PHYSICAL REVIEW B **97**, 195115 (2018)

Deconfined quantum critical point on the triangular lattice

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(Received 15 October 2017; revised manuscript received 14 April 2018; published 10 May 2018)

In this work we propose a theory for the deconfined quantum critical point (DQCP) for spin-1/2 systems on a triangular lattice, which is a direct unfine-tuned quantum phase transition between the standard "$\sqrt{3}\times\sqrt{3}$" noncollinear antiferromagnetic order (or the so-called 120° state) and the "$\sqrt{12}\times\sqrt{12}$" valence solid bond (VBS) order, both of which are very standard ordered phases often observed in numerical simulations. This transition is beyond the standard Landau-Ginzburg paradigm and is also fundamentally different from the original DQCP theory on the square lattice due to the very different structures of both the magnetic and VBS order on frustrated lattices. We first propose a topological term in the effective-field theory that captures the "intertwinement" between the $\sqrt{3}\times\sqrt{3}$ antiferromagnetic order and the $\sqrt{12}\times\sqrt{12}$ VBS order. Then using a controlled renormalization-group calculation, we demonstrate that an unfine-tuned direct continuous DQCP exists between the two ordered phases mentioned above. This DQCP is described by the $N_f = 4$ quantum electrodynamics (QED) with an emergent PSU(4)=SU(4)/Z_4 symmetry only at the critical point. The aforementioned topological term is also naturally derived from the $N_f = 4$ QED. We also point out that physics around this DQCP is analogous to the boundary of a 3d bosonic symmetry- protected topological state with only on-site symmetries.

DOI: 10.1103/PhysRevB.97.195115

I. INTRODUCTION

The deconfined quantum critical point (DQCP) [1,2] was proposed as the first explicit example of a direct unfine-tuned quantum critical point [3] beyond the standard Landau's paradigm because the DQCP is sandwiched between two very different ordered phases with completely unrelated broken symmetries [1]. More precisely, the symmetry that is spontaneously broken on one side of the transition is completely independent of the symmetry that is broken on the other side. This scenario was forbidden in the standard Landau's paradigm but was proposed to be possible in quantum spin systems [1,2]. A lot of numerical work has been devoted to investigating the DQCP with a full spin rotation symmetry [4–15], as well as spin models with only in-plane spin symmetry [16–19]. Recently developed duality between strongly interacting QCPs in $(2+1)$ dimensions have further improved our understanding of the DQCP [20–25], and the predictions made by duality have received numerical support [26,27].

Let us first summarize the key ingredients of the original DQCP on the square lattice [1,2]:

(1) This is a quantum phase transition sandwiched between the standard antiferromagnetic Néel state and the valence bond solid (VBS) state. The Néel state has a ground-state manifold (GSM) equivalent to a two-dimensional sphere (S^2); that is, all the configurations of the Néel vector form a manifold S^2. Although the VBS has only four-fold degeneracy on the square lattice, there is strong evidence that the fourfold rotation symmetry of the square lattice is enlarged to a U(1) rotation symmetry right at the DQCP, and the VBS state has an approximate GSM S^1 (one-dimensional ring), which is *not* a submanifold of the GSM of the Néel state on the other side of the DQCP. Thus we can view the DQCP on the square lattice as a S^2-to-S^1 transition.

In another proposed realization of the DQCP [28], the Néel order and the VBS order are replaced by the quantum spin Hall order parameter and the s-wave superconductor; thus in this realization the DQCP is literally a transition between S^2 and S^1.

(2) The vortex of the VBS order parameter carries a bosonic spinor (spin-1/2) of the spin symmetry, and the skyrmion of the Néel order carries lattice momentum. This physics can be described by the noncompact CP^1 (NCCP^1) model [1,2]: $\mathcal{L} = \sum_\alpha |(\partial_\mu - i a_\mu) z_\alpha|^2 + r|z_\alpha|^2 + \dots$, where the Neél order parameter is $\vec{N} = z^\dagger \vec{\sigma} z$, the flux of a_μ is the skyrmion density of $\vec{N}$, and the flux condensate (which is dual to the photon phase of a_μ [29–31] based on the standard photon-superfluid duality) is the VBS order. Thus there is an "intertwinement" between the Néel and VBS orders: the defect of one order parameter is decorated with the quantum number of the symmetry that defines the other order; thus the condensation of the defect leads to the other order. This unusual quantum phase transition is considered "deconfined" because the field theory above is not formulated in terms of the standard Landau order parameter, but in terms of "fractionalized" degrees of freedom such as the spinon field z_α.

(3) If we treat the Néel and VBS orders on equal footing, we can introduce a five-component unit vector $\vec{n} \sim (N_x, N_y, N_z, V_x, V_y)$, and the intertwinement between the two order parameters is precisely captured by a topological Wess-Zumino-Witten (WZW) term of the nonlinear σ model defined

in the target space S^4 (four-dimensional sphere) where $\vec{n}$ lives [28,32].

All the previous works on DQCP have focused on the example proposed in Refs. [1,2], which is a theory specially designed for the square lattice. In this work we propose a possible DQCP on the triangular lattice (and the kagome lattice) for spin-1/2 systems with a full SU(2) spin rotation symmetry. Soon we will see that due to the fundamentally different structure of the magnetic order and VBS order from that of the square lattice, the DQCP on frustrated lattices demands a completely different formalism, with a very different universality class and an unexpected emergent symmetry.

Let us first summarize the standard phases for spin-1/2 systems with a full spin rotation symmetry on the triangular lattice. On the triangular lattice, the standard antiferromagnetic order is no longer a collinear Néel order; it is the $\sqrt{3}\times\sqrt{3}$ noncollinear spin order (or the so-called 120° order) with GSM SO(3), which is fundamentally different from the GSM S^2 of the collinear magnetic order.

The VBS order most often discussed and observed in numerical simulations is the so-called $\sqrt{12}\times\sqrt{12}$ VBS pattern [33–35]. This VBS order is the most natural pattern that can be obtained from the condensate of the vison (or the m excitation) of a Z_2 spin liquid on the triangular lattice. The dynamics of visons on the triangular lattice is equivalent to a fully frustrated Ising model on the dual honeycomb lattice [36], and it has been shown that with nearest-neighbor hopping on the dual honeycomb lattice, there are four symmetry-protected degenerate minima of the vison band structure in the Brillouin zone and that the GSM of the VBS order can be most naturally embedded into manifold SO(3) (just like the VBS order on the square lattice can be embedded in S^1) [36]. Thus the $\sqrt{3}\times\sqrt{3}$ noncollinear spin order and the $\sqrt{12}\times\sqrt{12}$ VBS order have a "self-dual" structure; that is, the magnetic order and the VBS order are dual to each other. Conversely, on the square lattice, the self-duality between the Néel and VBS orders happens only in the easy-plane limit [37].

The self-duality structure on the triangular lattice was noticed in Ref. [38] and captured by a mutual Chern-Simons (CS) theory:

$$\mathcal{L} = |(\partial - ia)z|^2 + r_z|z|^2 + |(\partial - ib)v|^2 + r_v|v|^2 + \frac{i}{\pi}a\wedge db + \cdots . \quad (1)$$

Here z_α and v_β carry a spinor representation of $\mathrm{SO(3)}_e$ and $\mathrm{SO(3)}_m$ groups, respectively, and when they are both gapped ($r_z,r_v>0$), they are the e and m excitations of a symmetric Z_2 spin liquid on the triangular lattice, with a mutual semion statistics enforced by the mutual Chern-Simons (CS) term [38]. Physically, z_α is the Schwinger boson of the standard construction of spin liquids on the triangular lattice [39–41], while v_β is the low-energy effective modes of the vison.

Equation (1) already unifies much of the physics for spin-1/2 systems on the triangular lattice [38]. For example, when both z_α and v_β are gapped, the system is in the Z_2 spin liquid mentioned above. The $\sqrt{3}\times\sqrt{3}$ noncollinear spin order and the VBS order can be obtained from the self-dual Z_2 spin liquid by condensing z_α and v_β, respectively, and both transitions have an emergent O(4) symmetry [36,42].

The problem of finding a DQCP on the triangular lattice between the noncollinear magnetic order and the VBS order is equivalent to finding a *direct unfine-tuned* transition between two different orders each with GSM SO(3), or in our notation an "SO(3)-to-SO(3) transition."

II. TOPOLOGICAL TERM OF EFFECTIVE-FIELD THEORY

As we discussed in the Introduction, the physical picture of the DQCP is the intertwinement between the two ordered phases; namely, the defect of one order is decorated with the quantum number of the other order. Hence once we "melt" one ordered phase by proliferating its defects, the system will automatically be driven into the other order. On the square lattice, a five component unit vector $n\sim(N_x,N_y,N_z,V_x,V_y)$ can be introduced; then the intertwinement between the two order parameters is precisely captured by a topological WZW term of the nonlinear sigma model defined in the target space S^4 (four-dimensional sphere) where $\vec{n}$ lives [28,32]:

$$\mathcal{L}_{wzw} = \int d^3x \int_0^1 du\, \frac{2\pi i}{\Omega_4}\epsilon_{abcde}n^a\partial_x n^b\partial_y n^c\partial_\tau n^d\partial_u n^e, \quad (2)$$

where Ω_4 is the volume of S^4. $\vec{n}(x,\tau,u)$ is any smooth extension of $\vec{n}(x,\tau)$ such that $\vec{n}(x,\tau,0)=(1,0,0,0,0)$ and $\vec{n}(x,\tau,1)=\vec{n}(x,\tau)$.

In Eq. (1), v_β is the vison of the spin liquid, and it carries a π-flux of a_μ due to the mutual CS term in Eq. 1. The π flux of a_μ is bound by the Z_2 vortex of the $\mathrm{SO(3)}_e$ GSM of the $\sqrt{3}\times\sqrt{3}$ spin order. Due to the homotopy group $\pi_1[\mathrm{SO(3)}]=Z_2$, any ordered phase with GSM SO(3) has Z_2 vortex excitations; namely, two vortices can annihilate each other. Similarly, z_α is also the Z_2 vortex of the $\mathrm{SO(3)}_m$ GSM of the VBS order, analogous to the vortex of the VBS order on the square lattice. This mutual "decoration" of topological defects means that there is also an intertwinement between the noncollinear $\sqrt{3}\times\sqrt{3}$ magnetic order and the $\sqrt{12}\times\sqrt{12}$ VBS orders on the triangular lattice.

To capture the intertwinement of the two phases with GSM SO(3), i.e., to capture the mutual decoration of topological defects, we need to design a topological term for these order parameters, just like the O(5) WZW term for the DQCP on the square lattice [32]. The topological term we design is as follows:

$$\mathcal{L}_{wzw} = \int d^3x \int_0^1 du\, \frac{2\pi i}{256\pi^2}\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}[\mathcal{P}\partial_\mu\mathcal{P}\partial_\nu\mathcal{P}\partial_\rho\mathcal{P}\partial_\lambda\mathcal{P}]. \quad (3)$$

Here $\mathcal{P}$ is a 4×4 Hermitian matrix field:

$$\mathcal{P} = \sum_{a,b=1}^{3} N_e^a N_m^b \sigma^{ab} + \sum_{a=1}^{3} M_e^a\sigma^{a0} + \sum_{b=1}^{3} M_m^b\sigma^{0b}, \quad (4)$$

where $\sigma^{ab}=\sigma^a\otimes\sigma^b$ and $\sigma^0=\mathbf{1}_{2\times2}$. Vectors $\vec{N}_e$, $\vec{N}_m$, $\vec{M}_e$, and $\vec{M}_m$ transform as vectors under $\mathrm{SO(3)}_e$ and $\mathrm{SO(3)}_m$ depending on their subscripts. We also need to impose some extra constraints:

$$\mathcal{P}^2 = \mathbf{1}_{4\times4},\ \ \vec{N}_e\cdot\vec{M}_e = \vec{N}_m\cdot\vec{M}_m = 0. \quad (5)$$

Then $\vec{N}_e$ and $\vec{M}_e$ together will form a SO(3) "tetrad," which is equivalent to the SO(3) manifold. $\vec{N}_m$ and $\vec{M}_m$ form another SO(3) manifold. With the constraints in Eq. (5), the matrix field $\mathcal{P}$ is embedded in the manifold

$$\mathcal{M} = \frac{\mathrm{U}(4)}{\mathrm{U}(2)\times\mathrm{U}(2)}. \tag{6}$$

The maximal symmetry of the WZW term (3) is PSU(4) $=$ SU(4)$/Z_4$ [which contains both SO(3)$_e$ and SO(3)$_m$ as subgroups], as the WZW term is invariant under a SU(4) transformation: $\mathcal{P}\rightarrow U^\dagger \mathcal{P} U$ with $U\in$ SU(4), while the Z_4 center of SU(4) does not change any configuration of $\mathcal{P}$. The WZW term (3) is well defined based on its homotopy group $\pi_4[\mathcal{M}]=\mathbb{Z}$, just like $\pi_4[S^4]=\mathbb{Z}$. Obviously, the SU(4) symmetry contains both SO(3)$_e$ and SO(3)$_m$ as subgroups.

The topological WZW term in Eq. (3) is precisely the boundary theory of a $3d$ symmetry-protected topological (SPT) state with PSU(4) symmetry [43]. We will discuss this further later.

Let us test that this topological term captures the correct intertwinement. That is, it must capture the physics that the Z_2 vortex of SO(3)$_e$ carries the spinor of SO(3)$_m$ and vice versa. This effect is most conveniently visualized after breaking SO(3)$_m$ down to SO(2)$_m$, and the Z_2 vortex of the SO(3)$_m$ manifold becomes the ordinary vortex of an SO(2) order parameter. This symmetry breaking allows us to take $\vec{N}_m=(0,0,1)$, i.e., $N_m^1=N_m^2=0$, $N_m^3=1$. Because $\vec{N}_m\cdot\vec{M}_m=0$ [Eq. (5)], $\vec{M}_m=(M_m^1,M_m^2,0)$. Then one allowed configuration of $\mathcal{P}$ is

$$\mathcal{P}=\sum_{a=1}^{3}N_e^a\sigma^{a3}+\sum_{b=1}^{2}M_m^b\sigma^{0b}=\vec{n}\cdot\vec{\Gamma}, \tag{7}$$

where $\vec{n}$ is a five-component vector and $|\vec{n}|=1$ due to the constraint $\mathcal{P}^2=\mathbf{1}_{4\times4}$. $\vec{\Gamma}$ are five anticommuting Gamma matrices. Now the WZW term (3) reduces *precisely* to the standard O(5) WZW at level 1 in (2 + 1) dimensions, and it becomes manifest that the vortex of (M_m^1,M_m^2) [the descendant of the Z_2 vortex of SO(3)$_m$ under the assumed symmetry breaking] is decorated with a spinor of SO(3)$_e$. To explicitly visualize the effect of the decorated vortex, one can follow the procedure of Ref. [28] and create a vortex of (n_4,n_5). Then the physics in the vortex core becomes a zero-dimensional quantum mechanics problem, whose exact solution reveals that there is a spin-1/2 carried by each vortex.

III. FIELD THEORY AND RENORMALIZATION-GROUP ANALYSIS

Equation (3) is a topological term in the low-energy effective-field theory that describes the physics of the ordered phases. But a complete field theory which reduces to the WZW term in the infrared is still demanded. For example, the O(5) nonlinear σ model with a WZW term at level 1 can be derived as the low-energy effective-field theory of the $N=2$ QCD with the SU(2) gauge field, which has an explicit SO(5) global symmetry [25].

The WZW term in Eq. (3) can be derived in the same manner by coupling the matrix field $\mathcal{P}$ to the Dirac fermions of the $N_f=4$ QED:

$$\mathcal{L}=\sum_{j=1}^{4}\bar{\psi}_j\gamma\cdot(\partial-ia)\psi_j+m\sum_{i,j}\bar{\psi}_i\psi_j\mathcal{P}_{ij}. \tag{8}$$

The WZW term of $\mathcal{P}$ is generated after integrating out the fermions using the same method as in Ref. [44], and the PSU(4) global symmetry becomes explicit in the $N_f=4$ QED [45].

Our goal is to demonstrate that the $N_f=4$ QED corresponds to an unfine-tuned DQCP between the noncollinear magnetic order and the VBS order, or in our notation a SO(3)-to-SO(3) transition [as the DQCP is sandwiched between two ordered phases both with GSM SO(3)]. The PSU(4) global symmetry of $N_f=4$ QED must be explicitly broken down to the physical symmetry. The most natural terms that beak this PSU(4) global symmetry down to SO(3)$_e\times$ SO(3)$_m$ are four-fermion interaction terms. Under the assumption of an emergent Lorentz invariance, which often happens at quantum critical points and algebraic spin liquids (such as the original DQCP on the square lattice), there are *only two* such linearly independent terms that beak the PSU(4) global symmetry down to SO(3)$_e\times$ SO(3)$_m$:

$$\mathcal{L}_1=(\bar{\psi}\vec{\sigma}\psi)\cdot(\bar{\psi}\vec{\sigma}\psi),\quad \mathcal{L}_2=(\bar{\psi}\vec{\tau}\psi)\cdot(\bar{\psi}\vec{\tau}\psi), \tag{9}$$

where ψ carries both indices from the Pauli matrices $\vec{\sigma}$ and $\vec{\tau}$, so that ψ is a vector representation $(\frac{1}{2},\frac{1}{2})$ of SO(4) $\sim$ SO(3)$_e\times$ SO(3)$_m$.

One can think of some other four-fermion terms, for example, $\mathcal{L}'=\sum_\mu(\bar{\psi}\vec{\sigma}\gamma_\mu\psi)\cdot(\bar{\psi}\vec{\sigma}\gamma_\mu\psi)$, but we can repeatedly use the Fierz identity and reduce these terms to a linear combination of $\mathcal{L}_1$ and $\mathcal{L}_2$, as well as SU(4)-invariant terms: $\mathcal{L}'=-2\mathcal{L}_2-\mathcal{L}_1+\cdots$ (for more details refer to the Appendix C). The ellipses are SU(4)-invariant terms, which, according to Refs. [46–48], are irrelevant at the $N_f=4$ QED.

The renormalization-group (RG) flow of $\mathcal{L}_1$ and $\mathcal{L}_2$ can be most conveniently calculated by generalizing the two-dimensional space of Pauli matrices $\vec{\tau}$ to an N-dimensional space; that is, we generalize the QED$_3$ to an $N_f=2N$ QED$_3$ with SU(2) $\times$ SU(N) symmetry. And we consider the following independent four-fermion terms:

$$g\mathcal{L}=g(\bar{\psi}\vec{\sigma}\psi)\cdot(\bar{\psi}\vec{\sigma}\psi),\quad g'\mathcal{L}'=g'(\bar{\psi}\vec{\sigma}\gamma_\mu\psi)\cdot(\bar{\psi}\vec{\sigma}\gamma_\mu\psi). \tag{10}$$

One can check that all SU(2) $\times$ SU(N) four-fermion interactions in this QED$_3$ can be written in terms of the linear combinations of the two terms above up to SU(2N)-invariant terms, which, according to Refs. [46–48], are irrelevant under RG even for small N. At the first order of $1/N$ expansion, the RG equation reads

$$\beta(g)=\left(-1+\frac{128}{3(2N)\pi^2}\right)g+\frac{64}{(2N)\pi^2}g',$$
$$\beta(g')=-g'+\frac{64}{3(2N)\pi^2}g. \tag{11}$$

There are two RG flow eigenvectors: (1, − 1), with RG flow eigenvalue $-1-64/[3(2N)\pi^2]$, and (3,1), with eigenvalue $-1+64/[(2N)\pi^2]$. This means that when $N=2$, there is one irrelevant eigenvector with

$$\mathcal{L}-\mathcal{L}'=2(\mathcal{L}_1+\mathcal{L}_2)+\cdots \tag{12}$$

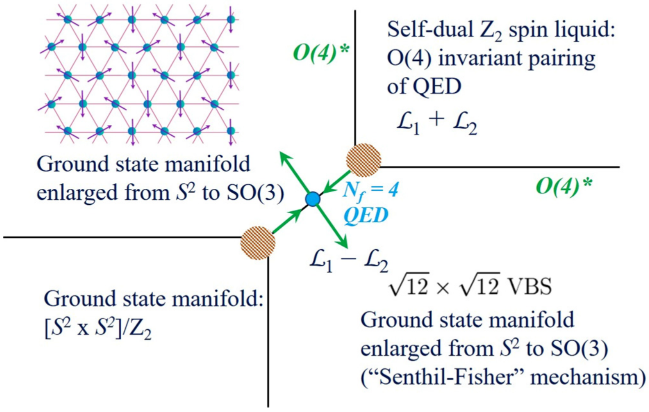

FIG. 1. The global phase diagram of spin-1/2 systems on the triangular lattice. The intertwinement between the order parameters is captured by the WZW term (3). Our RG analysis concludes that there is a direct unfine-tuned SO(3)-to-SO(3) transition, which is a direct unfine-tuned transition between the noncollinear magnetic order and the VBS order. The detailed structure of the shaded areas demands further studies.

and a relevant eigenvector with

$$3\mathcal{L} + \mathcal{L}' = 2(\mathcal{L}_1 - \mathcal{L}_2) + \cdots . \quad (13)$$

Again, the ellipses are SU(4)-invariant terms that are irrelevant. In fact, $\mathcal{L}_1 + \mathcal{L}_2$ preserves the exchange symmetry (duality) between $\mathrm{SO}(3)_e$ and $\mathrm{SO}(3)_m$; in other words $\mathcal{L}_1 + \mathcal{L}_2$ preserves the O(4) symmetry that contains an extra improper rotation in addition to SO(4), while $\mathcal{L}_1 - \mathcal{L}_2$ breaks the O(4) symmetry down to SO(4). Thus $\mathcal{L}_1 + \mathcal{L}_2$ and $\mathcal{L}_1 - \mathcal{L}_2$ both *must be* eigenvectors under RG. The RG flow is sketched in Fig. 1.

To have a complete story, we should also discuss other perturbations on the $N_f = 4$ QED. The fermion bilinear terms are forbidden either by the flavor symmetry or by discrete space-time symmetries, while higher-order fermion interactions (such as eight-fermion interactions) are very likely to be irrelevant. The monopoles of a_μ were ignored in this RG calculation. According to Ref. [49], monopoles of QED carry nontrivial quantum numbers. A multiple-monopole could be a singlet under the global symmetry and hence allowed in the action, but it will have a higher scaling dimension than the single monopole. It is known that with large N_f all the monopoles are irrelevant, but the scaling dimension of the multiple monopole for the current case with $N_f = 4$ needs further study.

Since $u(\mathcal{L}_1 - \mathcal{L}_2)$ is relevant, when the coefficient $u > 0$, a simple mean-field theory implies that this term leads to a nonzero expectation value for $\langle \bar{\psi}\vec{\sigma}\psi \rangle$. It appears that this order parameter is a three-component vector, so the GSM should be S^2. However, using the Senthil-Fisher mechanism of Ref. [32], the actual GSM is enlarged to SO(3) due to the gauge fluctuation of a_μ (for a review of the Senthil-Fisher mechanism, refer to the Appendix A). When $u < 0$, the condensed order parameter is $\langle \bar{\psi}\vec{\tau}\psi \rangle$, and the Senthil-Fisher mechanism again enlarges the GSM to SO(3). Based on our calculation, because $u(\mathcal{L}_1 - \mathcal{L}_2)$ is the *only* relevant perturbation allowed by symmetry, u drives a direct unfine-tuned continuous SO(3)-to-SO(3) transition, which is consistent with a transition between the $\sqrt{3} \times \sqrt{3}$ noncollinear magnetic order and the $\sqrt{12} \times \sqrt{12}$ VBS order. And our theory predicts that at the critical point, there is an emergent PSU(4) symmetry.

Now let us investigate the perturbation $\mathcal{L}_1 + \mathcal{L}_2$. First of all, let us think of a seemingly different term: $\mathcal{L}_3 = \sum_{a,b} (\bar{\psi}\sigma^a\tau^b\psi)(\bar{\psi}\sigma^a\tau^b\psi)$. This term also preserves the O(4) symmetry, and after some algebra we can show that $\mathcal{L}_3 = -(\mathcal{L}_1 + \mathcal{L}_2) + \cdots$. Another very useful way to rewrite $\mathcal{L}_3$ is that:

$$\mathcal{L}_3 = -(\bar{\psi}^t J\epsilon\bar{\psi})(\psi^t J\epsilon\psi) + \cdots = -\hat{\Delta}^\dagger\hat{\Delta} + \cdots \quad (14)$$

where $\hat{\Delta} = \psi^t J\epsilon\psi$, $J = \sigma^2 \otimes \tau^2$. ϵ is the antisymmetric tensor acting on the Dirac indices.

Thus although the O(4) invariant deformation in our system (at low energy it corresponds to $\mathcal{L}_1 + \mathcal{L}_2$) is perturbatively irrelevant at the $N_f = 4$ QED fixed point, when it is strong and nonperturbative, the standard Hubbard-Stratonovich transformation and mean field theory suggests that, depending on its sign, it may lead to either a condensate of $\hat{\Delta}$, or condensate of $(\bar{\psi}\sigma^a\tau^b\psi)$ over certain critical strength of $\mathcal{L}_3$. The condensate of $(\bar{\psi}\sigma^a\tau^b\psi)$ has GSM $[S^2 \times S^2]/Z_2$, and is identical to the submanifold of $\mathcal{P}$ when $\vec{M}_e = \vec{M}_m = 0$ in Eq. 4. The Z_2 in the quotient is due to the fact that $\mathcal{P}$ is unaffected when both $\vec{N}_e$ and $\vec{N}_m$ change sign simultaneously.

Now we show that the condensate of $\hat{\Delta}$ is precisely the self-dual Z_2 topological order described by Eq. 1. First of all, in the superconductor phase with $\hat{\Delta}$ condensate, there will obviously be a Bogoliubov fermion. This Bogoliubov fermion carries the (1/2,1/2) representation under $\mathrm{SO}(3)_e \times \mathrm{SO}(3)_m$. The deconfined $\pi -$ flux of the gauge field a_μ is bound to a $2\pi -$ vortex of the complex order parameter $\hat{\Delta}$, which then traps 4 Majorana zero modes. The 4 Majorana zero modes transform as a vector under the SO(4) action that acts on the flavor indices. The 4 Majorana zero modes define 4 different states that can be separated into two groups of states depending on their fermion parities. In fact, the two groups should be identified as the (1/2,0) doublet and the (0,1/2) doublet of $\mathrm{SO}(3)_e \times \mathrm{SO}(3)_m$. Therefore, the $\pi -$ flux with two different types of doublets should be viewed as two different topological excitations. Let us denote the (1/2,0) doublet as e and the (0,1/2) doublet as m. Both e and m have bosonic topological spins. And they differ by a Bogoliubov fermion. Therefore, their mutual statistics is semionic (which rises from the braiding between the fermion and the π flux). At this point, we can identify the topological order of the $\hat{\Delta}$ condensate as the Z_2 topological order described by Eq. (1).

IV. INTERPRETATION OF THE DQCP AS THE BOUNDARY OF A THREE-DIMENSIONAL SYSTEM

Decorating quantum numbers to topological defects is also a key physical picture of constructing SPT states. The analogy between the DQCP on the square lattice and a three-dimensional (3D) bulk SPT state with an SO(5) symmetry was discussed in Ref. [25]. Many 3D SPT states can be constructed by decorating the defects in the system with a lower-dimensional SPT state and then proliferating the defects [50,51].

The physics around the DQCP discussed in this work is equivalent to the boundary state of a 3D bosonic SPT state with $SO(3)_e \times SO(3)_m$ symmetry once we view both SO(3) groups as on-site symmetries. We have already mentioned that the topological WZW term (3) is identical to the boundary theory of a 3D SPT state with PSU(4) symmetry [43], which comes from a Θ term in the 3D bulk. By breaking the symmetry down to either $SO(3)_e \times SO(2)_m$ or $SO(2)_e \times SO(3)_m$, the bulk SPT state is reduced to a $SO(3) \times SO(2)$ SPT state, which can be interpreted as the decorated vortex line construction [51]; namely, one can decorate the SO(2) vortex line with the Haldane phase with the SO(3) symmetry and then proliferate the vortex lines. In our case, the bulk SPT state with $SO(3)_e \times SO(3)_m$ symmetry can be interpreted as a similar decorated vortex line construction; that is, we can decorate the Z_2 vortex line of one of the SO(3) manifolds with the Haldane phase of the other SO(3) symmetry, then proliferate the vortex lines. The Z_2 classification of the Haldane phase is perfectly compatible with the Z_2 nature of the vortex line of a SO(3) manifold. Using the method in Ref. [25], one can also derive that the (3 + 1)D bulk SPT state must have a topological response action $\mathcal{S} = i\pi \int w_2[\mathcal{A}_e] \cup w_2[\mathcal{A}_m]$ in the presence of background $SO(3)_e$ gauge field $\mathcal{A}_e$ and $SO(3)_m$ gauge field $\mathcal{A}_m$ (w_2 represents the second Stiefel-Whitney class). This topological response theory also matches exactly the decorated vortex line construction.

We have shown that the physics around the critical point has the same effective-field theory as the boundary of a 3D SPT state [43]. The anomaly (once we view all the symmetries as on-site symmetries) of the large-N generalizations of our theory will be analyzed in the future, and a Lieb-Shultz-Mattis theorem for SU(N) and SO(N) spin systems on the triangular and kagome lattices can potentially be developed in the same way as in Refs. [52,53].

V. SUMMARY

In summary, we proposed a theory for a potentially direct unfine-tuned continuous quantum phase transition between the noncollinear magnetic order and VBS order on the triangular lattice, and at the critical point the system has an emergent PSU(4) global symmetry. Our proposed DQCP is fundamentally different from the original example on the square lattice due to the very different structure of both the magnetic and VBS orders compared with the unfrustrated square lattice. Our conclusion is based on a controlled RG calculation and an effective nonlinear σ model with a topological WZW term.

A similar structure of noncollinear magnetic order and VBS orders can be found on the kagome lattice. For example, it was shown in Ref. [54] that the vison band structure could have symmetry-protected fourfold-degenerate minima just like the triangular lattice [although the emergence of O(4) symmetry in the infrared is less likely]. Indeed, algebraic spin liquids with $N_f = 4$ QED as their low-energy description have been discussed extensively for both the triangular and kagome lattices [41,55–57]. Reference [41] also made the observation that the noncollinear magnetic order, the VBS order, and the Z_2 spin liquid are all near an $N_f = 4$ QED (the so-called π-flux state from microscopic construction). The Z_2 spin liquid was shown to be equivalent to the one constructed from Schwinger boson [40], which can evolve into the $\sqrt{3} \times \sqrt{3}$ magnetic order and the $\sqrt{12} \times \sqrt{12}$ VBS order through an O(4)* transition. But we should stress that in this work we focus only on the field theory for the SO(3)-to-SO(3) DQCP, without fully determining the relation between the field theory and the microscopic degrees of freedom.

It is a challenge to find an antiferromagnetic spin model on a frustrated lattice without a sign problem. But we note that in Ref. [35] spin nematic phases with the GSM S^N/Z_2 [analogous to the spin-1/2 $\sqrt{3} \times \sqrt{3}$ state with the GSM SO(3)= S^3/Z_2] and the $\sqrt{12} \times \sqrt{12}$ VBS order were found in a series of sign-problem-free models on the triangular lattice. Thus it is possible to design a modified version of the models discussed in Ref. [35] to bring together the spin nematic order and VBS order and then access the DQCP that we are proposing.

ACKNOWLEDGMENTS

C.-M.J. is partly supported by the Gordon and Betty Moore Foundations EPiQS Initiative through Grant No. GBMF4304. A.T. was supported by the National Science Foundation under Grant No. NSF PHY-1125915 at KITP. A.R. and C.X. are supported by the David and Lucile Packard Foundation and NSF Grant No. DMR-1151208. The authors thank S. Sachdev, T. Senthil, A. Vishwanath, and C. Wang for very helpful discussions.

APPENDIX A: THE ORDERED PHASES AND THE SENTHIL-FISHER MECHANISM

Here we reproduce the discussion in Ref. [32] and demonstrate how the GSM of the order of $\bar{\psi}\vec{\sigma}\psi$ (and, similarly, $\bar{\psi}\vec{\tau}\psi$) is enlarged from S^2 to SO(3). First, we couple the $N_f = 4$ QED to a three-component dynamical unit vector field $\boldsymbol{N}(x,\tau)$:

$$\mathcal{L} = \bar{\psi}\gamma_\mu(\partial_\mu - ia_\mu)\psi + m\bar{\psi}\boldsymbol{\sigma}\psi \cdot \boldsymbol{N}. \tag{A1}$$

The flavor indices are hidden in the equation above for simplicity. Now following the standard $1/m$ expansion of Ref. [44], we obtain the following action after integrating out the fermion ψ_j:

$$\mathcal{L}_{\text{eff}} = \frac{1}{g}(\partial_\mu \boldsymbol{N})^2 + i2\pi \,\text{Hopf}[\boldsymbol{N}] + i2a_\mu J^T_\mu + \frac{1}{e^2} f^2_{\mu\nu}, \tag{A2}$$

where $1/g \sim m$. $J^T_0 = \frac{1}{4\pi}\epsilon_{abc}N^a\partial_x N^b \partial_y N^c$ is the skyrmion density of $\boldsymbol{N}$; thus J^T_μ is the skyrmion current. The second term of Eq. (A2) is the Hopf term of $\boldsymbol{N}$, which comes from the fact that $\pi_3[S^2] = \mathbb{Z}$.

Now if we introduce the CP^1 field $z_\alpha = (z_1,z_2)^t = (n_1 + in_2, n_3 + in_4)^t$ for $\boldsymbol{N}$ as $\boldsymbol{N} = z^\dagger \boldsymbol{\sigma} z$, the Hopf term becomes precisely the Θ term for the O(4) *unit* vector $\boldsymbol{n}$ with $\Theta = 2\pi$:

$$i2\pi \,\text{Hopf}[\boldsymbol{N}] = \frac{i2\pi}{2\pi^2}\epsilon_{abcd}n^a\partial_x n^b \partial_y n^c \partial_\tau n^d. \tag{A3}$$

In the CP^1 formalism, the skyrmion current $J^T_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\rho}\partial_\nu \alpha_\rho$, where α_μ is the gauge field that the CP^1 field z_α couples to. The coupling between a_μ and α_μ,

$$2ia_\mu J^T_\mu = \frac{i2}{2\pi}\epsilon_{\mu\nu\rho}a_\mu \partial_\nu \alpha_\mu, \tag{A4}$$

takes precisely the form of the mutual CS theory of a Z_2 topological order; it implies that the gauge charge z_α is an anyon of a Z_2 topological order, and the condensate of z_α (equivalently, the order of $\boldsymbol{N}$) has a GSM $= \mathrm{SO}(3) = S^3/Z_2$, where S^3 is the manifold of the unit vector $\vec{n}$.

APPENDIX B: DERIVING THE WZW TERM

Let us consider a theory of QED$_3$ with $N_f = 4$ flavors of Dirac fermions coupled to a matrix order parameter field $\mathcal{P}$:

$$\mathcal{L} = \sum_{i,j} \bar{\psi}_i[\gamma_\mu(\partial_\mu - ia_\mu)\delta_{ij} + m\mathcal{P}_{ij}]\psi_j. \tag{B1}$$

$\mathcal{P}$ takes values in the target manifold $\mathcal{P} \in \mathcal{M} = \frac{\mathrm{U}(4)}{\mathrm{U}(2)\times\mathrm{U}(2)}$. We can parametrize the matrix field $\mathcal{P} = U^\dagger \Omega U$, where $U \in \mathrm{SU}(4)$ and $\Omega = \sigma^z \otimes \mathbf{1}_{2\times 2}$. $\mathcal{P}$ satisfies $\mathcal{P}^2 = \mathbf{1}_{4\times 4}$ and $\mathrm{tr}\mathcal{P} = 0$.

The effective action after integrating over the fermion fields formally reads

$$\begin{aligned}\mathcal{S}_{\text{eff}}[a_\mu,\mathcal{P}] &= -\ln \int D\bar{\psi} D\psi \exp\left[-\int d^3x \mathcal{L}(\psi,a_\mu,\mathcal{P})\right] \\ &= -\ln\det[\mathcal{D}(a_\mu,\mathcal{P})] = -\mathrm{Tr}\ln[\mathcal{D}(a_\mu,\mathcal{P})].\end{aligned} \tag{B2}$$

The expansion of $\mathcal{S}_{\text{eff}}$ has the following structure:

$$\mathcal{S}_{\text{eff}}[a_\mu,\mathcal{P}] = \mathcal{S}_{\text{eff}}[a_\mu = 0,\mathcal{P}] + O(a), \tag{B3}$$

and we will look at the first term in the expansion. In general, all terms that respect the symmetry of the original action will appear in the expansion of the fermion determinant. Here we want to derive the topological term of $\mathcal{P}$. One way to obtain the effective action is the perturbative method developed in Ref. [44]. Let us vary the action over the matrix field $\mathcal{P}$,

$$\delta\mathcal{S}_{\text{eff}} = -\mathrm{Tr}[m\delta\mathcal{P}(\mathcal{D}^\dagger\mathcal{D})^{-1}\mathcal{D}^\dagger], \tag{B4}$$

and then expand $(\mathcal{D}^\dagger\mathcal{D})^{-1}$ in gradients of $\mathcal{P}$:

$$\begin{aligned}(\mathcal{D}^\dagger\mathcal{D})^{-1} &= (-\partial^2 + m^2 - m\gamma_\mu\partial_\mu\mathcal{P})^{-1} \\ &= (-\partial^2 + m^2)^{-1} \\ &\quad \times \sum_{n=0}^{\infty}[(-\partial^2 + m^2)^{-1} m\gamma_\mu\partial_\mu\mathcal{P}]^n.\end{aligned}$$

Since the coefficient of the WZW term is dimensionless, we will look at the following term in the expansion:

$$\begin{aligned}\delta W(\mathcal{P}) &= -\mathrm{Tr}\{m^2\delta\mathcal{P}(-\partial^2 + m^2)^{-1} \\ &\quad \times [(-\partial^2 + m^2)^{-1} m\gamma_\mu\partial_\mu\mathcal{P}]^3\mathcal{P}\} \\ &= -K\int d^3x\, \mathrm{Tr}[\delta\mathcal{P}(\gamma_\mu\partial_\mu\mathcal{P})^3\mathcal{P}],\end{aligned}$$

where $K = \int \frac{d^3p}{(2\pi)^3}\frac{m^5}{(p^2+m^2)^4} = \frac{1}{64\pi}$ is a dimensionless number and Tr is the trace over the Dirac and flavor indices. After tracing over the Dirac indices,

$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 2i\epsilon_{\mu\nu\rho}, \tag{B5}$$

we obtain the following term for the variation:

$$\delta W(\mathcal{P}) = -\frac{2\pi i}{64\pi^2}\epsilon_{\mu\nu\rho}\int d^3x\, \mathrm{tr}[\delta\mathcal{P}\partial_\mu\mathcal{P}\partial_\nu\mathcal{P}\partial_\rho\mathcal{P}\mathcal{P}], \tag{B6}$$

where tr is the trace for only the flavor indices.

We can restore the topological term of the nonlinear σ model by the standard method of introducing an auxiliary coordinate u. The field $\tilde{\mathcal{P}}(x,u)$ interpolates between $\tilde{\mathcal{P}}(x,u=0) = \Omega$ and $\tilde{\mathcal{P}}(x,u=1) = \mathcal{P}(x)$. The topological term reads

$$W(\tilde{\mathcal{P}}) = -\frac{2\pi i}{256\pi^2}\epsilon_{\mu\nu\rho\delta}\int_0^1 du \int d^3x \mathrm{tr}[\tilde{\mathcal{P}}\partial_\mu\tilde{\mathcal{P}}\partial_\nu\tilde{\mathcal{P}}\partial_\rho\tilde{\mathcal{P}}\partial_\delta\tilde{\mathcal{P}}] \tag{B7}$$

(the extra factor of $1/4$ comes from the antisymmetrization of the u coordinate with other indices).

APPENDIX C: LINEAR DEPENDENCE OF FOUR-FERMION INTERACTIONS IN $N_f = 2N$ QED$_3$

In this section, we study all the $\mathrm{SU}(2)\times\mathrm{SU}(N)$ symmetric four-fermion interactions in the $N_f = 2N$ QED$_3$ and their linear dependence up to $\mathrm{SU}(2N)$-invariant terms.

First of all, we write down all the $\mathrm{SU}(2)\times\mathrm{SU}(N)$ symmetric four-fermion terms:

$$(\bar{\psi}\psi)(\bar{\psi}\psi),\ \ (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi), \tag{C1}$$

$$(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi),\ \ (\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi), \tag{C2}$$

$$(\bar{\psi}T^a\psi)(\bar{\psi}T^a\psi),\ \ (\bar{\psi}\gamma^\mu T^a\psi)(\bar{\psi}\gamma^\mu T^a\psi), \tag{C3}$$

$$(\bar{\psi}\vec{\sigma}T^a\psi)(\bar{\psi}\vec{\sigma}T^a\psi),\ \ (\bar{\psi}\gamma^\mu\vec{\sigma}T^a\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}T^a\psi), \tag{C4}$$

where $\vec{\sigma}$ is the generator of the SU(2) symmetry and T^a (with $a = 1,2,\ldots,N^2-1$) is the generator of the $\mathrm{SU}(N)$ symmetry. Here we have also implicitly assumed the summation over repeated indices in these expressions. The two terms in the second line are exactly the terms introduced in Eq. (9) of the main text.

Since all the $\mathrm{SU}(2N)$-invariant four-fermion interactions are irrelevant under RG [46–48], we are concerned only with the linear dependence of all the $\mathrm{SU}(2)\times\mathrm{SU}(N)$-symmetric four-fermion interactions up to $\mathrm{SU}(2N)$-invariant ones. First, we notice that the terms in Eq. (C1) are $\mathrm{SU}(2N)$ invariant. Therefore we can ignore them for this analysis. Notice that we can rewrite the two terms in Eq. (C3) as

$$\begin{aligned}&(\bar{\psi}T^a\psi)(\bar{\psi}T^a\psi) \\ &\quad = -\frac{N}{4}(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi) - \frac{N}{4}(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi) \\ &\quad\quad - \frac{N}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) - \frac{N+4}{4}(\bar{\psi}\psi)(\bar{\psi}\psi),\end{aligned} \tag{C5}$$

$$\begin{aligned}&(\bar{\psi}T^a\gamma^\mu\psi)(\bar{\psi}T^a\gamma^\mu\psi) \\ &\quad = \frac{N}{4}(\bar{\psi}\gamma^\mu\vec{\sigma}\psi)(\bar{\psi}\gamma^\mu\vec{\sigma}\psi) - \frac{3N}{4}(\bar{\psi}\vec{\sigma}\psi)(\bar{\psi}\vec{\sigma}\psi) \\ &\quad\quad + \frac{N-4}{4}(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) - \frac{3N}{4}(\bar{\psi}\psi)(\bar{\psi}\psi).\end{aligned} \tag{C6}$$

Therefore up to $\mathrm{SU}(2N)$-invariant terms, the two terms in Eq. (C3) can be written as a linear combination of the two terms in Eq. (C2). In the rewriting given above, we have used the Fierz identity $\sum_a T^a_{ij}T^a_{kl} = N\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}$ for the $\mathrm{SU}(N)$ group, as well as the Fierz identities $\vec{\sigma}_{ab}\cdot\vec{\sigma}_{cd} = 2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd}$ for the Pauli matrices $\vec{\sigma}$ and $\gamma^\mu_{\alpha\beta}\gamma^\mu_{\eta\rho} = 2\delta_{\alpha\rho}\delta_{\beta\eta} - \delta_{\alpha\beta}\delta_{\eta\rho}$ for the gamma matrices γ^μ. Similarly, we can rewrite the two terms in Eq. (C4) as

$$\begin{aligned}(\bar\psi\vec\sigma T^a\psi)(\bar\psi\vec\sigma T^a\psi) &= \frac{N}{4}(\bar\psi\gamma^\mu\vec\sigma\psi)(\bar\psi\gamma^\mu\vec\sigma\psi) + \frac{N-4}{4}(\bar\psi\vec\sigma\psi)(\bar\psi\vec\sigma\psi) \\ &\quad - \frac{3N}{4}(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\mu\psi) - \frac{3N}{4}(\bar\psi\psi)(\bar\psi\psi), \end{aligned} \tag{C7}$$

$$\begin{aligned}(\bar\psi\gamma^\mu\vec\sigma T^a\psi)(\bar\psi\gamma^\mu\vec\sigma T^a\psi) &= -\frac{N+4}{4}(\bar\psi\gamma^\mu\vec\sigma\psi)(\bar\psi\gamma^\mu\vec\sigma\psi) + \frac{3N}{4}(\bar\psi\vec\sigma\psi)(\bar\psi\vec\sigma\psi) \\ &\quad + \frac{3N}{4}(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma^\mu\psi) - \frac{9N}{4}(\bar\psi\psi)(\bar\psi\psi). \end{aligned} \tag{C8}$$

Therefore, all the $SU(2)\times SU(N)$-symmetric four-fermion interactions can be written as linear combinations of $(\bar\psi\vec\sigma\psi)(\bar\psi\vec\sigma\psi)$ and $(\bar\psi\gamma^\mu\vec\sigma\psi)(\bar\psi\gamma^\mu\vec\sigma\psi)$, namely, the two terms in Eq. (C2) [as well as Eq. (9) in the main text].

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