PARAMETRIC MODELS OF SURFACES

Stewart Alexander Robertson

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1957

P A R A M E T R I C
M O D E L S O F
S U R F A C E S

By

Stewart A. Robertson

Thesis Submitted to the University of St. Andrews
for the Degree of Ph.D., Sept., 1957.

DECLARATION

I declare that this thesis is entirely my own composition, that it is a record of work done by me, and that it has not been accepted in any previous application for a Higher Degree.

STATEMENT OF RESEARCH TRAINING

Having obtained the degree of B.Sc. at St. Andrews in June, 1955, I was admitted to the University as a full time research student in July of the same year, and began work under the supervision of Dr. E.M. Patterson. After completing five terms in attendance at the University, I continued my study at the University of Leeds, where I am now in my fourth term of research. During my period of absence from St. Andrews, my work has been supervised by Professor E.T. Copson.

Leeds,

1st. September, 1957

CERTIFICATE

I certify that, in the preparation of this thesis, Stewart A. Robertson has complied with and fulfilled the Ordinances and Regulations of the University of St. Andrews pertaining to the degree of Doctor of Philosophy.

3/7/57

ACKNOWLEDGEMENTS

I wish to record my sincere thanks to Dr. E.M. Patterson of the University of Leeds for his guidance and encouragement throughout the preparation of this thesis, both at Leeds and St. Andrews, and to Professor E.T. Copson for undertaking the supervision of my work during the past year. Thanks are due also to the Carnegie Trust for the Universities of Scotland for their generosity in supporting this research.

S. A. R.

CONTENTS

CHAPTER ONE

Introduction

1.1 Surfaces in Topology

In topology, a surface X is a connected, metrisable topological space such that each point of X has a neighbourhood which is homeomorphic with the set of points interior to a circle in the Euclidean plane. This definition is essentially that used in Lefschetz's book (1), from which we shall borrow many terms and notations.

We are to be concerned only with those surfaces which are closed, these being defined as follows : A surface X is said to be closed if every covering of X by open sets admits a finite open sub-covering. In other words, X is closed if it is compact.

Surfaces possess the important property of admitting a triangulation : that is, for a given surface X, there can be found a simplicial complex K whose underlying polyhedron $|K|$ is homeomorphic with X. The complex K is said to cover X. If X is closed, then it can be covered by a finite complex K consisting of finite numbers α_2, α_1, α_0 of simplexes s_i^2, s_j^1, s_k^0 (faces, edges, and vertices).

We may give K an orientation by ordering the vertices of each simplex in either of the two possible ways. Suppose that, having done so, we form the 2-chain $\gamma^2 = \sum_{i=1}^{\alpha_2} s_i^2$. Then we say that X is an orientable surface if it is possible to choose the orientation of K so that $\delta\gamma^2 = 0$, where δ denotes the standard boundary operator of homology. In such a case, γ^2 is called the fundamental 2-cycle of X. It can be shown that orientability is independent of the particular covering complex K, and it is therefore an intrinsic property of X. The sphere and torus are probably the most familiar orientable surfaces. On the other hand, we may mention the projective plane and the klein bottle, both 'one-sided' surfaces, as examples of closed surfaces which are non-orientable. In fact, the intuitive distinction between one-sidedness and two-sidedness coincides exactly with the distinction between orientable and non-orientable surfaces. Orientability is, further, a property which is preserved under homeomorphism - topologically equivalent surfaces are orientable or non-orientable together.

Another topological invariant of a closed surface X is the alternating sum $\chi = \alpha_2 - \alpha_1 + \alpha_0$ of the numbers of simplexes of each dimension in any covering complex K. χ is the Euler Characteristic of X. The only other topological invariant which we need mention is associated more particularly with closed orientable surfaces. This is the genus, and like χ itself it is an integer. If p denotes the genus of a closed orientable surface X, and χ its characteristic, then $\chi = 2 - 2p$.

1.2 Classification of Surfaces

The topological classification of surfaces was completed many years ago (cf. von Kerekjarto (2)). Surfaces are collected together in equivalence classes under homeomorphism, and the various classes are described by the construction of particularly simple representative surfaces, known as canonical forms. As we have already indicated, we shall confine our attention to closed surfaces.

Perhaps the best-known set of canonical forms for closed surfaces is that which involves the attachment of 'handles' and 'cross-caps' to a sphere. A convenient description of these is afforded by the idea of the topological sum of surfaces (H. Hopf (3)) as follows.

Suppose that X_1, X_2 are two closed surfaces. From X_1 let us remove a 2-cell e_1 (the homeomorph of the interior of a circle), and from X_2 a 2-cell e_2, the frontier of each cell being homeomorphic with a circle. We may then identify points on the frontiers of e_1, e_2 which correspond under some homeomorphism, to obtain a closed surface X_3, which we call the topological sum of X_1 and X_2. If X_1, X_2 are covered by oriented complexes K_1, K_2; then e_1, e_2 may be taken to be simplexes s_i^2, s_j^{*2} of K_1, K_2, and the correspondence beween the simplexes composing the frontiers of these cells may be chosen to make the orientations coincide. X_3 is thereby associated with an oriented covering complex K_3.

The definition may be extended in the obvious way to include the case of any finite number of surfaces, and it is clear that there is no question of the ordering of the surfaces being relevant to the topological type of any sum (i.e., the 'adding' process is both commutative and associative). For convenience, we still speak of 'topological sum' when we are considering a single surface without any associated excisions or identifications.

It can be shown that any closed orientable surface is homeomorphic with the topological sum of p+1 closed surfaces $V_0, V_1, \ldots, V_p$, where V_0 is a topological sphere, and $V_1, V_2, \ldots, V_p$ are homeomorphs of a torus. The integer p is uniquely defined, and may take any one of the values 0,1,...; this number is called the genus of the surface. Thus a double torus, which has genus 2, is homeomorphic with the topological sum of a sphere and two anchor rings.

For the sake of completeness, we add that any closed non-orientable surface is homeomorphic with the topological sum of q closed surfaces $W_1, W_2, \ldots, W_q$, which are all homeomorphs of the projective plane; q is just the number of 'cross-caps' mentioned above, and may have any positive integral value. Quite often q, too, is called the genus of the corresponding surface, and we have the relation $\chi = 2 - q$.

Other well-known canonical forms are the convex polygons with pairs of directed sides identified. We shall not pause to consider these in detail. Instead, we end this section by describing a very simple canonical form for closed orientable surfaces for which we have some use later. This is called by Lefschetz a two-sided disk Δ_p with p holes, p again being the genus.

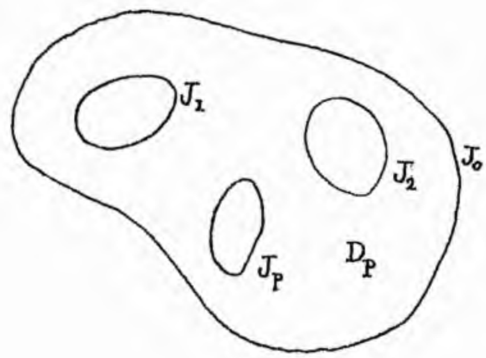


Fig. 1

Let D_p be a closed connected subset of the Euclidean plane E_2, bounded by p+1 Jordan curves $J_0, J_1, \ldots, J_p$, with J_0 enclosing $J_1, J_2, \ldots, J_p$, which are mutually exterior. The frontier of D_p is $J = J_0 + \bigcup_{i=1}^{p} J_i = J_0 + I$, say. We now perform a disjunction of each point $d \varepsilon D_p - J$ into two points d_+, d_-, and topologise the sets so formed to give two homeomorphs $D_p^+ = J + \bigcup \{d_+\}$ $D_p^- = J + \bigcup \{d_-\}$ of D_p. Then $\Delta_p = D_p^+ \cup D_p^-$ is the homeomorph of a closed orientable surface of genus p. We call Δ_p a p-holed, two-sided disk.

1.3 Surfaces in Three-Space

In the classical theory of surfaces, the term 'surface' is used to mean the set of points of Euclidean three-space E_3, determined either by some equation of the form $F(x,y,z) = 0$, connecting the cartesian co-ords. x, y, z of each point of the surface, or alternatively by a set of three parametric equations $x = X(u,v)$, $y = Y(u,v)$, $z = Z(u,v)$, X, Y, Z being real single-valued functions of the real variables, or parameters u,v, which are tacitly assumed to be differentiable, or even analytic, as the occasion demands. Similar assumptions are made concerning the function F. For most purposes, the latter definition of surface is more convenient. It is useful, too, to write these parametric equations in vector form $\underline{r}(u,v) = (X(u,v),Y(u,v),Z(u,v))$, where $\underline{r}$ is the position vector of a point of co-ords. x,y,z.

Suppose that $\underline{r} = \underline{r}(u,v)$ determines a subspace S of E_3. Then S may, or may not, be the homeomorph of a surface in the topological sense previously defined. For example, $\underline{r}$ = (cos v cos u, cos v sin u, sin v) is the position vector of a Euclidean sphere - a closed orientable surface of genus zero - while $\underline{r}$ = (v sin α cos u, v sin α sin u, v cos α), where α is fixed, is the position vector of a right circular cone C of semi-vertical angle α. There is in C no neighbourhood of the point (0,0,0) homeomorphic with the interior of a circle, so the origin is in this case a topological singularity.

If S is free of topological singularities, then it is homeomorphic with a surface X, and if X is closed, it is necessarily orientable, since closed non-orientable surfaces cannot be represented by subspaces of a Euclidean space of less than four dimensions. Such a subspace S associated with a position vector $\underline{r} = \underline{r}(u,v)$ we call a <u>parametric model</u> of X. (We shall not be concerned with the distinctions which must be made when the same subspace S is determined by two different sets of parameters.) We call the correspondence f beween pairs (u,v) and points of S a parametisation of S.

Of great importance are the two families of pafametric curves u = const., v = const. Through each point of S there passes at least one curve of each family. If through a point $\alpha \epsilon S$ there passes exactly one curve of each family, we call α an ordinary point of S with respect to f; otherwise α is called a singular point or singularity. We always assume that the singularities are isolated, in the sense that each such point is contained in a neighbourhood which contains no other. We see, therefore, by applying the Weierstrass theorem to the compact space S, that the singular points form a finite set. It is a well-known fact that if there are no singular points associated with f, then *S* is of genus one (the homeomorph of a torus).

Suppose that $s \epsilon S$ is a singularity with respect to some parametrisation f. Since s is isolated, there is a neighbourhood N of s containing no other singular point. Also, from the fact that S is a topological surface, it follows at once that we can construct a homeomorphism h of a neighbourhood $N^* \subset N$ of s onto the interior Δ of a circle Γ in the plane E_2. This homeomorphism may be modified if necessary so that the families u = const., v = const., of parametric curves in the neighbourhood of s are mapped onto continuous families of differentiable curves in Δ, having the same parameters u, v. We can choose polar co-ords. (ϱ, θ) for E_2 in such a way that h(s) is the origin of co-ords., and the circle Γ^*, given by $\varrho = 1$, lies in Δ. Since each point of Γ^* is

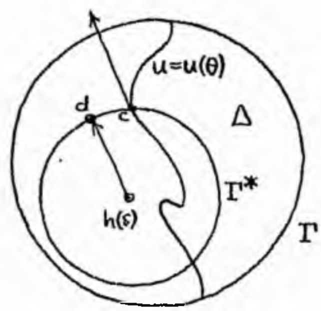


Fig. 2

the image of an ordinary point of S, a unique curve $u = u(\theta)$ passes through the point c of argument θ on Γ^*, and it has a uniquely defined unit tangent vector at c, sensed in the direction of increasing v. By constructing at the origin a unit vector strictly parallel to this tangent vector, we determine a second point d, say, of Γ^*. As the point c describes the curve Γ^* exactly once in the direction of increasing θ (the 'positive' direction), the point d describes Γ^* a finite number of times j, positive or negative. The number j is called the index of the singularity s. The same value of j is obtained using the curves v = const. (The above is only one of several ways of defining the index of a singular point;cf. Alexandroff-Hopf (4)).

The singular points of S, being finite in number, may be denoted by $s_1, s_2, \ldots, s_k$. Let j_m denote the index of s_m. Then, from a theorem of Poincaré (5), we have $\sum_{m=1}^{k} j_m = \chi (= 2 - 2p)$.

1.4 Parametric Models as Canonical Forms

As above, let $\underline{r} = \underline{r}(u,v)$ determine a parametrisation f of a closed orientable surface S of genus p, lying in E_3. We are interested in the problem of constructing parametric models which are sufficiently simple and symmetrical to merit the name 'canonical form'. Thus we are looking for subspaces S which are regular in form, and with the singularities of f distributed regularly over S. It is natural, too, to require that the families of parametric curves should be orthogonal. A third, and fairly strong, condition which we impose is that the singularities should be of common index; many of the subsequent results depend on this condition.

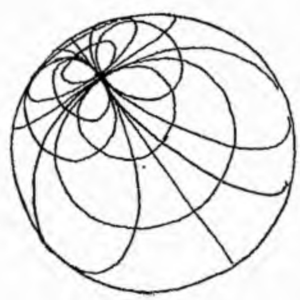
If f is associated with m singularities, each of index j, then we say that the model S is of *type* (m,j). We have $mj = 2 - 2p$. Let us consider first of all the simplest cases $p = 0$, $p = 1$.

For $p = 0$, there are two possibilities, namely $m = 2$, $j = 1$, and $m = 1$, $j = 2$; and corresponding to each of these there is a well-known parametric model of the 2-sphere, obtained in the following way.

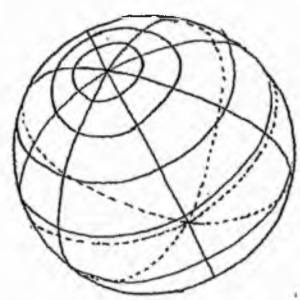
Let x,y,z be cartesian co-ords. for E_3, and let E_2 denote the plane $z + R = 0$, S the 2-sphere $x^2 + y^2 + z^2 = R^2$, $R > 0$. The two 'simplest possible' parametrisations of E_2 are by the cartesian co-ords. x,y, and by polars r,θ. If we close E_2 by adding a point ∞, and map the parametric curves x = const., y = const. (orthogonal families of st. lines meeting only at the point ∞) onto the sphere S using the inverse stereographic projection $h: E_2 \to S$, then we obtain a parametrisation of S having a single singularity, of index $+2$, at the vertex of projection $(0,0,R)$: from the elementary properties of h, it follows that the parametric curves are families of circles.

Similarly, h maps the straight lines r = const. and the circles Θ = const. onto the meridians and parallels of the 'latitude-longitude' parametrisations of S, which has singularities of index +1 at the two 'poles' $(0,0,\pm R)$.

These models, as far as the above conditions are concerned, are both canonical forms for genus zero, and certainly no simpler models could be expected. It will be seen later that only the second of these belongs to the general class of models with which we shall be concerned.



type (1,2)



type (2,1)

Fig. 3

When $p = 1$, we have $mj = 0$, so that $m = 0$ or $j = 0$. We may ignore the second of these possibilities, and so we have to consider only parametrisations free of singular points. As in the case of the sphere, we are already provided with a model which it is scarcely possible to simplify. This is of course the 'anchor-ring' imbedding of the torus (Fig. 4), given by the vector $\underline{r}(u,v) = ((R+r\cos v)\cos u, (R+r\cos v)\sin u, r\sin v)$, where $R > r > 0$. Both families of parametric curves are families of circles.

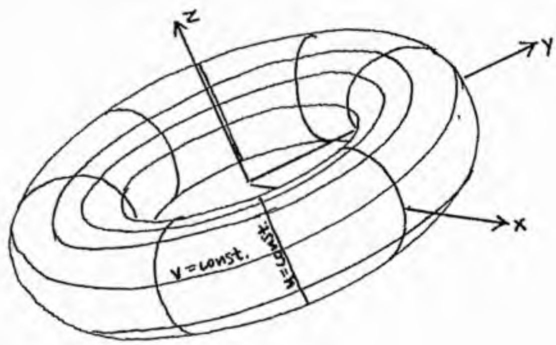


Fig. 4

For higher values of p, the number of possible types (m, j) may be larger, and we end this chapter by deriving a formula for the exact number of distinct types possible if m is even. We shall see that m does in fact satisfy this last condition whenever one family of parametric curves on S has a certain simple property.

Suppose then that $p > 1$, to give $mj < 0$. Then j itself is negative, say $j = -g$, where $g > 0$; and $mg = 2(p-1)$. If $m = 2n$, say, is even, then we have simply $ng = p-1$. Now $p-1$ is a positive integer, and may be expressed uniquely, therefore, as a product $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_q^{\alpha_q}$ of powers of primes. Thus for some integers $n_1, n_2, \ldots, n_q$, $\alpha_i \geqslant n_i \geqslant 0$, $n = p_1^{n_1} p_2^{n_2} \cdots p_q^{n_q}$, $g = p_1^{\alpha_1 - n_1} p_2^{\alpha_2 - n_2} \cdots p_q^{\alpha_q - n_q}$. Hence the number of possible types is

$$(\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_q + 1) \quad .$$

CHAPTER TWO

Surfaces Associated With a Family of Planes

2.1 Plane Intersections

In this section we shall prove the following result :

THEOREM I: If Π is a continuous, one-parameter family of planes filling three-space E_3, and $S \subset E_3$ is a parametric model of a closed surface, such that

i) through each point of S there passes exactly one plane of the family Π, and

ii) the singular points of the curves of intersection of Π with S are of common index, then the number of singularities is even.

We observe first of all that if S is of type (m, j), and j is odd, then m is necessarily even invi ew of the relation $mj = 2 - 2p$; our proof will hold both for j even and for j odd. But before proceeding to the proof itself, we shall discuss in some detail what is involved in the notion of a continuous one-parameter family of planes satisfying condition i).

***

We suppose that the parameter w of the family $\Pi = \{P(w)\}$ is unrestricted by inequalities, and so ranges through the real numbers. We also stipulate that w parametrises Π in such a way that there is a number $K > 0$ such that, given any two values w, w^* of the parameter, $P(w) = P(w^*)$, $w \neq w^*$, implies $|w - w^*| > K$. This condition ensures that w runs through Π without at any stage 'doubling back'.

We denote $P(w) \cap S$ by the symbol $S(w)$, and consider two cases:

a) there are planes $P(w) \varepsilon \Pi$ for which $S(w) = 0$;

b) $S(w) \neq 0$, for all values of w.

Let us examine each case in turn.

a) Suppose that, for $w = w_0$, $S(w) = 0$. Then, by increasing (or decreasing) w from this value, a further value $w = w_1$ can be reached for which $S(w_1) \neq 0$, $S(w_1 - \epsilon) = 0$, for all $\epsilon > 0$ and sufficiently small. A further increase in w traverses only values for which $S(w) \neq 0$, but because S is compact and condition 1) holds, a second value $w = w_2$ must be encountered, with the property that $S(w_2) \neq 0$, $S(w_2 + \epsilon) = 0$, for all $\epsilon > 0$ and sufficiently small (cf. Fig.5). The fact that S is connected guarantees that $\bigcup_{w_1 \leq w \leq w_2} S(w) = S$. If we increase w still further, it may happen eventually that a value w^* of the parameter is met for which $P(w^*) = P(w)$, where w takes some value in the range $w_1 \leq w \leq w_2$. In such a case, the parameter may be modified if necessary to give $P(\tilde{w}) = P(\tilde{w}')$ if and only if $\tilde{w} \equiv \tilde{w}' \bmod \pi$, say. Otherwise we modify Π itself by replacing that part of the family which does not meet S by another subfamily, in such a way that the above process may be carried out.

b) If every plane of Π meets S, then, increasing the parameter from any given value w, we reach a value w* such that $P(w) = P(w^*)$, and we then modify the parameter as above to give $P(w) = P(w^*)$ if and only if $w \equiv w^* \bmod \pi$.

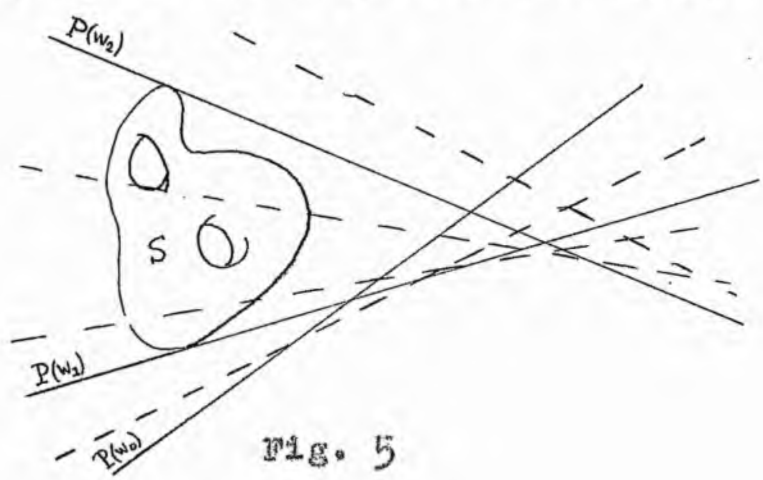


Fig. 5

In both cases, therefore, we arrange that each plane of Π is associated with a complete congruence class of values of w. Two values in the same class will be called equivalent. Simple examples of such families Π are formed by 1) the set of planes parallel to a given plane, and 2) the pencil of planes through a given st. line (the condition i) implying that S does not intersect this line). Any model S associated with the first of these belongs to category a), and in Fig. 6 we show how we should modify Π as described above.

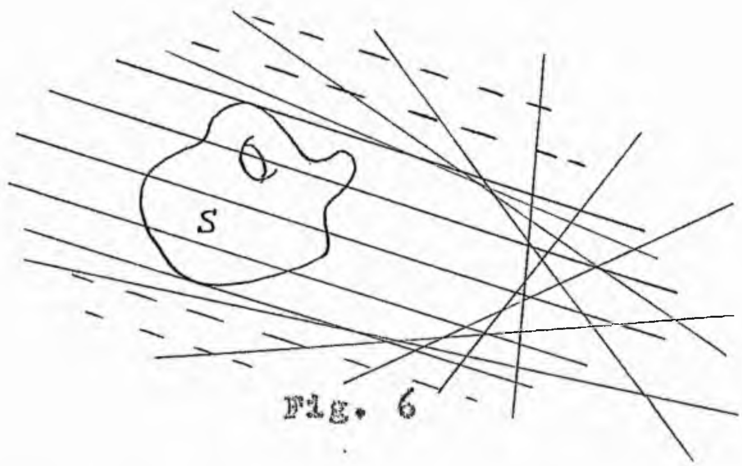


Fig. 6

Fig. 7 gives another specimen of a family Π, to illustrate the wide variety of possibilities. This family is composed of portions of three pencils with parallel axes. It is interesting to note that suitable families are not formed by combining four or more such pencils in a similar way.

Next, we describe some of the properties of these intersections, using a few results of Poincaré (6). In general, P(w) intersects S in a finite system S(w) of non-intersecting simple

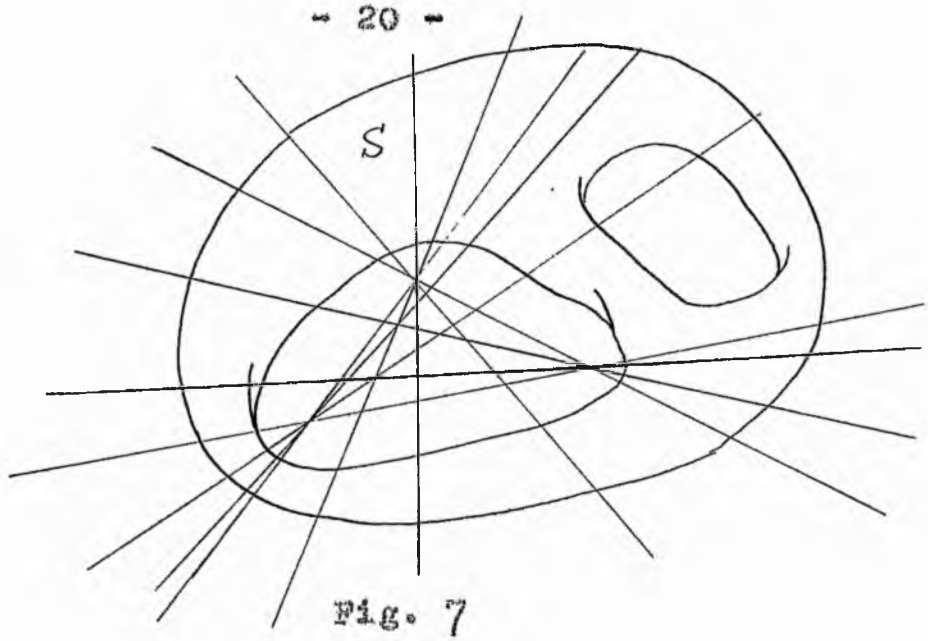


Fig. 7

closed curves (homeomorphs of a circle) which we denote by $\sigma_1(w)$, $\sigma_2(w)$, ..., $\sigma_t(w)$, but neighbouring sets $S(w)$, $S(w+\epsilon)$ are not necessarily homeomorphic. Poincaré describes two things which may happen to bring this about.

Firstly, there may be values of w at which one of the $\sigma_i(w)$ reduces to a single point. Suppose that, for $w = w_\alpha$, $\sigma_j(w)$ reduces to a single point α. Then, for sufficiently small $\epsilon>0$, either $\sigma_j(w_\alpha+\epsilon)$ is a simple closed curve, and $\sigma_j(w_\alpha-\epsilon)$ vanishes completely, or vice-versa. Such a point is called a <u>node</u>.

Secondly, there may be values of w where two of the $\sigma_i(w)$ meet to form a closed curve with a multiple point (of order one, in the sense of the theory of plane curves). If $\sigma_f(w)$ and $\sigma_g(w)$ are two distinct closed curves for $w = w_{\beta-\eta}$, $|\eta| > 0$ and as small as we please, which meet at a point β for $w = w_\beta$, then Poincaré proves that $\sigma_f(w_\beta+\eta) \cup \sigma_g(w_\beta+\eta)$ is in fact a single simple closed curve $\sigma_h(w)$, say, and the number of curves in $S(w)$ is thereby reduced by one. Points such as β are called saddle-points.

A possibility which Poincaré does not mention is that two curves $\sigma_{\tilde{c}}(w)$, $\sigma_{c^*}(w)$ may, for $w = w_c$ say, coincide completely to form a single closed curve σ_c. This happens, for instance, for the planes $z = \pm r$ in the case of the anchor-ring of Ch.1 (p.14). In such a case, both $\sigma_{\tilde{c}}(w)$ and $\sigma_{c^*}(w)$ vanish completely immediately to one side or other of the plane $P(w_c)$. We call σ_c a curve of contact of $P(w_c)$.

We do not of course envisage such 'degenerate' occurrences as the existence of a closed 2-cell common to S and some plane P(w), as might happen, for instance, if S had a 'flat' region.

Obviously, several nodes may lie in the same plane, which may also contain saddle-points and curves of contact. Two saddle-points may even lie on the same curve σ_c, which may itself be a curve of contact of the corresponding plane. More generally still, a curve $\sigma(w)$ may have a multiple point of multiplicity greater than one, where three or more simple curves meet. We shall now consider the situation immediately to either side of the plane P(w) in such a case.

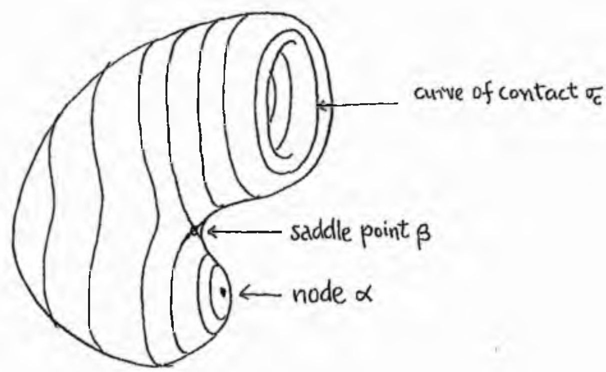


Fig. 8

A basic property of any closed orientable surface $S \subset E_3$ is that it is the common frontier of an unbounded subset A, and a bounded subset B of E_3, which are open, disjoint, and such that $A \cup B \cup S = E_3$. A is called the exterior, and B the interior, of S.

Suppose that P(w) has a non-null intersection with S, and contains no saddle-points, nodes, or curves of contact. Such a member of Π can always be found since we assume that the nodes and saddle-points are isolated in the sense that each is contained in a neighbourhood of S which contains no other node or saddle-point, while the curves of contact are 'isolated' too in the sense that if $P(w_c)$ contains a curve of contact, then there is an $\epsilon > 0$ such that $P(w)$ does not, $w_c - \epsilon < w < w_c + \epsilon$, $w \neq w_c$. The intersection $S(w)$ therefore consists of finite number of simple closed curves $\sigma_1(w)$, $\sigma_2(w), \ldots, \sigma_t(w)$.

Let us write $P(w) \cap A = A_w$, $P(w) \cap B = B_w$. Then it is clear that A_w, B_w are disjoint open sub-sets of $P(w)$ with common frontier $S(w)$, such that $A_w \cup B_w = P(w) - S(w)$. A_w is unbounded, B_w is bounded, and each has a finite number of components (maximal connected open sub-sets), say a_w for A_w, and b_w for B_w. ($a_w - 1$, $b_w - 1$ are the 1- and 0-dimensional Betti numbers of B_w; cf. Newman (7).)

Now consider a saddle-point Q of multiplicity k lying in the plane $P(\tilde{w})$. The form of $S(\tilde{w})$ in the neighbourhood of Q (the 'critical region') is illustrated in Fig. 9 for $k = 3$. There are $2(k + 1)$ segmental regions with common vertex Q belonging alternately to $A_{\tilde{w}}$, $B_{\tilde{w}}$. For ϵ small enough, $S(\tilde{w}-\epsilon)$, $S(\tilde{w}+\epsilon)$ are locally of the form shown in Figs. 10, 11, again for $k = 3$. In the critical region, both $S(\tilde{w}-\epsilon)$, $S(\tilde{w}+\epsilon)$ form k+1 'horns'; in $P(\tilde{w}-\epsilon)$, points on the convex side of each apex

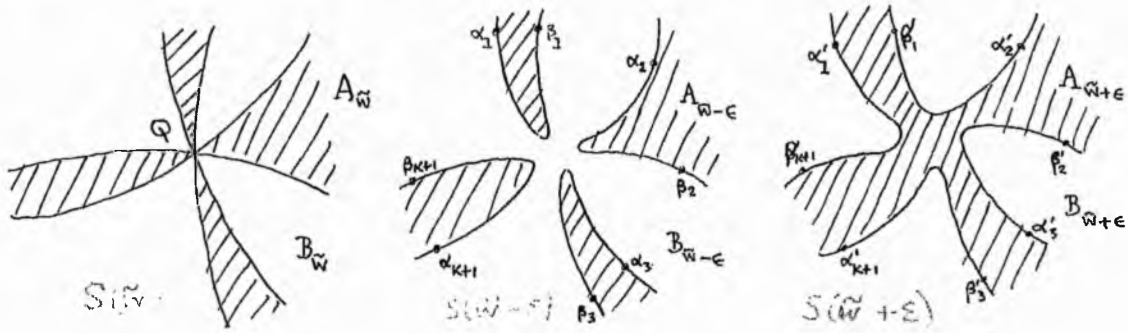


Fig. 9 Fig. 10 Fig. 11

are in $A_{\tilde{w}-\epsilon}$, while in $P(\tilde{w}+\epsilon)$, such points are in $B_{\tilde{w}+\epsilon}$. The situation is clear from the diagrams. Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{k+1}, \beta_{k+1}$ be points on $S(\tilde{w}-\epsilon)$ near these horns, arranged in the manner of Fig. 10, so that α_i, β_i belong to the same closed curve, and lie on the edges of the same horn. Since Q is isolated, we can guarantee that the plane $P(\tilde{w}-\epsilon)$ contains no singular points. The curves of $S(\tilde{w}-\epsilon)$ are therefore without multiple points. Suppose then that we start from the point α_1, and trace out $S(\tilde{w}-\epsilon)$ in the direction away from the critical region. Since α_1, β_1 lie on the same closed curve, we must eventually return to α_1, via the point β_1; but in describing this path we may also encounter some others of the α's and β's. Suppose that α_i, $i \neq 1$, is the first such point ~~next~~ after α_i itself. Then β_i is the next, and it is easy to see (by the Jordan Curve Theorem) that, since there are no cross-points, we cannot return to β_1 from this position along any curve without entering the critical region along some path not in $S(\tilde{w}-\epsilon)$. Hence the first point met after α_1 is some β, say β_j, which is of course followed by α_j, and then, by the above argument, by some point β_m; and so on, ending with β_1 and back to α_1. If all the α's and β's lie on the curve so described, then all the horns belong to the same component of $A_{\tilde{w}-\epsilon}$. Otherwise, suppose α_s does not lie on the curve. Then we may again set out from α_s, and determine a further set of points $\alpha_s, \beta_t, \alpha_t, \ldots, \beta_s$ lying on the same curve of $S(\tilde{w}-\epsilon)$.

By repeating this procedure a finite number of times, we exhaust all the α's, and determine a collection of sets of points α, β, each of which determines a component of $A_{\tilde{w}-\epsilon}$ to which the horns associated with that set belong. By counting the number of pairs α_a, β_a in any such set, we may form a partition $(\xi_1, \xi_2, \ldots, \xi_r)$ of k+1, r being equal to the number of components of $A_{\tilde{w}-\epsilon}$ which protrude into the critical region.

Now it is quite clear that, as we change the parameter from $\tilde{w}-\epsilon$ to $\tilde{w}+\epsilon$, these horns join together and so unite the r components of $A_{\tilde{w}-\epsilon}$ to form a single component of $A_{\tilde{w}+\epsilon}$. Assuming for the moment, then, that Q is the only singular point in $P(\tilde{w})$, we have $a_{\tilde{w}-\epsilon} = a_{\tilde{w}+\epsilon} + (r-1)$.

Similarly, we see that only one of the components of $B_{\tilde{w}-\epsilon}$ has points in the critical region. Consider a component of $A_{\tilde{w}-\epsilon}$ having ξ_s horns. As we pass to $\tilde{w}+\epsilon$, these horns unite and determine ('seal off') $\xi_s - 1$ components of $B_{\tilde{w}+\epsilon}$. Summing over the ξ's, we have therefore that $b_{\tilde{w}-\epsilon} = b_{\tilde{w}+\epsilon} - (\sum_{s=1}^{r} \xi_s - r) - 1$. Thus:

$$
(1) \qquad
\begin{aligned}
a_{\tilde{w}-\epsilon} &= a_{\tilde{w}+\epsilon} + (r-1), \\
b_{\tilde{w}-\epsilon} &= b_{\tilde{w}+\epsilon} + (r-1) - k.
\end{aligned}
$$

(Notice that ϵ is allowed to be either positive or negative, so that equations (1) each includes two cases.)

At a node, the situation is much simpler. We have:

$$(2)\qquad \begin{aligned} a_{\tilde{w}-\epsilon} &= a_{\tilde{w}+\epsilon}\,, \\ b_{\tilde{w}-\epsilon} &= b_{\tilde{w}+\epsilon} + 1; \end{aligned} \quad \text{or} \quad \begin{aligned} a_{\tilde{w}-\epsilon} &= a_{\tilde{w}+\epsilon} + 1, \\ b_{\tilde{w}-\epsilon} &= b_{\tilde{w}+\epsilon}\,. \end{aligned}$$

Likewise, at a simple curve of contact,

$$(3)\qquad a_{\tilde{w}-\epsilon} = a_{\tilde{w}+\epsilon} + 1\,, \qquad b_{\tilde{w}-\epsilon} = b_{\tilde{w}+\epsilon} + 1\,.$$

***

Proof of Theorem I

a) Suppose that $S(w_0) = 0$. Then, as we have already seen, we have only to increase (or decrease) w to reach eventually a value $w = w_1$ for which $S(w_1) \neq 0$, while $S(w_1 - \epsilon) = 0$, all $\epsilon > 0$ and sufficiently small. The associated $S(w_1)$ contains only nodes and curves of contact. Now, from the fact that each node α is surrounded (on S) by an arbitrarily small closed curve σ_α, we see at once (Fig. 12) that the index j_α of α is +1. Similarly, the index j_β of a 'simple' saddle-point β is -1, and of a point of multiplicity k, is $-k$.

If $S(w_1)$ contains any nodes at all, then,

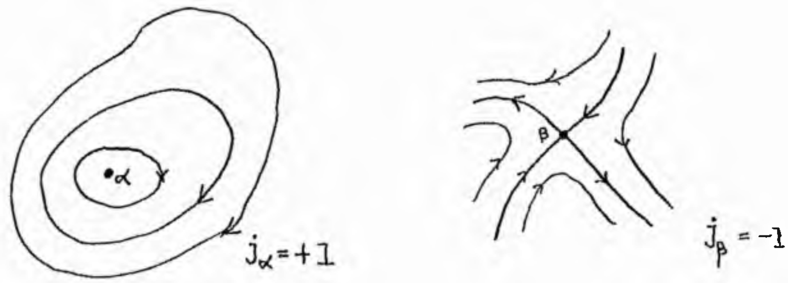


Fig. 12

by the condition ii) of the theorem, all the singular points are nodes. Hence, since $\Sigma j = 2 - 2p$, we have that S is of genus zero, and there are exactly two singularities. Let us suppose then that S is of genus $p > 0$. Then $S(w_1)$ contains no nodes. By a slight deformation of S (we merely 'press in' each such multiple point slightly: this can always be done without creating or destroying singular points), we can ensure that the curves of contact have no multiple points: that is, $S(w_1)$ consists of a number of simple closed curves. Now let w be increased slightly from w_1 to $\bar{w}$. Each closed curve thereby splits into two, one enclosed by the other, and it is evident that the interior of each annulus so formed is in $B_{\bar{w}}$. Hence the numbers of components in $A_{\bar{w}}$, $B_{\bar{w}}$ are respectively t+1, t, where t is the number of closed curves in $S(w_1)$. As w is increased continuously we reach a similar situation, in reverse as it were, approaching the value denoted earlier by $w = w_2$, and we have, for $\bar{\bar{w}}$ slightly less than w_2, that the numbers of components in $A_{\bar{\bar{w}}}$, $B_{\bar{\bar{w}}}$ are respectively $r+1$, r, where r is the number of closed curves in $S(w_2)$. As w varies in the range $w_1 < w < w_2$, the numbers of these components vary if there are singular points or curves of contact; again we can deform S slightly without altering the number of singular points, to render all curves of contact free of multiple points. If there are none, then ~~$r = t =$~~ $p = 1$ (~~because S is connected~~). Suppose in fact that there are m singular points $s_1, s_2, \ldots, s_m$, each of index $-k$, and n curves of contact. Then, by (1) and (3), we have:

$$r = t + \sum_{i=1}^{m_1} a_i - \sum_{j=1}^{m_2} b_j + n_1 - n_2 ,$$

$$r + 1 = t + 1 + \sum_{i=1}^{m_1} a_i - \sum_{j=1}^{m_2} b_j + n_1 - n_2 - m_1 k + m_2 k,$$

where a_i, b_j are the numbers of components by which A_w is increased and diminished at s_i, s_j respectively, and $m_1 + m_2 = m$, $n_1 + n_2 = n$. These two relations show at once that $(m_1 - m_2)k = 0$, i.e., $m_1 = m_2$. Thus m is even, and the theorem is proved for this case.

b) Suppose that every $P(w)$ intersects S, and choose a value w^* of the parameter such that $P(w^*)$ contains no singular points or curves of contact; that such values of w exist is now evident. Let the number of components in B_{w^*}, A_{w^*} be t, t'. As we have seen, by increasing w from $w = w^*$, we reach a value $w = \bar{w}^*$, where $w^* \equiv \bar{w}^* \bmod \pi$ and so $P(\bar{w}^*)=P(w^*)$, the plane $P(w)$ having 'swept through' S in the process. If again we have m singular points $s_1, s_2, \ldots, s_m$, each of index $-k$, and n curves of contact, then the change in t, t' in moving from w^* to $\bar{w}^*$ is zero, and, using the same notation as above, we have:

$$\sum_{i=1}^{m_1} a_i - \sum_{j=1}^{m_2} b_j + n_1 - n_2 = 0,$$

$$\sum_{i=1}^{m_1} a_i - \sum_{j=1}^{m_2} b_j + n_1 - n_2 + m_2 k - m_1 k = 0.$$

Thus once again $m_1 = m_2$. If S is of genus zero, then there are just two nodes as in case a). The proof of the theorem is now complete.

2.2 Pairs

Suppose that S, Π satisfy the conditions of Theorem I. Then, if in addition the family $\{S(w)\}$ coincides pointwise with one family of parametric curves on S, we say that S and Π form a pair (S,Π). This means that, if we denote the entire curve of parameter v by L(v), then $S(w) = L(v_1) \cup L(v_2) \cup \ldots \cup L(v_q)$, for some integer $q \geq 1$, which may change as w varies. We assume, as we did in the case of w itself, that v is subject to no inequalities, and runs therefore throughout the real numbers, parametrising continuously the family $\{L(v)\}$. Under this assumption, it is possible then that, for some values of v, we must consider L(v) which are empty; but this we shall see occurs if, and only if, p = 0. If necessary, we modify v to ensure that $L(v+\epsilon)$, for ϵ as small as we please, always lies entirely to one sideof $P(w) \supset L(v)$. A further condition, which we imposein order to make a certain mapping 'smooth' (in a sense to be explained later), is that only by choosing a value of w for which P(w) is sufficiently near a plane containing a curve of contact can we obtain L(v), L(v') in P(w), and such that $(v-v') \neq 0$ is as small as we please.

If $L(v\ ) \subset S(w)$ includes a curve of contact σ_c (which, as in 2.1, we may take to be a simple closed curve), then, for $w-\epsilon$, say, where ϵ is as small as we please, this curve vanishes completely.

Hence, for $\epsilon' \neq 0$ and sufficiently small, both $L(v_i \pm \epsilon')$ lie to the same side of $P(w)$. Clearly $L(v_i)$ consists entirely of curves of contact, all vanishing for $w-\epsilon$ (cf. Fig. 13).

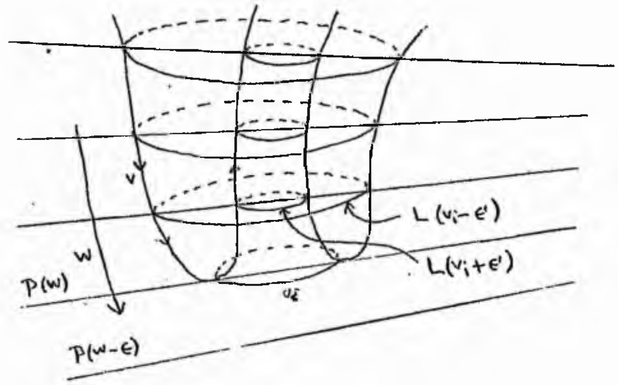


Fig. 13

We may, confining our attention to those values of w for which $S(w) \neq 0$, regard w as a single-valued real function of the real variable v. Thus $w = w(v)$, defined by the relation $L(v) \subset P(w(v))$ — $w(v)$ is the parameter of the plane containing $L(v)$. And we suppose that this function is everywhere differentiable: the derivative is certainly never infinite.

Suppose now that $S(\tilde{w}) = L(\tilde{v}_1) \cup L(\tilde{v}_2) \cup \ldots \cup L(\tilde{v}_q)$. With each $L(\tilde{v}_i)$ we associate a positive or negative sign according as $\left(\frac{dw}{dv}\right)_{v=\tilde{v}_i} \gtrless 0$. At curves of contact we have $\left(\frac{dw}{dv}\right)_{v=\tilde{v}_i} = 0$, and we may say that $L(\tilde{v}_i)$ is neutral in such a case. It is possible for $\left(\frac{dw}{dv}\right)_{v=\tilde{v}_i}$ to vanish without $L(\tilde{v}_i)$ consisting of curves of contact. In such circumstances, however, $\left(\frac{dw}{dv}\right)_{v=\tilde{v}_i+\tilde{\epsilon}}$, $\tilde{\epsilon} \neq 0$ and small as we please, is always definitely positive or negative, and this sign is therefore naturally assigned to $L(\tilde{v}_i)$ itself. Thus in the

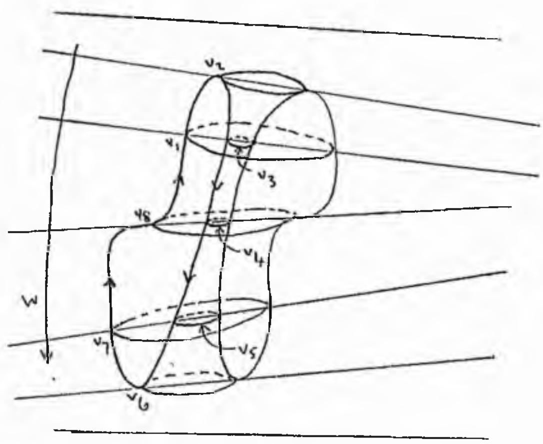


Fig. 14

pair illustrated in Fig. 14, $L(v_2)$, $L(v_6)$ are neutral; $L(v_1)$, $L(v_7)$, and $L(v_8)$ are negative; and $L(v_3)$, $L(v_4)$, $L(v_5)$ are positive.

Let N_+, N_- be the numbers of positive and negative $L(\tilde{v})$ respectively in $P(\tilde{w})$. Then we call the integer $D_{\tilde{w}} = |N_+ - N_-|$ the <u>signature</u> of $P(\tilde{w})$ with respect to S. It will appear shortly that, when $p > 0$, $D_{\tilde{w}}$ is in fact the modulus of the local degree (see H. Hopf (8) for a definition of this term) of a certain map of one circle on another. The following theorem, then, is essentially little else than a statement of the fact that, since this mapping is smooth, the local degree is equal to the global degree. The proof, however, is quite simple, and we include it in full.

THEOREM II: If $p > 0$, then D_w is independent of the value of w.

Proof: Under the hypotheses of Theorem I, a surface of genus $p > 0$ has no nodes, but may have curves of contact and saddle-points. Let us adopt our usual procedure, and start with some plane $P(\tilde{w})$ (which may, or may not, intersect S). By increasing, or decreasing, w, we sweep out the whole of S, and run through the entire family Π, thus allowing for all possible changes in D_w.

It is clear that D changes, if at all, only at values of w for which P(w) contains either curves of contact or saddle-points. But it follows at once from our conditions on v that the occurrence of a saddle-point does not affect the sign of $\frac{dw}{dv}$, and does not alter the number of distinct curves

L(v) in the corresponding P(w). Again, when a curve of contact is met, the number of distinct curves L(v) is either increased of decreased by exactly two, and the associated derivatives are always one positive, one negative, since curves of contact correspond to maxima and minima of the function w(v). Hence $D_w = D_{\tilde{w}}$ for all values of w, and the result is proved.

We may now speak of the signature of the pair (S, Π), and write this D(S, Π). We shall have occasion shortly to re-parametrise both the parametric curves v = const. and the family of planes of given pairs. We observe, therefore, that the signature D(S, Π), provided that the re-parametrisation does not alter the number of L(v)'s in any plane, is also independent of the way in which the parameters v, w are chosen.

Corollary 1 : If there is a value of w for which

$$S(w) = 0, \text{ then } D(S, \Pi) = 0.$$

From this we deduce

Corollary 2 : If D(S, Π) $\neq$ 0, then S(w) $\neq$ 0, for all w.

THEOREM III : If $p > 0$, then $D(S, \Pi)$ is even.

Proof: In view of Corollary 1 above, we may assume that every plane of Π intersects S. Further as we have already pointed out, the occurrence of saddle-points is irrelevant in the calculation of $D(S, \Pi)$. Suppose that $P(\bar{w}) \epsilon \Pi$ contains no curves of contact or saddle-points, and that $S(\bar{w}) = L(\bar{v}_1) \cup L(\bar{v}_2) \cup \ldots \cup L(\bar{v}_q)$. Let $D(S, \Pi) = D_{\bar{w}}$ $= |2\alpha - q|$, where α of the $L(v_r)$ are positive. Let us choose one value ~~of~~ $i \epsilon Q$, where Q denotes the set $(1, 2, \ldots, q)$; then $L(\bar{v}_i + \epsilon)$ lies entirely to one side of $P(\bar{w})$. As we increase v, we may encounter curves of contact or saddle-points, but no nodes. Hence v may be increased indefinitely without the vanishing of $L(v)$. Since S is closed, however, this implies that v repeats, and may be modified if necessary to have period 2π, say. (As in the case of w, we can speak of equivalent values of v, meaning thereby values of v which are congruent mod 2π.) Thus $L(v+2\pi) = L(v)$, and in particular $L(\bar{v}_i + 2\pi) = L(\bar{v}_i)$. Now, since $L(\bar{v}_i)$ is not a curve of contact, $L(\bar{v}_i + \epsilon)$, $L(\bar{v}_i + 2\pi - \epsilon)$, for $\epsilon > 0$ and sufficiently small, lie on opposite sides of $P(\bar{w})$. Hence $q \geqslant 2$, and, for some λ_i, $0 < \lambda_i < 2\pi$, $L(v_i + \lambda_i) = L(v_j)$, $j \epsilon Q$, $j \neq i$, λ_i being chosen

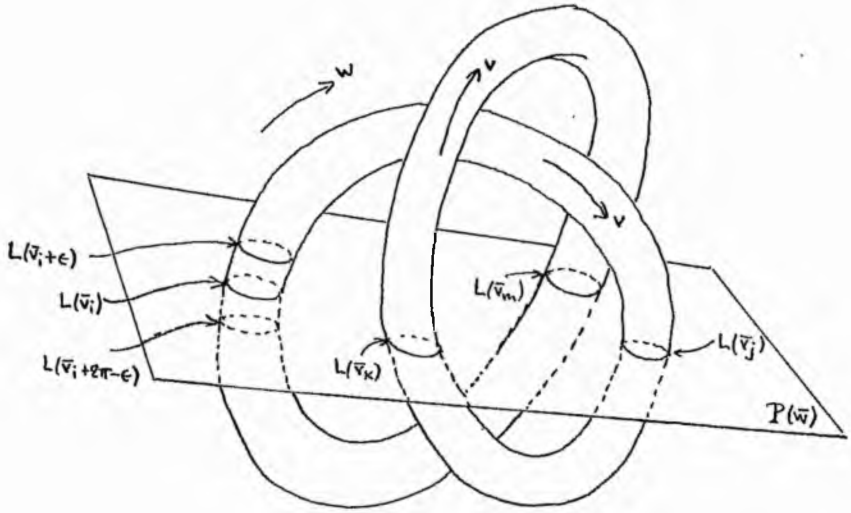


Fig. 15

as the smallest number with this property. On increasing $\bar{v}_j$, we must 'cross' $P(\bar{w})$ again, either in $L(\bar{v}_i)$ itself, in which case $q = 2$, and the theorem is proved, or else in some $L(\bar{v}_k)$, $k \in Q$, $k \neq i, j$. By continuing the process further, $L(\bar{v}_m)$, $L(\bar{v}_n)$ are obtained, where $m \neq i, j, k$. If $n = i$, then $q = 4$ and there is no more to prove. Otherwise we continue in the same way, at each step exhausting exactly two values of the set Q. After a finite number of steps, s, we must return to $L(\bar{v}_i)$, having used up all q values. Thus $q = 2s$, and $D(S, \Pi) = |2\alpha - 2s|$ is even.

For $p = 0$, there are no results analogous to the above theorems: in the process described above, for example, the parameter v runs (with non-null L(v)) through some finite closed interval [a,b], with nodes at v=a, v=b.

2.3 Classification of Pairs

Let (S,Π), (S*,Π*) be two pairs, such that there is a homeomorphism h:S→S* with the property that for all values of v, h|L(v) is a homeomorphism onto some corresponding curve L*(v*) on S*. Then we say that (S,Π) is compatible with (S*,Π*). Compatibility is obviously a proper equivalence relation, and it may be compared with the notion of bundle-map in the theory of fibre-bundles. It is too weak a relation to provide in itself an effective classification of pairs. For example, two pairs (S,Π), (S*,Π*) are always compatible if S, S* are of genus 0, or both of genus 1. If S, S* are both of genus 0, we have only to choose a homeomorphism h:S→S* such that, with the notation of 2.2, the node v=a is mapped onto the node v*=a*, and that given by v=b onto v*=b*; for intermediate values of v, since the L(v) are all simple closed curves, the definition of h may be completed in an obvious way. The compatibility of any two pairs in which the surfaces are of genus 1 is equally obvious. This relation, then, concentrates attention on the form of the L(v), but takes little account of how these are arranged in E_3. Nevertheless, it is natural to require that in our final classification, members of the same class are compatible. The relation which we do in fact propose as the basis of a classification of pairs is defined in the following way:

Let (S_0, Π_0), (S_1, Π_1) be compatible under a map $h: S_0 \rightarrow S_1$. If there is a family of pairs (S_t, Π_t), $t \in I = [0,1]$, compatible with (S_0, Π_0) under homeomorphisms $h_t: S_0 \rightarrow S_t$ which form a homotopy from $i: S_0 \rightarrow S_0$ to $h: S_0 \rightarrow S_1$, then we say that (S_0, Π_0) is <u>similar</u> to (S_1, Π_1). Again it is clear that similarity is a proper equivalence relation. It is intuitively obvious that any two pairs whose surfaces are of genus zero, besides being compatible, are also similar. We shall see below, however, that when the genus is non-zero, similar pairs have the same signature, and there is therefore at least an enumerable infinity of similarity classes in these cases.

We have seen that, for $p > 0$, the parameter v may be made periodic. Now the set of values of v has the topology of the real line, and so, if we denote the congruence class mod v to which v belongs by (v), and require that the natural map $v \rightarrow (v)$ be continuous, the set of congruence classes $\{(v)\}$ is given the topology of a circle. Thus if C is a circle, we may set up a homeomorphism $\phi: \{(v)\} \rightarrow C$ and write $\phi((v)) = c_{(v)}$. Likewise, we may construct a homeomorphism $\psi: \{(w)\} \rightarrow \Gamma$ onto some circumference Γ. We simplify our notation, and write $C = \{c_v\}, \Gamma = \{\gamma_w\}$.

Given a pair (S,Π), then, parametrised in a given way, there is a natural map $f: C \rightarrow \Gamma$ defined by setting $f(c_v) = \gamma_w$ if $L(v) \subset P(w)$ (cf. p.32). Invview of the conditions imposed earlier on v and w, this mapping is <u>smooth</u> at all points γ_w for which P(w) does not contain a curve of contact, in the sense that each points c_v such that $f(c_v) = \gamma_w$ has a neighbourhood in which f is one-one. Further, both C and Γ are given orientations in the obvious way by the directions of increasing v,w. Hence the local degree d_w of f is defined at these smooth points, and is given by $(N_+ - N_-)$ in our notation, so that $D_w = |d_w|$.

Also the local degree is equal to the degree d(f) of the mapping f, and we have $D(S,\Pi) = |d(f)|$. The corollaries to Theorem II are now seen to be paraphrases of the statements that if f is into, then d(f) = 0, while if the local degree at a smooth point is non-zero, and all the points of Γ are smooth (the exceptional points which occur in the case we are considering do not upset the result), then f is onto.

These observations are of use in demonstrating the following result, to which we referred above.

THEOREM IV: If (S_0,Π_0), (S_1,Π_1) are similar pairs, and the genus of S_0 is non-zero, then $D(S_0,\Pi_0) = D(S_1,\Pi_1)$.

Proof: By definition, if (S_0,Π_0) is similar to (S_0,Π_0), then there is a family (S_t,Π_t) of pairs compatible with (S_0,Π_0), with corresponding homeomorphisms $h_t: S_0 \to S_t$ forming a homotopy from $i: S_0 \to S_0$ to $h: S_0 \to S_1$. We suppose that the pair (S_0,Π_0) is parametrised in some way by v,w. Using suffices with the obvious meaning, we have that for each value $\bar{v}$ of v, $h_t(L_0(\bar{v}))$ coincides with some curve L_t of S_t, to which we assign the value

$\tilde{v}$ itself of the parameter. Since h_t is a homeomorphism, v parametrises S_t in accordance with our conditions. For each value of v, let us choose a point $\tilde{c}_0(v) \varepsilon L_0(v)$ in such a way that the totality of such points forms a simple closed curve $\tilde{C}_0$ in E_3. Then the set of points $h_t(\tilde{C}_0) = \tilde{C}_t$, say, is also a simple closed curve in E_3, and there is a 1-1 correspondence between points of $\tilde{C}_t$ and curves $L_t(v)$. Clearly the family of maps $\tilde{h}_t = h_t | \tilde{C}_0$ forms a homotopy from $1: \tilde{C}_0 \to \tilde{C}_0$ to $\tilde{h}_1 : \tilde{C}_0 \to \tilde{C}_1$.

Again, let a point $\tilde{y}_0(w)$ be chosen in each plane $P_0(w)$ in such a way that the totality of these points forms an arc $\tilde{\Gamma}_0$, closed at one end, open at the other, which is parametrised in the natural way by the values of w. Now, from the fact that at least that part of the family Π of any pair (S, Π) which intersects S has the structure of a product $E_2 \times I$ of a plane with a finite interval, or circle we see that we can in fact stipulate in all cases that the families Π_t are parametrised in such a way that a family of such arcs $\tilde{\Gamma}_t$ can be formed, with the property that the family of natural maps

$\tilde{K}_t : \tilde{\Gamma}_0 \to \tilde{\Gamma}_t$ defined by setting $\tilde{K}_t(\tilde{\gamma}_0(w)) = \tilde{\gamma}_t(w)$ forms a homotopy from $1 : \tilde{\Gamma}_0 \to \tilde{\Gamma}_0$ to $\tilde{K}_1 : \tilde{\Gamma}_0 \to \tilde{\Gamma}_1$.

Now for each value of t, there is a natural map $\tilde{F}_t : \tilde{C}_t \to \tilde{\Gamma}_t$ defined as usual by setting $\tilde{F}_t(\tilde{c}_t(v)) = \tilde{\gamma}_t(w)$ if $L_t(v) \subset P_t(w)$. We may use this map to define a map $\tilde{F}^*_t : \tilde{C}_0 \to \tilde{\Gamma}_0$ by setting $\tilde{F}^*_t = \tilde{H}_t \tilde{F}_t \tilde{K}_t^{-1}$. Clearly this family of maps $\tilde{F}^*_t$ constitutes a homotopy from $\tilde{F}^*_t : \tilde{C}_0 \to \tilde{\Gamma}_0$ to $\tilde{F}^*_1 : \tilde{C}_0 \to \tilde{\Gamma}_0$. Since this homotopy induces a similar homotopy of maps $f_t : C \to \Gamma$ from $f_0 : C \to \Gamma$ to $f_1 : C \to \Gamma$, we have $d(f_0) = d(f_1)$, and a fortiori, $D(S_0, \Pi_0) = D(S_1, \Pi_1)$.

***

It seems likely that (if $D(S,\Pi) \leq 2$ in all parametrisations) the equality of the signatures is also a sufficient condition for the similarity of two pairs which are compatible, and for which the surfaces are of genus 1 or 2; but we are not able to give a satisfactory proof of the existence of the requisite homotopy, and so we leave the matter here. For pairs involving surfaces whose genus is greater than 2, however, we can state with certainty that the equality of the signatures is <u>not</u> sufficient to guarantee the similarity of compatible pairs. In Fig.16 and Fig. 17, we indicate the form of two pairs which are compatible, and for both of which $D(S, \Pi) = 0$,

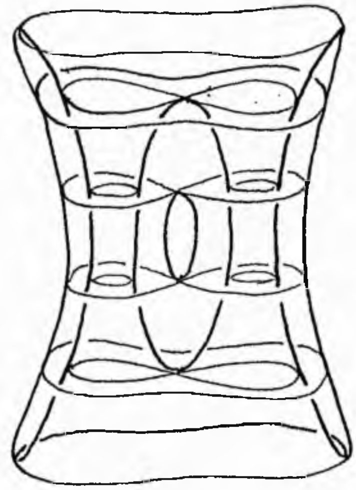


Fig. 16

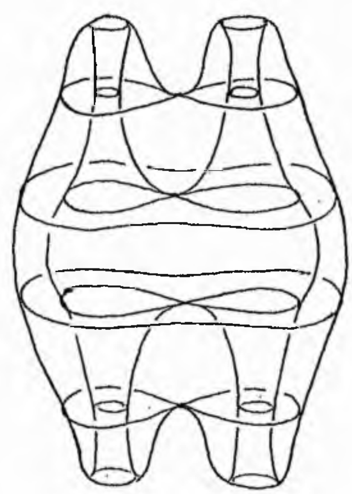


Fig. 17

although it is obvious that no homotopy of the kind required for similarity exists. Each surface is of genus three.

More complicated examples of any given signature and of any genus $p > 2$ can be constructed without difficulty.

The question of defining such models by parametric equations has been avoided so far. In the next chapter, we give explicit examples of the construction of such equations, and from these it is possible then to see how sets of parametric equations yielding surfaces such as those of Figs. 16 and 17 are formed.

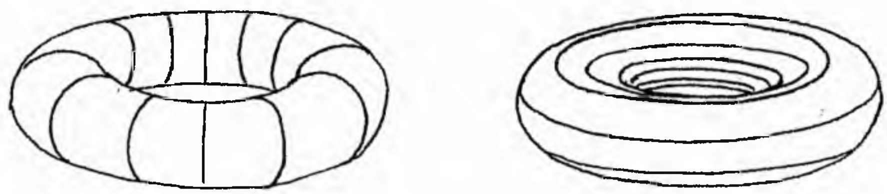
CHAPTER THREE

Construction of Some Simple Models

3.1 Introductory

The models which we shall construct are intended both to illustrate the foregoing theory, and to provide some analogues, for genus $p > 1$, of the standard models of sphere and torus. It is apparent from the results of the previous chapter that the sphere plays an exceptional role in the theory, and so our constructions are derived rather from the anchor-ring model of the torus (Fig. 4). This model is remarkable in that both families of parametric curves are families of plane curves, and in fact each may be associated with a family of planes Π to form a pair (T, Π), where T denotes the subspace of genus one itself. If Π_1, Π_2 denote the families of planes associated with the curves u = const., v = const. respectively, then, as is clear from Fig. 18, $D(T, \Pi_1) = 2$, $D(T, \Pi_2) = 0$. Thus (T, Π_1), (T, Π_2) are not similar, although they are of course compatible.

The intersections $T(w)$ are circles both for Π_1 and Π_2. The essential step in proceeding to cases of higher genus is just to replace these circles by a slightly more general type of curve.



$D(T, \Pi_1) = 2$ $\qquad$ $D(T, \Pi_2) = 0$

Fig. 18

3.2 Parametrisation of a Two-sided Disk

Let us identify the Euclidean plane E_2 with the unextended plane Z of complex numbers $x + iy = \zeta$. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be p distinct points of Z, $p \geq 2$, and consider the locus L_ϕ of points such that $|(\zeta - \alpha_1)(\zeta - \alpha_2)\ldots(\zeta - \alpha_p)| = \phi$. As ϕ varies, we obtain a family of such subsets of E_2, having the property that its members are not all mutually homeomorphic. L_0, of course, consists only of the points α_i themselves, and we consider only non-zero values of ϕ. For ϕ sufficiently small, L_ϕ consists of p Jordan curves, each enclosing one of the α_i, and mutually exterior. For ϕ sufficiently large, L_ϕ is a single Jordan curve which encloses all the α_i and approximates to the form of a circle as ϕ is increased. For some intermediate values of ϕ, L_ϕ takes the form of a closed curve with a number of loops, the values of ϕ in question being those for which L_ϕ passes through a critical point of the function $(\zeta - \alpha_1)(\zeta - \alpha_2)\ldots(\zeta - \alpha_p) = R(\zeta)$: that is, through a zero of $R'(\zeta)$. The number of critical points is therefore at most p-1.

The loci L_ϕ are known as lemniscates (see, for example, Walsh (9)). We have laid down that $p \geqslant 2$, but the curves are defined for p=1, and are of course circles in this case. The points α_i are called the poles of the locus.

Suppose now that $\alpha_m = \varrho_1\omega^m$, where $\varrho_1 > 0$, and $\omega = e^{2\pi i/p}$, so that $P(\zeta) = \zeta^p - \varrho_1{}^p$. The poles α_m are therefore situated at the vertices of a regular polygon of p sides, centre the origin, and every locus L_ϕ has the symmetry of this polygon. Fig. 19 indicates the three types of locus for the case p=3.

The family which we are considering is given by the equation $|\zeta^p - \varrho_1^p| = \phi$, where ϕ may assume any positive value. But for our purposes, it is necessary to consider only a finite range of values of ϕ, which must be chosen so as to include curves of all three types.

The curve which passes through the origin has p loops, and is given by $\phi = \rho_1^p$. L_ϕ, $\phi < \rho_1^p$, has p components, and for $\phi > \rho_1^p$ it is a simple Jordan curve. We make ϕ obey an inequality of the form $A \leq \phi \leq B$, therefore, where $0 < A < \rho_1^p < B$.

Setting $\rho_2 = \frac{1}{2}(B+A)$, $\rho_3 = \frac{1}{2}(B-A)$, we have

$$0 < \rho_2 - \rho_3 < \rho_1^p < \rho_2 + \rho_3 \quad ,$$
(1)
$$\rho_2 - \rho_3 \leq \phi \leq \rho_2 + \rho_3 \quad ,$$

where ρ_3, ρ_2 are both positive.

It is clear, then, that we may set $\phi = \rho_2 + \rho_3 \cos v$, where v is a new variable parameter, which obviously must have period 2π. Further, if we replace the symbol L_ϕ by the symbol $L(v)$, which will denote the locus $|\zeta^p - \rho_1^p| = \rho_2 + \rho_3 \cos v$, then we have $L(v) = L(v+2\pi)$, $L(v) = L(2\pi - v)$.

Referring now to 1.2, p.6, we make the following identifications: $J_0 = L(0)$, $I = L(\pi)$. The disjunction of points mentioned there is now brought about by distinguishing between corresponding points of $L(v)$ and $L(2\pi - v)$. Thus we parametrise a circle $C = \{c_v\}$ by values of v, and set $\Delta_p = \bigcup_v [L(v), c_v]$.

where Δ_p is topologised as a subspace of $E_2 \times C$. It is evident that Δ_p is a p-holed, two-sided disk.

In order to parametrise Δ_p, we require some information about the orthogonal trajectories of the L(v). Since the latter are loci of constant modulus, their orthogonal trajectories are loci of constant argument. But here we must be careful to avoid ambiguities, for the argument of a function is of course a multi-valued quantity. We proceed as follows.

The set of points ζ on the real axis such that $\zeta > \alpha_1$ is certainly part of the locus $\arg(\zeta^p - \varsigma_1^p) = 0$. We make the convention that this set of points, and no others, composes the entire locus, which we denote by M_0. For ψ small enough and positive, we say that the set of points ζ such that $\arg(\zeta^p - \varsigma_1^p) = \psi$ constituting a 'ray' issuing from α_1 and making asymptotic angle ψ/p with the positive direction of the realaxis is the locus $\arg(\zeta^p - \varsigma_1^p) = \psi$, denoted by M_ψ. Continuing in this way, for increasing ψ, we meet no difficulty until we approach $\psi = \pi$. We then say that the locus $\arg(\zeta^p - \varsigma_1^p) = \pi$ consists of the real axis $0 \leq \zeta < \alpha_1$.

together with the radial line issuing from the origin at an angle π/p to the real axis. And for $\psi = \pi + \epsilon$, $\epsilon > 0$ and as small as we please, we say that the locus $M_{\pi+\epsilon}$ consists of the ray of $\arg(\zeta^p - \frac{p}{1})$ $= \pi + \epsilon$ issuing from α_2 and making asymptotic angle $\pi + \epsilon/p$ with the positive direction of the real axis. We continue in this way for increasing ψ until $\psi = 2\pi$, when similar conventions are adopted, and so on.

In this way, we define a family of subsets M_ψ, with $M_\psi = M_{\psi+2p\pi}$, filling the whole ζ-plane. Every point which is neither the origin nor one of the poles belongs to a unique M_ψ. We set $\psi = pu$, and write $M(u) = M_{\psi/p}$. In particular, every point of $L(v)$, except for the origin on $L(\cos^{-1}\frac{\rho_1-\rho_2}{\rho_3})$, belongs to a unique $M(u)$. Hence, with the exception of the two points $[0, c_{\tilde{v}}]$, where $\tilde{v}$ denotes either of the values (or rather a representative of either of the congruence classes mod 2π) of $\cos^{-1}\frac{\rho_1-\rho_2}{\rho_3}$, every point of Δ_p is associated with a unique pair of loci $L(v)$, $M(u)$. We therefore say that the disk Δ_p is parametrised by u, v, with two singularities which are obviously both of index $1-p$. The parameter u, it is true, does not parametrise the family $\{M(u)\}$ in a continuous fashion, there being discontinuities along certain slits in the u-v plane. This we regard as having no great importance for us, however, since we are interested primarily in the geometrical form of the curves L(v), M(u) as point sets.

A point $[\zeta, c_v] \varepsilon \Delta_p$ which lies on L(v) and M(u) is given by the equations

$$x = r \cos \theta ,$$
(2)
$$y = r \sin \theta ,$$

where

$$\zeta = x+iy, \quad r = (A^2+B^2)^{1/2p} , \quad \theta = \frac{1}{p} (\xi + 2m\pi),$$

(3) $$A = \rho_1^p + (\rho_2 + \rho_3 \cos v)\cos pu, \quad B = (\rho_2 + \rho_3 \cos v)\sin p$$

$$\cos \xi = A/r^p , \quad \sin \xi = B/r^p , \quad 2m\pi/p \leq u < 2(m+1)\pi/p.$$

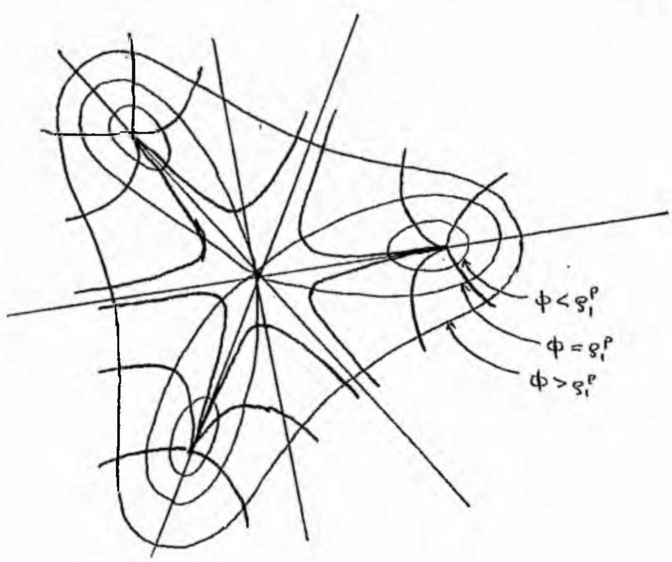


Fig. 19

The form of the curves L(v), M(u) is indicated in Fig.19, for p=3.

3.3 Parametric Models Derived from Δ_p

The final step in our construction is to find a suitable homeomorphism of Δ_p into E_3. One map suggests itself immediately. For equations (2) give us two co-ords. out of three as functions of u,v. It is obvious, too, that a natural choice for the third co-ord. z is $z = \rho_3 \sin v$. We then have the equations :

$$x = r \cos \theta,$$
$$(4) \qquad y = r \sin \theta,$$
$$z = \rho_3 \sin v,$$

which define a surface $S \subset E_3$. If we denote by Π the family of planes P(w), where P(w) is the plane z = w, then (S, Π) is a pair associated with the family of curves L(v) (which we regard now, without change of notation, as curves on S), such that $D(S, \Pi) = 0$. There are two saddle-points, each of index 1-p. We shall see shortly, too, that the families of parametric curves are orthogonal, so that we have in fact constructed a parametric model of genus p satisfying our conditions for a canonical form. The symmetry of this type of model is suggested in Fig. 20 (p=3). The two singular points have co-ords. $0, 0, \pm\sqrt{(\rho_3^2 - (\rho_1 - \rho_2)^2)}$.

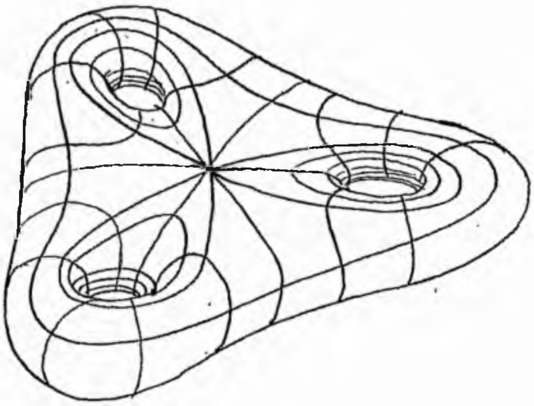


Fig. 20

We notice that if S(w) contains L(v), then it also contains L(π-v), and of these, one is enclosed by the other in the obvious sense. We may, however, envisage a situation in which L(v), L(π-v) still lie in the same plane P(w), but L(v) and L(π-v) are mutually exterior. This idea yields a second model of genus p, which again satisfies our conditions for a canonical form.

Let $\rho_4 > (\rho_1+\rho_2+\rho_3)^{1/p} > 0$. A homeomorphism from Δ_p into E_3 is then defined by the parametric equations

$$(5)\qquad \begin{aligned} x &= (\rho_4 + r\cos\theta)\cos v, \\ y &= (\rho_4 + r\cos\theta)\sin v, \\ z &= \quad r\sin\theta, \end{aligned}$$

and determines a model having the form shown for p=3 in Fig. 21. The two saddle-points now have co-ords. $\left(\frac{\rho_4}{\rho_3}(\rho_1-\rho_2),\ \pm\frac{\rho_4}{\rho_3}\sqrt{\{\rho_3^2-(\rho_1-\rho_2)^2\}}\right.$,0. The family Π of planes P(w) associated with this model is the pencil of planes through the z-axis, P(w) being given by the equation x sin w = y cos w. We see from the diagram that $D(S,\Pi) = 2$. Hence, like those of the torus in Fig. 18, the two models which we have now constructed are obviously compatible (for given p), but they are not similar.

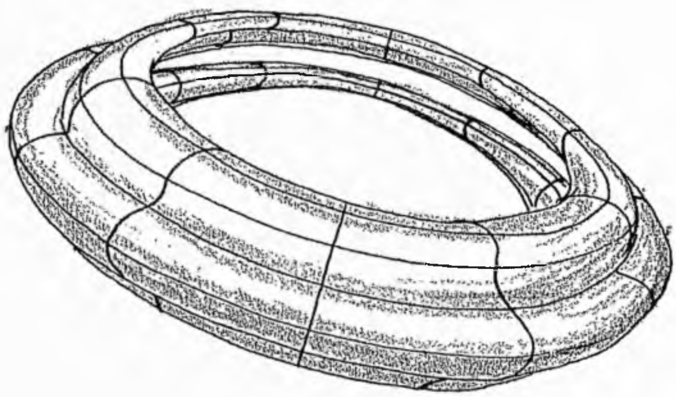


Fig. 21

On p.15 we showed that parametric model of genus p, and of type (2n,-k) is one of $(\alpha_1+1)(\alpha_2+1)\ldots(\alpha_q+1)$ possible types, where $p_1^{\alpha_1} p_2^{\alpha_2}\ldots p_q^{\alpha_q}$ is the prime-power exprssion of p-1. Here we observe that the disk Δ_p may be made to yield simple specimens of each type. Thus in order to form a model of type (2n,-k) we modify the equations (5) derived from a disk Δ_k, by the simple device of replacing v by v/n at two places. The resulting equations

(6)
$$x = (\varrho_4 + r\cos\theta)\cos v/n,$$
$$y = (\varrho_4 + r\cos\theta)\sin v/n,$$
$$z = r\sin\theta,$$

determine an n-fold covering surface of Δ_k, of genus $p = nk + 1$. The associated family of planes is again that for which $F(w)$ has equation $x \sin w = y \cos w$, and we have $D(S, \Pi) = 2$. For $n = 1$, we obtain the model of equations (5). Since models of the same type are not necessarily compatible, however, we have not constructed a representative of each compatibility class.

In order to decide whether the two families of parametric curves of all our models are orthogonal, we have only to calculate the second coefficient of the second differential form $Edu^2 + 2Fdudv + Gdv^2$ of each model. We find in fact that $F = 0$ in both cases, so that the families are orthogonal. The forms themselves are as follows:

$$ds^2 = r^{\chi}\left[\left(\frac{\varrho_3^2 \sin^2 v}{p^2} + \varrho_3^2\cos^2 v\right)dv^2 + (\varrho_2 + \varrho_3 \cos v)^2 du^2\right]$$

for the model with equations (4), and

$$ds^2 = r^{\chi}\left[\left(\frac{\rho_3^2 \sin^2 v}{k^2} + \frac{(\rho_4 + r\cos\theta)^2}{n^2}\right)dv^2 + (\rho_2 + \rho_3 \cos v)^2 du^2\right],$$

for those with equations (6), denoting the Euler Characteristic of the disks Δ_p, Δ_k in each case respectively.

REFERENCES

(The relevant portions of books and papers are indicated on the right in parentheses.)

(1) S. Lefschetz : (p.82)
Introduction to Topology ; Princeton, 1949.

(2) B. von Kerekjarto :
Vorlesungen uber Topologie, I; Springer, 1923

(3) H. Hopf : (p.316)
Beiträge über Homotopietheorie ; Comm. Math. Helv., 17, 1944-45, 307-331.

(4) P. Alexandroff, H. Hopf : (Ch.14)
Topologie, I ; Springer, 1935.

(5) H. Poincaré : (Ch.XIII)
Sur les Courbes Définies par une Équation Différentielle ; Journ. de Math.,(4),1, 1885, 167-244.

(6) H. Poincaré :
Cinquième Complément a L'Analysis Situs ; Rend.Circ.Mat.Palermo, 18, 1904, 45-110.

(7) M.H.A. Newman : (pp.153-4)
Topology of Plane Sets of Points ; 2nd.ed., Cambridge, 1951.

(8) H. Hopf : (pp.120-21)
Über den Rang geschlossener Liescher Gruppen ; Comm. Math. Helv., 13, 1941, 119-143.

(9) J.L. Walsh : (pp.18-21)
Location of Critical Points ; Amer. Math. Soc. Colloquium Publications, XXXIV.