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Exponential Stability of Highly Nonlinear Neutral Pantograph Stochastic Differential Equations

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ABSTRACT

In this paper, we investigate the exponential stability of highly nonlinear hybrid neutral pantograph stochastic differential equations(NPSDEs). The aim of this paper is to establish exponential stability criteria for a class of hybrid NPSDEs without the linear growth condition. The methods of Lyapunov functions and M-matrix are used to study exponential stability and boundedness of the hybrid NPSDEs.

Key Words: highly nonlinear, Itô's formula, exponential stability, neutral pantograph stochastic differential equations, M-matrix.

I. INTRODUCTION

Stochastic delay differential equations are widely used to model stochastic systems whose evolution depends on past history of the state. On the other hand, those systems may often experience abrupt changes in their structures and parameters caused by phenomena such as component failures or repairs, the hybrid systems driven by continuous-time Markov chains are often used to model such systems (see, e.g., [1–8]). Stability and boundedness are two of most popular topics in the area of systems and controls, most of the papers can only be applied to delay systems where their coefficients are either linear or nonlinear but bounded by linear functions (see, e.g., [9–12]). Recently, there are some progress on stability for highly nonlinear stochastic delay systems. For example, Hu *et al.* [13] investigated the stability and boundedness for nonlinear stochastic differential delay equations (SDDEs) with Markovian switching without the linear growth condition, the robust stability and boundedness of SDDEs without the linear growth condition were studied by Hu *et al.* [14].

Pantograph differential equation was used by Ockendon and Tayler [15] to investigate how the electric current is collected by the pantograph of an electric locomotive, from where it gets the name. Hybrid pantograph stochastic differential equations(PSDEs) are unbounded SDDEs which have been frequently applied in many practical areas, including biology, mechanic, engineering and finance. We refer the reader to [16–23] where pantograph stochastic differential equations and pantograph stochastic differential equations with Markovian switching are considered. In fact, many dynamical systems do not only depend on present and past states but also involve derivatives with delays. Neutral differential delay equations are often used to model such systems (see, e.g., [24–30]). And neutral pantograph differential equations have been studied by some researchers [31, 32].

To the best of our knowledge, there is so far little exponential stability theory on hybrid neutral pantograph stochastic differential equations (NPSDEs) without linear growth condition. Inspired by the works

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of You *et al.* [22], the paper shall develop exponential stability criteria by the methods of Lyapunov functions and M-matrices for a class hybrid neutral pantograph stochastic differential equations. The rest of the paper is organized as follows. Section 2 offers some necessary notations, assumptions and lemmas. Section 3 establishes exponential stability criterion and boundedness for hybrid neutral pantograph stochastic differential equations by the methods of Lyapunov functions. In Section 4, stability criterion will be proposed with the aid of M-matrices. In Section 5, two examples are discussed to illustrate the theory. Finally, a conclusion is drawn in Section 6.

II. Preliminary

Before stating our main results, we present essential notations and definitions which are necessary for further consideration. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous-left-limit Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. We also denote by $|x|$ the Euclidean norm for $x \in R^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. If both a and b are real numbers, then $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

We also need some notation on M-matrices. For a vector or matrix A, by $A > 0$ we mean all elements of A are positive. A Z-matrix is a square matrix $A = (a_{ij})_{N\times N}$ which has non-positive off-diagonal entries (namely $a_{ij} \leq 0$ for all $i \neq j$). The following lemma provides us with two useful criteria to verify if a given Z-matrix is a nonsingular M-matrix.

Lemma II.1 *Let $A = (a_{ij})_{N\times N}$ be a Z-matrix. Then A is a nonsingular M-matrix if and only if one of the the following statements holds:*

(1) A^{-1} exists and its elements are all nonnegative.

(2) There exists $x > 0$ in R^N such that $Ax > 0$.

We now cite the useful nonnegative semi-martingale convergence theorem as a lemma.

Lemma II.2 *([6, Theorem 1.10 on page 18]) Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ξ be a nonnegative $\mathcal{F}_0$-measurable random variable such that $E\xi < \infty$. Define $X(t) = \xi + A(t) - U(t) + M(t)$ for all $t \geq 0$. If $X(t)$ is nonnegative, then*

$$\left\{\lim_{t\to\infty} A(t) < \infty\right\} \subset \left\{\lim_{t\to\infty} X(t) < \infty\right\} \cap \left\{\lim_{t\to\infty} U(t) < \infty\right\} \quad \text{a.s.}$$

Consider an n-dimensional hybrid NPSDE

$$d[x(t) - D(x(\theta t), r(t), t)] = f(x(t), x(\theta t), r(t), t)dt + g(x(t), x(\theta t), r(t), t)dB(t), 0 < \theta < 1 \quad (1)$$

on $t \geq 0$ with initial data $x(0) = x_0, r(0) = r_0$, where the coefficients

$f : R^n \times R^n \times S \times R_+ \to R^n$ and $g : R^n \times R^n \times S \times R_+ \to R^{n\times m}$ are Borel measurable. As a standing hypothesis, we assume the coefficients are locally Lipschitz continuous (see, e.g., [6, 10]).

Assumption II.3 *For each integer $h \geq 1$ there is a positive constant K_h such that*

$$|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)|^2 \vee |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)|^2 \leq K_h(|x - \bar{x}|^2 + |y - \bar{y}|^2)$$

for those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq h$ and all $(t, i) \in R_+ \times S$.

Assume moreover that there is a constant $\kappa \in (0, 1)$ such that

$$|D(u, i, t) - D(v, i, t)| \leq \kappa \exp(-\frac{1}{p}(1 - \theta)t)|u - v| \quad (2)$$

for all $u, v \in R$, and $D(0, i, t) = 0$.

Remark II.4 *In order to overcome the difficulties caused by the unbounded delay θt, the factor $e^{-\frac{1}{p}(1-\theta)t}$ is used in the neutral part.*

For $V \in C^{2,1}(R^n \times S \times R_+; R_+)$, define an operator $LV : R^n \times R^n \times S \times R_+ \to R$ by

$$LV(x - D(y,i,t), y, i, t) = V_t(x - D(y,i,t), i, t)$$
$$+ V_x(x - D(y,i,t), i, t) f(x, y, i, t)$$
$$+ \frac{1}{2}\text{trace}[g^T(x,y,i,t)V_{xx}(x - D(y,i,t), i, t)g(x,y,i,t)]$$
$$+ \sum_{j=1}^{N} \gamma_{ij} V(x - D(y,j,t), j, t),$$

where

$$V_t(x,i,t) = \frac{\partial V(x,i,t)}{\partial t},$$
$$V_x(x,i,t) = \Big(\frac{\partial V(x,i,t)}{\partial x_1}, \cdots, \frac{\partial V(x,i,t)}{\partial x_n}\Big),$$
$$V_{xx}(x,i,t) = \Big(\frac{\partial^2 V(x,i,t)}{\partial x_k \partial x_l}\Big)_{n\times n}.$$

III. Criterion in terms of Lyapunov functions

In this section, we will give a criterion on exponential stability and boundedness for system (1) by Lyapunov functions.

Assumption III.1 *There exists the function $V(x,i,t) \in C^{2,1}(R^n \times S \times R_+, R_+)$, and positive constants c_1, c_2, β_k $(k = 1, \dots, 5)$ such that*

$$c_1|x|^p \le V(x,i,t) \le c_2|x|^p,$$
$$for\ all\ (x,i,t) \in R^n \times S \times R_+,$$

$$LV(x - D(y,i,t), y, i, t)$$
$$\le \beta_1 - \beta_2|x|^p + \beta_3\theta\exp(-(1-\theta)t)|y|^p$$
$$- \beta_4|x|^q + \beta_5\theta\exp(-(1-\theta)t)|y|^q, \qquad (3)$$

where $q > p \ge 2$, $c_2 > c_1, \beta_2 > \beta_3, \beta_4 > \beta_5$.

Theorem III.2 *Let Assumption II.3, III.1 hold, then for any given initial data, there exists a unique global solution $x(t)$ of (1), moreover, the solution has the properties that*

$$\int_0^t E|x(s)|^q ds \le \frac{c_2|x(0) - D(x(0), r(0), 0)|^p}{\beta_4 - \beta_5} + \frac{\beta_1 t}{\beta_4 - \beta_5} \qquad (4)$$

and

$$\limsup_{t\to\infty} E|x(t)|^p \le \frac{\beta_1}{(1-\kappa)^p c_1 \varepsilon}, \qquad (5)$$

where $\varepsilon := \min(1, \frac{\beta_2 - \beta_3}{2^{p-1}c_2(1+\frac{\kappa^p}{\theta})})$.

Proof: By the similar method used in Theorem 1 of [21], the existence and uniqueness of solution can be got. Fix the initial data $x_0 \in R^n$ and $r_0 \in S$ arbitrarily. Define $z(t) = x(t) - D(x(\theta t), r(t), t)$, then we have $|z(0)| \le |x_0| + |D(x_0, r(0), 0)| \le (1+\kappa)|x_0|$. Let $k_0 > 0$ be a sufficiently large integer such that $(1+\kappa)|x_0| < k_0$. For each integer $k > k_0$, define the stopping time

$$\sigma_k = \inf\{t \ge 0 : |z(t)| \vee |x(t)| \ge k\}.$$

Throughout this paper we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). It is easy to see that σ_k is increasing as $k \to \infty$ and $\lim_{k\to\infty}\sigma_k = \infty$ a.s. By the generalized Itô formula (see, e.g., [6, Theorem 1.45 on page 48]) we obtain that

$$EV(z(t\wedge\sigma_k), r(t\wedge\sigma_k), t\wedge\sigma_k)$$
$$= V(z(0)) + E\int_0^{t\wedge\sigma_k} LV(z(s), x(\theta s), r(s), s)ds.$$

By Assumption (III.1) and $\exp(-(1-\theta)t) \le 1$, we can obtain that

$$c_1E|z(t\wedge\sigma_k)|^p \le c_2|z(0)|^p + \beta_1 t$$
$$- \beta_2 E\int_0^{t\wedge\sigma_k}|x(s)|^p ds + \beta_3\theta E\int_0^{t\wedge\sigma_k}|x(\theta s)|^p ds$$
$$- \beta_4 E\int_0^{t\wedge\sigma_k}|x(s)|^q ds + \beta_5\theta E\int_0^{t\wedge\sigma_k}|x(\theta s)|^q ds$$

Noting that

$$\beta_3\theta E\int_0^{t\wedge\sigma_k}|x(\theta s)|^p ds = \beta_3 E\int_0^{\theta(t\wedge\sigma_k)}|x(u)|^p du$$
$$\le \beta_3 E\int_0^{t\wedge\sigma_k}|x(u)|^p du$$

and

$$\beta_5\theta E\int_0^{t\wedge\sigma_k}|x(\theta s)|^q ds = \beta_5 E\int_0^{\theta(t\wedge\sigma_k)}|x(u)|^q du$$
$$\le \beta_5 E\int_0^{t\wedge\sigma_k}|x(u)|^q du,$$

due to $\beta_2 > \beta_3, \beta_4 > \beta_5$, we can obtain

$$c_1E|z(t\wedge\sigma_k)|^p \le c_2|z(0)|^p$$
$$+ \beta_1 t - (\beta_4 - \beta_5)E\int_0^{t\wedge\sigma_k}|x(s)|^q ds. \qquad (6)$$

Noting that $\beta_5 < \beta_4$, we obtain

$$c_1E|z(t\wedge\sigma_k)|^p \le c_2|z(0)|^p + \beta_1 t.$$

Consequently

$$c_1 k^p P\{\sigma_k \le t\} \le c_2|z(0)|^p + \beta_1 t.$$

Letting $k \to \infty$ gives that $P\{\sigma_\infty \le t\} = 0$. This means that $\sigma_\infty > t$ a.s. Letting $t \to \infty$, we can get $\sigma_\infty = \infty$ a.s, which implies that there exists a global solution $x(t)$ to the system (1).

We shall show assertion (4). It follows from (6) that

$$(\beta_4 - \beta_5)E\int_0^{t\wedge\sigma_k} |x(s)|^q ds \le c_2|z(0)|^p + \beta_1 t,$$

let $k \to \infty$, we have

$$(\beta_4 - \beta_5)E\int_0^{t} |x(s)|^q ds \le c_2|z(0)|^p + \beta_1 t,$$

by which we can obtain

$$E\int_0^{t} |x(s)|^q ds \le \frac{c_2|z(0)|^p}{\beta_4 - \beta_5} + \frac{\beta_1 t}{\beta_4 - \beta_5}.$$

Using the well-known Fubini theorem to obtain

$$\int_0^{t} E|x(s)|^q ds \le \frac{c_2|z(0)|^p}{\beta_4 - \beta_5} + \frac{\beta_1 t}{\beta_4 - \beta_5},$$

which is the desired assertion (4).

We now claim the assertion (5). By the generalized Itô formula, we have

$$\begin{aligned}
&E[\exp(\varepsilon(t\wedge\sigma_k))V(z(t\wedge\sigma_k), r(t\wedge\sigma_k), t\wedge\sigma_k)]\\
&= V(z(0), r(0), 0) + E\int_0^{t\wedge\sigma_k} \varepsilon\exp(\varepsilon s)V(z(s), r(s), s)ds\\
&+ E\int_0^{t\wedge\sigma_k} \exp(\varepsilon s)LV(z(s), x(\theta s), r(s), s)ds. \qquad (7)
\end{aligned}$$

By (3) and inequality

$$\begin{aligned}
&|x(t) - D(x(\theta t), r(t), t)|^p\\
&\le 2^{p-1}(|x(t)|^p + |D(x(\theta t), r(t), t)|^p)\\
&\le 2^{p-1}(|x(t)|^p + \kappa^p \exp(-(1-\theta)t)|x(\theta t)|^p),
\end{aligned}$$

we can obtain that

$$\begin{aligned}
&E\int_0^{t\wedge\sigma_k} \varepsilon\exp(\varepsilon s)V(z(s), r(s), s)ds\\
&\le 2^{p-1}c_2 E\int_0^{t\wedge\sigma_k} \varepsilon\exp(\varepsilon s)\\
&\times(|x(s)|^p + \kappa^p\exp(-(1-\theta)s)|x(\theta s)|^p)ds\\
&= 2^{p-1}c_2 E\int_0^{t\wedge\sigma_k} \varepsilon\exp(\varepsilon s)|x(s)|^p ds\\
&+ 2^{p-1}\kappa^p c_2 E\int_0^{t\wedge\sigma_k} \varepsilon\exp((\varepsilon - 1 + \theta)s)|x(\theta s)|^p ds
\end{aligned}$$

and

$$\begin{aligned}
&E\int_0^{t\wedge\sigma_k} \exp(\varepsilon s)LV(z(s), r(s), s)ds\\
&\le E\int_0^{t\wedge\sigma_k} \exp(\varepsilon s)(\beta_1 - \beta_2|x(s)|^p\\
&+ \beta_3\theta\exp(-(1-\theta)s)|x(\theta s)|^p - \beta_4|x(s)|^q\\
&+ \beta_5\theta\exp(-(1-\theta)s)|x(\theta s)|^q)ds\\
&\le \frac{\beta_1}{\varepsilon}\exp(\varepsilon t) - \beta_2 E\int_0^{t\wedge\sigma_k} \exp(\varepsilon s)|x(s)|^p ds\\
&+ \beta_3 E\int_0^{t\wedge\sigma_k} \theta\exp((\varepsilon - 1 + \theta)s)|x(\theta s)|^p ds\\
&- \beta_4 E\int_0^{t\wedge\sigma_k} \exp(\varepsilon s)|x(s)|^q ds\\
&+ \beta_5 E\int_0^{t\wedge\sigma_k} \theta\exp((\varepsilon - 1 + \theta)s)|x(\theta s)|^q ds.
\end{aligned}$$

Note that $0 < \theta < 1$ and $\varepsilon \le 1$, we have $(\varepsilon - 1 + \theta)/\theta \le \varepsilon$. Thus, we get

$$\begin{aligned}
&2^{p-1}\kappa^p E\int_0^{t\wedge\sigma_k} \varepsilon\exp((\varepsilon - 1 + \theta)s)|x(\theta s)|^p ds\\
&\le 2^{p-1}\kappa^p \frac{1}{\theta} E\int_0^{t\wedge\sigma_k} \varepsilon\exp(\varepsilon s)|x(s)|^p ds.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\beta_3 E\int_0^{t\wedge\sigma_k} \theta\exp((\varepsilon - 1 + \theta)s)|x(\theta s)|^p ds\\
&\le \beta_3 E\int_0^{t\wedge\sigma_k} \exp(\varepsilon s)|x(s)|^p ds,\\
&\beta_5 E\int_0^{t\wedge\sigma_k} \theta\exp((\varepsilon - 1 + \theta)s)|x(\theta s)|^q ds\\
&\le \beta_5 E\int_0^{t\wedge\sigma_k} \exp(\varepsilon s)|x(s)|^q ds.
\end{aligned}$$

Substituting above inequalities into (7), we have

$$\begin{aligned}
&E[\exp(\varepsilon(t\wedge\sigma_k))V(z(t\wedge\sigma_k), x(\theta(t\wedge\sigma_k)),\\
&r(t\wedge\sigma_k), t\wedge\sigma_k] \le c_2|z(0)|^p + \frac{\beta_1}{\varepsilon}\exp(\varepsilon t)\\
&- (\beta_2 - \beta_3 - 2^{p-1}c_2\varepsilon(1 + \frac{\kappa^p}{\theta}))E\int_0^{t\wedge\sigma_k} |x(s)|^p ds\\
&- (\beta_4 - \beta_5)E\int_0^{t\wedge\sigma_k} |x(s)|^q ds \qquad (8)
\end{aligned}$$

For $\varepsilon \leq \frac{\beta_2-\beta_3}{2^{p-1}c_2(1+\frac{\kappa p}{\theta})}$, $\beta_4 > \beta_5$ and Assumption (III.1), we can deduce

$$E[\exp(\varepsilon(t\wedge\sigma_k))|z(t\wedge\sigma_k)|^p] \leq \frac{c_2}{c_1}|z(0)|^p + \frac{\beta_1}{c_1\varepsilon}\exp(\varepsilon t).$$

Letting $k \to \infty$, we obtain

$$\exp(\varepsilon t)E|z(t)|^p \leq \frac{c_2}{c_1}|z(0)|^p + \frac{\beta_1}{c_1\varepsilon}\exp(\varepsilon t), \qquad (9)$$

which shows

$$\limsup_{t\to\infty} E|z(t)|^p \leq \frac{\beta_1}{c_1\varepsilon} \qquad (10)$$

and

$$\limsup_{0\leq t<\infty} E|z(t)|^p < \infty. \qquad (11)$$

By the elementary inequality (see [33, Lemma 4.3])

$$(a+b)^p \leq (1+\epsilon)^{p-1}(a^p + \epsilon^{1-p}b^p), \quad \forall a,b \geq 0,\ p>1,\ \epsilon>0,$$

we have

$$E|x(t)|^p \leq (1+\epsilon)^{p-1}(E|z(t)|^p + \epsilon^{1-p}|D(x(\theta t), r(t), t)|^p),$$

setting $\epsilon = \frac{\kappa}{1-\kappa}$, we derive

$$\begin{aligned} &E|x(t)|^p \\ &\leq (1-\kappa)^{1-p}E|z(t)|^p + \kappa^{1-p}E|D(x(\theta t), r(t), t)|^p \\ &\leq (1-\kappa)^{1-p}E|z(t)|^p + \kappa\exp(-(1-\theta)t)E|x(\theta t)|^p \\ &\leq (1-\kappa)^{1-p}E|z(t)|^p + \kappa E|x(\theta t)|^p. \end{aligned} \qquad (12)$$

By inequality (12), we have that for any $T > 0$,

$$\begin{aligned} &\sup_{0\leq t\leq T} E|x(t)|^p \\ &\leq \sup_{0\leq t\leq T}(1-\kappa)^{1-p}E|z(t)|^p + \sup_{0\leq t\leq T}\kappa E|x(\theta t)|^p \\ &\leq \sup_{0\leq t\leq T}(1-\kappa)^{1-p}E|z(t)|^p + \sup_{0\leq t\leq T}\kappa E|x(t)|^p, \end{aligned} \qquad (13)$$

we can get

$$(1-\kappa)^p \sup_{0\leq t\leq T} E|x(t)|^p \leq \sup_{0\leq t\leq T} E|z(t)|^p. \qquad (14)$$

Letting $T \to \infty$, we have

$$(1-\kappa)^p \limsup_{0\leq t<\infty} E|x(t)|^p \leq \limsup_{0\leq t<\infty} E|z(t)|^p, \qquad (15)$$

from (11), we have

$$\limsup_{t\to\infty} E|x(t)|^p < \infty,$$

by inequality (12), we can obtain that

$$(1-\kappa)^p \limsup_{t\to\infty} E|x(t)|^p \leq \limsup_{t\to\infty} E|z(t)|^p,$$

this together with (10) imply

$$\limsup_{t\to\infty} E|x(t)|^p \leq \frac{\beta_1}{(1-\kappa)^p c_1\varepsilon}.$$

Thus the proof is complete. □

Theorem III.3 *Let Assumption II.3, III.1 hold, if* $\beta_1 = 0$*, the solution of the NPSDE (1) has the properties that*

$$\int_0^\infty E|x(t)|^q dt < \infty, \qquad (16)$$

$$\int_0^\infty |x(t)|^q dt < \infty \quad \text{a.s.} \qquad (17)$$

$$\limsup_{t\to\infty} \frac{1}{t}\log E|x(t)|^p \leq -\varepsilon$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq -\frac{\varepsilon}{p} \quad \text{a.s.}$$

where ε *is defined as in Theorem III.2.*

Proof: If $\beta_1 = 0$, (4) turns to be

$$\int_0^t E|x(s)|^q ds \leq \frac{c_2|x(0) - D(x(0), r(0), 0)|^p}{\beta_4 - \beta_5},$$

letting $t \to \infty$, we have the assertion (16) holds, this implies another assertion (17). Moreover, when $\beta_1 = 0$, (9) turns to be

$$\exp(\varepsilon t)E|z(t)|^p \leq \frac{c_2}{c_1}|z(0)|^p.$$

By the same method of (13), we have that for any $T > 0$

$$(1-\kappa)^p \sup_{0\leq t\leq T} E|x(t)|^p \leq \sup_{0\leq t\leq T} E|z(t)|^p,$$

which implies that

$$\sup_{0\leq t\leq T} \exp(\varepsilon t)E|x(t)|^p \leq \frac{c_2}{c_1(1-\kappa)^p}|z(0)|^p.$$

Letting $T \to \infty$, we have

$$\sup_{0\le t<\infty} \exp(\varepsilon t)E|x(t)|^p \le \frac{c_2}{c_1(1-\kappa)^p}|z(0)|^p,$$

that is

$$\limsup_{t\to\infty} \frac{1}{t}\log E|x(t)|^p \le -\varepsilon.$$

Applying the generalized Itô formula on $\exp(\varepsilon t)V(z(t), r(t), t)$ to get

$$\begin{aligned}&\exp(\varepsilon t)V(z(t)) = V(z(0), r(0), 0)\\&+\int_0^t \varepsilon\exp(\varepsilon s)V(z(s), r(s), s)ds\\&+\int_0^t \exp(\varepsilon s)LV(z(s), x(\theta s), r(s), s)ds + M(t),\end{aligned}$$

where $M(t)$ is a local martingale with the initial value $M(0) = 0$. For $\beta_1 = 0$, applying the same argument on deriving (8), we have

$$\begin{aligned}&\int_0^t \varepsilon\exp(\varepsilon s)V(z(s), r(s), s)\\&+\exp(\varepsilon s)LV(z(s), x(\theta s), r(s), s)ds\\&\le -(\beta_2-\beta_3-2^{p-1}c_2\varepsilon(1+\frac{\kappa^p}{\theta}))\int_0^t |x(s)|^p ds\\&\quad-(\beta_4-\beta_5)\int_0^t |x(s)|^q ds \le 0.\end{aligned}$$

By Assumption (III.1) we can get

$$c_1\exp(\varepsilon t)|z(t)|^p \le c_2|z(0)|^p + M(t).$$

Applying Lemma II.2 we immediately obtain that

$$c_1 \sup_{0\le t<\infty} \exp(\varepsilon t)|z(t)|^p < \infty \quad \text{a.s.}$$

Therefore, there is a finite positive random variable ξ such that

$$c_1 \sup_{0\le t<\infty} \exp(\varepsilon t)|z(t)|^p \le \xi \quad \text{a.s.} \tag{18}$$

Applying the similarly argument as (14), we have

$$(1-\kappa)^p \sup_{0\le t<\infty} \exp(\varepsilon t)|x(t)|^p \le \sup_{0\le t<\infty} \exp(\varepsilon t)|z(t)|^p,$$

this together with (18), we obtain that

$$c_1(1-\kappa)^p \sup_{0\le t<\infty} \exp(\varepsilon t)|x(t)|^p \le \xi \quad \text{a.s.}$$

which implies that

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \le -\frac{\varepsilon}{p} \quad \text{a.s.}$$

Which completes the proof. □

IV. Criterion on M-matrix

On the basis of Lyapunov function, we establish the above Lyapunov-type stability criteria for system (1). However, the condition (3) in Assumption (III.1) is not related to the coefficient f, g explicitly. In this section, we derive some sufficient conditions by using the coefficients of system (1) to guarantee its exponential stability.

Assumption IV.1 *Assume also that there are two constants $q > p > 2$. Assume furthermore that for each $i \in S$, there are constants $\alpha_{i2} \in R$ and $\alpha_{i1}, \alpha_{i3}, a_{i4}, \alpha_{i5} \in R_+$ such that*

$$\begin{aligned}&(x-D(y,i,t))^T f(x,y,i,t) + \frac{p-1}{2}|g(x,y,i,t)|^2\\&\le \alpha_{i1} + \alpha_{i2}|x|^2 + \alpha_{i3}\theta\exp\Big(-\frac{2}{p}(1-\theta)t\Big)|y|^2\\&-\alpha_{i4}|x|^{q-p+2}\\&+\alpha_{i5}\theta\exp\Big(-\frac{q-p+2}{q}(1-\theta)t\Big)|y|^{q-p+2}.\end{aligned} \tag{19}$$

Assumption IV.2 *Under Assumption IV.1, assume furthermore that*

$$A := -\mathrm{diag}(p\tilde{\alpha}_{12}, \cdots, p\tilde{\alpha}_{N2}) - (\tilde{\gamma}_{ij})_{i,j\in S}$$

is a nonsingular M-matrix, where $\underline{a} = 2^{3-p} \wedge 1$, $\bar{a} = 2^{p-3} \vee 1$,

$$\tilde{\alpha}_{i2} = \begin{cases} \bar{a}(1+\frac{2}{p}\kappa^{p-2})\alpha_{i2} & \text{if } \alpha_{i2} \ge 0,\\ (\underline{a}-\frac{2}{p}\kappa^{p-2})\alpha_{i2} & \text{if } \alpha_{i2} < 0,\end{cases}$$

$$\text{and}\quad \tilde{\gamma}_{ij} = \begin{cases} 2^{1-p}\gamma_{ii} & \text{if } i = j,\\ 2^{p-1}\gamma_{ij} & \text{if } i \ne j.\end{cases}$$

We also define

$$\hat{\alpha}_{i2} = \begin{cases} \bar{a}\frac{(p-2)}{p}\alpha_{i2} & \text{if } \alpha_{i2} \ge 0,\\ -\frac{(p-2)}{p}\alpha_{i2} & \text{if } \alpha_{i2} < 0,\end{cases}$$

$$\text{and}\quad \hat{\gamma}_{ij} = \begin{cases} -\gamma_{ii} & \text{if } i = j,\\ 2^{p-1}\gamma_{ij} & \text{if } i \ne j.\end{cases}$$

By properties of M-matrices, we have a vector with all positive entries defined by the nonsingular M-matrix A:

$$(\lambda_1, \cdots, \lambda_N)^T = A^{-1}(1, \cdots, 1)^T > 0. \tag{20}$$

Theorem IV.3 *Let Assumptions II.3, IV.1 and IV.2 hold. Set $c_1 = \min_{i\in S}\lambda_i$, $c_2 = \max_{i\in S}\lambda_i$, $\delta_1 = \max_{i\in S}\bar{a}p\lambda_i\alpha_{i1}$, $\delta_2 = \min_{i\in S}\{1-\bar{a}\theta(p-2)\lambda_i\alpha_{i3}\}$, $\delta_3 = \max_{i\in S}\{\bar{a}\lambda_i\alpha_{i3}(2+p\kappa^{p-2})+$*

$\frac{1}{\theta}\kappa^{p-2}p\lambda_i\hat{\alpha}_{i2} + \kappa^p\frac{1}{\theta}\sum_{j=1}^N\hat{\gamma}_{ij}\lambda_j\}$, $\delta_4 = \min_{i\in S}\{p\lambda_i\alpha_{i4}(\underline{a} - \kappa^{p-2}\frac{q-p+2}{q}) - \frac{p(p-2)}{q}\theta\alpha_{i5}\lambda_i\bar{a}\}$, $\delta_5 = \max_{i\in S}\{\frac{(p-2)}{\theta q}p\lambda_i\alpha_{i4}\kappa^{p-2} + \alpha_{i5}\lambda_i\bar{a}(\frac{q-p+2}{q} + \kappa^{p-2})\}$, *and* $\rho = \frac{\theta}{1+\theta}[\delta_2 - \delta_3]$, *if* $\alpha_{i1} = 0$ *for all* $i \in S$, *then* $\rho = 0$.

Assume that

$$\delta_3 < \delta_2 \tag{21}$$

and

$$\delta_5 \leq \delta_4. \tag{22}$$

Then for any initial data, there is a unique global solution $x(t)$ *to the hybrid NPSDE (1) on* $t \in [0,\infty)$. *Moreover, the solution has the properties that*

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t E|x(s)|^q ds \leq K_1, \tag{23}$$

and

$$\limsup_{t\to\infty} E|x(t)|^p \leq K_2, \tag{24}$$

where K_1 *and* K_2 *are positive constants.*

Proof: We will define a Lyapunov function $V : R^n \times S \times R_+ \to R_+$ by

$$V(x,i,t) = \lambda_i|x|^p, \quad i \in S.$$

It is easy to see that

$$c_1|x|^p \leq V(x,i,t) \leq c_2|x|^p.$$

Now we compute $LV(x - D(y,i,t), y, i, t)$. For any $i \in S$,

$$\begin{aligned}
&LV(x - D(y,i,t), y, i, t)\\
&= p\lambda_i|x - D(y,i,t)|^{p-2}(x - D(y,i,t))^T f(x,y,i,t)\\
&\quad + \frac{p(p-2)}{2}\lambda_i|x - D(y,i,t)|^{p-4}\\
&\quad \times |(x - D(y,i,t))^T g(x,y,i,t)|^2\\
&\quad + \frac{1}{2}p\lambda_i|x - D(y,i,t)|^{p-2}|g(x,y,i,t)|^2\\
&\quad + \sum_{j\in S}\lambda_j\gamma_{ij}|x - D(y,i,t)|^p\\
&\leq p\lambda_i|x - D(y,i,t)|^{p-2}\Big[(x - D(y,i,t))^T f(x,y,i,t)\\
&\quad + \frac{p-1}{2}|g(x,y,i,t)|^2\Big]\\
&\quad + \sum_{j\in S}\lambda_j\gamma_{ij}|x - D(y,i,t)|^p\\
&\leq p\lambda_i\alpha_{i1}|x - D(y,i,t)|^{p-2}\\
&\quad + p\lambda_i\alpha_{i2}|x - D(y,i,t)|^{p-2}|x|^2\\
&\quad + \sum_{j\in S}\gamma_{ij}\lambda_j|x - D(y,i,t)|^p\\
&\quad + p\lambda_i\alpha_{i3}\theta\exp(-\frac{2}{p}(1-\theta)t)|x - D(y,i,t)|^{p-2}|y|^2\\
&\quad - p\lambda_i\alpha_{i4}\theta|x - D(y,i,t)|^{p-2}|x|^{q-p+2}\\
&\quad + p\lambda_i\alpha_{i5}\theta\exp(-\frac{q-p+2}{q}(1-\theta)t)\\
&\quad \times |x - D(y,i,t)|^{p-2}|y|^{q-p+2}.
\end{aligned} \tag{25}$$

Noting that if $p > 2$,

$$|x|^{p-2} \leq \bar{a}(|x - D(y,i,t)|^{p-2} + |D(y,i,t)|^{p-2}),$$

$$|x - D(y,i,t)|^{p-2} \leq \bar{a}(|x|^{p-2} + |D(y,i,t)|^{p-2}),$$

we have that

$$\begin{aligned}
&|x - D(y,i,t)|^{p-2}\\
&\quad \geq \underline{a}|x|^{p-2} - \kappa^{p-2}\exp(-\frac{p-2}{p}(1-\theta)t)|y|^{p-2},\\
&|x - D(y,i,t)|^{p-2}\\
&\quad \leq \bar{a}(|x|^{p-2} + \kappa^{p-2}\exp(-\frac{p-2}{p}(1-\theta)t)|y|^{p-2}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&p\lambda_i\alpha_{i1}|x-D(y,i,t)|^{p-2} \le \bar{a}p\lambda_i\alpha_{i1}|x|^{p-2} \\
&+ \bar{a}p\lambda_i\alpha_{i1}\kappa^{p-2}\exp(-\frac{p-2}{p}(1-\theta)t)|y|^{p-2} \\
&\le \delta_1|x|^{p-2} + \delta_1\kappa^{p-2}\exp(-\frac{p-2}{p}(1-\theta)t)|y|^{p-2} \\
&= \delta_1\rho^{-\frac{p-2}{p}}(\rho|x|^p)^{\frac{p-2}{p}} \\
&+ \delta_1\rho^{-\frac{p-2}{p}}(\rho\kappa^p\exp(-(1-\theta)t)|y|^p)^{\frac{p-2}{p}} \\
&\le \frac{4}{p}\rho^{-\frac{p-2}{2}}\delta_1^{\frac{p}{2}} + \frac{\rho(p-2)}{p}|x|^p \\
&+ \frac{\rho(p-2)}{p}\kappa^p\exp(-(1-\theta)t)|y|^p,
\end{aligned}$$

$$p\lambda_i\alpha_{i2}|x-D(y,i,t)|^{p-2}|x|^2 \le \begin{cases} \bar{a}p\lambda_i\alpha_{i2}|x|^p + \bar{a}p\lambda_i\alpha_{i2}\kappa^{p-2} \\ \times\exp(-\frac{p-2}{p}(1-\theta)t)|x|^2|y|^{p-2} & \text{if } \alpha_{i2}\ge 0, \\ \underline{a}p\lambda_i\alpha_{i2}|x|^p - p\lambda_i\alpha_{i2}\kappa^{p-2} \\ \times\exp(-\frac{p-2}{p}(1-\theta)t)|x|^2|y|^{p-2} & \text{if } \alpha_{i2}<0, \end{cases}$$

by well-known Young inequality, we have

$$\begin{aligned}
&p\lambda_i\alpha_{i2}|x-D(y,i,t)|^{p-2}|x|^2 \\
&\le \begin{cases} \bar{a}p\lambda_i\alpha_{i2}(1+\frac{2}{p}\kappa^{p-2})|x|^p + \bar{a}p\lambda_i\frac{(p-2)}{p}\alpha_{i2}\kappa^{p-2} \\ \quad\times\exp(-(1-\theta)t)|y|^p \quad \text{if } \alpha_{i2}\ge 0 \\ p\lambda_i\alpha_{i2}(\underline{a}-\frac{2}{p}\kappa^{p-2})|x|^p - p\lambda_i\frac{(p-2)}{p}\alpha_{i2}\kappa^{p-2} \\ \quad\times\exp(-(1-\theta)t)|y|^p \quad \text{if } \alpha_{i2}<0 \end{cases} \\
&= p\lambda_i\tilde{\alpha}_{i2}|x|^p + p\lambda_i\hat{\alpha}_{i2}\kappa^{p-2}\exp(-(1-\theta)t)|y|^p.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&p\lambda_i\alpha_{i3}\exp(-\frac{2}{p}(1-\theta)t)|x-D(y,i,t)|^{p-2}|y|^2 \\
&\le \bar{a}(p-2)\lambda_i\alpha_{i3}|x|^p \\
&\quad + \bar{a}\lambda_i\alpha_{i3}(2+p\kappa^{p-2})\exp(-(1-\theta)t)|y|^p,
\end{aligned}$$

$$\begin{aligned}
&-p\lambda_i\alpha_{i4}|x-D(y,i,t)|^{p-2}|x|^{q-p+2} \\
&\le -p\lambda_i\alpha_{i4}\Big(\underline{a}-\kappa^{p-2}\frac{q-p+2}{q}\Big)|x|^q \\
&\quad + p\lambda_i\alpha_{i4}\kappa^{p-2}\frac{p-2}{q}\exp(-(1-\theta)t)|y|^q
\end{aligned}$$

and

$$\begin{aligned}
&p\lambda_i\alpha_{i5}\exp(-\frac{q-p+2}{q}(1-\theta)t)|x-D(y,i,t)|^{p-2}|y|^{q-p+2} \\
&\le \frac{p(p-2)}{q}\bar{a}\lambda_i\alpha_{i5}|x|^q \\
&+ \bar{a}p\lambda_i\alpha_{i5}(\frac{q-p+2}{q}+\kappa^{p-2})\exp(-\frac{q}{p}(1-\theta)t)|y|^q \\
&\le \frac{p(p-2)}{q}\bar{a}\lambda_i\alpha_{i5}|x|^q \\
&+ \bar{a}p\lambda_i\alpha_{i5}(\frac{q-p+2}{q}+\kappa^{p-2})\exp(-(1-\theta)t)|y|^q.
\end{aligned}$$

We also have

$$\begin{aligned}
&\sum_{j\in S}\gamma_{ij}\lambda_j|x-D(y,t,i)|^p \\
&\le |x|^p\sum_{j\in S}\tilde{\gamma}_{ij}\lambda_j + \kappa^p\exp(-(1-\theta)t)|y|^p\sum_{j\in S}\hat{\gamma}_{ij}\lambda_j.
\end{aligned}$$

Substituting above inequalities into (25), by condition (20) we can obtain

$$\begin{aligned}
&LV(x-D(y,i,t),y,t,i) \\
&\le \frac{4}{p}\rho^{-\frac{p-2}{2}}\delta_1^{\frac{p}{2}} - [1-\bar{a}\theta(p-2)\lambda_i\alpha_{i3} - \frac{\rho(p-2)}{p}]|x|^p \\
&+ [\hat{\alpha}_{i2}p\lambda_i + \bar{a}\theta\alpha_{i3}\lambda_i(2+p\kappa^{p-2}) + \frac{\rho(p-2)}{p}\kappa^p \\
&+ \kappa^p\sum_{j=1}^{N}\hat{\gamma}_{ij}\lambda_j]\exp(-(1-\theta)t)|y|^p \\
&- [\underline{a}p\lambda_i\alpha_{i4} - p\lambda_i\alpha_{i4}\kappa^{p-2}\frac{q-p+2}{q} \\
&- \frac{p(p-2)}{q}\theta\alpha_{i5}\lambda_i\bar{a}]|x|^q + [\frac{(p-2)}{q}p\lambda_i\alpha_{i4}\kappa^{p-2} \\
&+ \theta\alpha_{i5}\lambda_i\bar{a}(\frac{q-p+2}{q}+\kappa^{p-2})]\exp(-(1-\theta)t)|y|^q \\
&\le \frac{4}{p}\rho^{-\frac{p-2}{2}}\delta_1^{\frac{p}{2}} - [\delta_2 - \frac{\rho(p-2)}{p}]|x|^p + [\delta_3 + \frac{\rho(p-2)}{\theta p}\kappa^p]\theta \\
&\times\exp(-(1-\theta)t)|y|^p - \delta_4|x|^q + \delta_5\theta\exp(-(1-\theta)t)|y|^q.
\end{aligned}$$

Define $\beta_1 \sim \beta_5$ as follows $\beta_1 = \frac{4}{p}\rho^{-\frac{p-2}{2}}\delta_1^{\frac{p}{2}}, \beta_2 = \delta_2 - \frac{\rho(p-2)}{p}, \beta_3 = \delta_3 + \frac{\rho(p-2)}{\theta p}\kappa^p, \beta_4 = \delta_4, \beta_5 = \delta_5$.

We have that

$$\begin{aligned}
&LV(x-D(y,t,i),y,i,t) \\
&\le \beta_1 - \beta_2|x|^p + \beta_3\theta\exp(-(1-\theta)t)|y|^p \\
&- \beta_4|x|^q + \beta_5\theta\exp(-(1-\theta)t)|y|^q.
\end{aligned}$$

By condition (21) and definition of ρ, we have

$$\beta_2 - \beta_3 = \delta_2 - \frac{\rho(p-2)}{p} - \delta_3 - \frac{\rho(p-2)}{\theta p}\kappa^p$$
$$\geq \delta_2 - \delta_3 - \rho - \frac{\rho}{\theta}\kappa^p = (\delta_2 - \delta_3)(1 - \frac{\theta + \kappa^p}{1+\theta})$$
$$= (\delta_2 - \delta_3)\frac{1-\kappa^p}{1+\theta} > 0,$$

this together with (22), we have

$$\beta_2 > \beta_3 \quad \text{and} \quad \beta_4 \geq \beta_5.$$

Then the assertions (23) and (24) can be obtained from (4) and (5) in Theorem III.2.

Assumption IV.4 *If $q > p = 2$, assume furthermore that for each $i \in S$, there are constants $\bar{\alpha}_{i2} \in R$ and $\bar{\alpha}_{i1}, \bar{\alpha}_{i3}, \bar{\alpha}_{i4}, \bar{\alpha}_{i5} \in R_+$ such that*

$$(x - D(y,i,t))^T f(x,y,i,t) + \frac{1}{2}|g(x,y,i,t)|^2$$
$$\leq \bar{\alpha}_{i1} + \bar{\alpha}_{i2}|x|^2 + \bar{\alpha}_{i3}\theta\exp(-(1-\theta)t)|y|^2$$
$$- \bar{\alpha}_{i4}|x|^q + \bar{\alpha}_{i5}\theta\exp(-(1-\theta)t)|y|^q.$$

Assumption IV.5 *Under Assumption IV.4, assume furthermore that*

$$A := -\text{diag}(2\bar{\alpha}_{12}, \cdots, 2\bar{\alpha}_{N2}) - \Gamma$$

is an nonsingular M-matrix. By properties of M-matrices, we have a vector with all positive entries defined by the nonsingular M-matrix A:

$$(\bar{\lambda}_1, \cdots, \bar{\lambda}_N)^T = A^{-1}(1, \cdots, 1)^T > 0.$$

Corollary IV.6 *Let Assumptions II.3, IV.4 and IV.5 hold. Set $\bar{c}_1 = \min_{i\in S}\bar{\lambda}_i$, $\bar{c}_2 = \max_{i\in S}\bar{\lambda}_i$, $\bar{\delta}_1 = \max_{i\in S}\bar{\lambda}_i\bar{\alpha}_{i1}$, $\bar{\delta}_2 = \max_{i\in S}\sum_{j\in S}\kappa\bar{\lambda}_j|\gamma_{ij}|$, $\bar{\delta}_3 = \max_{i\in S}\{2\bar{\lambda}_i\bar{\alpha}_{i3} + \frac{1}{\theta}\sum_{j\in S}\kappa|\gamma_{ij}|\bar{\lambda}_j + \frac{1}{\theta}\sum_{j\in S, j\neq i}\kappa^2\gamma_{ij}\bar{\lambda}_j\}$, $\bar{\delta}_4 = \min_{i\in S}\bar{\lambda}_i\bar{\alpha}_{i4}$, $\bar{\delta}_5 = \max_{i\in S}\bar{\lambda}_i\bar{\alpha}_{i5}$. If, moreover,*

$$\bar{\delta}_2 + \bar{\delta}_3 < 1 \tag{26}$$

and

$$\bar{\delta}_5 \leq \bar{\delta}_4, \tag{27}$$

then the hybrid NPSDE (1) on $t \in [0,\infty)$ has a unique global solution. Moreover, the solution has the properties that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t E|x(s)|^q ds \leq \bar{K}_1, \tag{28}$$

and

$$\limsup_{t\to\infty} E|x(t)|^2 \leq \bar{K}_2, \tag{29}$$

where $\bar{K}_1$ and $\bar{K}_2$ are positive constants.

Proof: We will define a Lyapunov function $V : R^n \times S \times R_+ \to R_+$ by

$$V(x,i,t) = \bar{\lambda}_i|x|^2 \quad i \in S.$$

It is easy to see that

$$\bar{c}_1|x|^2 \leq V(x,i,t) \leq \bar{c}_2|x|^2.$$

Now we compute $LV(x - D(y,i,t), y, i, t)$. For any $i \in S$,

$$LV(x - D(y,i,t), y, t, i)$$
$$= 2\bar{\lambda}_i[(x - D(y,i,t))^T f(x,y,i,t) + \frac{1}{2}|g(x,y,i,t)|^2]$$
$$+ \sum_{j\in S}\bar{\lambda}_j\gamma_{ij}(x - D(y,i,t))^T(x - D(y,i,t))$$
$$\leq 2\bar{\lambda}_i\bar{\alpha}_{i1} + 2\bar{\lambda}_i\bar{\alpha}_{i2}|x|^2 + 2\bar{\lambda}_i\bar{\alpha}_{i3}\theta\exp(-(1-\theta)t)|y|^2$$
$$- 2\bar{\lambda}_i\bar{\alpha}_{i4}|x|^q + 2\bar{\lambda}_i\bar{\alpha}_{i5}\theta\exp(-(1-\theta)t)|y|^q$$
$$+ \sum_{j\in S}\bar{\lambda}_j\gamma_{ij}|x|^2 - 2\sum_{j\in S}\bar{\lambda}_j\gamma_{ij}x^T D(y,i,t)$$
$$+ \sum_{j\in S}\bar{\lambda}_j\gamma_{ij}|D(y,i,t)|^2$$
$$\leq 2\bar{\lambda}_i\bar{\alpha}_{i1} + [2\bar{\lambda}_i\bar{\alpha}_{i2} + \sum_{j\in S}\bar{\lambda}_j\gamma_{ij} + \sum_{j\in S}\kappa\bar{\lambda}_j|\gamma_{ij}|]|x|^2$$
$$+ [2\bar{\lambda}_i\bar{\alpha}_{i3} + \frac{1}{\theta}\sum_{j\in S}\kappa|\gamma_{ij}|\bar{\lambda}_j + \frac{1}{\theta}\sum_{j\in S, j\neq i}\kappa^2\gamma_{ij}\bar{\lambda}_j]$$
$$\times \theta\exp(-(1-\theta)t)|y|^2$$
$$- 2\bar{\lambda}_i\bar{\alpha}_{i4}|x|^q + 2\bar{\lambda}_i\bar{\alpha}_{i5}\theta\exp(-(1-\theta)t)|y|^q,$$

by definitions of $\bar{\delta}_1 \sim \bar{\delta}_5$, we have

$$LV(x - D(y,i,t), y, t, i)$$
$$\leq 2\bar{\delta}_1 - (1 - \bar{\delta}_2)|x|^2 + \bar{\delta}_3\theta\exp(-(1-\theta)t)|y|^2$$
$$- 2\bar{\delta}_4|x|^q + 2\bar{\delta}_5\theta\exp(-(1-\theta)t)|y|^q.$$

Define $\beta_1 \sim \beta_5$ as follows

$$\beta_1 = 2\bar{\delta}_1,\ \beta_2 = 1 - \bar{\delta}_2,\ \beta_3 = \bar{\delta}_3,\ \beta_4 = 2\bar{\delta}_4,\ \beta_5 = 2\bar{\delta}_5.$$

We have that

$$LV(x - D(y,t,i), y, i, t)$$
$$\leq \beta_1 - \beta_2|x|^2 + \beta_3\theta\exp(-(1-\theta)t)|y|^2$$
$$- \beta_4|x|^q + \beta_5\theta\exp(-(1-\theta)t)|y|^q.$$

By conditions (26) and (27), we have

$$\beta_2 > \beta_3 \quad \text{and} \quad \beta_4 \geq \beta_5.$$

Then the assertions (28) and (29) can be obtained from (4) and (5) in Theorem III.2.

V. Examples

In this section we will discuss two examples to illustrate our theory.

Example V.1 Consider a scalar hybrid NPSDE

$$d[x(t) - D(x(0.5t), r(t), t)] = f(x(t), r(t), t)dt + g(x(0.5t), r(t), t)dB(t), \quad (30)$$

where $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 10 & -10 \end{pmatrix}.$$

The coefficients D, f and g are defined as

$$D(y,1,t) = D(y,2,t) = 0.1\exp(-0.25t)y,$$
$$f(x,1,t) = -4x - 4x^3, \quad f(x,2,t) = x - 4x^3,$$
$$g(y,1,t) = \exp(-0.25t)y^2, \quad g(y,2,t) = \exp(-0.25t)y^2.$$

We will refer to $r(t)$ as the mode of the system. So the system is operated in two modes, 1 and 2. In mode 1, the system is described by the NPSDE

$$d[x(t) - 0.1\exp(-0.25t)x(0.5t)] = [-4x(t) - 4x^3(t)]dt + \exp(-0.25t)x^2(0.5t)dB(t),$$

while in mode 2

$$d[x(t) - 0.1\exp(-0.25t)x(0.5t)] = [x(t) - 4x^3(t)]dt + \exp(-0.25t)x^2(0.5t)dB(t).$$

When the system is being operated, it will switch from one NPSDE to the other according to the movement of the Markov chain. Let us define $V \in C^{2,1}(R \times S \times R_+; R_+)$ by

$$V(x,i,t) = \begin{cases} x^2 & \text{if } i = 1, \\ 2x^2 & \text{if } i = 2. \end{cases}$$

It is easy to see

$$\begin{aligned} &LV(x - D(y,1,t), y, 1, t) \\ &= 2(x - 0.1\exp(-0.25t)y)(-4x - 4x^3) \\ &+ \exp(-0.5t)y^4 + (x - 0.1\exp(-0.25t)y)^2 \end{aligned}$$

and

$$\begin{aligned} &LV(x - D(y,2,t), y, 2, t) \\ &= 4(x - 0.1\exp(-0.25t)y)(x - 4x^3) \\ &+ 2\exp(-0.5t)y^4 - 10(x - 0.1\exp(-0.25t)y)^2. \end{aligned}$$

Applying the inequality $a^\beta b^{1-\beta} \leq \beta a + (1-\beta)b$, we can obtain

$$\begin{aligned} LV(x - D(y,1,t), y, 1, t) &\leq -6.5x^2 + 0.51\exp(-0.5t)y^2 \\ &- 7.4x^4 + \exp(-0.5t)y^4 + 0.2\exp(-t)y^4 \\ &\leq -6.5x^2 + 0.51\exp(-0.5t)y^2 \\ &- 7.4x^4 + 1.2\exp(-0.5t)y^4, \end{aligned}$$

similarly,

$$LV(x - D(y,2,t), y, 2, t) \leq -4.8x^2 + 1.1\exp(-0.5t)y^2 - 14.8x^4 + 2.4\exp(-0.5t)y^4.$$

Thus, we have

$$LV(x - D(y,i,t), y, i, t) \leq -4.8x^2 + 1.1\exp(-0.5t)y^2 - 7.4x^4 + 2.4\exp(-0.5t)y^4.$$

Thus, we have $p = 2, q = 4$, $\beta_1 = 0, \beta_2 = 4.8, \beta_3 = 2.2, \beta_4 = 7.4$, $\beta_5 = 4.8$. Figures 4.1 illustrates the sample paths of the Markovian chain and $x(t)$ for the solution of the NPSDE (30) by the Euler–Maruyama method (see [34]).

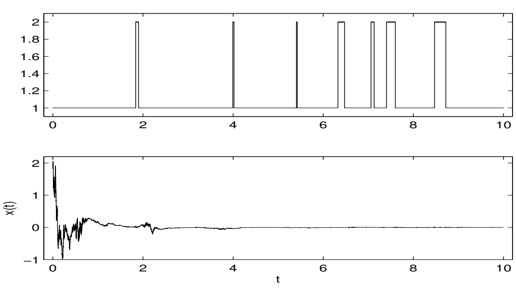

Figure 4.1 : The computer simulation of the sample paths of the Markovian chain and $x(t)$ for the solution of the NPSDE (30) using the Euler–Maruyama method with step size 10^{-3}.

Example V.2 Consider following hybrid NPSDEs

$$d[x(t) - D(x(0.9t), r(t), t)] = f(x(t), r(t), t)dt + g(x(0.9t), r(t), t)dB(t). \quad (31)$$

where $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -0.1 & 0.1 \\ 4 & -4 \end{pmatrix}.$$

Set $p = 3, q = 5$. The coefficients are

$$\begin{aligned} &D(y,1,t) = D(y,2,t) = 0.1\exp(-\frac{t}{30})y, \\ &f(x,1,t) = -6x - 8x^3, f(x,2,t) = 0.1x - x^3, \\ &g(y,1,t) = 0.2\exp(-0.04t)y^2, \\ &g(y,2,t) = 0.1\exp(-0.04t)y^2. \end{aligned}$$

It can be estimated that

$$\begin{aligned} &(x - D(y,1,t))^T f(x,y,1,t) + |g(x,y,1,t)|^2 \\ &= (x - 0.1\exp(-\frac{t}{30})y)(-6x - 8x^3) \\ &\quad + 0.04\exp(-0.08t)y^4 \\ &\leq -5.7x^2 + 0.3\exp(-\frac{t}{15})y^2 - 7.4x^4 \\ &\quad + 0.2\exp(-\frac{2t}{15})y^4 + 0.04\exp(-0.08t)y^4 \\ &\leq -5.7x^2 + 0.3\exp(-\frac{t}{15})y^2 \\ &\quad - 7.4x^4 + 0.24\exp(-0.08t)y^4 \end{aligned}$$

and

$$\begin{aligned} &(x - D(y,2,t))^T f(x,y,2,t) + |g(x,y,2,t)|^2 \\ &\quad \leq 0.105x^2 + 0.005\exp(-\frac{t}{15})y^2 \\ &\quad - 0.925x^4 + 0.035\exp(-0.08t)y^4. \end{aligned}$$

That means the quantities appeared in (19) are $\alpha_{11} = \alpha_{21} = 0, \alpha_{12} = -5.7, \alpha_{22} = 0.105, \alpha_{13} = 0.3, \alpha_{23} = 0.005, \alpha_{14} = 7.4, \alpha_{24} = 0.925, \alpha_{15} = 0.24, \alpha_{25} = 0.035$. For $\alpha_{11} = \alpha_{21} = 0$, we have $\rho = 0$. Thus, we can obtain that

$$A = \begin{pmatrix} 17.125 & -0.4 \\ -16 & 0.685 \end{pmatrix}$$

$$\text{and} \quad A^{-1} = \begin{pmatrix} 0.128 & 0.075 \\ 3.002 & 3.213 \end{pmatrix}.$$

By the definition in (20), we have $\lambda_1 = 0.203, \lambda_2 = 6.215$. Direct calculation gives the quantities involved in Theorem IV.3: $\delta_1 = 0, \delta_2 = 0.776, \delta_3 = 0.692, \delta_4 = 1.788, \delta_5 = 1.166$. It can be concluded that the hybrid system (31) with the above coefficients is stable in theory.

VI. Conclusion

In this paper, we have investigated exponential stability and asymptotic boundedness for a class of highly nonlinear hybrid NPSDEs by removing the linear growth condition. Both Lyapunov functions and M-matrix techniques have been used to study stability and boundedness. Two examples have been provided to illustrate our results.

REFERENCES

1. Costa, O.L.V., E.O. Assumpcao, and E.K. Boukas, *et al.*, Constrained quadratic state feedback control of discrete-time Markovian jump linear systems, *Automatica.* Vol. 35, No. 4, pp. 617–626(1999).
2. Fei, W. Y., L. J. Hu., X. R. Mao, and M. X. Shen, Delay dependent stability of highly nonlinear hybrid stochastic systems, *Automatica.* Vol. 82, pp. 165–170(2017).
3. Li, Y. Y., J. Q. Lu, X. R. Mao, and Q. W. Qiu, Stabilization of Hybrid Systems by Feedback Control Based on Discrete-Time State and Mode Observations, *Asian J. Control.* Vol. 19, No. 6, pp. 1943-1953(2017).
4. Mao, X. R., Exponential stability of stochastic delay interval systems with Markovian switching, *IEEE Trans. Automat. Control.* Vol. 47, No. 10, pp. 1604–1612(2002).
5. Mao, X. R., A. Matasov, and A.B. Piunovskiy, Stochastic differential delay equations with Markovian switching, *Bernoulli.* Vol. 6, No. 1, pp. 73–90(2000).
6. Mao, X. R., and C.Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London (2006).
7. Long, S. L., S. M. Zhong, and Z. J. Liu, Robust Stochastic Stability for a Class of Singular Systems with Uncertain Markovian Jump and Time-Varying Delay, *Asian J. Control.* Vol. 15, No. 4, pp. 1102–1111(2013).
8. Shen, M. X., W. Y. Fei., X. R. Mao, and Y. Liang, Stability of highly nonlinear neutral stochastic differential delay equations, *Syst. Control. Lett.* Vol. 115, No. 5, pp. 1–8(2018).
9. Haussmann, U. G., Asymptotic stability of the linear Itô equation in infinite dimensions, *J. Math. Anal. Appl.* Vol. 65, No. 1, pp. 219–235(1978).
10. Mao, X. R., *Stochastic Differential Equations and Applications*, 2nd Edition, Horwood Publishing, Chichester, Chichester (2007).

11. Wu, X. T., Y. Tang, and W. B. Zhang, Stability analysis of stochastic delayed systems with an application to multi-agent systems, *IEEE Trans. Automat. Control.* Vol. 61, No. 12, pp. 4143–4149(2016).
12. Xia, H. W., P. P. Zhao, L. Li, Y. M. Wang, and A. G. Wu, Improved stability criteria for linear neutral time-delay systems, *Asian J. Control.* Vol. 17, No. 1, pp. 343–351(2015).
13. Hu, L. J., X. R. Mao, and Y. Shen, Stability and boundedness of nonlinear hybrid stochastic differential delay equations, *Syst. Control. Lett.* Vol. 62, No. 2, pp. 178–187(2013).
14. Hu, L. J., X. R. Mao, and L. G. Zhang, Robust stability and boundedness of nonlinear hybrid stochastic differential delay equations, *IEEE Trans. Automat. Control.* Vol. 59, No. 9, pp. 2319–2332(2013).
15. Ockendon, J. R., and A. B. Tayler, The dynamics of a current collection system for an electric locomotive, *Proc. Roy. Soc. Lond. A.* Vol. 322, pp.447–468(1971).
16. Baker, C., and E. Buckwar, Continuous θ-methods for the stochastic pantograph equation, *Electron. Trans. Number. Anal.* Vol. 11, No. 3, 131–151(2000).
17. Fan, Z. C., M. H. Song, and M. Z. Liu, The αth moment stability for the stochastic pantograph equation, *J. Comput. Appl. Math.* Vol. 233, No. 2, pp. 109–120(2009).
18. Guo, Y., X. H. Ding, and Y. J. Li, Stochastic stability for pantograph multi-group models with dispersal and stochastic perturbation, *J. Frankl. Inst. - Eng. Appl. Math.* Vol. 353, No. 13, pp. 2980–2998(2016).
19. Li, R. H., M. Liu, and W. K. Pang, Convergence of numerical solutions to stochastic pantograph equations with Markovian switching, *Appl. Math. Comput.* Vol. 215, No. 1, pp. 414–422(2009).
20. Milošević, M., and M. Jovanović, A Taylor polynomial approach in approximations of solution to pantograph stochastic differential equations with Markovian switching, *Math. Comput. Model.* Vol. 53, pp. 280–293(2011).
21. Obradović, M., and M. Milošević, Stability of a class of neutral stochastic differential equations with unbounded delay and Markovian switching and the Euler-Maruyama method, *J. Comput. Appl. Math.* Vol. 309, pp. 244–266(2017).
22. You, S. R., W. Mao, X. R. Mao, and L. J. Hu, Analysis on exponential stability of hybrid pantograph stochastic differential equations with highly nonlinear coefficients, *Appl. Math. Comput.* Vol. 263, pp. 73–83(2015).
23. Zhou, S. B., Almost surely exponential stability of numerical solutions for stochastic pantograph equations, *Abstr. Appl. Anal.* 2014, Article ID 751209(2014).
24. Wu, F. K. and S. G. Hu, The LaSalle-type theorem for neutral stochastic functional differential equations with infinite delay, *Discrete Cont. Dyn. S.* Vol. 32, No. 3, pp.1065–1094(2012).
25. Zong, X. F., F. K. Wu and C. M. Huang, The boundedness and exponential stability criterions for nonlinear hybrid neutral stochastic functional differential equations, *Abstr. Appl. Anal.* 2013, Article ID 138031(2013).
26. Mao, X. R., Y. Shen and C. G. Yuan, Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching, *Stoch. Proc. Appl.* Vol. 118, No. 8, pp.1385–1406(2008).
27. Chen, H., P. Shi, C. L. Lim, and P. Hu, Exponential stability for neutral stochastic Markov systems with time-varying delay and its applications, *IEEE. Trans. Cybernetics.* Vol. 46, No. 6, pp. 1350–1362(2016).
28. Deng, F. Q., W. H. Mao, and A. H. Wang, A novel result on stability analysis for uncertain neutral stochastic time-varying delay systems, *Appl. Math. Comput.* Vol. 221, No. 2, pp. 132–143(2013).
29. Luo, Q., X. R. Mao, and Y. Shen, New criteria on exponential stability of neutral stochastic differential delay equations, *Syst. Control. Lett.* Vol. 55, No. 10, pp. 826–834(2006).
30. Li, M. L., and F. Q. Deng, Almost sure stability with general decay rate of neutral stochastic delayed hybrid systems with Lévy noise, *Nonlinear Anal. Hybrid Syst.* Vol. 24, pp. 171–185(2017).
31. Iserles, A., On the generalized pantograph functional-differential equation, *Eur. J. Appl. Math.* Vol.4, No. 1, pp.1–38(1993).
32. Yu, Z., Razumikhin-type theorem and mean square asymptotic behavior of the backward Euler method for neutral stochastic pantograph equations, *J. Inequal. Appl.* Vol. 299, No. 1, pp. 1–15(2013).
33. Kolmanovskii, V., N. Koroleva, T. Maizenberg, X. Y. Mao, and A. Matasov, Neutral stochastic differential delay equations with markovian switching, *Stoch. Anal. Appl.* Vol. 21, No. 4, pp. 819–847(2003).
34. Zhou, S. B., and M. G. Xue, Exponential stability for nonlinear hybrid stochastic pantograph equations and numberical approximation, *Acta Math. Sci.* Vol. 34, No. 4, pp.1254–1270(2014).