Search for an Immobile Hider in a Known Subset of a Network

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Abstract

A unit speed Searcher, constrained to start in a given closed set S, wishes to quickly find a point x known to be located in a given closed subset H of a metric network Q. This defines a game $G = G(Q, H, S)$, where the payoff to the maximizing Hider is the time for the Searcher path to reach x. Lengths on Q are defined by a measure λ, which then defines distance as least length of connecting path. For trees Q, we find that the minimax search time (value V of G) is given by $V = \lambda(H) - d_H(S)/2$, where $d_H(S)$ is what we call the '$H-$diameter of S', and equals the usual diameter $d(S)$ of S in the case $H = Q$. For the classical case of Gal where the S is a singleton and $H = Q$, our formula reduces to his result $V = \lambda(Q)$. If $S = H = Q$, our formula gives Dagan and Gal's result $V = \lambda(Q) - d(Q)/2$. In all other cases, our result is new. Optimal searches consist of minimum length paths covering H which start and end at points of S, traversed equiprobably in either direction.

keywords: network, search, zero-sum game, tree

1 Introduction

We are concerned with the problem of finding an unknown hiding point x on a known metric network Q, by tracing out a path at unit speed. The hiding point x might be say the midpoint of an arc; it does not have to be a node. Speed is well defined because arcs α have given lengths denoted by $\lambda(\alpha)$. The arc α can then be thought of metrically (topologically) as the closed interval $[0, \lambda(\alpha)]$, and we can extend λ to the Lebesgue measure on this interval, and then similarly to a measure λ on all of Q. The aim is to minimize the time T required to reach x. This is equivalent to minimizing the length of the search path which ends at x. An approach which goes back at least to the book of Isaacs [23] is to suppose that the point x is chosen by an adversarial Hider, who wishes to maximize the search time, thereby creating a zero-sum 'search game' where capture time T is the payoff to the maximizing Hider. In the original formulation of the problem, by Isaacs on general regions and by Gal [17] on networks, the Hider can hide (or place an object) anywhere on Q and the Searcher must start his search at a specified point known to the Hider. Roughly speaking, the Hider tends to place x away from the Searcher's start. There are two interpretations of the Hider placing x : it could be where she hides (herself) or where she places some object which the Searcher needs to find quickly.

If the network is some physical entity, say a road system, it may be that there are only a limited number of places where there is enough 'cover' to avoid the object at x being spotted from afar, circumventing the necessity of a search on foot. This is the first paper to model this as a parameter in the game. We specify that the point x must belong to an arbitrary given closed subset H (called the 'hiding set') of Q, not necessarily a node. Thus all previous work in this area can be said, in our terminology, to have the assumption $H = Q$. In addition, we model the possibility that the Searcher can only enter the network at some closed subset S of Q, known as the Searcher starting set. Together, these two new parameters define a more general search game $G = G(Q, H, S)$. The work of Gal for the simpler cases goes over to this general setting to establish that the infinite strategy game G has a value, an optimal mixed search strategy, and $\varepsilon-$optimal mixed hiding strategies (see Appendix 1 of [18]). Of course when Q is a tree all points other than leaf nodes are clearly dominated and so optimal search strategies are simply going to these in some order; so both players have finitely many undominated strategies and the standard finite minimax theorem applies. In this paper, we give a complete solution to the game G for the case where Q is a tree. Our solution has a simple formula for the value $V = V(G)$, the minimax search time, which subsumes the known cases for $H = Q$ and where S is a singleton or all of Q (fixed start or arbitrary start). For a zero-sum two person game, V is the expected value of the payoff, assuming optimal play on both sides.

Our solution involves a new concept of a signed metric called the $H-$distance $d_H(a,b)$ between points a and b of Q (again, not necessarily nodes). Roughly speaking, this distance measures the length of the shortest path P between a and b, however only the portion of P within the set H is counted positively; the portion lying outside of H is counted negatively. As long as this path intersects H, that is the definition. However if a and b are both outside H and can be connected outside of H, then we require that P is the shortest path between a and b which intersects H. So for example if a lies outside of H, then $d_H(a,a)$ is minus twice the usual distance of a to H. We then define the $H-$radius of S , denoted $\rho_H(S)$, by $\min_{a\in H}\max_{b\in S} d_H(a,b)$. If $H = Q$ then the 'distance' d_H and the 'radius' ρ_H revert to their usual definitions. Our main result, Theorem 3, says that the value V of the game $G(Q,H,S)$ is given by

$$V = \lambda(H) - d_H(S)/2 = \lambda(H) - \rho_H(S).$$

In the original paper of Gal [17], where $H = Q$ and S is a singleton, the value was found to be $\lambda(Q)$, which agrees with our result. In the paper of Dagan and Gal [15], where the Searcher was allowed to begin his search anywhere in Q (so $S = H = Q$), the value was found to be $\lambda(Q) - \rho(Q)$, which agrees with our more general result. In the cases where H is not Q and S is not a singleton or all of Q, our result is new.

2 Literature

The idea of studying worst case analysis of search problems by positing an adversarial immobile Hider goes back at least to the pioneering work of Isaacs [23]. Such search games on networks were introduced by Gal [17], who gave a complete solution to search games on a tree when the Hider could locate anywhere and the Searcher began his search at a specified point known the Hider. In our terminology, this is the case where $H = Q$ and S is a singleton set. This paper in fact contained the main ideas that were subsequently extended to cover more cases. Under the same assumptions on Q, H and S, Gal's solution was extended to weakly cyclic networks by Reijnierse and Potter [27] and to weakly Eulerian networks by Gal [19]. Another important and surprisingly difficult case is the network consisting of two nodes connected by an odd number of equal length arcs, solved by Pavlovic [26]. Search on lattices has been considered by Zoroa et al [28]. An algorithmic approach to the problem is given by Anderson and Aramendia [13]. These development has been surveyed several times, see [4], [20], [21], [25].

Enlarging the Searcher's starting set S to all of Q (called the 'arbitrary start' game) was first considered by Dagan and Gal [15] for trees, a result which is subsumed by our Theorem 3. This was extended by Alpern [1] to trees to which disjoint Eulerian networks are attached at single points. The limits to such topological extensions of the Dagan-Gal result were explored by Alpern, Baston and Gal [6], who showed that the qualitative solution to a game on a certain network changed as the length of one of its arcs changed. The arbitrary start problem was solved for symmetric networks by Alpern, Baston and Gal [7]. There is no literature on restricting the Hider to a specified subset H of the network.

Variations on the basic search methods and costs have also been considered. The arcs might have different travel times when traversed in different direction (see Alpern [2], Alpern and Lidbetter [10]; there might be a cost to turning (Demaine, Fekete and Gal [16]); the search might be able to jump to previously searched points (Alpern and Lidbetter [9]); there might be varying costs to search nodes, in addition to travel costs (Baston and Kikuta[14]); there might be several Hiders (Lidbetter [24]); there might be different speeds for travelling on the network and stooping to discover a Hider (Alpern and Lidbetter [11]); the Searcher might not be able to turn around inside an arc (Alpern [5]); the Searcher might have to find the Hider and bring her back to the start point (Alpern [3]).

The subject of search games is covered in several books. See Gal [18], Garnaev [22], Alpern and Gal [8] and Alpern et al [12].

3 Definitions and Main Result

3.1 The $H-$distance d_H

The usual 'shortest path' distance on a tree Q is defined as the minimum length of a path between the two points. On a tree, such a path (called a geodesic) will always be unique. In the presence of a given set H, part of the shortest path will lie within H and some of it will be outside. The $H-$distance d_H counts the part inside H positively and the part outside H negatively. Of course if $H = Q$, then this reduces to the usual distance. In general, we should think of d_H as a signed distance. A more formal definition, which includes the possibility that the shortest path between a and b does not pass through H, is given below.

Definition 1 *Suppose Q is a tree and $H \subset Q$ is connected, hence also a tree. Let $P_H(a,b)$ be the minimum length path starting at a, ending at b, and intersecting H. Define the $H-$distance $d_H(a,b)$ between points $a, b \in Q$ by*

$$d_H(a,b) = \lambda(P_H(a,b) \cap H) - \lambda\left(P_H(a,b) \cap \bar{H}\right), \tag{1}$$

where $\bar{H}$ denotes the complement of H in Q.

Note that if $H = Q$ then d_H reduces to the usual distance d; otherwise it is not a metric.

To illustrate the calculation of d_H, consider the family of trees $Q = Q_{r,s}$ drawn below in Figure 1. Here H is the subtree which is a star with three rays of length 2 and the Searcher start set S consists of the three points A, B and C. We have drawn H in thick red lines and S as green disks. Note that the network Q has total length given by $\lambda(Q) = 6 + r + s$.

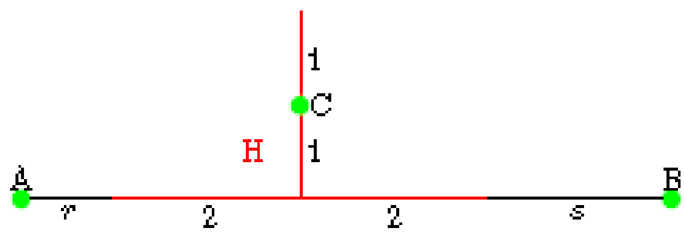

Figure 1. Tree $Q = Q_r, s$ with Hider tree H (thick red).
Searcher start $S = \{A, B, C\}$.

The function for d_H on the set $\{A, B, C\}$ is given by the symmetric matrix. Note that some entries may be negative.

d_H	A	B	C
A	$-2r$	$4 - r - s$	$3 - r$
B	$4 - r - s$	$-2s$	$3 - s$
C	$3 - r$	$3 - s$	0

(2)

Table 1: The 'distances' d_H for $Q_{r,s}$.

There are several observations to be made. For points like C, which lie inside H, self distances are 0. However for points like A, lying outside H, the path $P_H(A, A)$ goes to H and back, so the self distance is twice the distance to H. The $H-$distance $d_H(C, A)$ is based on the path $P_H(C, A)$ which consists of a path (red) in H of length 3 followed by a path outside H (blue) of length r, so $d_H(C, A) = 3 - r$ by the definition (1).

Recall the usual definitions of the diameter and radius of a compact metric space (X, d) as $d(X) = \max_{x,y} d(x, y)$ and $\rho(X) = \min_x \max_y d(x, y) = d(X)/2$. We use the analog to define the $H-$versions of these concepts. We will only consider the case where Q is a tree in this paper, which makes the center c unique (but not the antipodal pairs). A center is a point which minimizes the maximum distance to other points.

Definition 2 *Let Q be a tree. Define the H-diameter of S, denoted $d_H(S)$ as*

$$d_H(S) = \max_{a,b \in S} d_H(a,b) = d_H(a^*, b^*),$$

where such a pair a^* and b^* are called antipodal points. Note that it is possible that $a^* = b^*$. It is also possible that the pair is not unique. Define the H-radius of S by

$$\rho_H(S) = \min_{x \in H} \max_{y \in S} d_H(x,y) = \max_{y \in S} d_H(c,y) = d_H(S)/2,$$

where c is called the $H-$center of S.

Note that if $Q = H = S$ is a star with m equal length rays, $m > 2$, then any pair of distinct leaf nodes form an antipodal pair. To illustrate these concepts, take $r = 1$ and $s = 2$ in the network depicted in Figure 1. The $H-$diameter will be simply the largest element of the 9 entry matrix given in Table 1, where the antipodal points a^* and b^* are A and C. We thus have

$$d_H(\{A,B,C\}) = d_H(A,C) = 3 - r = 3 - 1 = 2.$$

The $H-$center of S is the node of degree 3, which has respective distances from the points A, B, and C of S of $1, 0$ and 1, so the $H-$radius is $1 = \max\{1,0,1\}$ (which follows from the above).

3.2 Main Result

Now that we have defined and explained the notion of $H-$diameter or $H-$radius, we can easily state our main result in terms of either of these. The assumption that the hiding set H is connected is not required, because we can replace H with its connected hull, without helping the Hider. That will be explained later. At the moment, we give the optimal Hider mixed strategy in terms of the (known) optimal strategy found by Gal [17] for the game $G(Q,H,S)$ when the Hider can hide anywhere ($H = Q$) and the Searcher starts at a known point c (the only point in the singleton set S). We will explicitly describe this strategy h_c later in Section 5. The distribution h_c is a probability measure on H concentrated on the leaf nodes of the tree H and is uniquely defined by the property that at every branch node, the measure of each branch (away from the root c) is proportional to its total length. For that reason it has been called the Equal Branch Density (EBD) distribution, see Alpern [2]. Note that the EBD distribution is a function of both the tree Q and the chosen root c.

Theorem 3 *Let Q be a tree and let H be a connected closed subset of Q (a subtree). Then the value V of the game $G(Q, H, S)$ is given by*

$$V = \lambda(H) - d_H(S)/2 = \lambda(H) - \rho_H(S). \tag{3}$$

An optimal strategy for the Hider is to adopt Gal's optimal strategy h_c for the game where $H = Q$ and $S = \{c\} \in H$ (the EBD strategy), where c is the $H-$center of S. An optimal strategy for the Searcher is to pick a minimum length path $\bar{P}$ which starts and ends at antipodal points of S and covers H, and to traverse $\bar{P}$ with probability $1/2$ in each direction. (A path covers a set if every point in the set belongs to the path.)

A trivial example is when H is a singleton $\{x\}$ in which case it is obvious that the Searcher starts at the closest point $y \in S$ to x and the value of the game is $d(x, y)$. But this is the same as $-\rho_H(S)$ and since $\lambda(H) = 0$ in this case our formula is trivially correct. For the more interesting example given by the network of Figure 1 with $r = 1$ and $s = 2$ the Theorem says that the expected search time is given by $V = \lambda(H) - \rho_H(S) = 6 - 1 = 5$ and an optimal searcher strategy is to traverse a shortest path between A and C which covers H equiprobably in either direction. Taking the root of H to be the $H-$center given by the node of degree 3, we see that it has three equal length branches. So the optimal strategy for the Hider is the EBD distribution which gives the same probability to each of the three leaf nodes of H. In Section 6 we will consider a range of r and s.

We will prove Theorem 1 in two parts. In Section 4 we will establish in (5) that V is not more than the stated value and in Section 5 we will show in (8) that it is not less than the stated value.

Two well known results can be immediately obtained as corollaries when $Q = H$.

Corollary 4 *If the Hider can choose the hiding point x anywhere in Q, that is $H = Q$, then*

$$V = \lambda(H) - d(S)/2 = \begin{cases} \lambda(H), & \textit{if } S \textit{ is a singleton [17], and} \\ \lambda(H) - d(H)/2 & \textit{if } H = S = Q \textit{ [15].} \end{cases}$$

In Theorem 3 we assumed that the Hider set H is a tree. Suppose the Hider is forced to locate in some closed set $\hat{H}$. Define H to be the tree spanned by $\hat{H}$, the intersection of all connected supersets of H. Clearly, since the Hider has more strategies, we have

$$V(Q,H,S) \geq V\left(Q,\hat{H},S\right).$$

On the other hand, by Theorem 3, an optimal Hider strategy for the game $G(Q,H,S)$ is supported on the leaves of the tree H, which all must be in $\hat{H}$. So restricting the Hider to $\hat{H}$ does not decrease the value. So Theorem 3 holds in the case where H is the spanning tree of the Hider set.

4 Searcher Strategies

We begin with an elementary result which has appeared in various forms and contexts, beginning perhaps with Theorem 12.3.1 of Isaacs [23].

Lemma 5 *Let P be a path in Q of length L which covers H. Then the equiprobable mixture P^* of P and its time reversed path P^{-1} (defined by $P^{-1}(t) = P(L-t)$) reaches any point x of H in expected time not exceeding $L/2$.*

Proof. Fix any point $x \in H$. Since P covers H it reaches x at some time $t \leq L$. At time $L-t$ the reverse path P^{-1} is at $P^{-1}(L-t) = P(L-(L-t)) = P(t) = x$ and so $T(x,P^{-1}) \leq L-t$. So P^* reaches x in expected time not exceeding $(t+(L-t))/2 = L/2$. ■

Obviously the Searcher, who wants to minimize the capture time, wants to find such a covering path with minimum length L. Let $\bar{P} = P_H(a,b)$ be a path starting and ending at points a and b of S (possibly the same, in which case $\bar{P}$ is a tour), which covers H. The length $L = \lambda\left(\bar{P}\right)$ is given by

$$\begin{aligned}\lambda\left(\bar{P}\right) &= 2\lambda(H) - \lambda(P_H(a,b)\cap H) + \lambda\left(P_H(a,b)\cap \bar{H}\right)\\ &= 2\lambda(H) - d_H(a,b)\\ &\leq 2\lambda(H) - d_H(S).\end{aligned}$$

This hold with equality when a and b are antipodal points a^*, b^*. Let $T(x,P)$ denote the time taken for the path P to reach the point x, that is, the payoff if the hiding point is x and the search path is P. So taking a and b to be antipodal points and letting P^* denote the equiprobable mixture of $\bar{P}$ and its time reversed path $\bar{P}^{-1}$, we have for any $x \in H$ by Lemma 5 the estimate

$$
\begin{aligned}
T\left(x, P^*\right) &= \frac{1}{2} T\left(x, \bar{P}\right) + \frac{1}{2} T\left(x, \bar{P}^{-1}\right) \\
&\leq \frac{1}{2} \lambda\left(\bar{P}\right) \\
&= \frac{1}{2}\left(2\lambda\left(H\right) - d_H\left(S\right)\right) \\
&= \lambda\left(H\right) - \frac{d_H\left(S\right)}{2}.
\end{aligned}
\tag{4}
$$

Hence we have

$$V \leq \lambda\left(H\right) - \frac{d_H\left(S\right)}{2}. \tag{5}$$

This establishes the Searcher part of Theorem 3.

5 Hider Strategies

The optimal strategy for the hider is a variant of the optimal mixed strategy h_c for the game $G\left(Q, H, \{c\}\right)$, where $Q = H$ and the searcher starts at a given point c, found by Gal [17]. (The strategy h_c considered Q as a tree with root c and has all its weight on leaf nodes. At every branch node of Q, it assigns total weight to each branch proportional to its total length. This strategy is known as the Equal Branch Density (EBD) strategy.) For our purposes here, we only need to know Gal's result in [17] that against any pure search strategy starting at a known point c, the expected search time is at least $\lambda\left(H\right)$.

Suppose the hider adopts the strategy h_c, taking c to be the H-center of Q, as defined earlier. Let P be any searcher strategy, a path starting at some point $a \in S$ and covering H. Let a' be the first point of H reached by P, at some time $t_0 = d\left(a, a'\right)$. Once reaching a', the expected additional time to find the hider is at least $\lambda\left(H\right) - d\left(c, a'\right)$; otherwise in the game where the searcher starts at c, he could first go to a' and then find the hider in expected additional time less than $\lambda\left(H\right) - d\left(c, a'\right)$, or in total time less than $\lambda\left(H\right)$, contradicting Gal's 1979 result. So we have shown that the expected time to find a hider distributed according to h_c, when starting at a point $a \in S$, is at least $d(a,a')$ plus the time needed to find the Hider when starting at a'.

$$\begin{aligned}
T(h_c,P) &\geq d(a,a') + [\lambda(H) - d(c,a')] \\
&= \lambda(H) + [d(a,a') - d(c,a')] \qquad (6)\\
&= \lambda(H) - d_H(c,a) \;\; \text{(because } d_H(c,a) = [d(c,a') - d(a',a)] \\
&\geq \lambda(H) - \rho_H(S) \;\; \text{(by definition of } H\text{-radius } \rho_H) \\
&= \lambda(H) - \frac{d_H(S)}{2}, \text{ for any searcher path } P. \qquad (7)
\end{aligned}$$

It follows that

$$V \geq \lambda(H) - \frac{d_H(S)}{2}. \qquad (8)$$

We note that combining our two estimates for the value, (8) and (5), establishes our main result, Theorem 3.

6 Examples: The $Q_{r,s}$ network of Figure 1

We now give a family of examples, based on the network of Figure 1, which illustrate various cases of optimal strategies in Theorem 3. Without loss of generality we may assume $r \leq s$, so we have $d_H(C,A) \geq d_H(C,B)$. The case $r=1$ and $s=2$ has already been covered in the main text, showing in particular that $V=5$.

6.1 The network $Q_{3.5,3.5}$

Consider the case with $r=s=3.5$. In this case all the elements of the matrix (2) are negative except for $d_H(C,C)=0$. So the antipodal points are C and C itself. The $H-$radius of S, $\rho_H(S)=0$ taking the center at $c=C$. Thus $V=6-0=6$. An optimal mixed strategy is to pick a Chinese Postman tour of H starting at C, and to traverse it equiprobably in either direction. For the Hider, the EBD strategy on H taking C as the root hides at the top leaf of H with probability $1/6$, with the remaining probability being split equally at $5/12$ for the left and right leaves of H. This structure clearly holds for all lengths $3 \leq r \leq s$.

6.2 The network $Q_{1/2,2}$

Consider the case $r = 1/2$ and $s = 2$. Here the largest entry of the matrix (2) is $d_H(A, C) = 5/2 = d_H(S)$, so by Theorem 3 we have $V = 6 - 5/4 = 19/4$. The $H-$radius is $5/4$ and the $H-$center is the point at distance $1/4$ to the left of the degree 3 node. An optimal search strategy is to go between A and C covering H in a random direction. The Hider should locate at the left leaf of H with probability $(2 - 1/4)/6 = 7/24$ and then at the other two nodes with the same probability $17/48$. More generally, for $s \geq 1$ and $r \leq 3$, we have $V = 6 - d_H(A, C) = 6 - (3 - r)/2 = (9 + r)/2$.

6.3 The Network $Q_{1/4,1/2}$

Consider the case $r = 1/4$ and $s = 1/2$. Now the largest element of the matrix (2) is $d(A, B) = 4 - r - s = 4 - 3/4 = 13/4$. So $V = 6 - 13/8 = 35/8$. The optimal search strategy is to go between A and B covering H equiprobably in either direction. The $H-$center of S is at distance y to the left of the degree 3 node, where y satisfies the equation

$$(2 - y) - 1/4 = (2 + y) - 1/2, \text{ or } y = 1/8.$$

So the Hider locates at the left leaf of H with probability $(2 - 1/8)/6 = 5/16$. As above, the remaining probability is split equally between the other two leaf nodes of H. More generally, for $r \leq s \leq 1$ we have $V = 6 - (4 - r - s)/2 = (8 + r + s)/2$.

6.4 Summary for $Q_{r,s}$

The region of the space of the parameters r and s where the optimal endpoints of the Searcher path are AB, AC or CC (path is a tour) is drawn in Figure 2, together with the value $V = V(r, s)$. By symmetry, it is enough to consider the region $r \leq s$. There are other possibilities for other networks; for example the optimal search path could start and end at distinct points of H.

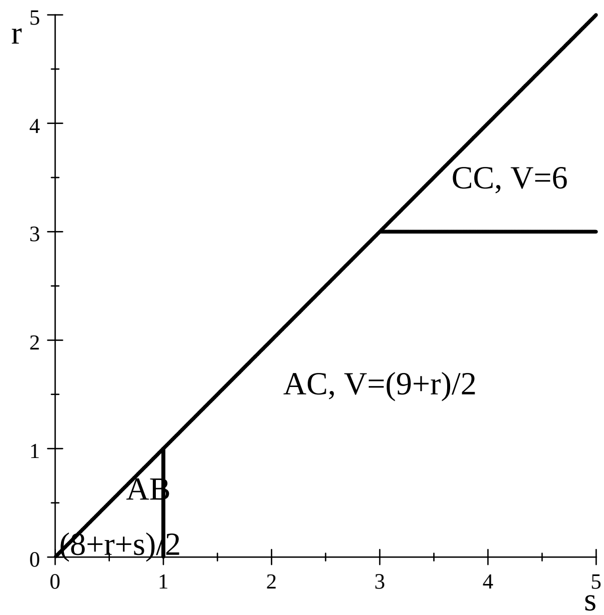

Figure 2. Regions where search endpoints AB,AC,CC are optimal.

7 Conclusions

This paper generalizes previous assumptions about the start of a network search game with immobile Hider by allowing restricted hiding places and restricted places for the Searcher to begin. Despite this, we are able to give a complete solution to the problem for trees. Many aspects of the solution mirror that of Gal's original paper [17] with specified searcher start and hiding anywhere. In particular, there exist optimal strategies for the Searcher which consist of only two equiprobable pure strategies (paths) which are time reversals of each other. Also, the Hider's optimal distribution over H is an EBD distribution for a root determined by the problem, as in Dagan and Gal [15]. It remains to be seen whether this degree of generalization is amenable to other variations of the problem such as expanding search, variable speed search, turning costs, and combined search and travel costs. These variations were discussed in the literature section. In addition, it is possible that some of these results can be extended to other networks than trees; at least to networks consisting of trees to which disjoint Eulerian networks are attached at single points (as in [1]).

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