SOME EXACT SOLUTIONS IN THE ONE-DIMENSIONAL UNSTEADY MOTION OF A GAS

David Gordon Weir

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SOME EXACT SOLUTIONS IN THE ONE-DIMENSIONAL UNSTEADY MOTION OF A GAS.

being a THESIS presented by

DAVID GORDON WEIR

to the University of St. Andrews
in application for the degree of
DOCTOR OF PHILOSOPHY.

DECLARATION

I hereby declare that the following thesis is a record of original work, that it has been composed by me, and that it has not been accepted for any other degree.

DECLARATION

I certify that Mr. D. G. Weir has fulfilled the conditions of the Ordinance and Regulations for the presentation of the following thesis.

Research Supervisor

PERSONAL FOREWORD

The work presented in this thesis was begun at St. Andrews University in October 1957 when I was admitted as a research student having graduated the previous June with a first class honours M.A. degree in Mathematics and Applied Mathematics.

It was completed at the Royal College of Science and Technology, Glasgow, where I obtained a post as an Assistant Lecturer in January 1960.

I would like to take this opportunity to thank Dr. A. G. Mackie for introducing me to the subject, and for his help and encouragement while acting as my supervisor. My thanks are also due to the Caird Trust for providing a Scholarship during the course of this work.

The theory of the constant strength shock given in § 3 of chapter 3, together with the example of § 5 of that chapter has been published as a joint paper with Dr. A. G. Mackie [7], while the substance of § 2 of chapter 5 has been accepted for publication in the Proceedings of the Cambridge Philosophical Society.

CONTENTS

Chapter 1

Introduction

§ 1. In this thesis, we present certain exact solutions of the mathematical equations governing the one-dimensional unsteady flow of a compressible fluid. This fluid will be assumed to have the following idealised properties. At each point of it at any instant there is a definite state defined by the pressure p, the temperature θ, the specific volume τ, the density ρ, with $\rho\tau = 1$, the specific entropy S, and the specific internal energy e. Also at each point during any motion there is a velocity vector $\underline{q}$, and the trajectories, or particle paths, are defined by the equation $\frac{d\underline{r}}{dt} = \underline{q}$. Except in very narrow regions where the motion may be represented mathematically by a discontinuity, viscosity, heat conduction and deviation of the medium from thermodynamical equilibrium at any point can be neglected. This implies that, except when the motion is discontinuous, the specific entropy of any element of the fluid is constant.

From thermodynamics it is known that only two of p, ρ, θ, S are independent. We assume that the relationshipsbetween them are the ideal gas equation

$$p\tau = R\theta$$

where R is a constant for any gas, and an equation of state of the form

$$p = p(\rho, S)$$

which unless otherwise stated will be that of a polytropic gas, namely

$$p = k\rho^{\gamma} \qquad (1.1.1)$$

where $k = \frac{p_o}{\rho_o^{\gamma}} \exp \frac{S - S_o}{c_v}$ is a function of the entropy only. Here γ and c_v are the adiabatic exponent and the specific heat at constant volume respectively, both constants for the gas, and p_o , ρ_o , S_o represent an arbitrary reference state.

We define the sound speed c of the gas at any point as a positive quantity given by the equation

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S , \qquad (1.1.2)$$

which for a polytropic gas becomes

$$c^2 = \frac{\gamma p}{\rho} . \qquad (1.1.3)$$

A justification of the term sound speed will be given later, but for the present we may note that c has the dimensions of a velocity.

For problems in gas flow there are two possible methods of procedure. These are the Eulerian method, in which the independent variables are the time t and the co-ordinates x, y, z of points in space, and the Lagrangian method in which co-ordinates a, b, c are attached to each particle of gas and employed with t as independent variables. In general the former method is much more convenient.

The mathematical equations governing the flow of such a fluid are well known and will not be derived here. They are the equation expressing conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q}) = 0 , \qquad (1.1.4)$$

and the equation expressing conservation of momentum

$$\frac{\partial \underline{q}}{\partial t} + (\underline{q} \cdot \nabla)\underline{q} + \frac{1}{\rho}\nabla p = \underline{0} , \qquad (1.1.5)$$

where we assume that the only forces acting on the gas are pressure forces and the condition that the specific entropy is constant for any particle, which is

$$\frac{\partial S}{\partial t} + \underline{q} \cdot \underline{\nabla S} = 0 . \qquad (1.1.6)$$

In the case of one-dimensional flow, with which we shall be solely concerned from now on, where the state of the medium depends on a single Cartesian co-ordinate, x, and the time, t, these equations (1.1.4), (1.1.5), (1.1.6) reduce to

$$\rho_t + u\rho_x + \rho u_x = 0 , \qquad (1.1.7)$$

$$\rho u_t + \rho u u_x + p_x = 0 , \qquad (1.1.8)$$

$$S_t + u S_x = 0 , \qquad (1.1.9)$$

where u is the x-component of the velocity vector $\underline{q}$, and the subscripts denote partial differentiation.

We may now eliminate p and ρ from these equations in terms of the sound speed c of equation (1.1.2). When

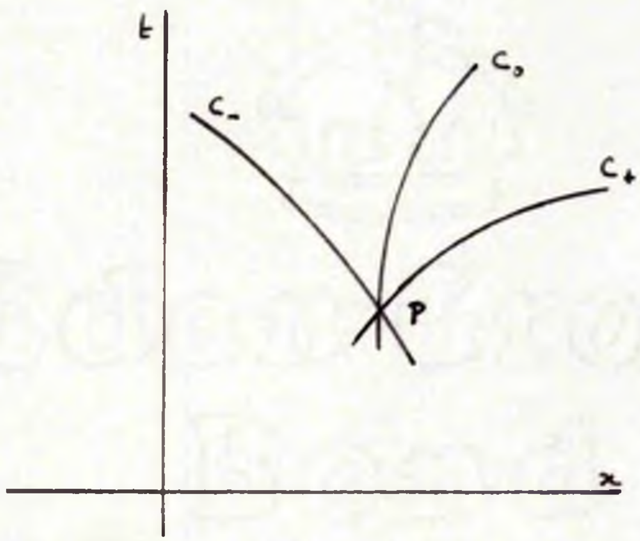

Figure I.

this is done they become

$$(\gamma - 1)\, cu_x + 2c_t + 2uc_x = 0 \ , \qquad (1.1.10)$$

$$u_t + uu_x + \frac{2}{\gamma-1} cc_x - [\gamma(\gamma-1)c_v]^{-1} c^2 S_x = 0 \ , \qquad (1.1.11)$$

$$S_t + u S_x = 0 \ . \qquad (1.1.12)$$

where the c_v in (1.1.11) is the specific heat at constant volume not a partial derivative of the sound speed.

Equations (1.1.10), (1.1.11) and (1.1.12) form a set of partial differential equations of hyperbolic type, which has three sets of real characteristics known as C_+, C_0 and C_- characteristics respectively, given by

$$C_+ : \quad \frac{dx}{dt} = u + c$$

$$C_0 : \quad \frac{dx}{dt} = u$$

$$C_- : \quad \frac{dx}{dt} = u - c \ .$$

Since c is by definition a positive quantity, it is clear that if we sketch the three characteristic lines through some point P in the $x - t$ plane, as in figure I, they will always be in the same order from left to right. But from the theory of hyperbolic differential equations, the properties of the gas holding at P will only affect the region between the characteristics at P . Thus a small disturbance at P , such as a sound pulse, will travel into the gas along C_+ and C_-, which means that it will have velocity c with respect to the gas. This forms the justification for the term sound speed used for c .

As with all hyperbolic equations a problem may be solved by finding solutions which hold in different regions of the $x - t$ plane and suitably 'patching' these together across characteristics. This technique will be widely used in the following chapters.

We now introduce the Lagrange variable h for one-dimensional flow. Since the motion depends only on one space co-ordinate we require h to be constant on each plane section of particles normal to the x-axis. Then the changing position of each section is given by a function $x(h,t)$. The numbers h can of course be chosen in infinitely many ways. Here we use the method employed by Courant and Friedrichs [3] whereby we attach the value $h = 0$ to some definite zero section moving with the medium, and let h for any other section be equal in magnitude to the mass of the medium per unit cross-section between it and the zero. The sign of h will be taken as positive to the right and negative to the left of the zero.

Thus h satisfies

$$h = \int_{x(0,t)}^{x(h,t)} \rho(x,t)\, dx$$

where ρ is regarded from the Eulerian standpoint as a function of x and t . An equivalent equation is

$$dh = \rho\, dx - \rho\, u\, dt \qquad (1.1.13)$$

since we require h to be constant on a particle path $\frac{dx}{dt} = u$.

As will be seen, many problems are more easily solved if we treat u and c , or simple functions of these quantities, as the independent variables and find x or h and t as functions of them. Such a transformation is known as a 'hodograph transformation', and the solution in this form is said to be in the 'hodograph plane', as opposed to the 'physical plane' where x and t are independent. The images of the C characteristics in the u - v plane are usually referred to as Γ characteristics.

The physical model we use for one-dimensional flows is that of a gas moving in a tube which stretches along the x-axis from $-\infty$ to $+\infty$. At any time the gas occupies some region of the tube. This region may be infinite, semi-infinite, or finite. We will usually assume that the initial state of the gas is known. That is, we assume that at $t = 0$ we know u, c, p, ρ , S, etc. as functions of x . A second set of data may arise from the end conditions of the gas. We may assume that the gas is initially in contact at one end with a piston which is set in motion along a known path. Then this path is taken as the trajectory of the gas particles initially in contact with it. A special case of this condition is the assumption that a rigid wall exists at some point $x = x_0$. This requires that $u = 0$ at $x = x_0$ for all time. Alternatively we may have the case of an expansion into a vacuum, when we assume that the pressure, and hence the sound speed, at the boundary of the gas is zero. In general motion in the gas may arise due to the end conditions or to pressure gradients in the initial state of the gas, or both.

We now summarise briefly the contents of the following chapters. In chapter 2 we introduce the well-known simplification of equations (1.1.10), (1.1.11) and (1.1.12) which occurs when the entropy is assumed to be constant, and conditions for patching solutions of this equation along characteristics are obtained. These results are used to generalise a problem solved by Mackie [5]. In chapter 3 we meet the concept of a shock, and exact solutions are obtained for two problems in which shocks occur in non-uniform flows. In chapter 4 the case of waves in shallow water which has differential equations similar to those of gas flow is discussed. The results of the previous section are applied to this case and a problem attacked which permits a comparison to be made of the results obtained by this theory and a simpler linearised theory. Finally in chapter 5 we examine a method introduced by Martin for dealing with certain non-isentropic flows. Some new exact solutions of non-isentropic flows are thus obtained.

Chapter 2

Isentropic Flow

§ 1. An obvious simplification of equations (1.1.10) to (1.1.12) occurs if we assume the gas to have a constant entropy throughout the region under consideration. This is the case of isentropic flow. Here equation (1.1.12) is satisfied trivially, and the flow equations reduce to

$$\frac{\gamma-1}{2} c u_x + c_t + u c_x = 0 \tag{2.1.1}$$

$$u_t + u u_x + \frac{2}{\gamma-1} c c_x = 0 \ . \tag{2.1.2}$$

After simple manipulation these may be written

$$\left[\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right]\left(\frac{c}{\gamma-1} + \frac{u}{2}\right) = 0$$

$$\left[\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right]\left(\frac{c}{\gamma-1} - \frac{u}{2}\right) = 0 \ .$$

From these equations it follows immediately that the expressions

$$r = \frac{u}{2} + \frac{c}{\gamma-1} \tag{2.1.3}$$

$$s = -\frac{u}{2} + \frac{c}{\gamma-1} \tag{2.1.4}$$

must be constant along the characteristics $\frac{dx}{dt} = u + c$ and $\frac{dx}{dt} = u - c$ respectively. The quantities r and s are known as the Riemann Invariants, and the C_+ and C_- characteristics may now be called r-characteristics and s-characteristics respectively.

If we now invert the equations of flow taking r and s as independent variables we will obtain differential equations for x and t. These equations will be linear and consequently easier to deal with. We are now working in the hodograph plane, and this method naturally assumes a one-one correspondence between this plane and the physical plane. We therefore first consider the most important cases where this does not hold - those cases where one or both of the Riemann invariants are constant in a region of the physical plane.

Clearly if both r and s are constant we will have from (2.1.3) and (2.1.4) that both u and c are constant. Thus the region is simply one of uniform motion, and the families of characteristics become simply the sets of parallel straight lines

$$x = (u + c)t + \text{constant},$$

$$x = (u - c)t + \text{constant}.$$

Suppose now we have a region in which only one of the invariants, say s, is constant. Then along the r-characteristics in this region both r and s, and hence both u and c, are constant. Thus throughout the region any r-characteristic has a constant slope, given by $u + c$, and so the family of these characteristics again consists of straight lines. In this case however the lines are not parallel since $u + c$ depends on the value of r

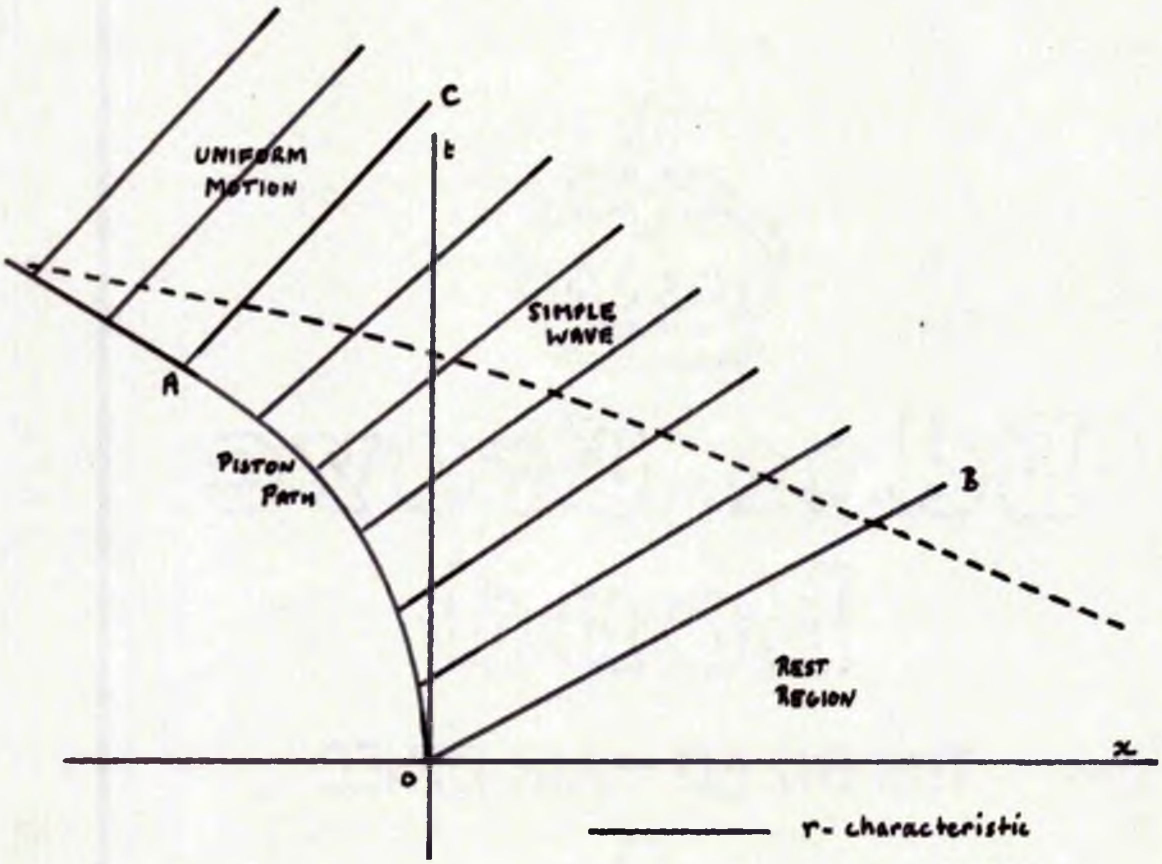
C
t
UNIFORM
MOTION
SIMPLE
WAVE
A
PISTON
PATH
B
REST
REGION
x
O
r - characteristic
s - characteristic

Figure II

on that particular characteristic. When we solve the equation $\frac{dx}{dt} = u + c$ for these characteristics, the constant of integration which arises will also be a function of r which must be obtained from the boundary conditions. When this is found we have an equation which may be solved in theory for r in terms of x and t, and this solution together with the relation $s =$ constant is used to give u and c as functions of x and t from (2.1.3) and (2.1.4).

Such a flow, with either r or s constant, is known as a simple wave. It is easy to see that if there are no discontinuities in the flow, any region bounding a region of uniform motion must be a simple wave. For suppose the bounding characteristic along which the patching is done is an r-characteristic. Along its length it will be crossed by s-characteristics which all pass through the uniform region and therefore carry the same value of s. Hence it is clear that in the neighbourhood of this characteristic in the unknown region s must be constant, which is the condition for a simple wave.

The simplest example of this type of flow is that in which a uniform isentropic gas is at rest in a semi-infinite tube held there by a piston at the origin. At time $t = 0$ this piston is withdrawn with a constant acceleration a. The resulting flow in the $x - t$ plane is shown in figure II.

The withdrawal of the piston along OA creates a disturbance at O which is propagated into the gas along OB , the r-characteristic through O . The region to the right of this characteristic, denoted by the subscript o , is at rest with $u_o = 0$, $c_o =$ constant. This gives for OB the straight line $x = c_o t$.

To the left of this line we have a simple wave in which s is constant. This region, denoted by the subscript 1 , thus has

$$-\frac{u_1}{2} + \frac{c_1}{\gamma-1} = s_1 = s_o = \frac{c_o}{\gamma-1} \tag{2.1.5}$$

$$\frac{u_1}{2} + \frac{c_1}{\gamma-1} = r_1 \tag{2.1.6}$$

Thus the r-characteristics are given by the equation

$$\frac{dx}{dt} = u_1 + c_1 = \frac{\gamma+1}{2} r_1 - \frac{3-\gamma}{2(\gamma-1)} c_o$$

which integrates to give

$$x = \left[\frac{\gamma+1}{2} r - \frac{3-\gamma}{2(\gamma-1)} c_o\right] t + f(r) . \tag{2.1.7}$$

Now on the piston path $u_1 = -at$, and so using (2.1.5) and (2.1.6) $r_1 = \frac{c_o}{\gamma-1} - at$, or $t = \frac{1}{a}\left(\frac{c_o}{\gamma-1} - r\right)$. Also on the piston path $x = -\frac{a}{2}t^2$. Thus we find that the constant of integration is given by

$$f(r) = \frac{1}{2a}\left(r - \frac{c_o}{\gamma-1}\right)\left(\gamma r - \frac{2-\gamma}{\gamma-1} c_o\right) .$$

From (2.1.7) we may now obtain r , and hence u and c , in terms of x and t , and the flow in the simple wave

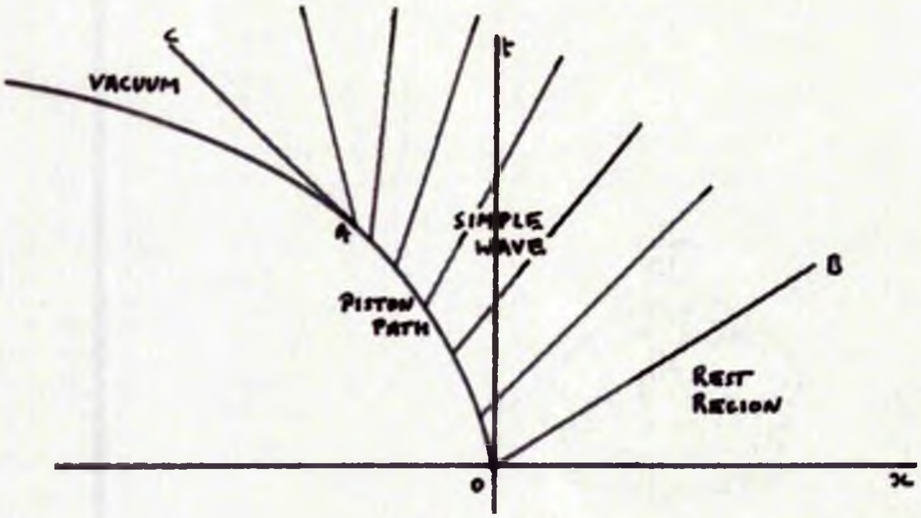

Figure III

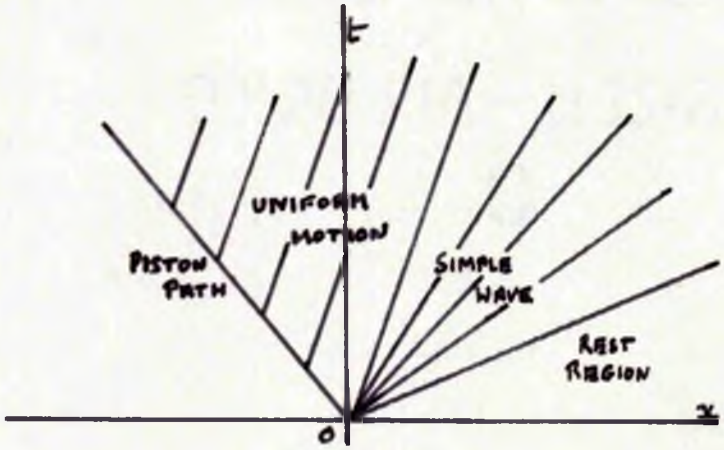

Figure IV

is thus completely known. We note that, since u_1 is negative, r_1 decreases along OA and that hence the slope of the r-characteristic, defined as $\frac{dx}{dt}$, also decreases so that the characteristics tend to diverge, as shown.

If at the point A the piston stops accelerating but continues in motion with constant velocity $-u_A$, then on this part of the piston path u, c, and r will all be constant. Thus to the left of the characteristic through A, AC in the figure, both r and s will be constant, and we have a region of uniform flow with $u = -u_A$, and $c = c_0 - \frac{\gamma-1}{2} u_A$.

It is clear from (2.1.5) that, since c must remain non-negative, there is an upper bound to the (negative) velocity of the simple wave given by $u = \frac{-2c_0}{\gamma-1}$. If the piston continues to accelerate to a speed greater than this, reached at A in figure III, then at A, c_1 is zero, so that $u + c = u$, and the characteristic AC is tangential to the piston path. Along this characteristic u_1 remains at this maximum and c_1 remains zero. Clearly this characteristic is also a particle path representing the front of the gas in the $x - t$ plane.

Thus as the piston continues to accelerate it leaves a vacuum between itself and AC, the front of the gas.

If instead of being accelerated the piston is suddenly withdrawn at a constant speed u_p from the gas at rest, the part OA of the piston path in figure II is reduced to a point. The simple wave becomes a 'point centred' simple wave with each r-characteristic passing through the origin. This case, illustrated in figure IV, shows how an initial discontinuity may be smoothed out immediately leaving a continuous flow.

Hence if the speed of withdrawal of the piston is greater than the maximum possible velocity of the gas the problem is identical with that in which the gas is suddenly permitted to expand into a vacuum to its left. The region of uniform flow vanishes, and the front of the gas is the straight r-characteristic through the origin on which $c = 0$.

§ 2. We now proceed to the case in which r and s both vary. Taking r and s as independent variables we have by their definitions

$$\left.\begin{aligned} x_s &= (u + c)t_s = \left(\frac{\gamma+1}{2} r - \frac{3-\gamma}{2} s\right) t_s \\ x_r &= (u - c)t_r = \left(-\frac{\gamma+1}{2} s + \frac{3-\gamma}{2} r\right) t_r . \end{aligned}\right\} \qquad (2.2.1)$$

These may be rewritten as

$$x_s = \frac{2}{2N+1}\,[(N+1)r - Ns]t_s \qquad (2.2.2)$$

$$x_r = \frac{2}{2N+1}\,[Nr - (N+1)s]t_r \qquad (2.2.3)$$

by putting $\gamma = \frac{2N+3}{2N+1}$. If we restrict our attention to positive values of N we find that γ must lie between 1 and 3. However this range contains all values of physical interest. In fact putting $N = 1$, $N = 2$ gives the values of γ usually taken to correspond to monatomic and diatomic gases respectively.

If we now eliminate either x or t from (2.2.2) and (2.2.3) we obtain the respective linear hyperbolic differential equations

$$t_{rs} + \frac{N+1}{r+s}(t_r + t_s) = 0 \qquad (2.2.4)$$

$$x_{rs} + \frac{N+1}{r+s}\left\{\frac{(N+1)r - Ns}{Nr - (N+1)s}\, x_r + \frac{Nr - (N+1)s}{(N+1)r - Ns}\, x_s\right\} = 0 .$$

Then any problem in one-dimensional unsteady isentropic flow requires the solution of one of these equations for suitable boundary conditions. It is preferable, however, to introduce a new dependent variable w by the equations

$$\frac{\partial w}{\partial r} = x - \frac{2}{2N+1}[(N+1)r - Ns]t \qquad (2.2.5)$$

$$-\frac{\partial w}{\partial s} = x + \frac{2}{2N+1}[(N+1)s - Nr)t \quad . \qquad (2.2.6)$$

We may now eliminate both x and t from these equations giving for w the Euler-Poisson equation

$$w_{rs} + \frac{N}{r+s}(w_r + w_s) = 0 . \qquad (2.2.7)$$

When we have solved this equation for any particular problem we may then find x and t in terms of r and s by solving equations (2.2.5) and (2.2.6) to give

$$x = \frac{1}{r+s}\left\{[(N+1)s - Nr]w_r - [(N+1)r - Ns]w_s\right\} \qquad (2.2.8)$$

$$t = -\frac{2N+1}{2(r+s)}(w_r + w_s) . \qquad (2.2.9)$$

There are two main methods of solving the equation (2.2.7) for any particular problem. These are Riemann's method (or the similar method due to Martin [8]) and the method of arbitrary functions. The first pair may be used when boundary conditions are given along a single non-characteristic curve. Both require the determination of a subsidiary function. For Riemann's method the appropriate Riemann-Green function is

$$\left(\frac{r + s}{\bar{r} + \bar{s}}\right)^N F(1-N,N;\ 1;\ \theta) \qquad (2.2.10)$$

where $\theta = -\frac{(r-\bar{r})(s-\bar{s})}{(r+s)(\bar{r}+\bar{s})}$, and F is a hypergeometric function in the usual notation. This particular function will simplify to a polynomial in θ of degree $N-1$ whenever N is a positive integer. The value of w at the general point $(\bar{r},\bar{s})$ in the hodograph plane is now found by integration round the contour given by $r = \bar{r}$, $s = \bar{s}$, and a section of the curve on which the boundary conditions are given.

Martin's method is essentially similar. The subsidiary function, which he calls the resolvent, is made to satisfy simpler conditions, but the function itself is is more complex. It is given by equations of the form

$$\bar{v} = (r-\bar{r})\left(\frac{r-\bar{s}}{\bar{r}-\bar{s}}\right)^{N} F\left(1-N;\, N,-N;\, 2;\, \frac{r-\bar{r}}{\bar{s}-\bar{r}},\, \frac{r-\bar{r}}{s-\bar{r}}\right)$$

$$- (s-\bar{s})\left(\frac{r-\bar{s}}{\bar{r}-\bar{s}}\right)^{N} F\left(1-N;-N,N;\, 2;\, \frac{s-\bar{s}}{\bar{r}-\bar{s}},\, \frac{s-\bar{s}}{r-s}\right)$$

and the analytic continuation of this expression into the other regions of the flow. Here F is Appell's hypergeometric function of two variables.

Both these methods will give a formal solution, but, except for low integral values of N, the integration is difficult to perform. For general values of N an explicit solution is often more easily obtained by the method of arbitrary functions.

As is well known, a general solution of (2.2.7) for integral N is given by

$$w = \left(\frac{\partial}{\partial r}\right)^{N-1} \frac{R(r)}{(r+s)^{N}} + \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{S(s)}{(r+s)^{N}} \qquad (2.2.11)$$

where $R(r)$, $S(s)$ are arbitrary functions.

This form of solution was generalised to all N by Copson [1] using contour integral theory as follows.

For real r, s, the expression

$$w = \frac{z^{2N}}{(z-r)^{N}(z+s)^{N}}$$

is an analytic function of the complex variable z which, if N is not an integer, is analytic in the complex plane cut along the segment of the real axis joining 0, r, $-s$.

We fix the branch of the function as that which is real and positive when z is real and greater than $|r|$ or $|s|$.

This function is easily seen to be a solution of (2.2.7) and it follows that if $f(z)$ is an analytic function of z regular in a region containing the real axis, then

$$w = \frac{1}{2\pi i} \int_C \frac{z^{2N} f(z)}{(z-r)^N (z+s)^N} \, dz$$

is also a solution for any contour C in the cut plane.

Similarly, for any contour in the plane cut through $0, -r, s$,

$$w = \frac{1}{2\pi i} \int_C \frac{z^{2N} g(z)}{(z+r)^N (z-s)^N} \, dz$$

is also a solution for suitable $g(z)$.

By combining these two, we obtain a solution, in terms of two arbitrary functions, for w which holds for all values of N. If we assume r and s to be positive we may employ contours similar to those introduced by Mackie [6]. The contours Γ_1 and Γ_2 both start at the origin below the cut and finish at the origin above the cut, going in an anti-clockwise direction about $z = r$ and $z = s$ respectively. If we assimilate the z^{2N} term in the arbitrary function, and write

$$w = \frac{\Gamma(N)}{2\pi i} \int_{\Gamma_1} \frac{R(z)}{(z-r)^N (z+s)^N} \, dz + \frac{\Gamma(N)}{2\pi i} \int_{\Gamma_2} \frac{S(z)}{(z+r)^N (z-s)^N} \, dz$$

(2.2.12)

we see that this is a solution of (2.2.7) which, in the case where N is an integer, reduces to (2.2.11) by Cauchy's integral formula, since for this case the origin is no longer a singularity, and the contours become closed loops surrounding $z = r$ and $z = s$ respectively. Results obtained by this method will generally be capable of extension to regions where r or s is negative by analytic continuation.

We now consider the equation for the trajectories in the hodograph plane. In terms of r, s, and w, their differential equation, $\frac{dx}{dt} = u$, becomes, using (2.2.8), and (2.2.9),

$$\left\{Nw_{rr} + (N+1)w_{rs}\right\} dr - \left\{Nw_{ss} + (N+1)w_{rs}\right\} ds = 0 \quad ,$$

which has an integrating factor $(r+s)^{2N+1}$. Thus in the hodograph plane the trajectories are given by $\Phi(r,s) =$ constant where Φ satisfies

$$\frac{\partial\Phi}{\partial r} = (r+s)^{2N+1}[Nw_{rr} + (N+1)w_{rs}] \quad , \qquad (2.2.13)$$

and

$$\frac{\partial\Phi}{\partial s} = (r+s)^{2N+1}[Nw_{ss} + (N+1)w_{rs}] \quad . \qquad (2.2.14)$$

If we now eliminate w from these equations we find that Φ satisfies an equation of the Euler-Poisson type namely

$$\Phi_{rs} - \frac{N+1}{r+s}(\Phi_r + \Phi_s) = 0 \quad . \qquad (2.2.15)$$

We now prove the following theorem.

Theorem I If w is given by (2.2.12), then is given by

$$\Phi = \frac{N\Gamma(N+2)}{2\pi i}(r+s)^{2N+3}\left\{\int_{\Gamma_1}\frac{F(\zeta)\,d\zeta}{(\zeta-r)^{N+2}(\zeta+s)^{N+2}} - \int_{\Gamma_2}\frac{G(\zeta)\,d\zeta}{(\zeta+r)^{N+2}(\zeta-s)^{N+2}}\right\} \qquad (2.2.16)$$

where $F'(z) = R(z)$, $G'(z) = S(z)$.

To prove this result we show that the values of $\frac{\partial\Phi}{\partial r}$ given by (2.2.16) and by substitution of (2.2.12) in (2.2.13) are identical, and similarly for $\frac{\partial}{\partial s}$. This will be a sufficient proof since is only defined to within an arbitrary constant.

Differentiation of (2.2.16) gives

$$\frac{\partial\Phi}{\partial r} = \frac{N\Gamma(N+2)}{2\pi i}(r+s)^{2N+2}\left\{\int_{\Gamma_1}\frac{F(\zeta)\left[(2N+3)\zeta+(N+2)s-(N+1)r\right]}{(\zeta-r)^{N+3}(\zeta+s)^{N+2}}d\zeta - \int_{\Gamma_2}\frac{G(\zeta)\left[(2N+3)\zeta-(N+2)s+(N+1)r\right]}{(\zeta+r)^{N+3}(\zeta-s)^{N+2}}d\zeta\right\}$$

while substitution of (2.2.12) in (2.2.13) gives

$$\frac{\partial\Phi}{\partial r} = \frac{N\Gamma(N+2)}{2\pi i}(r+s)^{2N+2}\left\{\int_{\Gamma_1}\frac{F'(\zeta)\,d\zeta}{(\zeta-r)^{N+2}(\zeta+s)^{N+1}} - \int_{\Gamma_2}\frac{G'(\zeta)\,d\zeta}{(\zeta+r)^{N+2}(\zeta-s)^{N+1}}\right\}$$

on putting $R(z) = F'(z)$, $S(z) = G'(z)$.

If we now subtract the second of these expressions from the first we obtain

$$\frac{N\Gamma(N+2)}{2\pi i}(r+s)^{2N+2}\left\{-\int_{\Gamma_1}\frac{d}{d\zeta}\left[\frac{F(\zeta)}{(\zeta-r)^{N+2}(\zeta+s)^{N+1}}\right]d\zeta + \int_{\Gamma_2}\frac{d}{d\zeta}\left[\frac{G(\zeta)}{(\zeta+r)^{N+2}(\zeta-s)^{N+1}}\right]d\zeta\right.$$

and, as a term in z^{2N} was assimilated in $R(z)$ and $S(z)$ it is seen that this expression is identically zero. Thus (2.2.16) gives a correct form for $\frac{\partial\Phi}{\partial r}$, and similarly for

$\frac{\partial \xi}{\partial s}$.

Finally we may note that (2.2.16) is a solution of (2.2.15) for all N, F(z), G(z) . This completes the proof.

§ 3. In this section we examine the conditions which must be satisfied by two regions if they are to be patched together along a characteristic. We treat only the case where N is an integer. In the more general case, the presence of contour integrals makes an exact statement of these conditions difficult, and moreover, as will be seen in the examples of chapters 3 and 4, the easiest method of solving a particular problem in this general case is first to obtain a solution for integral N , and then generalise this solution by means of equation (2.2.12).

For the integral case w is given by (2.2.11) which we repeat here for convenience

$$w = \left(\frac{\partial}{\partial r}\right)^{N-1} \frac{R(r)}{(r+s)^N} + \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{S(s)}{(r+s)^N} \qquad (2.3.1)$$

We now prove:

Theorem II If the functions $R_1(r)$ and $S_1(s)$, on substitution into (2.3.1), define a region of the flow, then the functions $R_2(r)$ and $S_2(s)$ where

$$\begin{aligned} R_2(r) &= R_1(r) + \sum_{n=0}^{2N-1} a_n r^n \\ S_2(s) &= S_1(s) - \sum_{n=0}^{2N-1} (-1)^n a_n s^n \end{aligned} \qquad (2.3.2)$$

define the identical region for any constants a_n .

We write

$$w_i = \left(\frac{\partial}{\partial r}\right)^{N-1} \frac{R_i(r)}{(r+s)^N} + \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{S_i(s)}{(r+s)^N} \text{ for } i = 1,2.$$

Then the regions defined by w_1 and w_2 will be identical if the expressions for x and t in terms of r and s given by substitution of w_1 and w_2 in turn in equations (2.2.8) and (2.2.9) are identical. This requires the first derivatives of w_1 and w_2 to be identical, and hence that w_1 and w_2 should differ only by an arbitrary constant. Thus we require to show that

$$w_2 = w_1 + k$$

for some constant k .

Substituting for R_2 and S_2 in w_2 reduces this equation to

$$\left(\frac{\partial}{\partial r}\right)^{N-1} \frac{\Sigma\, a_n\, r^n}{(r+s)^N} - \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{\Sigma\,(-1)^n\, a_n\, s^n}{(r+s)^N} = k$$

and the left side may be written as

$$\frac{(N-1)!}{2\pi i} \int_C \frac{\Sigma\, a_n\, z^n}{(z-r)^N (z+s)^N}\, dz$$

where C is any contour surrounding $z = r$ and $z = -s$. By taking $C = \rho e^{i\theta}$, and letting $\rho \to \infty$, this integral is seen to be equal to $(N-1)!\, a_{2N-1}$ which is a constant. This proves the theorem.

We now find the conditions that two regions given by

w_1 and w_2 may be patched together along a characteristic, which we take to be an s-characteristic $s = s_0$. These conditions have been previously employed, although not actually stated, by Pack [10]. Then the values of x and t on $s = s_0$ must be identical in both regions and this implies

$$[w_2]_{s=s_0} \equiv [w_1]_{s=s_0} + c \quad ,$$

that is,

$$\left[\left(\frac{\partial}{\partial r}\right)^{N-1}\frac{R_2(r)}{(r+s)^N} + \left(\frac{\partial}{\partial s}\right)^{N-1}\frac{S_2(s)}{(r+s)^N}\right]_{s=s_0} \equiv \left[\left(\frac{\partial}{\partial r}\right)^{N-1}\frac{R_1(r)}{(r+s)^N} + \left(\frac{\partial}{\partial s}\right)^{N-1}\frac{S_1(s)}{(r+s)^N}\right]_{s=s_0} + c.$$

for some constant c .

If we multiply both sides of this expression through by $(r + s_0)^{2N-1}$, then differentiate the result N times with respect to r , the result, on cancelling a term $(r + s_0)^{N-1}$, is simply

$$R_2^{(2N-2)}(r) = R_1^{(2N-2)}(r) + \frac{(2N-1)!}{(N-1)!}\, c \quad .$$

Integrating this expression it will be seen that $R_2(r)$ and $R_1(r)$ are connected by an equation of the form (2.3.2). Then by our theorem we may add suitable terms to $R_2(r)$ and $S_2(s)$ so as to make $R_2(r) \equiv R_1(r)$. We have thus shown that if we patch an unknown region onto a known one along an s-characteristic we may take the function $R(r)$ which is to specify the unknown region as identical with that specifying the known.

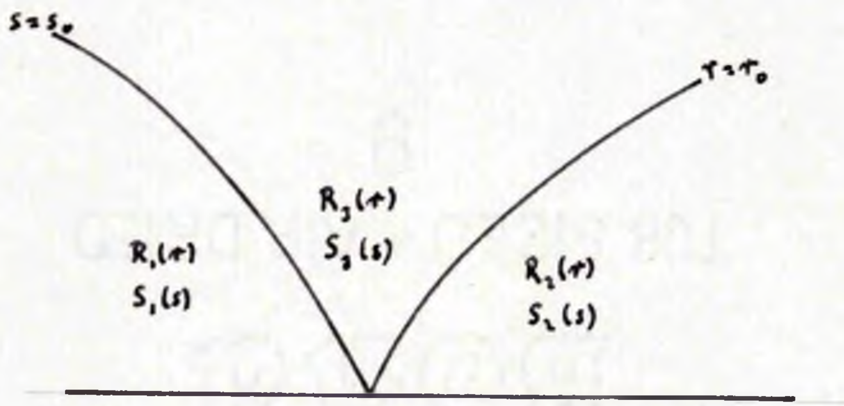

Figure V

If we now substitute this identity into (2.33) and compare co-efficients of r we find firstly that c must be zero (which happens automatically when the term a_{2N-1} is removed) and also that the functions $S_1(s)$ and $S_2(s)$, and their first $N-1$ derivatives must be equal when $s = s_0$.

These two requirements clearly constitute a sufficient condition for two regions to patch together. The necessary condition is that one of the regions may be reduced to that prescribed by the sufficiency condition on the application of theorem II.

We now consider the case shown in figure V where a region $R_3(r)$, $S_3(s)$ is to be patched on to two known regions $R_1(r)$, $S_1(s)$ and $R_2(r)$, $S_2(s)$ along an s and an r-characteristic respectively.

Under these conditions we may make either $R_3(r) \equiv R_1(r)$ or $S_3(s) \equiv S_2(s)$, but, in general, not both. The most general possible expression for region (3) is

$$R_3(r) = R_1(r) + \sum_{n=0}^{2N-1} b_n r^n$$

$$S_3(s) = S_2(s) - \sum_{n=0}^{2N-1} (-1)^n c_n s^n$$

for some sets of constants b_n and c_n.

Application of theorem II now enables us to write either

$$R_3(r) = R_1(r)$$

$$S_3(s) = S_2(s) - \sum_{n=0}^{2N-2} (-1)^n a_n s^n \qquad (2.3.4)$$

or

$$R_3(r) = R_1(r) + \sum_{n=0}^{2N-2} (-a_n) r^n$$

$$S_3(s) = S_2(s) \qquad (2.3.5)$$

where in either case $a_n = c_n - b_n$, and we have made a_{2N-1} zero. These alternative expressions will of course give identical values for w in the region (3). The values of the $2N-1$ constants a_n are found by applying the second condition. If we use the representation (2.3.4) we require $S_3(s_0) = S_1(s_0)$, $S_3'(s_0) = S_1'(s_0)$, ... $S_3^{(N-1)}(s_0) = S_1^{(N-1)}(s_0)$. This gives N equations for the a_n namely

$$\sum_{n=p}^{2N-2} (-1)^n a_n \frac{n!}{(n-p)!} s_0^{n-p} = S_2^{(p)}(s_0) - S_1^{(p)}(s_0) \quad , \quad p = 0, 1, \ldots N-1. \qquad (2.3.6)$$

But we also know that the representation (2.3.5) is compatible with region (2) across $r = r_0$. This leads to the further N equations,

$$\sum_{n=p}^{2N-2} a_n \frac{n!}{(n-p)!} r_0^{n-p} = R_1^{(p)}(r_0) - R_2^{(p)}(r_0) \quad , \quad p = 0, 1, \ldots N-1 \qquad (2.3.7)$$

This gives $2N$ equations for $2N-1$ unknowns, but elimination of all the a_n would leave an equation equivalent to the statement that the expressions for the regions (1) and (2) are identical at (r_0, s_0), which is assumed. We thus have as many independent equations as unknowns, and the representation of region (3) is therefore obtained.

§ 4 As an illustration of this theory we now generalise to the case where N is any integer a problem solved by Mackie [5] for N unity.

The problem is that in which a gas is initially at rest in the region $-h_2 < x < h_1$ bounded on both sides by a vacuum into which it is allowed to expand. It is assume that at $t = 0$ the density is continuous, rising steadily from 0 at $x = -h_2$ to a maximum at $x = 0$ then falling steadily to 0 at $x = h_1$. Specifically, at $t = 0$ we have

$$c = c_1(x) \qquad 0 \leqslant x \leqslant h_1$$

$$c = c_2(x) \qquad -h_2 \leqslant x \leqslant 0$$

where the functions $c_1(x)$, $c_2(x)$ are required to satisfy

$$c_1(h_1) = c_2(-h_2) = 0$$

$$c_1(0) = c_2(0) = \frac{2p}{N+1} \text{ (say)} \qquad (2.4.1)$$

$$c_1'(x) < 0 \quad \text{for} \quad 0 < x < h_1$$

$$c_2'(x) > 0 \quad \text{for} \quad -h_2 < x < 0 .$$

If we assume that there will be no breakdown in the continuity of the flow, the solution of the problem requires that we patch together solutions for the seven regions shown

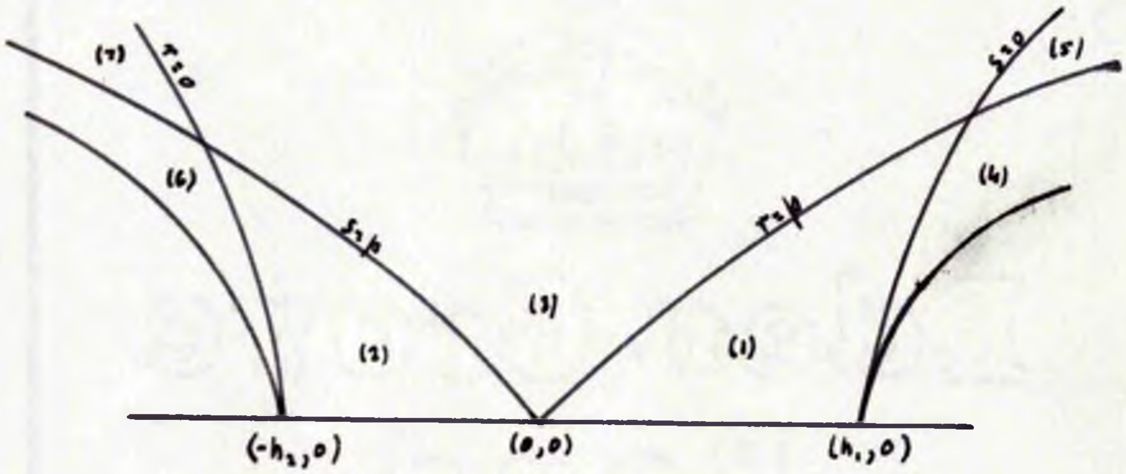

Figure VI

in figure VI.

We require first to find the solution in region (1). Boundary conditions are given along OA . They are

$$t = 0$$

$$u = r - s = 0$$

$$c = \frac{2r}{2v+1} = c_1(x) \ .$$

If we now write these in terms of the dependent variable w with independent variables r and s they become respectively, using (2.2.8) and (2.2.9)

$$\frac{\partial w_1}{\partial r} + \frac{\partial w_1}{\partial s} = 0$$

$$r = s$$

$$\frac{1}{r+s}\left\{[(N+1)s - Nr]\frac{\partial w_1}{\partial r} - [(N+1)r - Ns]\frac{\partial w_1}{\partial s}\right\} = \phi_1(r)$$

or much more simply,

$$\frac{\partial w_1}{\partial r} = -\frac{\partial w_1}{\partial s} = \phi_1(r) \quad \text{on} \quad r = s \qquad (2.4.2)$$

where w_1 is of course the solution of (2.2.7) corresponding to region (1), and where $x = \phi_1(r)$ is the inverse function obtained from $r = \frac{2N+1}{2} c_1(x)$. From the conditions imposed on $c_1(x)$ this inverse function will be continuous with continuous derivatives, one-valued, and monotonic in $0 < r < p$, where p is the constant defined in (2.4.1)

Then

$$w_1 = \left(\frac{\partial}{\partial r}\right)^{N-1} \frac{R_1(r)}{(r+s)^N} + \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{S_1(s)}{(r+s)^N}$$

where

$$R_1(r) = \frac{2}{(N-1)!} \int^r (r^2 - \xi^2)^{N-1} \xi \, \phi_1(\xi) \, d\xi \ , \qquad (2.4.3)$$

$$S_1(s) = -R_1(s) \ .$$

These results have been given for integral N, by Pack [10], with a different numerical factor due to the slightly different form of (2.3.1) used in his paper. An alternative proof based on induction is given in Appendix I. Similar results for all values of N have been given by Copson [1] and Mackie [6].

We now look for the solution in region (4). Since this region is adjacent to region (1) along the characteristic $s = 0$ we may put

$$R_4(r) = R_1(r) \ .$$

The other condition on this region is that both x and t, and hence both $\frac{\partial w_4}{\partial r}$ and $\frac{\partial w_4}{\partial s}$ are to be finite on the front of expanding gas, on which $c = r + s = 0$. Thus in the case $N = 1$, where $w_4 = \frac{R_4(r) + S_4(s)}{r + s}$, we must clearly put $S_4(s) = -R_4(-s)$. If we now put $r + s = \varepsilon$ and expand in powers of ε we obtain the results given by Mackie

$$w_4 = 2r\, \phi_1(r) - \varepsilon(\phi + r\phi') + O(\varepsilon^2) \ ,$$

$$\frac{\partial w_4}{\partial r} = \phi + r\phi' - \tfrac{1}{3}s(2\phi' + r\phi'') + O(s^2) ,$$

$$\frac{\partial w_4}{\partial s} = -\phi - r\phi' + \tfrac{2}{3}s(2\phi' + r\phi'') + O(s^2) ,$$

$$t = -\tfrac{1}{2}(2\phi' + r\phi'') + O(s) \qquad (2.4.4)$$

$$x = \phi - r\phi' + r^2\phi'' + O(s) . \qquad (2.4.5)$$

The equation in the physical plane of the front of the gas is given parametrically by the last two of these equations with $s = 0$.

Guided by this result, we try for general values of N

$$w_4 = \left(\frac{\partial}{\partial r}\right)^{N-1} \frac{R_4(r)}{(r+s)^N} - \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{R_4(-s)}{(r+s)^N} .$$

We see that for this particular value of the s-function w_4 may be expressed as a single contour integral

$$w_4 = \frac{(N-1)!}{2\pi i} \int_C \frac{R_4(z)\,dz}{(z-r)^N(z+s)^N}$$

where C is any closed contour surrounding $z = r$ and $z = -s$.

Substitution of this value in (2.2.8) and (2.2.9) gives the corresponding values for t and x as

$$t = -\frac{(2N+1)N!}{4\pi i} \int_C \frac{R_4(z)\,dz}{(z-r)^{N+1}(z+s)^{N+1}}$$

$$x = \frac{N!}{2\pi i}\int_C \frac{[z-(N+1)(r-s)]\,R_4(z)}{(z-r)^{N+1}(z+s)^{N+1}}\,dz\ .$$

For the values of t and x on $r+s=0$ we may simply substitute $s=-r$ in these equations, and express the result in real variable terms. Substituting for $R_4(z)$ from (2.4.3) we get

$$t = -\frac{1}{2(2N-1)!}\left(\frac{d}{dr}\right)^{2N+1}\int^r (r^2-\xi^2)^{N-1}\,\xi\,\phi_1(\xi)\,d\xi$$

$$x = 2rt + \frac{1}{(2N-1)!}\left(\frac{d}{dr}\right)^{2N}\int^r (r^2-\xi^2)^{N-1}\,\xi\,\phi_1(\xi)\,d\xi\ ,$$

which are the parametric equations for the front of the gas, and are clearly finite. If we put $N=1$ we may recover equations (2.4.4) and (2.4.5).

Finally we note that $S_4(s) = -R_4(-s) = S_1(-s)$. Clearly the even derivatives of $S_4(s)$ and $S_1(s)$ are equal at $s=0$, and from the form of (2.4.3) it will be seen that the odd derivatives (at least up to the $N-1^{th}$) contain a term in S, and hence are zero at $S=0$. Thus the second patching condition on region (4) is satisfied.

The solutions in regions (2) and (6) are obtained similarly. For the corresponding inverse function $\phi_2(r)$ we get

$$R_2(r) = \frac{2}{(N-1)!}\int^r (r^2-\xi^2)^{N-1}\,\xi\,\phi_2(\xi)\,d\xi$$

$$S_2(s) = -R_2(s)$$

$$S_6(s) = S_2(s) = -R_2(s)$$

$$R_6(r) = -S_6(-r) = R_2(-r) \; .$$

Region (3) is now bounded by regions (1) and (2) along an r and an s-characteristic respectively. We may therefore write

$$R_3(r) = R_2(r)$$

$$S_3(s) = -R_1(s) - \sum_{n=0}^{2N-2} (-1)^n a_n s^n$$

where the co-efficients a_n are given by equations similar to (2.3.6) and (2.3.7).

In the same way regions (5) and (7) are found to be given by

$$R_5(r) = R_2(r) + \Sigma \alpha_n r^n$$

$$S_5(s) = -R_1(s)$$

$$R_7(r) = R_2(-r) + \Sigma \beta_n r^n$$

$$S_7(s) = -R_1(s)$$

for some constants α_n, β_n .

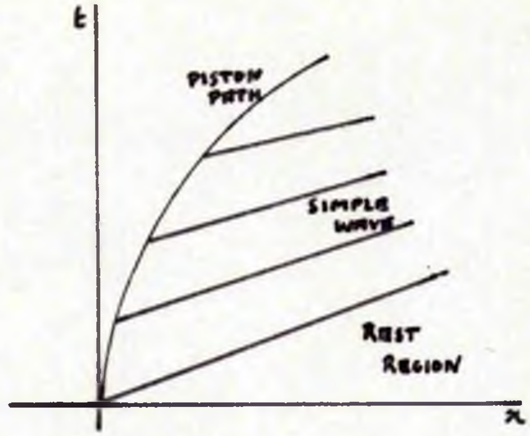

Figure VII

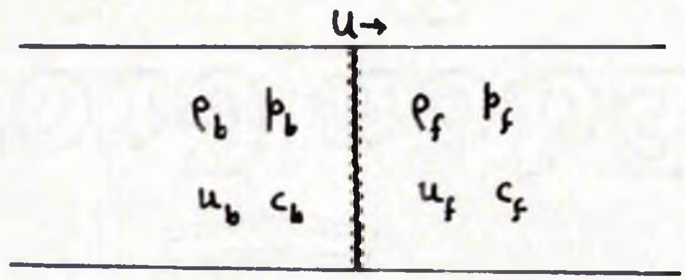

Figure VIII

Chapter 3

Shocks

§ 1. So far we have dealt only with flows in which the characteristics have been assumed to diverge. We go on now to consider cases in which this does not hold. Suppose, for example, that a piston is moved with constant acceleration into a gas at rest, as in figure VII.

Considering this flow mathematically, we have that the region to the right of $x = c_0 t$ remains at rest while the region to the left is a simple wave as before. Again the r-characteristics are straight lines, but this time the quantity $u + c$ which determines their slope is increasing along the piston path. The r-characteristics will therefore tend to converge, as shown, forming a simple compression, as opposed to expansion, wave. It is clear that they will intersect, and as the value of r cannot be the same on different characteristics, there must be a breakdown in the continuity of the flow. We say that this breakdown indicates the presence of a shock wave which we treat mathematically as a sudden jump discontinuity occurring across certain sharply defined lines in the fluid.

The physical picture is as follows. Since the influence of the piston motion is propagated into the gas through waves travelling with sound speed c relative to the gas, and since to greater piston speed there corresponds a greater sound speed, the later influences travel faster, overtaking those sent out earlier. The resulting build up leads to a region in which our previous idealised assumptions are no longer valid. The effect of friction, for example, can no longer be ignored, and since friction will cause irreversible thermodynamic processes we will expect the entropy of a particle of gas to increase on passing through such a region. However these effects have been shown to occur only in very narrow regions of width comparable with the mean free path of a particle. Thus our assumption of a sharp mathematical discontinuity may be considered a reasonable idealisation.

In the simple compression wave illustrated in figure VII, the shock would be assumed to form at the first point on the envelope of the r-characteristics. We note that this point need not be on a boundary of the region.

§ 2. We now obtain the mathematical relationships which must hold across such a discontinuity. These are the Rankine-Hugoniot shock relations. We use the basic assumptions of conservation of mass, momentum, and energy remembering that in this case some of the energy is internal. We define the two regions at the 'front' and the 'back' of the shock by assuming that the particles pass through the shock from the front to the back. Quantities in these regions will be denoted by the subscripts f and b respectively. We will always assume that the front of the shock is to the right, that is to the side of x-increasing. Then the situation in the tube at any instant t is as in figure VIII, where U is the velocity of the shock.

Clearly for the flow to exist as specified, the relative velocities of the gas with respect to the shock must be negative on both sides. That is, we require $v_b = u_b - U < 0$, and $v_f = u_f - U < 0$.

Now let the position of the shock be specified by $\xi(t)$, and consider a column of gas which at time t covers the interval $a_0(t) < x < a_1(t)$ where $a_0(t)$ and $a_1(t)$ denote the positions of the particles at the ends of the column, and $a_0(t) < \xi(t) < a_1(t)$.

Then the basic principles are expressed by the relations

$$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho(x,t)\,dx = 0 \ , \quad \text{(Conservation of mass)},$$

$$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho\, u\, dx = p(a_0,t) - p(a_1,t), \quad \text{(Conservation of momentum)}, \qquad (3.2.1)$$

$$\frac{d}{dt}\int_{a_0(t)}^{a_1(t)} \rho\left(\tfrac{1}{2}u^2+e\right)dx = p(a_0,t)\,u(a_0,t) - p(a_1,t)\,u(a_1,t) \qquad \text{(Conservation of Energy)}$$

where e is the internal energy of the gas. Of these equations, the first is obvious while the others are based on the respective assumptions that the only forces acting are pressure forces, and that the only gain in total energy is due to the action of these forces.

All these integrals are of the form

$$I = \int_{a_0(t)}^{a_1(t)} \Phi(x,t)\,dx$$

where Φ is discontinuous at an interior point $\xi(t)$ of the interval of integration. Then

$$\frac{d}{dt}I = \frac{d}{dt}\int_{a_0}^{\xi} \Phi\,dx + \frac{d}{dt}\int_{\xi}^{a_1} \Phi\,dx$$

$$= \int_{a_0}^{a_1} \Phi_t\,dx + \Phi_b^0\,\xi'(t) - \Phi(a_0,t)\,u(a_0,t) - \Phi_f^0\,\xi'(t) + \Phi(a_1,t)\,u(a_1,t)$$

where Φ_b^0 is the value of Φ_b on the back of the shock itself, and similarly for Φ_f^0, and $u(a_1,t)$ is $a_1'(t)$, the velocity at the point $a_1(t)$.

Taking limits, we get

$$\lim_{(a_1-a_0)\to 0} \frac{d}{dt}I = \Phi_f^0\,v_f^0 - \Phi_b^0\,v_b^0 .$$

Thus from equations (3.2.1) at the limit, we obtain the relationships (dropping the affix o) which hold between the quantities on the back and the front of the shock. These are

$$\rho_f v_f = \rho_b v_b \tag{3.2.2}$$

$$\rho_f v_f^2 + p_f = \rho_b v_b^2 + p_b \tag{3.2.3}$$

$$\rho_f(\tfrac{1}{2}u_f^2 + e_f)v_f + p_f u_f = \rho_b(\tfrac{1}{2}u_b^2 + e_b)v_b + p_b u_b \tag{3.2.4}$$

For a polytropic gas $e = \frac{1}{\gamma-1}\frac{p}{\rho}$, and (3.2.4) may be written, with the help of (3.2.2) and (3.2.3), as

$$\tfrac{1}{2}v_f^2 + \frac{\gamma}{\gamma-1}\frac{p_f}{\rho_f} = \tfrac{1}{2}v_b^2 + \frac{\gamma}{\gamma-1}\frac{p_b}{\rho_b} \quad . \tag{3.2.5}$$

If now we are given the conditions in front of the shock, u_f, p_f, ρ_f, and one other conditions from U, u_b, p_b, ρ_b, the above equations will be sufficient to determine the three unknowns. We find it convenient to introduce the shock strength, z, defined by

$$p_b = z\, p_f \quad . \tag{3.2.6}$$

Then in terms of z we may obtain the relations

$$\rho_b = \frac{(\gamma+1)z + (\gamma-1)}{(\gamma-1)z + (\gamma+1)}\,\rho_f \tag{3.2.7}$$

$$u_b = u_f + Ac_f \tag{3.2.8}$$

$$c_b = Bc_f \tag{3.2.9}$$

$$U = u_f + Dc_f \tag{3.2.10}$$

where A, B, and D are the functions of z

$$A = 2(z-1)\left[2\gamma\left\{(\gamma+1)z + (\gamma-1)\right\}\right]^{-\frac{1}{2}} \tag{3.2.11}$$

$$B = \left[\frac{z\left\{(\gamma-1)z + (\gamma+1)\right\}}{(\gamma+1)z + (\gamma-1)} \right]^{\frac{1}{2}}$$

$$D = \left[\frac{(\gamma+1)z + (\gamma-1)}{2\gamma} \right]^{\frac{1}{2}}$$

If the gas is polytropic, the change of entropy across the shock is given, from equation (1.1.1), by

$$S_b - S_f = c_v \log \frac{p_b \rho_f^{\gamma}}{p_f \rho_b^{\gamma}} \qquad (3.2.12)$$

where the right side, from equations (3.2.6) and (3.2.7) is a function of z only. Since entropy cannot decrease, by the laws of thermodynamics, we find from this equation that we must have $z > 1$.

As a check, we note that as $z \to 1$, $A \to 0$ and $B \to 1$, and we also see that $D \to 1$ showing that the limiting form of the shock is a characteristic. Since $D > 1$, we have $|v_f| > c_f$, and since $D - A < B$ we have $|v_b| < c_b$. Thus the shock is supersonic relative to the gas in front of it, and subsonic relative to the gas behind. From this it follows that a shock must always catch up with a shock or a disturbance ahead of it, or vice-versa.

Since the entropy change is a function of z which need not be constant, the flow behind the shock will contain particles at different entropy levels and consequently the isentropic theory will no longer hold. One well-known problem which can be solved is that in which

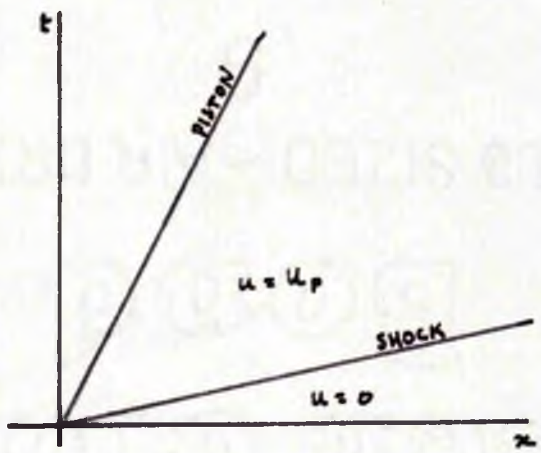

Figure IX

a piston is suddenly pushed into a gas at rest with a constant velocity u_p . Since there is an immediate discontinuity in $u + c$ a shock will form at once and move into the gas ahead of the piston. The boundary condition will be satisfied if we can find a shock which will separate two regions of constant velocity 0 and U_p respectively, as in figure IX .

Putting $u_b = U_p$ in (3.2.8) gives a constant value for A , and hence for z by solving (3.2.11) and taking the root greater than unity. This value of z in (3.2.6), (3.2.7) and (3.2.9) gives the values for the pressure, density, and sound speed in the region behind the shock. Finally from (3.2.10) we see that the shock moves into the gas at constant speed so that the picture in figure IX is qualitively correct.

§ 3. The reason why this problem could be solved exactly is that the shock propagated under the given conditions happened to be of constant strength. In general, a problem posed by giving boundary conditions in the form of an initial gas state and some compressive piston motion will lead to non-isentropic flows and hence be incapable of exact solution. In order to obtain such solutions to problems involving shocks other than those with uniform conditions on each side, we employ the following indirect procedure. We know from (3.2.12) that a constant strength shock passing through an isentropic region will leave behind a region which is also isentropic, although at a different entropy level. Starting with a given isentropic flow, we find the possible paths through it for a shock of constant strength. This may be done since if z is constant, D is constant, and the shock paths then form a one-parameter family of solution of the differential equation of the shock paths given by (3.2.10), namely

$$U = \frac{dx}{dt} = u_f(x,t) + Dc_f(x,t) \ . \tag{3.3.1}$$

This is the equation of the shock paths in the physical plane. If we are dealing with the hodograph plane, with the region in front of the shock known by having x and t in terms of r and s, the equation for the front of the shock will then be

$$\frac{dr}{ds} = -\frac{D-1}{D+1}\frac{t_r}{t_s} \tag{3.3.2}$$

which may be obtained from (3.3.1) using (2.2.2) and (2.2.3).

Equations (3.2.8) and (3.2.9) give the values of u_b and c_b, and hence of r_b and s_b along the back of the shock, in terms both of r_f and s_f, and also of x and t. The former solutions together with (3.3.2) may be solved to give the equation for the back of the shock in the hodograph plane which owing to the jump in u and c will not be the same equation as for the front. The latter solutions substituted in (2.2.5) and (2.2.6) give values for $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ along the back of the shock. Thus we have boundary conditions which hold for w along a known curve in the $r - s$ plane, and hence we obtain a solution of (2.2.7) for the region behind the shock. We may now find the particle paths in this region, and relate a suitable one to the motion of a piston. That is to say, we start with the shock path and find the piston motion which would give rise to that shock.

§ 4. For our first example we consider the case in which a piston is withdrawn from a gas at rest with a constant velocity less than the escape velocity of the gas, causing a point-centred simple wave. The region next to the receding piston is one of uniform velocity, so if the piston is stopped after a finite time it will have the same effect as pushing a piston at constant speed into a gas at rest. A shock of constant strength will move into the gas leaving behind it a region of rest. As has been pointed out, the shock must catch up with the simple wave. As it moves into the wave it will tend to decrease in strength. To keep the strength constant the piston must be moved again, being accelerated smoothly from rest causing a simple compression wave which will be required to catch up with the shock at the moment it enters the original wave. The problem is to find the future motion of the piston in order that the shock be

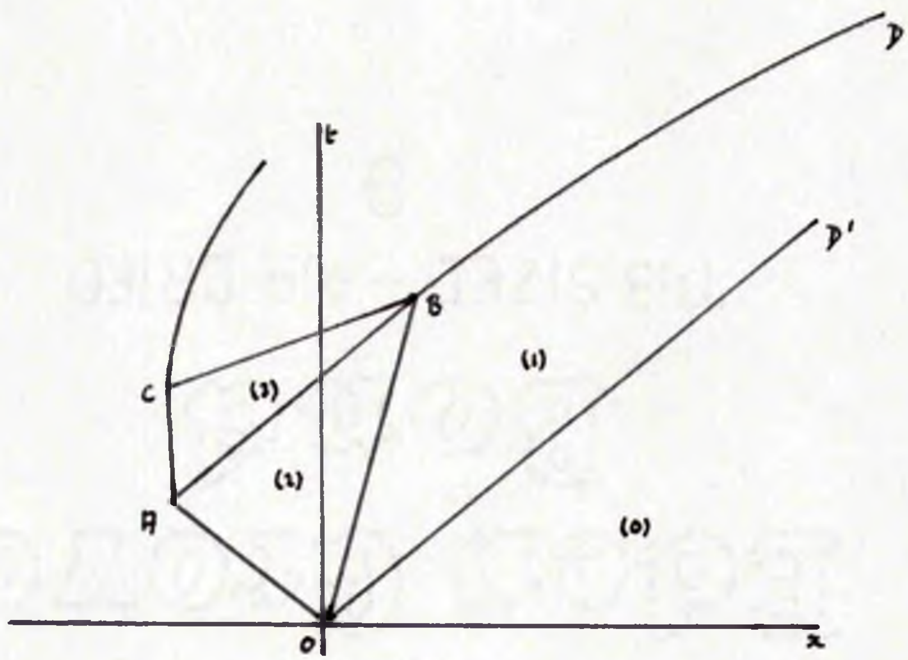

Figure X

kept constant.

This problem has previously been attempted by Gundersen in an unpublished Brown University report. However his solution is not correct, the most important error being in restarting the piston when the s-characteristic from the point of intersection of the shock and the wave reaches it, instead of at a time such that the r-characteristic from the piston will catch up with the shock as it enters the wave. One consequence of this error is that the piston is restarted with a finite velocity which would clearly lead to the formation of a second shock wave.

Since it is not possible to complete this problem without recourse to numerical work, and since in our second example the method whereby a general solution could be found as far as such a solution is possible is clearly shown, we do this example only for the case $N = 1$.

The first draft of the problem is shown in figure X.

The region (o) is at rest bounded by the r-characteristic through the origin OD' . Region (1) is the simple wave bounded by another r-characteristic OB behind which is region (2) which is in uniform motion with velocity equal to that of the piston. The piston is stopped at A causing the shock wave ABD to move into the gas. To keep the shock constant the piston moves forward again at C such that CB is an r-characteristic, which bounds the rest region (3).

The first step is to solve for the regions (1), (2) and (3), then to find the equation of the shock through (1), BD, and discover whether it meets OD', that is, whether it passes completely through the simple wave. Once this has been done we can split up the flow behind the shock into regions and solve these in turn.

Numerical suffices will refer to the regions in the figure, capital letters to the respective points.

We assume c_0 to be known, so the line OD' is $x = c_0 t$.

The point A at which the piston is stopped is also assumed to be known, so we may put $A = (-x_A, t_A)$ and hence $u_2 = -\frac{x_A}{t_A}$. Also $s_2 = s_0 = \frac{3c_0}{2}$ (since $N = 1$, $\gamma = 5/3$) and so we may obtain $c_2 = c_0 - \frac{x_A}{3t_A}$. Then OB, an r-characteristic is given by

$$x = (c_2 + u_2)t = (c_0 - \frac{4x_A}{3t_A})t \quad . \qquad (3.4.1)$$

Region (1) is a point centred simple wave with $s_1 = s_0$, so that

$$u_1 = \frac{3}{4}\left(\frac{x}{t} - c_0\right) \qquad (3.4.2)$$

$$c_1 = \frac{1}{4}\left(\frac{x}{t} + 3c_0\right) \qquad (3.4.3)$$

Region (3) has $u_3 = 0$ by hypothesis, so (3.2.8) becomes $u_2 + Ac_2 = 0$. This equation may be solved for z in terms of c_o, x_A, and t_A which in turn gives values for the constants B and D, and hence we get $c_3 = Bc_2$, and also the equation for the shock path AB which is

$$(x + x_A) = (u_2 + Dc_2)(t - t_A)$$

or

$$x = (u_2 + Dc_2)t - Dc_2 t_A \ .$$

This is turn gives, with (3.4.1), the coordinates of the point B,

$$x_B = \frac{D}{D-1} \left(c_o - \frac{4}{3} \frac{x_A}{t_A} \right) t_A \ .$$

$$t_B = \frac{D}{D-1} t_A \ .$$

The line BC, an r-characteristic of region (3) is

$$x - x_B = (u_3 + c_3)(t - t_B)$$

from which equation the point C is found to be

$$x_c = - x_A$$

$$t_c = \frac{BD + A - D}{BD - B} t_A \ .$$

If we now substitute equations (3.4.2) and (3.4.3) in the differential equation (3.3.1) and integrate, we obtain as the path of the shock through the simple wave the equation

$$x + 3c_o t = Kt^{\frac{D+3}{4}} \qquad (3.4.4)$$

where the constant of integration K is known since the curve must pass through B. This curve will meet the

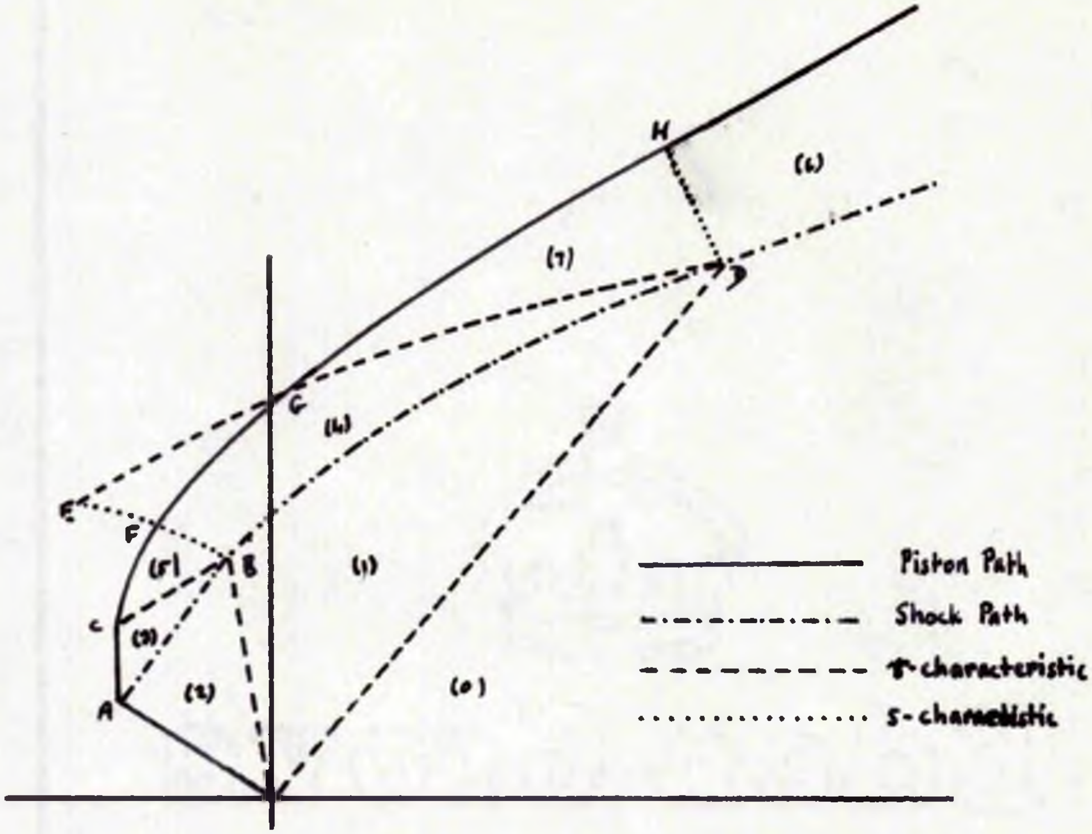

Figure XI

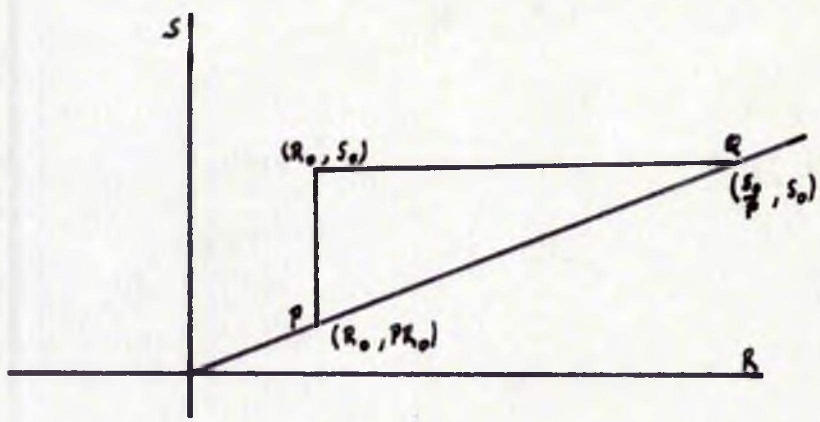

Figure XII

line $x = c_0 t$ at a point D given by

$$x_D = c_0 t_D$$

$$t_D = \left(\frac{4c_0}{K}\right)^a$$

where $a = \frac{4}{D-1}$. We thus see that the shock will pass through the simple wave in a finite time.

We now look for solutions to the regions behind the shock. Such regions will be bounded by characteristics and may be assumed to have a distribution as shown in figure XI.

The region (4) will be a non-uniform region bounded by BE and DE, s and r characteristics respectively. Region (6) is separated from a rest region by a constant strength shock, and is hence a region of uniform motion. It follows that region (7), like region (5), will be a simple wave. These regions have r constant and s constant respectively. The piston path is now OACFGH, and this is the path we require to determine. The first step is to find a solution of (2.2.7) holding in region (4).

On the back of the shock the relations $u_4 = u_1 + Ac_1$, $c_4 = Bc_1$ hold, enabling us to express u_4 and c_4, and hence r_4 and s_4, in terms of x and t from (3.4.2) and (3.4.3). The resulting equations, together with (3.4.4) enable us to find the equation for the back of the shock in the hodograph plane, which is

$$(1 - E)r_4 = (1 + E)s_4 - 3c_o \tag{3.4.5}$$

where $E = \frac{A+3}{3B}$, and also the values for x and t in terms of r_4 along this line, namely

$$t = G^a(r_4 + \tfrac{3}{2}c_o)^a$$

$$x = [KGr_4 + \tfrac{3}{2}c_o(KG - 2)]t$$

where $G = \frac{2}{3BK(1+E)}$.

Substituting these in (2.2.5) and (2.2.6) we obtain as the boundary conditions along (3.4.5)

$$\frac{\partial w}{\partial r_4} = L(r_4 + \tfrac{3}{2}c_o)^{a+1}$$

$$\frac{\partial w}{\partial s_4} = M(r_4 + \tfrac{3}{2}c_o)^{a+1}$$

where $L = 2G^a \frac{1 - A - B}{3 + A + 3B}$, $M = 2G^a \frac{A - B - 1}{3 + A + 3B}$.

For simplicity we introduce new independent variables R_4 and S_4 defined by

$$\begin{aligned} R_4 &= r_4 + \tfrac{3}{2}c_o \\ S_4 &= s_4 - \tfrac{3}{2}c_o . \end{aligned} \tag{3.4.6}$$

In terms of these variables our problem is to find a solution of

$$\frac{\partial^2 w}{\partial R_4\, \partial S_4} + \frac{1}{R_4 + S_4}\left(\frac{\partial w}{\partial R_4} + \frac{\partial w}{\partial S_4}\right) = 0$$

under the conditions that $\frac{\partial w}{\partial R_4} = LR_4^{a+1}$, $\frac{\partial w}{\partial S_4} = MR_4^{a+1}$ along $S_4 = PR_4$, where $P = \frac{1-E}{1+E}$.

We solve this problem as an example of Riemann's method, although it may be done as easily by the method of arbitrary functions used in the next section. For $N = 1$, the Riemann-Green function (2.2.10) has the particularly simple form $v = \frac{R + S}{R_o + S_o}$. When we integrate round the contour shown in figure XII, the value for w at (R_o, S_o) is simply

$$2w(R_o, S_o) = (vw)_P + (vw)_Q - \int_{PQ} (K\, dr - H\, ds)$$

where H and K are the functions

$$H = v\frac{\partial w}{\partial S_4} - w\frac{\partial v}{\partial S_4} + \frac{vw}{R_4 + S_4}$$

$$K = v\frac{\partial w}{\partial R_4} - w\frac{\partial v}{\partial R_4} + \frac{vw}{R_4 + S_4} .$$

This gives for the solution

$$w(R_o, S_o) = \frac{\alpha R_o^{a+3} + \beta S_o^{a+3}}{R_o + S_o}$$

where

$$\alpha = \frac{(L+MP)[(a+1) + (a+2)P]}{2(a+2)(a+3)} + \frac{(1+P)(L-MP)}{2(a+3)}$$

$$\beta = \frac{(L+MP)[(a+2) + (a+1)P]}{2(a+2)(a+3)P^{n+3}} - \frac{(1+P)(L-MP)}{2(a+3)P^{a+3}} .$$

Since (R_o, S_o) was a general point the final solution for w in region (4) is thus

$$w_4 = \frac{\alpha R_4^{a+3} + \beta S_4^{a+3}}{R_4 + S_4}$$

which leads from (2.2.8) and (2.2.9) to the equations for x_4 and t_4,

$$x_4 = \frac{1}{(R_4+S_4)^3}\left\{\alpha R_4^{a+2}\left[2(a+3)S_4^2 + aR_4S_4 - aR_4^2\right] - \beta S_4^{a+2}\left[2(a+3)R_4^2 + aR_4S_4 - aS_4^2\right]\right\} - 3c_0t_4 .$$

$$t_4 = \frac{-3}{2(R_4+S_4)^3}\left\{\alpha R_4^{a+2}\left[(a+1)R_4 + (a+3)S_4\right] + \beta S_4^{a+2}\left[(a+1)S_4 + (a+3)R_4\right]\right\}$$

The characteristics BE and DE are given parametrically by putting $S_4 = \frac{3(B-1)}{2}c_0 + \frac{Bu_2}{2}$ and $R_4 = \frac{3c_0}{2}(1 + B + \frac{A}{3})$ respectively into these equations. The trajectories in this region are obtained by eliminating R_4 and S_4 from these equations and that obtained from (2.2.13) and (2.2.14), namely

$$\Phi \equiv \alpha R_4^{a+2}\left[\frac{a(a+1)}{a+4}R_4^2 + 2aR_4S_4 + (a+3)S_4^2\right] - \beta S_4^{a+2}\left[\frac{a(a+1)}{a+4}S_4^2 + 2aR_4S_4 + (a+3)R_4^2\right] = \text{Constant} \qquad (3.4.7)$$

One of these trajectories will be the required piston path.

Region (5) is a simple wave with s constant. Defining R_5 and S_5 similarly to (3.4.6), the region is $S_5 = \frac{3(B-1)}{2}c_0 + \frac{Bu_2}{2}$. Then the r-characteristics are the straight lines given by

$$x = (u + c)t + f(r)$$

$$= \frac{1}{3}(4R_5 - 2S_5 - 9c_0)t + f(R_5) \qquad (3.4.8)$$

where $f(R_5)$ may be found since x and t are known in terms of R along BE. This gives

$$f(R_5) = \frac{1}{(R_5+S_5)^2}\left\{\alpha R_5^{a+2}[(a+2)R_5 + (a+3)S_5] - \beta S_5^{a+3}\right\}$$

where S_5 is the above constant.

If this value for x is substituted in the differential equation for the trajectories,

$$\frac{dx}{dt} = u = R_5 - S_5 - 3c_0 \quad ,$$

we may integrate to get

$$(R_5+S_5)^4 t = -3\alpha R_5^{a+2}\left[\frac{(a+1)(a+2)}{(a+4)}R_5^2 + 2(a+1)R_5S_5 + (a+3)S_5^2\right] - 6\beta S_5^{a+3}R_5 + \text{Constant} \qquad (3.4.9)$$

and this together with (3.4.8) gives the parametric equation for the trajectories. The particular value of the constant in (3.4.9) required to give the piston path is found since this path passes through C at which $R_5 = R_3$ and $t = t_C$.

Once this is known the position of the point F may be found and hence the value of the constant in (3.4.7) for the piston path. This in turn enables us to find x_G and t_G.

Region (7) is a simple wave with $R_7 = \frac{3c_0}{2}(1 + \frac{A}{3} + B)$ and may be solved in a similar fashion to region (5). Corresponding to (3.4.8) and (3.4.9) we have

$$x = \frac{1}{3}(2R_7 - 4S_7 - 9c_0)t + \frac{1}{(R_7+S_7)^2}\left\{\alpha R_7^{a+3} - \beta S_7^{a+2}\left[(a+2)S_7 + (a+3)R_7\right]\right\}$$

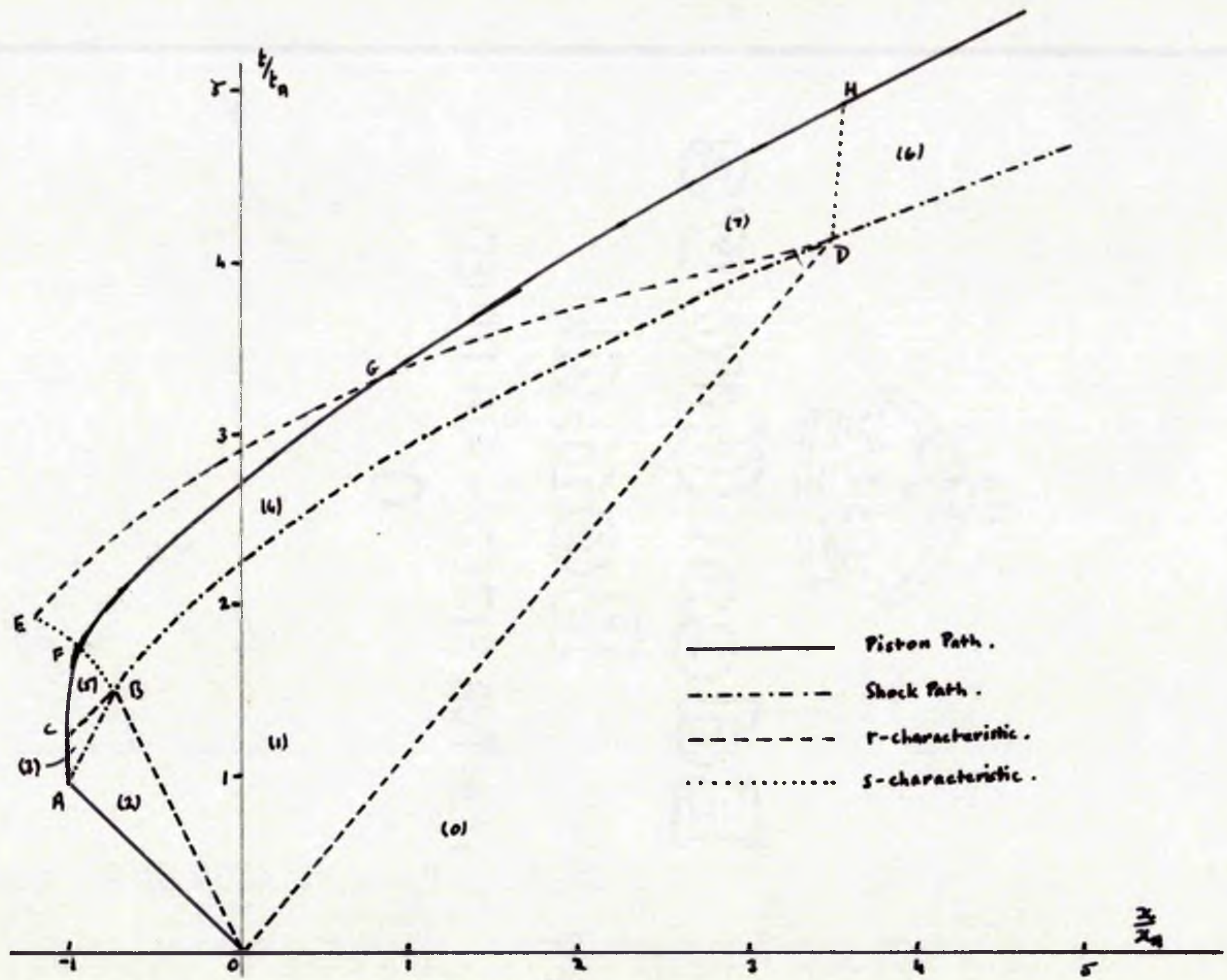
t/t_A
x/x_A
Piston Path.
Shock Path.
r-characteristic.
s-characteristic.
A
B
C
D
E
F
G
H
(0)
(1)
(2)
(3)
(4)
(5)
(6)
(7)

Figure XIII

$$(R_7+S_7)^4 t = -6\alpha R_7^{a+3} S_7 - 3\beta S_7^{a+2}\left[\frac{(a+1)(a+2)}{a+4} S_7^2 + 2(a+1)R_7 S_7 + (a+3)R_7^2\right] + \text{constant}$$

where S_7 is now the parameter, and the value of the constant corresponding to the piston is found from conditions at G .

Region (6) is a region of uniform motion behind a constant shock so that $u_6 = Ac_0$ and $c_6 = Bc_0$. The shock path beyond D is simply a straight line given by

$$(x - x_D) = Dc_0(t - t_D) \quad .$$

The s-characteristic DH has equation

$$(x - x_D) = (A - B)c_0 \; (t - t_D)$$

and the point H is the intersection of this line with the piston path in region (7).

Finally the path of the piston beyond H is also a straight line given by

$$(x - x_H) = Ac_0(t - t_H) \quad .$$

To illustrate the results we give a numerical example. For simplicity we take $z = 11$, since this gives $A = 2$, $D = 3$, $a = 2$. Then the diagram points are $A = (-x_A, t_A)$ $B = (-0.75x_A,\ 1.5t_A)$ $C = (-x_A,\ 1.24t_A)$ $D = (3.47x_A,\ 4.17t_A)$ $F = (-0.96x_A,\ 1.78t_A)$ $G = (0.92x_A,\ 3.39t_A)$ and $H = (3.52x_A,\ 4.93t_A)$. Then the complete flow in the physical plane is shown in figure XIII.

§ 5. For our second example in which the value of N is quite general, we treat a case in which a shock passes through a flow which is neither uniform, nor represented by a simple wave. This flow is chosen to be that in which u is a function of t only. In other words we seeks a solution of (2.2.4) of the form $t = f(r-s)$. Substitution leads immediately to the equation $f''(r-s) = 0$ showing that the only possible case of this type has t linear in u - a result which is generalised to non-isentropic flows in chapter 5. We select the origin of time so that the velocity is zero at $t = 0$, giving the condition, which holds throughout the flow

$$t = -au = a(s-r) \quad . \qquad (3.5.1)$$

Integration of equations (2.2.2) and (2.2.3) give for x

$$x = \frac{a}{2N+1}[2(N+1)rs - N(r^2+s^2)] \qquad (3.5.2)$$

$$= \frac{a}{2}[(2N+1)c^2 - u^2] \qquad (3.5.3)$$

and from (2.2.5) and (2.2.6) we get

$$w = \frac{a(r-s)}{3(2N+1)}\left[(N+2)(r^2+s^2) - 2(N-1)rs\right] \quad . \qquad (3.5.4)$$

Inverting equations (3.5.1) and (3.5.3) gives

$$u = -\frac{t}{a}$$

$$c = +\left[\frac{2ax+t^2}{a^2(2N+1)}\right]^{\frac{1}{2}} \qquad (3.5.5)$$

showing that the initial conditions were $u = 0$, $c = \mu x^{\frac{1}{2}}$ at $t = 0$ a flow which has been employed in many papers, in particular one by Copson [2] where the solution of this problem was obtained with $N = 1$ for one special case, although not be this method. The work that follows is intended to present this in a more systematic way, to generalise the result, and to include all values of N.

For this flow the differential equation of the shock (3.3.1) becomes

$$\frac{dx}{dt} = -\frac{t}{a} + D\left[\frac{2ax + t^2}{a^2(2N+1)}\right]^{\frac{1}{2}}$$

which has as its solution

$$x = \frac{D^2 - (2N+1)}{2a(2N+1)}\,t^2 + \frac{Dk}{a\sqrt{2N+1}}\,t + \frac{k^2}{2a}$$

where k is a constant of integration.

We assume the shock to arise at the origin due to the motion of a piston situated there at $t = 0$. We thus require $k = 0$, and we write the shock path as

$$x = \lambda t^2 \qquad (3.5.6)$$

with $\lambda = \dfrac{D^2 - 2N - 1}{2a(2N+1)}$.

Using the subscripts 1 and 2 to refer to regions in front of, and behind the shock respectively, the equation of the front of the shock in the hodograph plane is given either by substituting (3.5.1) and (3.5.2) in (3.5.6), or by using (3.3.2) as

$$r_1 + s_1 = -D(r_1 - s_1)$$

and use of the jump conditions gives for the back of the shock

$$s_2 = Pr_2 \tag{3.5.7}$$

where

$$P = \frac{(2N+1)(BD+1) - AD}{(2N+1)(BD-1) + AD} \tag{3.5.8}$$

Use of the jump conditions and (3.5.7) in (3.5.1) gives on the back of the shock

$$t = a(PF - E)r_2$$

so that from (3.5.6)

$$x = \lambda a^2(PF - E)^2 r_2^2$$

where $E = 1 - \frac{A}{(2N+1)B}$, $F = 1 + \frac{A}{(2N+1)B}$.

Thus we can state the boundary value problem for the region behind the shock, using (2.2.5) and (2.2.6). Dropping the suffix 2, we require a solution of (2.2.7) satisfying

$$\frac{\partial w}{\partial r} = Lr^2 \quad , \quad \frac{\partial w}{\partial s} = Mr^2 \tag{3.5.9}$$

on $s = Pr$ where $L = \frac{2a}{(2N+1)(BDF-1)^2}[(2N+1) - D(2A+2B-D)]$

and $M = -\frac{2a}{(2N+1)(BDF-1)^2}[(2N+1) + D(-2A+2B+D)]$.

Before proceeding to the general case we discuss the case $N = 1$ in order to display the motivation for the general solution. For $N = 1$, the most general solution (2.2.7) is given by (2.2.11) as

$$w = \frac{R(r) + S(s)}{r + s} .$$

From the homogeneous nature of the boundary conditions, it appears reasonable to try a solution of the form

$$w = \frac{\bar{\alpha} r^4 + \bar{\beta} s^4}{r + s} \qquad (3.5.10)$$

and it is elementary to show that this does satisfy equation (3.5.9) on $s = Pr$ provided $\bar{\alpha}$ and $\bar{\beta}$ satisfy

$$\bar{\alpha}(3 + 4P) - \bar{\beta} P^4 = (1 + P)^2 L \ ,$$

$$-\bar{\alpha} + \bar{\beta} P^3 (4 + 3P) = (1 + P)^2 M \ .$$

Using equations (2.2.8), (2.2.9), (2.2.13) and (2.2.14), we find that region (2) is given by

$$x = \frac{1}{(r+s)^3}\left[\bar{\alpha} r^3 (8s^2 + rs - r^2) - \bar{\beta} s^3 (8r^2 + rs - s^2)\right] \qquad (3.5.11)$$

$$t = -\frac{3}{(r+s)^3}\left[\bar{\alpha} r^3 (r + 2s) + \bar{\beta} s^3 (s + 2r)\right] \qquad (3.5.12)$$

$$\Phi = \frac{2\bar{\alpha}}{5} r^3 (r^2 + 5rs + 10s^2) - \frac{2\bar{\beta}}{5} s^3 (s^2 + 5rs + 10r^2) \qquad (3.5.13)$$

In theory the particle paths may now be obtained by setting   = constant, and eliminating r and s from these three equations. For the piston path we are interested in the particle path through the origin. This may be investigated because of the important relation

$$x - \frac{r - s}{2} t = \frac{5}{4(r+s)^3} \Phi \ ,$$

which may be verified from equations (3.5.11) to (3.5.13). Before discussing the significance of this result we show that the more general relation

$$x - \frac{r - s}{2} t = \frac{2N + 3}{4N(r+s)^{2N+1}} \Phi \qquad (3.5.14)$$

holds for all values of N. This result was first established for integral values of N using a solution of the type of (2.2.11) with $R(r) = \alpha r^{2N+2}$ and $S(s) = \beta s^{2N+2}$, functions chosen by again considering the homogeneity of the boundary conditions. This method could be shown to give the correct form for w when α and β were determined by a pair of simultaneous equations. We omit the details as a proof will be given for all N, but the case of integral N is important as it enables the correct form of solution to be surmised for the general case.

Before proceeding we point out that on the shock locus $s = Pr$, P can be shown to be always positive. Since c is non-negative, $r + s > 0$, and hence the mapping of the back of the shock in the $r - s$ plane is in the first quadrant. The solution is determined between $s = Pr$ and $s = 0$ which is the bounding characteristic through the origin behind the shock, and consequently r and s are non-negative throughout. Thus we may use equation (2.2.12) with the contours given there throughout the region.

Going by the solution for integral N, we try a solution of the form

$$w = \frac{\alpha\,\Gamma(N)}{2\pi i}\int_{\Gamma_1} \frac{z^{2N+2}\,dz}{(z-r)^N(z+s)^N} + \frac{\beta\,\Gamma(N)}{2\pi i}\int_{\Gamma_2} \frac{z^{2N+2}\,dz}{(z+r)^N(z-s)^N} \ .$$

It is shown in Appendix II that contour integrals of this type may be expressed more conveniently in terms of the hypergeometric function. Accordingly we write

$$w = \frac{\Gamma(2N+3)}{\Gamma(N+4)}\left[\frac{\alpha r^{N+3}}{(r+s)^N} F\left(N, 1-N; N+4; \frac{r}{r+s}\right) + \frac{\beta s^{N+3}}{(r+s)^N} F\left(N, 1-N; N+4; \frac{s}{r+s}\right)\right] \qquad (3.5.15)$$

in the usual notation for hypergeometric functions. Again considerations of homogeneity show that $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ will be multiples of r^2 on $s = Pr$, and so the constants α and β may be determined from the boundary conditions.

Equation (2.2.16) may be used to give the corresponding value of Φ as

$$\Phi = (r+s)^{N+1}\frac{N\,\Gamma(2N+3)}{\Gamma(N+3)}\left[\alpha r^{N+2} F\left(N+2, -N-1; N+3; \frac{r}{r+s}\right) - \beta s^{N+2} F\left(N+2, -N-1; N+3; \frac{s}{r+s}\right)\right] \qquad (3.5.16)$$

If we put $N = 1$ in equations (3.5.15) and (3.5.16), they reduce immediately to the form of (3.5.10) and (3.5.13).

Differentiating (3.5.15) leads to the determination of x and t from (2.2.8) and (2.2.9), and it is now a matter of algebraic manipulation to establish equation (3.5.14).

Since Φ is constant on a particle path it follows at once from (3.5.14) that $\Phi = 0$ on the particle path through the origin, that is, on the piston path. Thus for all N this path is given by

$$x - \frac{r-s}{2}\,t = 0$$

and since $r - s = u = \frac{dx}{dt}$ on a particle path, we have a simple differential equation for the piston path whose integral is

$$x = bt^2 \qquad (3.5.17)$$

where b is a constant. The exact value of this constant is rather difficult to determine. In theory it may be obtained by eliminating the ratio r/s from the equations $\Phi(r,s) = 0$ and $2bt = r - s$ together with the expression for t as a function of r and s.

These results may now be interpreted as follows. Corresponding to a given value of the shock strength z a shock wave is sent into the gas on its right along a path $x = \lambda t^2$. This shock may be looked on as being caused by the motion of a piston along a path $x = bt^2$. In other words, if a piston at the origin at $t = 0$ beside a gas whose initial conditions are given in equations (3.5.1) and (3.5.3) is moved with constant acceleration it causes a uniform shock to move into the gas also with constant acceleration. It should be pointed out that λ and b are not restricted to positive values. Clearly $b < \lambda$, and b is bounded below by the value corresponding to the free expansion of the gas into a vacuum. There are two special cases. One in which a uniform retardation of the piston causes a shock which remains stationary at the origin, corresponding to $\lambda = 0$.

The other is the case in which the piston is held stationary at the origin, that is, where $b = 0$. This is the special case which was discovered by Copson when $N = 1$.

It is of interest to examine the value of b as the shock strength decreases. To avoid excessive algebra we treat only the case $N = 1$. Putting $s = 1 + \delta$, the expansions of α and β are found to be

$$\alpha = -\frac{25}{48}a\left(\delta^{-2} + \frac{34}{15}\delta^{-1} + \ldots\right)$$

$$\beta = -\frac{1}{3}a\left(1 - \frac{1}{75}\delta^{2} - \frac{4}{1125}\delta^{2} + \ldots\right).$$

Since $\alpha \to \infty$ as $\delta \to 0$, it follows that in the $r-s$ plane, the piston path, $\Phi(r,s) = 0$ where Φ is given by (3.5.13), must tend to simply $r = 0$.

If in general we use $\Phi = 0$ in (3.5.13) to eliminate α from (3.5.11) and (3.5.12) these become

$$x = \frac{18\beta s^3(s - r)}{r^2 + 5rs + 10s^2},$$

$$t = -\frac{36\beta s^3}{r^2 + 5rs + 10s^2}.$$

Letting $\delta \to 0$, so that $\beta \to -\frac{1}{3}a$, and $r \to 0$ on the piston path, we may eliminate s from these equations to get the limiting value for the piston motion,

$$x = -\frac{5}{12a}t^2. \qquad (3.5.18)$$

It may also be shown that P, given by (3.5.8) also tends to infinity as $\delta \to 0$. Thus the mapping of the back of the shock in the $r-s$ plane also tends to $r = 0$, while in the physical plane $\lambda \to -\frac{1}{3a}$, and the shock becomes

$$x = -\frac{1}{3a}t^2$$

which is simply the equation of the r characteristic through the origin in the original flow, as may be seen by putting $r = 0$ and eliminating s in equations (3.5.1) and (3.5.2). This is of course the expected result.

The equation (3.5.18) will now be obtained in another way. Since in the $r-s$ plane both the piston path, and the back of the shock path are represented in the limit by the characteristic $r = 0$, it follows that the region between them must be the simple wave characterised by $r = 0$. Then since we have $x = -\frac{a}{3}s^2$, $t = as$ on the limit of the shock path, the s-characteristics, which are straight lines for the simple wave have the equation

$$x + \frac{a}{3}s^2 = (u - c)(t - as) \quad . \qquad (3.5.19)$$

But $r = 0$ in the region, so $-u = 3c = s$ and s is found as a function of x and t from (3.5.19). Then the particle paths are $\frac{dx}{dt} = u = -s$, and on integrating this, the path through the origin is found to be, as before,

$$x = -\frac{5}{12a}t^2 .$$

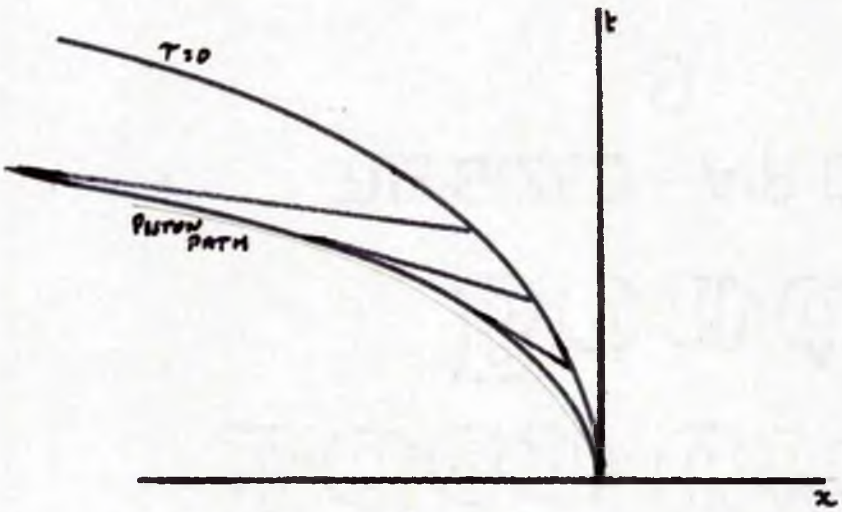

Figure XIV

The simple wave between the piston and the characteristic is shown in figure XIV.

The situation here is different from the classical simple wave problem in which a piston is withdrawn from a gas at rest as now the straight characteristics meet the piston instead of originating on it. The s-characteristics tend to converge, but a simple calculation shows that their envelope is formed to the left of the piston, and so the motion is continuous.

The value of b for free expansion of the gas into a vacuum is found by eliminating r and s from (3.5.11), (3.5.12) and the condition for such an expansion, namely $r + s = 0$. It is $b = -\frac{1}{2a}$. For values of b between this and $-\frac{5}{12a}$ the motion to the right of $r = 0$ will be unaltered. To the left the solution will have the form of (3.5.10) with $\bar{\beta} = -\frac{1}{3}a$ while $\bar{a}$ takes some value between $\frac{1}{3}a$ (the value for expansion into a vacuum) and $+\infty$ (the value for the simple wave case of figure VII).

§ 6. It will be noticed that both the problems of § 4 and § 5 have involved the solution of the Euler-Poisson equation with boundary conditions of the form

$$\frac{\partial w}{\partial R} = LR^{n}, \quad \frac{\partial w}{\partial S} = MR^{n} \quad \text{on} \quad S = PR \qquad (3.6.1)$$

where $R = r + \nu$, $S = s - \nu$, r and s being the Riemann Invariants, and n, ν, L, M and P are constants.

Under these boundary conditions, a contour integral solution of the type of (2.2.12) may be obtained and expressed as shown in Appendix   in the form

$$\omega = \frac{\alpha R^{N+m+1}}{(R+S)^N} F\left(N, 1-N; N+m+2; \frac{R}{R+S}\right) + \frac{\beta S^{N+m+1}}{(R+S)^N} F\left(N, 1-N; N+m+2; \frac{S}{R+S}\right) \qquad (3.6.2)$$

where additional constants have been absorbed in α and β .

To this corresponds the expression for Φ ,

$$\Phi = n(N+m+1)(R+S)^{N+1}\left[\alpha R^{N+m} F\left(N+2, -N-1; N+m+1; \frac{R}{R+S}\right) - \beta S^{N+m} F\left(N+2, -N-1; N+m+1; \frac{S}{R+S}\right)\right]$$

In terms of R and S equations (2.2.8) and (2.2.9) for x and t become

$$x = \frac{1}{R+S}\left\{[(N+1)S - NR]w_R - [(N+1)R - NS]w_S\right\} - 2\nu t$$

$$t = -\frac{2N+1}{2(R+S)}(w_R + w_S) \quad .$$

Algebraic manipulation will now give a generalisation of (3.5.14) namely

$$\frac{x+2\nu t}{n-1} - \frac{R-S}{n} t = \frac{(2N+n+1)}{2Nn(n-1)(R+S)^{2N+1}} \Phi \quad .$$

In this case $R - S = u + 2$, and we again have an equation for the particle path through the origin which becomes

$$x + 2\nu t = bt^{\frac{n}{n-1}} \qquad (3.6.3)$$

where b is a constant which may be found in a manner similar to the b in (3.5.17).

Clearly the above equations are identical with those of § 5 when we put $n = 2$ and $\nu = 0$. They also hold for region (4) of § 4 on putting $n = a + 1$, $\nu = \frac{3c_0}{2}$. Unfortunately for this case, equation (3.6.3) is of little value as the curve represented by this equation passes behind the piston path we employed. Thus to relate this equation to a piston path would require ahead of the shock a piston moving in such a way as to generate a point-centred simple wave a finite distance ahead of it. Such a flow, if possible, would be excessively artificial.

Since flows corresponding to these boundary conditions are capable of such complete solution, it is of interest to discover what type of flow in front of the constant shock will give rise to such conditions behind. Using the shock jump conditions in reverse on equations (3.6.1) we find for the front of the shock

$$S = P'R \qquad (3.6.4)$$

where $P' = \dfrac{(2N+1+A)(P+1) + (2N+1)(P-1)B}{(2N+1-A)(P+1) - (2N+1)(P-1)B}$.

The special cases $P' = 0$, $P' = \infty$ correspond respectively to the simple waves $S = 0$ and $R = 0$, that is, to $s = \nu$ and $r = -\nu$. For general P', the conditions on w along (3.6.4) turn out to be

$$\frac{\partial w}{\partial R} = L'R^n , \quad \frac{\partial w}{\partial S} = M'R^n$$

where $L' = \frac{1}{2B}[(L+M)(1-A) + B(L-M)]J^n$,

$$M' = \frac{1}{2B}[(L+M)(1-A) - B(L-M)]J^n \text{ , and}$$

$$J = \frac{2(2N+1)B}{(2N+1-A)(P+1) - (2N+1)(P-1)B} \text{ ,}$$ and we see that the boundary conditions for the flow in front of the shock are identical with those for the flow behind.

Thus we have as a final result that if we start with any region of the form of (3.6.2), and force a constant shock through it, the resulting region behind the shock will also be of the form of (3.6.2) with, of course, different values of the constants α and β .

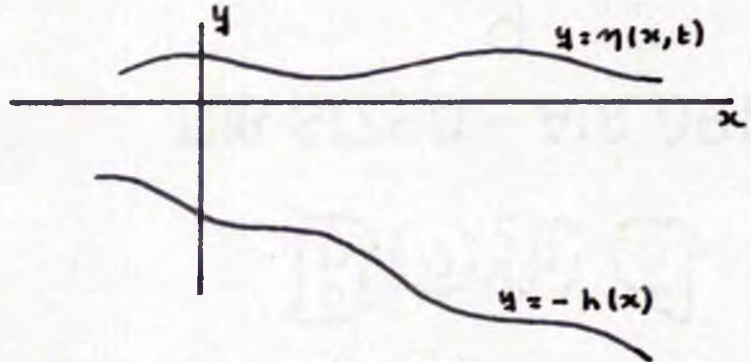

Figure XV

Chapter IV

The Hydraulic Analogy.

§ 1. In this chapter we apply some of our previous results to the hydraulic analogy. This is an analogue to one-dimensional isentropic gas flow encountered in the motion of any liquid - generally water - if we assume that it is shallow. The type of problem considered is shown in figure XV. The x-axis is taken to lie in the undistrubed free surface of the liquid, and the y-axis is vertically upward. The bottom is given by $y = -h(x)$, while $y = \eta(x,t)$ is the water surface.

A completely general solution of this type of problem demands that we find a potential function $\phi(x,y,t)$ such that the velocity components are given by $u = \phi_x$, $v = \phi_y$. Then ϕ will satisfy Laplace's equation in the region $-\infty < x < \infty$ and $-h(x) < y < \eta(x,t)$. The pressure p is given by Bernoulli's equation

$$\frac{p}{\rho} + \frac{1}{2}(u^2 + v^2) + gy + \phi_t = C(t) \quad ,$$

in which $C(t)$ depends only on t, and where the density ρ is now of course constant.

At the free surface $y = \eta(x,t)$ we require the conditions

$$v = \eta_t + u\eta_x \tag{4.1.1}$$

$$p = 0 \tag{4.1.2}$$

while at the bottom $y = -h(x)$ we require

$$\phi_n = 0 \tag{4.1.3}$$

where n is the normal to the bottom profile.

It is clear that this is a very difficult problem as it features non-linear boundary conditions on a curve $y = \eta(x,t)$ whose location must be determined as part of the problem. We therefore seek an approximation which in this case leads to the shallow water theory. The conventional derivation of this theory requires the assumption that the y-component of the acceleration of the water particles has a negligible effect on the pressure, so that the pressure is assumed to be given as in hydrostatics by

$$p = g\rho(\eta - y) \; . \tag{4.1.4}$$

Unfortunately this derivation does not make it clear that the accuracy of the approximation depends on the depth. An alternative method has been given by Friedrichs in an appendix to a paper by Stoker [13] in which the equations of this theory, including (4.1.4), are derived by a perturbation procedure using as parameter a quantity consisting of the depth multiplied by the maximum initial curvature of the water surface. This procedure was however challenged by Ursell [14], and as the results are in any case well known we shall derive them here by the simpler conventional process, as given by Stoker in the aforementioned paper.

The equation of continuity, Laplace's equation, may be expressed in terms of u and v as

$$u_x + v_y = 0 \qquad (4.1.5)$$

and this may be integrated to give

$$\int_{-h}^{\eta} u_x \, dy + v\Big|_{y=\eta} - v\Big|_{y=-h} = 0 \ . \qquad (4.1.6)$$

Expressing the condition at the bottom (4.1.3) as

$$(uh_x + v)_{y=-h} = 0$$

and substituting this, together with (4.1.1), in (4.1.6) gives

$$\int_{-h}^{\eta} u_x \, dy + [\eta_t + u\eta_x]_{y=\eta} + [uh_x]_{y=-h} = 0 \ .$$

This is equivalent to

$$\frac{\partial}{\partial x}\int_{-h(x)}^{\eta(x,t)} u \, dy + \eta_t = 0 \ . \qquad (4.1.7)$$

We now introduce the approximation (4.1.4), which clearly satisfies (4.1.2). The immediate consequence is that

$$p_x = g\rho\eta_x \qquad (4.1.8)$$

so that p_x is independent of y. Thus the x-component of the acceleration of the water particles is also independent of y and hence u will remain independent of y for all time if it ever was at any time, say $t = 0$. We shall assume that this was true so that from now on $u = u(x,t)$. Then using (4.1.8), and putting $u_y = 0$, the Eulerian equation of motion in the x-direction becomes simply

$$u_t + uu_x = -g\eta_x \tag{4.1.9}$$

while (4.1.7) is

$$[u(\eta + h)]_x = -\eta_t \quad . \tag{4.1.10}$$

This pair of first order differential equations for u and η forms the equations of the shallow water theory. We see that the variable y is not involved in these equations at all, so that we are now dealing with what amounts to a one-dimensional unsteady motion.

Before going on to develop the theory of these equations we mention here the approximations which arise if we assume u, η, and their derivatives to be so small that quadratics terms are negligible. Then if we also take h as constant, we obtain the so-called linear shallow water theory. Elimination of u from (4.1.9) and (4.1.10) under these conditions gives the wave equation

$$\eta_{tt} - gh\,\eta_{xx} = 0 \tag{4.1.11}$$

for the surface displacement.

Returning to (4.1.9) and (4.1.10), we now introduce new dependent variables. These are the mass per unit area,

$$\bar{\rho} = \rho(\eta + h) \tag{4.1.12}$$

and the excess force per unit length

$$\bar{p} = \int_{-h}^{\eta} p\,dy$$

$$= \frac{g}{2\rho}\bar{\rho}^2 \tag{4.1.13}$$

In terms of these, we may rewrite (4.1.9) and (4.1.10) as

$$\bar{\rho} u_t + \bar{\rho} u u_x + \bar{p}_x = g \bar{\rho} h_x \qquad (4.1.14)$$

$$u \bar{\rho}_x + \bar{\rho} u_x + \bar{\rho}_t = 0 \ . \qquad (4.1.15)$$

On comparing these equations with (1.1.7) and (1.1.8) we see that apart from the term in h_x they are identical. Corresponding to (1.1.2) we may introduce a quantity c defined by

$$c^2 = \frac{d\bar{p}}{d\bar{\rho}} = g(\eta + h) \qquad (4.1.16)$$

so that in this theory c is a measure of the depth of the water. We may now introduce c into (4.1.14) and (4.1.15) in place of $\bar{p}$ and $\bar{\rho}$ getting

$$u_t + uu_x + 2cc_x - gh_x = 0 \qquad (4.1.17)$$

$$2c_t + 2uc_x + cu_x = 0 \qquad (4.1.18)$$

and these equations are identical, except for the h_x term, with those of the isentropic gas flow, (2.1.1) and (2.1.2), provided we put $\gamma = 2$. For this value of γ, also, (4.1.13) corresponds to (1.1.1). Here however we find the chief difference between this theory and that of a compressible fluid. In (4.1.13) the co-efficient of ρ^γ is a constant, not a function of entropy. In fact, entropy as such does not appear in this theory and there is no equation corresponding to (1.1.9).

Returning to (4.1.17) and (4.1.18), it is clear that if $h_x = 0$, that is, if the bottom is flat, we have the equations of isentropic flow with $\gamma = 2$ and $N = \frac{1}{2}$, and the theory is identical with that of chapter 2. It is also possible to develop a similar theory in the case where the bottom has constant slope, that is, $h_x = m$.

When this holds, (4.1.18) and (4.1.19) may be combined to give

$$\left\{\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right\}(u + 2c - gmt) = 0$$

$$\left\{\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right\}(u - 2c - gmt) = 0 .$$

Thus the curves $\frac{dx}{dt} = u \pm c$ are still characteristics, but quantities corresponding to r and s must be defined as

$$2r = u + 2c - gmt \qquad (4.1.19)$$

$$2s = -u + 2c + gmt . \qquad (4.1.20)$$

These equations may be inverted to give

$$c = \frac{1}{2}(r + s) \qquad (4.1.21)$$

$$u = r - s + gmt . \qquad (4.1.22)$$

This last equation shows the difference caused by the sloping bottom. Instead of being simply equal to $r - s$, u will satisfy some differential equation in r and s.

As before we may treat x and t as functions of r and s and write, as in (2.2.1),

$$\begin{aligned} x_s &= (u + c)t_s \\ x_r &= (u - c)t_r \end{aligned} \quad . \tag{4.1.23}$$

Substituting for u and c from (4.1.21) and (4.1.22), and eliminating x gives the differential equation for t

$$t_{rs} + \frac{3}{2(r+s)}(t_r + t_s) = 0 \tag{4.1.24}$$

which we see is identical with (2.2.4) for $N = \frac{1}{2}$. From (4.1.22) it follows at once that u satisfies an identical equation

$$u_{rs} + \frac{3}{2(r+s)}(u_r + u_s) = 0 .$$

§ 2. It is clear that, as in chapter 3 § 5, the only flow of the form $t = f(r - s)$ that is possible as a solution of (4.1.24) is

$$t = a(s - r) . \tag{4.2.1}$$

We now consider the flow for which this holds.

From (4.1.22) we have immediately that

$$u = (1 - amg)(r - s) \tag{4.2.2}$$

so that again u is a function of t only.

The right sides of equations (4.1.23) may now be expressed in terms of r and s only, and the resulting equations integrated to give

$$x = \frac{a}{2}\left[(amg-\tfrac{1}{2})(r^2+s^2) + 2rs(\tfrac{3}{2}-amg)\right] . \tag{4.2.3}$$

We may now find r, s, and then, from (4.1.21), c, in terms of x and t as

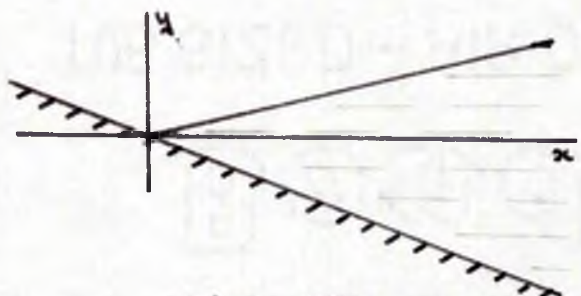

Figure XVI

$$r = \frac{1}{2a}[-t + \{4ax + 2(1-amg)t^2\}^{\frac{1}{2}}]$$

$$s = \frac{1}{2a}[t + \{4ax + 2(1-amg)t^2\}^{\frac{1}{2}}]$$

$$c = \frac{1}{2a}\{4ax + 2(1-amg)t^2\}^{\frac{1}{2}} \qquad (4.2.4)$$

Thus the initial conditions on u and c required to give rise to this flow are $u = 0$ and $c = \left(\frac{x}{a}\right)^{\frac{1}{2}}$. This means that at $t = 0$, from (4.1.16),

$$\eta(x,0) = (\frac{1}{ag} - m)x \qquad (4.2.5)$$

so that the free surface starts with a constant slope as in figure XVI.

For all t, the free surface is given by (4.2.4) and (4.1.16) as

$$\eta(x,t) = (1-amg)\left(\frac{x}{ag} + \frac{t^2}{2a^2g}\right) \qquad (4.2.6)$$

from which it is clear that the surface remains parallel to its original position. The front of the advancing water is given by $\eta + h = 0$, that is, by $c = 0$ as in the case of a gas advancing into a vacuum. Then in the $x-t$ plane the front of the water has the path

$$2ax = (amg - 1)t^2$$

so that the whole advances with a constant acceleration.

We observe that if $amg = 1$, the original surface is level, and that the velocity remains zero, and the front of the water stays at $x = 0$, as would be expected. We may also note that with $m = 0$ (4.2.2), (4.2.3) and (4.2.4) give the same expressions as (3.5.1), (3.5.2) and (3.5.5) with $N = \frac{1}{2}$. This is the case when the bottom is level. If the bottom does not coincide with the x-axis for this case, being at $y = -h$, then in equations η must be replaced by $\eta + h$, so that (4.2.5) for instance becomes

$$\eta = \frac{x}{ag} - h \qquad (4.2.7)$$

The most interesting feature of this flow is the fact that it is also a solution of the exact theory. If we take

$$\phi = (mg - \frac{1}{a})(x - my)t$$

then we have $\nabla^2\phi = 0$, and also

$$\phi_x = (mg - \frac{1}{a})t = u$$

$$\phi_y = -m(mg - \frac{1}{a})t = -mu$$

and it is obvious that $v = -mu$ as the whole flow moves bodily up a slope m. In this case (4.1.1) becomes a differential equation for η which has a solution, if $\eta = \left(\frac{1}{ag} - m\right)x$ at $t = 0$, identical with (4.2.6). The other boundary conditions are also satisfied.

A possible explanation for this flow satisfying the exact theory and the shallow water theory is that since the latter theory may be expressed as a perturbation in terms of powers of the curvature of the surface and since for this case the curvature is always zero, the first order solution must be exact.

§ 3. For the remainder of this chapter we deal only with the case in which the bottom is flat. Thus we have $h_x = 0$, and the equations of motion are identical with those of an isentropic compressible fluid. Mathematically it is clear that under similar conditions to those of chapter 3 discontinuities will arise in this type of flow also. Such discontinuities may be observed in nature in the form of bores or hydraulic jumps. For simplicity we shall refer to them even in this context as shocks. When we seek conditions that must hold across such shocks, we find that the equations which arise from consideration of the conservation of mass and of momentum are identical in terms of the variables $\bar{p}$ and $\bar{\rho}$ with the corresponding equations for a gas, (3.22) and (3.2.3). That is, we have

$$\bar{\rho}_f v_f = \bar{\rho}_b v_b \qquad (4.3.1)$$

$$\bar{\rho}_f v_f^2 + \bar{p}_f = \bar{\rho}_b v_b^2 + \bar{p}_b \qquad (4.3.2)$$

where v is again the relative velocity of the water with respect to the shock, and the subscripts f and b refer to conditions in front of, and behind, the shock respectively.

At this point the two theories part company. That they must do so is clear from the fact that, for the gas, equation (3.2.4) involved the entropy, and in the shallow water theory there is nothing to correspond to entropy in a gas. The third shock condition for this theory arises simply from (4.1.13) since $\frac{g}{2\rho}$ is constant throughout the flow, even across the shock. Hence

$$\bar{p}_f \bar{\rho}_f^{-2} = \bar{p}_b \bar{\rho}_b^{-2} \quad . \qquad (4.3.3)$$

We see that there is here no equation corresponding to the conservation of energy equation (3.2.4), and, indeed, energy would not seem to be conserved across a jump discontinuity defined by (4.3.1), (4.3.2) and (4.3.3). The reason for this is that the above conditions are all mechanical, and there will be a loss of mechanical energy across the shock in the form of heat due to turbulence which corresponds to the entropy increase of the gas theory. Once the shock has been specified, an equation analogous to (3.2.4) may be used to measure this energy loss. It will be sufficient here to state that for the change in energy to be a loss requires $\bar{p}_b > \bar{p}_f$.

We may now define a quantity $z > 1$ by

$$\bar{p}_b = z^2 \bar{p}_f \qquad (4.3.4)$$

where we use z^2 for convenience. It will be seen from (4.1.12) and (4.1.13) that z is the ratio of the total depth (i.e. of $\eta + h$) behind and in front of the shock. This definition of z leads to equations corresponding to (3.2.8), (3.2.9) and (3.2.10) namely

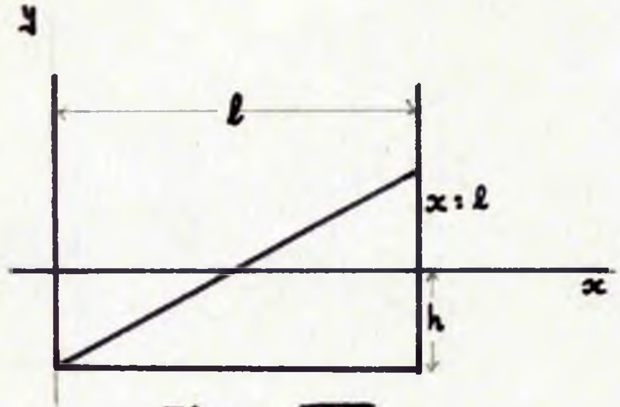

Figure XVII

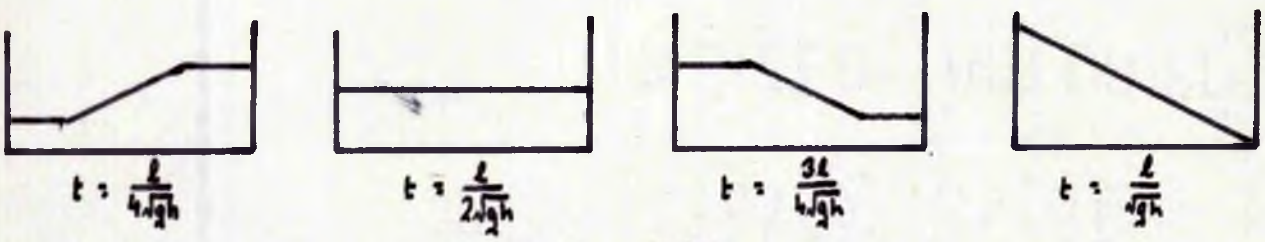

Figure XVIII

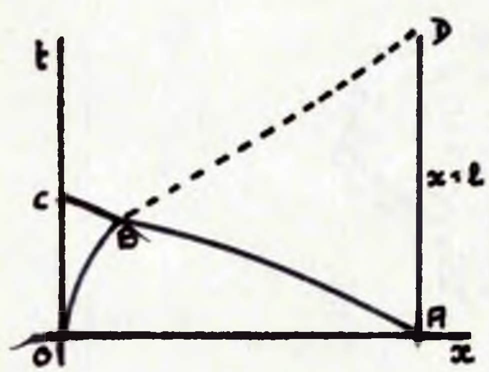

Figure XIX

$$u_b = u_f + Ac_f \qquad (4.3.5)$$

$$c_b = Bc_f \qquad (4.3.6)$$

$$U = u_f + Dc_f \qquad (4.3.7)$$

where A, B, and D are functions of z given by

$$A = (z-1)\left(\frac{z+1}{2z}\right)^{1/2}$$

$$B = \sqrt{z} \qquad (4.3.8)$$

$$D = \left[\frac{z(z+1)}{2}\right]^{1/2}$$

and it will be seen that (4.3.5) to (4.3.7) differ from (3.2.8) to (3.2.10) only in the form taken by A, B and D as functions of z. The quantities A, B and D will still be constant whenever z is constant, so the theory of chapter 3 will still hold for shallow water, and it will be possible to repeat the solution of a problem similar to that of § 5 of that chapter and get the same result for the region behind the shock except that all the constants, P, M, N, etc. will be different functions of z.

§ 4. We shall now consider a particular shallow water problem. The initial conditions are those shown in figure XVII. The water is constrained between two vertical walls at $x = 0$ and $x = \ell$, and is released from the position shown at $t = 0$. At $t = 0$ the surface of the water is given by $\eta(x,0) = \frac{2h}{\ell}x - h$.

This problem has an immediate solution in terms of the linear shallow water theory. The wave equation (4.1.11) yields the solution

$$\eta + h = \tfrac{1}{2}f(x + \sqrt{gh}\ t) + \tfrac{1}{2}f(x - \sqrt{gh}\ t) \qquad (4.4.1)$$

where $f(\xi) = \begin{cases} \frac{2n}{\ell}(\xi - 2n\ell), & 2n\ell \leq \xi \leq (2n+1)\ell \\ \frac{2n}{\ell}(2n\ell - \xi), & (2n-1)\ell \leq \xi \leq 2n\ell \end{cases}$.

This equation gives the usual series of profiles for the wave equation as shown in figure XVIII.

If we now take the point of view of the hydraulic analogy we see that the initial position of the water is that already investigated in § 2 corresponding to the gas flow of § 5 of chapter 3. We also have that the wall at $x = 0$ corresponds to one of the piston paths which were found to give rise to a constant strength shock. Thus for the hydraulic analogy also a constant strength shock will move into the water, and the solution for the region behind the shock will be the same as that for the gas which is given by equation (3.5.15). In the absence of the wall at $x = \ell$, with the water originally extending to infinity this would be the complete solution. In the presence of this wall we will require the solution of another region of the flow, namely that bounded by this wall and the s-characteristic through $x = \ell$, $t = 0$. The flow in the x - t plane is shown in figure XIX.

It is obvious that once the shock moves into the region of flow behind the characteristic AB it will cease to be of constant strength, and the complicated interaction will make it extremely hard, if not impossible, to determine an exact solution. So for this theory we will only have a solution for all x up to the time $t = t_B$ at which the shock meets the characteristic. The solution in region (1) of the figure will hold up to the s-characteristic BC, and that in region (2) up to the (undetermined) path, BD, of the shock. The interest in this solution lies in the comparison of the results with those given by equation (4.4.1).

This problem may equally well be formulated in terms of compressible flow and for the present we will treat it in this fashion, retaining general values of N. We now require solutions to equation (2.2.7) applicable to the three regions numbered in figure XIX.

Regions (0) and (1) have already been solved in chapter 3. For region (0) we have

$$w_0 = \frac{a}{3(2N+1)}\left[(N+2)r^3 - 3Nr^2s + 3Nrs^2 - (N+2)s^3\right] \qquad (4.4.2)$$

$$t_0 = a(s-r) .$$

This region is separated from (1) by the shock wave whose path has the equation $x = t^2$ as in (3.5.6). Then for region (1) we have as in (3.5.15)

$$w_1 = \frac{\Gamma(2N+3)}{\Gamma(N+4)}\left[\frac{\alpha r^{N+3}}{(r+s)^N} F\left(N, 1-N; N+4; \frac{r}{r+s}\right) + \frac{\beta s^{N+3}}{(r+s)^N} F\left(N, 1-N; N+4; \frac{s}{r+s}\right)\right]$$

and a corresponding equation for t_1,

$$t_1 = -\frac{(2N+1)\Gamma(2N+3)}{2\Gamma(N+3)}\left[\frac{\alpha r^{N+2}}{(r+s)^N} F\left(N+1, -N; N+3; \frac{r}{r+s}\right) + \frac{\beta s^{N+2}}{(r+s)^N} F\left(N+1, -N; N+3; \frac{s}{r+s}\right)\right] \qquad (4.4.3).$$

for some α, β which are functions of z. But we also require to satisfy the condition that the piston path which gives rise to the shock is simply $x = 0$, on which path $u = 0$, that is, $r = s$. But the piston path is $\Phi(r,s) = 0$ where Φ is given by (3.5.16) so $r = s$ on this path implies $\alpha = \beta$, and this relationship amounts to an equation for z. For a gas the solution for z will depend on N. For water it will be different again owing to the altered definition of z.

The region (2) is separated from (0) by the s-characteristic AB on which $s = \left[\frac{(2N+1)\ell}{2a}\right]^{1/2} = \mu$ (say). The equation of this characteristic in the x - t plane is

$$x = \ell - \frac{2}{2N+1}\mu t - \frac{2N}{(2N+1)^2\ell}\mu^2 t^2$$

and this curve meets the shock path at the point

$$x_B = \frac{D^2 - 2N - 1}{(D+1)^2}\ell \qquad t_B = \frac{2a}{D+1}\mu$$

and we notice that x_B does not depend upon a at all. Thus whatever the ratio of c^2 to x the shock and characteristic meet at the same point between the walls, although the time which elapses before they do is different.

The solution to region (2) is required to patch on to (0) along this characteristic, and also to satisfy the condition that $r = s$ on $x = \ell$. It is proved in Appendix III that the required solution is

$$\omega_2 = \omega_0 + \frac{2a(2\mu)^{N+1}(\mu-s)^2}{(2N+1)(\mu+r)^N} F\left(N+2; -N-1, N; 3; \frac{\mu-s}{2\mu}, \frac{\mu-s}{\mu+r}\right)$$

with the corresponding equation for t_2

$$t_2 = a(s-r) + \frac{2a\,(2\mu)^{N+1}(\mu-s)}{(\mu+r)^{N+1}}\, F\left(N+2; -N-1, N+1; 2; \frac{\mu-s}{2\mu}, \frac{\mu-s}{\mu+r}\right) \qquad (4.4.4)$$

where F is Appells hypergeometric function of two variables.

We may note that in the absence of the wall at $x = 0$ we have here the solution in the region next to $x = \ell$ of the problem of a gas bounded by a wall at $x = \ell$ and under initial conditions $u = 0$, $c = \left[\frac{2x}{(2N+1)a}\right]^{\frac{1}{2}}$ expanding into a vacuum at its left. If we inspect the values of t at the wall, where $r = s$, we see that as r and s decrease from μ, t increases, but the hypergeometric series remains convergent up to the point where $r = s = 0$. But this means that t becomes infinite only when, c and hence ρ, has finally become zero, which agrees with our intuitive physical picture of what would happen in such conditions.

We now apply the above results to the shallow water theory, for which $N = \frac{1}{2}$ and $a = \frac{\ell}{2gh}$ so that $\mu = \sqrt{2gh}$. For this case the equation for z, $\alpha = \beta$, becomes

$$M\left\{7(1+P)\,F\left(\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{7}{2}; \tfrac{1}{1+P}\right) - P^{7/2}\,F\left(\tfrac{3}{2}, \tfrac{1}{2}; \tfrac{9}{2}; \tfrac{P}{1+P}\right)\right\} =$$

$$L\left\{7(1+P)\,P^{5/2}\,F\left(\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{7}{2}; \tfrac{P}{1+P}\right) - F\left(\tfrac{3}{2}, \tfrac{1}{2}; \tfrac{9}{2}; \tfrac{1}{1+P}\right)\right\}$$

where L, M, P are defined in (3.5.7) and (3.5.8) in terms of A, B and D now given by (4.3.8). This equation may be solved numerically for z and yields the value

$\frac{\sqrt{gh}}{l}t$	$\frac{\eta+h}{h}$	
	exact	linear
0·0	0·0	0·0
0·045272	0·011133	0·090544
0·092882	0·046860	0·185764
0·143136	0·111286	0·286272
0·196397	0·209514	0·392794
0·253092	0·347936	0·506184
0·313730	0·534630	0·627460
0·378920	0·779896	0·757860
0·449398	1·096994	0·898796
0·536603 $= t_B$	1·564038	1·073206
0·601735 $= t_c$	1·966759	1·203470

Table I - Conditions at $x = 0$.

$\frac{\sqrt{gh}}{l}t$	$\frac{\eta+h}{h}$	
	exact	linear
0·0	2·0	2·0
0·045272	1·876953	1·909456
0·092882	1·757813	1·814237
0·143136	1·662578	1·713728
0·196397	1·531250	1·607205
0·253092	1·423828	1·493815
0·313730	1·320313	1·372540
0·378920	1·220703	1·242159
0·449398	1·125000	1·112030
0·536603 $= t_B$	1·021481	0·926794
0·610002	0·945313	0·779996
0·702585	0·861328	0·594830
0·805509	0·781250	0·388982
0·920925	0·705078	0·158149

Table II Conditions at $x = l$.

$\frac{\sqrt{gh}}{l}t$	$\frac{x}{l}$	$\frac{\eta+h}{h}$ - exact			$\frac{\eta+h}{h}$
		Foot of shock	top of shock	mean	linear
0·0	0·0	0·0	0·0	0·0	0·0
0·045272	0·000692	0·00[illegible]75	0·004993	0·003834	0·090544
0·092882	0·002911	0·023076	0·043068	0·033072	0·185764
0·143136	0·006913	0·054802	0·102280	0·078541	0·286272
0·196397	0·011301	0·103173	0·192558	0·147866	0·392794
0·253092	0·021614	0·171338	0·319779	0·245559	0·506184
0·313730	0·033211	0·263275	0·491367	0·377321	0·627460
0·378920	0·048447	0·384054	0·716784	0·550419	0·757840
0·449398	0·068144	0·560206	1·008220	0·774213	0·888796
0·536603 $= t_B$	0·097157	0·770200	1·437473	1·103837	1·073206

Table III Conditions at the shock.

$z = 1.866363$. For this value the point B becomes $x_B = 0.097157\, \ell$, $t_B = 0.536603\, \ell (gh)^{-1/2}$.

We now seek to compare the two solutions. For brevity they will be referred to as the linear and the exact solution respectively, although of course the shallow water theory does not yield an exact solution in the terms of § 1.

There are three points at which comparisons may be made with comparative ease. These are at the walls $x = 0$, on both of which $r = s$, and at the shock.

Consider first the wall $x = 0$. In the linear theory $\eta + h$ at this point will be linear in t , increasing from 0 to 2h as t goes from 0 to $\frac{\ell}{\sqrt{gh}}$, then decreasing again to 0 in a similar length of time, and so on. In the exact theory putting $r = s$ in (4.4. 3) shows that t is simply proportional to r , and hence to c , and hence to $(\eta + h)^{\frac{1}{2}}$. In fact the equation for the total depth in terms of the time may be calculated as

$$\eta + h = 5.431765 \frac{g^2h^2}{\ell^2} t^2$$

and this holds at $x = 0$ from $t = 0$ to $t = t_2$. The characteristic BC carries along itself the value, from the shock jump conditions,

$$s = \frac{2BD + 2 - AD}{2(D+1)} \mu = 1.402412 \sqrt{gh} .$$

which corresponds to the values $\eta + h = 1.966759\, h$, $t = 0.601735 \frac{\ell}{\sqrt{gh}}$. In table I we list the depths

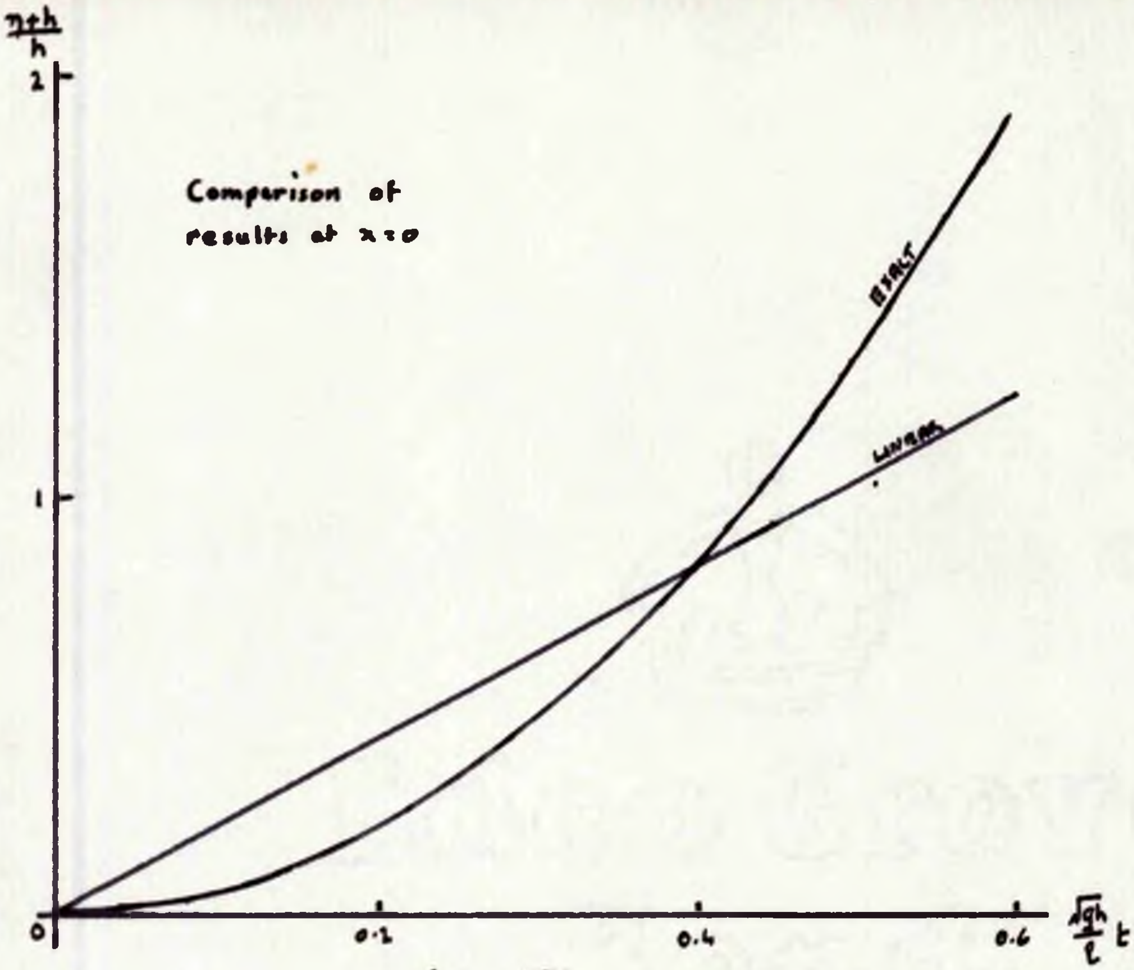

Figure **XX**

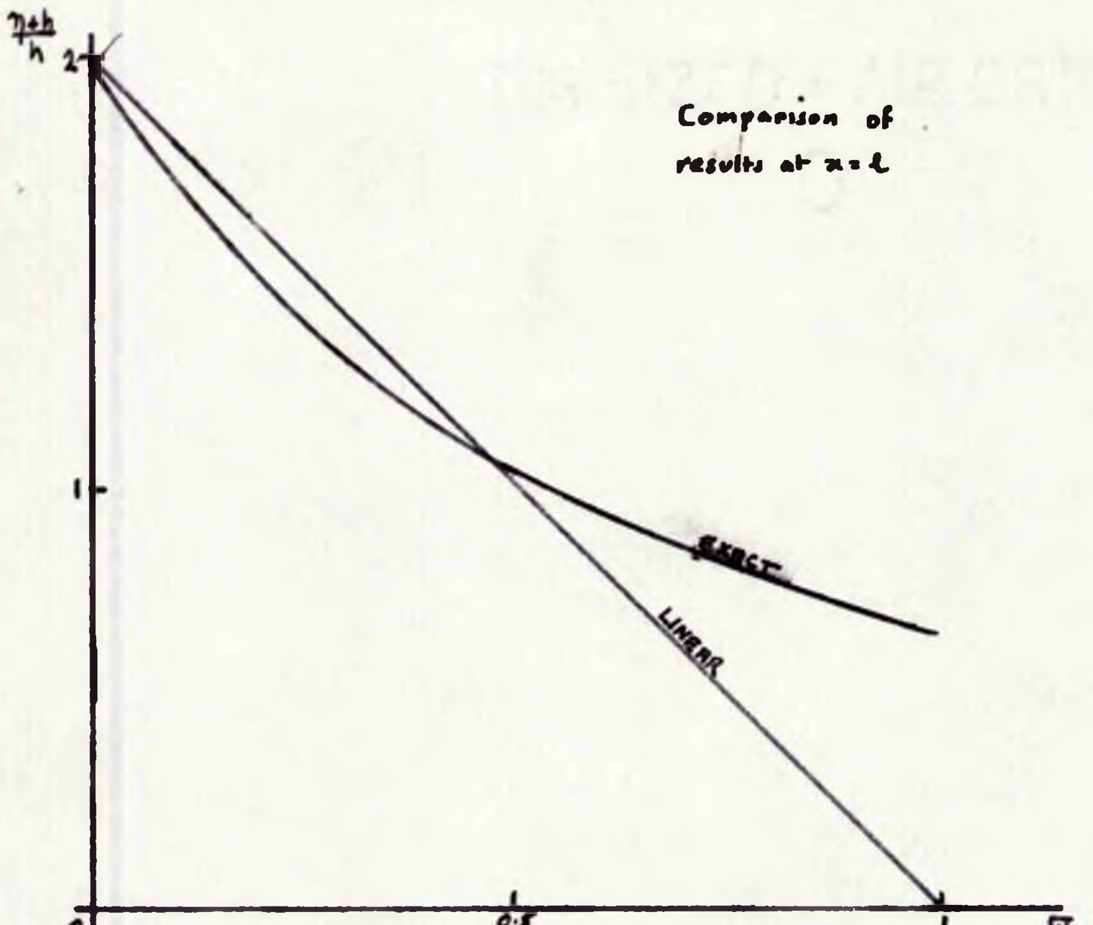

Figure **XXI**

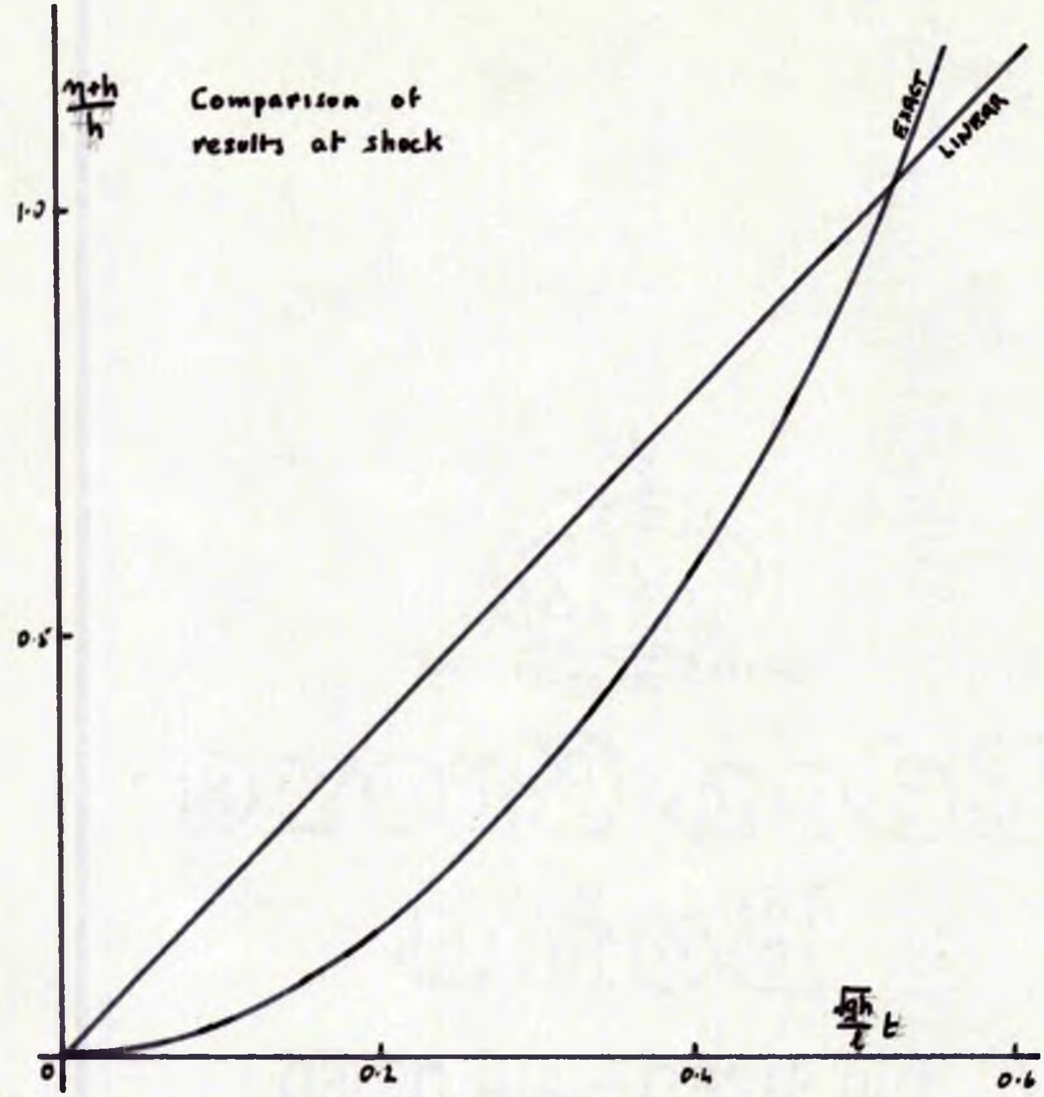

Figure XXII

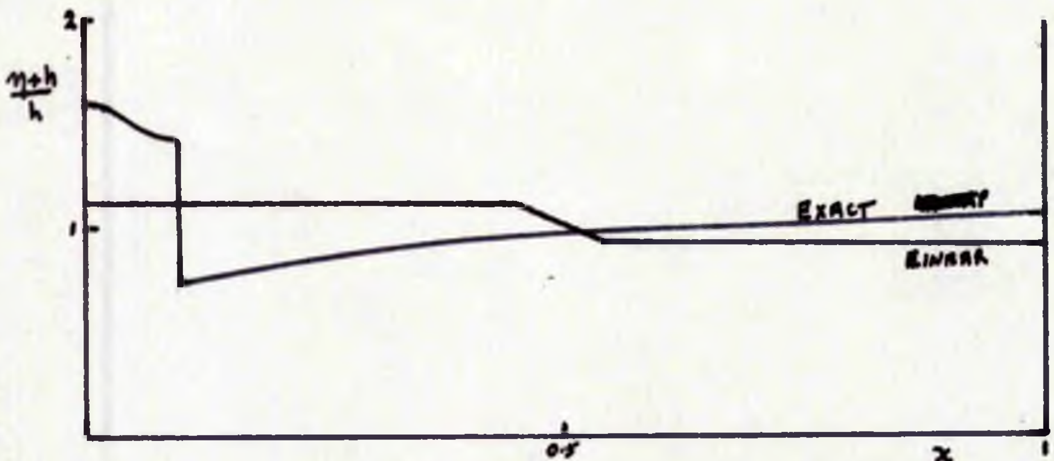

Figure XXIII

predicted at the same times by the two theories.

At $x = l$ the linear theory predicts the opposite results from those at $x = 0$, $\eta + h$ going linearly from 2h to 0 to 2h, etc. as t goes from 0 to $\frac{l}{\sqrt{gh}}$ to $\frac{2l}{\sqrt{gh}}$ etc. For the exact theory we put $s = \mu\sigma$ in (4.4.4) getting

$$t = \frac{4(1-\sigma)}{(1+\sigma)^{3/2}} F\left(5/2; -3/2, 3/2; 2; \frac{1-\sigma}{2}, \frac{1-\sigma}{1+\sigma}\right)\frac{l}{\sqrt{gh}}$$

This equation may be tabulated in terms of σ for t, and from σ we may find the depth. These results are compared with the corresponding results from the linear theory in table II.

Finally the foot of the shock is in region (0) up to $t = t_B$. Its position x at various times is easily found as is the corresponding depth. The depth at the top of the shock is then found by multiplying by z. The most suitable point for comparison with the linear theory would appear to be the mean of these depths. In table III these results are listed together with those of the linear theory which, on account of the form of solution which the wave equations give, figure XVIII, are the same as the result at $x = 0$.

The results given in tables I, II, and III are graphed in figures XX, XXI and XXII, from which it may be seen that within its limitations the linear theory is a reasonable approximation to the exact theory as far as this problem can be solved in the latter theory.

In figure XXIII we show the corresponding profiles of the water in the two theories at the time $t = t_B$. It is curious to note that the disturbances in the linear theory, travelling along characteristics $x = \sqrt{gh}\, t$ and $\ell - x = \sqrt{gh}\, t$ have already met while those of the exact theory, one of which is a shock which is by definition 'faster' than a corresponding characteristic are only just meeting.

Chapter 5

Anisentropic Flows.

§ 1. In this chapter we consider a method of attacking problems in anisentropic flow due to Martin [9, 4]. By taking p, the pressure, and h, the Lagrange variable constant on a trajectory as independent variables he shows that all problems in this type of flow may be reduced to the solution of a Monge-Ampere partial differential equation of the form

$$\xi_{pp}\xi_{hh} - \xi_{ph}^2 = \tau_p$$

where τ the specific volume is a function of $S = S(h)$ the specific entropy. Martin and Ludford [4] have shown that this equation has intermediate integrals for only one entropy distribution $S(h)$, and we shall be mainly concerned with this case. The result obtained in this chapter do not involve shocks directly, but are presented for general interest and to indicate some of the difficulties which may arise in this type of problem.

For convenience we repeat the one-dimensional gas equations,

$$\rho_t + (u\rho)_x = 0 , \tag{5.1.1}$$

$$\rho(uu_x + u_t) + p_x = 0 , \tag{5.1.2}$$

$$S_t + uS_x = 0 . \tag{5.1.3}$$

Equations (5.1.1) and (5.1.2) are together equivalent to the systems

$$(p + \rho u^2)_x + (\rho u)_t = 0$$

$$(\rho u)_x + \rho_t = 0$$

and from the first of these we may deduce the existence of a function $\bar{\xi}$ of x and t defined by

$$d\bar{\xi} = \rho u \, dx - (p + \rho u^2) dt .$$

In terms of h, defined as in (1.1.13) by

$$dh = \rho dx - \rho u \, dt \qquad (5.1.4)$$

the expression for $\bar{\xi}$ becomes

$$d\bar{\xi} = - p \, dt + u \, dh$$

and if we now define $\xi = \bar{\xi} + pt$ we get

$$d\xi = u \, dh + t \, dp . \qquad (5.1.5)$$

Those considerations led Martin to use p and h as independent variables. Then in terms of $\xi(h,p)$ we have immediately

$$u = \xi_h \qquad t = \xi_p \quad . \qquad (5.1.6)$$

Substitution in (5.1.4) gives

$$dh = \rho dx - \rho \xi_h (\xi_{pp} dp + \xi_{ph} dh)$$

from which we have

$$dx = (\tau + \xi_h \xi_{ph}) dh + \xi_h \xi_{pp} \, dp$$

or

$$x_h = \tau + \zeta_h \zeta_{ph} \qquad x_p = \zeta_h \zeta_{pp} . \qquad (5.1.7)$$

On eliminating x from these equations we see that satisfies the Monge-Ampere equation

$$\zeta_{hh} \zeta_{pp} - \zeta_{hp}^2 = \tau_p . \qquad (5.1.8)$$

When dealing with a polytropic gas we have, from (1.1.1) with $\rho = \tau^{-1}$,

$$\tau_p = -\frac{1}{\gamma} k(h) p^{-(1+\frac{1}{\gamma})}$$

where k(h) is a function of h which depends on the entropy distribution.

Thus we have Martin's result that once the entropy distribution has been specified, suitable boundary value problems reduce to the solution of a Cauchy problem for the Monge-Ampere equation.

We note that

$$\frac{\partial(x,t)}{\partial(h,p)} = \tau t_p \qquad (5.1.9)$$

so the mapping of the h - p plane on the x - t plane is locally one-one provided τ and t_p remain finite and non-zero.

If we write $\lambda^2 = -\tau_p$, then since $c^2 = \gamma p \tau$ for a polytropic gas, by equation (1.1.3), we have $\lambda = \tau/c$, and using (5.1.4) we may immediately express the characteristics $\frac{dx}{dt} = u \pm c$ in the form

$$\frac{dt}{dh} = \pm \lambda \qquad (5.1.10)$$

If we now use this equation together with $t = \xi_p$ we find that on the characteristics

$$\frac{dp}{dh} = \frac{-\xi_{ph} \pm \lambda}{\xi_{pp}}$$

and since $\frac{du}{dp} = \xi_{ph} + \xi_{hh}\frac{dh}{dp}$, we get as a third expression for the characteristics

$$\frac{du}{dp} = \mp \lambda .$$

§ 2. Martin's method naturally requires that the trajectories (h = constant) and the isobars (p = constant) do not coincide. The case in which this does occur turns out to be extremely simple, but nevertheless is of some interest as it provides generalisations of some previous results.

We suppose $p = p(h)$. Then from (5.1.4) p satisfies

$$p_t + u\,p_x = 0 . \qquad (5.2.1)$$

But S satisfies (5.1.3) and ρ is a function of p and S, so ρ must satisfy

$$\rho_t + u\rho_x = 0 \qquad (5.2.2)$$

and thus is also constant along a trajectory.

Together, (5.2.2) and (5.1.1) imply that

$$\rho u_x = 0 .$$

Now ρ cannot be zero throughout any region of flow, so this equation clearly implies that u must be a function of t only which may be written as

$$u = -f'(t) .$$

Under this condition (5.2.1) becomes

$$p_t - f'(t)p_x = 0$$

which may be integrated at once to give the result that p, and similarly ρ, S, and h are all functions of X where

$$X = x + f(t) .$$

This means that $p_x = \frac{dp}{dX}$, and hence that (5.1.2) may be written

$$-\rho(X)\, f''(t) + \frac{dp}{dX} = 0 . \qquad (5.2.3)$$

But this equation implies that $f''(t)$ is in fact a constant, or that $f(t) = A_1t^2 + A_2t + A_3$ for some constants A_1, A_2, A_3, and we may take $A_3 = 0$ without loss of generality.

Thus we have

$$u = -2A_1t - A_2$$

which generalises to anisentropic flow the result obtained in chapter 3 § 5 for isentropic flows, namely that the only possible flow in which u is a function of t only has u as a linear function of t. This result thus holds whatever the entropy distribution and also is not restricted to polytropic gases as we have not assumed the form (1.1.1) for the equation of state.

We may now rename $A_2 = -u_0$, and drop the subscript on A_1 . Then

$$u = -2At + u_0 \tag{5.2.4}$$

$$X = u + At^2 - u_0 t \tag{5.2.5}$$

Finally, from (5.1.4) $dh = \rho\, dX$ and from (5.2.3)

$$\rho(X) = \frac{1}{2A}\frac{dp}{dX} \tag{5.2.6}$$

so we have at once that

$$h = \frac{1}{2A}p + B \tag{5.2.7}$$

for some constant B . Thus the only possible flow in which h is a function of p has h a linear function of p .

Since h is a function of X , the trajectories are given by X = constant, that is, they are the family of parabolae

$$x + At^2 - u_0 t = \text{constant} \ . \tag{5.2.8}$$

For a polytropic gas the sound speed c is also a function of X given by

$$c^2 = \frac{\gamma p}{\rho} = \frac{2A\gamma p}{dp/dX} \tag{5.2.9}$$

so that the characteristics may be expressed as

$$\frac{dx}{dt} = u \pm c = -2At + u_0 \pm \left(\frac{2A\gamma p}{dp/dX}\right)^{1/2}$$

or

$$t = \pm \int \left[\frac{1}{2A\gamma p(X)}\frac{dp}{dX}\right]^{\frac{1}{2}} dX \tag{5.2.10}$$

Suppose we are given an entropy distribution $S = S(h)$ and require a flow of this type. Then since $p = k(h)\rho^{\gamma}$ where $k(h)$ is now a known function, we have, from (5.2.6)

$$X = \frac{1}{2A} \int \left[\frac{k(h)}{p}\right]^{\frac{1}{\gamma}} dp \qquad (5.2.11)$$

and with h given by (5.2.7) this may be integrated to give p, implicitly at least, as a function of X. Then, once p is known, ρ and h are found from (5.2.6) and (5.2.7), the characteristics from (5.2.10) and the complete flow is found. For example if we put $k(h) = k$, a constant, we recover the constant entropy case of chapter 3, §5.

Alternatively we may take initial conditions in terms of $p(x)$ at $t = 0$. This together with the requirement that the flow be of the above type is sufficient to determine the complete flow pattern, including the necessary entropy distribution. We complete this section by giving two examples of such flows. In each case we assume that $u = 0$ at $t = 0$, that is, $u_0 = 0$.

For the first example we take $p = Pe^x$, at $t = 0$ for some constant P. Then since p is a function of x only we require $p = Pe^X$ throughout the flow. Then (5.2.6) gives $\rho = \frac{P}{2A}e^X$, and (5.2.9) gives $c = (2A\gamma)^{\frac{1}{2}}$, a constant. We thus have an example of a gas in motion which still possesses a constant sound speed due to an entropy gradient. Integration of (5.2.10) gives as the equations of the characteristic curves

$$\sqrt{A\gamma}\, t = \pm \left(x + \frac{A}{2}t^2\right) + \text{const.}$$

while the trajectories are given by (5.2.8) with $u_0 = 0$.

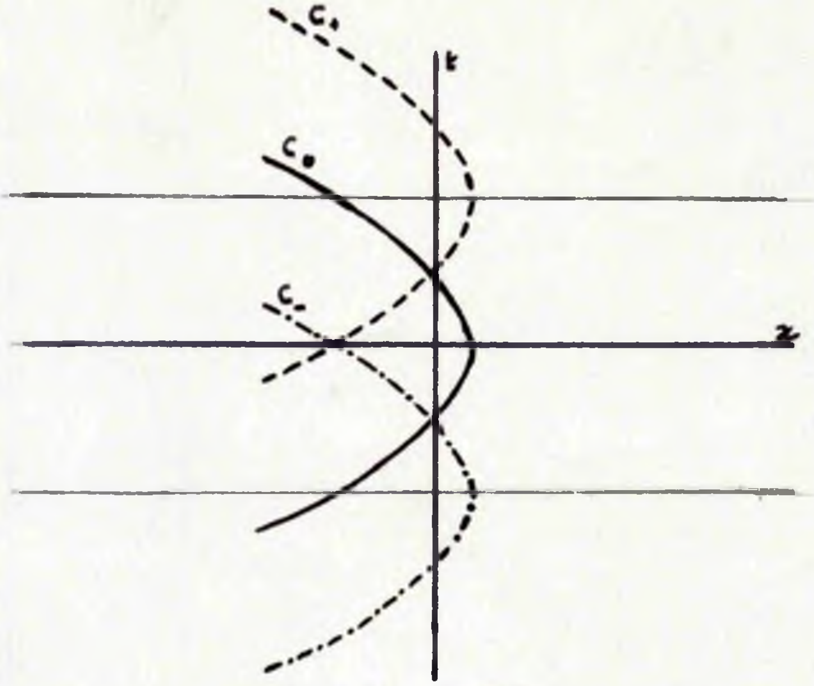

Figure XXIV

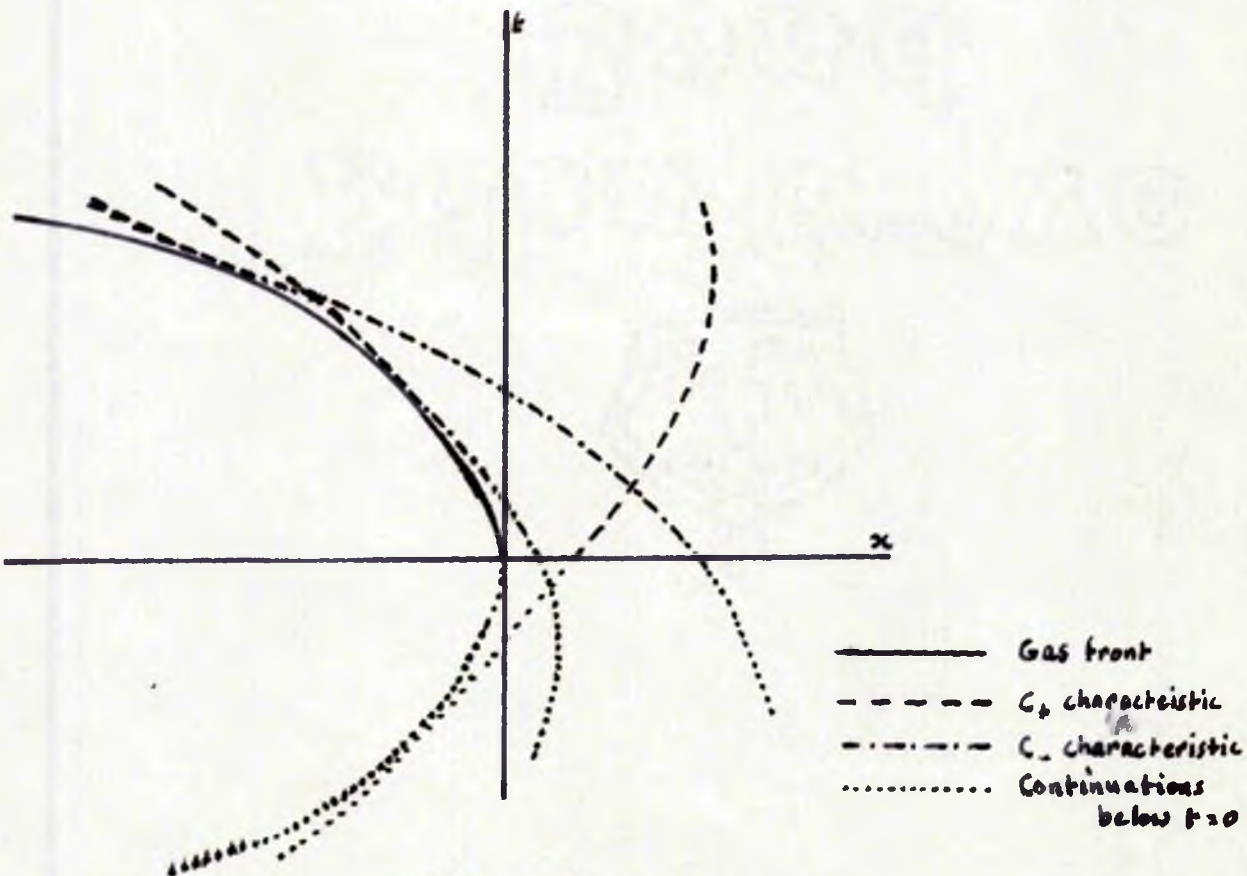

Figure XXV

Thus the characteristics form two families of parabolae congruent to the trajectories but with the lines $t = \pm\left(\frac{X}{A}\right)^{1/2}$ as their axes. The flow in the physical plane is shown in figure XXIV.

The entropy distribution for this flow is found from (1.1.1) to be $S = (1-\gamma)c_v X +$ constant. This is to say that the initial conditions $u = 0$, $p = Pe^X$ will give rise to the type of flow we are discussing if the entropy distribution has this form.

For the second example we suppose $p = \alpha x$ at $t = 0$ for some constant α so that $p = \alpha X$ throughout the flow. We note that this condition means that at $t = 0$ the gas is situated in the region $x > 0$, so that we have a case of an expansion into a vacuum. For this flow we get $\rho = \frac{\alpha}{2A}$, a constant, so that here we have an example of a non-uniform motion having a constant density throughout. The equation for c is now $c = (2\gamma Ax)^{\frac{1}{2}}$ and the characteristics are

$$t = \pm\left(\frac{2X}{\gamma A}\right)^{\frac{1}{2}} + a$$

where a is a constant of integration. Thus the two families of characteristics are represented by the same family of parabolae

$$At^2(2-\gamma) + 2a\gamma At + 2x = \gamma a^2 A \quad .$$

The envelope of this family is $x + At^2 = X = 0$, and it is clear that the point at which any member of the family touches the envelope is also the point which divides the part of the curve representing a C_+ characteristic from that representing a C_- characteristic. But $x + At^2 = 0$ is the trajectory through the origin, and hence the front of the expanding gas. Thus we have an example in anisentropic flow of the phenomenon, observed by Pack [11] for an isentropic flow, where a C_- characteristic meets the gas front at a tangent and is reflected off it as a C_+ characteristic. This is due to the fact that at the front c is zero so that u and $u \pm c$ are the same. Typical characteristics of this flow are shown in figure XXV. Here the entropy distribution is given by $S = c_v \log X +$ constant.

§ 3. We now return to equation (5.1.8). Martin and Ludford [4] have shown that for a polytropic gas this equation will possess intermediate integrals only in the case where

$$-\tau_p = \lambda^2 = k^2 h^{1/\gamma - 3} p^{-(1+\frac{1}{\gamma})}$$

or in terms of N,

$$\lambda = \frac{k}{ph}\left(\frac{p}{h}\right)^{\frac{1}{2N+3}} \qquad (5.3.1)$$

where k is an arbitrary constant.

This means that the equation of state must be

$$p = (\gamma k^2)^{\gamma} h^{1-3\gamma} \rho^{\gamma} \qquad (5.3.2)$$

and we may thus find the relationship between h and ρ.

Consider now the expression

$$r = \zeta - uh - tp - (2N+3)k\left(\frac{p}{h}\right)^{\frac{1}{2N+3}} \qquad (5.3.3)$$

Then

$$dr = \zeta_h dh + \zeta_p dp - u dh - t dp - h du - p dt - k\left(\frac{p}{h}\right)^{\frac{1}{2N+3}-1}\left(\frac{dp}{h} - \frac{p\,dh}{h^2}\right)$$

and from (5.1.6) the first four terms of this expression cancel. Using the condition (5.3.1) leaves us with

$$dr = -h\,du - p\,dt - \lambda h\,dp + \lambda p\,dh$$

and from (5.1.10) and (5.1.11) we have immediately that the left side of this equation is zero on a C_+ characteristic, and hence that the expression (5.3.3) is constant on such a characteristic. Similarly the expression

$$s = -\zeta + uh + tp - (2N+3)k\left(\frac{p}{h}\right)^{\frac{1}{2N+3}} \qquad (5.3.4)$$

may be shown to be constant along a C_- characteristic, provided λ^2 is defined by equation (5.3.1).

These quantities r and s may be referred to as "Generalised Riemann Invariants."

As in the isentropic theory we may now take r and s as independent variables. Then by (5.1.10) and (5.1.11) we may write

$$u_r = \lambda p_r \qquad t_r = -\lambda h_r$$

$$u_s = -\lambda p_s \qquad t_s = \lambda h_s \quad .$$

We also have that

$$r + s = -2(2N+3)k\left(\frac{p}{h}\right)^{\frac{1}{2N+3}} \qquad (5.3.5)$$

and using this together with (5.3.1), we may substitute for λ , getting

$$u_r = -\beta(r+s)^{2N+4}\pi_r \qquad u_s = \beta(r+s)^{2N+4}\pi_s \qquad (5.3.6)$$

where $\pi = p^{-1}$, and $\beta = k[2k(2N+3)]^{-(2N+4)}$, and

$$w_r = \alpha(r+s)^{2N+2}t_r \qquad w_s = -\alpha(r+s)^{2N+2}t_s \qquad (5.2.7)$$

where $w = h^{-1}$ and $\alpha = k^{-1}[2k(2N+3)]^{-(2N+2)}$.

From these equations it is immediate to show that t, w, π, and u all satisfy equations of the Euler-Poisson type, which are

$$t_{rs} + \frac{N+1}{r+s}(t_r + t_s) = 0 \qquad (5.3.8)$$

$$w_{rs} - \frac{N+1}{r+s}(w_r + w_s) = 0 \qquad (5.3.9)$$

$$\pi_{rs} + \frac{N+2}{r+s}(\pi_r + \pi_s) = 0 \qquad (5.3.10)$$

$$u_{rs} - \frac{N+2}{r+s}(u_r + u_s) = 0 \qquad (5.3.11)$$

where the solutions of these equations are associated by (5.3.5), (5.3.6) and (5.3.7).

Thus suppose we start with any solution $t(r,s)$ of (5.3.8). We find w by integrating (5.3.7), then π from (5.3.5), u from (5.3.6), from (5.3.2), c from (5.2.1), since $\lambda = {}^{t}/c$ for a polytropic gas, and finally x from the equations, similar to (2.2.1),

$$x_r = (u-c)t_r \ , \quad x_s = (u+c)t_s \ .$$

For integral N the general solution to (5.3.8) may be written

$$t = \left(\frac{\partial}{\partial r}\right)^N \frac{R''(r)}{(r+s)^{N+1}} + \left(\frac{\partial}{\partial s}\right)^N \frac{S''(s)}{(r+s)^{N+1}} \qquad (5.3.12)$$

giving as the solution for the other unknowns

$$w = \alpha(r+s)^{2N+3}\left[\left(\frac{\partial}{\partial r}\right)^{N+1} \frac{R'(r)}{(r+s)^{N+2}} - \left(\frac{\partial}{\partial s}\right)^{N+1} \frac{S'(s)}{(r+s)^{N+2}}\right] \qquad (5.3.13)$$

$$\pi = -2(2N+3)\left[\left(\frac{\partial}{\partial r}\right)^{N+1} \frac{R'(r)}{(r+s)^{N+2}} - \left(\frac{\partial}{\partial s}\right)^{N+1} \frac{S'(s)}{(r+s)^{N+2}}\right] \qquad (5.3.14)$$

$$u = 2\beta(2N+3)(r+s)^{2N+5}\left[\left(\frac{\partial}{\partial r}\right)^{N+2} \frac{R(r)}{(r+s)^{N+3}} + \left(\frac{\partial}{\partial s}\right)^{N+2} \frac{S(s)}{(r+s)^{N+3}}\right] \qquad (5.3.15)$$

$$c = -\frac{r+s}{2(2N+1)}\, w \qquad (5.3.16)$$

$$\tau = -\gamma k^2 \left[\frac{2k(2N+3)}{r+s}\right]^{2N+1} w^3 \ . \qquad (5.3.17)$$

Thus far we have followed the Martin and Ludford paper. We may here add that for integral N it is possible to give conditions similar to those of chapter 2 §3 under which flows of this type may be patched together. The proofs of the following results are identical with those of chapter 2 and are therefore omitted. Corresponding to equations (2.3.2) we have the result that the regions of flow given by substitution of the functions $R_1(r)$, $S_1(s)$ and $R_2(r)$, $S_2(s)$ into equations (5.3.12) to (5.3.15) will be identical provided the functions are connected by equations of the type

$$R_2(r) = R_1(r) + \sum_{n=0}^{2N+2} a_n r^n$$

$$S_2(s) = S_1(s) - \sum_{n=0}^{2N+2} (-1)^n a_n s^n$$

for some constants a_n .

Then the conditions corresponding to those of equations (2.3.2) that the two regions $R_1(r)$, $S_1(s)$ and $R_2(r)$, $S_2(s)$ may be patched along and s-characteristic $s = s_o$, are that a transformation of this type can be made to reduce the equations to the form

$$R_2(r) = R_1(r) \qquad (5.3.18)$$

$$S_2(s) = (s-s_o)^{N+3} f(s) + S_1(s) \qquad (5.3.19)$$

for some function $f(s)$ finite at $s = s_o$. The second of these equations is equivalent to the condition that $S_2(s)$ and $S_1(s)$ and their first $N+2$ derivatives take the same value at $s = s_o$.

We will now complete the theory in this section by giving Martin's results for what he terms the anisentropic simple wave, that is, for the flow given by r = constant. Since $u = \zeta_h$, $t = \zeta_p$ (5.2.3) under this condition is simply a first order partial differential equation for which may be integrated to give

$$\zeta = p\,K(\kappa) + (2N+3)k\,\kappa^{\frac{1}{2N+3}} + r$$

where $\kappa = {}^p/h$ and K is an arbitrary function. Then substitution in (5.1.6) and (5.1.7) gives

$$t = K + \kappa K' + \frac{k}{p}\kappa^{\frac{1}{2N+3}} \tag{5.3.20}$$

$$u = -\kappa^2 K - \frac{k}{h}\kappa^{\frac{1}{2N+3}} \tag{5.3.21}$$

$$x = \frac{1}{3}\int \kappa^3 K' K'' d\kappa - \frac{2}{3}\kappa^3 K'^2 - \frac{2N+2}{2N+1}\frac{k^2}{ph}\kappa^{\frac{2}{2N+3}} - \frac{k}{h}\kappa^{\frac{2N+4}{2N+3}} K' \tag{5.3.22}$$

which, when K is known, are the parametric equations for the simple wave. We also have, from (5.3.4),

$$s = -r - 2(2N+3)k\kappa^{\frac{1}{2N+3}} . \tag{5.3.23}$$

The Jacobian $J = \tau t_p$ of equation (5.1.8) is zero along

$$t_p = \frac{2\kappa}{p}K' + \frac{\kappa^2}{p}K'' - \frac{2N+2}{2N+3}\frac{k}{p^2}\kappa^{\frac{1}{2N+3}} = 0$$

and by substituting for p in terms of κ from this expression into (5.3.20) and (5.3.22) (first putting $h = p/\kappa$) we may find the parametric equation of the 'limit line' in the $x-t$ plane.

The characteristic curves may be found by integrating

$$t_h + t_p\frac{dp}{dh} = \pm\lambda \tag{5.3.24}$$

which comes from (5.1.9), and in the Ludford and Martin paper it is shown that both these families of curves are enveloped by the limit line.

§ 4. It would seem that as well as these simple waves there might be other flows which, although possessing the required entropy distribution, could not be defined by substituting functions of r and s into equations (5.3.12) to (5.3.17). This would mean that it would not be possible to patch other regions on to them by using the patching conditions (5.3.18) and (5.3.19). Such a flow would be that in which the special $p = p(h)$ condition of § 2 is present, or the special case of this in which p is constant throughout a region with this entropy distribution due to a density gradient. For this case the gas will of course be at rest. In neither of these flows can p and h be taken as independent variables so that we cannot seek a solution of (5.1.7). In this section we show that by defining ξ formally it is still possible to define r and s by (5.3.3) and (5.3.4) and to obtain functions of these variables which give the flow on substitution into (5.3.12) to (5.3.15).

If we consider first the case $p = p_0$, we get from (5.3.2)

$$p_0 = (\gamma k^2)^{\gamma} h^{1-3\gamma} \rho^{\gamma} .$$

Then since $u = 0$ we may use (1.1.13) to get h in terms of x as

$$h^{-(2 - 1/\gamma)} = - \frac{2\gamma-1}{\gamma^2 k^2} p_0^{1/\gamma} x + \text{const.}$$

We see that as $x \to \infty$, $h \to 0-$, and also that h must be infinite for some value of x depending on the constant. The reason why h is not positive despite its definition in chapter 1 is that the above flow considered between $x = -\infty$ and $x = +\infty$ contains a point at which the density is infinite. To avoid this point we select the constant so that $h = -a$ at $x = 0$, and consider the gas as lying in the region $x \geq 0$ only. To simplify the working we define a new variable $\psi = -h$, and assume N to be an integer.

Then

$$\psi^{-(2-\frac{1}{\gamma})} - a^{-(2-\frac{1}{\gamma})} = \frac{2\gamma-1}{\gamma^2 k^2} p_o^{\frac{1}{\gamma}} x \qquad (5.4.2)$$

for the region $0 \leq x < \infty$, $a \geq \psi > 0$.

Since this is a region at rest the characteristic curves are simply given by

$$\frac{dx}{dt} = \pm c = \pm \left(\frac{\gamma p_o}{\rho}\right)^{\frac{1}{2}}$$

where ρ is found in terms of x by (5.4.1) and (5.4.2).

Alternatively we find x and ρ in terms of ψ, when this equation integrates to give

$$\frac{2\gamma}{\gamma-1} \psi^{-\frac{\gamma-1}{2\gamma}} = \pm \frac{p_o^{\frac{\gamma+1}{2\gamma}}}{k} t + \text{constant}$$

or in terms of N

$$\psi^{-\frac{1}{2N+3}} = \pm \frac{p_o^{\frac{2N+2}{2N+3}}}{(2N+3)k} t + \text{constant}. \qquad (5.4.3)$$

Now suppose we use equation (5.1.5) to give a formal definition for ζ in this region. We will get simply

$\zeta = \zeta_o$, a constant. Then if we define r and s by means of (5.3.3) and (5.3.4) we get

$$r = (2N+3)k\left(\frac{p_o}{\psi}\right)^{\frac{1}{2N+3}} - tp_o + \zeta_o \qquad (5.4.4)$$

$$s = (2N+3)k\left(\frac{p_o}{\psi}\right)^{\frac{1}{2N+3}} + tp_o - \zeta_o \qquad (5.4.5)$$

and by inspecting (5.4.3) we can see that these quantities are constant on their respective characteristics.

In terms of this r and s we may write

$$t = \frac{s - r + 2\zeta_o}{2p_o}$$

$$\psi = p_o\left[\frac{2k(2N+3)}{r + s}\right]^{2N+3}$$

and ρ and x are given by (5.4.1) and (5.4.2).

If we now consider the functions

$$R(r) = \frac{(2N+1)!}{N!\,p_o}\left[-\frac{r^{2N+4}}{(2N+4)!} + \zeta_o\frac{r^{2N+3}}{(2N+3)!}\right]$$

$$S(s) = \frac{(2N+1)!}{N!\,p_o}\left[\frac{s^{2N+4}}{(2N+4)!} + \zeta_o\frac{s^{2N+3}}{(2N+3)!}\right]$$

it is not difficult to show that these, substituted in equations (5.3.12) onwards give the correct form of t, w, π, etc., as functions of r and s for this flow.

Considering now the general case of the flow of § 2, we have from (5.2.7)

$$h = \frac{1}{2A}p + B \ .$$

From (5.3.2) the function $k(h)$ that must be substituted in equation (5.2.11) to give X as a function of p

$$X = \frac{\gamma^2 k^2}{4A^2B^2}\left(\frac{p}{h}\right)^{\frac{\gamma-1}{\gamma}}\left[\frac{1}{1-2\gamma}\frac{p}{h} - \frac{2A}{1-\gamma}\right] + \text{constant} \qquad (5.4.6)$$

and with this value the equation for the characteristics, (5.2.10) gives

$$t = \pm\frac{\gamma k}{(\gamma-1)AB}\left(\frac{p}{h}\right)^{\frac{\gamma-1}{\gamma}} + \text{constant}$$

$$= \pm\frac{(2N+3)k}{AB}\left(\frac{p}{k}\right)^{\frac{1}{2N+3}} + \text{constant} \ . \qquad (5.4.7)$$

As with the last flow ξ cannot exist as a solution of (5.1.8) but may be defined formally by (5.1.5). This leads to

$$\xi = \frac{u_o}{2A}p + D$$

for some constant D .

We now define r and s as in (5.3.3) and (5.3.4) for this flow. Then using (5.2.4), (5.2.7) and (5.4.8) we get

$$r = 2ABt + (D - Bu_o) - (2N+3)k\left(\frac{p}{h}\right)^{\frac{1}{2N+3}}$$

$$s = -2ABt - (D - Bu_o) - (2N+3)k\left(\frac{p}{h}\right)^{\frac{1}{2N+3}}$$

and inspection of (5.4.7) again shows that these quantities are constant on their respective characteristics.

Expressing the other variables in terms of r and s we get

$$t = \frac{r - s - 2(D - Bu_o)}{4AB}$$

$$u = \frac{s - r + 2D}{2B} \qquad \text{from (5.2.4)}$$

$$\frac{p}{h} = -\left[\frac{r+s}{2k(2N+3)}\right]^{2N+3} , \quad h = \frac{1}{A}p + B$$

$$x = X - At^2 - u_o t$$

where X is given by (5.4.6). Again we may find functions which when substituted in (5.3.12) onward give this result. In this case they are

$$R(r) = \frac{(2N+1)!}{2ABN!}\left[\frac{r^{2N+4}}{(2N+4)!} - (D - Bu_o)\frac{r^{2N+3}}{(2N+3)!} + \lambda(r - D)\right]$$

$$S(s) = -\frac{(2N+1)!}{2ABN!}\left[\frac{s^{2N+4}}{(2N+4)!} + (D - Bu_o)\frac{s^{2N+3}}{(2N+3)!}\right]$$

where $\lambda = \frac{(-1)^{N+1}A}{k}\left[\frac{(N+1)!}{(2N+3)!}\right]^2 [2k(2N+3)]^{2N+4}$.

Npw suppose we seek to patch another region onto one of these special regions, along a common characteristic. Then, although r and s have only been defined formally for these special regions, their values on this boundary characteristic will be the same as those for the general region provided we choose correct values for ζ_o and D . But the compatibility conditions (5.3.18) and (5.3.19) were found by considering conditions at the common characteristic only so these conditions will have to hold

hold for the neighbouring region. Thus we may treat these special regions just like any other.

§ 5. We complete this chapter by giving an example of a complete flow with this entropy distribution. We choose the simplest possible example, in which an anisentropic simple wave is patched on to a region at rest with this entropy gradient, that is, the first of the special flows of the last section. Comparing the resulting picture with those of § 1 of chapter 2 will give an idea of the difficulties which may arise in anisentropic flow problems.

We again use the variable $\psi = -h$, and we assume the gas to occupy the region $x > 0$, and with $\psi = a$ at the origin at $t = 0$. The r-characteristic through the origin is then the member of the family of equation (5.4.3) given by

$$\left(\frac{1}{\psi}\right)^{\frac{1}{2N+3}} = \frac{1}{(2N+3)k}\, p_o^{\frac{2N+2}{2N+3}} t + \left(\frac{1}{a}\right)^{\frac{1}{2N+3}} . \qquad (5.5.1)$$

The value of r on this characteristic is, from (5.4.4)

$$r = \zeta_o + (2N+3)k \left(\frac{p_o}{a}\right)^{\frac{1}{2N+3}} = r_o \quad \text{(say)}$$

and along it the variation of s is given by

$$s = -r_o + 2(2N+3)h \left(\frac{p_o}{\psi}\right)^{\frac{1}{2N+3}} \qquad (5.5.2)$$

from (5.4.5).

We now seek to patch a simple wave type of solution onto the left of this characteristic, which, since s must vary in this region, by (5.5.2), must be given by $r = r_0$. When this has been done there will be a curve in the x - t plane corresponding to the particle path $\psi = a$, which passes through the origin, and we will be able to equate this curve to the path of a piston originally at the origin and to say that it was the motion of this piston that caused the disturbance being propagated into the rest region by the simple wave.

The boundary conditions on the simple wave are that along the characteristic (5.5.1) s is given by (5.5.2) and also $p = p_0$, $u = 0$. But throughout the simple wave s must be given by (5.3.23) which is identical with (5.5.2) showing that the formal definition of s for the rest region does lead to a correct result.

From (5.3.21), the boundary condition $u = 0$ on $p = p_0$ gives

$$\kappa^2 K' + \frac{k}{p_0} \kappa^{\frac{2N+4}{2N+3}} = 0$$

for all ψ, where κ is now $-{}^{p_0}/\psi$.

Integration of this equation gives

$$K(\kappa) = -(2N+3) \frac{k}{p_0} \kappa^{\frac{1}{2N+3}} + \mu$$

for some constant μ. Substituting this in (5.3.20) gives

$$t = k \kappa^{\frac{1}{2N+3}} \left(\frac{1}{p} - \frac{2N+4}{p_0} \right) + \mu$$

and since t is known at $p = p_o$, from (5.5.1), we get a value for μ. We have thus determined the simple wave completely, and we get

$$K = (2N+3)\frac{k}{p_o}\left[\left(\frac{p}{\psi}\right)^{\frac{1}{2N+3}} - \left(\frac{p_o}{a}\right)^{\frac{1}{2N+3}}\right]$$

$$t = k\left(\frac{p}{\psi}\right)^{\frac{1}{2N+3}}\left(\frac{2N+4}{p_o} - \frac{1}{p}\right) - \frac{(2N+3)k}{p_o}\left(\frac{p_o}{a}\right)^{\frac{1}{2N+3}} \qquad (5.5.3)$$

$$x = k^2\left(\frac{p}{\psi}\right)^{\frac{2N+5}{2N+3}}\left[\frac{2N+2}{2N+1}\frac{1}{p^2} - \frac{1}{pp_o} + \frac{2N+4}{2N+5}\frac{1}{p_o^2}\right] - \frac{(2N+3)k^2}{(2N+1)(2N+5)p_o^2}\left(\frac{p_o}{a}\right)^{\frac{2N+5}{2N+3}} \qquad (5.5.4)$$

from (5.3.22)

$$u = k^2\left(\frac{p}{\psi}\right)^{\frac{2N+4}{2N+3}}\left(\frac{1}{p_o} - \frac{1}{p}\right) \qquad (5.5.5)$$

from (5.3.21).

These equations (5.5.3), (5.5.4), and (5.5.5) give the simple wave parametrically in terms of p and ψ. On a trajectory, ψ = constant, we see from (5.5.3) that that the pressure increases with the time, and hence, from (5.5.5) that the velocity also increases with time.

The isobars p = constant are given by

$$X = P(p)T^{2N+5} \qquad (5.5.6)$$

where $X = x + \dfrac{(2N+3)^2k^2}{(2N+1)(2N+5)p_o^2}\left(\dfrac{p_o}{a}\right)^{\frac{2N+5}{2N+3}}$,

$T = t + \dfrac{(2N+3)k}{p_o}\left(\dfrac{p_o}{a}\right)^{\frac{1}{2N+3}}$ and $P(p) = \dfrac{\left[\dfrac{2N+2}{(2N+1)p^2} - \dfrac{1}{pp_o} + \dfrac{2N+4}{(2N+5)p_o^2}\right]}{k^{2N+3}\left(\dfrac{2N+4}{p_o} - \dfrac{1}{p}\right)^{2N+5}}$

We note that as $p \to \infty$,

$$P(p) \to \left(\frac{p_o}{k}\right)^{2N+3} (2N+5)^{-1}(2N+4)^{-(2N+4)}$$ which value,

substituted in (5.5.6) must be expected to give some sort of limiting line for the flow.

We may now integrate (5.3.24) for this case to find the equations for the characteristics, which are, for the r - characteristic

$$\left(\frac{p}{\psi}\right)^{\frac{2N+1}{2N+3}} \left(\frac{1}{p_o} - \frac{1}{p}\right) = \text{constant}$$

and, for the s - characteristic,

$$\frac{p}{\psi} = \text{constant} \quad . \qquad (5.5.7)$$

From (5.1.9) the limit line for this flow will be given by $t_p = 0$. From (5.5.3) this gives either $p = \infty$, as expected, or $p = -\frac{N+1}{N+2} p_o$, neither of which values can occur in a real flow. As a check on this we may obtain the equation of the s - characteristic in terms of X and T from (5.5.3), (5.5.4) and (5.5.7) getting

$$(2N+1)(2N+5)p_o^2 X = (2N+5)(2N+2)\lambda^{2N+3} p_o^2 T^2 - (2N+3)(2N+5)(4N+5)\lambda^{2N+4} k p_o T + (2N+4)^2(2N+3)^2\lambda^{2N+5} k^2$$

where $\lambda = \left(\frac{p}{\psi}\right)^{\frac{1}{2N+3}}$ is a constant for each characteristic. When we find the envelope of this family of curves we get two curves of the type of (5.5.6), having respectively

$$P(p) = \left(\frac{p_o}{k}\right)^{2N+3} (2N+5)^{-1} (2N+4)^{-(2N+4)} \quad \text{i.e. } p = \infty$$

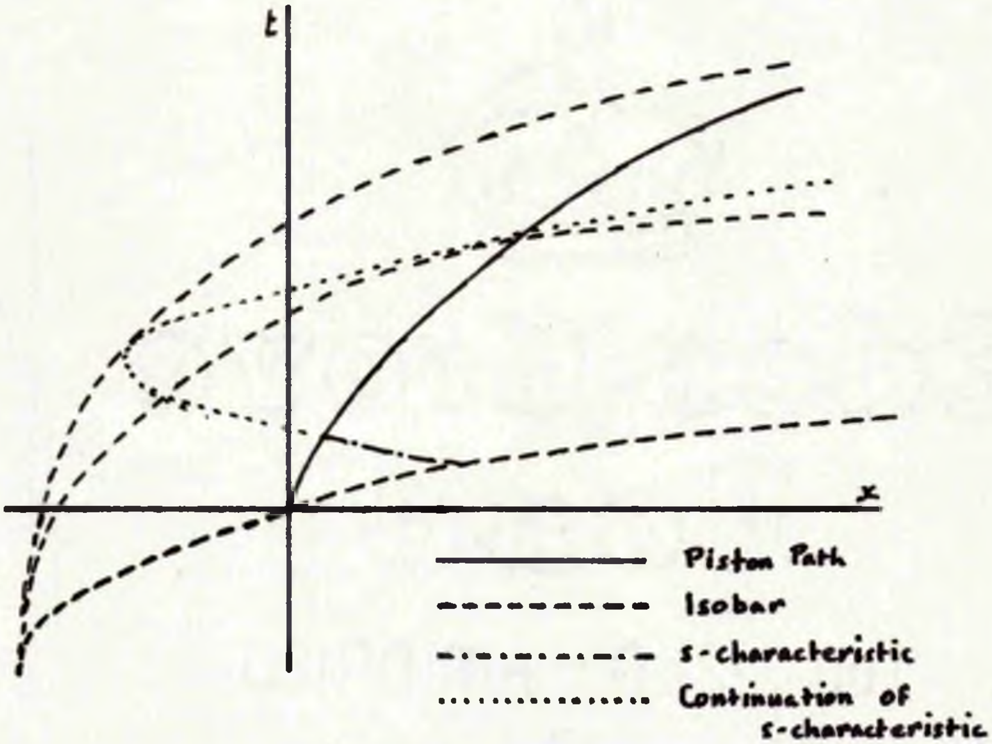
t
x
Piston Path
Isobar
s-characteristic
Continuation of
s-characteristic

Figure XXVI.

$$P(p) = \frac{3(N+1)^{2N+4} p_o^{2N+3}}{(2N+1)(2N+5)(2N+3)^{2N+3}(N+2)^{2N+4}k^{2N+3}} \quad \text{i.e. } p = -\frac{N+1}{N+2} p_o$$

so that each of these curves corresponds to a curve given by the limit line. However we may notice that the second of them also corresponds to some value of p between p_o and $2p_o$, that is, to a value of p which does occur inside the region of the flow, so it would seem that an envelope for the s-characteristic must be present within the region of flow indicating a breakdown. However, as may be seen in figure XXVI. It is not the s-characteristic, but its continuation beyond the flow that envelopes this curve.

We have already seen that the velocity increases with time along a trajectory. Thus, when we equate the trajectory through the origin with a piston path, the motion of the piston will be compressive. But we have shown that no shock arises in the resultant flow. Thus we have here a flow in which a piston, moving into a gas originally at rest gives rise to a continuous shock-free motion. This as we have seen is an impossibility in the isentropic flow.

Of course, the entropy distribution dealt with in this example has been extremely artificial, although it could arise in a gas after the passage of a shock. It does, however, serve to indicate the care with which ideas from the isentropic flow must be carried over to the anisentropic case.

Finally, we may say that while most of the exact solutions presented throughout this thesis are unlikely to occur in practice it is hoped that they may be of value in providing an opportunity for testing possible approximate theories which may then be used in solving problems with more physical meaning.

Appendix I.

To show that the form of the unknown functions in

$$w = \left(\frac{\partial}{\partial r}\right)^{N-1} \frac{R(r)}{(r+s)^N} + \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{S(s)}{(r+s)^N}$$

when w is required to satisfy $\frac{\partial w}{\partial r} = - \frac{\partial w}{\partial s} = \phi(r)$ on $r = s$ is given by

$$R(r) = \frac{2}{(N-1)!} \int_0^r (r^2 - \xi^2)^{N-1} \xi \phi(\xi)\, d\xi \qquad (A.1.1)$$

$$S(s) = - R(s) \quad . \qquad (A.1.2)$$

As stated in the text, this result has been proved by Pack [10] using the methods of the θ operator. The following method based on induction has the merit of requiring considerably less algebra.

If we expand the terms of w using Leibniz's rule, the condition that $\frac{\partial w}{\partial r} = - \frac{\partial w}{\partial s}$ on $r = s$ requires that the differential equation

$$\sum_{p=0}^{N} \frac{(-1)^p (2N-p)!}{p!(N-p)!} 2^p r^p T^{(p)}(r) = 0 \qquad (A.1.3)$$

be satisfied by $T(r) = R(r) + S(r)$.

This equation may be differentiated n times to give, as may be verified by induction,

$$\sum_{p=0}^{N-n} \frac{(-1)^p (2N-p-2n)!}{p!(N-n-p)!} 2^p r^p T^{(p+2n)}(r) = 0 .$$

From which it follows that on differentiating (A.1.3) N

times, it reduces to

$$T^{(2N)}(r) = 0$$

which gives immediately

$$T(r) = \sum_{n=0}^{2N-1} \alpha_n r^n \qquad (A.1.4)$$

where only N of the α_n may be independent since T satisfies the N^{th} order differential equation (A.1.3). Substituting (A.1.4) into (A.1.3) gives

$$\sum_{p=0}^{N} \sum_{n=p}^{2N-1} \alpha_n (-1)^p 2^p \frac{n!(2N-p)!}{(n-p)!\,p!(N-p)!} r^p r^{n-p} = 0$$

or, reversing the order of the summations

$$\sum_{m=0}^{N} \alpha_m r^m m! \sum_{p=0}^{m} \frac{(-1)^p 2^p (2N-p)!}{(N-p)!(m-p)!\,p!} + \sum_{m=N+1}^{2N-1} \alpha_m r^m m! \sum_{p=0}^{N} \frac{(-1)^p 2^p (2N-p)!}{(N-p)!(m-p)!\,p!}$$

and in both these terms the second summation may be written as

$$kF(-N,-n;\ -2N;\ 2)$$

where k is some constant. Applying the formula (see, for example, Sneddon [12] p.23, equation (7.5))

$$F(\alpha,\beta;\ \gamma;\ x) = (1-x)^{-\beta} F(\gamma-\alpha,\beta;\ \gamma;\ \frac{x}{x-1}) \qquad (A.1.6)$$

we get

$$F(-N,-n;\ -2N;\ 2) = (-1)^n F(-N,-n;\ -2N;\ 2)$$

so that clearly (A.1.5) must be zero when n is odd. It follows that (A.1.4) will satisfy (A.1.3) for any values of α_n when n is odd provided $\alpha_n = 0$ for n even. We may thus write the final solution of (A.1.3) as

$$T(r) = \sum_{n=0}^{N-1} \beta_n r^{2n+1} .$$

Then, from the definition of $T(r)$ we may write

$$R(r) = \psi(r) + \sum_{n=0}^{N-1} \frac{\beta_n}{2} r^{2n+1}$$

$$S(s) = -\psi(s) + \sum_{n=0}^{N-1} \frac{\beta_n}{2} s^{2n+1}$$

for some function $\psi(r)$, and from the result of Theorem II, chapter 2, equation (2.3.2), the summed terms may in fact be ignored. Hence, we have proved the result (A.1.2).

The remaining condition, $\frac{\partial w}{\partial r} = \phi(r)$ on $r = s$ may now be expressed as

$$\sum_{p=1}^{N} \frac{(-1)^{N-p}(2N-1-p)!}{(p-1)!(N-p)!(2r)^{2N-p}} R^{(p)}(r) = \phi(r) . \qquad (A.1.7)$$

For this linear differential equation a complementary function may be found by a method identical with that used to find $T(r)$ above. This function is $\sum_{n=0}^{N-1} \gamma_n r^{2n}$, and these terms must occur both in $R(r)$ and $-S(s)$ and again, from Theorem II, they may be ignored. It therefore remains only to prove that (A.1.1) is a solution of (A.1.7). For the case $N = 1$, this is immediate. For the general case, write

$$R_N = \frac{2}{(N-1)!}\int_0^r (r^2 - \xi^2)^{N-1} \xi \phi(\xi) d\xi .$$

Then we require to show that if this satisfies (A.1.7), R_{N+1} will satisfy

$$\sum_{p=1}^{N+1} \frac{(-1)^{N+1-p}(2N+1-p)!}{(p-1)!(N+1-p)!(2r)^{2N+2-p}} R_{N+1}^{(p)}(r) = \phi(r) . \quad (A.1.8)$$

Now $\frac{d}{dr} R_{N+1} = 2rR_n$, and, by induction

$R_{N+1}^{(p)}(r) = 2[rR_N^{(p-1)} + (p-1)R_N^{(p-2)}]$. When this result is substituted on the right of (A.1.8), the two resulting summations may be combined to give

$$\sum_{p=1}^{N} \frac{(-1)^{N-p}(2N-1-p)}{(p-1)!(N-p)!(2r)^{2N-p}} R_N^{(p)}(r)$$

and this is the left side of (A.1.7), equal to $\phi(r)$ by the induction assumption. Thus the proof of (A.1.1) is complete.

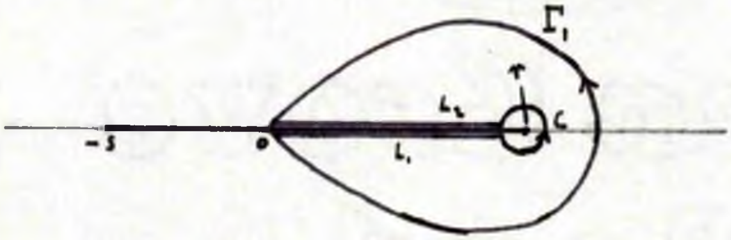

Figure XXVII

Appendix II.

To show that

$$\frac{\kappa\Gamma(N)}{2\pi i}\int_{\Gamma_1}\frac{z^{2N+m}}{(z-r)^N(z+s)^N}\,dz = \frac{\Gamma(2N+m+1)}{\Gamma(N+m+2)}\,\frac{\kappa r^{N+m+1}}{(r+s)^N}\,F(N,1-N;N+m+2;\tfrac{r}{r+s}) \qquad (A.2.1)$$

where Γ_1 is a contour starting at 0 below a cut joining 0, r and -s and finishing again at 0 above the cut after going anti-clockwise around r, assuming both r and s to be positive.

We prove this result for $0 < N < 1$. It may then be extended immediately to all positive values of N by analytic continuation, since both sides of (A.2.1) are meaningful for all positive N.

The first step is to collapse the contour Γ_1 as shown in figure XXVII. The contour now consists of three parts, the line L_1 from 0 to $r - \varepsilon$ below the cut, the circular arc C radius ε surrounding $z = r$, and the line L_2 from $r - \varepsilon$ to 0 above the cut.
Considering first the integral around C, we get

$$\int_C \frac{(r+\varepsilon e^{i\theta})^{2N+n}}{\varepsilon^N e^{iN\theta}(r+s+\varepsilon e^{i\theta})^N}\,\varepsilon i e^{i\theta}\,d\theta \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad 0 < N < 1 .$$

We are left with the two line integrals, and these may be expressed in terms of real variables. Remembering that the argument of $(z - r)$ increases by 2π on going round C, while those of $(z + s)$ and z remain the same, we get

$$\frac{a\,\Gamma(N)}{2\pi i}\left\{\int_{L_1}\frac{x^{2N+n}}{(r-x)^N e^{-iN\pi}(x+s)^N}\,dx+\int_{L_2}\frac{x^{2N+n}}{(r-x)^N e^{iN\pi}(x+s)^N}\,dx\right\}$$

$$=\frac{a\,\Gamma(N)}{2\pi i}\left(e^{iN\pi}-e^{-iN\pi}\right)\int_0^r\frac{x^{2N+n}\,dx}{(r-x)^N(x+s)^N}\ .$$

If we now put $x = r\xi$, and $t = r/s$ this expression becomes

$$\frac{a\,\Gamma(N)}{\pi\,(\ast)}\sin N\pi\,\frac{r^{N+n+1}}{s^N}\int_0^1\xi^{2N+n}(1-\xi)^{-N}(1+\xi t)^{-N}\,d\xi \qquad (A.2.2)$$

and if we compare this integral with that given by Sneddon [12] equation (7.1) which says

$$\int_0^1\xi^{\beta-1}(1-\xi)^{\gamma-\beta-1}(1-\xi t)^{-\alpha}\,d\xi = B(\beta,\gamma-\beta)F(\alpha,\beta;\gamma;t)$$

where B is a beta function, we see that (A.2.2) may be expressed as a hyptergeometric function which is, using the formula $\Gamma(N)\,\Gamma(1-N) = \pi\operatorname{cosec} N\pi$,

$$\frac{a\,\Gamma(2N+n+1)}{\Gamma(N+n+2)}\,\frac{r^{N+n+1}}{s^N}\,F(N,2N+n+1;\,N+n+2;\,-t)\ . \qquad (A.2.3)$$

When both r and s are positive this hyptergeometric will only converge for $r \leqslant s$. The continuation of this function into the range of all positive r and s may be found by using the transformation (A.1.6) already used in Appendix I. In this case it gives

$$\frac{1}{s^N}F(N,2N+n+1;\,N+n+2;\,-\tfrac{r}{s}) = \frac{1}{(r+s)^N}F(N,1-N;\,N+n+2;\,\tfrac{r}{r+s})$$

and substitution of this in (A.2.3) gives the required result, (A.2.1).

Similarly we may prove that

$$\frac{\beta\,\Gamma(N)}{2\pi i}\int_{\Gamma_2}\frac{\eta^{2N+m}}{(\eta+r)^N(\eta-s)^N}\,d\eta = \frac{\beta\,\Gamma(2N+m+1)}{\Gamma(N+m+2)}\,\frac{s^{N+m+1}}{(r+s)^N}\,F\left(N, 1-N; N+m+2; \frac{s}{r+s}\right)$$

for the contour Γ_2 corresponding to Γ_1.

We may note here that when N is a positive integer the hypergeometric co-efficients become zero after N terms so that the right side of (A.2.1) may be expressed as the finite sum

$$\frac{a\,\Gamma(2N+n+1)}{\Gamma(N+n+2)}\,\frac{r^{N+n+1}}{(r+s)^N}\sum_{p=0}^{N-1}\frac{(N-1+p)!\,(N-1)!\,\Gamma(N+n+2)(-1)^p\,r^p}{(N-1)!\,(N-1-p)!\,\Gamma(N+n+2+p)\,p!\,(r+1)^p}$$

$$= a\sum_{p=0}^{N-1}\frac{(N-1+p)!\,\Gamma(2N+n+1)(-1)^p\,r^{N+n+1+p}}{(N-1-p)!\,\Gamma(N+n+2+p)\,p!\,(r+s)^{N+p}}$$

$$= a\left(\frac{\partial}{\partial r}\right)^{N-1}\frac{r^{2N+n}}{(r+s)^N}$$

which is the required result for integral N .

Appendix III.

To show that if we are given a region defined by

$$w_0 = \frac{a}{3(2N+1)}[(N+2)r^3 - 3Nr^2s + 3Nrs^2 - (N+2)s^3]$$

then another region defined by

$$w_2 = w_0 + \frac{2a(2\mu)^{N+1}(\mu-s)^2}{(2N+1)(\mu+r)^N} F(N+2; -N-1, N; 3; \frac{\mu-s}{2\mu}, \frac{\mu-s}{\mu+r}) \tag{A.3.1}$$

patches on to the first along the characteristic $s = \mu$, and also satisfies the condition that $u = 0$ at $x = \ell$, assuming the curves $s = \mu$ and $x = \ell$ meet in the $x - t$ plane at $t = 0$.

We first seek w_2 in the case where N is an integer. For this case w_0 may be expressed as

$$w_0 = \frac{2a}{(2N+1)(N+1)!} \left\{ \left(\frac{\partial}{\partial r}\right)^{N-1} \frac{r^{2N+2}}{(r+s)^N} - \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{s^{2N+2}}{(r+s)^N} \right\}$$

so that, using the first patching condition of chapter 2, § 3, (2.3.2) we may try a solution for w_2 of the form

$$w_2 = \frac{2a}{(2N+1)(N+1)!} \left\{ \left(\frac{\partial}{\partial r}\right)^{N-1} \frac{r^{2N+2}}{(r+s)^N} + \left(\frac{\partial}{\partial s}\right)^{N-1} \frac{S(s)}{(r+s)^N} \right\} \tag{A.3.2}$$

and we seek a function $S(s)$ such that this w_2 will satisfy $r = s$ on $x = \ell$. In terms of w, x is given by (2.2.8), and putting $r = s$ this becomes

$$\left(\frac{\partial w_2}{\partial r}\right)_{r=s} - \left(\frac{\partial w_2}{\partial s}\right)_{r=s} = 2\ell .$$

Substituting (A.3.2) in this equation, expressing the result as a sum, and introducing a new function $F(r)$ defined by

$$F(r) = r^{2N+1} - \frac{1}{2N+2} S'(r) \qquad \text{(A.3.3)}$$

we obtain a differential equation for $F(r)$ namely

$$\sum_{p=0}^{N-1} \frac{(2N-2-p)!(-1)^{N-1-p}2^p}{p!(N-1-p)!} r^p F^{(p)}(r) = (2r)^{2N-1} \frac{(2N+1)N!\ell}{2a}.$$

This differential equation is similar to those treated in the first half of Appendix I, and a similar method to that used there gives a solution for $F(r)$ which in turn gives from (A.3.3) the result for $S(s)$ as

$$S(s) = s^{2N+2} - \frac{(2N+2)!\ell}{(2N)!\,2a} s^{2N} - \sum_{n=0}^{N-1} \beta_n s^{2n}$$

for some constants β_n.

We may now apply the second patching condition that the $S(s)$ functions of w_0 and w_2, and their first $N - 1$ derivatives must have the same values on $s = \mu = [\frac{(2N+1)\ell}{2a}]^{\frac{1}{2}}$. This is equivalent to saying that the difference of the two S-functions must have a factor $(s-\mu)^N$. Since the co-efficients β_n are non-zero only for even powers of s it is easy to see that the β_n must be chosen to give

$$S(s) = 2(s^2 - \mu^2)^{N+1} - s^{2N+2}.$$

Thus for integral N we have the result that w_2 is given by

$$w_2 = \frac{2a}{(2N+1)(N+1)!}\left\{\left(\frac{\partial}{\partial r}\right)^{N-1}\frac{r^{2N+2}}{(r+s)^N} - \left(\frac{\partial}{\partial s}\right)^{N-1}\frac{s^{2N+2}}{(r+s)^N} + 2\left(\frac{\partial}{\partial s}\right)^{N-1}\frac{(s^2-\mu^2)^{N+1}}{(r+s)^N}\right\}$$

$$= w_0 + \frac{4a}{(2N+1)(N+1)!}\left(\frac{\partial}{\partial s}\right)^{N-1}\frac{(s^2-\mu^2)^{N+1}}{(r+s)^N}$$

We now seek to generalise this result to all N. The term w_0 generalises immediately. For the other term we employ a process similar to that of chapter 3 §5 and Appendix II.

We write

$$v = \frac{4a}{(2N+1)(N+1)!}\left(\frac{\partial}{\partial s}\right)^{N-1}\frac{(s^2-\mu^2)^{N+1}}{(r+s)^N} \qquad (A.3.4)$$

$$= \frac{4a}{(2N+1)(N+1)!}\left(\frac{\partial}{\partial \sigma}\right)^{N-1}\frac{\sigma^{N+1}(\sigma+2\mu)^{N+1}}{(\rho+\sigma)^N}$$

on introducing new variables $\rho = r + \mu$ and $\sigma = s - \mu$. The expression for v will also be a solution of an Euler-Poisson equation in terms of ρ and σ, so we may attempt to generalise v by means of the contour integral

$$v = \frac{4a\,\Gamma(N)}{(2N+1)\,\Gamma(N+2)2\pi i}\int_{\Gamma_2}\frac{z^{N+1}(z+2\mu)^{N+1}}{(\rho+z)^N(z-\sigma)^N}\,dz$$

where Γ_2 is a contour of the usual type surrounding $z = \sigma$. This contour integral may be evaluated by the same method as that used in Appendix II. Corresponding to equation (A.2.2) we may obtain

$$v = \frac{4a\,\Gamma(N)}{(2N+1)\pi\,\Gamma(N+2)}\,\sigma^3 \sin N\pi \int_0^1 \zeta^{N+1}(\zeta+p)^{N+1}(\zeta+q)^{-N}(1-\zeta)^{-N}\,d\zeta$$

where $p = \frac{2\mu}{\sigma}$ and $q = \frac{\rho}{\sigma}$, and this holds for values of N between 0 and 1.

This integral is a representation of the hypergeometric function of two variables, (see Whittaker and Watson [15] p.298) and so we may write

$$v = \frac{2a(2\mu)^{N+1}(\mu-s)^2}{(2N+1)(r+\mu)^N}\,F\left(N+2;\ -N-1, N;\ 3;\ \frac{\mu-s}{2\mu}, \frac{\mu-s}{\mu+r}\right)$$

and this result is now extended to all N by analytic continuation.

The foregoing does not amount to a proof that this value of v is that required for all N, merely a strong indication. That it is the required value must be checked. It is easy to see that $w_2 = w_0 + v$ will be equal to w_0 at $s = \mu$. Also, if we put $x = \frac{\mu-s}{2\mu}$, $y = \frac{\mu-s}{\mu+r}$, then substitute $w = A\,\frac{(\mu-s)^2}{(\mu+r)^N}\,F(x,y)$ into the Euler-Poisson equation the resulting differential equation in x and y is

$$y(y-1)F_{yy} + x(y-1)F_{xy} + \{(2N+3)y-3\}F_y + NxF_x + N(N+2)F = 0$$

and this is the differential equation for the above hypergeometric, as in Whittaker and Watson p.300. Finally we may check that the remaining condition, $u = 0$ on $x = \ell$ is satisfied by the w_2 given by this method simply by manipulation.

As a final note we may point out that by using the transformation corresponding for these functions to (A.1.5), namely

$$F(\alpha;\beta,\beta';\gamma;x,y) = (1-x)^{-\beta}(1-y)^{-\beta'}F(\gamma-\alpha;\beta,\beta';\gamma;\frac{x}{x-1},\frac{y}{y-1})$$

(which may be proved by a simple transformation in the integral representation used in (A.3.5)), v reduces to a form in which the series is finite for integral N , that is

$$v = \frac{2a(\mu-s)^2(\mu+s)^{N+1}}{(2N+1)(r+s)^N} \quad F(1-N;-N-1,N;3;\frac{s-\mu}{s+\mu},\frac{s-\mu}{r+s})$$

and this series may be shown to be equal to that given by expanding (A.3.4).

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