finite* causal set $([n], \lhd)$ the analog of order-invariant measure is the uniform distribution on the extensions of $\lhd$. The uniqueness assertion follows from the fact that $P^{(\alpha,1)}$ is a weak limit of such measures as $n \to \infty$ along (β_i), and condition (4) ensures that the limit is a bijection. We omit details, see [3, Section 9] for a more general result. $\square$

The Young-Fibonacci lattice The *Young-Fibonacci graph* (lattice) was introduced by Stanley [26] and Fomin [6]. They found out that it shares lots of the features with the Young graph, which is the object naturally arising in

the theory of group representations and combinatorics. In particular, Stanley proved that both graphs are *differential posets.*

The vertices of the Young-Fibonacci graph at level n are labeled by words in the alphabet $\{1, 2\}$, with the sum of digits equal n. For instance 1111, 211, 121, 112, 22 are all the words on level $n = 4$. The number of vertices on nth level is the nth Fibonacci number. Successors of a word are obtained by either inserting a 1 in any position within the leftmost contiguous block of 2's, or by replacing the leftmost 1 with 2. For instance, 2212 has successors $12212, 21212, 22112, 2222$.

Goodman and Kerov [12] studied the Martin boundary of the Young-Fibonacci graph. Comparing with their result, it is seen that the Martin boundary of the Young-Fibonacci graph has the same conical structure as our Ω. The apex is the *Plancherel measure*, which (like our P^*) appears as a pushforward of the uniform distribution on permutations. The base is a discrete space comprised of the measures which (like our $P^{(\alpha,1)}$'s) are parametrized by infinite words in the alphabet $\{1, 2\}$ with 'rare' occurrences of 2's, to satisfy a condition similar to (4). The Plancherel measure of the Young-Fibonacci graph was further studied in [8].

The arguments of [12] are very much different from the present paper. Goodman and Kerov intensively use the relation to a certain non-commutative algebra introduced by Okada [21]. Note also that unlike the Young-Fibonacci graph, the graph of record-sets $\mathcal{R}$ is not a differential poset. Thus, it seems that no direct connection of central measures on $\mathcal{R}$ and the Young-Fibonacci graph exist. This makes the coincidence of the Martin boundaries even more intriguing.

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