SOME CONSEQUENCES OF SYMMETRY IN STRONG STIELTJES DISTRIBUTIONS

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A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1998

Some consequences of symmetry in strong Stieltjes distributions

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A thesis submitted for the degree of Doctor of Philosophy

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Scotland, U.K.

15th December 1998

to my parents

and

to my sisters

Acknowledgements

I would like to begin by expressing my sincere thanks to my supervisor, Dr. John H. McCabe, for his invaluable guidance, support and assistance over the past three years. My sincere thanks also to my external co-supervisor, Dr. Alagacone Sri Ranga, for his considerable incentive and enthusiasm. Without their knowledge, encouragement and patience this thesis would not have been completed.

Special thanks go to my former supervisor Dr. Odelar L. Linhares and to my colleague and friend Dr. Eliana X.L. de Andrade, whose constant encouragement and confidence in my work have helped me pursue my 'mathematical life'. Thanks to Dr. George M. Phillips and Mrs. Rona Phillips for making me feel at home in Scotland.

I would like to thank my family, whose love and support have helped me not only over the last three years, but throughout my life. Thanks also to all my friends in St. Andrews for the unforgettable dinner parties.

I am deeply grateful to the State University of São Paulo (UNESP) and my colleagues at the Department of Computational Sciences and Statistics (DCCE), in particular the Numerical Analysis group, for granting me leave to undertake my Ph.D. program.

Finally, the financial support for this research from the National Council for Scientific and Technological Development of Brazil (CNPq) is gratefully acknowledged.

Abstract

The main purpose of this work is to study a class of strong Stieltjes distributions $\psi(t)$, defined on an interval $(a, b) \subseteq (0, \infty)$, where $0 < \beta < b \leq \infty$ and $a = \beta^2/b$, which satisfy the symmetric property

$$\frac{d\psi(t)}{t^\omega} = -\frac{d\psi(\beta^2/t)}{(\beta^2/t)^\omega}, \quad t \in (a, b), \quad 2\omega \in \boldsymbol{Z}.$$

We investigate the consequences of this symmetric property on the orthogonal L-polynomials related to distributions $\psi(t)$, and which are the denominators of the two-point Padé approximants for the power series that arise in the moment problem. We examine relations involving the coefficients of the continued fractions that correspond to these power series. We also study the consequences of the symmetry on the associated quadrature formulae.

Declarations

I, Cleonice Fátima Bracciali, hereby certify that this thesis, which is approximately 30,000 words in length, has been written by me, that it is the record of work carried out by me, and that it has not been submitted in any previous application for a higher degree.

C.F. Bracciali Date 15th December 1998.

I was admitted as a research student in January 1996 and as a candidate for the degree of Doctor of Philosophy in January 1997; the higher study for which this is a record was carried in the University of St. Andrews between 1996 and 1998.

C.F. Bracciali Date 15th December 1998

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St. Andrews, and that the candidate is qualified to submit this thesis in application for that degree.

Signed on behalf of Dr. John H. McCabe (supervisor) and Dr. A. Sri Ranga (external co-supervisor) by the former

Date 15" December 1998

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Contents

Overview

This thesis describes an investigation into the strong Stieltjes distributions that satisfy certain symmetric properties. We organized this work in 5 chapters as follows.

In Chapter 1 we give a historic summary of the topics related to this thesis, such as, continued fractions, moment problems, Padé approximants, orthogonal polynomials and quadrature formulae.

Chapter 2 contains a study of the orthogonal L-polynomials, which are the denominator polynomials of the two-point Padé table for the power series that arise in the moment problem. It also includes a study of the continued fractions related to strong Stieltjes distributions, and of the orthogonal Laurent polynomials.

In Chapter 3 we present a symmetric property that is satisfied by some strong Stieltjes distributions and we give some relations satisfied by the orthogonal L-polynomials that are consequences of this symmetry. We construct extensions of the continued fractions related to these distributions. Some examples are also given.

Further in Chapter 4 we deal with some polynomials derived from the orthogonal L-polynomials and the consequences of the symmetric property.

Finally in Chapter 5 we consider the associated polynomials and the quadrature formulae related to these symmetric strong Stieltjes distributions.

Chapter 1

Introduction

1.1 History

The mathematical topics orthogonal polynomials and moment problems are two of many that arose in the analytic theory of continued fractions, but then developed as subjects in their own right. These topics, and others, can now be profitably studied and developed without any reference to the theory of continued fractions. However there is great value, both in terms of progress and mathematical beauty, of developing results in the combined areas of these three particular topics. Quadrature formulae, a related topic which is a natural consequence of studying orthogonal polynomials, is frequently included in this connected development. Many authors have taken this approach, and continue to do so. This is the method adopted for the work that is described in this thesis.

Continued fractions

A *continued fraction* is an finite or infinite expansion of the form

$$q_0 + \cfrac{p_1}{q_1 + \cfrac{p_2}{q_2 + \cfrac{p_3}{q_3 + \cfrac{p_4}{q_4 + \cdots}}}} \quad , \tag{1.1.1}$$

where q_n and p_n are real or complex numbers, real or complex variables or functions of real or complex variables. The continued fraction is usually written as

$$q_0 + \frac{p_1}{q_1} \, \frac{p_2}{+ \, q_2} \, \frac{p_3}{+ \, q_3} \, \frac{p_4}{+ \, q_4} + \cdots ,$$

or, alternatively, as

$$q_0 + \mathrm{K}_{n=1}^{\infty}(p_n/q_n) \qquad \text{or} \qquad q_0 + \mathrm{K}(p_n/q_n) \, .$$

The finite continued fraction

$$\frac{P_n}{Q_n} = q_0 + \frac{p_1}{q_1} \, \frac{p_2}{+ \, q_2} \, \frac{p_3}{+ \, q_3} \, \frac{p_4}{+ \, q_4} + \cdots \frac{p_n}{+ \, q_n}, \quad n = 0, 1, 2, \ldots , \tag{1.1.2}$$

obtained by truncation of (1.1.1), is called the *nth convergent* or *nth approximant* of the continued fraction (1.1.1). The limit of the sequence $\left\{ \frac{P_n}{Q_n} \right\}$, $n = 0, 1, 2, \ldots$, when it exists, is the value of the continued fraction.

As Brezinski in [9] pointed out, continued fractions were used implicitly for many centuries before their real discovery, but since their discovery they have played a leading role in the development of many branches of mathematics. These branches are mainly, but not exclusively, in number theory and in analysis. In the former, the partial quotients of the continued fractions, $\frac{p_n}{q_n}$, are rational numbers or rational functions. Many of the applications are based on the *regular* continued fraction expansions of real numbers, in which all of the partial numerators p_n are unity and q_n are real numbers. These regular continued fractions are the extensions to irrational numbers of the finite expansions for rationals that are yielded as a by-product of Euclid's algorithm for finding the greatest common divisor of two integers. One of the beauties of such expansions is that no convergence theory is necessary. One of their many properties is their ability to provide best rational approximations to irrational numbers. Some examples of regular continued fractions are

1. The golden ratio,
$$\frac{1+\sqrt{5}}{2} = 1+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$$

2. The number e,
$$e = 2+\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{8+}\cdots$$

3. The number π,
$$\pi = 3+\frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{14+}\cdots$$

We can see that the first convergent of the continued fraction for π is $\frac{22}{7}$, and the third convergent is $\frac{355}{113}$ which is the best rational approximation to π with denominator less than 33102. That is $|113\pi - 355| < |q\pi - p|$ for $q < 33102$. Although e and π are transcendental numbers, the coefficients in the continued fraction for the number e follow a pattern, but this does not happen in the continued fraction for the number π.

An excellent text on the arithmetical and metrical properties of regular continued fractions is the classic work of Khintchine [28], which is the starting point for the more recent book by Rocket and Szüsz [40]. The comprehensive and classic text by Perron [39] treats both the arithmetic and analytic properties of many kinds of continued fractions.

It was in the work of Euler that it first became clear that continued fractions also have a major role in analysis. The main concern is now the expansions and convergence theory, of infinite continued fractions whose partial numerators and denominators are polynomials of a real or complex variable. Hence the finite continued fractions obtained by truncation are rational functions, which can possibly approximate the function being expanded. The importance of continued fractions in nineteenth century analysis is indicated by the long list of major analysts who contributed to the development of the subject. These include mathematicians such as Laplace, Legendre, Jacobi, Laguerre, Riemann, Stieltjes and Gauss. The latter also applied continued fractions to number theory.

In addition to Perron's book mentioned above the analytic theory of continued fractions is very well covered by three excellent texts in particular. These are the classic book by Wall [54], the later work by Jones and Thron [25], and the very recent text by Lorentzen and Waadeland [31].

The continued fractions that appear in this thesis are all particular cases of those that correspond to power series expansions, and the form and properties of such continued fractions are described below. First, a few standard results that apply to all continued fractions are presented.

In number theory the convergents are generally rational numbers, while in this thesis they will be rational functions. The numerators P_n and denominators Q_n satisfy, respectively, the three-term recurrence relations

$$\begin{aligned} P_{n+1} &= q_{n+1}P_n + p_{n+1}P_{n-1}, \\ Q_{n+1} &= q_{n+1}Q_n + p_{n+1}Q_{n-1}, \end{aligned} \tag{1.1.3}$$

for $n = 1, 2, 3, ...$ with $P_0 = q_0$, $Q_0 = 1$ and $P_1 = q_0q_1 + p_1$, $Q_1 = q_1$.

In addition, the numerators and denominators satisfy the *determinant* formula

$$P_{n+1}Q_n - P_nQ_{n+1} = (-1)^n p_1p_2\cdots p_{n+1}, \tag{1.1.4}$$

while the convergents themselves satisfy

$$\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n} = \frac{(-1)^n p_1 p_2 \cdots p_{n+1}}{Q_{n+1} Q_n}\,, \tag{1.1.5}$$

and

$$\frac{P_{n+1}}{Q_{n+1}} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n+1} p_1 p_2 \cdots p_n q_{n+1}}{Q_{n+1} Q_{n-1}}\,. \tag{1.1.6}$$

The continued fractions that appear in this thesis are those that correspond to a series expansion of the form

$$c_{-1} + c_{-2}z + c_{-3}z^2 + c_{-4}z^3 + \cdots\,, \tag{1.1.7}$$

or to one of the form

$$\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \frac{c_3}{z^4} + \cdots\,, \tag{1.1.8}$$

or to a series of each form simultaneously.

The partial numerators of a continued fraction of the form

$$c_{-1} + \frac{a_1 z}{1} \, {}_{+} \, \frac{a_2 z}{1} \, {}_{+} \, \frac{a_3 z}{1} \, {}_{+} \, \frac{a_4 z}{1} \, {}_{+} \cdots \tag{1.1.9}$$

can, under certain conditions, be chosen so that the nth convergent corresponds to the series (1.1.7) when expanded as a power series in z. The nth convergent is a ratio of polynomials of degree r and s respectively, where $r = [\![\frac{n+1}{2}]\!]$ and $s = [\![\frac{n}{2}]\!]$.

For example the continued fraction corresponding to the exponential series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

is

$$1 + \frac{z}{1} \, {}_{-} \, \frac{z/2}{1} \, {}_{+} \, \frac{z/6}{1} \, {}_{-} \, \frac{z/2}{1} \, {}_{+} \, \frac{z/10}{1} \, {}_{-} \, \frac{z/2}{1} \, {}_{+} \, \frac{z/14}{1} \, {}_{-} \, \frac{z/2}{1} \, {}_{+} \cdots.$$

Continued fractions of the above form, and with the above correspondence properties, are known as *regular C-fractions.*

Similarly the partial numerators of a continued fraction of the form

$$\frac{a_1}{z} + \frac{a_2}{1} + \frac{a_3}{z} + \frac{a_4}{1} + \frac{a_5}{z} + \frac{a_6}{1} + \cdots \tag{1.1.10}$$

can, under certain conditions, be chosen so that the nth convergent corresponds to the series (1.1.8) when expanded as a series in inverse powers of z. The nth convergent is also a ratio of polynomials of degree r and s respectively, where $r = [\![\frac{n-1}{2}]\!]$ and $s = [\![\frac{n+1}{2}]\!]$.

If c_{-1} is zero in (1.1.7) then clearly a series of the form (1.1.8) can be obtained from (1.1.7) by replacing z by $1/z$. The corresponding fraction (1.1.9) becomes, under the same substitution and after simple manipulations, of the form (1.1.10).

The even contraction of (1.1.10), that is the continued fraction whose successive convergents are the even order convergents of (1.1.10), is of the form

$$\frac{a_1}{z + a_2} + \frac{a_2 a_3}{z + a_3 + a_4} + \frac{a_4 a_5}{z + a_5 + a_6} + \frac{a_6 a_7}{z + a_7 + a_8} + \cdots$$

or, in simpler form,

$$\frac{\lambda_1}{z + \gamma_1} + \frac{\lambda_2}{z + \gamma_2} + \frac{\lambda_3}{z + \gamma_3} + \frac{\lambda_4}{z + \gamma_4} + \cdots.$$

In certain circumstances this continued fraction is a *J-fraction*, and it was in the study of these fractions that orthogonal polynomials first arose. For an arbitrary J-fraction, with all $\lambda_k \neq 0$, the denominators of the convergents form a sequence of orthogonal polynomials. For example, the denominators of the convergents of the continued fraction

$$\frac{1}{z-1} - \frac{1}{z-3} - \frac{4}{z-5} - \frac{9}{z-7} - \frac{16}{z-9} - \cdots$$

are the classical Laguerre polynomials, in their monic form.

A third type of corresponding continued fraction, and one that is central to a study of the class of moment problems known as strong moment problems, which we introduce in the next subsection is of the form

$$\frac{c_0}{1+d_1 z} \; + \; \frac{n_2 z}{1+d_2 z} \; + \; \frac{n_3 z}{1+d_3 z} \; + \; \frac{n_4 z}{1+d_4 z} \; + \cdots .$$

This continued fraction can be written in the equivalent form

$$\frac{c_0}{z-\beta_1} \; - \; \frac{\alpha_2 z}{z-\beta_2} \; - \; \frac{\alpha_3 z}{z-\beta_3} \; - \; \frac{\alpha_4 z}{z-\beta_4} \; - \cdots .$$

In certain circumstances the nth convergent of this continued fraction, for $n = 1, 2, 3...$, is a ratio of polynomials of degree $n-1$ and n respectively and the continued fraction corresponds to the two series (1.1.7) and (1.1.8) simultaneously. That is, when the nth convergent is expanded in powers of z and in inverse powers of z, the expansions will agree with n terms of (1.1.7) and (1.1.8) respectively. These fractions are called *M-fractions*. They were studied by Murphy in [37], by McCabe in [32, 33] and by McCabe and Murphy in [36]. Independently, Jones and Thron [24] and Thron [53] introduced the general *T-fractions*, equivalent to the M-fractions. The general T-fractions are of the form

$$\frac{f_1 z}{1+g_1 z} \; + \; \frac{f_2 z}{1+g_2 z} \; + \; \frac{f_3 z}{1+g_3 z} \; + \; \frac{f_4 z}{1+g_4 z} \; + \cdots .$$

For example, the continued fraction

$$\frac{1}{z+1} - \frac{2z}{z+3} - \frac{4z}{z+5} - \frac{6z}{z+7} - \frac{8z}{z+9} - \cdots$$

corresponds to the two series

$$1 - \frac{1}{3}z + \frac{1}{15}z^2 - \frac{1}{105}z^3 + \frac{1}{945}z^4 - \cdots$$

and

$$\frac{1}{z} + \frac{1}{z^2} + \frac{3}{z^3} + \frac{15}{z^4} + \frac{105}{z^5} + \frac{945}{z^6} + \cdots .$$

These series where studied by McCabe in [32, 33]. They are related to, respectively, the Maclaurin series and an asymptotic expansion for Dawson's integral

$$F(z) = e^{-z^2} \int_0^z e^{t^2} dt .$$

Further details of all of these corresponding fractions, their properties and their related orthogonal polynomials, will be given in the following chapters of this thesis. An algorithm for transforming the series expansions into a corresponding continued fraction is also described.

Moment problems

A recent and excellent account of the history of moment problems is given by Kjeldsen [29]. He describes how the concept of a moment problem first arose in the work of Stieltjes, which led to the integral that now bears his name, and how the moment problems then became a topic in their own right. The texts by Shohat and Tamarkin [41] and by Akhiezer [1] are classic sources on moment problems.

In its simplest terms a moment problem is related to the existence or non existence of a distribution $\psi(t)$, a bounded real valued function that is defined on $(a, b) \subseteq \mathbb{R}$ and with infinitely many points of increase and for which the moments

$$\int_a^b t^n d\psi(t), \quad n = 0, 1, 2, ..., \tag{1.1.11}$$

all exist. Then, given a sequence $\{\mu_n\}_{n=0}^{\infty}$ of real numbers, is there a unique distribution $\psi(t)$ such that

$$\mu_n = \int_a^b t^n d\psi(t),$$

for $n = 0, 1, 2, ...$?

A moment problem is said to be determinate when such a unique distribution exists. There are many variations of moment problems, depending on the interval (a, b). In recent years extensions to doubly infinite sequences of real numbers have resulted in further moment problems. In all of them, as suggested above, there are two questions to be answered, namely existence, or solvability, and uniqueness, or determinability. There are different ways to answer these questions. For example those based on continued fractions and those based on Hankel determinants.

Three particular cases of the above general moment problem have come to be called the *classical* moment problems, though strictly the term describes a much wider class. These are

(i) the *Stieltjes* moment problem, where the integration is over $(0, \infty)$,

(ii) the *Hamburger* moment problem, where the integration is over $(-\infty, \infty)$,

(iii) the *Hausdorff* moment problem, where the integration is over $(0, 1)$.

Stieltjes, in his memoir of 1894-95, introduced the concept of the moment problem and solved the moment problem which was named after him. Earlier Chebyshev, in 1874, also dealt with moments. Kjeldsen in [29] describes how Chebyshev was interested in the following problem. Given the values of the integrals

$$\int_A^B f(x)dx, \quad \int_A^B xf(x)dx, \; \ldots, \; \int_A^B x^m f(x)dx,$$

can upper or lower bounds for the value of $\int_a^b f(x)dx$ be found, where $f(x)$ is an unknown function and $A < a < b < B$? Chebyshev worked only with a finite number of moments.

Stieltjes was able to solve the moment problem on the positive real axis by making extensive use of continued fractions. In particular he used the continued fractions

$$\frac{1}{a_1 z} \; + \; \frac{1}{a_2} \; + \; \frac{1}{a_3 z} \; + \; \frac{1}{a_4} \; + \; \cdots,$$

associated with the integral

$$\int_0^\infty \frac{d\psi(t)}{z+t} \quad \sim \quad \frac{\mu_0}{z} - \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} - \frac{\mu_3}{z^4} + \cdots .$$

He showed that all of the parameters a_i's are positive, and that this is a necessary and sufficient condition for the existence of a solution of the Stieltjes moment problem. This is equivalent to the positivity of the Hankel determinants

$$\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{n+2} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{vmatrix} \quad n = 0, 1, 2, \ldots .$$

Hamburger, in 1920, in solving the moment problem on the whole real axis, showed that it was not just a trivial extension of Stieltjes work. Later, with the work of Hausdorff, Nevanlinna and Riesz, moment problems then became an important topic in their own right, independent of continued fractions.

Recent variations of these problems are the *strong* moment problems. In these the sequence $\{\mu_n\}_{n=0}^{\infty}$ is replaced by the doubly infinite sequence $\{\mu_n\}_{n=-\infty}^{\infty}$ of real numbers. Given such a sequence $\{\mu_n\}_{n=-\infty}^{\infty}$ of real numbers, find a unique distribution $\psi(t)$ such that the elements μ_n are the moments of this distribution, that is,

$$\mu_n = \int t^n d\psi(t), \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots .$$

These strong problems are also distinguished by the interval of integration, and they are

(i) the *strong Stieltjes* moment problem, where the integration is over $(0, \infty)$,
(ii) the *strong Hamburger* moment problem, where the integration is over $(-\infty,\infty)$,
(iii) the *strong Hausdorff* moment problem, where the integration is over $(0, 1)$.

The same questions about existence and uniqueness arise in connection with these problems. The necessary and sufficient conditions are given in terms of Hankel determinants involving the moments $\{\mu_n\}_{n=-\infty}^{\infty}$. Jones, Thron and Waadeland [27] proposed and solved the strong Stieltjes moment problem, while Jones, Njåstad and Thron [20] solved the strong Hamburger moment problem.

Another variation of this problem is the *trigonometric moment problem.* This is, given a sequence of numbers $\{\mu_n\}_{n=-\infty}^{\infty}$, find a distribution function $\psi(\theta)$, where $-\pi \leq \theta \leq \pi$, such that

$$\mu_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} d\psi(\theta), \quad \text{for} \quad n = 0, \pm 1, \pm 2, \dots . \tag{1.1.12}$$

This problem was first considered by Akhiezer and Krein in 1934. The existence and uniqueness conditions can be found in Akhiezer [1].

Orthogonal polynomials

As we stated earlier, the theory of orthogonal polynomials originated from the analysis of certain types of continued fractions associated with moment problems. One of the possible starting points of the study of orthogonal polynomials was when Chebyshev considered the problem of finding the properties of the denominator polynomials of a J-fraction. Gauss, Jacobi, Legendre and others had also worked with orthogonal polynomials.

To define, in the modern way, a sequence of orthogonal polynomials we consider a distribution $\psi(t)$, on $(a, b) \subseteq \mathbb{R}$, and the moments

$$\mu_n = \int_a^b t^n d\psi(t), \quad n = 0, 1, 2, \dots .$$

If there exists a sequence of polynomials, $\{P_n(z)\}_{n=0}^{\infty}$, where $P_n(z)$ is of degree n, such that

$$\int_a^b P_m(t) P_n(t) d\psi(t) \begin{cases} = 0, & \text{if} \quad m \neq n, \\ \neq 0, & \text{if} \quad m = n. \end{cases} \quad n, m = 0, 1, 2, \dots , \tag{1.1.13}$$

or, alternatively, such that

$$\int_a^b t^s P_n(t) d\psi(t) \begin{cases} = 0, & \text{if} \quad 0 \leq s \leq n-1, \\ \neq 0, & \text{if} \quad s = n. \end{cases} \quad n = 0, 1, 2, \dots ,$$

then $\{P_n(z)\}_{n=0}^{\infty}$ is called a *sequence of orthogonal polynomials* related to the distribution $\psi(t)$.

The existence of such a sequence of orthogonal polynomials is guaranteed if the Hankel determinants,

$$H_n^{(m)} = \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+n-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+n} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{m+n-1} & \mu_{m+n} & \cdots & \mu_{m+2n-2} \end{vmatrix},$$

satisfy

$$H_n^{(0)} \neq 0, \quad n = 0, 1, 2\ldots .$$

The proof of this result can be found in the classical sources such as Chihara [10], Freud [15] and Szegö [52], as can all the fundamental properties of the orthogonal polynomials.

Recent studies of the strong moment problems and the denominators of the general T-fractions and M-fractions gave origin to the study of orthogonal Laurent polynomials. A Laurent polynomial is a function of a real or complex variable z of the form $\sum_{i=k}^{m} r_i z^i$, where $k, m \in \boldsymbol{Z}$, $k \leq m$ and $r_i \in \mathbb{C}$.

In a similar way to that for the classical orthogonal polynomials, we consider a strong distribution $\psi(t)$, defined on $(a, b) \subseteq \mathbb{R}$, and the moments

$$\mu_n = \int_a^b t^n d\psi(t), \quad n = 0, \pm 1, \pm 2, \ldots .$$

According to Jones, Njåstad and Thron [19], the Laurent polynomials

$$\begin{aligned} Q_{2n}(z) &= q_{2n,-n} z^{-n} + \cdots + q_{2n,n} z^n, & q_{2n,-n} &= 1, \\ Q_{2n+1}(z) &= q_{2n+1,-n-1} z^{-n-1} + \cdots + q_{2n+1,n} z^n, & q_{2n+1,-n-1} &= 1, \end{aligned}$$

are *orthogonal Laurent polynomials* if

$$\int_a^b Q_n(t)Q_m(t)d\psi(t) \begin{cases} = 0, & n \neq m, \\ > 0, & n = m. \end{cases}$$

Further details of the theory of orthogonal Laurent polynomials are given in Chapter 2 of this thesis.

A natural use of the orthogonal polynomials is in the theory of the quadrature formulae. Newton and Cotes, independently, developed similar methods to approximate integrals with formulae of the form

$$\int_a^b f(t)dt = \sum_{i=1}^{n} w_i f(t_i) + \mathbb{E}_n(f), \qquad (1.1.14)$$

where $\mathbb{E}_n(f)$ is the error term, using interpolating polynomials. The weights w_i can be expressed as

$$w_i = \int_a^b L_i(t)dt, \quad i = 1, 2, ..., n \ ,$$

where, $L_i(t), i = 1, 2, ..., n$, are the Lagrange polynomials, defined such that $L_i(t) \in \mathbb{P}_{n-1}$ and $L_i(t_j) = \delta_{i,j}$.

A quadrature formula of the form (1.1.14) is called *interpolatory* and $\mathbb{E}_n(f) = 0$ for all $f(t) \in \mathbb{P}_{n-1}$. In other words, the quadrature formula has order $n - 1$.

Gauss, in 1814, based on the methods of Newton and Cotes and on his own work in both hypergeometric functions and continued fractions, proposed a new quadrature method of the form (1.1.14) but with order $2n - 1$.

Later, Jacobi proved Gauss' results without using continued fractions, but based on polynomial orthogonality. As Gautschi [16] pointed out, the term "orthogonal" came into use later, probably in the work of Murphy in 1835.

Christoffel then generalized the Gaussian quadrature for weighted integrals such as

$$\int_a^b f(t)\omega(t)dt,$$

where $\omega(t)$ is a nonnegative weight function.

Heine, Chebyshev, Stieltjes and other mathematicians then extended the Gaussian quadrature formula during the nineteenth century. This powerful tool to approximate integrals became very popular with the advances of computers. For further details see Gautschi [16], Krylov [30] and Stroud and Secrest [51].

Following Stieltjes, in this thesis we consider quadrature formulae to approximate integrals of the form

$$\int_a^b f(t)d\psi(t) = \sum_{i=1}^{n} w_i f(t_i) + \mathbb{E}_n(f), \tag{1.1.15}$$

where $\psi(t)$ is a distribution defined on $(a, b) \subseteq (0, \infty)$. In Chapter 5 we will study the quadrature formulae related to the strong Stieltjes distributions.

Padé approximants

The two-point Padé table is the extension to the two series (1.1.7) and (1.1.8), namely

$$c_{-1} + c_{-2}z + c_{-3}z^2 + c_{-4}z^3 + \cdots$$

and

$$\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \frac{c_3}{z^4} + \cdots,$$

of the classical Padé table associated with the single series (1.1.7). Padé approximations are rational functions and they are closely connected with continued fractions and orthogonal polynomials. An excellent account of the history of Padé approximants, accompanying a history of continued fractions, is given by

Brezinski [9]. An excellent text about the Padé approximants is given by Baker and Graves-Morris [4]. The classical Padé approximants, and the Padé table containing these rational functions are, respectively, subsets of the more recent two-point Padé approximants and the two-point Padé table. Hence it is these approximants, and this table that are now defined.

A *two-point Padé approximant* is a rational function $\frac{A_n^{(r)}(z)}{B_n^{(r)}(z)}$, where $n \in \mathbb{N}$ and $r \in \boldsymbol{Z}$. The two-point Padé table is the doubly infinite array of these approximants, seen below. The significance of the "staircase" lines will be explained below.

$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\frac{A_0^{(-2)}(z)}{B_0^{(-2)}(z)}$	$\frac{A_1^{(-2)}(z)}{B_1^{(-2)}(z)}$	$\frac{A_2^{(-2)}(z)}{B_2^{(-2)}(z)}$	$\frac{A_3^{(-2)}(z)}{B_3^{(-2)}(z)}$	$\frac{A_4^{(-2)}(z)}{B_4^{(-2)}(z)}$	$\cdots$
$\frac{A_0^{(-1)}(z)}{B_0^{(-1)}(z)}$	$\frac{A_1^{(-1)}(z)}{B_1^{(-1)}(z)}$	$\frac{A_2^{(-1)}(z)}{B_2^{(-1)}(z)}$	$\frac{A_3^{(-1)}(z)}{B_3^{(-1)}(z)}$	$\frac{A_4^{(-1)}(z)}{B_4^{(-1)}(z)}$	$\cdots$
$\frac{A_0^{(0)}(z)}{B_0^{(0)}(z)}$	$\frac{A_1^{(0)}(z)}{B_1^{(0)}(z)}$	$\frac{A_2^{(0)}(z)}{B_2^{(0)}(z)}$	$\frac{A_3^{(0)}(z)}{B_3^{(0)}(z)}$	$\frac{A_4^{(0)}(z)}{B_4^{(0)}(z)}$	$\cdots$
$\frac{A_0^{(1)}(z)}{B_0^{(1)}(z)}$	$\frac{A_1^{(1)}(z)}{B_1^{(1)}(z)}$	$\frac{A_2^{(1)}(z)}{B_2^{(1)}(z)}$	$\frac{A_3^{(1)}(z)}{B_3^{(1)}(z)}$	$\frac{A_4^{(1)}(z)}{B_4^{(1)}(z)}$	$\cdots$
$\frac{A_0^{(2)}(z)}{B_0^{(2)}(z)}$	$\frac{A_1^{(2)}(z)}{B_1^{(2)}(z)}$	$\frac{A_2^{(2)}(z)}{B_2^{(2)}(z)}$	$\frac{A_3^{(2)}(z)}{B_3^{(2)}(z)}$	$\frac{A_4^{(2)}(z)}{B_4^{(2)}(z)}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Two-point Padé table

Given that certain determinant conditions on the coefficients are satisfied, the approximants in the table are all unique and they correspond to the series as follows.

For $|r| < n$ and $n \in \mathbb{N}$ the rational functions $\dfrac{A_n^{(r)}(z)}{B_n^{(r)}(z)}$ are strictly two-point Padé approximants. Each is a ratio of polynomials of degree $n-1$ and n respectively with $A_n^{(r)}(0)B_n^{(r)}(0) \neq 0$. When expanded as series in powers of z, it will agree with $(n-r)$ terms of (1.1.7) and also, when expanded as series in powers of $\dfrac{1}{z}$, it will agree with $(n+r)$ terms of (1.1.8). These are the approximants lying between the staircase lines in the table above. The remaining approximants above and below these lines will still agree with $(n-r)$ terms of (1.1.7) and $(n+r)$ terms of (1.1.8) when expanded accordingly, where negative numbers of terms indicates no agreement.

When $r \leq -n$, that is the approximants above the upper staircase, the denominator $B_n^{(r)}(z)$ is still a polynomial of degree n but the numerator is now a polynomial of degree $|r|-1$. These are the classical Padé approximants for the series (1.1.7). Specifically,

$$\frac{A_n^{(r)}(z)}{B_n^{(r)}(z)} = P_{n,|r|-1}(z) = c_{-1} + c_{-2}z + \cdots + c_{-(n+|r|)}z^{n+|r|-1} + \text{ higher order terms.}$$

That is, $(n-r)$ terms of (1.1.7) are "fitted" by the approximant. In the classical Padé table for the series (1.1.7) these elements would appear in the lower half of the table or on the first upper diagonal.

The lower part of the two-point table, the approximants when $r \geq n$, is the triangular array known as the E array described by Wynn [55]. The denominator $B_n^{(r)}(z)$ is still a polynomials of degree n but the numerator can now have positive and negative powers of z, it is a Laurent polynomial. Specifically,

$$A_n^{(r)}(z) = z^{n-r}\sum_{j=0}^{r-1} a_{n,j}^{(r)} z^j$$

and

$$\frac{A_n^{(r)}(z)}{B_n^{(r)}(z)} = E_n^{r-n} = \frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots \frac{c_{n+r-1}}{z^{n+r}} + \text{ lower order terms,}$$

that is, $(n+r)$ terms of (1.1.8) are "fitted" by the approximant.

The approximants in the first column of the two-point Padé table are simply the partial sums of one of the series, (1.1.7) when $r<0$ and (1.1.8) when $r>0$. The first approximant on the central row $\frac{A_0^{(0)}(z)}{B_0^{(0)}(z)}$, is identically zero.

Since the classical Padé table is a subset of the two-point Padé table many of the results that have been established for the classical Padé table extend to the two-point Padé table. One such result is Wynn's identity for example. This identity links any approximant, denoted by C say, with its immediate neighbours in each of the four directions N, S, E and W by the relation

$$\frac{1}{N-C}+\frac{1}{S-C}=\frac{1}{E-C}+\frac{1}{W-C}.$$

Finally, the link with continued fractions. In many cases, ordered sequences of approximants in the two-point table form the convergents of a continued fraction. The particular continued fractions that arise for row sequences, staircase sequences, sawtooth sequences and some other sequences, and methods for obtaining these continued fractions from the series coefficients, will be given in the following chapters.

1.2 Symmetries in distributions

In this work we consider the strong Stieltjes distributions $\psi(t)$ that satisfy the following symmetric property

$$\frac{d\psi(t)}{t^{\omega}}=-\frac{d\psi(\beta^2/t)}{(\beta^2/t)^{\omega}}, \tag{1.2.1}$$

with $0<\beta<b\leq\infty$, $a=\beta^2/b$, $t\in(a,b)$, and $2\omega\in\boldsymbol{Z}$. We denote this kind of distribution by $S^3[\omega,\beta,b]$ distribution. The moments of an $S^3[\omega,\beta,b]$ distribution

satisfy the "symmetric" relations

$$\mu_n = \beta^{2(n+\omega)}\mu_{-n-2\omega}, \quad n = 0, \pm1, \pm2, \dots .$$

Similar relations can be found in a study of the trigonometric moment problem. From (1.1.12) we see that the moments of a solution of the trigonometric moment problem satisfy

$$\mu_n = \overline{\mu}_{-n} \quad n = 0, \pm1, \pm2, \dots .$$

Jones, Njåstad and Thron in [23] provide a survey on this moment problem and the related orthogonal polynomials, quadrature formulae and continued fractions.

Sri Ranga in [43] studied strong distributions $\psi(t)$ defined on $\mathbb{R}$ that satisfy

$$\frac{d\psi(t)}{t} = -\frac{d\psi(\beta^2/t)}{\beta^2/t}, \quad t \in \mathbb{R}. \tag{1.2.2}$$

Specifically he considered a sequence of monic polynomials $\{Q_n(z)\}_{n=0}^{\infty}$, where $Q_n(z)$ is of degree n, that satisfy

$$\int_{\mathbb{R}} t^{-2[\![n/2]\!]+s} Q_n(t)d\psi(t) = 0, \quad 0 \le s \le n-1, \quad \text{for } n \ge 1. \tag{1.2.3}$$

These polynomials were first studied by Sri Ranga and McCabe in [49]. They also satisfy the three-term recurrence relations

$$\begin{aligned} Q_{2n}(z) &= (z - \beta_{2n})Q_{2n-1}(z) - \alpha_{2n}Q_{2n-2}(z), & n \ge 1, \\ Q_{2n+1}(z) &= \{(1+\alpha_{2n})z - \beta_{2n+1}\}Q_{2n}(z) - \alpha_{2n}z^2Q_{2n-1}(z), & n \ge 1, \end{aligned} \tag{1.2.4}$$

where $Q_0(z) = 1$, $Q_1(z) = z - \beta_1$, and $\alpha_{n+1} > 0$ and $\beta_n \in \mathbb{R}$. It was shown that if the distribution $\psi(t)$ satisfies (1.2.2), then the polynomials $Q_n(z)$ defined by the conditions (1.2.3) satisfy the recurrence relations (1.2.4) with $\beta_n = 0$, $\alpha_{2n} = \beta^2$ for $n \ge 1$. These distributions were called *strong c-symmetric* distributions.

In [44] Sri Ranga dealt with the case when $\omega = 1/2$ and considered a sequence of monic polynomials $\{B_n(z)\}_{n=0}^{\infty}$, where $B_n(z)$ is of degree n, that satisfy

$$\int_a^b t^{-n+s} B_n(t)d\psi(t) = 0, \quad 0 \le s \le n-1, \quad \text{for } n \ge 1. \tag{1.2.5}$$

They satisfy the three-term recurrence relation

$$B_{n+1}(z) = (z - \beta_{n+1})B_n(z) - \alpha_{n+1} z B_{n-1}(z), \quad n \geq 1, \tag{1.2.6}$$

where $B_0(z) = 1$ and $B_1(z) = z - \beta_1$, with $\beta_n, \alpha_{n+1} > 0$, for $n \geq 1$. The polynomials that satisfy (1.2.5) are called *orthogonal L-polynomials* because of their relations with the orthogonal Laurent polynomials defined in section 1.1.

In this case, the distribution $\psi(t)$ is a strong Stieltjes distribution that satisfies

$$\frac{d\psi(t)}{\sqrt{t}} = -\frac{d\psi(\beta^2/t)}{\sqrt{\beta^2/t}}, \quad t \in (a,b). \tag{1.2.7}$$

In [44] this distribution was called $ScS(a,b)$ distribution. The polynomials $B_n(z)$ defined by (1.2.5), for an $ScS(a,b)$ distribution, satisfy the recurrence relations (1.2.6) with $\beta_n = \beta$, for all values of n.

In [48] Sri Ranga, de Andrade and McCabe have investigated the polynomials $B_n(z)$ defined by (1.2.5) which are associated with a distribution that satisfy (1.2.1) with $\omega = 0$, that is

$$d\psi(t) = -d\psi(\beta^2/t), \quad t \in (a,b). \tag{1.2.8}$$

These distributions were called $S\bar{c}S(a,b)$ distributions. The authors in [48] also studied the real polynomials $B_n(\lambda, z)$, $n \geq 1$ defined by

$$B_n(\lambda, z) = B_n(z) - \lambda B_{n-1}(z), \quad \lambda \in \mathbb{R},$$

and some consequences of the symmetric property (1.2.8) in these polynomials.

Common and McCabe [14] considered distributions that satisfy (1.2.1) for the specific case $\omega = 0$ and $\beta = 1$. They studied the polynomials defined by (1.2.3), the denominators of the two-point Padé table and the related continued fractions.

If we consider the classical orthogonal polynomials, it is well known that if a weight function $v(t)$ defined on $(-d, d)$, $0 < d \leq \infty$, satisfies $v(t) = v(-t)$ then the classical orthogonal polynomials, $P_n(z)$, $n = 0, 1, 2, \ldots$, defined by (1.1.13), associated with $v(t)$, satisfy the symmetric property

$$P_n(z) = (-1)^n P_n(-z), \quad n \geq 0.$$

The polynomials $P_n(z)$, $n = 0, 1, 2, \ldots$ that satisfy the above property are called symmetric orthogonal polynomials. In monic form they also satisfy the three-term recurrence relation

$$P_{n+1}(z) = zP_n(z) - a_{n+1}P_{n-1}(z), \quad n \geq 1,$$

with $P_0(z) = 1$, and $P_1(z) = z$. The coefficients a_{n+1}, $n \geq 1$ are given by

$$a_{n+1} = \frac{\int_{-d}^{d} t^n P_n(t)v(t)dt}{\int_{-d}^{d} t^{n-1} P_{n-1}(t)v(t)dt}.$$

See Chihara [10] and Szegö [52].

Sri Ranga in [46] has given some relations that exist between the symmetric orthogonal polynomials $P_n(z)$ and the orthogonal L-polynomials $B_n(z)$ related to distributions that satisfy (1.2.7). In Bracciali, Capela and Sri Ranga [6], these results were extended to the complex plane.

The objective of this work is to investigate the symmetric property (1.2.1) that occurs in some strong Stieltjes distributions, and the consequent symmetries that appear in the orthogonal L-polynomials, the associated polynomials and also in the monic polynomials $B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; z)$, $n \geq 0$, defined by

$$B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; z) = B_n^{(0)}(z) + \lambda_{n,1}^{(r)} B_{n-1}^{(0)}(z) + \cdots + \lambda_{n,r}^{(r)} B_{n-r}^{(0)}(z),$$

where $r \geq 1$ and $\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)} \in \mathbb{R}$.

We also study the quadrature formulae related to the orthogonal L-polynomials, the polynomials $B_n(\lambda_{n,1}^{(r)}, \ldots, \lambda_{n,r}^{(r)}; z)$ and some consequences of the symmetry (1.2.1) that appear in these quadrature formulae.

Chapter 2

Strong Stieltjes distributions and orthogonal L-polynomials

In the first four sections of this chapter we restate some existing results that are necessary for the development of our work. In section 2.5 we define and study a new sequence of monic polynomials.

2.1 Introduction

Let $0 < \beta < b \leq \infty$. Set $a = \beta^2/b$ and let $\psi(t)$ be a real, bounded and nondecreasing function defined on (a, b), with infinitely many points of increase in (a, b) and such that the moments

$$\mu_m = \int_a^b t^m d\psi(t), \quad m = 0, 1, 2, \dots,$$

all exist. With these conditions $\psi(t)$ is a distribution function on (a, b). Since $(a, b) \subseteq (0, \infty)$, $\psi(t)$ is called a Stieltjes distribution.

If the moments

$$\mu_m = \int_a^b t^m d\psi(t), \quad m = 0, \pm 1, \pm 2, ..., \tag{2.1.1}$$

all exist, then the distribution $\psi(t)$ is referred to as a *strong Stieltjes distribution.*

For strong Stieltjes distributions, the *Hankel determinants*, $H_n^{(m)}$, of order n, defined by $H_{-1}^{(m)} = 0$, $H_0^{(m)} = 1$, and

$$H_n^{(m)} = \begin{vmatrix} \mu_m & \mu_{m+1} & \cdots & \mu_{m+n-1} \\ \mu_{m+1} & \mu_{m+2} & \cdots & \mu_{m+n} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{m+n-1} & \mu_{m+n} & \cdots & \mu_{m+2n-2} \end{vmatrix}, \tag{2.1.2}$$

for $n = 1, 2, ...$ and $m = 0, \pm 1, \pm 2, ...$, are positive. A direct proof of this result can be found in de Andrade [2], where the author used similar ideas that those given by Chihara [10], Szegö [52] and Wall [54].

2.2 The orthogonal L-polynomials

Let $\psi(t)$ be a strong Stieltjes distribution and, for $r \in \boldsymbol{Z}$, let $\{B_n^{(r)}(z)\}_{n=0}^{\infty}$ be sequences of monic polynomials, where $B_n^{(r)}(z)$ is of degree n, defined by the conditions

$$\int_a^b t^{-n+s+r} B_n^{(r)}(t) d\psi(t) = \begin{cases} 0, & 0 \le s \le n-1, \\ \rho_{n,r}^{(r)} > 0, & s = n. \end{cases} \tag{2.2.1}$$

The numbers $\rho_{n,k}^{(r)}$, which we will use later in this work, are given by

$$\rho_{n,k}^{(r)} = \int_a^b t^k B_n^{(r)}(t) d\psi(t). \tag{2.2.2}$$

The polynomials $B_n^{(r)}(z)$ have been studied by Sri Ranga in [42]. In this section we give some results that can be deduced from the results in [42]. These polynomials are called *orthogonal L-polynomials* since they are related to the orthogonal Laurent polynomials defined in section 1.1. In section 2.4 we will present the relations between these polynomials.

The polynomials $B_n^{(r)}(z)$, $n \geq 0$, $r = 0, \pm 1, \pm 2, \ldots$, can be written as

$$B_n^{(r)}(z) = \sum_{j=0}^{n} b_{n,n-j}^{(r)} z^{n-j}, \tag{2.2.3}$$

where $b_{n,n}^{(r)} = 1$, for all values of n.

Using the definition (2.1.1), we can write the equations (2.2.1) as the linear system

$$\left\{ \begin{array}{ccccccccc} \mu_{r-n} b_{n,0}^{(r)} & + & \mu_{r-n+1} b_{n,1}^{(r)} & + & \cdots & + & \mu_r b_{n,n}^{(r)} & = & 0 \\ \mu_{r-n+1} b_{n,0}^{(r)} & + & \mu_{r-n+2} b_{n,1}^{(r)} & + & \cdots & + & \mu_{r+1} b_{n,n}^{(r)} & = & 0 \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\ \mu_{r-1} b_{n,0}^{(r)} & + & \mu_r b_{n,1}^{(r)} & + & \cdots & + & \mu_{r+n-1} b_{n,n}^{(r)} & = & 0 \\ \mu_r b_{n,0}^{(r)} & + & \mu_{r+1} b_{n,1}^{(r)} & + & \cdots & + & \mu_{r+n} b_{n,n}^{(r)} & = & \rho_{n,r}^{(r)} \end{array} \right. . \tag{2.2.4}$$

We use Cramer's rule and the definition (2.1.2), to then obtain

$$b_{n,n}^{(r)} = \frac{\rho_{n,r}^{(r)} H_n^{(-n+r)}}{H_{n+1}^{(-n+r)}}.$$

Then, with the normalization $b_{n,n}^{(r)} = 1$, it follows that

$$\rho_{n,r}^{(r)} = \frac{H_{n+1}^{(-n+r)}}{H_n^{(-n+r)}}. \tag{2.2.5}$$

If we now substitute the last row in the system (2.2.4) by the equation (2.2.3), we obtain

$$\left\{ \begin{array}{ccccccccc} \mu_{r-n} b_{n,0}^{(r)} & + & \mu_{r-n+1} b_{n,1}^{(r)} & + & \cdots & + & \mu_r b_{n,n}^{(r)} & = & 0 \\ \mu_{r-n+1} b_{n,0}^{(r)} & + & \mu_{r-n+2} b_{n,1}^{(r)} & + & \cdots & + & \mu_{r+1} b_{n,n}^{(r)} & = & 0 \\ \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\ \mu_{r-1} b_{n,0}^{(r)} & + & \mu_r b_{n,1}^{(r)} & + & \cdots & + & \mu_{r+n-1} b_{n,n}^{(r)} & = & 0 \\ b_{n,0}^{(r)} & + & z\, b_{n,1}^{(r)} & + & \cdots & + & z^n\, b_{n,n}^{(r)} & = & B_n^{(r)}(z) \end{array} \right. .$$

Once again, using Cramer's rule, we find

$$B_n^{(r)}(z) = \frac{1}{H_n^{(-n+r)}} \begin{vmatrix} \mu_{r-n} & \mu_{r-n+1} & \cdots & \mu_r \\ \mu_{r-n+1} & \mu_{r-n+2} & \cdots & \mu_{r+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{r-1} & \mu_r & \cdots & \mu_{r+n-1} \\ 1 & z & \cdots & z^n \end{vmatrix}.$$

Setting $z = 0$ in the equation above, we see that

$$B_n^{(r)}(0) = (-1)^n \frac{H_n^{(-n+1+r)}}{H_n^{(-n+r)}} = b_{n,0}^{(r)}.$$

Hence, the existence and uniqueness of the polynomials $B_n^{(r)}(z)$, with $B_n^{(r)}(0) \neq 0$, depend on $H_n^{(-n+r)}$ and $H_n^{(-n+1+r)}$ being non zero. But from (2.1.2), we know that $H_n^{(m)} > 0$ for $n \geq 0$, $m = 0, \pm 1, \pm 2, \ldots$.

Proceeding, we consider

$$\sigma_{n,k}^{(r)} = \int_a^b t^{k-(n+1)} B_n^{(r)}(t) d\psi(t),$$

and we see that

$$\sigma_{n,r}^{(r)} = \int_a^b t^{r-(n+1)} B_n^{(r)}(t) d\psi(t) = \mu_{-n-1+r} b_{n,0}^{(r)} + \mu_{-n+r} b_{n,1}^{(r)} + \cdots + \mu_{-1+r} b_{n,n}^{(r)} \,.$$

Then, using Cramer's rule on the linear system containing this equation followed by the first n equations in (2.2.4), we find that

$$\sigma_{n,r}^{(r)} = (-1)^n \frac{H_{n+1}^{(-(n+1)+r)}}{H_n^{(-n+r)}}. \tag{2.2.6}$$

The polynomials $B_n^{(r)}(z)$, $r \in \boldsymbol{Z}$, satisfy the three-term recurrence relation

$$B_{n+1}^{(r)}(z) = (z - \beta_{n+1}^{(r)}) B_n^{(r)}(z) - \alpha_{n+1}^{(r)} z B_{n-1}^{(r)}(z), \quad n \geq 1, \tag{2.2.7}$$

with $B_0^{(r)}(z) = 1$, $B_1^{(r)}(z) = z - \beta_1^{(r)}$. The coefficients $\alpha_{n+1}^{(r)}$ and $\beta_{n+1}^{(r)}$, $n \geq 1$, are given by

$$\alpha_{n+1}^{(r)} = \frac{\int_a^b t^r B_n^{(r)}(t) d\psi(t)}{\int_a^b t^r B_{n-1}^{(r)}(t) d\psi(t)} = \frac{\rho_{n,r}^{(r)}}{\rho_{n-1,r}^{(r)}},$$

$$\beta_{n+1}^{(r)} = -\alpha_{n+1}^{(r)} \frac{\int_a^b t^{-n+r} B_{n-1}^{(r)}(t) d\psi(t)}{\int_a^b t^{-(n+1)+r} B_n^{(r)}(t) d\psi(t)} = -\alpha_{n+1}^{(r)} \frac{\sigma_{n-1,r}^{(r)}}{\sigma_{n,r}^{(r)}},$$

and $\beta_1^{(r)} = \dfrac{\mu_r}{\mu_{r-1}}$.

We prove this result as follows. Since $B_n^{(r)}(z)$, $n \geq 0$, $r = 0, \pm 1, \pm 2, \ldots$, are monic polynomials of degree n, the polynomial $B_{n+1}^{(r)}(z) - zB_n^{(r)}(z)$ is also a polynomial of degree n. We then write

$$B_{n+1}^{(r)}(z) - zB_n^{(r)}(z) = -\beta_{n+1}^{(r)} B_n^{(r)}(z) - \alpha_{n+1}^{(r)} z B_{n-1}^{(r)}(z) + P_{n-1}^{(r)}(z), \qquad (2.2.8)$$

where $P_{n-1}^{(r)}(z)$ is a polynomial of degree $n-1$. We can write $P_{n-1}^{(r)}(z) = \sum_{j=0}^{n-1} p_{n-1,j}^{(r)} z^j$.

Multiplying both sides of the equation (2.2.8) by z^{-n+s+r} and integrating over (a, b), we obtain

$$\begin{aligned} &\int_a^b t^{-n+s+r} B_{n+1}^{(r)}(t) d\psi(t) - \int_a^b t^{-n+s+1+r} B_n^{(r)}(t) d\psi(t) = \\ &-\beta_{n+1}^{(r)} \int_a^b t^{-n+s+r} B_n^{(r)}(t) d\psi(t) - \alpha_{n+1}^{(r)} \int_a^b t^{-n+s+1+r} B_{n-1}^{(r)}(t) d\psi(t) \qquad (2.2.9) \\ &+ \sum_{j=0}^{n-1} p_{n-1,j}^{(r)} \int_a^b t^{-n+s+j+r} d\psi(t). \end{aligned}$$

Setting $s = 0, 1, \ldots, n-1$ in (2.2.9) and using the definition (2.2.1) we obtain the following linear system

$$\left\{ \begin{array}{ccccccccc} \mu_{r-n} p_{n-1,0}^{(r)} & + & \mu_{r-n+1} p_{n-1,1}^{(r)} & + \cdots + & \mu_{r-1} p_{n-1,n-1}^{(r)} & = & 0 \\ \mu_{r-n+1} p_{n-1,0}^{(r)} & + & \mu_{r-n+2} p_{n-1,1}^{(r)} & + \cdots + & \mu_r p_{n-1,n-1}^{(r)} & = & 0 \\ \cdots & & \cdots & \cdots & \cdots & & \cdots \\ \mu_{r-2} p_{n-1,0}^{(r)} & + & \mu_{r-1} p_{n-1,1}^{(r)} & + \cdots + & \mu_{r+n-3} p_{n-1,n-1}^{(r)} & = & 0 \\ \mu_{r-1} p_{n-1,0}^{(r)} & + & \mu_r p_{n-1,1}^{(r)} & + \cdots + & \mu_{r+n-2} p_{n-1,n-1}^{(r)} & = & \alpha_{n+1}^{(r)} \rho_{n-1,r}^{(r)} - \rho_{n,r}^{(r)} \end{array} \right.$$

The determinant of this system, $H_n^{(-n+r)}$, is positive, and if we choose

$$\alpha_{n+1}^{(r)} = \frac{\rho_{n,r}^{(r)}}{\rho_{n-1,r}^{(r)}},$$

then the solution of the linear system is $p_{n-1,j}^{(r)} = 0$, for $j = 0, 1, ..., n-1$. This means that $P_{n-1}^{(r)}(z) \equiv 0$ and the first part of the result holds.

In order to find $\beta_{n+1}^{(r)}$, we set $s = -1$ in the equation (2.2.9) and we see that

$$0 = -\beta_{n+1}^{(r)}\sigma_{n,r}^{(r)} - \alpha_{n+1}^{(r)}\sigma_{n-1,r}^{(r)}.$$

The proof is now complete.

Now, from the relations (2.2.5) and (2.2.6), we can also write the coefficients $\alpha_{n+1}^{(r)}$ and $\beta_{n+1}^{(r)}$, $n \geq 0$, $r = 0, \pm 1, \pm 2, ...$, as

$$\alpha_{n+1}^{(r)} = \frac{H_{n+1}^{(-n+r)} H_{n-1}^{(-(n-1)+r)}}{H_n^{(-n+r)} H_n^{(-(n-1)+r)}}, \quad \beta_{n+1}^{(r)} = \frac{H_n^{(-n+r)} H_{n+1}^{(-n+r)}}{H_{n+1}^{(-(n+1)+r)} H_n^{(-(n-1)+r)}}. \tag{2.2.10}$$

The zeros of the polynomial $B_n^{(r)}(z)$ are all real and distinct, and they lie inside (a, b).

We prove this result by contradiction. Since $0 \leq a < b \leq \infty$, then from

$$\int_a^b t^{-n+r} B_n^{(r)}(t) d\psi(t) = 0,$$

we can see that the polynomial $B_n^{(r)}(z)$ changes sign at least once in (a, b).

We now assume that $B_n^{(r)}(z)$ changes sign exactly m, $1 \leq m < n$, times in (a, b), at the points $z_1^{(r)}, z_2^{(r)}, ..., z_m^{(r)}$. Consider the polynomial

$$\pi(z) = (z - z_1^{(r)})(z - z_2^{(r)}) \cdots (z - z_m^{(r)}).$$

We then obtain

$$\int_a^b t^{-n+r} \pi(t) B_n^{(r)}(t) d\psi(t) \neq 0.$$

This contradicts the definition (2.2.1). Hence $m = n$, and $B_n^{(r)}(z)$ has n real distinct zeros inside (a, b).

We arrange the polynomials $B_n^{(r)}(z)$, $n = 0, 1, 2, \ldots$, $r = 0, \pm 1, \pm 2, \ldots$, in the following table

$$\begin{array}{ccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
B_0^{(-2)}(z) & B_1^{(-2)}(z) & B_2^{(-2)}(z) & B_3^{(-2)}(z) & B_4^{(-2)}(z) & B_5^{(-2)}(z) & \cdots \\
B_0^{(-1)}(z) & B_1^{(-1)}(z) & B_2^{(-1)}(z) & B_3^{(-1)}(z) & B_4^{(-1)}(z) & B_5^{(-1)}(z) & \cdots \\
B_0^{(0)}(z) & B_1^{(0)}(z) & B_2^{(0)}(z) & B_3^{(0)}(z) & B_4^{(0)}(z) & B_5^{(0)}(z) & \cdots \\
B_0^{(1)}(z) & B_1^{(1)}(z) & B_2^{(1)}(z) & B_3^{(1)}(z) & B_4^{(1)}(z) & B_5^{(1)}(z) & \cdots \\
B_0^{(2)}(z) & B_1^{(2)}(z) & B_2^{(2)}(z) & B_3^{(2)}(z) & B_4^{(2)}(z) & B_5^{(2)}(z) & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} \quad (2.2.11)$$

We recall that the polynomials in the row of the table with superscript (r) satisfy the conditions

$$\int_a^b t^{-n+s+r} B_n^{(r)}(t) d\psi(t) = 0, \quad 0 \le s \le n-1.$$

Setting $n = r$ in the orthogonality conditions above, we obtain

$$\int_a^b t^s B_n^{(n)}(t) d\psi(t) = 0, \quad 0 \le s \le n-1. \qquad (2.2.12)$$

Thus, the polynomials $B_0^{(0)}(z), B_1^{(1)}(z), \ldots, B_n^{(n)}(z), \ldots$ are the classical orthogonal polynomials with respect to the distribution $\psi(t)$. They satisfy the three-term recurrence relation

$$B_{n+1}^{(n+1)}(z) = (z - b_{n+1}^{(0)}) B_n^{(n)}(z) - a_{n+1}^{(0)} B_{n-1}^{(n-1)}(z), \quad n \ge 1,$$

with $B_0^{(0)}(z) = 1$, $B_1^{(1)}(z) = z - b_1^{(0)}$. The coefficients $a_{n+1}^{(0)}$ and $b_{n+1}^{(0)}$ are given by

$$a_{n+1}^{(0)} = \frac{\int_a^b (B_n^{(n)}(t))^2 d\psi(t)}{\int_a^b (B_{n-1}^{(n-1)}(t))^2 d\psi(t)}, \quad b_{n+1}^{(0)} = \frac{\int_a^b t (B_n^{(n)}(t))^2 d\psi(t)}{\int_a^b (B_n^{(n)}(t))^2 d\psi(t)}, \quad n \ge 1,$$

and with $b_1^{(0)} = \mu_1/\mu_0$.

The polynomials, $B_n^{(r)}(z)$ in the diagonal paths of the table, such that $r = n+l$ for $l = 0, \pm1, \pm2, \ldots$, satisfy the conditions

$$\int_a^b t^s B_n^{(n+l)}(t) t^l d\psi(t) = 0, \quad 0 \leq s \leq n-1. \qquad (2.2.13)$$

They are the classical orthogonal polynomials with respect to the distribution $\psi_l(t)$ such that $d\psi_l(t) = t^l d\psi(t)$. They satisfy the three-term recurrence relation

$$B_{n+1}^{(n+1+l)}(z) = (z - b_{n+1}^{(l)}) B_n^{(n+l)}(z) - a_{n+1}^{(l)} B_{n-1}^{(n-1+l)}(z), \quad n \geq 1, \qquad (2.2.14)$$

with $B_0^{(l)}(z) = 1,\ B_1^{(1+l)}(z) = z - b_1^{(l)}$. The coefficients $a_{n+1}^{(l)}$ and $b_{n+1}^{(l)}$ are given by

$$a_{n+1}^{(l)} = \frac{\int_a^b t^l (B_n^{(n+l)}(t))^2 d\psi(t)}{\int_a^b t^l (B_{n-1}^{(n-1+l)}(t))^2 d\psi(t)}, \quad b_{n+1}^{(l)} = \frac{\int_a^b t^{1+l} (B_n^{(n+l)}(t))^2 d\psi(t)}{\int_a^b t^l (B_n^{(n+l)}(t))^2 d\psi(t)}, \quad n \geq 1,$$

with $b_1^{(l)} = \mu_{1+l}/\mu_l$, for $l = 0, \pm1, \pm2, \ldots$.

2.3 Continued fractions

Since the distribution function $\psi(t)$ has the moments $\mu_m, m = 0, \pm1, \pm2, \ldots$ defined in (2.1.1), the Stieltjes function given by

$$G(z) = \int_a^b \frac{1}{z-t} d\psi(t)$$

has the following two formal power series expansions

$$L_0 = -\mu_{-1} - \mu_{-2} z - \mu_{-3} z^2 - \mu_{-4} z^3 - \cdots,$$

and

$$L_\infty = \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \frac{\mu_3}{z^4} + \frac{\mu_4}{z^5} + \cdots.$$

We consider the continued fractions

$$M^{(r)}(z) + \frac{\mu_r/z^r}{z-\beta_1^{(r)}} - \frac{\alpha_2^{(r)}z}{z-\beta_2^{(r)}} - \frac{\alpha_3^{(r)}z}{z-\beta_3^{(r)}} - \frac{\alpha_4^{(r)}z}{z-\beta_4^{(r)}} - \cdots, \qquad (2.3.1)$$

for $r = 0, \pm1, \pm2, \ldots$, which are the corresponding M-fractions (see Murphy [37], McCabe and Murphy [36] and McCabe [33]). The term $M^{(r)}(z)$ is given by

$$M^{(r)}(z) = z^{-r}\int_a^b \frac{z^r - t^r}{z-t} d\psi(t) = \begin{cases} \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \cdots + \frac{\mu_{r-1}}{z^r}, & r \geq 0 \\ -\mu_{-1} - \mu_{-2}z - \cdots - \mu_r z^{-(r+1)}, & r < 0 \end{cases}.$$

The coefficients $\alpha_n^{(r)}$ in the partial numerators and the coefficients $\beta_n^{(r)}$ in the partial denominators for $n \geq 0$ and $r = 0, \pm1, \pm2, \ldots$, are the coefficients of the recurrence relations for the polynomials $B_n^{(r)}(z)$.

For strong Stieltjes distributions, the Hankel determinants $H_n^{(m)}$, for $n \geq 0$ and $m = 0, \pm1, \pm2, \ldots$, are non zero. Hence there exist rational functions $\frac{A_n^{(r)}(z)}{B_n^{(r)}(z)}$, for $n \geq 0$ and $r = 0, \pm1, \pm2, \ldots$, such that

$$\frac{A_n^{(r)}(z)}{B_n^{(r)}(z)} = \begin{cases} -\mu_{-1} - \mu_{-2}z - \cdots - \mu_{-n-r}z^{n-r-1} + \text{h.o.t.} \\ \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \cdots + \frac{\mu_{n+r-1}}{z^{n+r}} + \text{l.o.t.} \end{cases},$$

when expanded at $z = 0$ and at $z = \infty$ respectively. These rational functions form the two-point Padé table for the two series. The two-point Padé approximants were described in section 1.1. The rational functions $\frac{A_n^{(r)}(z)}{B_n^{(r)}(z)}$, for $r \in \boldsymbol{Z}$, are the nth convergents of the M-fractions (2.3.1). The M-fractions (2.3.1) correspond to L_0 at $z = 0$ and to L_∞ at $z = \infty$.

We then see that the numerators of the nth convergent of the M-fraction (2.3.1), $A_n^{(r)}(z)$, for $n \geq 0$ and $r = 0, \pm1, \pm2, \ldots$ satisfy the same three-term recurrence relation that the denominators of the nth convergent, $B_n^{(r)}(z)$, but with different starting values. Thus

$$A_{n+1}^{(r)}(z) = (z - \beta_{n+1}^{(r)})A_n^{(r)}(z) - \alpha_{n+1}^{(r)} z A_{n-1}^{(r)}(z), \quad n \geq 1, \qquad (2.3.2)$$

with $A_0^{(r)}(z) = M^{(r)}(z)$, $A_1^{(r)}(z) = (z - \beta_1^{(r)})M^{(r)}(z) + \mu_r z^{-r}$.

As we mentioned in the section 1.1, the numerators $A_n^{(r)}(z)$, for $n > |r|$, are polynomials of degree $n-1$, while for $r < 0$ and $n \leq -r$, they are polynomials of degree $|r|-1$. For $r > 0$ and $n \leq r$ they are Laurent polynomials of the form

$$A_n^{(r)}(z) = z^{n-r} \sum_{j=0}^{r-1} a_{n,j}^{(r)} z^j.$$

We also consider the M-fractions,

$$\frac{\mu_r}{z - \beta_1^{(r)}} - \frac{\alpha_2^{(r)} z}{z - \beta_2^{(r)}} - \frac{\alpha_3^{(r)} z}{z - \beta_3^{(r)}} - \frac{\alpha_4^{(r)} z}{z - \beta_4^{(r)}} - \cdots \tag{2.3.3}$$

for $r = 0, \pm 1, \pm 2, ...$, that correspond to the two series

$$-\mu_{r-1} - \mu_{r-2} z - \mu_{r-3} z^2 - \mu_{r-4} z^3 - \cdots \tag{2.3.4}$$

and

$$\frac{\mu_r}{z} + \frac{\mu_{r+1}}{z^2} + \frac{\mu_{r+2}}{z^3} + \frac{\mu_{r+3}}{z^4} + \frac{\mu_{r+4}}{z^5} + \cdots. \tag{2.3.5}$$

The denominator polynomials of the nth convergent of the M-fraction (2.3.3), are also the polynomials $B_n^{(r)}(z)$, $n \geq 0$, $r = 0, \pm 1, \pm 2, ...$. The numerator polynomials $C_n^{(r)}(z)$, $n \geq 0$, $r = 0, \pm 1, \pm 2, ...$, satisfy the three-term recurrence relation

$$C_{n+1}^{(r)}(z) = (z - \beta_{n+1}^{(r)}) C_n^{(r)}(z) - \alpha_{n+1}^{(r)} z C_{n-1}^{(r)}(z), \quad n \geq 1, \tag{2.3.6}$$

with $C_0^{(r)}(z) = 0$, $C_1^{(r)}(z) = \mu_r$. The polynomials $C_n^{(r)}(z)$, for $n \geq 1$, are of degree $n-1$, and

$$\frac{C_n^{(r)}(z)}{B_n^{(r)}(z)} = \begin{cases} -\mu_{r-1} - \mu_{r-2} z - \cdots - \mu_{-n-r} z^{n-1} + \text{h.o.t.} \\ \dfrac{\mu_r}{z} + \dfrac{\mu_{r+1}}{z^2} + \dfrac{\mu_{r+2}}{z^3} + \cdots + \dfrac{\mu_{n+r-1}}{z^n} + \text{l.o.t.} \end{cases},$$

when expanded accordingly.

Computing the coefficients $\alpha_n^{(r)}$ and $\beta_n^{(r)}$

Starting with the moments defined by (2.1.1), we may construct the $\alpha - \beta$ table (see McCabe [33]), for the coefficients of the recurrence relations for the polynomials $B_n^{(r)}(z)$, $A_n^{(r)}(z)$, and $C_n^{(r)}(z)$ for $n \geq 1$ and $r = 0, \pm 1, \pm 2, \ldots$. The table is shown below

$$\begin{array}{c|ccccccccc}
r & \beta_1^{(r)} & \alpha_2^{(r)} & \beta_2^{(r)} & \alpha_3^{(r)} & \beta_3^{(r)} & \alpha_4^{(r)} & \beta_4^{(r)} & \alpha_5^{(r)} & \cdots \\
\hline
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-2 & \beta_1^{(-2)} & \alpha_2^{(-2)} & \beta_2^{(-2)} & \alpha_3^{(-2)} & \beta_3^{(-2)} & \alpha_4^{(-2)} & \beta_4^{(-2)} & \alpha_5^{(-2)} & \cdots \\
-1 & \beta_1^{(-1)} & \alpha_2^{(-1)} & \beta_2^{(-1)} & \alpha_3^{(-1)} & \beta_3^{(-1)} & \alpha_4^{(-1)} & \beta_4^{(-1)} & \alpha_5^{(-1)} & \cdots \\
0 & \beta_1^{(0)} & \alpha_2^{(0)} & \beta_2^{(0)} & \alpha_3^{(0)} & \beta_3^{(0)} & \alpha_4^{(0)} & \beta_4^{(0)} & \alpha_5^{(0)} & \cdots \\
1 & \beta_1^{(1)} & \alpha_2^{(1)} & \beta_2^{(1)} & \alpha_3^{(1)} & \beta_3^{(1)} & \alpha_4^{(1)} & \beta_4^{(1)} & \alpha_5^{(1)} & \cdots \\
2 & \beta_1^{(2)} & \alpha_2^{(2)} & \beta_2^{(2)} & \alpha_3^{(2)} & \beta_3^{(2)} & \alpha_4^{(2)} & \beta_4^{(2)} & \alpha_5^{(2)} & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array} . \tag{2.3.7}$$

From the algorithm given by McCabe in [33], we deduce that the elements in the $\alpha - \beta$ table can be obtained using the following algorithm.

Algorithm 2.1

for $r = \ldots, -2, -1, 0, 1, 2, \ldots$

$$\alpha_1^{(r)} = 0, \qquad \beta_1^{(r)} = \frac{\mu_r}{\mu_{r-1}},$$

for $n = 2, 3, \ldots$

for $r = \ldots, -2, -1, 0, 1, 2, \ldots$

$$\alpha_n^{(r)} = \beta_{n-1}^{(r+1)} + \alpha_{n-1}^{(r+1)} - \beta_{n-1}^{(r)}, \tag{2.3.8}$$

$$\beta_n^{(r)} = \frac{\alpha_n^{(r)}\,\beta_{n-1}^{(r-1)}}{\alpha_n^{(r-1)}}. \tag{2.3.9}$$

These two relations link, respectively, the coefficients in the two rhombii

$$\begin{array}{ccc} & \beta_{n-1}^{(r)} & \rightarrow & \alpha_n^{(r)} \\ \nearrow & & \nearrow & \\ \alpha_{n-1}^{(r+1)} & \rightarrow & \beta_{n-1}^{(r+1)} & \end{array} \quad \text{and} \quad \begin{array}{cccc} \beta_{n-1}^{(r-1)} & \rightarrow & \alpha_n^{(r-1)} & \\ & \searrow & & \searrow \\ & \alpha_n^{(r)} & \rightarrow & \beta_n^{(r)} \end{array}.$$

Bracciali and McCabe in [7] have given some examples to show the advantages of using this algorithm in symbolic computation.

The equations (2.3.8) and (2.3.9) allow the construction of the $\alpha - \beta$ table column by column. However if one row in this table is known, we can construct the table row by row, as we show below.

We set $\gamma_n^{(r)} = \beta_n^{(r)} + \alpha_{n+1}^{(r)}$. Then from the relations (2.3.8) and (2.3.9), we obtain

$$\gamma_n^{(r)} = \beta_n^{(r)} + \alpha_{n+1}^{(r)} = \beta_n^{(r+1)} + \alpha_n^{(r+1)}, \tag{2.3.10}$$

and

$$\beta_n^{(r+1)} = \frac{\alpha_n^{(r+1)}\,\beta_{n-1}^{(r)}}{\alpha_n^{(r)}}. \tag{2.3.11}$$

Substituting (2.3.11) in (2.3.10) we obtain

$$\beta_n^{(r)} + \alpha_{n+1}^{(r)} = \frac{\alpha_n^{(r+1)}\,\beta_{n-1}^{(r)}}{\alpha_n^{(r)}} + \alpha_n^{(r+1)}\frac{\alpha_n^{(r)}}{\alpha_n^{(r)}},$$

$$\beta_n^{(r)} + \alpha_{n+1}^{(r)} = (\beta_{n-1}^{(r)} + \alpha_n^{(r)})\frac{\alpha_n^{(r+1)}}{\alpha_n^{(r)}},$$

$$\gamma_n^{(r)} = \gamma_{n-1}^{(r)}\frac{\alpha_n^{(r+1)}}{\alpha_n^{(r)}}.$$

From (2.3.11), we can see that

$$\frac{\gamma_n^{(r)}}{\gamma_{n-1}^{(r)}} = \frac{\alpha_n^{(r+1)}}{\alpha_n^{(r)}} \quad \text{and} \quad \frac{\gamma_n^{(r)}}{\gamma_{n-1}^{(r)}} = \frac{\beta_n^{(r+1)}}{\beta_{n-1}^{(r)}}. \tag{2.3.12}$$

If we know the row with superscript (r), namely

$$\beta_1^{(r)} \quad \alpha_2^{(r)} \quad \beta_2^{(r)} \quad \alpha_3^{(r)} \quad \beta_3^{(r)} \quad \alpha_4^{(r)} \quad \beta_4^{(r)} \quad \cdots ,$$

then, from (2.3.8) with $n = 2$ and $\alpha_1^{(r+1)} = 0$, we obtain

$$\beta_1^{(r+1)} = \beta_1^{(r)} + \alpha_2^{(r)}.$$

We can calculate $\alpha_n^{(r+1)}$ and $\beta_n^{(r+1)}$, for $n \geq 2$, from the elements $\beta_{n-1}^{(r)}$, $\alpha_n^{(r)}$, $\beta_n^{(r)}$ and $\alpha_{n+1}^{(r)}$, from (2.3.12). We obtain

$$\begin{aligned} \beta_n^{(r+1)} &= \beta_{n-1}^{(r)} \frac{\gamma_n^{(r)}}{\gamma_{n-1}^{(r)}} = \beta_{n-1}^{(r)} \frac{\beta_n^{(r)} + \alpha_{n+1}^{(r)}}{\beta_{n-1}^{(r)} + \alpha_n^{(r)}}, \\ \alpha_n^{(r+1)} &= \alpha_n^{(r)} \frac{\gamma_n^{(r)}}{\gamma_{n-1}^{(r)}} = \alpha_n^{(r)} \frac{\beta_n^{(r)} + \alpha_{n+1}^{(r)}}{\beta_{n-1}^{(r)} + \alpha_n^{(r)}}. \end{aligned} \tag{2.3.13}$$

Similarly, we can calculate $\beta_{n-1}^{(r-1)}$ and $\alpha_n^{(r-1)}$, for $n \geq 2$, from the elements $\alpha_{n-1}^{(r)}$, $\beta_{n-1}^{(r)}$, $\alpha_n^{(r)}$ and $\beta_n^{(r)}$. Again, from (2.3.12), we obtain

$$\begin{aligned} \beta_{n-1}^{(r-1)} &= \beta_n^{(r)} \frac{\gamma_{n-1}^{(r-1)}}{\gamma_n^{(r-1)}} = \beta_n^{(r)} \frac{\beta_{n-1}^{(r)} + \alpha_{n-1}^{(r)}}{\beta_n^{(r)} + \alpha_n^{(r)}}, \\ \alpha_n^{(r-1)} &= \alpha_n^{(r)} \frac{\gamma_{n-1}^{(r-1)}}{\gamma_n^{(r-1)}} = \alpha_n^{(r)} \frac{\beta_{n-1}^{(r)} + \alpha_{n-1}^{(r)}}{\beta_n^{(r)} + \alpha_n^{(r)}}. \end{aligned} \tag{2.3.14}$$

These relations can also be derived from the algorithms given by Jones and Magnus in [17].

We now return to the table (2.2.11) of the polynomials $B_n^{(r)}(z)$, $n \geq 0$, $r = 0, \pm 1, \pm 2, \dots$.

Most of the results established for the classical Padé approximants can be extended to the two-point Padé approximants. For example, there are several links involving the polynomials in the table (2.2.11) and they follow from the theory of the Padé approximants, (see McCabe [35]). In particular we consider the five polynomials $B_n^{(r-1)}(z)$, $B_{n-1}^{(r)}(z)$, $B_n^{(r)}(z)$, $B_{n+1}^{(r)}(z)$ and $B_n^{(r+1)}(z)$ displayed as follows

$$
\begin{array}{ccc}
 & B_n^{(r-1)}(z) & \\
 & & \\
B_{n-1}^{(r)}(z) & B_n^{(r)}(z) & B_{n+1}^{(r)}(z)\ . \\
 & & \\
 & B_n^{(r+1)}(z) &
\end{array}
$$

Firstly, the link involving the polynomials

$$
\begin{array}{ccc}
B_{n-1}^{(r)}(z) & \rightarrow & B_n^{(r)}(z) \\
 & & \downarrow \\
 & & B_n^{(r+1)}(z)
\end{array}
$$

is given by

$$B_n^{(r+1)}(z) = B_n^{(r)}(z) - \alpha_{n+1}^{(r)} B_{n-1}^{(r)}(z). \tag{2.3.15}$$

While that involving the polynomials

$$B_{n-1}^{(r)}(z) \ \rightarrow \ B_n^{(r)}(z) \ \rightarrow \ B_{n+1}^{(r)}(z)$$

is given by

$$B_{n+1}^{(r)}(z) = (z - \beta_{n+1}^{(r)}) B_n^{(r)}(z) - \alpha_{n+1}^{(r)} z B_{n-1}^{(r)}(z). \tag{2.3.16}$$

Similarly the polynomials

$$
\begin{array}{ccc}
B_n^{(r)}(z) & \leftarrow & B_{n+1}^{(r)}(z) \\
\downarrow & & \\
B_n^{(r+1)}(z) & &
\end{array}
$$

are related by

$$B_n^{(r+1)}(z) = \frac{1}{z}(B_{n+1}^{(r)}(z) + \beta_{n+1}^{(r)} B_n^{(r)}(z)). \tag{2.3.17}$$

Continuing, the link between the polynomials

$$\begin{array}{c} B_n^{(r-1)}(z) \\ \uparrow \\ B_n^{(r)}(z) \\ \uparrow \\ B_n^{(r+1)}(z), \end{array}$$

is given by

$$\begin{aligned} B_n^{(r-1)}(z) &= \frac{1}{\alpha_{n+1}^{(r)} + \beta_{n+1}^{(r)}} \left((\beta_{n+1}^{(r)} + z) B_n^{(r)}(z) - z B_n^{(r+1)}(z) \right) \\ &= \frac{1}{\gamma_{n+1}^{(r-1)}} \left((\beta_{n+1}^{(r)} + z) B_n^{(r)}(z) - z B_n^{(r+1)}(z) \right), \end{aligned} \tag{2.3.18}$$

while the polynomials

$$\begin{array}{ccc} B_n^{(r-1)}(z) & & \\ \uparrow & & \\ B_n^{(r)}(z) & \leftarrow & B_{n+1}^{(r)}(z) \end{array}$$

are related by

$$\begin{aligned} B_n^{(r-1)}(z) &= \frac{1}{\alpha_{n+1}^{(r)} + \beta_{n+1}^{(r)}} \left(z B_n^{(r)}(z) - B_{n+1}^{(r)}(z) \right) \\ &= \frac{1}{\gamma_{n+1}^{(r-1)}} \left(z B_n^{(r)}(z) - B_{n+1}^{(r)}(z) \right). \end{aligned} \tag{2.3.19}$$

Finally, the three polynomials

$$\begin{array}{ccc} & & B_n^{(r-1)}(z) \\ & & \uparrow \\ B_{n-1}^{(r)}(z) & \rightarrow & B_n^{(r)}(z) \end{array}$$

satisfy

$$\begin{aligned} B_n^{(r-1)}(z) &= \frac{1}{\alpha_{n+1}^{(r)} + \beta_{n+1}^{(r)}} \left(\beta_{n+1}^{(r)} B_n^{(r)}(z) + \alpha_{n+1}^{(r)} z B_{n-1}^{(r)}(z) \right) \\ &= \frac{1}{\gamma_{n+1}^{(r-1)}} \left(\beta_{n+1}^{(r)} B_n^{(r)}(z) + \alpha_{n+1}^{(r)} z B_{n-1}^{(r)}(z) \right) \\ &= \frac{1}{\gamma_n^{(r-1)}} \left(\beta_n^{(r-1)} B_n^{(r)}(z) + \alpha_{n+1}^{(r-1)} z B_{n-1}^{(r)}(z) \right). \end{aligned} \tag{2.3.20}$$

The first of these results, (2.3.15), can be proved by considering the monic polynomial of degree n given by

$$B_n^{(r-1)}(z) - \tau_{n,r-1}B_{n-1}^{(r-1)}(z).$$

From (2.2.1) we can see that

$$\int_a^b t^{-n+s+r}\left(B_n^{(r-1)}(z) - \tau_{n,r-1}B_{n-1}^{(r-1)}(z)\right)d\psi(t) = 0, \quad 0 \le s \le n-2.$$

However, if we choose $\tau_{n,r-1} = \alpha_{n+1}^{(r-1)} = \dfrac{\rho_{n,r-1}^{(r-1)}}{\rho_{n-1,r-1}^{(r-1)}}$, then the integral vanishes when $s = n-1$. Hence

$$\int_a^b t^{-n+s+r}\left(B_n^{(r-1)}(z) - \alpha_{n+1}^{(r-1)}B_{n-1}^{(r-1)}(z)\right)d\psi(t) = 0, \quad 0 \le s \le n-1.$$

The monic polynomial of degree n that satisfies the above condition is unique and equal to $B_n^{(r)}(z)$. Hence

$$B_n^{(r)}(z) = B_n^{(r-1)}(z) - \alpha_{n+1}^{(r-1)}B_{n-1}^{(r-1)}(z).$$

The relation (2.3.16) is the three-term recurrence relation (2.2.7). Then, writing (2.3.16) as

$$B_{n+2}^{(r)}(z) + \beta_{n+2}^{(r)}B_{n+1}^{(r)}(z) = z(B_{n+1}^{(r)}(z) - \alpha_{n+2}^{(r)}B_n^{(r)}(z)),$$

and using (2.3.15) gives

$$B_{n+2}^{(r)}(z) + \beta_{n+2}^{(r)}B_{n+1}^{(r)}(z) = zB_{n+1}^{(r+1)}(z).$$

Hence (2.3.17) follows.

From (2.3.15) and (2.3.17), we obtain, respectively

$$B_{n-1}^{(r-1)}(z) = \frac{1}{\alpha_{n+1}^{(r-1)}}\left(B_n^{(r-1)}(z) - B_n^{(r)}(z)\right)$$

and

$$B_{n-1}^{(r)}(z) = \frac{1}{z}\left(B_n^{(r-1)}(z) + \beta_n^{(r-1)} B_{n-1}^{(r-1)}(z)\right).$$

Substituting the last two relations in (2.3.15), we obtain

$$B_n^{(r+1)}(z) = B_n^{(r)}(z) - \frac{\alpha_{n+1}^{(r)}}{z}\left[B_n^{(r-1)}(z) + \frac{\beta_n^{(r-1)}}{\alpha_{n+1}^{(r-1)}}\left(B_n^{(r-1)}(z) - B_n^{(r)}(z)\right)\right],$$

or

$$zB_n^{(r+1)}(z) = zB_n^{(r)}(z) - \alpha_{n+1}^{(r)} B_n^{(r-1)}(z) - \frac{\alpha_{n+1}^{(r)}\beta_n^{(r-1)}}{\alpha_{n+1}^{(r-1)}}\left(B_n^{(r-1)}(z) - B_n^{(r)}(z)\right)$$

and from (2.3.9) the result (2.3.18) follows.

From (2.3.17)

$$zB_{n+1}^{(r+1)}(z) = B_{n+2}^{(r)}(z) + \beta_{n+2}^{(r)} B_{n+1}^{(r)}(z),$$

while from (2.3.18)

$$zB_{n+1}^{(r+1)}(z) = (\beta_{n+2}^{(r)} + z)B_{n+1}^{(r)}(z) - (\beta_{n+2}^{(r)} + \alpha_{n+2}^{(r)})B_{n+1}^{(r-1)}(z).$$

It then follows that

$$B_{n+1}^{(r-1)}(z) = \frac{1}{\alpha_{n+2}^{(r)} + \beta_{n+2}^{(r)}}\left(zB_{n+1}^{(r)}(z) - B_{n+2}^{(r)}(z)\right),$$

which is (2.3.19).

Finally, from (2.3.16),

$$zB_n^{(r)}(z) - B_{n+1}^{(r)}(z) = \beta_{n+1}^{(r)} B_n^{(r)}(z) + \alpha_{n+1}^{(r)} zB_{n-1}^{(r)}(z)),$$

and from (2.3.19) we see that (2.3.20) holds. □

As a consequence of (2.2.13) we know that the classical orthogonal polynomials with respect to the distribution $d\psi_l(t) = t^l d\psi(t)$ for $l = 0, \pm1, \pm2, ...$ satisfy the recurrence relations (2.2.14), namely

$$B_{n+1}^{(n+1+l)}(z) = (z - b_{n+1}^{(l)})B_n^{(n+l)}(z) - a_{n+1}^{(l)} B_{n-1}^{(n-1+l)}(z), \quad n \geq 1,$$

with $B_0^{(0+l)}(z) = 1,\ B_1^{(1+l)}(z) = z - b_1^{(l)}$.

From the links (2.3.15) and (2.3.19) between the orthogonal L-polynomials we can see that the "staircase sequence" of polynomials

$$\begin{array}{ccccc} B_{n-1}^{(n-1+l)}(z) & \rightarrow & B_n^{(n-1+l)}(z) & & \\ & & \downarrow & & \\ & & B_n^{(n+l)}(z) & \rightarrow & B_{n+1}^{(n+l)}(z) \\ & & & & \downarrow \\ & & & & B_{n+1}^{(n+1+l)}(z) \end{array}$$

are related by the relations

$$B_n^{(n-1+l)}(z) = B_n^{(n+l)}(z) + \alpha_{n+1}^{(n-1+l)} B_{n-1}^{(n-1+l)}(z), \tag{2.3.21}$$

$$B_{n+1}^{(n+l)}(z) = zB_n^{(n+l)}(z) - (\alpha_{n+1}^{(n+l)} + \beta_{n+1}^{(n+l)}) B_n^{(n-1+l)}(z), \tag{2.3.22}$$

and

$$B_{n+1}^{(n+1+l)}(z) = B_{n+1}^{(n+l)}(z) - \alpha_{n+2}^{(n+l)} B_n^{(n+l)}(z). \tag{2.3.23}$$

Substituting (2.3.21) and (2.3.22) in (2.3.23), we obtain

$$B_{n+1}^{(n+1+l)}(z) = \left(z - (\alpha_{n+1}^{(n+l)} + \beta_{n+1}^{(n+l)} + \alpha_{n+2}^{(n+l)})\right) B_n^{(n+l)}(z)$$

$$-(\alpha_{n+1}^{(n+l)} + \beta_{n+1}^{(n+l)}) \alpha_{n+1}^{(n-1+l)} B_{n-1}^{(n-1+l)}(z).$$

Hence, the coefficients of the recurrence relation (2.2.14) can be given in terms of the coefficients of the recurrence relation (2.2.7) as follows

$$b_{n+1}^{(l)} = \alpha_{n+1}^{(n+l)} + \beta_{n+1}^{(n+l)} + \alpha_{n+2}^{(n+l)} \quad \text{and} \quad a_{n+1}^{(l)} = (\alpha_{n+1}^{(n+l)} + \beta_{n+1}^{(n+l)}) \alpha_{n+1}^{(n-1+l)}.$$

2.4 The orthogonal Laurent polynomials

Jones, Thron *et al.* in [19, 25, 26, 27] studied strong moment problems and the orthogonal Laurent polynomials that are associated with them.

As it was mentioned in section 1.1, a Laurent polynomial is a function of the form

$$R(z) = \sum_{i=k}^{m} r_i z^i, \quad r_i \in \mathbb{C}, \quad -\infty < k \leq m < \infty. \tag{2.4.1}$$

The set $\mathcal{R}$ of all Laurent polynomials forms a linear space over $\mathbb{R}$. The authors denote by $\mathcal{R}_{k,m}$ the set of all polynomials of the form (2.4.1). For $m \geq 0$, they define

$$\mathcal{R}_{2m} = \{R(z) \in \mathcal{R}_{-m,m}\}$$

and

$$\mathcal{R}_{2m+1} = \{R(z) \in \mathcal{R}_{-m-1,m}\}.$$

The moments of the distribution $\psi(t)$ are given by

$$c_k = \int_a^b (-t)^k d\psi(t), \quad k = 0, \pm 1, \pm 2, \dots . \tag{2.4.2}$$

Let the function $\tilde{G}(z)$ be defined by the Stieltjes integral

$$\tilde{G}(z) = z \int_a^b \frac{1}{z+t} d\psi(t). \tag{2.4.3}$$

Then the two series

$$\tilde{L}_0 = -c_{-1}z - c_{-2}z^2 - c_{-3}z^3 - c_{-4}z^4 - \cdots \tag{2.4.4}$$

and

$$\tilde{L}_\infty = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \frac{c_4}{z^4} + \cdots, \tag{2.4.5}$$

are asymptotic expansions of $\tilde{G}(z)$ at $z = 0$ and $z = \infty$ respectively. The (n, n) two-point Padé approximant of $(\tilde{L}_0, \tilde{L}_\infty)$ is the nth approximant of the positive T-fraction

$$\frac{F_1 z}{1 + G_1 z} + \frac{F_2 z}{1 + G_2 z} + \frac{F_3 z}{1 + G_3 z} + \frac{F_4 z}{1 + G_4 z} + \cdots \quad F_k, G_k > 0. \tag{2.4.6}$$

The continued fraction (2.4.6) corresponds simultaneously to the series $\tilde{L}_0$ and $\tilde{L}_\infty$. That is, the nth approximant, $\frac{U_n(z)}{V_n(z)}$, which is a ratio of polynomials of degree n, agrees with n terms of each series when expanded accordingly.

We recall the moments defined by (2.1.1), namely

$$\mu_k = (-1)^k c_k = \int_a^b t^k d\psi(t), \quad k = 0, \pm 1, \pm 2, ...,$$

and the two series

$$L_0 = -\mu_{-1} - \mu_{-2} z - \mu_{-3} z^2 - \mu_{-4} z^3 - \cdots$$

and

$$L_\infty = \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \frac{\mu_2}{z^3} + \frac{\mu_3}{z^4} + \cdots.$$

From section 2.3, we know that the M-fraction (2.3.1) with $r = 0$, namely

$$\frac{\alpha_1^{(0)}}{z - \beta_1^{(0)}} - \frac{\alpha_2^{(0)} z}{z - \beta_2^{(0)}} - \frac{\alpha_3^{(0)} z}{z - \beta_3^{(0)}} - \frac{\alpha_4^{(0)} z}{z - \beta_4^{(0)}} - \cdots \quad \alpha_k^{(0)}, \beta_k^{(0)} > 0, \tag{2.4.7}$$

corresponds to L_0 and to L_∞. We recall that the numerator and denominator of the nth convergent of the M-fraction (2.4.7) are the polynomials $A_n^{(0)}(z)$ and $B_n^{(0)}(z)$ respectively.

We can see from the T-fraction (2.4.6) and the M-fraction (2.4.7) that

$$V_n(z) = \frac{B_n^{(0)}(-z)}{B_n^{(0)}(0)} \quad \text{and} \quad U_n(z) = \frac{-z A_n^{(0)}(-z)}{B_n^{(0)}(0)}, \tag{2.4.8}$$

where

$$F_1 = -c_{-1}; \quad G_n = \frac{1}{\beta_n^{(0)}}, \quad \text{and} \quad F_{n+1} = \frac{\alpha_{n+1}^{(0)}}{\beta_n^{(0)}\beta_{n+1}^{(0)}}, \quad n \geq 1.$$

The coefficients F_{n+1} and G_{n+1}, $n \geq 1$, can also be computed from

$$F_{n+1}^{(r)} = F_n^{(r+1)} + G_n^{(r+1)} - G_n^{(r)}, \quad r = ..., -1, 0, 1, ...$$

and

$$G_{n+1}^{(r)} = \frac{F_{n+1}^{(r)}}{F_{n+1}^{(r-1)}} G_n^{(r-1)}, \quad r = ..., -1, 0, 1, ... ,$$

with

$$F_1^{(r)} = 0, \quad G_1^{(r)} = -\frac{c_{-r-1}}{c_{-r}}, \quad r = ..., -1, 0, 1, ... ,$$

and

$$F_1 = -c_{-1}; \quad F_{n+1} = F_{n+1}^{(0)} \quad \text{and} \quad G_n = G_n^{(0)}, \quad n \geq 1 .$$

Jones, Njåstad and Thron in [19] defined a sequence of Laurent polynomials $\{Q_n(z)\}_{n=0}^{\infty}$, where $Q_{2n}(z) \in \mathcal{R}_{2n}$ and $Q_{2n+1}(z) \in \mathcal{R}_{2n+1}$,

$$Q_{2n}(z) = \frac{(-1)^n}{H_{2n}^{(-2n+1)}} \begin{vmatrix} c_{-2n} & \cdots & c_{-1} & (-z)^{-n} \\ \vdots & & \vdots & \vdots \\ c_{-1} & \cdots & c_{2n-2} & (-z)^{n-1} \\ c_0 & \cdots & c_{2n-1} & (-z)^n \end{vmatrix}, \quad n \geq 1,$$

and

$$Q_{2n+1}(z) = \frac{(-1)^n}{H_{2n+1}^{(-2n)}} \begin{vmatrix} c_{-2n-1} & \cdots & c_{-1} & (-z)^{-n-1} \\ \vdots & & \vdots & \vdots \\ c_{-2} & \cdots & c_{2n-1} & (-z)^{n-1} \\ c_{-1} & \cdots & c_{2n} & (-z)^n \end{vmatrix}, \quad n \geq 0,$$

with $Q_0(z) = 1$. Here

$$H_n^{(m)} = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+n-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m+n-1} & c_{m+n} & \cdots & c_{m+2n-2} \end{vmatrix}, \quad n \geq 1, \quad m = 0, \pm 1, \pm 2, ...,$$

and $H_0^{(m)} = 1$.

The even and odd order polynomials can, respectively, be written as

$$\begin{aligned} Q_{2n}(z) &= q_{2n,-n}z^{-n} + \cdots + q_{2n,n}z^n, & q_{2n,-n} = 1, \\ Q_{2n+1}(z) &= q_{2n+1,-n-1}z^{-n-1} + \cdots + q_{2n+1,n}z^n, & q_{2n+1,-n-1} = 1. \end{aligned}$$

The authors proved that the polynomials $Q_n(z)$, $n \geq 0$ satisfy the orthogonality conditions

$$\int_a^b Q_n(t)Q_m(t)d\psi(t) = \begin{cases} 0, & \text{if } n \neq m, \\ ||Q_n(t)||^2 > 0, & \text{if } n = m. \end{cases}$$

Hence they are called *orthogonal Laurent polynomials.*

The even and odd order orthogonal Laurent polynomials satisfy, respectively, the three-term recurrence relations

$$\begin{aligned} Q_{2n}(z) &= (1 - G_{2n}z)Q_{2n-1}(z) - F_{2n}Q_{2n-2}(z), & n \geq 1, \\ Q_{2n+1}(z) &= (z^{-1} - G_{2n+1})Q_{2n}(z) - F_{2n+1}Q_{2n-1}(z), & n \geq 1, \end{aligned}$$

with $Q_0(z) = 1$ and $Q_1 = z^{-1} + c_{-1}/c_0$.

They also defined the associated polynomials $P_n(z)$ by

$$P_n(z) = z\int_a^b \frac{Q_n(z) - Q_n(t)}{z - t} d\psi(t), \quad n \geq 0. \tag{2.4.9}$$

The polynomials $P_n(z)$ satisfy the same recurrence relations as the polynomials $Q_n(z)$, but with initials conditions $P_0(z) = 0$ and $P_1(z) = -F_1$.

The polynomials $Q_n(z)$ and $P_n(z)$, and the denominator $V_n(z)$ and numerator $U_n(z)$ of the nth convergent of the T-fraction (2.4.6), satisfy the relations

$$Q_{2n}(z) = \frac{V_{2n}(-z)}{z^n} \quad \text{and} \quad Q_{2n+1}(z) = \frac{V_{2n+1}(-z)}{z^{n+1}}, \quad n \geq 0,$$

$$P_{2n}(z) = \frac{U_{2n}(-z)}{z^n} \quad \text{and} \quad P_{2n+1}(z) = \frac{U_{2n+1}(-z)}{z^{n+1}}, \quad n \geq 0.$$

Hence, using (2.4.8), we obtain the relations between the orthogonal Laurent polynomials $Q_n(z)$ and the L-polynomials $B_n^{(0)}(z)$. Specifically, for $n \geq 0$,

$$Q_{2n}(z) = \frac{B_{2n}^{(0)}(z)}{z^n B_{2n}^{(0)}(0)} \quad \text{and} \quad Q_{2n+1}(z) = \frac{B_{2n+1}^{(0)}(z)}{z^{n+1} B_{2n+1}^{(0)}(0)}. \tag{2.4.10}$$

and

$$P_{2n}(z) = \frac{A_{2n}^{(0)}(z)}{z^{n-1} B_{2n}^{(0)}(0)} \quad \text{and} \quad P_{2n+1}(z) = \frac{A_{2n+1}^{(0)}(z)}{z^{n} B_{2n+1}^{(0)}(0)}. \tag{2.4.11}$$

In the literature, the orthogonal Laurent polynomials are sometimes called orthogonal L-polynomials.

2.5 The polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$

We define $B_n^{(0)}(z) = 0$, for $n < 0$. Then, for a fixed integer $r \geq 1$, we define the real monic polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$, $n \geq 0$, by

$$B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z) = B_n^{(0)}(z) + \lambda_{n,1}^{(r)} B_{n-1}^{(0)}(z) + \cdots + \lambda_{n,r}^{(r)} B_{n-r}^{(0)}(z), \tag{2.5.1}$$

where $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)} \in \mathbb{R}$.

From (2.2.1) and (2.5.1) it is easy to see that the conditions

$$\int_a^b t^{-n+s} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; t) d\psi(t) = 0, \quad r \leq s \leq n-1, \quad \text{for } n \geq r+1, \tag{2.5.2}$$

are satisfied.

The polynomial $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$, $n \geq r$, has at least $n - r$ zeros inside the interval (a, b) with odd multiplicity, otherwise equation (2.5.2) is contradicted. The remaning zeros, which may be inside or outside (a, b), can have even or odd multiplicity.

Using (2.2.2) and (2.5.1) we obtain

$$\int_a^b t^{-n+s} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; t) d\psi(t) = \rho_{n,-n+s}^{(0)} + \lambda_{n,1}^{(r)} \rho_{n-1,-n+s}^{(0)} + \cdots + \lambda_{n,r}^{(r)} \rho_{n-r,-n+s}^{(0)},$$

for all values of n.

For $n \geq r+1$, we set

$$\rho_{n,-n+s}^{(0)} + \lambda_{n,1}^{(r)} \rho_{n-1,-n+s}^{(0)} + \cdots + \lambda_{n,r}^{(r)} \rho_{n-r,-n+s}^{(0)} = 0, \quad n \leq s \leq n+r-1.$$

We obtain the r x r linear system

$$\begin{pmatrix} \rho_{n-1,0}^{(0)} & \rho_{n-2,0}^{(0)} & \cdots & \rho_{n-r,0}^{(0)} \\ \rho_{n-1,1}^{(0)} & \rho_{n-2,1}^{(0)} & \cdots & \rho_{n-r,1}^{(0)} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n-1,r-1}^{(0)} & \rho_{n-2,r-1}^{(0)} & \cdots & \rho_{n-r,r-1}^{(0)} \end{pmatrix} \begin{pmatrix} \lambda_{n,1}^{(r)} \\ \lambda_{n,2}^{(r)} \\ \vdots \\ \lambda_{n,r}^{(r)} \end{pmatrix} = \begin{pmatrix} -\rho_{n,0}^{(0)} \\ -\rho_{n,1}^{(0)} \\ \vdots \\ -\rho_{n,r-1}^{(0)} \end{pmatrix} . \tag{2.5.3}$$

For $1 \leq n \leq r$, we set

$$\rho_{n,-n+s}^{(0)} + \lambda_{n,1}^{(r)} \rho_{n-1,-n+s}^{(0)} + \cdots + \lambda_{n,n}^{(n)} \rho_{n-r,-n+s}^{(0)} = 0, \quad r \leq s \leq n+r-1,$$

and we obtain the n x n linear system

$$\begin{pmatrix} \rho_{n-1,r-n}^{(0)} & \rho_{n-2,r-n}^{(0)} & \cdots & \rho_{0,r-n}^{(0)} \\ \rho_{n-1,r+1-n}^{(0)} & \rho_{n-2,r+1-n}^{(0)} & \cdots & \rho_{0,r+1-n}^{(0)} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n-1,r-1}^{(0)} & \rho_{n-2,r-1}^{(0)} & \cdots & \rho_{0,r-1}^{(0)} \end{pmatrix} \begin{pmatrix} \lambda_{n,1}^{(r)} \\ \lambda_{n,2}^{(r)} \\ \vdots \\ \lambda_{n,n}^{(r)} \end{pmatrix} = \begin{pmatrix} -\rho_{n,r-n}^{(0)} \\ -\rho_{n,r+1-n}^{(0)} \\ \vdots \\ -\rho_{n,r-1}^{(0)} \end{pmatrix} . \tag{2.5.4}$$

Suppose that $\hat{\lambda}_{n,1}^{(r)}, ..., \hat{\lambda}_{n,r}^{(r)}$, for $n \geq r+1$, is a solution of the system (2.5.3) and for $1 \leq n \leq r$, is a solution of the system (2.5.4). Then

$$\int_a^b t^{-n+s} B_n(\hat{\lambda}_{n,1}^{(r)}, ..., \hat{\lambda}_{n,r}^{(r)}; t) d\psi(t) = 0, \quad r \leq s \leq n+r-1, \quad \text{for } n \geq 1. \tag{2.5.5}$$

From (2.2.1) we know that these conditions define a unique monic polynomial. The system thus has a unique solution and

$$B_n^{(r)}(z) = B_n(\hat{\lambda}_{n,1}^{(r)}, ..., \hat{\lambda}_{n,r}^{(r)}; z), \quad \text{for } n \geq 0. \tag{2.5.6}$$

We have included $n=0$ because $B_0(\hat\lambda_{0,1}^{(r)},...,\hat\lambda_{0,r}^{(r)};z) = B_0^{(0)}(z) = 1$.

From (2.3.15) we know that

$$B_n^{(r)}(z) = B_n^{(r-1)}(z) - \alpha_{n+1}^{(r-1)} B_{n-1}^{(r-1)}(z)\,,$$

and using again the relation (2.3.15) we obtain

$$B_n^{(r)}(z) = B_n^{(r-2)}(z) - [\alpha_{n+1}^{(r-1)} + \alpha_{n+1}^{(r-2)}]\, B_{n-1}^{(r-2)}(z) + \alpha_{n+1}^{(r-1)}\alpha_n^{(r-2)}\, B_{n-2}^{(r-2)}(z)$$

and

$$\begin{aligned} B_n^{(r)}(z) = B_n^{(r-3)}(z) &- [\alpha_{n+1}^{(r-1)} + \alpha_{n+1}^{(r-2)} + \alpha_{n+1}^{(r-3)}]\, B_{n-1}^{(r-3)}(z) \\ &+ [\alpha_{n+1}^{(r-1)}\alpha_n^{(r-2)} + \alpha_{n+1}^{(r-1)}\alpha_n^{(r-3)} + \alpha_{n+1}^{(r-2)}\alpha_n^{(r-3)}]\, B_{n-2}^{(r-3)}(z) \\ &- \alpha_{n+1}^{(r-1)}\alpha_n^{(r-2)}\alpha_{n-1}^{(r-3)}\, B_{n-3}^{(r-3)}(z). \end{aligned}$$

Continuing in this way we see that

$$\begin{aligned} B_n^{(r)}(z) = B_n^{(0)}(z) &- [\alpha_{n+1}^{(r-1)} + \alpha_{n+1}^{(r-2)} + \cdots + \alpha_{n+1}^{(0)}]\, B_{n-1}^{(0)}(z) \\ &+ [\alpha_{n+1}^{(r-1)}\alpha_n^{(r-2)} + \alpha_{n+1}^{(r-1)}\alpha_n^{(r-3)} + \cdots + \alpha_{n+1}^{(1)}\alpha_n^{(0)}]\, B_{n-2}^{(0)}(z) \\ &- [\alpha_{n+1}^{(r-1)}\alpha_n^{(r-2)}\alpha_{n-1}^{(r-3)} + \alpha_{n+1}^{(r-1)}\alpha_n^{(r-2)}\alpha_{n-1}^{(r-4)} + \cdots + \alpha_{n+1}^{(2)}\alpha_n^{(1)}\alpha_{n-1}^{(0)}]\, B_{n-3}^{(0)}(z) \\ &+ \cdots (-1)^r\, \alpha_{n+1}^{(r-1)}\alpha_n^{(r-2)}\cdots\alpha_{n+2-r}^{(0)}\, B_{n-r}^{(0)}(z)\,. \end{aligned}$$

Since $B_n^{(r)}(z) = B_n(\hat\lambda_{n,1}^{(r)},...,\hat\lambda_{n,r}^{(r)};z)$ we obtain

$$\hat\lambda_{n,i}^{(r)} = (-1)^i \sum_1^p \prod_{k=1}^i \alpha_{n+2-k}^{(m_k)}, \quad \text{for} \quad i=1,2,...,r, \tag{2.5.7}$$

where $p = \binom{r}{i}$ and the summation is over the $\binom{r}{i}$ ways of choosing the i integers $m_1, m_2, ..., m_i$, satisfying $0 \le m_i < m_{i-1} < ... < m_1 \le r-1$.

Thus, for example, we also have a way of expressing the classical orthogonal polynomials $B_n^{(n)}(z)$, $n \geq 0$, see (2.2.12), in terms of the orthogonal L-polynomials, $B_n^{(0)}(z)$. Specifically,

$$B_n^{(n)}(z) = B_n(\hat{\lambda}_{n,1}^{(n)}, ..., \hat{\lambda}_{n,n}^{(n)}; z) = B_n^{(0)}(z) + \hat{\lambda}_{n,1}^{(n)} B_{n-1}^{(0)}(z) + \cdots + \hat{\lambda}_{n,n}^{(n)} B_0^{(0)}(z),$$

where

$$\hat{\lambda}_{n,i}^{(n)} = (-1)^i \sum_{1}^{p} \prod_{k=1}^{i} \alpha_{n+2-k}^{(m_k)}, \quad \text{for} \quad i = 1, 2, ..., n,$$

with $p = \binom{n}{i}$ and the summation is over the $\binom{n}{i}$ ways of choosing the i integers $m_1, m_2, ..., m_i$, satisfying $0 \leq m_i < m_{i-1} < ... < m_1 \leq n-1$.

Chapter 3

Symmetric strong Stieltjes distributions

3.1 Symmetric strong Stieltjes distributions

In this chapter we consider strong Stieltjes distributions, $\psi(t)$, defined on the interval (a, b), where $0 < \beta < b \leq \infty$ and $a = \beta^2/b$, that satisfy the following inversive symmetric property

$$\frac{d\psi(t)}{t^\omega} = -\frac{d\psi(\beta^2/t)}{(\beta^2/t)^\omega}, \quad t \in (a, b), \quad 2\omega \in \boldsymbol{Z}. \tag{3.1.1}$$

We denote these symmetric strong Stieltjes distributions by $S^3[\omega, \beta, b]$ distributions.

When $d\psi(t)$ can be given in the form $v(t)dt$ the property (3.1.1) becomes

$$t^{1-\omega}v(t) = (\beta^2/t)^{1-\omega}v(\beta^2/t), \quad t \in (a, b), \quad 2\omega \in \boldsymbol{Z}.$$

Substituting t for β^2/t in (2.1.1) we obtain

$$\mu_m = -\int_a^b \beta^{2m} t^{-m} d\psi(\beta^2/t), \quad m = 0, \pm 1, \pm 2, \dots .$$

Hence, using the property (3.1.1) we can conclude that the moments of an $S^3[\omega,\beta,b]$ distribution satisfy the relation

$$\mu_m = \beta^{2(m+\omega)} \mu_{-m-2\omega}, \quad m = 0, \pm 1, \pm 2, \dots . \tag{3.1.2}$$

It follows from property (3.1.1), since

$$\frac{t^m d\psi(t)}{t^{\omega+m}} = -\frac{(\beta^2/t)^m d\psi(\beta^2/t)}{(\beta^2/t)^{\omega+m}}, \quad t \in (a,b), \quad 2\omega \in \boldsymbol{Z},$$

that for $m \in \boldsymbol{Z}$, $t^m d\psi(t)$ is an $S^3[\omega+m,\beta,b]$ distribution. Furthermore, multiplying both sides of (3.1.1) by $(t+\beta)/\sqrt{t}$, we obtain

$$\frac{(t+\beta) d\psi(t)}{t^{\omega+1/2}} = -\frac{(\beta^2/t+\beta) d\psi(\beta^2/t)}{(\beta^2/t)^{\omega+1/2}}, \quad t \in (a,b), \quad 2\omega \in \boldsymbol{Z}.$$

Hence, $(t+\beta)d\psi(t)$ is an $S^3[\omega+1/2,\beta,b]$ distribution.

From (3.1.2) it is easily verified that for an $S^3[\omega,\beta,b]$ distribution the Hankel determinants, $H_n^{(m)}$, defined by (2.1.2), satisfy the relations

$$H_n^{(m)} = \beta^{2n(m+\omega+n-1)} H_n^{(-m-2\omega-2n+2)}, \quad n \geq 1, \quad m = 0, \pm 1, \pm 2, \dots . \tag{3.1.3}$$

3.2 The $S^3[\omega,\beta,b]$ distributions and the polynomials $B_n^{(r)}(z)$

The polynomials $B_n^{(r)}(z)$, $n \geq 0$, $r = 0, \pm 1, \pm 2, \dots$, defined by (2.2.1), satisfy certain symmetric inversive relations when the distribution $\psi(t)$ satisfies the property (3.1.1). We present these relations in the following theorem and its corollary.

Theorem 3.2.1 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $n \geq 0$ and $j = 1 - 2\omega$, the related polynomials $B_n^{(l)}(z)$ satisfy*

$$\frac{z^n B_n^{(l)}(\beta^2/z)}{B_n^{(l)}(0)} = B_n^{(j-l)}(z), \quad \text{for } l = 0, \pm1, \pm2, \dots . \tag{3.2.1}$$

Proof: From the definition (2.2.1) we know that

$$\int_a^b t^{-n+s+l} B_n^{(l)}(t) d\psi(t) = 0, \quad 0 \leq s \leq n-1, \quad l = 0, \pm1, \pm2, \dots .$$

Setting $t = \beta^2/t$, dividing by $B_n^{(l)}(0)$ and using (3.1.1) we obtain

$$\int_a^b t^{-s-2\omega-l} \left(\frac{t^n B_n^{(l)}(\beta^2/t)}{B_n^{(l)}(0)} \right) d\psi(t) = 0, \quad 0 \leq s \leq n-1, \quad l = 0, \pm1, \pm2, \dots .$$

Since $j = 1-2\omega$, the substitution of the exponent $-s-2\omega-l$ by $-n+s+j-l$ then yields

$$\int_a^b t^{-n+s+j-l} \left(\frac{t^n B_n^{(l)}(\beta^2/t)}{B_n^{(l)}(0)} \right) d\psi(t) = 0, \quad 0 \leq s \leq n-1, \quad l = 0, \pm1, \pm2, \dots .$$

Using (2.2.1) for $B_n^{(j-l)}(z)$ and the fact that both polynomials are monic, we see that (3.2.1) holds. □

The following corollary brings out the particular details of the symmetric inversive property (3.2.1) when (i) 2ω is odd and (ii) 2ω is even.

Corollary 3.2.1.1 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $n \geq 0$, the associated polynomials $B_n^{(l)}(z)$ satisfy*

(i) for 2ω odd and $j = \frac{1}{2} - \omega$,

$$\frac{z^n B_n^{(j+l)}(\beta^2/z)}{B_n^{(j+l)}(0)} = B_n^{(j-l)}(z), \quad \text{for } l = 0, \pm1, \pm2, \dots , \tag{3.2.2}$$

(ii) for 2ω even and $j = -\omega$,

$$\frac{z^n B_n^{(j+l)}(\beta^2/z)}{B_n^{(j+l)}(0)} = B_n^{(j+1-l)}(z), \quad \text{for } l = 0, \pm1, \pm2, \dots . \tag{3.2.3}$$

We denote the zeros of the polynomial $B_n^{(r)}(z)$ by $z_{n,1}^{(r)}, z_{n,2}^{(r)}, ..., z_{n,n}^{(r)}$, in increasing order. From (3.2.1) we can see that for $j = 1-2\omega$ the zeros of the polynomial $B_n^{(l)}(z)$ and the zeros of the polynomial $B_n^{(j-l)}(z)$, satisfy

$$z_{n,i}^{(l)} = \frac{\beta^2}{z_{n,n+1-i}^{(j-l)}}, \quad i = 1,2,..,n, \quad l = 0, \pm 1, \pm 2, \dots . \tag{3.2.4}$$

When the distribution $\psi(t)$ satisfies (3.1.1), the coefficients $\beta_n^{(r)}$ and $\alpha_{n+1}^{(r)}$, $n \geq 1$, $r = 0, \pm 1, \pm 2, ...$ also satisfy particular symmetric relations. We present these relations in the following theorem and its corollary.

Theorem 3.2.2 *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $j = 1-2\omega$, the coefficients $\beta_n^{(r)}$ and $\alpha_{n+1}^{(r)}$ satisfy*

$$\beta_n^{(l)}\beta_n^{(j-l)} = \beta^2, \quad \text{and} \quad \frac{\alpha_{n+1}^{(j-l)}}{\alpha_{n+1}^{(l)}} = \frac{\beta_n^{(j-l)}}{\beta_{n+1}^{(l)}}, \quad n \geq 1, \quad l = 0, \pm 1, \pm 2, \dots . \tag{3.2.5}$$

Proof: From the recurrence relation (2.2.7) for $l = 0, \pm 1, \pm 2, ...$, we obtain

$$B_{n+1}^{(l)}(z) = (z - \beta_{n+1}^{(l)})B_n^{(l)}(z) - \alpha_{n+1}^{(l)} z B_{n-1}^{(l)}(z), \quad n \geq 1,$$

with $B_0^{(l)}(z) = 1$, and $B_1^{(l)}(z) = z - \beta_1^{(l)}$.

Replacing z by β^2/z and multiplying by z^{n+1} we obtain

$$z^{n+1}B_{n+1}^{(l)}(\beta^2/z) = (\beta^2 - z\beta_{n+1}^{(l)})z^n B_n^{(l)}(\beta^2/z) - \alpha_{n+1}^{(l)}\beta^2 z z^{n-1} B_{n-1}^{(l)}(\beta^2/z), \quad n \geq 1.$$

Since $B_n^{(l)}(0) = -\beta_n^{(l)} B_{n-1}^{(l)}(0)$, then dividing by $B_{n+1}^{(l)}(0)$ yields

$$\frac{z^{n+1}B_{n+1}^{(l)}(\beta^2/z)}{B_{n+1}^{(l)}(0)} = \left(z - \frac{\beta^2}{\beta_{n+1}^{(l)}}\right)\frac{z^n B_n^{(l)}(\beta^2/z)}{B_n^{(l)}(0)} - \frac{\alpha_{n+1}^{(l)}\beta^2}{\beta_{n+1}^{(l)}\beta_n^{(l)}} z \frac{z^{n-1}B_{n-1}^{(l)}(\beta^2/z)}{B_{n-1}^{(l)}(0)}, n \geq 1.$$

Finally, using (3.2.1), we obtain

$$B_{n+1}^{(j-l)}(z) = \left(z - \frac{\beta^2}{\beta_{n+1}^{(l)}}\right) B_n^{(j-l)}(z) - \frac{\alpha_{n+1}^{(l)}\beta^2}{\beta_{n+1}^{(l)}\beta_n^{(l)}} z B_{n-1}^{(j-l)}(z), \quad n \geq 1, \tag{3.2.6}$$

with $B_0^{(j-l)}(z) = 1$, and $B_1^{(j-l)}(z) = z - \beta^2/\beta_1^{(l)}$.

The result follows from substituting r by $j - l$ in (2.2.7) and comparing the result with (3.2.6). □

Once again the following corollary is a restatement of the symmetric relations in (3.2.5), where we can see the behaviour of the coefficients $\beta_n^{(r)}$ and $\alpha_{n+1}^{(r)}$, $n \geq 1$, $r = 0, \pm 1, \pm 2, \ldots$, when (i) 2ω is odd and (ii) 2ω is even.

Corollary 3.2.2.1 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then the coefficients $\beta_n^{(r)}$ and $\alpha_{n+1}^{(r)}$, for $n \geq 1$, satisfy*
(i) for 2ω odd and $j = \frac{1}{2} - \omega$,

$$\beta_n^{(j+l)} \beta_n^{(j-l)} = \beta^2 \quad \text{and} \quad \frac{\alpha_{n+1}^{(j+l)}}{\alpha_{n+1}^{(j-l)}} = \frac{\beta_n^{(j+l)}}{\beta_{n+1}^{(j-l)}}, \quad l = 0, \pm 1, \pm 2, \ldots, \tag{3.2.7}$$

consequently, when $l = 0$, $\beta_n^{(j)} = \beta$, $n \geq 1$.
(ii) for 2ω even and $j = -\omega$

$$\beta_n^{(j+l)} \beta_n^{(j+1-l)} = \beta^2 \quad \text{and} \quad \frac{\alpha_{n+1}^{(j+l)}}{\alpha_{n+1}^{(j+1-l)}} = \frac{\beta_n^{(j+l)}}{\beta_{n+1}^{(j+1-l)}}, \quad l = 0, \pm 1, \pm 2, \ldots. \tag{3.2.8}$$

As a consequence of the above corollary the algorithm 2.1 to construct the $\alpha - \beta$ table can be modified. We now use the moments μ_r, $r = j-1, j, j+1, \ldots$, to construct the elements $\beta_1^{(r)}$, for $r = j, j+1, \ldots$. Then we use the equations (3.2.7) and (3.2.8) to generate the remaining elements of the table. The new algorithm can be given as follows.

Algorithm 3.1

$$\text{Setting } j = \begin{cases} \frac{1}{2} - \omega, & \text{if } 2\omega \text{ odd} \\ -\omega, & \text{if } 2\omega \text{ even,} \end{cases}$$

Given μ_r, for $r = j-1, j, j+1, \ldots,$

for $r = j, j+1, \ldots$

$$\alpha_1^{(r)} = 0, \quad \beta_1^{(r)} = \frac{\mu_r}{\mu_{r-1}},$$

for $n = 2, 3, \ldots$

for $r = j, j+1, \ldots$

$$\alpha_n^{(r)} = \beta_{n-1}^{(r+1)} + \alpha_{n-1}^{(r+1)} - \beta_{n-1}^{(r)},$$

for $r = j+1, j+2, \ldots$

$$\beta_n^{(r)} = \frac{\alpha_n^{(r)}\,\beta_{n-1}^{(r-1)}}{\alpha_n^{(r-1)}},$$

$$\beta_n^{(j)} = \begin{cases} \beta, & \text{if } 2\omega \text{ odd} \\ \dfrac{\beta^2}{\beta_n^{(j+1)}}, & \text{if } 2\omega \text{ even,} \end{cases}$$

for $n = 1, 2, \ldots$

for $l = 1, 2, \ldots$

$$\beta_n^{(j-l)} = \begin{cases} \dfrac{\beta^2}{\beta_n^{(j+l)}}, & \text{if } 2\omega \text{ odd} \\ \dfrac{\beta^2}{\beta_n^{(j+1+l)}}, & \text{if } 2\omega \text{ even,} \end{cases}$$

$$\alpha_{n+1}^{(j-l)} = \begin{cases} \dfrac{\alpha_{n+1}^{(j+l)}\,\beta_{n+1}^{(j-l)}}{\beta_n^{(j+l)}}, & \text{if } 2\omega \text{ odd} \\ \dfrac{\alpha_{n+1}^{(j+1+l)}\,\beta_n^{(j-l)}}{\beta_{n+1}^{(j+l)}}, & \text{if } 2\omega \text{ even.} \end{cases}$$

The last two relations link the coefficients in the rhomboid

$$\begin{array}{ccccc} \beta_n^{(j-l)} & \rightarrow & \alpha_{n+1}^{(j-l)} & & \\ & \nwarrow & & \nwarrow & \\ & & \nwarrow & & \nwarrow \\ & & \alpha_{n+1}^{(j+l)} & \rightarrow & \beta_{n+1}^{(j+l)} \end{array},$$

for 2ω odd, $j = \frac{1}{2} - \omega$ and $l = 1, 2, \ldots$, and in the rhomboid

$$
\begin{array}{cccccc}
\beta_n^{(j-l)} & \rightarrow & \alpha_{n+1}^{(j-l)} & & & \\
& \nwarrow & & \nwarrow & & \\
& & \nwarrow & & \nwarrow & \\
& & \alpha_{n+1}^{(j+1+l)} & \rightarrow & \beta_{n+1}^{(j+1+l)} &
\end{array},
$$

for 2ω even, $j = -\omega$ and $l = 1, 2, \ldots$.

See Bracciali, McCabe and Sri Ranga [8] for a published version of the main results in these last two sections.

3.3 The $S^3[\omega, \beta, b]$ distributions when 2ω odd and when 2ω even

We now use the notation ω^o and $\psi^o(t)$ when 2ω is odd and ω^e and $\psi^e(t)$ when 2ω is even.

We recall that if $\psi(t)$ is an $S^3[\omega, \beta, b]$ distribution, then, for $m \in \boldsymbol{Z}$,

$$t^m d\psi(t) \text{ is an } S^3[\omega + m, \beta, b] \text{ distribution} \tag{3.3.1}$$

and

$$(t + \beta)^m d\psi(t) \text{ is an } S^3[\omega + m/2, \beta, b] \text{ distribution.} \tag{3.3.2}$$

Using the properties (3.3.1) and (3.3.2) we see that the following result holds.

Theorem 3.3.1 *Let $\psi^o(t)$ be an $S^3[\omega^o, \beta, b]$ distribution where $2\omega^o$ is odd and $\psi^e(t)$ be an $S^3[\omega^e, \beta, b]$ distribution where $2\omega^e$ is even. They satisfy the relation*

$$d\psi^o(t) = (t + \beta) d\psi^e(t), \quad t \in (a, b), \tag{3.3.3}$$

or the relation

$$d\psi^e(t) = \frac{t+\beta}{t} d\psi^o(t), \quad t \in (a, b), \tag{3.3.4}$$

where $\beta = \sqrt{ab}$, *if and only if*

$$\omega^o = \omega^e + \frac{1}{2}. \tag{3.3.5}$$

From Corollary 3.2.1.1, for the case $2\omega^o$ odd, $j = j^o = \frac{1}{2} - \omega^o$, while for $2\omega^e$ even, $j = j^e = -\omega^e$. From the above theorem $\omega^o = \omega^e + \frac{1}{2}$ hence $j^o = j^e$. To simplify the notation we set $j = j^o = j^e$.

If we set

$$d\psi_j^o(t) = t^j d\psi^o(t) \quad \text{and} \quad d\psi_j^e(t) = t^j d\psi^e(t)$$

then, from the properties (3.3.1) and (3.3.2), we obtain
(*i*) $d\psi_j^o(t)$ is an $S^3[\omega^o + j, \beta, b] = S^3[1/2, \beta, b]$ distribution,
(*ii*) $d\psi_j^e(t)$ is an $S^3[\omega^e + j, \beta, b] = S^3[0, \beta, b]$ distribution.

Then, from (3.3.3) and (3.3.4),

$$d\psi_j^o(t) = (t+\beta) d\psi_j^e(t) \quad \text{or} \quad d\psi_j^e(t) = \frac{t+\beta}{t} d\psi_j^o(t).$$

Since $d\psi_j^o(t)$ is an $S^3[1/2, \beta, b]$ distribution and $d\psi_j^e(t)$ is an $S^3[0, \beta, b]$ distribution, we may extend some results given by Sri Ranga and McCabe [50] and by Sri Ranga [45]. If we consider the distributions $d\psi_j^o(t)$ and $d\psi_j^e(t)$, which are related by

$$d\psi_j^o(t) = (t+\beta) d\psi_j^e(t),$$

then the coefficients $\beta_n^{o(j)}$ and $\alpha_{n+1}^{o(j)}$, $n \geq 1$, with respect to $d\psi_j^o(t)$, and the coefficients $\beta_n^{e(j)}$ and $\alpha_{n+1}^{e(j)}$ $n \geq 1$, with respect to $d\psi_j^e(t)$, are related by

$$\beta_n^{o(j)} = \beta, \qquad \alpha_{n+1}^{o(j)} = \beta(l_{n+1}^{(j)} + 1)(l_n^{(j)} - 1), \quad n \geq 1, \tag{3.3.6}$$

where $l_n^{(j)} = \sqrt{\dfrac{\gamma_n^{e(j)}}{\beta_n^{e(j)}}}$ and $\gamma_n^{e(j)} = \beta_n^{e(j)} + \alpha_{n+1}^{e(j)}$, $n \geq 1$.

They are also related for $n \geq 1$, by

$$\beta_n^{e(j)} = \beta\,\frac{l_{n-1}^{(j)}}{l_n^{(j)}}, \qquad \alpha_{n+1}^{e(j)} = \beta_n^{e(j)}(l_n^{(j)2} - 1), \tag{3.3.7}$$

where $l_{n+1}^{(j)} = \dfrac{\alpha_{n+1}^{o(j)}/\beta}{l_n^{(j)} - 1} - 1$, $n \geq 1$ and $l_0^{(j)} = 1$. We need to calculate the coefficient $l_1^{(j)}$ from $\beta_1^{e(j)}$.

Further, using Corollary 3.2.1.1, we may prove that, for $n \geq 1$,

$$\beta_n^{e(j+1)} = \beta\,\frac{l_n^{(j)}}{l_{n-1}^{(j)}}, \qquad \alpha_n^{e(j+1)} = \beta_{n+1}^{e(j+1)}(l_n^{(j)2} - 1). \tag{3.3.8}$$

On the other hand, if we consider the relation

$$d\psi_j^e(t) = \frac{t+\beta}{t}\,d\psi_j^o(t),$$

then the coefficients $\beta_n^{o(j)}$, $\alpha_{n+1}^{o(j)}$, $\beta_n^{e(j)}$ and $\alpha_{n+1}^{e(j)}$, $n \geq 1$, are related by

$$\beta_n^{o(j)} = \beta, \qquad \alpha_{n+1}^{o(j)} = \beta(l_{n-1}^{(j)} + 1)(l_n^{(j)} - 1), \quad n \geq 1, \tag{3.3.9}$$

where $l_n^{(j)} = \sqrt{\dfrac{\gamma_n^{e(j)}}{\beta_n^{e(j)}}}$ and $\gamma_n^{e(j)} = \beta_n^{e(j)} + \alpha_{n+1}^{e(j)}$, $n \geq 1$.

They are also related by

$$\beta_n^{e(j)} = \beta\,\frac{l_{n-1}^{(j)}}{l_n^{(j)}}, \qquad \alpha_{n+1}^{e(j)} = \beta l_{n-1}^{(j)} l_n^{(j)} - \beta_n^{e(j)} = \beta_n^{e(j)}(l_n^{(j)2} - 1), \tag{3.3.10}$$

for $n \geq 1$, where $l_n^{(j)} = \dfrac{\alpha_{n+1}^{o(j)}/\beta}{l_{n-1}^{(j)} + 1} + 1$, $n \geq 1$ and $l_0^{(j)} = 1$.

The results in this section, and in the first example in the next section, have been submitted for publication. See Bracciali [5].

3.4 Some examples of $S^3[\omega,\beta,b]$ distributions

In this section we present some examples of $S^3[\omega,\beta,b]$ distributions.

Example 3.1 *The distributions defined by*

$$(i)\quad d\psi(t)=\frac{(1+\beta/t)^{1-2\omega}}{\sqrt{b-t}\sqrt{t-a}}dt, \qquad\qquad (ii)\quad d\psi(t)=\frac{1+(\beta/t)^{1-2\omega}}{\sqrt{b-t}\sqrt{t-a}}dt,$$

on (a,b), *where* $0<\beta<b<\infty$ *and* $\beta=\sqrt{ab}$, *are* $S^3[\omega,\beta,b]$ *distributions.*

These distributions are deduced from an example of an strong distribution given by Sri Ranga and McCabe in [49]. They also were considered by Sri Ranga *et al.* in [44, 47, 48, 50] as examples of $S^3[1/2,\beta,b]$ and $S^3[0,\beta,b]$ distributions.

First we consider the distribution defined in example 3.1.(i), namely

$$d\psi(t)=\frac{(1+\beta/t)^{1-2\omega}}{\sqrt{b-t}\sqrt{t-a}}dt,$$

where $0<\beta<b<\infty$, $a=\beta^2/b$ and $2\omega\in\boldsymbol{Z}$.

Theorem 3.4.1 *For the* $S^3[\omega,\beta,b]$ *distribution defined in example 3.1.(i), and for any* $\omega\geq 1/2$, *the coefficients* $\alpha_n^{(r)}$ *and* $\beta_n^{(r)}$, *from the recurrence relation (2.2.7), satisfy*

$$\begin{aligned}\beta_n^{(r)}&=\beta,\quad \alpha_n^{(r)}=\alpha,\quad \text{for } r>j,\quad n\geq r+2\omega+1,\\ \beta_n^{(r)}&=\beta,\quad \alpha_{n+1}^{(r)}=\alpha,\quad \text{for } r\leq j,\quad n\geq -r+2,\end{aligned}\qquad(3.4.1)$$

where

$$\beta=\sqrt{ab},\quad \alpha=\frac{(\sqrt{b}-\sqrt{a})^2}{4},$$

and

$$j=\begin{cases}-\omega+\frac{1}{2}, & \text{for } 2\omega \text{ odd}\\ -\omega, & \text{for } 2\omega \text{ even.}\end{cases}$$

This means that some of the coefficients $\alpha_n^{(r)}$ are equal to α and some of the coefficients $\beta_n^{(r)}$ are equal to β. They appear in a region enclosed by "staircase" lines in the $\alpha - \beta$ table. It is easy to visualize this behaviour from examples of the $\alpha - \beta$ table.

For example, for $\omega = 1/2$ and $j = 0$, the coefficients $\alpha_n^{(r)}$ and $\beta_n^{(r)}$ related to the distribution

$$d\psi(t) = \frac{1}{\sqrt{b-t}\sqrt{t-a}}dt,$$

satisfy

$$\begin{aligned} \beta_n^{(r)} &= \beta, \quad \alpha_n^{(r)} = \alpha, \quad \text{for } r > 0, \quad n \geq r+2, \\ \beta_n^{(r)} &= \beta, \quad \alpha_{n+1}^{(r)} = \alpha, \quad \text{for } r \leq 0, \quad n \geq -r+2. \end{aligned}$$

The $\alpha - \beta$ table is

r	$\beta_1^{(r)}$	$\alpha_2^{(r)}$	$\beta_2^{(r)}$	$\alpha_3^{(r)}$	$\beta_3^{(r)}$	$\alpha_4^{(r)}$	$\beta_4^{(r)}$	$\alpha_5^{(r)}$	$\beta_5^{(r)}$	$\alpha_6^{(r)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-3	$\beta_1^{(-3)}$	$\alpha_2^{(-3)}$	$\beta_2^{(-3)}$	$\alpha_3^{(-3)}$	$\beta_3^{(-3)}$	$\alpha_4^{(-3)}$	$\beta_4^{(-3)}$	$\alpha_5^{(-3)}$	β	α	$\cdots$
-2	$\beta_1^{(-2)}$	$\alpha_2^{(-2)}$	$\beta_2^{(-2)}$	$\alpha_3^{(-2)}$	$\beta_3^{(-2)}$	$\alpha_4^{(-2)}$	β	α	β	α	$\cdots$
-1	$\beta_1^{(-1)}$	$\alpha_2^{(-1)}$	$\beta_2^{(-1)}$	$\alpha_3^{(-1)}$	β	α	β	α	β	α	$\cdots$
0	β	2α	β	α	β	α	β	α	β	α	$\cdots$
1	$\beta_1^{(1)}$	$\alpha_2^{(1)}$	$\beta_2^{(1)}$	α	β	α	β	α	β	α	$\cdots$
2	$\beta_1^{(2)}$	$\alpha_2^{(2)}$	$\beta_2^{(2)}$	$\alpha_3^{(2)}$	$\beta_3^{(2)}$	α	β	α	β	α	$\cdots$
3	$\beta_1^{(3)}$	$\alpha_2^{(3)}$	$\beta_2^{(3)}$	$\alpha_3^{(3)}$	$\beta_3^{(3)}$	$\alpha_4^{(3)}$	$\beta_4^{(3)}$	α	β	α	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

For $\omega = 1$ and $j = -1$, the distribution

$$d\psi(t) = \frac{(1+\sqrt{ab}/t)^{-1}}{\sqrt{b-t}\sqrt{t-a}}dt,$$

yields the $\alpha - \beta$ table

r	$\beta_1^{(r)}$	$\alpha_2^{(r)}$	$\beta_2^{(r)}$	$\alpha_3^{(r)}$	$\beta_3^{(r)}$	$\alpha_4^{(r)}$	$\beta_4^{(r)}$	$\alpha_5^{(r)}$	$\beta_5^{(r)}$	$\alpha_6^{(r)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-3	$\beta_1^{(-3)}$	$\alpha_2^{(-3)}$	$\beta_2^{(-3)}$	$\alpha_3^{(-3)}$	$\beta_3^{(-3)}$	$\alpha_4^{(-3)}$	$\beta_4^{(-3)}$	$\alpha_5^{(-3)}$	β	α	$\cdots$
-2	$\beta_1^{(-2)}$	$\alpha_2^{(-2)}$	$\beta_2^{(-2)}$	$\alpha_3^{(-2)}$	$\beta_3^{(-2)}$	$\alpha_4^{(-2)}$	β	α	β	α	$\cdots$
-1	$\beta_1^{(-1)}$	$\alpha_2^{(-1)}$	$\beta_2^{(-1)}$	$\alpha_3^{(-1)}$	β	α	β	α	β	α	$\cdots$
0	$\beta_1^{(0)}$	$\alpha_2^{(0)}$	$\beta_2^{(0)}$	α	β	α	β	α	β	α	$\cdots$
1	$\beta_1^{(1)}$	$\alpha_2^{(1)}$	$\beta_2^{(1)}$	$\alpha_3^{(1)}$	$\beta_3^{(1)}$	α	β	α	β	α	$\cdots$
2	$\beta_1^{(2)}$	$\alpha_2^{(2)}$	$\beta_2^{(2)}$	$\alpha_3^{(2)}$	$\beta_3^{(2)}$	$\alpha_4^{(2)}$	$\beta_4^{(2)}$	α	β	α	$\cdots$
3	$\beta_1^{(3)}$	$\alpha_2^{(3)}$	$\beta_2^{(3)}$	$\alpha_3^{(3)}$	$\beta_3^{(3)}$	$\alpha_4^{(3)}$	$\beta_4^{(3)}$	$\alpha_5^{(3)}$	$\beta_5^{(3)}$	α	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

That is

$$\beta_n^{(r)} = \beta, \quad \alpha_n^{(r)} = \alpha, \quad \text{for } r > -1, \quad n \geq r+3,$$
$$\beta_n^{(r)} = \beta, \quad \alpha_{n+1}^{(r)} = \alpha, \quad \text{for } r \leq -1, \quad n \geq -r+2.$$

Further, for $\omega = 3/2$ and $j = -1$, the distribution

$$d\psi(t) = \frac{(1+\sqrt{ab}/t)^{-2}}{\sqrt{b-t}\sqrt{t-a}}dt,$$

yields the $\alpha - \beta$ table

r	$\beta_1^{(r)}$	$\alpha_2^{(r)}$	$\beta_2^{(r)}$	$\alpha_3^{(r)}$	$\beta_3^{(r)}$	$\alpha_4^{(r)}$	$\beta_4^{(r)}$	$\alpha_5^{(r)}$	$\beta_5^{(r)}$	$\alpha_6^{(r)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-3	$\beta_1^{(-3)}$	$\alpha_2^{(-3)}$	$\beta_2^{(-3)}$	$\alpha_3^{(-3)}$	$\beta_3^{(-3)}$	$\alpha_4^{(-3)}$	$\beta_4^{(-3)}$	$\alpha_5^{(-3)}$	β	α	$\cdots$
-2	$\beta_1^{(-2)}$	$\alpha_2^{(-2)}$	$\beta_2^{(-2)}$	$\alpha_3^{(-2)}$	$\beta_3^{(-2)}$	$\alpha_4^{(-2)}$	β	α	β	α	$\cdots$
-1	β	$\alpha_2^{(-1)}$	β	$\alpha_3^{(-1)}$	β	α	β	α	β	α	$\cdots$
0	$\beta_1^{(0)}$	$\alpha_2^{(0)}$	$\beta_2^{(0)}$	$\alpha_3^{(0)}$	$\beta_3^{(0)}$	α	β	α	β	α	$\cdots$
1	$\beta_1^{(1)}$	$\alpha_2^{(1)}$	$\beta_2^{(1)}$	$\alpha_3^{(1)}$	$\beta_3^{(1)}$	$\alpha_4^{(1)}$	$\beta_4^{(1)}$	α	β	α	$\cdots$
2	$\beta_1^{(2)}$	$\alpha_2^{(2)}$	$\beta_2^{(2)}$	$\alpha_3^{(2)}$	$\beta_3^{(2)}$	$\alpha_4^{(2)}$	$\beta_4^{(2)}$	$\alpha_5^{(2)}$	$\beta_5^{(2)}$	α	$\cdots$
3	$\beta_1^{(3)}$	$\alpha_2^{(3)}$	$\beta_2^{(3)}$	$\alpha_3^{(3)}$	$\beta_3^{(3)}$	$\alpha_4^{(3)}$	$\beta_4^{(3)}$	$\alpha_5^{(3)}$	$\beta_5^{(3)}$	$\alpha_6^{(3)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

That is

$$\beta_n^{(r)}=\beta,\quad \alpha_n^{(r)}=\alpha,\quad \text{for } r>-1,\quad n\ge r+4,$$
$$\beta_n^{(r)}=\beta,\quad \alpha_{n+1}^{(r)}=\alpha,\quad \text{for } r\le -1,\quad n\ge -r+2.$$

Similarly, the $\alpha-\beta$ table

r	$\beta_1^{(r)}$	$\alpha_2^{(r)}$	$\beta_2^{(r)}$	$\alpha_3^{(r)}$	$\beta_3^{(r)}$	$\alpha_4^{(r)}$	$\beta_4^{(r)}$	$\alpha_5^{(r)}$	$\beta_5^{(r)}$	$\alpha_6^{(r)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-3	$\beta_1^{(-3)}$	$\alpha_2^{(-3)}$	$\beta_2^{(-3)}$	$\alpha_3^{(-3)}$	$\beta_3^{(-3)}$	$\alpha_4^{(-3)}$	$\beta_4^{(-3)}$	$\alpha_5^{(-3)}$	β	α	$\cdots$
-2	$\beta_1^{(-2)}$	$\alpha_2^{(-2)}$	$\beta_2^{(-2)}$	$\alpha_3^{(-2)}$	$\beta_3^{(-2)}$	$\alpha_4^{(-2)}$	β	α	β	α	$\cdots$
-1	$\beta_1^{(-1)}$	$\alpha_2^{(-1)}$	$\beta_2^{(-1)}$	$\alpha_3^{(-1)}$	$\beta_3^{(-1)}$	α	β	α	β	α	$\cdots$
0	$\beta_1^{(0)}$	$\alpha_2^{(0)}$	$\beta_2^{(0)}$	$\alpha_3^{(0)}$	$\beta_3^{(0)}$	$\alpha_4^{(0)}$	$\beta_4^{(0)}$	α	β	α	$\cdots$
1	$\beta_1^{(1)}$	$\alpha_2^{(1)}$	$\beta_2^{(1)}$	$\alpha_3^{(1)}$	$\beta_3^{(1)}$	$\alpha_4^{(1)}$	$\beta_4^{(1)}$	$\alpha_5^{(1)}$	$\beta_5^{(1)}$	α	$\cdots$
2	$\beta_1^{(2)}$	$\alpha_2^{(2)}$	$\beta_2^{(2)}$	$\alpha_3^{(2)}$	$\beta_3^{(2)}$	$\alpha_4^{(2)}$	$\beta_4^{(2)}$	$\alpha_5^{(2)}$	$\beta_5^{(2)}$	$\alpha_6^{(2)}$	$\cdots$
3	$\beta_1^{(3)}$	$\alpha_2^{(3)}$	$\beta_2^{(3)}$	$\alpha_3^{(3)}$	$\beta_3^{(3)}$	$\alpha_4^{(3)}$	$\beta_4^{(3)}$	$\alpha_5^{(3)}$	$\beta_5^{(3)}$	$\alpha_6^{(3)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

is related to the distribution

$$d\psi(t)=\frac{(1+\sqrt{ab}/t)^{-3}}{\sqrt{b-t}\sqrt{t-a}}dt,$$

where $\omega=2$. That is

$$\beta_n^{(r)}=\beta,\quad \alpha_n^{(r)}=\alpha,\quad \text{for } r>-2,\quad n\ge r+5,$$
$$\beta_n^{(r)}=\beta,\quad \alpha_{n+1}^{(r)}=\alpha,\quad \text{for } r\le -2,\quad n\ge -r+2.$$

We now prove Theorem 3.4.1.

In Sri Ranga [44] we can find that, for $\omega=1/2$

$$\beta_n^{(0)}=\beta,\ n\ge 1,\qquad \alpha_2^{(0)}=2\alpha,\quad \text{and}\quad \alpha_{n+1}^{(0)}=\alpha,\ \ n\ge 2.$$

From the equations (2.3.13), we see that for $r>0$ and $n\ge r+2$,

$$\beta_n^{(r)}=\beta,\quad \text{and}\quad \alpha_n^{(r)}=\alpha.$$

In addition, from the equations (2.3.14), we see that for $r < 0$ and $n \geq -r+2$,

$$\beta_n^{(r)} = \beta, \quad \text{and} \quad \alpha_{n+1}^{(r)} = \alpha.$$

Hence, the result (3.4.1) holds for $\omega = 1/2$.

To show the result for $\omega = 1$, we use the relation

$$d\psi^o(t) = (t+\beta)d\psi^e(t).$$

We set

$$d\psi^o(t) = \frac{1}{\sqrt{b-t}\sqrt{t-a}}dt, \quad \omega^o = 1/2$$

and

$$d\psi^e(t) = \frac{d\psi^o(t)}{t+\beta} = t^{-1}\frac{(1+\beta/t)^{-1}}{\sqrt{b-t}\sqrt{t-a}}dt, \quad \omega^e = 0.$$

Since we know that

$$\beta_n^{o(0)} = \beta,\ n \geq 1, \qquad \alpha_2^{o(0)} = 2\alpha, \quad \text{and} \quad \alpha_{n+1}^{o(0)} = \alpha, \ \ n \geq 1,$$

then, by using the relations (3.3.7), we obtain

$$\beta_1^{e(0)} = \frac{\beta}{2l-1}, \qquad \alpha_2^{e(0)} = \beta\left((2l-1) - \frac{1}{2l-1}\right),$$

$$\beta_2^{e(0)} = \frac{\beta(2l-1)}{l}, \qquad \alpha_3^{e(0)} = \beta(2l-1)\left(l - \frac{1}{l}\right),$$

and

$$\beta_n^{e(0)} = \beta, \qquad \alpha_{n+1}^{e(0)} = \alpha, \quad n \geq 3.$$

where $l = \sqrt{1+\alpha/\beta}$.

This last result can be found in Sri Ranga and McCabe [50].

We also know that for $\omega = 1$,

$$d\psi(t) = \frac{(1+\sqrt{ab}/t)^{-1}}{\sqrt{b-t}\sqrt{t-a}}dt \quad \Rightarrow \quad d\psi^e(t) = t^{-1}d\psi(t).$$

We thus conclude that for $\omega = 1$ and $j = -1$,

$$\beta_1^{(-1)} = \frac{\beta}{2l-1}, \qquad \alpha_2^{(-1)} = \beta\left((2l-1) - \frac{1}{2l-1}\right),$$
$$\beta_2^{(-1)} = \frac{\beta(2l-1)}{l}, \qquad \alpha_3^{(-1)} = \beta(2l-1)\left(l - \frac{1}{l}\right),$$

and

$$\beta_n^{(-1)} = \beta, \qquad \alpha_{n+1}^{(-1)} = \alpha, \quad n \geq 3.$$

From the equations (2.3.13), (2.3.14), and the above coefficients $\beta_n^{(-1)}$ and $\alpha_n^{(-1)}$, we can see that

$$\beta_n^{(r)} = \beta, \;\; \alpha_n^{(r)} = \alpha, \quad \text{for } r > -1, \;\; n \geq r+3,$$
$$\beta_n^{(r)} = \beta, \;\; \alpha_{n+1}^{(r)} = \alpha, \;\; \text{for } r < -1, \;\; n \geq -r+2.$$

Hence the result (3.4.1) holds for $\omega = 1$.

To show the result for $\omega = 3/2$, we use the relation

$$d\psi^e(t) = \frac{(t+\beta)}{t} d\psi^o(t).$$

We now set

$$d\psi^e(t) = \frac{(1+\beta/t)^{-1}}{\sqrt{b-t}\sqrt{t-a}} dt, \quad \omega^e = 1$$

and

$$d\psi^o(t) = \frac{t}{t+\beta} d\psi^e(t) = \frac{(1+\beta/t)^{-2}}{\sqrt{b-t}\sqrt{t-a}} dt, \quad \omega^o = 3/2.$$

Since we know the coefficients $\beta_n^{e(-1)}$ and $\alpha_{n+1}^{e(-1)}$, $n \geq 1$, we can use the relations (3.3.9) to obtain

$$\beta_1^{(-1)} = \beta, \qquad \alpha_2^{(-1)} = 4\beta(l-1),$$
$$\beta_2^{(-1)} = \beta, \qquad \alpha_3^{(-1)} = 2\beta l(l-1),$$

and

$$\beta_n^{(-1)} = \beta, \qquad \alpha_{n+1}^{(-1)} = \alpha, \quad n \geq 3,$$

where $l = \sqrt{1+\alpha/\beta}$.

Again by using (2.3.13) and (2.3.14) we can see that

$$\begin{aligned} &\beta_n^{(r)} = \beta, \quad \alpha_n^{(r)} = \alpha, \quad \text{for } r > -1, \quad n \geq r+4, \\ &\beta_n^{(r)} = \beta, \quad \alpha_{n+1}^{(r)} = \alpha, \quad \text{for } r < -1, \quad n \geq -r+2, \end{aligned}$$

and then conclude that the result (3.4.1) holds for $\omega = 3/2$.

Similarly we prove the result for $\omega = 2, 5/2, 3, 7/2, ...$. This completes the proof.

When we set $\omega = 0$ in the distribution of example 3.1.(i), we obtain

$$d\psi(t) = \frac{1+\sqrt{ab}/t}{\sqrt{b-t}\sqrt{t-a}}dt. \tag{3.4.2}$$

In the following theorem we give the behaviour of the coefficients $\alpha_{n+1}^{(r)}$ and $\beta_n^{(r)}$, $n \geq 1$, $r = 0, \pm1, \pm2, ...$, related to the above distribution.

Theorem 3.4.2 *For the $S^3[\omega,\beta,b]$ distribution defined by (3.4.2), the coefficients $\alpha_n^{(r)}$ and $\beta_n^{(r)}$, from the recurrence relations (2.2.7), satisfy*

$$\begin{aligned} &\beta_n^{(r)} = \beta_n^{(0)}, \quad \alpha_n^{(r)} = \alpha_n^{(0)}, \quad \text{for } r > 0, \quad n \geq r+1 \\ &\beta_n^{(r)} = \beta_n^{(0)}, \quad \alpha_{n+1}^{(r)} = \alpha_{n+1}^{(0)}, \quad \text{for } r < 0, \quad n \geq -r+2 \end{aligned}, \tag{3.4.3}$$

for r even, and

$$\begin{aligned} &\beta_n^{(r)} = \beta_n^{(1)}, \quad \alpha_n^{(r)} = \alpha_n^{(1)}, \quad \text{for } r > 1, \quad n \geq r+1 \\ &\beta_n^{(r)} = \beta_n^{(1)}, \quad \alpha_{n+1}^{(r)} = \alpha_{n+1}^{(1)}, \quad \text{for } r < 1, \quad n \geq -r+2 \end{aligned}, \tag{3.4.4}$$

for r odd.

To emphasize these common elements in the $\alpha - \beta$ table we write them in "bold" characters. Then the table for the distribution (3.4.2) is

r	$\beta_1^{(r)}$	$\alpha_2^{(r)}$	$\beta_2^{(r)}$	$\alpha_3^{(r)}$	$\beta_3^{(r)}$	$\alpha_4^{(r)}$	$\beta_4^{(r)}$	$\alpha_5^{(r)}$	$\beta_5^{(r)}$	$\alpha_6^{(r)}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-3	$\beta_1^{(-3)}$	$\alpha_2^{(-3)}$	$\beta_2^{(-3)}$	$\alpha_3^{(-3)}$	$\beta_3^{(-3)}$	$\alpha_4^{(-3)}$	$\beta_4^{(-3)}$	$\alpha_5^{(-3)}$	$\beta_{\mathbf{5}}^{(\mathbf{1})}$	$\alpha_{\mathbf{6}}^{(\mathbf{1})}$	$\cdots$
-2	$\beta_1^{(-2)}$	$\alpha_2^{(-2)}$	$\beta_2^{(-2)}$	$\alpha_3^{(-2)}$	$\beta_3^{(-2)}$	$\alpha_4^{(-2)}$	$\beta_{\mathbf{4}}^{(\mathbf{0})}$	$\alpha_{\mathbf{5}}^{(\mathbf{0})}$	$\beta_{\mathbf{5}}^{(\mathbf{0})}$	$\alpha_{\mathbf{6}}^{(\mathbf{0})}$	$\cdots$
-1	$\beta_1^{(-1)}$	$\alpha_2^{(-1)}$	$\beta_2^{(-1)}$	$\alpha_3^{(-1)}$	$\beta_{\mathbf{3}}^{(\mathbf{1})}$	$\alpha_{\mathbf{4}}^{(\mathbf{1})}$	$\beta_{\mathbf{4}}^{(\mathbf{1})}$	$\alpha_{\mathbf{5}}^{(\mathbf{1})}$	$\beta_{\mathbf{5}}^{(\mathbf{1})}$	$\alpha_{\mathbf{6}}^{(\mathbf{1})}$	$\cdots$
0	$\beta_{\mathbf{1}}^{(\mathbf{0})}$	$\alpha_{\mathbf{2}}^{(\mathbf{0})}$	$\beta_{\mathbf{2}}^{(\mathbf{0})}$	$\alpha_{\mathbf{3}}^{(\mathbf{0})}$	$\beta_{\mathbf{3}}^{(\mathbf{0})}$	$\alpha_{\mathbf{4}}^{(\mathbf{0})}$	$\beta_{\mathbf{4}}^{(\mathbf{0})}$	$\alpha_{\mathbf{5}}^{(\mathbf{0})}$	$\beta_{\mathbf{5}}^{(\mathbf{0})}$	$\alpha_{\mathbf{6}}^{(\mathbf{0})}$	$\cdots$.
1	$\beta_{\mathbf{1}}^{(\mathbf{1})}$	$\alpha_{\mathbf{2}}^{(\mathbf{1})}$	$\beta_{\mathbf{2}}^{(\mathbf{1})}$	$\alpha_{\mathbf{3}}^{(\mathbf{1})}$	$\beta_{\mathbf{3}}^{(\mathbf{1})}$	$\alpha_{\mathbf{4}}^{(\mathbf{1})}$	$\beta_{\mathbf{4}}^{(\mathbf{1})}$	$\alpha_{\mathbf{5}}^{(\mathbf{1})}$	$\beta_{\mathbf{5}}^{(\mathbf{1})}$	$\alpha_{\mathbf{6}}^{(\mathbf{1})}$	$\cdots$
2	$\beta_1^{(2)}$	$\alpha_2^{(2)}$	$\beta_2^{(2)}$	$\alpha_{\mathbf{3}}^{(\mathbf{0})}$	$\beta_{\mathbf{3}}^{(\mathbf{0})}$	$\alpha_{\mathbf{4}}^{(\mathbf{0})}$	$\beta_{\mathbf{4}}^{(\mathbf{0})}$	$\alpha_{\mathbf{5}}^{(\mathbf{0})}$	$\beta_{\mathbf{5}}^{(\mathbf{0})}$	$\alpha_{\mathbf{6}}^{(\mathbf{0})}$	$\cdots$
3	$\beta_1^{(3)}$	$\alpha_2^{(3)}$	$\beta_2^{(3)}$	$\alpha_3^{(3)}$	$\beta_3^{(3)}$	$\alpha_{\mathbf{4}}^{(\mathbf{1})}$	$\beta_{\mathbf{4}}^{(\mathbf{1})}$	$\alpha_{\mathbf{5}}^{(\mathbf{1})}$	$\beta_{\mathbf{5}}^{(\mathbf{1})}$	$\alpha_{\mathbf{6}}^{(\mathbf{1})}$	$\cdots$
4	$\beta_1^{(4)}$	$\alpha_2^{(4)}$	$\beta_2^{(4)}$	$\alpha_3^{(4)}$	$\beta_3^{(4)}$	$\alpha_4^{(4)}$	$\beta_4^{(4)}$	$\alpha_{\mathbf{5}}^{(\mathbf{0})}$	$\beta_{\mathbf{5}}^{(\mathbf{0})}$	$\alpha_{\mathbf{6}}^{(\mathbf{0})}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Using (3.3.10), we see that

$$\beta_n^{(0)} = \beta \frac{l_{n-1}}{l_n}, \quad \alpha_{n+1}^{(0)} = \beta_n^{(0)}(l_n^2 - 1), \quad n \geq 1, \tag{3.4.5}$$

where

$$l_0 = 1, \quad l_1 = \frac{2\alpha/\beta}{l_0 + 1} + 1, \quad \text{and} \quad l_n = 1 + \frac{\alpha/\beta}{1 + l_{n-1}}, \quad n \geq 2. \tag{3.4.6}$$

In Sri Ranga and Bracciali [47] it was proved that in this special case

$$l_n = \frac{(1+l)^n - (1-l)^n}{(1+l)^n + (1-l)^n} l, \quad n \geq 1, \quad l = \sqrt{1 + \alpha/\beta}. \tag{3.4.7}$$

From the relations (3.2.8) with $j = 0$ and $l = 0$, we obtain

$$\beta_n^{(1)} = \beta \frac{l_n}{l_{n-1}}, \quad \alpha_{n+1}^{(1)} = \beta_{n+1}^{(1)}(l_n^2 - 1), \quad n \geq 1. \tag{3.4.8}$$

We now prove the following property.

Property 3.4.1 *The coefficients l_n, $n \geq 0$, defined by (3.4.6) or (3.4.7), satisfy*

$$l_{n+1}^2 + l_{n+2}l_n\,(l_{n+1}^2 - 1) = \begin{cases} l^2(3l^2-2), & n = 0, \\ l^4, & n > 0, \end{cases} \tag{3.4.9}$$

where $l = \sqrt{1+\alpha/\beta}$. For $n < 0$, we define $l_{n+1}^2 + l_{n+2}l_n\,(l_{n+1}^2 - 1) = 1$.

Proof: For $n = 0$, the result follows directly from (3.4.7). For $n > 0$, from the relation (3.4.6), we obtain

$$l_{n+2} = 1 + \frac{\alpha/\beta}{1 + l_{n+1}}$$

or

$$l_{n+2} = 1 + \frac{\alpha/\beta}{2 + \frac{\alpha/\beta}{1+l_n}}$$

or

$$l_{n+2} = \frac{(2+\alpha/\beta)(1+l_n) + \alpha/\beta}{2(1+l_n) + \alpha/\beta}.$$

Thus,

$$l_{n+1}^2 + l_{n+2}l_n\,(l_{n+1}^2 - 1) =$$

$$\frac{(1+l_n+\alpha/\beta)^2}{(1+l_n)^2} + \frac{(2+\alpha/\beta)(1+l_n)+\alpha/\beta}{2(1+l_n)+\alpha/\beta}\, l_n\, \frac{(1+l_n+\alpha/\beta)^2 - (1+l_n)^2}{(1+l_n)^2},$$

and after some simple manipulations we obtain the result. □

We now can prove Theorem 3.4.2.

From the relations (2.3.13) we construct the row with superscript (2) in the $\alpha - \beta$ table using the row with superscript (1) given by (3.4.8). We then obtain

$$\beta_n^{(2)} = \beta_n^{(0)} \frac{l_n^2 + l_{n+1}l_{n-1}\,(l_n^2 - 1)}{l_{n-1}^2 + l_n l_{n-2}\,(l_{n-1}^2 - 1)}, \quad n \geq 2,$$

$$\alpha_n^{(2)} = \alpha_n^{(0)} \frac{l_n^2 + l_{n+1}l_{n-1}\,(l_n^2 - 1)}{l_{n-1}^2 + l_n l_{n-2}\,(l_{n-1}^2 - 1)}, \quad n \geq 2,$$

with

$$\beta_1^{(2)} = \beta_1^{(1)} + \alpha_2^{(1)}.$$

Hence, using the Property 3.4.1, we obtain

$$\beta_1^{(2)} = \beta_1^{(0)}\, l^2\, (3l^2 - 2), \qquad \alpha_2^{(2)} = \alpha_2^{(0)} \frac{l^2}{3l^2 - 2}, \qquad \beta_2^{(2)} = \beta_2^{(0)} \frac{l^2}{3l^2 - 2},$$

and

$$\beta_n^{(2)} = \beta_n^{(0)}, \qquad \alpha_n^{(2)} = \alpha_n^{(0)}, \quad n \geq 3.$$

Similarly, using (2.3.13), we obtain the result for the rows with superscript $(3), (4), (5), \ldots$.

From the relations (2.3.14) we construct the row with superscript (-1) in the $\alpha - \beta$ table using the row with superscript (0), given by (3.4.5). We then obtain

$$\beta_n^{(-1)} = \beta_n^{(1)} \frac{l_{n-1}^2 + l_n l_{n-2}\, (l_{n-1}^2 - 1)}{l_n^2 + l_{n+1} l_{n-1}\, (l_n^2 - 1)}, \quad n \geq 2,$$

$$\alpha_{n+1}^{(-1)} = \alpha_{n+1}^{(1)} \frac{l_{n-1}^2 + l_n l_{n-2}\, (l_{n-1}^2 - 1)}{l_n^2 + l_{n+1} l_{n-1}\, (l_n^2 - 1)}, \quad n \geq 2.$$

Hence, using the Property 3.4.1, we obtain

$$\beta_1^{(-1)} = \beta_1^{(1)} \frac{1}{l^2(3l^2 - 2)}, \qquad \alpha_2^{(-1)} = \alpha_2^{(1)} \frac{1}{l^2(3l^2 - 2)},$$

$$\beta_2^{(-1)} = \beta_2^{(1)} \frac{3l^2 - 2}{l^2}, \qquad \alpha_3^{(-1)} = \alpha_3^{(1)} \frac{3l^2 - 2}{l^2},$$

and

$$\beta_n^{(-1)} = \beta_n^{(1)}, \qquad \alpha_{n+1}^{(-1)} = \alpha_{n+1}^{(1)}, \quad n \geq 3.$$

Similarly, using (2.3.14) we obtain the result for the rows with superscript $(-2), (-3), (-4), \ldots$. This completes the proof.

In addition, for the distribution defined by (3.4.2), the coefficients $\alpha_{n+1}^{(0)}$ and $\alpha_{n+1}^{(1)}$ satisfy the property

$$\alpha_{n+1}^{(0)} + \alpha_{n+1}^{(1)} = 2\alpha, \quad n \geq 2. \tag{3.4.10}$$

We can prove this result by substituting (3.4.5) and (3.4.8) in (3.4.10), that is, for $n \geq 1$

$$\alpha_{n+1}^{(0)} + \alpha_{n+1}^{(1)} = \beta_n^{(0)}(l_n^2 - 1) + \beta_{n+1}^{(1)}(l_n^2 - 1)$$

or

$$\alpha_{n+1}^{(0)} + \alpha_{n+1}^{(1)} = \beta \left(\frac{l_{n-1}}{l_n} + \frac{l_{n+1}}{l_n} \right) (l_n^2 - 1) = \beta \left(\frac{l_{n-1} + l_{n+1}}{l_n} \right) (l_n^2 - 1).$$

Then, using the relation (3.4.6), we see that the right hand side can be written as

$$\beta \left(\frac{l_{n-1} + \dfrac{(2+\alpha/\beta)(1+l_{n-1}) + \alpha/\beta}{2(1+l_{n-1}) + \alpha/\beta}}{\dfrac{1 + l_{n-1} + \alpha/\beta}{1 + l_{n-1}}} \right) \cdot \left(\left(\frac{1 + l_{n-1} + \alpha/\beta}{1 + l_{n-1}} \right)^2 - 1 \right) = 2\alpha.$$

When $\omega < 0$, we do not find similar behaviour for the coefficients $\alpha_n^{(r)}$ and $\beta_n^{(r)}$, but it seems that

$$\alpha_n^{(r)} \to \alpha \quad \text{and} \quad \beta_n^{(r)} \to \beta, \quad r = 0, \pm 1, \pm 2, \ldots,$$

when $n \to \infty$.

We now consider the example 3.1.(ii), namely

$$d\psi(t) = \frac{1 + (\beta/t)^{1-2\omega}}{\sqrt{b-t}\sqrt{t-a}} dt.$$

Comparing the examples 3.1.(i) and 3.1.(ii), we see that for $\omega = 0$ the distributions are the same, while for $\omega = 1/2$ the distributions are, respectively,

$$d\psi(t) = \frac{1}{\sqrt{b-t}\sqrt{t-a}} dt \quad \text{and} \quad d\psi(t) = \frac{2}{\sqrt{b-t}\sqrt{t-a}} dt.$$

Hence, they have the same $\alpha - \beta$ table.

The distribution in the case $\omega = 1$ in the example 3.1.(ii), namely

$$d\psi(t) = \frac{1 + (\beta/t)^{-1}}{\sqrt{b-t}\sqrt{t-a}} dt,$$

can be written as

$$d\psi(t) = (t/\beta) \cdot \frac{1 + \beta/t}{\sqrt{b-t}\sqrt{t-a}} dt.$$

It thus corresponds to the example 3.1.(ii), case $\omega = 0$, with a shifted $\alpha - \beta$ table. This behaviour is illustrated in the tables 3.1.1 and 3.1.2 below.

r	$\beta_1^{(r)}$	$\alpha_2^{(r)}$	$\beta_2^{(r)}$	$\alpha_3^{(r)}$	$\beta_3^{(r)}$	$\alpha_4^{(r)}$	$\beta_4^{(r)}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
-4	$\frac{14936}{13183}$	$\frac{1222552}{24612661}$	$\frac{32233806032}{17020891969}$	$\frac{437089852944}{1393206047033}$	$\frac{1865861721448}{859799579759}$	$\frac{42082719512}{143936635163}$	$\frac{207766566208}{102754656743}$
-3	$\frac{2208}{1867}$	$\frac{11944}{128823}$	$\frac{22299448}{10544511}$	$\frac{79847856}{228158767}$	$\frac{80688432}{38195419}$	$\frac{166267072}{615450231}$	$\frac{27834304}{13880889}$
-2	$\frac{88}{69}$	$\frac{136}{759}$	$\frac{37536}{16423}$	$\frac{25760}{76143}$	$\frac{836080}{408969}$	$\frac{861461}{3367980}$	$\frac{19041}{9520}$
-1	$\frac{16}{11}$	$\frac{32}{99}$	$\frac{352}{153}$	$\frac{121}{408}$	$\frac{561}{280}$	$\frac{289}{1155}$	$\frac{38080}{19041}$
0	$\frac{16}{9}$	$\frac{17}{36}$	$\frac{17}{8}$	$\frac{35}{136}$	$\frac{1120}{561}$	$\frac{577}{2310}$	$\frac{19041}{9520}$
1	$\frac{9}{4}$	$\frac{1}{2}$	$\frac{32}{17}$	$\frac{33}{136}$	$\frac{561}{280}$	$\frac{289}{1155}$	$\frac{38080}{19041}$
2	$\frac{11}{4}$	$\frac{17}{44}$	$\frac{153}{88}$	$\frac{35}{136}$	$\frac{1120}{561}$	$\frac{577}{2310}$	$\frac{19041}{9520}$
3	$\frac{69}{22}$	$\frac{1493}{6072}$	$\frac{16423}{9384}$	$\frac{29403}{101524}$	$\frac{408969}{209020}$	$\frac{289}{1155}$	$\frac{38080}{19041}$
4	$\frac{1867}{552}$	$\frac{152819}{1030584}$	$\frac{10544511}{5574862}$	$\frac{142983387}{456317534}$	$\frac{38195419}{20172108}$	$\frac{861461}{3376956}$	$\frac{13880889}{6958576}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 3.1.1: $\alpha - \beta$ table for the example 3.1.(ii) when $\omega = 0$, $a = 1$ and $b = 4$.

r	$\beta_1^{(r)}$	$\alpha_2^{(r)}$	$\beta_2^{(r)}$	$\alpha_3^{(r)}$	$\beta_3^{(r)}$	$\alpha_4^{(r)}$	$\beta_4^{(r)}$
...	...	...	...	...	...	...	...
-4	$\frac{2208}{1867}$	$\frac{11944}{128823}$	$\frac{22299448}{10544511}$	$\frac{79847856}{228158767}$	$\frac{80688432}{38195419}$	$\frac{166267072}{615450231}$	$\frac{27834304}{13880889}$
-3	$\frac{88}{69}$	$\frac{136}{759}$	$\frac{37536}{16423}$	$\frac{25760}{76143}$	$\frac{836080}{408969}$	$\frac{861461}{3367980}$	$\frac{19041}{9520}$
-2	$\frac{16}{11}$	$\frac{32}{99}$	$\frac{352}{153}$	$\frac{121}{408}$	$\frac{561}{280}$	$\frac{289}{1155}$	$\frac{38080}{19041}$
-1	$\frac{16}{9}$	$\frac{17}{36}$	$\frac{17}{8}$	$\frac{35}{136}$	$\frac{1120}{561}$	$\frac{577}{2310}$	$\frac{19041}{9520}$
0	$\frac{9}{4}$	$\frac{1}{2}$	$\frac{32}{17}$	$\frac{33}{136}$	$\frac{561}{280}$	$\frac{289}{1155}$	$\frac{38080}{19041}$
1	$\frac{11}{4}$	$\frac{17}{44}$	$\frac{153}{88}$	$\frac{35}{136}$	$\frac{1120}{561}$	$\frac{577}{2310}$	$\frac{19041}{9520}$
2	$\frac{69}{22}$	$\frac{1493}{6072}$	$\frac{16423}{9384}$	$\frac{29403}{101524}$	$\frac{408969}{209020}$	$\frac{289}{1155}$	$\frac{38080}{19041}$
3	$\frac{1867}{552}$	$\frac{152819}{1030584}$	$\frac{10544511}{5574862}$	$\frac{142983387}{456317534}$	$\frac{38195419}{20172108}$	$\frac{861461}{3376956}$	$\frac{13880889}{6958576}$
4	$\frac{13183}{3734}$	$\frac{9116707}{98450644}$	$\frac{17020891969}{8058451508}$	$\frac{850842572247}{2786412094066}$	$\frac{859799579759}{466465430362}$	$\frac{613800722699}{2302986162608}$	$\frac{102754656743}{51941641552}$
...	...	...	...	...	...	...	...

Table 3.1.2: $\alpha-\beta$ table for the example 3.1.(ii) when $\omega=1$, $a=1$ and $b=4$.

The $\alpha-\beta$ table for the distribution of example 3.1.(ii), for a particular value $1-\omega$, is the $\alpha-\beta$ table for the value ω, raised by $1-2\omega$ lines. This is because for $1-\omega$ we obtain the distribution

$$d\psi(t)=\frac{1+(\beta/t)^{-1+2\omega}}{\sqrt{b-t}\sqrt{t-a}}dt$$

and

$$\frac{1+(\beta/t)^{-1+2\omega}}{\sqrt{b-t}\sqrt{t-a}}dt=(t/\beta)^{1-2\omega}\cdot\frac{1+(\beta/t)^{1-2\omega}}{\sqrt{b-t}\sqrt{t-a}}dt.$$

The numerical results suggest that

$$\alpha_n^{(r)}\to\alpha=\frac{(\sqrt{b}-\sqrt{a})^2}{4}\quad\text{and}\quad\beta_n^{(r)}\to\beta=\sqrt{ab},\quad r=0,\pm1,\pm2,\dots,$$

when $n\to\infty$.

Example 3.2 *The generalized log-normal distribution is defined by*

$$d\psi(t) = \frac{\sqrt{q}}{2\lambda\sqrt{\pi}} t^p e^{-\left(\frac{ln(t)}{2\lambda}\right)^2} dt,$$

in $(0, \infty)$, *with* $0 < q < 1$, $q = e^{-2\lambda^2}$. *This is an* $S^3[\omega, \beta, b]$ *distribution with* $\beta = q^{\omega-p-1}$.

The log-normal distribution was given by Pastro in [38] as the first explicit example where orthogonal Laurent polynomials appeared. The orthogonal Laurent polynomials related to the log-normal distribution have been also studied by Jones, Thron *et al.* in [11, 12, 18]. In the symmetric case with $p = -1$ it has been considered as an $S^3[0, 1, \infty]$ distribution by Sri Ranga, de Andrade and McCabe in [48], and by Common and McCabe in [13, 14]. It has also been treated as an $S^3[1/2, q^{-1/2}, \infty]$ distribution with $p = 0$ by Sri Ranga in [44].

It is an interesting example since we can obtain explicit expressions for the orthogonal L-polynomials, the associated polynomials and the coefficients in the recurrence relation.

The classical log-normal distribution is when $p = 0$. Then $\beta = q^{(\omega-1)}$ and

$$d\psi_0(t) = \frac{\sqrt{q}}{2\kappa\sqrt{\pi}} e^{-\left(\frac{ln(t)}{2\kappa}\right)^2} dt.$$

This is an $S^3[\omega, q^{(\omega-1)}, \infty]$ distribution for any ω such that $2\omega \in \boldsymbol{Z}$. This means, it is also, for example, an $S^3[0, q^{-1}, \infty]$ distribution, an $S^3[1/2, q^{-1/2}, \infty]$ distribution, an $S^3[1, 1, \infty]$ distribution, etc. Hence the coefficients $\beta_n^{(r)}$, $n \geq 0$, are constants for a fixed value of r.

The moments are given by

$$\mu_m = q^{-(m+m^2/2)}, \quad m = 0, \pm 1, \pm 2, \dots .$$

Using the algorithm 2.1 we obtain the following table

r	$\beta_1^{(r)}$	$\alpha_2^{(r)}$	$\beta_2^{(r)}$	$\alpha_3^{(r)}$	$\beta_3^{(r)}$	$\alpha_4^{(r)}$	$\beta_4^{(r)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-3	$q^{5/2}$	$q^{5/2}(q^{-1}-1)$	$q^{5/2}$	$q^{5/2}(q^{-2}-1)$	$q^{5/2}$	$q^{5/2}(q^{-3}-1)$	$\cdots$
-2	$q^{3/2}$	$q^{3/2}(q^{-1}-1)$	$q^{3/2}$	$q^{3/2}(q^{-2}-1)$	$q^{3/2}$	$q^{3/2}(q^{-3}-1)$	$\cdots$
-1	$q^{1/2}$	$q^{1/2}(q^{-1}-1)$	$q^{1/2}$	$q^{1/2}(q^{-2}-1)$	$q^{1/2}$	$q^{1/2}(q^{-3}-1)$	$\cdots$.
0	$q^{-1/2}$	$q^{-1/2}(q^{-1}-1)$	$q^{-1/2}$	$q^{-1/2}(q^{-2}-1)$	$q^{-1/2}$	$q^{-1/2}(q^{-3}-1)$	$\cdots$
1	$q^{-3/2}$	$q^{-3/2}(q^{-1}-1)$	$q^{-3/2}$	$q^{-3/2}(q^{-2}-1)$	$q^{-3/2}$	$q^{-3/2}(q^{-3}-1)$	$\cdots$
2	$q^{-5/2}$	$q^{-5/2}(q^{-1}-1)$	$q^{-5/2}$	$q^{-5/2}(q^{-2}-1)$	$q^{-5/2}$	$q^{-5/2}(q^{-3}-1)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

In general

$$\beta_n^{(r)} = q^{-(\frac{1}{2}+r)} \quad \text{and} \quad \alpha_{n+1}^{(r)} = q^{-(\frac{1}{2}+r)}(q^{-n}-1), \quad n \geq 1. \tag{3.4.11}$$

By using the recurrence relation (2.2.7) and the relations (3.4.11) we can prove, by mathematical induction, that

$$B_n^{(r)}(z) = \sum_{j=0}^{n} (-1)^j q^{-j(\frac{1}{2}+r)} q^{j(j-n)} \begin{bmatrix} n \\ j \end{bmatrix} z^{n-j}, \quad n \geq 0,$$

where $\begin{bmatrix} n \\ j \end{bmatrix}$ are the q-binomial coefficients given, for $n \geq 0$, by

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{\prod_{k=1}^{n}(1-q^k)}{\prod_{k=1}^{j}(1-q^k) \prod_{k=1}^{n-j}(1-q^k)}, \; 1 \leq j \leq n-1, \quad \text{and} \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1.$$

By mathematical induction we can prove also that, for the classical log-normal distribution the associated polynomials $C_n^{(r)}(z)$, defined in section 2.3, are given by

$$C_n^{(r)}(z) = \left(\mu_r q^{(1-n)(\frac{3}{2}+r)} \sum_{s=0}^{n-1} q^{s(s-n+r+\frac{5}{2})} (-1)^{n+s+1} z^s \right) \times \left(\sum_{k=0}^{n-1-s} (-1)^k q^{\frac{k^2}{2}+(2s-n+\frac{3}{2})k} \begin{bmatrix} n \\ n-1-s-k \end{bmatrix} \right).$$

Since $d\psi_0(t)$ is an $S^3[\omega, q^{\omega-1}, \infty]$ distribution, then from (3.2.1) we see that, for $s = 0, \pm1, \pm2, ...,$

$$\frac{z^n B_n^{(l)}(q^s/z)}{B_n^{(l)}(0)} = B_n^{(-s-1-l)}(z), \quad l = 0, \pm1, \pm2, \dots .$$

In the table 3.2.1 we present the polynomials $B_n^{(r)}(z)$ for $n = 1, 2, 3$ and $r = -3, -2, ..., 3$.

r	$B_1^{(r)}(z)$	$B_2^{(r)}(z)$	$B_3^{(r)}(z)$
-3	$z - q^{\frac{5}{2}}$	$z^2 - q^{\frac{3}{2}}(q+1)z + q^5$	$z^3 - q^{\frac{1}{2}}(q^2+q+1)z^2 + q^3(q^2+q+1)z - q^{1\frac{5}{2}}$
-2	$z - q^{\frac{3}{2}}$	$z^2 - q^{\frac{1}{2}}(q+1)z + q^3$	$z^3 - q^{-\frac{1}{2}}(q^2+q+1)z^2 + q(q^2+q+1)z - q^{\frac{9}{2}}$
-1	$z - q^{\frac{1}{2}}$	$z^2 - q^{-\frac{1}{2}}(q+1)z + q$	$z^3 - q^{-\frac{3}{2}}(q^2+q+1)z^2 + q^{-1}(q^2+q+1)z - q^{\frac{3}{2}}$
0	$z - q^{-\frac{1}{2}}$	$z^2 - q^{-\frac{3}{2}}(q+1)z + q^{-1}$	$z^3 - q^{-\frac{5}{2}}(q^2+q+1)z^2 + q^{-3}(q^2+q+1)z - q^{-\frac{3}{2}}$
1	$z - q^{-\frac{3}{2}}$	$z^2 - q^{-\frac{5}{2}}(q+1)z + q^{-3}$	$z^3 - q^{-\frac{7}{2}}(q^2+q+1)z^2 + q^{-5}(q^2+q+1)z - q^{-\frac{9}{2}}$
2	$z - q^{-\frac{5}{2}}$	$z^2 - q^{-\frac{7}{2}}(q+1)z + q^{-5}$	$z^3 - q^{-\frac{9}{2}}(q^2+q+1)z^2 + q^{-7}(q^2+q+1)z - q^{-\frac{15}{2}}$
3	$z - q^{-\frac{7}{2}}$	$z^2 - q^{-\frac{9}{2}}(q+1)z + q^{-7}$	$z^3 - q^{-\frac{11}{2}}(q^2+q+1)z^2 + q^{-9}(q^2+q+1)z - q^{-\frac{21}{2}}$

Table 3.2.1: Polynomials $B_n^{(r)}(z)$, for the example 3.2 when $p = 0$.

The generalized log-normal distribution can be given, in terms of the classical one, by

$$d\psi_p(t) = t^p d\psi_0(t).$$

For the distribution $d\psi_p(t)$ we denote the moments by $\mu_m^{(p)}$, the orthogonal L-polynomials by $B_n^{(p,r)}(z)$, and the coefficients in the recurrence relation for the orthogonal L-polynomials by $\alpha_n^{(p,r)}$ and $\beta_n^{(p,r)}$. Hence the moments for the generalized distribution are

$$\mu_m^{(p)} = \mu_{m+p}^{(0)} = q^{(1-(1+m+p)^2)/2}, \quad m = 0, \pm1, \pm2, \dots .$$

Consequently the coefficients in the recurrence relations satisfy

$$\beta_n^{(p,r)} = \beta_n^{(0,r+p)} \quad \text{and} \quad \alpha_{n+1}^{(p,r)} = \alpha_{n+1}^{(0,r+p)}, \quad n \geq 1,$$

while the polynomials satisfy

$$B_n^{(p,r)}(z) = B_n^{(0,r+p)}(z), \quad n \geq 0.$$

Example 3.3 *The distribution defined by*

$$d\psi(t) = t^{\omega-1}dt,$$

on (a, b), *where* $0 < \beta < b < \infty$ *and* $\beta = \sqrt{ab}$ *is an* $S^3[\omega, \beta, b]$ *distribution.*

First we consider the case when 2ω is odd and set

$$d\psi_p(t) = t^p d\psi(t) = t^{p+\omega-1}dt, \quad p \in \boldsymbol{Z}.$$

In a similar way to the log-normal distribution, for the distribution $d\psi_p(t)$ we denote the moments by $\mu_m^{(p)}$, the orthogonal L-polynomials by $B_n^{(p,r)}(z)$, and the coefficients in the recurrence relation for the orthogonal L-polynomials by $\alpha_n^{(p,r)}$ and $\beta_n^{(p,r)}$. We then obtain

$$\mu_m^{(p)} = \mu_{m+p}^{(0)}, \quad m = 0, \pm 1, \pm 2, \ldots ,$$

$$\beta_n^{(p,r)} = \beta_n^{(0,r+p)} \quad \text{and} \quad \alpha_{n+1}^{(p,r)} = \alpha_{n+1}^{(0,r+p)}, \quad n \geq 1,$$

and the orthogonal L-polynomials satisfy

$$B_n^{(p,r)}(z) = B_n^{(0,r+p)}(z), \quad n \geq 0.$$

Hence, the $\alpha - \beta$ table for any ω where 2ω is odd will be a shifted $\alpha - \beta$ table for another value of ω where 2ω is odd. Similarly, when we consider the case when 2ω is even, the $\alpha - \beta$ table for any such value of ω will be a shifted $\alpha - \beta$ table for another such value of ω.

We now give examples to illustrate the behaviour of the coefficients $\alpha_{n+1}^{(r)}$ and $\beta_n^{(r)}$ for $n \geq 1$ and $r = 0, \pm 1, \pm 2, \ldots$.

For $\omega = 1/2$, we obtain the distribution $\psi(t)$ defined by

$$d\psi(t) = t^{-1/2}dt.$$

We already know from (3.2.7) that for $\omega = 1/2$,

$$\beta_n^{(0)} = \beta, \quad n \geq 1.$$

Sri Ranga in [44] showed that

$$\alpha_{n+1}^{(0)} = \frac{4n^2}{4n^2 - 1}\alpha, \quad n \geq 1,$$

where $\beta = \sqrt{ab}$ and $\alpha = (\sqrt{b} - \sqrt{a})^2/4$.

It is easy to see that $\alpha_n^{(0)} \to \alpha$ when $n \to \infty$. From (2.3.13) and (2.3.14) we can also see that, when $n \to \infty$,

$$\alpha_n^{(r)} \to \alpha \quad \text{and} \quad \beta_n^{(r)} \to \beta, \quad r = 0, \pm 1, \pm 2, \dots .$$

In the tables 3.3.1 and 3.3.2 we present the coefficients $\beta_n^{(r)}$ and $\alpha_n^{(r)}$, for $n = 10, 20, ..., 50$, and $r = -3, -2, ..., 3$, when $\omega = 1/2$, $a = 4$ and $b = 9$.

r	$\beta_{10}^{(r)}$	$\beta_{20}^{(r)}$	$\beta_{30}^{(r)}$	$\beta_{40}^{(r)}$	$\beta_{50}^{(r)}$
...	...	...	...	...	...
-3	6.0004277327683	6.0000487609309	6.0000140492222	6.0000058474392	6.0000029701348
-2	6.0002838261955	6.0000324726666	6.0000093618187	6.0000038972901	6.0000019797659
-1	6.0001415161093	6.0000162259518	6.0000046796105	6.0000019483442	6.0000009897857
0	6.0000000000000	6.0000000000000	6.0000000000000	6.0000000000000	6.0000000000000
1	5.9998584872285	5.9999837740921	5.9999953203932	5.9999980516564	5.9999990102144
2	5.9997161872301	5.9999675275092	5.9999906381960	5.9999961027124	5.9999980202348
3	5.9995722977221	5.9999512394654	5.9999859508107	5.9999941525665	5.9999970298667
...	...	...	...	...	...

Table 3.3.1: Coefficients $\beta_n^{(r)}$, example 3.3, when $\omega = 1/2$, $a = 4$ and $b = 9$.

r	$\alpha_{10}^{(r)}$	$\alpha_{20}^{(r)}$	$\alpha_{30}^{(r)}$	$\alpha_{40}^{(r)}$	$\alpha_{50}^{(r)}$
...	...	...	...	...	...
-3	.25080619446260	.25017598046900	.25007505171163	.25004138081603	.25002617293383
- 2	.25079297934951	.25017495233517	.25007479243191	.25004128004614	.25002612384579
-1	.25078227427672	.25017404270093	.25007455470721	.25004118583224	.25002607738152
0	.25077399380805	.25017325017325	.25007433838834	.25004109814236	.25002603353119
1	.25076807918757	.25017257362523	.25007414334675	.25004101694804	.25002599228583
2	.25076449720541	.25017201219204	.25007396947425	.25004094222420	.25002595363734
3	.25076323969871	.25017156526813	.25007381668294	.25004087394921	.25002591757850
...	...	...	...	...	...

Table 3.3.2: Coefficients $\alpha_n^{(r)}$, example 3.3, when $\omega = 1/2$, $a = 4$ and $b = 9$.

In the tables 3.3.3 and 3.3.4 we present the values of $\beta_n^{(r)}$ and $\alpha_n^{(r)}$, for $n = 10, 20, ..., 50.$ and $r = -3, -2, ..., 3$, when $\omega = 1/2$, $a = 1/4$ and $b = 4$. The numerical results suggest that, when $n \to \infty$, $\alpha_n^{(r)} \to \alpha = 0.5625$ and $\beta_n^{(r)} \to \beta = 1$, for $r = 0, \pm 1, \pm 2, \dots$.

r	$\beta_{10}^{(r)}$	$\beta_{20}^{(r)}$	$\beta_{30}^{(r)}$	$\beta_{40}^{(r)}$	$\beta_{50}^{(r)}$
...	...	...	...	...	...
-3	1.0006804352024	1.0000741300543	1.0000211969925	1.0000087996439	1.0000044643992
-2	1.0004348924003	1.0000489481922	1.0000140726399	1.0000058528617	1.0000029718863
-1	1.0002121040724	1.0000243340579	1.0000070187916	1.0000029223702	1.0000014846311
0	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000	1.0000000000000
1	.99978794090620	.99997566653418	.99999298125762	.99999707763838	.99999851537111
2	.99956529664886	.99995105420360	.99998592755809	.99999414717260	.99999702812250
3	.99932002747484	.99992587544059	.99997880345682	.99999120043355	.99999553562069
...	...	...	...	...	...

Table 3.3.3: Coefficients $\beta_n^{(r)}$, example 3.3, when $\omega = 1/2$, $a = 1/4$ and $b = 4$.

r	$\alpha_{10}^{(r)}$	$\alpha_{20}^{(r)}$	$\alpha_{30}^{(r)}$	$\alpha_{40}^{(r)}$	$\alpha_{50}^{(r)}$
...	...	...	...	...	...
-3	.56493675655964	.56294586764330	.56268179169528	.56259821370979	.56256140421217
-2	.56463504070771	.56292445640747	.56267647570192	.56259615879197	.56256040570711
-1	.56440899962162	.56290585801726	.56267164141395	.56259424644685	.56255946338478
0	.56424148606811	.56288981288981	.56266726137377	.56259247082032	.56255857544517
1	.56412183352989	.56287611582979	.56266331215722	.56259082672168	.56255774025446
2	.56404401310588	.56286460936173	.56265977405824	.56258930958562	.56255695633741
3	.56400582254732	.56285517913667	.56265663084153	.56258791544044	.56255622237092
...	...	...	...	...	...

Table 3.3.4: Coefficients $\alpha_n^{(r)}$, example 3.3, when $\omega = 1/2$, $a = 1/4$ and $b = 4$.

We now consider $\omega = 0$, then we obtain the distribution $\psi(t)$ defined by

$$d\psi(t) = t^{-1}dt.$$

In the table 3.3.5 and table 3.3.6 we give the coefficients $\beta_n^{(r)}$ and $\alpha_n^{(r)}$, for $n = 10, 20, ..., 50$, and $r = -3, -2, ..., 3$, when $\omega = 0$, $a = 4$ and $b = 9$.

Since $\beta = \sqrt{ab} = 6$ and $\alpha = \dfrac{(\sqrt{b} - \sqrt{a})^2}{4} = \dfrac{1}{4}$, it seems that in this example

$$\alpha_n^{(r)} \to \frac{1}{4} \quad \text{and} \quad \beta_n^{(r)} \to 6, \quad r = 0, \pm 1, \pm 2, ... ,$$

when $n \to \infty$.

r	$\beta_{10}^{(r)}$	$\beta_{20}^{(r)}$	$\beta_{30}^{(r)}$	$\beta_{40}^{(r)}$	$\beta_{50}^{(r)}$
...	...	...	...	...	...
-3	6.0005005417463	6.0000569271749	6.0000163956860	6.0000068231532	6.0000034655258
-2	6.0003555289721	6.0000406103014	6.0000117047084	6.0000048721767	6.0000024748896
-1	6.0002125221664	6.0000243454170	6.0000070202276	6.0000029227044	6.0000014847394
0	6.0000707083616	6.0000081116763	6.0000023396427	6.0000009741344	6.0000004948807
1	5.9999292924717	5.9999918883347	5.9999976603582	5.9999990258657	5.9999995051193
2	5.9997874853609	5.9999756546818	5.9999929797806	5.9999970772971	5.9999985152610
3	5.9996444920934	5.9999593899735	5.9999882953144	5.9999951278273	5.9999975251114
...	...	...	...	...	...

Table 3.3.5: Coefficients $\beta_n^{(r)}$, example 3.3, when $\omega = 0$, $a = 4$ and $b = 9$.

r	$\alpha_{10}^{(r)}$	$\alpha_{20}^{(r)}$	$\alpha_{30}^{(r)}$	$\alpha_{40}^{(r)}$	$\alpha_{50}^{(r)}$
...	...	...	...	...	...
-3	.25081377771014	.25017653948266	.25007518948693	.25004143367039	.25002619846508
-2	.25079926672437	.25017545149204	.25007491936726	.25004132960946	.25002614806119
-1	.25078731904095	.25017448279896	.25007467088499	.25004123212177	.25002610028632
0	.25077783524796	.25017363187539	.25007444388056	.25004114117366	.25002605513019
1	.25077074343226	.25017289746194	.25007423821512	.25004105673490	.25002601258341
2	.25076599771230	.25017227856318	.25007405377033	.25004097877874	.25002597263745
3	.25076357742537	.25017177444420	.25007389044815	.25004090728182	.25002593528463
...	...	...	...	...	...

Table 3.3.6: Coefficients $\alpha_n^{(r)}$, example 3.3, when $\omega = 0$, $a = 4$ and $b = 9$.

In the table 3.3.7 and table 3.3.8 we give the coefficients $\beta_n^{(r)}$ and $\alpha_n^{(r)}$, for $n = 10, 20, ..., 50.$ and $r = -3, -2, ..., 3$, when $\omega = 0$, $a = 1/4$ and $b = 4$. Once again the numerical results suggest that

$$\alpha_n^{(r)} \to \alpha = \frac{9}{16} = 0.5625 \quad \text{and} \quad \beta_n^{(r)} \to \beta = 1, \quad r = 0, \pm 1, \pm 2, \dots,$$

when $n \to \infty$.

r	$\beta_{10}^{(r)}$	$\beta_{20}^{(r)}$	$\beta_{30}^{(r)}$	$\beta_{40}^{(r)}$	$\beta_{50}^{(r)}$
...	...	...	...	...	...
-3	1.0008163320560	1.0000870281054	1.0000247969158	1.0000102817265	1.0000052134564
-2	1.0005539649702	1.0000614496302	1.0000176237583	1.0000073237021	1.0000037173202
-1	1.0003214596830	1.0000365884261	1.0000105391337	1.0000043860921	1.0000022277665
0	1.0001053978130	1.0000121495821	1.0000035072068	1.0000014606776	1.0000007421515
1	.99989461329457	.99998785056547	.99999649280545	.99999853932451	.99999925784901
2	.99967864362016	.99996341291257	.99998946097732	.99999561392709	.99999777223845
3	.99944634173711	.99993855414563	.99998237655225	.99999267635156	.99999628269361
...	...	...	...	...	...

Table 3.3.7: Coefficients $\beta_n^{(r)}$, example 3.3, when $\omega = 0$, $a = 1/4$ and $b = 4$.

r	$\alpha_{10}^{(r)}$	$\alpha_{20}^{(r)}$	$\alpha_{30}^{(r)}$	$\alpha_{40}^{(r)}$	$\alpha_{50}^{(r)}$
...	...	...	...	...	...
-3	.56512394489101	.56295772626259	.56268464010943	.56259929665521	.56256192514325
-2	.56477503203195	.56293479184027	.56267907160164	.56259716803631	.56256089781758
-1	.56451381640192	.56291482319919	.56267400015959	.56259518517974	.56255992763959
0	.56431871687925	.56289753030602	.56266939616739	.56259334188325	.56255901272305
1	.56417614984291	.56288268174884	.56266523422810	.56259163262960	.56255815135204
2	.56407791688054	.56287009688839	.56266149281162	.56259005254514	.56255734197292
3	.56401995898576	.56285964026859	.56265815397421	.56258859736497	.56255658318726
...	...	...	...	...	...

Table 3.3.8: Coefficients $\alpha_n^{(r)}$, example 3.3, when $\omega = 0$, $a = 1/4$ and $b = 4$.

The numerical results presented in this section were obtained by using the symbolic computation program Maple V.

3.5 Extensions of M-fractions related to $S^3[\omega,\beta,b]$ distributions

We now consider extensions of the M-fractions that are related to $S^3[\omega,\beta,b]$ distributions. An even (odd) extension of a continued fraction is a continued fraction whose even (odd) order convergents are the successive convergents of the original continued fraction. A continued fraction is called an even (odd) contraction of another continued fraction if its convergents are the even (odd) order convergents of this other continued fraction. Clearly contractions are unique, but extensions are not. The even (odd) contractions are also called even (odd) parts of the continued fraction. See Lorentzen and Waadeland [31] and Jones and Thron [25] for more details.

Even and odd extensions of the M-fractions

Firstly we consider even and odd extensions of the M-fraction (2.3.1), extending the results given by McCabe in [34]. We also show some properties regarding the coefficients in the even and odd extensions when the distribution is an $S^3[\omega,\beta,b]$ distribution.

We begin by recalling the M-fraction (2.3.1), namely

$$M^{(r)}(z)+\frac{\mu_r z^{-r}}{z-\beta_1^{(r)}}\ \underset{-}{}\ \frac{\alpha_2^{(r)}z}{z-\beta_2^{(r)}}\ \underset{-}{}\ \frac{\alpha_3^{(r)}z}{z-\beta_3^{(r)}}\ \underset{-}{}\ \frac{\alpha_4^{(r)}z}{z-\beta_4^{(r)}}\ \underset{-}{}\ \cdots, \tag{3.5.1}$$

where $M^{(r)}(z)$ is

$$M^{(r)}(z)=z^{-r}\int_a^b\frac{z^r-t^r}{z-t}d\psi(t)=\begin{cases}\dfrac{\mu_0}{z}+\dfrac{\mu_1}{z^2}+\dfrac{\mu_2}{z^3}+\cdots+\dfrac{\mu_{r-1}}{z^r}, & r\ge 0\\ -\mu_{-1}-\mu_{-2}z-\cdots-\mu_r z^{-(r+1)}, & r<0\end{cases}.$$

The continued fraction

$$M^{(r)}(z) - \frac{\mu_{r-1}z^{-r}}{1} - \frac{m_1^{(r)}z}{1} - \frac{l_2^{(r)}}{1} - \frac{m_2^{(r)}z}{1} - \frac{l_3^{(r)}}{1} - \frac{m_3^{(r)}z}{1} - \dots, \qquad (3.5.2)$$

where

$$m_n^{(r)} = \frac{1}{\alpha_n^{(r)} + \beta_n^{(r)}} = \frac{1}{\gamma_n^{(r-1)}}, \quad l_n^{(r)} = \frac{\alpha_n^{(r)}}{\alpha_n^{(r)} + \beta_n^{(r)}} = \frac{\alpha_n^{(r)}}{\gamma_n^{(r-1)}}, \quad n \geq 2, \qquad (3.5.3)$$

with $m_1^{(r)} = \dfrac{1}{\beta_1^{(r)}}$, is an even extension of the M-fraction (3.5.1). It is also an odd extension of the M-fraction

$$M^{(r-1)}(z) + \frac{\mu_{r-1}z^{-r+1}}{z - \beta_1^{(r-1)}} - \frac{\alpha_2^{(r-1)}z}{z - \beta_2^{(r-1)}} - \frac{\alpha_3^{(r-1)}z}{z - \beta_3^{(r-1)}} - \frac{\alpha_4^{(r-1)}z}{z - \beta_4^{(r-1)}} - \dots \,. \qquad (3.5.4)$$

Hence the even order convergents $\dfrac{R_{2n}^{(r)}(z)}{S_{2n}^{(r)}(z)}$, and the odd order convergents $\dfrac{R_{2n+1}^{(r)}(z)}{S_{2n+1}^{(r)}(z)}$, of the continued fraction (3.5.2) are the successive convergents of the continued fractions (3.5.1) and (3.5.4), respectively. That is

$$\frac{R_{2n}^{(r)}(z)}{S_{2n}^{(r)}(z)} = \frac{A_n^{(r)}(z)}{B_n^{(r)}(z)} \quad \text{and} \quad \frac{R_{2n+1}^{(r)}(z)}{S_{2n+1}^{(r)}(z)} = \frac{A_n^{(r-1)}(z)}{B_n^{(r-1)}(z)}, \quad n \geq 0\,.$$

The numerators and denominators of the convergents of the continued fraction (3.5.2) satisfy, respectively, the recurrence relations

$$R_{2n}^{(r)}(z) = R_{2n-1}^{(r)}(z) - m_n^{(r)} z R_{2n-2}^{(r)}(z),$$

$$R_{2n+1}^{(r)}(z) = R_{2n}^{(r)}(z) - l_{n+1}^{(r)} R_{2n-1}^{(r)}(z),$$

where $R_0^{(r)}(z) = M^{(r)}(z)$ and $R_1^{(r)}(z) = M^{(r-1)}(z)$ for $r = 0, \pm1, \pm2, ...$, and

$$S_{2n}^{(r)}(z) = S_{2n-1}^{(r)}(z) - m_n^{(r)} z S_{2n-2}^{(r)}(z),$$

$$S_{2n+1}^{(r)}(z) = S_{2n}^{(r)}(z) - l_{n+1}^{(r)} S_{2n-1}^{(r)}(z),$$

where $S_0^{(r)}(z) = 1$ and $S_1^{(r)}(z) = 1$ for $r = 0, \pm1, \pm2, \dots$. The denominators $S_{2n}^{(r)}(z)$ and $S_{2n+1}^{(r)}(z)$ are polynomials of degree n.

By using the relations (2.3.12), namely

$$\frac{\alpha_n^{(r)}}{\gamma_n^{(r-1)}} = \frac{\alpha_n^{(r-1)}}{\gamma_{n-1}^{(r-1)}} \quad \text{and} \quad \frac{\beta_n^{(r)}}{\gamma_n^{(r-1)}} = \frac{\beta_{n-1}^{(r-1)}}{\gamma_{n-1}^{(r-1)}},$$

and the relations (3.5.3), we can prove that the coefficients $m_n^{(r)}$ and $l_n^{(r)}$, satisfy

$$m_n^{(r-1)} = \frac{m_{n-1}^{(r)}\, l_n^{(r-1)}}{l_n^{(r)}},$$

$$1 - l_n^{(r)} = \frac{m_{n-1}^{(r)}\,(1 - l_{n-1}^{(r-1)})}{m_{n-1}^{(r-1)}},$$

for $n = 2, 3, \ldots$ and $r = 0, \pm 1, \pm 2, \ldots$, with

$$l_1^{(r)} = 0, \quad m_1^{(r)} = \frac{\mu_{r-1}}{\mu_r}, \quad r = 0, \pm 1, \pm 2, \ldots .$$

These relations link, respectively, the coefficients in the two rhombii

$$\begin{array}{ccc} & l_n^{(r-1)} & \to & m_n^{(r-1)} \\ \nearrow & & \nearrow & \\ m_{n-1}^{(r)} & \to & l_n^{(r)} & \end{array} \qquad \text{and} \qquad \begin{array}{cccc} 1 - l_{n-1}^{(r-1)} & \to & m_{n-1}^{(r-1)} & \\ & \searrow & & \searrow \\ & & m_{n-1}^{(r)} & \to & 1 - l_n^{(r)} \end{array}$$

in the $l - m$ table

$$\begin{array}{ccccccccc}
l_1^{(-3)} & m_1^{(-3)} & l_2^{(-3)} & m_2^{(-3)} & l_3^{(-3)} & m_3^{(-3)} & l_4^{(-3)} & m_4^{(-3)} & \cdots \\
l_1^{(-2)} & m_1^{(-2)} & l_2^{(-2)} & m_2^{(-2)} & l_3^{(-2)} & m_3^{(-2)} & l_4^{(-2)} & m_4^{(-2)} & \cdots \\
l_1^{(-1)} & m_1^{(-1)} & l_2^{(-1)} & m_2^{(-1)} & l_3^{(-1)} & m_3^{(-1)} & l_4^{(-1)} & m_4^{(-1)} & \cdots \\
l_1^{(0)} & m_1^{(0)} & l_2^{(0)} & m_2^{(0)} & l_3^{(0)} & m_3^{(0)} & l_4^{(0)} & m_4^{(0)} & \cdots \\
l_1^{(1)} & m_1^{(1)} & l_2^{(1)} & m_2^{(1)} & l_3^{(1)} & m_3^{(1)} & l_4^{(1)} & m_4^{(1)} & \cdots \\
l_1^{(2)} & m_1^{(2)} & l_2^{(2)} & m_2^{(2)} & l_3^{(2)} & m_3^{(2)} & l_4^{(2)} & m_4^{(2)} & \cdots \\
l_1^{(3)} & m_1^{(3)} & l_2^{(3)} & m_2^{(3)} & l_3^{(3)} & m_3^{(3)} & l_4^{(3)} & m_4^{(3)} & \cdots
\end{array} .$$

The continued fraction

$$M^{(r)}(z) + \frac{\mu_r z^{-r}}{z} \,-\, \frac{v_1^{(r)}}{1} \,-\, \frac{u_2^{(r)} z}{z} \,-\, \frac{v_2^{(r)}}{1} \,-\, \frac{u_3^{(r)} z}{z} \,-\, \frac{v_3^{(r)}}{1} \,-\, \frac{u_4^{(r)} z}{z} \,-\, \cdots, \tag{3.5.5}$$

where

$$v_n^{(r)} = \frac{\beta_n^{(r)}\beta_{n-1}^{(r)}}{\alpha_n^{(r)}+\beta_{n-1}^{(r)}} = \frac{\beta_n^{(r)}\beta_{n-1}^{(r)}}{\gamma_{n-1}^{(r)}}, \quad u_n^{(r)} = \frac{\alpha_n^{(r)}}{\alpha_n^{(r)}+\beta_{n-1}^{(r)}} = \frac{\alpha_n^{(r)}}{\gamma_{n-1}^{(r)}}, \quad n \geq 2, \qquad (3.5.6)$$

with $v_1^{(r)} = \beta_1^{(r)}$, is an even extension of the M-fraction (3.5.1). It is also an odd extension of the M-fraction

$$M^{(r+1)}(z) + \frac{\mu_{r+1}z^{-r-1}}{z-\beta_1^{(r+1)}} - \frac{\alpha_2^{(r+1)}z}{z-\beta_2^{(r+1)}} - \frac{\alpha_3^{(r+1)}z}{z-\beta_3^{(r+1)}} - \frac{\alpha_4^{(r+1)}z}{z-\beta_4^{(r+1)}} - \cdots . \qquad (3.5.7)$$

The even order convergents $\dfrac{U_{2n}^{(r)}(z)}{V_{2n}^{(r)}(z)}$ and the odd order convergents $\dfrac{U_{2n+1}^{(r)}(z)}{V_{2n+1}^{(r)}(z)}$ of the continued fraction (3.5.5) are the successive convergents of the continued fractions (3.5.1) and (3.5.7), respectively. That is

$$\frac{U_{2n}^{(r)}(z)}{V_{2n}^{(r)}(z)} = \frac{A_n^{(r)}(z)}{B_n^{(r)}(z)} \quad \text{and} \quad \frac{U_{2n+1}^{(r)}(z)}{V_{2n+1}^{(r)}(z)} = \frac{A_n^{(r+1)}(z)}{B_n^{(r+1)}(z)}, \quad n \geq 0 .$$

The numerators and denominators of the convergents of the continued fraction (3.5.5) satisfy, respectively, the three-term recurrence relations

$$U_{2n}^{(r)}(z) = U_{2n-1}^{(r)}(z) - v_n^{(r)}U_{2n-2}^{(r)}(z),$$

$$U_{2n+1}^{(r)}(z) = zU_{2n}^{(r)}(z) - u_{n+1}^{(r)}zU_{2n-1}^{(r)}(z),$$

where $U_0^{(r)}(z) = M^{(r)}(z)$ and $U_1^{(r)}(z) = zM^{(r)}(z) + \mu_r z^{-r}$, for $r = 0, \pm 1, \pm 2, ...$, and

$$V_{2n}^{(r)}(z) = V_{2n-1}^{(r)}(z) - v_n^{(r)}V_{2n-2}^{(r)}(z),$$

$$V_{2n+1}^{(r)}(z) = zV_{2n}^{(r)}(z) - u_{n+1}^{(r)}zV_{2n-1}^{(r)}(z),$$

where $V_0^{(r)}(z) = 1$ and $V_1^{(r)}(z) = z$ for $r = 0, \pm 1, \pm 2, ...$ The denominators $V_{2n}^{(r)}(z)$ and $V_{2n+1}^{(r)}(z)$ are polynomials of degree n and $n+1$, respectively.

From the relations (2.3.12) and (3.5.6) we can prove that the coefficients $v_n^{(r)}$ and $u_n^{(r)}$, satisfy the equations

$$v_n^{(r+1)} = \frac{v_{n-1}^{(r)}\, u_n^{(r+1)}}{u_n^{(r)}},$$

$$1 - u_n^{(r)} = \frac{v_{n-1}^{(r)}\,(1 - u_{n-1}^{(r+1)})}{v_{n-1}^{(r+1)}},$$

for $n = 2, 3, \dots$ and $r = 0, \pm 1, \pm 2, \dots$, and

$$u_1^{(r)} = 0, \quad v_1^{(r)} = \frac{\mu_r}{\mu_{r-1}}, \quad r = 0, \pm 1, \pm 2, \dots .$$

The rhombii

$$\begin{array}{ccc} v_{n-1}^{(r)} & \rightarrow & u_n^{(r)} & & \\ & \searrow & & \searrow & \\ & & u_n^{(r+1)} & \rightarrow & v_n^{(r+1)} \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & v_{n-1}^{(r)} & \rightarrow & 1 - u_n^{(r)} \\ & \nearrow & & \nearrow & \\ 1 - u_{n-1}^{(r+1)} & \rightarrow & v_{n-1}^{(r+1)} & & \end{array},$$

in the $u - v$ table

$$\begin{array}{ccccccccc} u_1^{(-3)} & v_1^{(-3)} & u_2^{(-3)} & v_2^{(-3)} & u_3^{(-3)} & v_3^{(-3)} & u_4^{(-3)} & v_4^{(-3)} & \cdots \\ u_1^{(-2)} & v_1^{(-2)} & u_2^{(-2)} & v_2^{(-2)} & u_3^{(-2)} & v_3^{(-2)} & u_4^{(-2)} & v_4^{(-2)} & \cdots \\ u_1^{(-1)} & v_1^{(-1)} & u_2^{(-1)} & v_2^{(-1)} & u_3^{(-1)} & v_3^{(-1)} & u_4^{(-1)} & v_4^{(-1)} & \cdots \\ u_1^{(0)} & v_1^{(0)} & u_2^{(0)} & v_2^{(0)} & u_3^{(0)} & v_3^{(0)} & u_4^{(0)} & v_4^{(0)} & \cdots \\ u_1^{(1)} & v_1^{(1)} & u_2^{(1)} & v_2^{(1)} & u_3^{(1)} & v_3^{(1)} & u_4^{(1)} & v_4^{(1)} & \cdots \\ u_1^{(2)} & v_1^{(2)} & u_2^{(2)} & v_2^{(2)} & u_3^{(2)} & v_3^{(2)} & u_4^{(2)} & v_4^{(2)} & \cdots \\ u_1^{(3)} & v_1^{(3)} & u_2^{(3)} & v_2^{(3)} & u_3^{(3)} & v_3^{(3)} & u_4^{(3)} & v_4^{(3)} & \cdots \end{array},$$

show the elements connected by the above relations.

We now present some relations involving the coefficients in the even and odd extensions. Since

$$\gamma_n^{(r)} \;=\; \alpha_{n+1}^{(r)} + \beta_n^{(r)} \;=\; \alpha_n^{(r+1)} + \beta_n^{(r+1)},$$

we can see that

$$l_n^{(r)} = \frac{\alpha_n^{(r)}}{\alpha_n^{(r)} + \beta_n^{(r)}} = \frac{\alpha_n^{(r)}}{\gamma_n^{(r-1)}} = \frac{\alpha_n^{(r-1)}}{\gamma_{n-1}^{(r-1)}} = \frac{\alpha_n^{(r-1)}}{\alpha_n^{(r-1)} + \beta_{n-1}^{(r-1)}} = u_n^{(r-1)}.$$

Hence, for $n \geq 1$ and $r = 0, \pm 1, \pm 2, \ldots$, it follows that

$$l_n^{(r)} = u_n^{(r-1)} . \tag{3.5.8}$$

In the remainder of this subsection the distribution $\psi(t)$ is assumed be an $S^3[\omega, \beta, b]$ distribution.

Proposition 3.5.1 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $j = 1 - 2\omega$, the coefficients of the even and the odd extensions (3.5.2) and (3.5.5) are related as following*

$$\begin{aligned} u_n^{(j-l)} &= l_n^{(l)}, \\ v_n^{(j-l)} &= \beta^2 m_n^{(l)}, \end{aligned} \tag{3.5.9}$$

for $n \geq 1$ and $l = 0, \pm 1, \pm 2, \ldots$.

Proof: From (3.2.5), we know that, for $l = 0, \pm 1, \pm 2, \ldots$

$$\beta_n^{(l)} \beta_n^{(j-l)} = \beta^2, \quad n \geq 1, \quad \text{and} \quad \frac{\beta_{n-1}^{(j-l)}}{\alpha_n^{(j-l)}} = \frac{\beta_n^{(l)}}{\alpha_n^{(l)}}, \quad n \geq 2.$$

Hence, for $n = 1$, $u_1^{(j-l)} = l_1^{(l)} = 0$ and

$$v_1^{(j-l)} = \beta_1^{(j-l)} = \frac{\beta_1^{(l)} \beta_1^{(j-l)}}{\beta_1^{(l)}} = \frac{\beta^2}{\beta_1^{(l)}} = \beta^2 m_1^{(l)}.$$

For $n \geq 2$ and $l = 0, \pm 1, \pm 2, \ldots$

$$u_n^{(j-l)} = \frac{\alpha_n^{(j-l)}}{\alpha_n^{(j-l)} + \beta_{n-1}^{(j-l)}} = \frac{1}{1 + \frac{\beta_{n-1}^{(j-l)}}{\alpha_n^{(j-l)}}} = \frac{1}{1 + \frac{\beta_n^{(l)}}{\alpha_n^{(l)}}} = \frac{\alpha_n^{(l)}}{\alpha_n^{(l)} + \beta_n^{(l)}} = l_n^{(l)} ,$$

and

$$v_n^{(j-l)} = \frac{\beta_n^{(j-l)} \beta_{n-1}^{(j-l)}}{\alpha_n^{(j-l)} + \beta_{n-1}^{(j-l)}} = \frac{\beta_n^{(j-l)}}{\frac{\alpha_n^{(j-l)}}{\beta_{n-1}^{(j-l)}} + 1} = \frac{\beta_n^{(j-l)}}{\frac{\alpha_n^{(l)}}{\beta_n^{(l)}} + 1} = \frac{\beta_n^{(j-l)} \beta_n^{(l)}}{\alpha_n^{(l)} + \beta_n^{(l)}} = \beta^2 m_n^{(l)}. \qquad \square$$

If we consider the separate cases when (i) 2ω is odd and (ii) 2ω is even, we obtain the following corollary.

Corollary 3.5.1.1 *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0<\beta<b\le\infty$, $a=\beta^2/b$ and $2\omega\in\mathbf{Z}$. The coefficients of the even and odd extensions (3.5.2) and (3.5.5) are related as following*

(i) for 2ω odd and $j=\frac{1}{2}-\omega$, $n\ge 1$, and $l=0,\pm1,\pm2,\ldots$,

$$u_n^{(j+l)} = l_n^{(j-l)} \quad \text{and} \quad v_n^{(j+l)} = \beta^2 m_n^{(j-l)} ,$$

(ii) for 2ω even and $j=-\omega$, $n\ge 1$, and $l=0,\pm1,\pm2,\ldots$,

$$u_n^{(j+l)} = l_n^{(j+1-l)} \quad \text{and} \quad v_n^{(j+l)} = \beta^2 m_n^{(j+1-l)} .$$

The coefficients $l_n^{(r)}$ of the even extension (3.5.2) and the coefficients $u_n^{(r)}$ of the odd extension (3.5.5) satisfy some symmetric properties as we can see in the following proposition and its corollary.

Proposition 3.5.2 *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0<\beta<b\le\infty$, $a=\beta^2/b$ and $2\omega\in\mathbf{Z}$. Then for $j=1-2\omega$, $n\ge1$ and $l=0,\pm1,\pm2,\ldots$, the coefficients of the even and the odd extensions (3.5.2) and (3.5.5) satisfy*

$$u_n^{(j-l)} = u_n^{(l-1)}, \tag{3.5.10}$$

$$l_n^{(j+1-l)} = l_n^{(l)}. \tag{3.5.11}$$

Proof: For $n=1$, $u_1^{(j-l)} = u_1^{(l-1)} = 0$.

From (3.2.5), for $n\ge2$ and $l=0,\pm1,\pm2,\ldots$

$$u_n^{(j-l)} = \frac{\alpha_n^{(j-l)}}{\alpha_n^{(j-l)}+\beta_{n-1}^{(j-l)}} = \frac{1}{1+\frac{\beta_{n-1}^{(j-l)}}{\alpha_n^{(j-l)}}} = \frac{1}{1+\frac{\beta_n^{(l)}}{\alpha_n^{(l)}}} = \frac{\alpha_n^{(l)}}{\alpha_n^{(l)}+\beta_n^{(l)}} = \frac{\alpha_n^{(l-1)}}{\alpha_n^{(l-1)}+\beta_{n-1}^{(l-1)}} = u_n^{(l-1)}$$

The relation (3.5.11) follows by using (3.5.8). □

Corollary 3.5.2.1 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then the coefficients of the even and the odd extensions (3.5.2) and (3.5.5) satisfy*

(i) for 2ω odd and $j = \frac{1}{2} - \omega$, $n \geq 1$, and $l = 0, \pm 1, \pm 2, \ldots$,

$$u_n^{(j-1+l)} = u_n^{(j-l)} \quad \text{and} \quad l_n^{(j+l)} = l_n^{(j+1-l)} ,$$

(ii) for 2ω even and $j = -\omega$, $n \geq 1$, and $l = 0, \pm 1, \pm 2, \ldots$,

$$u_n^{(j+l)} = u_n^{(j-l)} \quad \text{and} \quad l_n^{(j+1+l)} = l_n^{(j+1-l)} .$$

In the following proposition and its corollary we give symmetric relations involving the coefficients $\beta_n^{(r)}$ from (3.5.1), the coefficients $m_n^{(r)}$ from (3.5.2), and the coefficients $v_n^{(r)}$ from (3.5.5).

Proposition 3.5.3 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $j = 1 - 2\omega$, $n \geq 2$ and $l = 0, \pm 1, \pm 2, \ldots$, the coefficients of the even and the odd extensions (3.5.2) and (3.5.5) satisfy*

$$m_n^{(l)} = \frac{\beta_{n-1}^{(j-l)}}{\beta_n^{(l)}} m_{n-1}^{(j+1-l)}, \tag{3.5.12}$$

$$v_n^{(j-l)} = \frac{\beta_{n-1}^{(j-l)}}{\beta_n^{(l)}} v_{n-1}^{(l-1)}. \tag{3.5.13}$$

Proof: From (3.2.5), for $n \geq 2$ and $l = 0, \pm 1, \pm 2, \ldots$

$$m_n^{(l)} = \frac{1}{\alpha_n^{(l)} + \beta_n^{(l)}} = \frac{\frac{1}{\beta_n^{(l)}}}{\frac{\alpha_n^{(l)}}{\beta_n^{(l)}} + 1} = \frac{\frac{1}{\beta_n^{(l)}}}{\frac{\alpha_n^{(j-l)}}{\beta_{n-1}^{(j-l)}} + 1} = \frac{\beta_{n-1}^{(j-l)} / \beta_n^{(l)}}{\alpha_n^{(j-l)} + \beta_{n-1}^{(j-l)}} =$$

$$= \frac{\beta_{n-1}^{(j-l)} / \beta_n^{(l)}}{\alpha_{n-1}^{(j+1-l)} + \beta_{n-1}^{(j+1-l)}} = \frac{\beta_{n-1}^{(j-l)}}{\beta_n^{(l)}} m_{n-1}^{(j+1-l)}.$$

The result (3.5.13) follows by using (3.5.9). □

Corollary 3.5.3.1 *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0<\beta<b\le\infty$, $a=\beta^2/b$ and $2\omega\in\boldsymbol{Z}$. Then the coefficients of the even and the odd extensions (3.5.2) and (3.5.5) satisfy*

(i) for 2ω odd, $j=\frac{1}{2}-\omega$, $n\ge 2$, and $l=0,\pm1,\pm2,\ldots$,

$$m_n^{(j+l)}=\frac{\beta_{n-1}^{(j-l)}}{\beta_n^{(j+l)}}m_{n-1}^{(j+1-l)}\quad\text{and}\quad v_n^{(j-l)}=\frac{\beta_{n-1}^{(j-l)}}{\beta_n^{(j+l)}}v_{n-1}^{(j-1+l)},$$

(ii) for 2ω even, $j=-\omega$, $n\ge 2$, and $l=0,\pm1,\pm2,\ldots$,

$$m_n^{(j+1-l)}=\frac{\beta_{n-1}^{(j+l)}}{\beta_n^{(j+1-l)}}m_{n-1}^{(j+1+l)}\quad\text{and}\quad v_n^{(j+l)}=\frac{\beta_{n-1}^{(j+l)}}{\beta_n^{(j+1-l)}}v_{n-1}^{(j-l)}.$$

We now consider the example 3.2 from section 3.4, namely,

Example 3.2 *The classical log-normal distribution is given by*

$$d\psi_0(t)=\frac{\sqrt{q}}{2\kappa\sqrt{\pi}}e^{-\left(\frac{ln(t)}{2\kappa}\right)^2}dt.$$

This is an $S^3[\omega,\beta,\infty]$ distribution with $\beta=q^{(\omega-1)}$.

We recall from section 3.4 that the coefficients $\beta_n^{(r)}$ and $\alpha_{n+1}^{(r)}$ are given by

$$\beta_n^{(r)}=q^{-(\frac{1}{2}+r)}\quad\text{and}\quad\alpha_{n+1}^{(r)}=q^{-(\frac{1}{2}+r)}(q^{-n}-1),$$

for $n\ge1$ and $r=0,\pm1,\pm2,\ldots$. Hence, from (3.5.3) we see that the coefficients $m_n^{(r)}$ and $l_n^{(r)}$ are given by

$$m_n^{(r)}=q^{\frac{3}{2}+n+r}\quad\text{and}\quad l_n^{(r)}=1-q^{n+1},$$

for $n\ge2$ and $r=0,\pm1,\pm2,\ldots$, with $m_1^{(r)}=q^{\frac{1}{2}+r}$. Further, from (3.5.6) we see that the coefficients $v_n^{(r)}$ and $u_n^{(r)}$ are given by

$$v_n^{(r)}=q^{\frac{1}{2}+n-r}\quad\text{and}\quad u_n^{(r)}=1-q^{n+1},$$

for $n\ge2$ and $r=0,\pm1,\pm2,\ldots$, with $v_1^{(r)}=q^{-(\frac{1}{2}+r)}$.

Perron-Carathéodory continued fractions

Secondly we consider odd extensions of M-fractions of the form

$$\beta_0 + \frac{\alpha_1}{1} + \frac{1}{\beta_2 z} + \frac{\alpha_3 z}{\beta_3} + \frac{1}{\beta_4 z} + \frac{\alpha_5 z}{\beta_5} + \frac{1}{\beta_6 z} + \cdots, \qquad (3.5.14)$$

where $\alpha_1 \neq 0$, $\alpha_{2n+1} = 1 - \beta_{2n}\beta_{2n+1} \neq 0$, $n = 1, 2, 3, \ldots$, $\alpha_{2n+1}, \beta_n \in \mathbb{C}$.

These continued fractions are called Perron-Carathéodory continued fractions, or PC-fractions.

The continued fraction (3.5.14) is the usual notation for PC-fractions in the literature and, since the coefficients α_n's and β_n's have no superscript, they are not to be confused with the coefficients $\alpha_n^{(r)}$'s and $\beta_n^{(r)}$'s encountered earlier.

A subclass of PC-fractions, where the conditions

$$\alpha_1 = -2\beta_0 < 0, \quad \alpha_{2n+1} = 1 - \beta_{2n}\beta_{2n+1} > 0, \quad \beta_{2n} = \bar{\beta}_{2n+1}, \quad n = 1, 2, 3, \ldots,$$

are satisfied are called positive PC-fractions, or PPC-fractions. These have been studied by Jones, Njåstad and Thron in connection with the trigonometric moment problem and the Szegö polynomials that are orthogonal on the unit circle, see [21, 23].

Here we consider another subclass of PC-fractions, namely those for which

$$\beta_0 = 0, \quad \alpha_1 > 0, \quad \alpha_{2n+1} = 1 - \beta_{2n}\beta_{2n+1} < 0, \quad \beta_{n+1} > 0, \quad n = 1, 2, 3, \ldots.$$

These are called strong Stieltjes PC-fractions, or SSPC-fractions and they have been studied in connection with strong Stieltjes moment problems. They can be regarded as even extensions of T-fractions and as odd extensions of M-fractions, see Jones, Thron *et al.* [18, 22] and Common and McCabe [13, 14].

Theorem 3.5.1 *Given $r \in \boldsymbol{Z}$, let*

$$\frac{\alpha_{r,1}}{1} + \frac{1}{\beta_{r,2}z} + \frac{\alpha_{r,3}z}{\beta_{r,3}} + \frac{1}{\beta_{r,4}z} + \frac{\alpha_{r,5}z}{\beta_{r,5}} + \frac{1}{\beta_{r,6}z} + \cdots, \tag{3.5.15}$$

be an SSPC-fraction, where

$$\alpha_{r,1} = \mu_{r-1} > 0, \quad \beta_{r,2n} = \frac{H_n^{(-n+1+r)}}{H_n^{(-n+r)}}, \quad \beta_{r,2n+1} = \frac{H_n^{(-n-1+r)}}{H_n^{(-n+r)}}, \quad n \geq 1, \tag{3.5.16}$$

and $\alpha_{r,2n+1} = 1 - \beta_{r,2n}\beta_{r,2n+1} < 0$ *for* $n \geq 1$ *and* $r = 0, \pm 1, \pm 2, \ldots$.

The even part of the SSPC-fraction (3.5.15) is the T-fraction

$$\frac{F_1^{(r)}z}{1+G_1^{(r)}z} + \frac{F_2^{(r)}z}{1+G_2^{(r)}z} + \frac{F_3^{(r)}z}{1+G_3^{(r)}z} + \frac{F_4^{(r)}z}{1+G_4^{(r)}z} + \cdots, \tag{3.5.17}$$

where for $n = 2, 3, \ldots$

$$F_n^{(r)} = \frac{H_n^{(-n+1+r)}H_{n-2}^{(-n+2+r)}}{H_{n-1}^{(-n+1+r)}H_{n-1}^{(-n+2+r)}} = \alpha_n^{(r)}, \quad G_n^{(r)} = \frac{H_n^{(-n+1+r)}H_{n-1}^{(-n+1+r)}}{H_n^{(-n+r)}H_{n-1}^{(-n+2+r)}} = \beta_n^{(r)},$$

and

$$F_1^{(r)} = \mu_r, \quad G_1^{(r)} = \frac{\mu_r}{\mu_{r-1}} = \beta_1^{(r)}.$$

The odd part of the SSPC-fraction (3.5.15) is the M-fraction

$$\mu_{r-1} + \frac{U_1^{(r-1)}}{z+V_1^{(r-1)}} + \frac{U_2^{(r-1)}z}{z+V_2^{(r-1)}} + \frac{U_3^{(r-1)}z}{z+V_3^{(r-1)}} + \frac{U_4^{(r-1)}z}{z+V_4^{(r-1)}} + \cdots, \tag{3.5.18}$$

where for $n = 2, 3, \ldots$

$$U_n^{(r-1)} = \frac{H_n^{(-n-1+r)}H_{n-2}^{(-n+2+r)}}{H_{n-1}^{(-n+r)}H_{n-1}^{(-n+1+r)}} = \frac{\alpha_n^{(r-1)}}{\beta_{n-1}^{(r-1)}\beta_n^{(r-1)}},$$

$$V_n^{(r-1)} = \frac{H_n^{(-n-1+r)}H_{n-1}^{(-n+1+r)}}{H_n^{(-n+r)}H_{n-1}^{(-n+r)}} = \frac{1}{\beta_n^{(r-1)}},$$

and

$$U_1^{(r-1)} = -\mu_{r-2}, \quad V_1^{(r-1)} = \frac{\mu_{r-2}}{\mu_{r-1}}.$$

Proof: From (3.5.16), we obtain

$$\alpha_{r,2n+1} = \frac{(H_n^{(-n+r)})^2 - H_n^{(-n+1+r)}H_n^{(-n-1+r)}}{(H_n^{(-n+r)})^2}.$$

By using the Jacobi's identity,

$$(H_n^{(-n+r)})^2 - H_n^{(-n-1+r)}H_n^{(-n+1+r)} + H_{n+1}^{(-n-1+r)}H_{n-1}^{(-n+1+r)} = 0,$$

we obtain

$$\alpha_{r,2n+1} = -\frac{H_{n+1}^{(-n-1+r)}H_{n-1}^{(-n+1+r)}}{H_n^{(-n+r)}H_n^{(-n+r)}} < 0, \quad n = 1,2,3,\ldots .$$

We know from the theory of continued fraction that a continued fraction of the form

$$\frac{\alpha_{r,1}}{1} + \frac{1}{\beta_{r,2}z} + \frac{\alpha_{r,3}z}{\beta_{r,3}} + \frac{1}{\beta_{r,4}z} + \frac{\alpha_{r,5}z}{\beta_{r,5}} + \frac{1}{\beta_{r,6}z} + \cdots,$$

has an even part of the form

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \cdots,$$

where

$$a_n = -\frac{\alpha_{r,2n-1}\beta_{r,2n}}{\beta_{r,2n-2}}z, \quad b_n = 1 + \frac{\beta_{r,2n}}{\beta_{r,2n-2}}z \quad n \geq 2,$$

with $a_1 = \alpha_{r,1}\beta_{r,2}z$ and $b_1 = 1 + \beta_{r,2}z$.

From (3.5.16), we obtain, for $n \geq 2$,

$$a_n = -\frac{\alpha_{r,2n-1}\beta_{r,2n}}{\beta_{r,2n-2}}z = \frac{H_n^{(-n+r)}H_{n-2}^{(-n+2+r)}}{H_{n-1}^{(-n+1+r)}H_{n-1}^{(-n+1+r)}}\frac{H_n^{(-n+1+r)}}{H_n^{(-n+r)}}\frac{H_{n-1}^{(-n+1+r)}}{H_{n-1}^{(-n+2+r)}}z = F_n^{(r)}z$$

and

$$b_n = 1 + \frac{\beta_{r,2n}}{\beta_{r,2n-2}}z = 1 + \frac{H_n^{(-n+1+r)}}{H_n^{(-n+r)}}\frac{H_{n-1}^{(-n+1+r)}}{H_{n-1}^{(-n+2+r)}}z = 1 + G_n^{(r)}z,$$

with

$$a_1 = \alpha_{r,1}\beta_{r,2}z = \mu_{r-1}\frac{H_1^{(r)}}{H_1^{(r-1)}}z = \mu_r z = F_1^{(r)}z$$

and

$$b_1 = 1 + \beta_{r,2}z = 1 + \frac{H_1^{(r)}}{H_1^{(r-1)}}z = 1 + G_1^{(r)}z.$$

This proves that the T-fraction (3.5.17) is the even part of the SSPC-fraction (3.5.15).

The SSPC-fraction (3.5.15) has an odd part of the form

$$b_0 + \frac{a_1}{b_1} \underset{+}{} \frac{a_2}{b_2} \underset{+}{} \frac{a_3}{b_3} \underset{+}{} \frac{a_4}{b_4} \underset{+}{} \cdots,$$

where

$$a_n = -\frac{\alpha_{r,2n-1}\beta_{r,2n+1}}{\beta_{r,2n-1}}z, \quad b_n = z + \frac{\beta_{r,2n+1}}{\beta_{r,2n-1}}z, \quad n \geq 2,$$

with $b_0 = \alpha_{r,1}$, $a_1 = -\alpha_{r,1}\beta_{r,3}$, and $b_1 = z + \beta_{r,3}$.

Once again using (3.5.16) we obtain, for $n \geq 2$,

$$a_n = -\frac{\alpha_{r,2n-1}\beta_{r,2n+1}}{\beta_{r,2n-1}}z = \frac{H_n^{(-n+r)}H_{n-2}^{(-n+2+r)}}{H_{n-1}^{(-n+1+r)}H_{n-1}^{(-n+1+r)}} \frac{H_n^{(-n-1+r)}}{H_n^{(-n+r)}} \frac{H_{n-1}^{(-n+1+r)}}{H_{n-1}^{(-n+r)}}z = U_n^{(r-1)}z$$

and

$$b_n = z + \frac{\beta_{r,2n+1}}{\beta_{r,2n-1}} = z + \frac{H_n^{(-n-1+r)}}{H_n^{(-n+r)}} \frac{H_{n-1}^{(-n+1+r)}}{H_{n-1}^{(-n+r)}} = z + V_n^{(r-1)},$$

with

$$b_0 = \mu_{r-1},$$

$$a_1 = -\mu_{r-1}\frac{H_1^{(r-2)}}{H_1^{(r-1)}} = -\mu_{r-2} = U_1^{(r-1)}$$

and

$$b_1 = z + \frac{H_1^{(r-2)}}{H_1^{(r-1)}} = z + \frac{\mu_{r-2}}{\mu_{r-1}} = z + V_1^{(r-1)}.$$

This proves that the M-fraction (3.5.18) is the even part of the SSPC-fraction (3.5.15). □

This theorem is a generalization of a result given by Jones, Njåstad and Thron in [21].

If we consider an $S^3[\omega, \beta, b]$ distribution where $\omega \in \boldsymbol{Z}$ and $\beta = 1$, from (3.1.3) we obtain

$$H_n^{(m)} = H_n^{(-m-2\omega-2n+2)}, \quad m = 0, \pm 1, \pm 2, ..., \quad n \geq 1,$$

hence

$$H_n^{(-n+1+r)} = H_n^{(-n+1-r-2\omega)}, \quad n \geq 1.$$

If $r = 1 - \omega$ then

$$H_n^{(-n+1+r)} = H_n^{(-n+1-r-2(1-r))} = H_n^{(-n-1+r)},$$

and in the equations (3.5.16)

$$\beta_{r,2n} = \beta_{r,2n+1}, \quad n = 1, 2, ...$$

This proves the following result.

Theorem 3.5.2 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution where $\omega \in \boldsymbol{Z}$ and $\beta = 1$. Consider the SSPC-fraction (3.5.15) which is an even extension of the T-fraction (3.5.17) and an odd extension of the M-fraction (3.5.18). If $r = 1 - \omega$ then*

$$\beta_{r,2n} = \beta_{r,2n+1} = \frac{H_n^{(-n+1+r)}}{H_n^{(-n+r)}}, \quad n = 1, 2, 3, ...$$

The particular PC-fractions, in which

$$\beta_0 = 0, \;\; \alpha_1 > 0, \;\; \alpha_{2n+1} = 1 - \beta_{2n}\beta_{2n+1} < 0, \;\; \beta_{2n} = \beta_{2n+1} > 0, \;\; n = 1, 2, ...,$$

are called symmetric strong Stieltjes PC-fractions, or SSSPC-fractions. These continued fractions were studied by Common and McCabe in [13] in connection with symmetric strong Stieltjes moment problems.

We now show examples of SSSPC-fractions. First we return to the example 3.1.(i) from section 3.4, namely,

Example 3.1 *The $S^3[\omega,\beta,b]$ distribution, $\psi(t)$, defined by*

$$d\psi(t) = \frac{(1+\beta/t)^{1-2\omega}}{\sqrt{b-t}\sqrt{t-a}}dt,$$

on (a,b), where $0<\beta<b<\infty$ and $\beta=\sqrt{ab}$.

In this example, we set $\omega \in \boldsymbol{Z}$ and $r = 1-\omega$. We also set $a = 1/b$ then $\beta = 1$.

We recall from (3.4.1) that, for any $\omega \geq 1/2$,

$$\begin{aligned} \beta_n^{(r)} &= \beta, \quad \alpha_n^{(r)} = \alpha, \quad \text{for } r > -\omega, \quad n \geq r+2\omega+1, \\ \beta_n^{(r)} &= \beta, \quad \alpha_{n+1}^{(r)} = \alpha, \quad \text{for } r \leq -\omega, \quad n \geq -r+2, \end{aligned}$$

where

$$\beta = \sqrt{ab} = 1 \quad \text{and} \quad \alpha = \frac{(\sqrt{b}-\sqrt{a})^2}{4} = \frac{(b-1)^2}{4b}.$$

Since $r = 1-\omega$, it follows that

$$\beta_n^{(r)} = \beta = 1, \quad \alpha_n^{(r)} = \alpha = \frac{(b-1)^2}{4b}, \quad \text{for } n \geq \omega+2. \tag{3.5.19}$$

From the proof of Theorem 3.5.1 we know that the coefficients of an SSPC-fraction satisfy

$$\frac{\beta_{r,2n}}{\beta_{r,2n-2}} = \beta_n^{(r)} \quad \text{and} \quad -\frac{\alpha_{r,2n-1}\beta_{r,2n}}{\beta_{r,2n-2}} = \alpha_n^{(r)}.$$

Hence, using (3.5.19), for $n \geq \omega+2$,

$$\frac{\beta_{r,2n}}{\beta_{r,2n-2}} = \beta \quad \text{and} \quad \frac{\alpha_{r,2n-1}\beta_{r,2n}}{\beta_{r,2n-2}} = -\alpha.$$

Since $\beta_{r,2n} = \beta_{r,2n+1}$, for $n = 1,2,\ldots$, and $\beta = 1$, then

$$\begin{aligned} \beta_{r,n} &= \gamma, \qquad \text{for } n \geq 2\omega+2, \\ \alpha_{r,2n-1} &= -\alpha, \quad \text{for } n \geq \omega+2, \end{aligned}$$

where γ is constant.

From $\alpha_{r,2n-1} = 1 - \beta_{r,2n-2}\beta_{r,2n-1}$, we then find $\gamma = \sqrt{1+\alpha}$, hence

$$\beta_{r,n} = \sqrt{1+\alpha}, \quad \text{for } n \geq 2\omega + 2,$$
$$\alpha_{r,2n-1} = -\alpha, \quad \text{for } n \geq \omega + 2.$$

For example, for $\omega = 1$, then $r = 0$,

$$\beta_{0,n} = \sqrt{1+\alpha}, \quad n = 4, 5, 6, ...,$$
$$\alpha_{0,2n-1} = -\alpha, \quad n = 3, 4, 5, ...$$

The SSSPC-fraction is

$$\frac{\alpha_{0,1}}{1} + \frac{1}{\beta_{0,2}\, z} + \frac{\alpha_{0,3}\, z}{\beta_{0,3}} + \frac{1}{\sqrt{1+\alpha}\, z} + \frac{-\alpha\, z}{\sqrt{1+\alpha}} + \frac{1}{\sqrt{1+\alpha}\, z} + \frac{-\alpha\, z}{\sqrt{1+\alpha}} + \cdots$$

where

$$\alpha_{0,1} = \mu_{-1}, \quad \beta_{0,2} = \beta_{0,3} = \frac{\mu_0}{\mu_{-1}}, \quad \text{and} \quad \alpha_{0,3} = \frac{(\mu_{-1})^2 - (\mu_0)^2}{(\mu_{-1})^2}.$$

We now consider the example 3.2 from section 3.4, namely,

Example 3.2 *The classical log-normal distribution*

$$d\psi_0(t) = \frac{\sqrt{q}}{2\kappa\sqrt{\pi}} e^{-\left(\frac{ln(t)}{2\kappa}\right)^2} dt,$$

which is an $S^3[\omega, \beta, \infty]$ *distribution with* $\beta = q^{(\omega-1)}$.

We recall from section 3.4, that the corresponding moments are given by

$$\mu_m = q^{-(m+m^2/2)}, \quad m = 0, \pm 1, \pm 2, \ldots .$$

The coefficients $\alpha_{n+1}^{(r)}$ and $\beta_n^{(r)}$, $r = 0, \pm 1, \pm 2, ...$, are given by

$$\alpha_{n+1}^{(r)} = q^{-(\frac{1}{2}+r)}(q^{-n} - 1) \quad \text{and} \quad \beta_n^{(r)} = q^{-(\frac{1}{2}+r)}, \quad n \geq 1.$$

We consider $\omega = 1$, then $\beta = 1$, that is an $S^3[1, 1, \infty]$ distribution and the moments satisfy

$$\mu_m = \mu_{-m-2}, \quad m = 0, \pm 1, \pm 2, \dots .$$

The corresponding Hankel determinants are given by

$$H_n^{(m)} = (\mu_{m+n-1})^n q^{-n(n^2-1)/6} \prod_{j=1}^{n-1} (1 - q^j)^{n-j}, \quad n \geq 1, \quad m = 0, \pm 1, \pm 2, \dots .$$

Since $\omega = 1$, we find the SSSPC-fraction with $r = 0$, and the coefficients $\beta_{0,2n}$, $\beta_{0,2n+1}$ and $\alpha_{0,2n+1}$ are given by

$$\beta_{0,2n} = \beta_{0,2n+1} = \frac{H_n^{(-n+1)}}{H_n^{(-n)}} = q^{-\frac{n}{2}}, \quad n \geq 1,$$

with

$$\alpha_{0,1} = \mu_{-1} = q^{\frac{1}{2}}, \quad \text{and} \quad \alpha_{0,2n+1} = 1 - q^{-n}.$$

Hence, the SSSPC-fraction becomes

$$\frac{q^{\frac{1}{2}}}{1} + \frac{1}{q^{-\frac{1}{2}} z} + \frac{(1 - q^{-1}) z}{q^{-\frac{1}{2}}} + \frac{1}{q^{-1} z} + \frac{(1 - q^{-2}) z}{q^{-1}} + \frac{1}{q^{-\frac{3}{2}} z} + \frac{(1 - q^{-3}) z}{q^{-\frac{3}{2}}} + \cdots .$$

The same SSSPC-fraction has been given by Common and McCabe in [13], where a log-normal distribution with $p = -1$, $\omega = 0$ and $\beta = 1$ was considered.

Chapter 4

The $S^3[\omega, \beta, b]$ distributions and the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$

4.1 Introduction

In this chapter we study the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$, for an integer $r \geq 1$ and $n \geq 0$, as defined in section 2.5, when $\psi(t)$ is an $S^3[\omega, \beta, b]$ distribution. We recall that

$$B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z) = B_n^{(0)}(z) + \lambda_{n,1}^{(r)} B_{n-1}^{(0)}(z) + \cdots + \lambda_{n,r}^{(r)} B_{n-r}^{(0)}(z),$$

where $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)} \in \mathbb{R}$ and

$$\int_a^b t^{-n+s} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; t) d\psi(t) = 0, \quad r \leq s \leq n-1, \quad \text{for } n \geq r+1.$$

First we prove the following result.

Lemma 4.1.1 *Given an integer $r \geq 1$, let $Q_n(t)$ be a monic polynomial of degree $n \geq r+1$, such that*

$$\int_a^b t^{-n+s} Q_n(t) d\psi(t) = 0, \quad r \leq s \leq n-1.$$

Then, there exist real parameters $\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}$ such that

$$Q_n(z) = B_n(\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}; z).$$

Proof: We can write the polynomial $Q_n(z)$ as a linear combination of the polynomials $B_0^{(0)}(z), B_1^{(0)}(z), ..., B_n^{(0)}(z)$. Thus

$$Q_n(z) = \sum_{j=0}^{n} c_j B_j^{(0)}(z), \quad c_n = 1.$$

Hence,

$$\int_a^b t^{-n+s} Q_n(t) d\psi(t) = \sum_{j=0}^{n} c_j \int_a^b t^{-n+s} B_j^{(0)}(t) d\psi(t) = 0, \quad r \leq s \leq n-1.$$

Setting $s = r, r+1, ..., n-1$, and using the definition (2.2.1), we obtain a homogeneous triangular system of $n-r$ linear equations in the $n-r$ unknowns $c_0, c_1, ..., c_{n-r-1}$, in which the diagonal elements are non zero. This system is

$$\left\{ \begin{array}{lllllll} \sigma_{0,r+1-n}^{(0)} c_0 & \sigma_{1,r+2-n}^{(0)} c_1 & \cdots & \sigma_{n-r-2,-1}^{(0)} c_{n-r-2} & \sigma_{n-r-1,0}^{(0)} c_{n-r-1} & = 0 \\ \sigma_{0,r+2-n}^{(0)} c_0 & \sigma_{1,r+3-n}^{(0)} c_1 & \cdots & \sigma_{n-r-2,0}^{(0)} c_{n-r-2} & & = 0 \\ \vdots & \vdots & & & & \vdots \\ \sigma_{0,-1}^{(0)} c_0 & \sigma_{1,0}^{(0)} c_1 & & & & = 0 \\ \sigma_{0,0}^{(0)} c_0 & & & & & = 0 \end{array} \right. ,$$

where

$$\sigma_{j,k}^{(0)} = \int_a^b t^{k-(j+1)} B_j^{(0)}(t) d\psi(t).$$

Hence the only solution is $c_0 = c_1 = \cdots = c_{n-r-1} = 0$. Then setting $c_{n-j} = \eta_{n,j}^{(r)}$ for $j = 1, 2, ..., r$, we obtain the result. □

Using the above lemma we can prove the following result.

Theorem 4.1.1 *Given an integer $r \geq 1$, let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $\omega = (1-r)/2$, $0 < \beta < b \leq \infty$, and $a = \beta^2/b$. If $n \geq r$ then, for any $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)} \in \mathbb{R}$, there exist corresponding $\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)} \in \mathbb{R}$ such that*

$$\frac{z^n B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/z)}{B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0)} = B_n(\eta_{n,1}^{(r)} ..., \eta_{n,r}^{(r)}; z). \tag{4.1.1}$$

Proof: If $n = r$, the result is obvious. For $n \geq r+1$, we know that

$$\int_a^b t^{-n+s} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; t) d\psi(t) = 0, \quad r \leq s \leq n-1,$$

then setting $t = \beta^2/t$, and using (3.1.1) it follows that

$$\int_a^b t^{n-s-2\omega} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/t) d\psi(t) = 0, \quad r \leq s \leq n-1.$$

Hence, diving by $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0)$,

$$\int_a^b t^{-s-2\omega} \left(\frac{t^n B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/t)}{B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0)} \right) d\psi(t) = 0, \quad r \leq s \leq n-1,$$

since $r = 1 - 2\omega$ then $t^{-s-2\omega}$ for $r \leq s \leq n-1$ implies the same that t^{-n+s} for $r \leq s \leq n-1$. Thus for $n \geq r+1$

$$\int_a^b t^{-n+s} \left(\frac{t^n B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/t)}{B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0)} \right) d\psi(t) = 0, \quad r \leq s \leq n-1,$$

using Lemma 4.1.1, the result follows. □

These results can be found in Sri Ranga, de Andrade and McCabe [48], for the special case when $r = 1$ and $\omega = 0$.

We denote the zeros of the polynomial $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ by $z_{n,i}^{\lambda^{(r)}}$, for $i = 1, 2, ..., n$. Assume that $z_{n,1}^{\lambda^{(r)}} < z_{n,2}^{\lambda^{(r)}} < \cdots < z_{n,m}^{\lambda^{(r)}}$, $1 \leq m \leq n$, are the positive distinct zeros of $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ and $z_{n,1}^{\eta^{(r)}} < z_{n,2}^{\eta^{(r)}} < \cdots < z_{n,m}^{\eta^{(r)}}$, are the positive distinct zeros of $B_n(\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}; z)$. Then from (4.1.1) these zeros satisfy the relation

$$z_{n,i}^{\lambda^{(r)}} = \frac{\beta^2}{z_{n,m+1-i}^{\eta^{(r)}}}, \quad i = 1, 2, .., m. \tag{4.1.2}$$

The negative distinct zeros and the conjugate complex zeros will satisfy similar relations. From (4.1.2) it is easy to see that if $z = \beta$ or $z = -\beta$ is a zero of $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ then it is also a zero of $B_n(\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}; z)$.

In the remainder of this chapter we will seek the real parameters $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}$, such that

$$\frac{z^n B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/z)}{B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0)} = B_n(\lambda_{n,1}^{(r)} ..., \lambda_{n,r}^{(r)}; z). \tag{4.1.3}$$

We also will study the behaviour of the zeros of these polynomials. We call the polynomials that satisfy (4.1.3) *symmetric inversive polynomials.* From Theorem 4.1.1, we can find symmetric inversive polynomials when $r \geq 1$ and $\omega = (1-r)/2$.

If the relation (4.1.3) holds then the positive distinct zeros of the polynomial $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ satisfy

$$z_{n,i}^{\lambda^{(r)}} = \frac{\beta^2}{z_{n,m+1-i}^{\lambda^{(r)}}}, \quad i = 1, 2, .., m. \tag{4.1.4}$$

The negative distinct zeros of $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ satisfy the same relations.

Due to the symmetry, it is easy to see that if the number of positive zeros or the number of negative zeros is odd then $z = \beta$ or $z = -\beta$, respectively, is a zero of odd multiplicity of $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$.

If a symmetric inversive polynomial $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ has m complex zeros, these complex zeros also satisfy

$$z_{n,i}^{\lambda^{(r)}} = \frac{\beta^2}{z_{n,m+1-i}^{\lambda^{(r)}}}, \quad i = 1, 2, ..., m.$$

The zero $z_{n,m+1-i}^{\lambda^{(r)}}$ is the conjugate complex of the zero $z_{n,i}^{\lambda^{(r)}}$, and its modulus is equal to β, for $i = 1, 2, ..., m$.

In de Andrade, Bracciali and Sri Ranga [3] we find that if $Q_n(z)$ is a monic polynomial of degree n that satisfies

$$\frac{z^n Q_n(\beta^2/z)}{(-\beta)^n} = Q_n(z), \quad z \in (a,b). \tag{4.1.5}$$

Then there exist a unique set of real numbers $\gamma_0, ..., \gamma_s$, $0 \leq s \leq [\![n/2]\!]$ such that

$$Q_n(z) = \sum_{k=0}^{s} \gamma_k t^k B_{n-2k}^{(0)}(z), \tag{4.1.6}$$

where $B_n^{(0)}(z)$ are the polynomials related to an $S^3[1/2,\beta,b]$ distribution.

4.2 The $S^3[0,\beta,b]$ distributions and the polynomials $B_n(\lambda_{n,1}^{(1)};z)$

Here we consider the special case of Theorem 4.1.1 with $r=1$, and hence $\omega = 0$. That is, we are considering the polynomials

$$B_n(\lambda_{n,1}^{(1)};z) = B_n^{(0)}(z) + \lambda_{n,1}^{(1)} B_{n-1}^{(0)}(z), \quad n \geq 0,$$

where the polynomials $B_n^{(0)}(z)$ are related to $S^3[0,\beta,b]$ distributions. According to Theorem 4.1.1 there exist $\eta_{n,1}^{(1)} \in \mathbb{R}$ such that

$$\frac{z^n B_n(\lambda_{n,1}^{(1)};\beta^2/z)}{B_n(\lambda_{n,1}^{(1)};0)} = B_n(\eta_{n,1}^{(1)};z), \quad \text{for } n \geq 1.$$

From (2.5.7) we know that if we choose

$$\lambda_{n,1}^{(1)} = \hat{\lambda}_{n,1}^{(1)} = -\alpha_{n+1}^{(0)}$$

then

$$B_n(\hat{\lambda}_{n,1}^{(1)};z) = B_n(-\alpha_{n+1}^{(0)};z) = B_n^{(1)}(z).$$

From Theorem 3.2.1 it then follows that

$$\frac{z^n B_n^{(1)}(\beta^2/z)}{B_n^{(1)}(0)} = B_n^{(0)}(z).$$

Using the above results, Sri Ranga, de Andrade and McCabe in [48] showed that if

$$\eta_{n,1}^{(1)} = \frac{-\beta_n^{(0)}\,(\alpha_{n+1}^{(0)} + \lambda_{n,1}^{(1)})}{\beta_n^{(0)} - \lambda_{n,1}^{(1)}}$$

then

$$\frac{z^n B_n(\lambda_{n,1}^{(1)};\beta^2/z)}{B_n(\lambda_{n,1}^{(1)};0)} = B_n(\eta_{n,1}^{(1)};z), \quad \text{for } n \geq 1.$$

Setting $\lambda_{n,1}^{(1)} = \eta_{n,1}^{(1)}$ then it gives

$$\lambda_{n,1}^{(1)} = \frac{-\beta_n^{(0)}\,(\alpha_{n+1}^{(0)} + \lambda_{n,1}^{(1)})}{\beta_n^{(0)} - \lambda_{n,1}^{(1)}} \tag{4.2.1}$$

and

$$\frac{z^n B_n(\lambda_{n,1}^{(1)};\beta^2/z)}{B_n(\lambda_{n,1}^{(1)};0)} = B_n(\lambda_{n,1}^{(1)};z), \quad \text{for } n \geq 1.$$

The equation (4.2.1) has the two solutions

$$\lambda_{n,1}^{(1)} = \sqrt{\beta_n^{(0)}}\left(\sqrt{\beta_n^{(0)}} \pm \sqrt{\gamma_n^{(0)}}\right),$$

where $\gamma_n^{(0)} = \alpha_{n+1}^{(0)} + \beta_n^{(0)}$.

When $\lambda_{n,1}^{(1)} = \sqrt{\beta_n^{(0)}}\left(\sqrt{\beta_n^{(0)}} + \sqrt{\gamma_n^{(0)}}\right)$, then the polynomial $B_n(\lambda_{n,1}^{(1)};z)$ has one negative zero, and it is equal to $-\beta$. Also

$$B_n(\lambda_{n,1}^{(1)};z) = (z+\beta)\tilde{B}_{n-1}^{(0)}(z),$$

where $\tilde{B}_n^{(0)}(z)$ are the polynomials associated with the distribution

$$d\tilde{\psi}(t) = (t+\beta)d\psi(t).$$

In the second case, with $\lambda_{n,1}^{(1)} = \sqrt{\beta_n^{(0)}}\left(\sqrt{\beta_n^{(0)}} - \sqrt{\gamma_n^{(0)}}\right)$, all the zeros of the polynomial $B_n(\lambda_{n,1}^{(1)};z)$ lie inside the interval (a,b). Also, in this case

$$B_n(\lambda_{n,1}^{(1)};z) = \tilde{B}_n^{(0)}(z),$$

where $\tilde{B}_n^{(0)}(z)$ are the polynomials related to the distribution

$$d\tilde{\psi}(t) = \frac{t}{t+\beta}d\psi(t).$$

This result agrees with (4.1.6), because $d\tilde{\psi}(t)$ is an $S^3[1/2,\beta,b]$ distribution and $B_n(\lambda_{n,1}^{(1)};0) = (-\beta)^n$.

Other choices for the parameters $\lambda_{n,1}^{(1)}$

In general the polynomial $B_n(\lambda_{n,1}^{(1)};z)$, $n \geq 1$, has at least $n-1$ real and distinct zeros inside (a,b). Assume that $z_{n,1}^{\lambda^{(1)}} < z_{n,2}^{\lambda^{(1)}} < \ldots < z_{n,n}^{\lambda^{(1)}}$, are the zeros of the polynomial $B_n(\lambda_{n,1}^{(1)};z)$, then Sri Ranga, de Andrade and McCabe in [48] proved that

$$\lambda_{n,1}^{(1)} > \beta_n^{(0)} \Rightarrow z_{n,1}^{\lambda^{(1)}} < 0,$$

$$\lambda_{n,1}^{(1)} < \beta_n^{(0)} \Rightarrow z_{n,1}^{\lambda^{(1)}} > 0,$$

$$\lambda_{n,1}^{(1)} = \beta_n^{(0)} \Rightarrow z_{n,1}^{\lambda^{(1)}} = 0.$$

In addition, if $-\alpha_{n+1}^{(0)} \leq \lambda_{n,1}^{(1)} \leq 0$ then all of the zeros of the polynomial $B_n(\lambda_{n,1}^{(1)};z)$ lie inside the interval (a,b).

4.3 The $S^3[-1/2,\beta,b]$ distributions and the polynomials $B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)$

We now consider the particular case of Theorem 4.1.1 with $r=2$, and hence $\omega=-1/2$. That is, we are considering the polynomials

$$B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)=B_n^{(0)}(z)+\lambda_{n,1}^{(2)}B_{n-1}^{(0)}(z)+\lambda_{n,2}^{(2)}B_{n-2}^{(0)}(z),\quad n\geq 0. \tag{4.3.1}$$

The polynomials $B_n^{(0)}(z)$ are associated with $S^3[-1/2,\beta,b]$ distributions. From Theorem 4.1.1 we know that there exist $\eta_{n,1}^{(2)},\eta_{n,2}^{(2)}\in\mathbb{R}$ such that

$$\frac{z^nB_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};\beta^2/z)}{B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};0)}=B_n(\eta_{n,1}^{(2)},\eta_{n,2}^{(2)};z),\quad \text{for } n\geq 2. \tag{4.3.2}$$

The polynomial $B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)$, $n\geq 2$, has at least $n-2$ zeros inside the interval (a,b), and they have odd multiplicity.

From (2.5.7) we know that if we choose

$$\lambda_{n,1}^{(2)}=\hat{\lambda}_{n,1}^{(2)}=-(\alpha_{n+1}^{(0)}+\alpha_{n+1}^{(1)}),$$
$$\lambda_{n,2}^{(2)}=\hat{\lambda}_{n,2}^{(2)}=\alpha_n^{(0)}\alpha_{n+1}^{(1)},$$

we obtain

$$B_n(-(\alpha_{n+1}^{(0)}+\alpha_{n+1}^{(1)})\,,\,\alpha_n^{(0)}\alpha_{n+1}^{(1)};z)=B_n^{(2)}(z). \tag{4.3.3}$$

From Theorem 3.2.1 and Theorem 3.2.2, we know that, for an $S^3[-1/2,\beta,b]$ distribution the related polynomials satisfy

$$\frac{z^nB_n^{(0)}(\beta^2/z)}{B_n^{(0)}(0)}=B_n^{(2)}(z),\qquad \frac{z^nB_n^{(1)}(\beta^2/z)}{B_n^{(1)}(0)}=B_n^{(1)}(z),$$

and

$$\beta_n^{(1)}=\beta,\quad n=1,2,\ldots\,.$$

In addition, from the relations (2.3.15) and (2.3.17), namely

$$B_n^{(1)}(z) = B_n^{(0)}(z) - \alpha_{n+1}^{(0)} B_{n-1}^{(0)}(z)$$

and

$$zB_{n-1}^{(2)}(z) = B_n^{(1)}(z) + \beta_n^{(1)} B_{n-1}^{(1)}(z)$$

we obtain

$$zB_{n-1}^{(2)}(z) = B_n^{(0)}(z) + (\beta_n^{(1)} - \alpha_{n+1}^{(0)}) B_{n-1}^{(0)}(z) - \beta_n^{(1)} \alpha_n^{(0)} B_{n-2}^{(0)}(z). \qquad (4.3.4)$$

From the relation (2.3.17) we can obtain

$$zB_{n-2}^{(2)}(z) = B_{n-1}^{(1)}(z) + \beta_{n-1}^{(1)} B_{n-2}^{(1)}(z)$$

or

$$zB_{n-1}^{(1)}(z) = B_n^{(0)}(z) + \beta_n^{(0)} B_{n-1}^{(0)}(z)$$

or

$$zB_{n-2}^{(1)}(z) = B_{n-1}^{(0)}(z) + \beta_{n-1}^{(0)} B_{n-2}^{(0)}(z).$$

From the last three relations we see that

$$zB_{n-2}^{(2)}(z) = \frac{1}{z}\left(B_n^{(0)}(z) + \beta_n^{(0)} B_{n-1}^{(0)}(z)\right) + \beta_{n-1}^{(1)} \frac{1}{z}\left(B_{n-1}^{(0)}(z) + \beta_{n-1}^{(0)} B_{n-2}^{(0)}(z)\right),$$

or

$$z^2 B_{n-2}^{(2)}(z) = B_n^{(0)}(z) + (\beta_n^{(0)} + \beta_{n-1}^{(1)}) B_{n-1}^{(0)}(z) + \beta_{n-1}^{(1)} \beta_{n-1}^{(0)} B_{n-2}^{(0)}(z). \qquad (4.3.5)$$

Replacing z by β^2/z in the equation (4.3.1) and multiplying both sides of the equation by z^n, we can write

$$\begin{aligned} z^n B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; \beta^2/z) &= z^n B_n^{(0)}(\beta^2/z) + \lambda_{n,1}^{(2)} z z^{n-1} B_{n-1}^{(0)}(\beta^2/z) \\ &\quad + \lambda_{n,2}^{(2)} z^2 z^{n-2} B_{n-2}^{(0)}(\beta^2/z). \end{aligned}$$

Since

$$z^n B_n^{(0)}(\beta^2/z) = B_n^{(0)}(0) B_n^{(2)}(z),$$

then

$$\begin{aligned} z^n B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; \beta^2/z) &= B_n^{(0)}(0) B_n^{(2)}(z) + \lambda_{n,1}^{(2)} B_{n-1}^{(0)}(0) z B_{n-1}^{(2)}(z) \\ &\quad + \lambda_{n,2}^{(2)} B_{n-2}^{(0)}(0) z^2 B_{n-2}^{(2)}(z). \end{aligned}$$

Using the relations (4.3.3), (4.3.4) and (4.3.5) we then obtain

$$\begin{aligned} z^n B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)};\beta^2/z) = B_n^{(0)}(0) &\Big[B_n^{(0)}(z) - (\alpha_{n+1}^{(1)} + \alpha_{n+1}^{(0)}) B_{n-1}^{(0)}(z) + \alpha_{n+1}^{(1)} \alpha_n^{(0)} B_{n-2}^{(0)}(z)\Big] \\ + \lambda_{n,1}^{(2)} B_{n-1}^{(0)}(0) &\Big[B_n^{(0)}(z) + (\beta_n^{(1)} - \alpha_{n+1}^{(0)}) B_{n-1}^{(0)}(z) - \alpha_n^{(0)} \beta_n^{(1)} B_{n-2}^{(0)}(z)\Big] \\ + \lambda_{n,2}^{(2)} B_{n-2}^{(0)}(0) &\Big[B_n^{(0)}(z) + (\beta_n^{(0)} + \beta_{n-1}^{(1)}) B_{n-1}^{(0)}(z) + \beta_{n-1}^{(1)} \beta_{n-1}^{(0)} B_{n-2}^{(0)}(z)\Big]. \end{aligned}$$

Rearranging the above equation we have

$$\begin{aligned} z^n B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; \beta^2/z) &= \Big[B_n^{(0)}(0) + \lambda_{n,1}^{(2)} B_{n-1}^{(0)}(0) + \lambda_{n,2}^{(2)} B_{n-2}^{(0)}(0)\Big] B_n^{(0)}(z) \\ &\quad + \Big[-(\alpha_{n+1}^{(1)} + \alpha_{n+1}^{(0)}) B_n^{(0)}(0) + \lambda_{n,1}^{(2)} (\beta_n^{(1)} - \alpha_{n+1}^{(0)}) B_{n-1}^{(0)}(0) \\ &\qquad + \lambda_{n,2}^{(2)} (\beta_n^{(0)} + \beta_{n-1}^{(1)}) B_{n-2}^{(0)}(0)\Big] B_{n-1}^{(0)}(z) \\ &\quad + \Big[\alpha_{n+1}^{(1)} \alpha_n^{(0)} B_n^{(0)}(0) - \lambda_{n,1}^{(2)} \alpha_n^{(0)} \beta_n^{(1)} B_{n-1}^{(0)}(0) \\ &\qquad + \lambda_{n,2}^{(2)} \beta_{n-1}^{(1)} \beta_{n-1}^{(0)} B_{n-2}^{(0)}(0)\Big] B_{n-2}^{(0)}(z). \end{aligned}$$

Since

$$B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; 0) = B_n^{(0)}(0) + \lambda_{n,1}^{(2)} B_{n-1}^{(0)}(0) + \lambda_{n,2}^{(2)} B_{n-2}^{(0)}(0)$$

or

$$B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; 0) = \beta_n^{(0)} \beta_{n-1}^{(0)} B_{n-2}^{(0)}(0) - \lambda_{n,1}^{(2)} \beta_{n-1}^{(0)} B_{n-2}^{(0)}(0) + \lambda_{n,2}^{(2)} B_{n-2}^{(0)}(0).$$

We then write

$$\begin{aligned} &\frac{z^n B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; \beta^2/z)}{B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; 0)} = B_n^{(0)}(z) \\ &+ \frac{-(\alpha_{n+1}^{(1)} + \alpha_{n+1}^{(0)}) \beta_n^{(0)} \beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)} (\beta_n^{(1)} - \alpha_{n+1}^{(0)}) \beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)} (\beta_n^{(0)} + \beta_{n-1}^{(1)})}{\beta_n^{(0)} \beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)} \beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)}} B_{n-1}^{(0)}(z) \\ &+ \frac{\alpha_{n+1}^{(1)} \alpha_n^{(0)} \beta_n^{(0)} \beta_{n-1}^{(0)} + \lambda_{n,1}^{(2)} \beta_n^{(1)} \alpha_n^{(0)} \beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)} \beta_{n-1}^{(0)} \beta_{n-1}^{(1)}}{\beta_n^{(0)} \beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)} \beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)}} B_{n-2}^{(0)}(z). \end{aligned}$$

By comparing the last equation with the equation (4.3.2), we see that

$$\eta_{n,1}^{(2)} = \frac{-\beta_n^{(0)}\beta_{n-1}^{(0)}(\alpha_{n+1}^{(1)} + \alpha_{n+1}^{(0)}) - \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)}(\beta_n^{(1)} - \alpha_{n+1}^{(0)}) + \lambda_{n,2}^{(2)}(\beta_n^{(0)} + \beta_{n-1}^{(1)})}{\beta_n^{(0)}\beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)}}$$

and

$$\eta_{n,2}^{(2)} = \frac{\beta_n^{(0)}\beta_{n-1}^{(0)}\alpha_{n+1}^{(1)}\alpha_n^{(0)} + \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)}\beta_n^{(1)}\alpha_n^{(0)} + \lambda_{n,2}^{(2)}\beta_{n-1}^{(0)}\beta_{n-1}^{(1)}}{\beta_n^{(0)}\beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)}}.$$

Setting

$$\lambda_{n,1}^{(2)} = \eta_{n,1}^{(2)} \quad \text{and} \quad \lambda_{n,2}^{(2)} = \eta_{n,2}^{(2)}$$

we obtain

$$\frac{z^n B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; \beta^2/z)}{B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; 0)} = B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; z).$$

The coefficients $\lambda_{n,1}^{(2)}$ and $\lambda_{n,2}^{(2)}$ then satisfy

$$\lambda_{n,1}^{(2)} = \frac{-\beta_n^{(0)}\beta_{n-1}^{(0)}(\alpha_{n+1}^{(1)} + \alpha_{n+1}^{(0)}) - \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)}(\beta_n^{(1)} - \alpha_{n+1}^{(0)}) + \lambda_{n,2}^{(2)}(\beta_n^{(0)} + \beta_{n-1}^{(1)})}{\beta_n^{(0)}\beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)}} \tag{4.3.6}$$

and

$$\lambda_{n,2}^{(2)} = \frac{\beta_n^{(0)}\beta_{n-1}^{(0)}\alpha_{n+1}^{(1)}\alpha_n^{(0)} + \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)}\beta_n^{(1)}\alpha_n^{(0)} + \lambda_{n,2}^{(2)}\beta_{n-1}^{(0)}\beta_{n-1}^{(1)}}{\beta_n^{(0)}\beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)}}. \tag{4.3.7}$$

To find the values of $\lambda_{n,1}^{(2)}$ and $\lambda_{n,2}^{(2)}$ that satisfy (4.3.6) and (4.3.7) simultaneously we multiply (4.3.6) by $-\beta_{n-1}^{(0)}$ and add the result to (4.3.7). We then obtain

$$(\beta_n^{(0)}\beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)})^2 = \beta_n^{(0)}\beta_{n-1}^{(0)}\gamma_{n-1}^{(0)}\gamma_n^{(1)}. \tag{4.3.8}$$

Now, from (2.3.12), we obtain $\gamma_{n-1}^{(0)} = \dfrac{\gamma_n^{(0)}\beta_{n-1}^{(0)}}{\beta_n^{(1)}}$, and

$$\sqrt{\beta_n^{(0)}\beta_{n-1}^{(0)}\gamma_{n-1}^{(0)}\gamma_n^{(1)}} = \beta_{n-1}^{(0)}\sqrt{\frac{\gamma_n^{(0)}\beta_n^{(0)}\gamma_n^{(1)}}{\beta_n^{(1)}}} = \beta_{n-1}^{(0)}\sqrt{\frac{\gamma_n^{(0)}\beta_n^{(0)}(\beta_n^{(1)} + \alpha_{n+1}^{(1)})}{\beta_n^{(1)}}}.$$

Since $\beta_n^{(1)} = \beta$, $\alpha_{n+1}^{(0)} = \dfrac{\beta_n^{(0)}\alpha_{n+1}^{(1)}}{\beta_{n+1}^{(1)}}$, and $\gamma_n^{(0)} = \beta_n^{(0)} + \alpha_{n+1}^{(0)}$, then

$$\sqrt{\beta_n^{(0)}\beta_{n-1}^{(0)}\gamma_{n-1}^{(0)}\gamma_n^{(1)}} = \gamma_n^{(0)}\beta_{n-1}^{(0)}.$$

From (4.3.8) we obtain

$$(\beta_n^{(0)}\beta_{n-1}^{(0)} - \lambda_{n,1}^{(2)}\beta_{n-1}^{(0)} + \lambda_{n,2}^{(2)}) = \pm\gamma_n^{(0)}\beta_{n-1}^{(0)}.$$

Hence, the parameters $\lambda_{n,1}^{(2)}$ and $\lambda_{n,2}^{(2)}$ that satisfy the equations (4.3.6) and (4.3.7) simultaneously also satisfy

$$\lambda_{n,1}^{(2)} = \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} + \beta_n^{(0)} \pm \gamma_n^{(0)}.$$

Substituting $\lambda_{n,1}^{(2)} = \dfrac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} + \beta_n^{(0)} + \gamma_n^{(0)}$ in (4.3.6), we obtain

$$\begin{aligned} \lambda_{n,1}^{(2)} &= \beta_n^{(0)} + \beta_{n-1}^{(1)} = \beta_n^{(0)} + \beta, \\ \lambda_{n,2}^{(2)} &= -\beta_n^{(1)}\alpha_n^{(0)} \quad = -\beta\alpha_n^{(0)}. \end{aligned}$$

Substituting $\lambda_{n,1}^{(2)} = \dfrac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} + \beta_n^{(0)} - \gamma_n^{(0)}$ in (4.3.6), we obtain

$$\begin{aligned} \lambda_{n,1}^{(2)} &= \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} - \alpha_{n+1}^{(0)}, \\ \lambda_{n,2}^{(2)} &\in \mathbb{R}. \end{aligned}$$

Hence, we have proved the following result.

Theorem 4.3.1 *The polynomials $B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)$ related to an $S^3[-1/2,\beta,b]$ distribution satisfy*

$$\frac{z^n B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};\beta^2/z)}{B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};0)} = B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z), \quad \text{for } n \geq 2,$$

if either

$$\begin{aligned} \lambda_{n,1}^{(2)} &= \beta_n^{(0)} + \beta, \\ \lambda_{n,2}^{(2)} &= -\beta\alpha_n^{(0)}, \end{aligned} \tag{4.3.9}$$

or

$$\begin{aligned} \lambda_{n,1}^{(2)} &= \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} - \alpha_{n+1}^{(0)}, \\ \lambda_{n,2}^{(2)} &\in \mathbb{R}. \end{aligned} \tag{4.3.10}$$

For the solution (4.3.9) we use the relations (2.3.15) and (2.3.17) and we obtain

$$B_n(\beta_n^{(0)}+\beta,-\beta\alpha_n^{(0)};z) = (z+\beta)B_{n-1}^{(1)}(z), \quad \text{for } n \geq 2.$$

Hence one zero of the polynomial $B_n(\beta_n^{(0)}+\beta,-\beta\alpha_n^{(0)};z)$ is at $z=-\beta$ and the other $n-1$ zeros, all distinct lie inside the interval (a,b).

The solution (4.3.10), depending on the parameter $\lambda_{n,2}^{(2)}$, gives other possible locations of the zeros of the polynomial $B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)$, such as, n real and distinct zeros inside the interval (a,b), or $n-2$ real and distinct zeros in (a,b), one zero in $(0,a)$ and one zero in (b,∞), or two negative zeros, or two complex zeros.

Proposition 4.3.1 *If we consider the solution (4.3.10), namely*

$$\begin{aligned} \lambda_{n,1}^{(2)} &= \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} - \alpha_{n+1}^{(0)}, \\ \lambda_{n,2}^{(2)} &\in \mathbb{R}, \end{aligned}$$

then

$$B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z) = B_n^{(1)}(z) + \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} z B_{n-2}^{(1)}(z), \quad \text{for } n \geq 2.$$

Proof: From the recurrence relation (2.2.7), with $r=1$, we obtain

$$zB_{n-1}^{(1)}(z) - B_n^{(1)}(z) = \beta_n^{(1)} B_{n-1}^{(1)}(z) + \alpha_n^{(1)} z B_{n-2}^{(1)}(z). \tag{4.3.11}$$

From the relation (2.3.20), we can write

$$B_n^{(0)}(z) = \frac{1}{\gamma_n^{(0)}}\left(\beta_n^{(0)} B_n^{(1)}(z) + \alpha_{n+1}^{(0)} z B_{n-1}^{(1)}(z)\right).$$

Adding and subtracting the term $\alpha_{n+1}^{(0)} B_n^{(1)}(z)$ to the right hand side of the above equation we obtain

$$B_n^{(0)}(z) = \frac{1}{\gamma_n^{(0)}}\left((\beta_n^{(0)} + \alpha_{n+1}^{(0)}) B_n^{(1)}(z) + \alpha_{n+1}^{(0)}(zB_{n-1}^{(1)}(z) - B_n^{(1)}(z))\right).$$

Using the equation (4.3.11) we can write

$$B_n^{(0)}(z) = B_n^{(1)}(z) + \frac{\alpha_{n+1}^{(0)}}{\gamma_n^{(0)}}\left(\beta_n^{(1)} B_{n-1}^{(1)}(z) + \alpha_n^{(1)} z B_{n-2}^{(1)}(z)\right). \tag{4.3.12}$$

While from the relations (2.3.20) and (2.3.19), we obtain

$$B_{n-1}^{(0)}(z) = \frac{1}{\gamma_n^{(0)}}\left(\beta_n^{(1)} B_{n-1}^{(1)}(z) + \alpha_n^{(1)} z B_{n-2}^{(1)}(z)\right), \tag{4.3.13}$$

and

$$B_{n-2}^{(0)}(z) = \frac{1}{\gamma_{n-1}^{(0)}}\left(zB_{n-2}^{(1)}(z) - B_{n-1}^{(1)}(z)\right), \tag{4.3.14}$$

respectively. Since

$$B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z) = B_n^{(0)}(z) + \lambda_{n,1}^{(2)} B_{n-1}^{(0)}(z) + \lambda_{n,2}^{(2)} B_{n-2}^{(0)}(z)$$

using (4.3.12), (4.3.13) and (4.3.14), we obtain

$$\begin{aligned} B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z) = B_n^{(1)}(z) &+ \frac{\alpha_{n+1}^{(0)}}{\gamma_n^{(0)}}\left(\beta_n^{(1)} B_{n-1}^{(1)}(z) + \alpha_n^{(1)} z B_{n-2}^{(1)}(z)\right) \\ &+ \frac{\lambda_{n,1}^{(2)}}{\gamma_n^{(0)}}\left(\beta_n^{(1)} B_{n-1}^{(1)}(z) + \alpha_n^{(1)} z B_{n-2}^{(1)}(z)\right) \\ &+ \frac{\lambda_{n,2}^{(2)}}{\gamma_{n-1}^{(0)}}\left(zB_{n-2}^{(1)}(z) - B_{n-1}^{(1)}(z)\right) \end{aligned}$$

or

$$B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z) = B_n^{(1)}(z) + \left[\frac{(\alpha_{n+1}^{(0)}+\lambda_{n,1}^{(2)})\beta_n^{(1)}}{\gamma_n^{(0)}} - \frac{\lambda_{n,2}^{(2)}}{\gamma_{n-1}^{(0)}}\right] B_{n-1}^{(1)}(z) + \left[\frac{(\alpha_{n+1}^{(0)}+\lambda_{n,1}^{(2)})\alpha_n^{(1)}}{\gamma_n^{(0)}} + \frac{\lambda_{n,2}^{(2)}}{\gamma_{n-1}^{(0)}}\right] zB_{n-2}^{(1)}(z).$$

From (4.3.10), $\alpha_{n+1}^{(0)}+\lambda_{n,1}^{(2)} = \dfrac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}}$, hence

$$B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z) = B_n^{(1)}(z) + \left[\frac{\lambda_{n,2}^{(2)}\ \beta_n^{(1)}}{\beta_{n-1}^{(0)}\ \gamma_n^{(0)}} - \frac{\lambda_{n,2}^{(2)}}{\gamma_{n-1}^{(0)}}\right] B_{n-1}^{(1)}(z) + \left[\frac{\lambda_{n,2}^{(2)}\ \alpha_n^{(1)}}{\beta_{n-1}^{(0)}\ \gamma_n^{(0)}} + \frac{\lambda_{n,2}^{(2)}}{\gamma_{n-1}^{(0)}}\right] zB_{n-2}^{(1)}(z).$$

From (2.3.12) the result holds. □

Since

$$B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};0) = B_n^{(1)}(0) = (-\beta)^n,$$

we can see that the Property 4.3.1 agrees with the result (4.1.6) because for $n \geq 0$, the polynomials $B_n^{(1)}(z)$ are the orthogonal L-polynomials related to the distribution $td\psi(t)$, which is an $S^3[1/2,\beta,b]$ distribution.

From (2.3.16) we can write

$$B_n^{(1)}(z) + \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} zB_{n-2}^{(1)}(z) = (z-\beta_n^{(1)})B_{n-1}^{(1)}(z) - \left(\alpha_n^{(1)} - \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}}\right) zB_{n-2}^{(1)}(z).$$

Here, if $\lambda_{n,2}^{(2)} \leq \beta\alpha_n^{(0)}$ then

$$\alpha_n^{(1)} - \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} \geq \alpha_n^{(1)} - \frac{\beta\alpha_n^{(0)}}{\beta_{n-1}^{(0)}} = \alpha_n^{(1)} - \alpha_n^{(1)} = 0.$$

Hence, for the solution (4.3.10), if $\lambda_{n,2}^{(2)} \leq \beta\alpha_n^{(0)}$ then all zeros of the polynomial $B_n^{(1)}(z) + \dfrac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} zB_{n-2}^{(1)}(z)$ are real and positive.

For example, if we choose

$$\lambda_{n,2}^{(2)} = \beta\alpha_n^{(0)},$$

then

$$\begin{aligned} B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z) &= B_n(\alpha_n^{(1)} - \alpha_{n+1}^{(0)}, \beta\alpha_n^{(0)}; z) \\ &= B_n^{(1)}(z) + \alpha_n^{(1)} z B_{n-2}^{(1)}(z) \\ &= (z-\beta)B_{n-1}^{(1)}(z). \end{aligned}$$

Hence the polynomial $B_n(\alpha_n^{(1)} - \alpha_{n+1}^{(0)}, \beta\alpha_n^{(0)}; z)$, $n \geq 2$, has one zero at $z = \beta$ and the other $n-1$ zeros are the same as the zeros of the polynomial $B_{n-1}^{(1)}(z)$. This means all of the zeros lie inside the interval (a,b). When n is even, the zero at $z = \beta$ has multiplicity 2.

For the location of the zeros of the polynomials

$$B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z) = B_n^{(1)}(z) + \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} z B_{n-2}^{(1)}(z), \quad n \geq 2,$$

we have the following conjecture.

Conjecture 4.1 *If we choose* $\lambda_{n,2}^{(2)}$ *in (4.3.10) such that*

$$-\beta\alpha_n^{(0)} < \lambda_{n,2}^{(2)} < \beta\alpha_n^{(0)}$$

then the polynomial $B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)$ *has all zeros distinct and they all lie inside the interval* (a,b). *For*

$$\lambda_{n,2}^{(2)} \leq -\beta\alpha_n^{(0)}$$

all zeros are distinct and positive with one zero in $(0,a)$, $n-2$ *zeros in* (a,b) *and one zero in* (b,∞). *For*

$$\lambda_{n,2}^{(2)} > \beta\alpha_n^{(0)}$$

two of the zeros can be complex or negative.

Other choices for the parameters $\lambda_{n,1}^{(2)}$ and $\lambda_{n,2}^{(2)}$

We recall that for the case $r=1$, if $-\alpha_{n+1}^{(0)} \le \lambda_{n,1}^{(1)} \le 0$, then all of the zeros of the polynomial

$$B_n(\lambda_{n,1}^{(1)};z) = B_n^{(0)}(z) + \lambda_{n,1}^{(1)} B_{n-1}^{(0)}(z), \quad n \ge 1,$$

lie inside the interval (a,b).

For the case $r=2$, if we choose

$$\begin{aligned} \lambda_{n,1}^{(2)} &= \lambda - \alpha_{n+1}^{(0)}, \\ \lambda_{n,2}^{(2)} &= -\lambda\alpha_n^{(0)}, \end{aligned}$$

for $\lambda \in \mathbb{R}$, we then obtain

$$\begin{aligned} B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z) &= B_n^{(0)}(z) + (\lambda - \alpha_{n+1}^{(0)})B_{n-1}^{(0)}(z) - \lambda\alpha_n^{(0)} B_{n-2}^{(0)}(z) \\ &= B_n^{(0)}(z) - \alpha_{n+1}^{(0)} B_{n-1}^{(0)}(z) + \lambda[B_{n-1}^{(0)}(z) - \alpha_n^{(0)} B_{n-2}^{(0)}(z)] \\ &= B_n^{(1)}(z) + \lambda B_{n-1}^{(1)}(z). \end{aligned}$$

Thus the polynomial $B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)$, $n \ge 2$ has at least $n-2$ zeros inside of the interval (a,b). Further if $-\alpha_{n+1}^{(1)} \le \lambda \le 0$ then all of the zeros lie inside the interval (a,b).

Since

$$-\alpha_{n+1}^{(1)} \le \lambda \le 0 \Rightarrow \begin{cases} -\alpha_{n+1}^{(1)} - \alpha_{n+1}^{(0)} \le \lambda_{n,1}^{(2)} \le -\alpha_{n+1}^{(0)}, \\ 0 \le \lambda_{n,2}^{(2)} \le \alpha_{n+1}^{(1)}\alpha_n^{(0)} \end{cases}$$

we conclude that all of the zeros of the polynomial $B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)$, $n \ge 1$, lie inside the interval (a,b) for

$$\begin{aligned} -\alpha_{n+1}^{(1)} - \alpha_{n+1}^{(0)} \le \lambda_{n,1}^{(2)} &\le -\alpha_{n+1}^{(0)}, \\ \lambda_{n,2}^{(2)} &= -(\lambda_{n,1}^{(2)} + \alpha_{n+1}^{(0)})\alpha_n^{(0)}, \end{aligned}$$

or equivalently for

$$\begin{aligned} 0 \le \lambda_{n,2}^{(2)} &\le \alpha_{n+1}^{(1)}\alpha_n^{(0)} \\ \lambda_{n,1}^{(2)} &= -\frac{\lambda_{n,2}^{(2)}}{\alpha_n^{(0)}} - \alpha_{n+1}^{(0)}. \end{aligned}$$

Remarks

In the previous section, we saw that, for $r=1$ and $\omega=0$, the symmetric relation

$$\frac{z^n B_n(\lambda_{n,1}^{(1)};\beta^2/z)}{B_n(\lambda_{n,1}^{(1)};0)} = B_n(\lambda_{n,1}^{(1)};z), \quad n\geq 1,$$

holds when either

$$\lambda_{n,1}^{(1)} = \sqrt{\beta_n^{(0)}}\left(\sqrt{\beta_n^{(0)}}+\sqrt{\gamma_n^{(0)}}\right)$$

or

$$\lambda_{n,1}^{(1)} = \sqrt{\beta_n^{(0)}}\left(\sqrt{\beta_n^{(0)}}-\sqrt{\gamma_n^{(0)}}\right).$$

In the first case, $-\beta$ is a zero of $B_n(\lambda_{n,1}^{(1)};z)$. In the second case, all of the zeros of the polynomial $B_n(\lambda_{n,1}^{(1)};z)$ lie inside the interval (a,b).

In this section, we saw that, for $r=2$ and $\omega=-1/2$, we obtained the symmetric relation

$$\frac{z^n B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};\beta^2/z)}{B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};0)} = B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z), \quad n\geq 2,$$

when either

$$\lambda_{n,1}^{(2)} = \beta_n^{(0)}+\beta,$$
$$\lambda_{n,2}^{(2)} = -\beta\alpha_n^{(0)},$$

or

$$\lambda_{n,1}^{(2)} = \frac{\lambda_{n,2}^{(2)}}{\beta_{n-1}^{(0)}} - \alpha_{n+1}^{(0)},$$
$$\lambda_{n,2}^{(2)} \in \mathbb{R}.$$

In the first case, $-\beta$ is a zero of $B_n(\lambda_{n,1}^{(2)},\lambda_{n,2}^{(2)};z)$, and the remaining $n-1$ distinct zeros lie inside the interval (a,b). In the second case the other possibilities in the location of the two zeros that can be outside of the interval (a,b) depends on the value of $\lambda_{n,2}^{(2)}$.

The polynomials $B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)$ for $r=3,4,5,...$, that are related to the $S^3[\omega,\beta,b]$ distributions with $\omega=(1-r)/2$, present similar behaviour to those studied above.

We obtain parameters $\lambda_{n,i}^{(r)}$, $i=1,2,..,r$, such that

$$\frac{z^n B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};\beta^2/z)}{B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};0)} = B_n(\lambda_{n,1}^{(r)}...,\lambda_{n,r}^{(r)};z), \quad \text{for } n\geq r.$$

These parameters $\lambda_{n,i}^{(r)}$, $i=1,2,..,r$, will determine all of the possibilities for the location of the r zeros that can lie outside of the interval (a,b), and satisfy the relations (4.1.4), namely

$$z_{n,i}^{\lambda^{(r)}} = \frac{\beta^2}{z_{n,m+1-i}^{\lambda^{(r)}}}, \quad i=1,2,..,m.$$

Chapter 5

Quadrature formulae

5.1 Introduction

In this chapter we consider quadrature formulae that are related to strong Stieltjes distributions $\psi(t)$, and to the polynomials $B_n^{(r)}(z)$ and $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$.

First we consider a quadrature rule of the form

$$\int_a^b f(t)d\psi(t) = \sum_{i=1}^{n} w_{n,i}^{(r)} f(z_{n,i}^{(r)}) + \mathbb{E}_n(f), \tag{5.1.1}$$

where the points $z_{n,i}^{(r)}$, $i = 1, ..., n$ are the zeros of the polynomial $B_n^{(r)}(z)$ and where $f(t)$ is a real function defined on (a, b). The coefficients $w_{n,i}^{(r)}$, $i = 1, ..., n$ are called the *weights* and the points $z_{n,i}^{(r)}$, $i = 1, ..., n$ are called the *nodes* of the quadrature formula.

We can approximate the function $F(z) = z^{n-r} f(z)$ by the interpolating polynomial on the n zeros of the polynomials $B_n^{(r)}(z)$. We then obtain

$$F(z) = z^{n-r} f(z) = P_n(z) + R_n(z),$$

where $P_n(z)$ is the interpolating polynomial for $F(z)$, of degree less than or equal to $n-1$, and $R_n(z)$ is the remainder.

Using the Lagrange form for $P_n(z)$ and the divided difference form for $R_n(z)$ we obtain

$$F(z) = \sum_{i=1}^{n} \frac{B_n^{(r)}(z)}{(z - z_{n,i}^{(r)}) B_n^{(r)'}(z_{n,i}^{(r)})} (z_{n,i}^{(r)})^{n-r} f(z_{n,i}^{(r)}) + B_n^{(r)}(z) F[z_{n,1}^{(r)}, ..., z_{n,n}^{(r)}, z].$$

Multiplying by z^{-n+r} and integrating over (a,b), we obtain

$$\int_a^b t^{-n+r} F(t) d\psi(t) = \int_a^b t^{-n+r} \sum_{i=1}^{n} \frac{B_n^{(r)}(t)}{(t - z_{n,i}^{(r)}) B_n^{(r)'}(z_{n,i}^{(r)})} (z_{n,i}^{(r)})^{n-r} f(z_{n,i}^{(r)}) d\psi(t)$$

$$+ \int_a^b t^{-n+r} B_n^{(r)}(t) F[z_{n,1}^{(r)}, ..., z_{n,n}^{(r)}, t] d\psi(t),$$

or

$$\int_a^b f(t) d\psi(t) = \sum_{i=1}^{n} \frac{(z_{n,i}^{(r)})^{n-r}}{B_n^{(r)'}(z_{n,i}^{(r)})} \int_a^b \frac{t^{-n+r} B_n^{(r)}(t)}{(t - z_{n,i}^{(r)})} d\psi(t)\, f(z_{n,i}^{(r)})$$

$$+ \int_a^b t^{-n+r} B_n^{(r)}(t) F[z_{n,1}^{(r)}, ..., z_{n,n}^{(r)}, t] d\psi(t).$$

Hence,

$$\int_a^b f(t) d\psi(t) = \sum_{i=1}^{n} w_{n,i}^{(r)} f(z_{n,i}^{(r)}) + \mathbb{E}_n(f),$$

where the weights $w_{n,i}^{(r)}$ are given by

$$w_{n,i}^{(r)} = \frac{(z_{n,i}^{(r)})^{n-r}}{B_n^{(r)'}(z_{n,i}^{(r)})} \int_a^b \frac{t^{-n+r} B_n^{(r)}(t)}{t - z_{n,i}^{(r)}} d\psi(t), \quad i = 1, 2..., n. \tag{5.1.2}$$

The remainder $\mathbb{E}_n(f)$ is given by

$$\mathbb{E}_n(f) = \int_a^b t^{-n+r} B_n^{(r)}(t) F[z_{n,1}^{(r)}, ..., z_{n,n}^{(r)}, t] d\psi(t). \tag{5.1.3}$$

Theorem 5.1.1 *For the quadrature formula (5.1.1), with the weights* $w_{n,i}^{(r)}$, $i = 1, 2..., n$, *given by (5.1.2),*

$$\mathbb{E}_n(f) = 0 \quad \textit{whenever} \quad z^{n-r} f(z) \in \mathbb{P}_{2n-1}.$$

Proof: Since

$$F(z) = z^{n-r} f(z) \in \mathbb{P}_{2n-1},$$

then $F[z_{n,1}^{(r)}, ..., z_{n,n}^{(r)}, z]$ is a polynomial of degree less than or equal to $n-1$. Hence

$$\mathbb{E}_n(f) = \int_a^b t^{-n+r} B_n^{(r)}(t) \sum_{s=0}^{n-1} v_s t^s d\psi(t) = \sum_{s=0}^{n-1} v_s \int_a^b t^{-n+s+r} B_n^{(r)}(t) d\psi(t).$$

From the definition (2.2.1), the above integral vanishes for $0 \leq s \leq n-1$. Thus the result follows. □

We can prove that the weights $w_{n,i}^{(r)}$, $i = 1, 2, ..., n$, are positive by considering the special cases where

$$f(t) = t^{-n+r} \left\{ \frac{B_n^{(r)}(t)}{t - z_{n,i}^{(r)}} \right\}^2, \quad i = 1, 2, ..., n.$$

Using $f(t) = 1$ in (5.1.1) we can see that $\sum_{i=1}^{n} w_{n,i}^{(r)} = \mu_0$.

We observe that if $r = n$ we obtain the Gassian quadrature formula, which is expected since the polynomials $B_n^{(n)}(z)$ are then the classical orthogonal polynomials on (a, b) with respect to the distribution $\psi(t)$.

5.2 The associated polynomials

We recall that the polynomials $A_n^{(r)}(z)$, $n \geq 0$, are the successive numerators of the convergents of the M-fractions (2.3.1), namely

$$M^{(r)}(z) + \frac{\mu_r / z^r}{z - \beta_1^{(r)}} - \frac{\alpha_2^{(r)} z}{z - \beta_2^{(r)}} - \frac{\alpha_3^{(r)} z}{z - \beta_3^{(r)}} - \frac{\alpha_4^{(r)} z}{z - \beta_4^{(r)}} - \cdots \quad r = 0, \pm 1, \pm 2, ...,$$

where

$$M^{(r)}(z) = z^{-r} \int_a^b \frac{z^r - t^r}{z - t} d\psi(t) = \begin{cases} \dfrac{\mu_0}{z} + \dfrac{\mu_1}{z^2} + \dfrac{\mu_2}{z^3} + \cdots + \dfrac{\mu_{r-1}}{z^r}, & r \geq 0 \\ -\mu_{-1} - \mu_{-2} z - \cdots - \mu_r z^{-(r+1)}, & r < 0 \end{cases}.$$

These polynomials satisfy the recurrence relation (2.3.2), namely

$$A_{n+1}^{(r)}(z) = (z - \beta_{n+1}^{(r)})A_n^{(r)}(z) - \alpha_{n+1}^{(r)} z A_{n-1}^{(r)}(z), \quad n \geq 1,$$

with $A_0^{(r)}(z) = M^{(r)}(z)$, $A_1^{(r)}(z) = (z - \beta_1^{(r)})M^{(r)}(z) + \mu_r z^{-r}$.

Further, the polynomials $C_n^{(r)}(z)$, $n \geq 0$, are the successive numerators of the convergents of the M-fractions (2.3.3), namely

$$\frac{\mu_r}{z - \beta_1^{(r)}} - \frac{\alpha_2^{(r)} z}{z - \beta_2^{(r)}} - \frac{\alpha_3^{(r)} z}{z - \beta_3^{(r)}} - \frac{\alpha_4^{(r)} z}{z - \beta_4^{(r)}} - \cdots \quad r = 0, \pm 1, \pm 2, \ldots.$$

These polynomials satisfy the recurrence relation (2.3.6), namely

$$C_{n+1}^{(r)}(z) = (z - \beta_{n+1}^{(r)})C_n^{(r)}(z) - \alpha_{n+1}^{(r)} z C_{n-1}^{(r)}(z), \quad n \geq 1,$$

with $C_0^{(r)}(z) = 0$, $C_1^{(r)}(z) = \mu_r$.

From the three-term recurrence relations (2.3.2) and (2.3.6), we can see that $A_n^{(0)}(z) = C_n^{(0)}(z)$, for $n \geq 0$.

In this section we present particular properties involving the associated polynomials $A_n^{(r)}(z)$ and $C_n^{(r)}(z)$, the weights $w_{n,i}^{(r)}$, $i = 1, ..., n$, and the nodes $z_{n,i}^{(r)}$, $i = 1, ..., n$, of the quadrature formula (5.1.1). These properties can be deduced from the theory of classical orthogonal polynomials and Padé approximants. Sri Ranga in [42] has also studied these associated polynomials.

Property 5.2.1 *The associated polynomials $A_n^{(r)}(z)$ and $C_n^{(r)}(z)$, for $n \geq 0$ and $r = 0, \pm 1, \pm 2, ...$, can be defined from the polynomials $B_n^{(r)}(z)$ as*

$$A_n^{(r)}(z) = \int_a^b z^{-r} \frac{z^r B_n^{(r)}(z) - t^r B_n^{(r)}(t)}{z - t} d\psi(t), \tag{5.2.1}$$

and as

$$C_n^{(r)}(z) = \int_a^b t^r \frac{B_n^{(r)}(z) - B_n^{(r)}(t)}{z - t} d\psi(t). \tag{5.2.2}$$

Proof: We prove (5.2.1) by mathematical induction. It is easy to see that for $n=0$ the equation (5.2.1) satisfies the initial condition of the recurrence relation. For $n=1$,

$$\int_a^b z^{-r}\frac{z^r B_1^{(r)}(z)-t^r B_1^{(r)}(t)}{z-t}d\psi(t)=\int_a^b z^{-r}\frac{z^r(z-\beta_1^{(r)})-t^r(t-\beta_1^{(r)})}{z-t}d\psi(t).$$

By adding and subtracting the term $t^r z$ to the numerator of the integrand we obtain

$$\int_a^b z^{-r}\frac{z^r B_1^{(r)}(z)-t^r B_1^{(r)}(t)}{z-t}d\psi(t)=(z-\beta_1^{(r)})z^{-r}\int_a^b\frac{z^r-t^r}{z-t}d\psi(t)+z^{-r}\int_a^b t^r d\psi(t)$$

or

$$\int_a^b z^{-r}\frac{z^r B_1^{(r)}(z)-t^r B_1^{(r)}(t)}{z-t}d\psi(t)=(z-\beta_1^{(r)})M^{(r)}(z)+\mu_r z^{-r}=A_1^{(r)}(z).$$

We assume (5.2.1) valid for n and $n-1$, and we know that

$$\int_a^b z^{-r}\frac{z^r B_{n+1}^{(r)}(z)-t^r B_{n+1}^{(r)}(t)}{z-t}d\psi(t)=$$

$$\int_a^b z^{-r}\frac{z^r\left[(z-\beta_{n+1}^{(r)})B_n^{(r)}(z)-\alpha_{n+1}^{(r)}zB_{n-1}^{(r)}(z)\right]-t^r\left[(t-\beta_{n+1}^{(r)})B_n^{(r)}(t)-\alpha_{n+1}^{(r)}tB_{n-1}^{(r)}(t)\right]}{z-t}d\psi(t).$$

We add and subtract the term $t^r z\left(B_n^{(r)}(t)+\alpha_{n+1}^{(r)}B_{n-1}^{(r)}(t)\right)$ to the numerator of the integrand, and we obtain

$$\begin{aligned}\int_a^b z^{-r}\frac{z^r B_{n+1}^{(r)}(z)-t^r B_{n+1}^{(r)}(t)}{z-t}d\psi(t)&=(z-\beta_{n+1}^{(r)})\int_a^b z^{-r}\frac{z^r B_n^{(r)}(z)-t^r B_n^{(r)}(t)}{z-t}d\psi(t)\\&-\alpha_{n+1}^{(r)}z\int_a^b z^{-r}\frac{z^r B_{n-1}^{(r)}(z)-t^r B_{n-1}^{(r)}(t)}{z-t}d\psi(t)\\&+\int_a^b z^{-r}t^r\left(B_n^{(r)}(t)-\alpha_{n+1}^{(r)}B_{n-1}^{(r)}(t)\right)d\psi(t).\end{aligned}$$

From the induction hypothesis and (2.3.15)

$$\begin{aligned}\int_a^b z^{-r}\frac{z^r B_{n+1}^{(r)}(z)-t^r B_{n+1}^{(r)}(t)}{z-t}d\psi(t)&=(z-\beta_{n+1}^{(r)})A_n^{(r)}(z)-\alpha_{n+1}^{(r)}zA_{n-1}^{(r)}(z)\\&+z^{-r}\int_a^b t^r B_n^{(r+1)}(t)d\psi(t)\end{aligned}$$

or

$$\int_a^b z^{-r}\frac{z^r B_{n+1}^{(r)}(z) - t^r B_{n+1}^{(r)}(t)}{z-t}d\psi(t) = A_{n+1}^{(r)}(z) + z^{-r}\int_a^b t^r B_n^{(r+1)}(t)d\psi(t).$$

From (2.2.1) we can see that the integral on the right is equal to zero, hence the requited results. Similarly the result (5.2.2) is proved by mathematical induction. □

From (5.2.1), (5.2.2) and the definition of $M^{(r)}(z)$ we can easily see that

$$A_n^{(r)}(z) = M^{(r)}(z)B_n^{(r)}(z) + z^{-r}C_n^{(r)}(z), \quad n \geq 0, \quad r = 0, \pm1, \pm2, \dots . \qquad (5.2.3)$$

Property 5.2.2 *The associated polynomials $A_n^{(r)}(z)$ and $C_n^{(r)}(z)$, for $n \geq 0$ and $r = 0, \pm1, \pm2, ...$, can be written as*

$$A_n^{(r)}(z) = \int_a^b z^{-r+p}\,\frac{z^{r-p}B_n^{(r)}(z) - t^{r-p}B_n^{(r)}(t)}{z-t}d\psi(t), \quad \text{for} \quad 0 \leq p \leq n, \quad (5.2.4)$$

and as

$$C_n^{(r)}(z) = \int_a^b t^{r-p}\,\frac{t^p B_n^{(r)}(z) - z^p B_n^{(r)}(t)}{z-t}d\psi(t), \quad \text{for} \quad 0 \leq p \leq n. \qquad (5.2.5)$$

Proof: First, from (5.2.1)

$$A_n^{(r)}(z) = \int_a^b z^{-r+p}\frac{z^{r-p}B_n^{(r)}(z) - z^{-p}t^r B_n^{(r)}(t)}{z-t}d\psi(t).$$

By adding and subtracting the term $t^{r-p}B_n^{(r)}(t)$ to the numerator of the integral we obtain

$$\begin{aligned} A_n^{(r)}(z) = &\int_a^b z^{-r+p}\frac{z^{r-p}B_n^{(r)}(z) - t^{r-p}B_n^{(r)}(t)}{z-t}d\psi(t) \\ &-\int_a^b z^{-r+p}\frac{z^{-p}t^r - t^{r-p}}{z-t}B_n^{(r)}(t)d\psi(t), \end{aligned}$$

or

$$\begin{aligned} A_n^{(r)}(z) = &\int_a^b z^{-r+p}\frac{z^{r-p}B_n^{(r)}(z) - t^{r-p}B_n^{(r)}(t)}{z-t}d\psi(t) \\ &-z^{-r+p}\int_a^b t^r\frac{z^{-p} - t^{-p}}{z-t}B_n^{(r)}(t)d\psi(t). \end{aligned}$$

The second integral is equal to

$$\int_a^b t^r \left(-\sum_{j=0}^{p-1} t^{-1-j} z^{j-p}\right) B_n^{(r)}(t)d\psi(t) = -\sum_{j=0}^{p-1} z^{j-p} \int_a^b t^{r-(j+1)} B_n^{(r)}(t)d\psi(t).$$

From (2.2.1) we see that

$$\int_a^b t^{r-(j+1)} B_n^{(r)}(t)d\psi(t) = 0, \quad \text{for} \quad 0 \le j \le n-1,$$

and hence

$$\sum_{j=0}^{p-1} z^{j-p} \int_a^b t^{r-(j+1)} B_n^{(r)}(t)d\psi(t) = 0, \quad \text{for} \quad 1 \le p \le n.$$

The case $p = 0$ is covered by the relation (5.2.1) and so the result holds.

Now from (5.2.2),

$$C_n^{(r)}(z) = \int_a^b t^{r-p} \frac{t^p B_n^{(r)}(z) - t^p B_n^{(r)}(t)}{z-t} d\psi(t).$$

Again adding and subtracting the term $z^p B_n^{(r)}(t)$ to the numerator of the integrand we obtain

$$C_n^{(r)}(z) = \int_a^b t^{r-p} \frac{t^p B_n^{(r)}(z) - z^p B_n^{(r)}(t)}{z-t} d\psi(t) + \int_a^b t^{r-p} \frac{z^p - t^p}{z-t} B_n^{(r)}(t)d\psi(t).$$

The second integral is equal to

$$\int_a^b t^{r-p} \sum_{j=0}^{p-1} t^{p-1-j} z^j B_n^{(r)}(t)d\psi(t) = \sum_{j=0}^{p-1} z^j \int_a^b t^{r-(j+1)} B_n^{(r)}(t)d\psi(t).$$

Once again from (2.2.1) we obtain

$$\int_a^b t^{r-(j+1)} B_n^{(r)}(t)d\psi(t) = 0, \quad \text{for} \quad 0 \le j \le n-1,$$

and then

$$\sum_{j=0}^{p-1} z^j \int_a^b t^{r-(j+1)} B_n^{(r)}(t)d\psi(t) = 0, \quad \text{for} \quad 1 \le p \le n.$$

The case $p = 0$ is covered by the relation (5.2.2) and the second result also holds. □

From the three-term recurrence relations for the polynomials $B_n^{(r)}(z)$, $A_n^{(r)}(z)$ and $C_n^{(r)}(z)$, for $n \geq 1$ and $r = 0, \pm1, \pm2, ...$, it is easy to see that

$$A_n^{(r)}(z)B_{n-1}^{(r)}(z) - A_{n-1}^{(r)}(z)B_n^{(r)}(z) = \alpha_n^{(r)}\alpha_{n-1}^{(r)}\cdots\alpha_2^{(r)}\mu_r z^{n-1-r}\,, \tag{5.2.6}$$

and

$$C_n^{(r)}(z)B_{n-1}^{(r)}(z) - C_{n-1}^{(r)}(z)B_n^{(r)}(z) = \alpha_n^{(r)}\alpha_{n-1}^{(r)}\cdots\alpha_2^{(r)}\mu_r z^{n-1}\,. \tag{5.2.7}$$

From section 2.2 we know that $z = 0$ is not a zero of the polynomials $B_n^{(r)}(z)$. Substituting $z = z_{n,i}^{(r)}$ in (5.2.6) and (5.2.7) we obtain

$$A_n^{(r)}(z_{n,i}^{(r)})B_{n-1}^{(r)}(z_{n,i}^{(r)}) = \alpha_n^{(r)}\alpha_{n-1}^{(r)}\cdots\alpha_2^{(r)}\mu_r(z_{n,i}^{(r)})^{n-1-r} \neq 0,$$

and

$$C_n^{(r)}(z_{n,i}^{(r)})B_{n-1}^{(r)}(z_{n,i}^{(r)}) = \alpha_n^{(r)}\alpha_{n-1}^{(r)}\cdots\alpha_2^{(r)}\mu_r(z_{n,i}^{(r)})^{n-1} \neq 0,$$

respectively. This proves that, for $n \geq 1$ and $r = 0, \pm1, \pm2, ...$, the zeros of the polynomials $B_n^{(r)}(z)$ are different from the zeros of any of the polynomials $B_{n-1}^{(r)}(z)$, $A_n^{(r)}(z)$ and $C_n^{(r)}(z)$.

Theorem 5.2.1 *The weights $w_{n,i}^{(r)}$, $i = 1, 2..., n$, for $n \geq 1$ and $r = 0, \pm1, \pm2, ...$, can be given by either*

$$w_{n,i}^{(r)} = \frac{A_n^{(r)}(z_{n,i}^{(r)})}{B_n^{(r)\prime}(z_{n,i}^{(r)})}, \tag{5.2.8}$$

or

$$w_{n,i}^{(r)} = (z_{n,i}^{(r)})^{-r}\frac{C_n^{(r)}(z_{n,i}^{(r)})}{B_n^{(r)\prime}(z_{n,i}^{(r)})}. \tag{5.2.9}$$

Proof: From (5.2.4) with $p = n$ and $z = z_{n,i}^{(r)}$, $i = 1, 2..., n$,

$$\frac{A_n^{(r)}(z_{n,i}^{(r)})}{B_n^{(r)\prime}(z_{n,i}^{(r)})} = \frac{1}{B_n^{(r)\prime}(z_{n,i}^{(r)})}\int_a^b (z_{n,i}^{(r)})^{n-r}\,\frac{(z_{n,i}^{(r)})^{r-n}B_n^{(r)}(z_{n,i}^{(r)}) - t^{r-n}B_n^{(r)}(t)}{z_{n,i}^{(r)} - t}d\psi(t),$$

as $B_n^{(r)}(z_{n,i}^{(r)}) = 0$, we obtain

$$\frac{A_n^{(r)}(z_{n,i}^{(r)})}{B_n^{(r)'}(z_{n,i}^{(r)})} = \frac{(z_{n,i}^{(r)})^{n-r}}{B_n^{(r)'}(z_{n,i}^{(r)})} \int_a^b \frac{t^{r-n} B_n^{(r)}(t)}{t - z_{n,i}^{(r)}} d\psi(t) = w_{n,i}^{(r)}.$$

This proves the result (5.2.8). Similarly from (5.2.5) we prove the result (5.2.9).

□

Substituting (5.2.4) with $0 \le p \le n$, in the relation (5.2.8) we obtain

$$w_{n,i}^{(r)} = \frac{1}{B_n^{(r)'}(z_{n,i}^{(r)})} \int_a^b (z_{n,i}^{(r)})^{-r+p} \frac{(z_{n,i}^{(r)})^{r+p} B_n^{(r)}(z_{n,i}^{(r)}) - t^{r+p} B_n^{(r)}(t)}{z_{n,i}^{(r)} - t} d\psi(t)$$

or

$$w_{n,i}^{(r)} = \frac{(z_{n,i}^{(r)})^{p-r}}{B_n^{(r)'}(z_{n,i}^{(r)})} \int_a^b \frac{t^{r-p} B_n^{(r)}(t)}{t - z_{n,i}^{(r)}} d\psi(t), \quad \text{for} \quad 0 \le p \le n, \tag{5.2.10}$$

where $n \ge 1$ and $r = 0, \pm 1, \pm 2, \dots$. We can also obtain (5.2.10) by substituting (5.2.5) with $0 \le p \le n$, in the relation (5.2.9).

Theorem 5.2.2 *The polynomials $B_n^{(r)}(z)$ and $C_n^{(r)}(z)$ for $r = 0, \pm 1, \pm 2, \dots$ and $n \ge 1$, satisfy*

$$\frac{C_n^{(r)}(z)}{B_n^{(r)}(z)} = \sum_{i=1}^{n} \frac{(z_{n,i}^{(r)})^r w_{n,i}^{(r)}}{z - z_{n,i}^{(r)}}. \tag{5.2.11}$$

Equivalently, when $A_n^{(r)}(z)$ is a polynomial of degree $n-1$, the polynomials $B_n^{(r)}(z)$ and $A_n^{(r)}(z)$ satisfy

$$\frac{A_n^{(r)}(z)}{B_n^{(r)}(z)} = \sum_{i=1}^{n} \frac{w_{n,i}^{(r)}}{z - z_{n,i}^{(r)}}. \tag{5.2.12}$$

Proof: The polynomial $C_n^{(r)}(z)$ has degree $n-1$, and hence coincides with its own interpolating polynomial on the n roots of $B_n^{(r)}(z)$. Hence,

$$C_n^{(r)}(z) = \sum_{i=1}^{n} \frac{B_n^{(r)}(z)}{(z - z_{n,i}^{(r)}) B_n^{(r)'}(z_{n,i}^{(r)})} C_n^{(r)}(z_{n,i}^{(r)}),$$

and so

$$\frac{C_n^{(r)}(z)}{B_n^{(r)}(z)} = \sum_{i=1}^{n} \frac{C_n^{(r)}(z_{n,i}^{(r)})}{(z - z_{n,i}^{(r)}) B_n^{(r)'}(z_{n,i}^{(r)})} = \sum_{i=1}^{n} \frac{(z_{n,i}^{(r)})^r w_{n,i}^{(r)}}{z - z_{n,i}^{(r)}}.$$

When the polynomial $A_n^{(r)}(z)$ has degree $n-1$, then it coincides with its own interpolating polynomial on the n roots of $B_n^{(r)}(z)$. Hence,

$$A_n^{(r)}(z) = \sum_{i=1}^{n} \frac{B_n^{(r)}(z)}{(z - z_{n,i}^{(r)}) B_n^{(r)'}(z_{n,i}^{(r)})} A_n^{(r)}(z_{n,i}^{(r)}),$$

and so

$$\frac{A_n^{(r)}(z)}{B_n^{(r)}(z)} = \sum_{i=1}^{n} \frac{A_n^{(r)}(z_{n,i}^{(r)})}{(z - z_{n,i}^{(r)}) B_n^{(r)'}(z_{n,i}^{(r)})} = \sum_{i=1}^{n} \frac{w_{n,i}^{(r)}}{z - z_{n,i}^{(r)}}.$$

The proof is complete. □

From the recurrence relations of the polynomials $B_n^{(r)}(z)$ and $C_n^{(r)}(z)$ we easily prove that

$$C_n^{(r)}(0) = -\mu_{r-1} B_n^{(r)}(0), \tag{5.2.13}$$

for $n \geq 0$ and $r = 0, \pm 1, \pm 2, \ldots$.

Further, from the recurrence relations (2.3.2) we see that $A_n^{(r)}(0)$ are defined when $r < 0$ and $n \geq 0$ and when $r \geq 0$ and $n > r$.

For $r < 0$ and $n \geq 0$, since $-r > 0$ from (5.2.1) with $z = 0$, we obtain

$$A_n^{(r)}(0) = \int_a^b \frac{B_n^{(r)}(0) - 0^{-r} t^r B_n^{(r)}(t)}{0 - t} d\psi(t) = -\mu_{-1} B_n^{(r)}(0).$$

While $r \geq 0$ and $n > r$, since $-r + n > 0$ from (5.2.4) with $p = n$ and $z = 0$, we also obtain

$$A_n^{(r)}(0) = \int_a^b \frac{B_n^{(r)}(0) - 0^{-r+n} t^{r-n} B_n^{(r)}(t)}{0 - t} d\psi(t) = -\mu_{-1} B_n^{(r)}(0).$$

Hence,

$$A_n^{(r)}(0) = -\mu_{-1} B_n^{(r)}(0), \tag{5.2.14}$$

for $r < 0$ and $n \geq 0$ or for $r \geq 0$ and $n > r$.

In the next section we use the results (5.2.13) and (5.2.14) to prove some symmetric properties satisfied by the associated polynomials related to $S^3[\omega, \beta, b]$ distributions.

5.3 The $S^3[\omega, \beta, b]$ distributions and the quadrature formulae

In this section we consider some properties involving the associated polynomials, the weights and the nodes of the quadrature rules when $\psi(t)$ is an $S^3[\omega, \beta, b]$ distribution. The first theorem gives some symmetric relations satisfied by the weights $w_{n,i}^{(r)}$, and the nodes $z_{n,i}^{(r)}$, for $i = 1, 2, ...n$, $n \geq 1$, and $r = 0, \pm 1, \pm 2, \dots$.

Theorem 5.3.1 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $n \geq 1$ and $j = 1 - 2\omega$, the weights $w_{n,i}^{(l)}$ and nodes $z_{n,i}^{(l)}$, for $i = 1, ..., n$ and $l = 0, \pm 1, \pm 2, ...$, are related by*

$$\frac{w_{n,i}^{(l)}}{(z_{n,i}^{(l)})^{\omega}} = \frac{w_{n,n+1-i}^{(j-l)}}{(z_{n,n+1-i}^{(j-l)})^{\omega}}. \tag{5.3.1}$$

Proof: From (3.2.1) we find that for $l = 0, \pm 1, \pm 2, \dots$,

$$B_n^{(l)'}(\beta^2/z) = \beta^{-2} B_n^{(l)}(0)[nz^{-n+1} B_n^{(j-l)}(z) - z^{-n+2} B_n^{(j-l)'}(z)]. \tag{5.3.2}$$

Then, using the relations (5.2.5) and (3.2.1), we can deduce that

$$C_n^{(l)}(\beta^2/z) = -\beta^{2(l-1+\omega)} z^{-n+1} B_n^{(l)}(0) C_n^{(j-l)}(z). \tag{5.3.3}$$

From the relations (3.2.4) and (5.2.9) we obtain

$$w_{n,i}^{(l)} = (z_{n,i}^{(l)})^{-l} \frac{C_n^{(l)}(z_{n,i}^{(l)})}{B_n^{(l)'}(z_{n,i}^{(l)})} = (z_{n,i}^{(l)})^{-l} \frac{C_n^{(l)}(\beta^2/z_{n,n+1-i}^{(j-l)})}{B_n^{(l)'}(\beta^2/z_{n,n+1-i}^{(j-l)})},$$

for $i = 1, ..., n$ and $l = 0, \pm 1, \pm 2, ...$ Substituting z by $z_{n,n+1-i}^{(j-l)}$ in (5.3.2) and (5.3.3) we then obtain

$$w_{n,i}^{(l)} = (z_{n,i}^{(l)})^{-l} \frac{\beta^{2(l+\omega)} C_n^{(j-l)}(z_{n,n+1-i}^{(j-l)})}{z_{n,n+1-i}^{(j-l)} B_n^{(j-l)'}(z_{n,n+1-i}^{(j-l)})}.$$

Since $\beta^2 = z_{n,i}^{(l)} z_{n,n+1-i}^{(j-l)}$, for $i = 1, ..., n$ and $l = 0, \pm 1, \pm 2, ...$, it follows that

$$w_{n,i}^{(l)} = \frac{(z_{n,i}^{(l)})^{-l}(z_{n,i}^{(l)})^{l+\omega}(z_{n,n+1-i}^{(j-l)})^{l+\omega} C_n^{(j-l)}(z_{n,n+1-i}^{(j-l)})}{z_{n,n+1-i}^{(j-l)} B_n^{(j-l)\prime}(z_{n,n+1-i}^{(j-l)})},$$

or

$$w_{n,i}^{(l)} = \frac{(z_{n,i}^{(l)})^{\omega}(z_{n,n+1-i}^{(j-l)})^{l+2\omega} C_n^{(j-l)}(z_{n,n+1-i}^{(j-l)})}{(z_{n,n+1-i}^{(j-l)})^{1+\omega} B_n^{(j-l)\prime}(z_{n,n+1-i}^{(j-l)})}.$$

With $j = 1 - 2\omega$ and using again (5.2.9), we see that

$$w_{n,i}^{(l)} = \frac{(z_{n,i}^{(l)})^{\omega}(z_{n,n+1-i}^{(j-l)})^{1-(j-l)} C_n^{(j-l)}(z_{n,n+1-i}^{(j-l)})}{(z_{n,n+1-i}^{(j-l)})^{1+\omega} B_n^{(j-l)\prime}(z_{n,n+1-i}^{(j-l)})} = \frac{(z_{n,i}^{(l)})^{\omega}}{(z_{n,n+1-i}^{(j-l)})^{\omega}} w_{n,n+1-i}^{(j-l)},$$

for $i = 1, ..., n$ and $l = 0, \pm 1, \pm 2, ...$. This completes the proof. □

The following corollary gives some relations satisfied by the weights and the nodes of the quadrature rules when (i) 2ω is odd and (ii) 2ω is even.

Corollary 5.3.1.1 *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $n \geq 1$ the weights $w_{n,i}^{(l)}$ and nodes $z_{n,i}^{(l)}$, for $i = 1, ..., n$ and $l = 0, \pm 1, \pm 2, ...$, are related by*

(i) for 2ω odd and $j = \frac{1}{2} - \omega$,

$$\frac{w_{n,i}^{(j+l)}}{(z_{n,i}^{(j+l)})^{\omega}} = \frac{w_{n,n+1-i}^{(j-l)}}{(z_{n,n+1-i}^{(j-l)})^{\omega}}, \tag{5.3.4}$$

(ii) for 2ω even and $j = -\omega$,

$$\frac{w_{n,i}^{(j+l)}}{(z_{n,i}^{(j+l)})^{\omega}} = \frac{w_{n,n+1-i}^{(j+1-l)}}{(z_{n,n+1-i}^{(j+1-l)})^{\omega}}. \tag{5.3.5}$$

As with the polynomials $B_n^{(r)}(z)$, the associated polynomials possess some symmetric properties when $\psi(t)$ is an $S^3[\omega,\beta,b]$ distribution. In the following theorem and its corollary we show this behaviour of the associated polynomials $A_n^{(r)}(z)$.

Theorem 5.3.2 *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $n \geq 1$, $j = 1 - 2\omega$, the associated polynomials $A_n^{(l)}(z)$ satisfy*

$$\frac{z^{n-1}A_n^{(l)}(\beta^2/z)}{B_n^{(l)}(0)} = -\frac{z^j}{\beta^{j+1}}\left[A_n^{(j-l)}(z) - M^{(j)}(z)B_n^{(j-l)}(z)\right], \tag{5.3.6}$$

for $l = 0, \pm 1, \pm 2, \ldots$.

Proof: From (5.2.4) with $p = n$

$$A_n^{(l)}(z) = \int_a^b z^{-l+n}\,\frac{z^{l-n}B_n^{(l)}(z) - t^{l-n}B_n^{(l)}(t)}{z-t}\,d\psi(t)\,.$$

Substituting z by β^2/z, t by β^2/t and using the property (3.1.1) we obtain

$$A_n^{(l)}(\beta^2/z) = \int_a^b z^{l-n+1}t^{1-2\omega}\beta^{2(\omega-1)}\,\frac{z^{n-l}B_n^{(l)}(\beta^2/z) - t^{n-l}B_n^{(l)}(\beta^2/t)}{t-z}\,d\psi(t)\,.$$

Since $j = 1 - 2\omega$ and (3.2.1) we obtain

$$z^{n-1}A_n^{(l)}(\beta^2/z) = -B_n^{(l)}(0)\beta^{-j-1}\int_a^b z^l t^j\,\frac{z^{-l}B_n^{(j-l)}(z) - t^{-l}B_n^{(j-l)}(t)}{z-t}\,d\psi(t)\,.$$

Adding and subtracting the term $z^jB_n^{(j-l)}(z)$ to the numerator of the integrand, we obtain

$$\begin{aligned}\frac{z^{n-1}A_n^{(l)}(\beta^2/z)}{B_n^{(l)}(0)} &= -\frac{1}{\beta^{j+1}}\left[\int_a^b \frac{z^jB_n^{(j-l)}(z) - z^lt^{j-l}B_n^{(j-l)}(t)}{z-t}\,d\psi(t)\right.\\ &\qquad\left. -B_n^{(j-l)}(z)\int_a^b \frac{z^j - t^j}{z-t}\,d\psi(t)\right]\end{aligned}$$

or

$$\begin{aligned}\frac{z^{n-1}A_n^{(l)}(\beta^2/z)}{B_n^{(l)}(0)} &= -\frac{1}{\beta^{j+1}}\left[z^j\int_a^b z^{-j+l}\frac{z^{j-l}B_n^{(j-l)}(z) - t^{j-l}B_n^{(j-l)}(t)}{z-t}\,d\psi(t)\right.\\ &\qquad\left. -z^jB_n^{(j-l)}(z)z^{-j}\int_a^b \frac{z^j - t^j}{z-t}\,d\psi(t)\right].\end{aligned}$$

Finally using the definition (5.2.1) and the definition of $M^{(j)}(z)$ the result holds.

□

Corollary 5.3.2.1 *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0<\beta<b\leq\infty$, $a=\beta^2/b$ and $2\omega\in\boldsymbol{Z}$. Then for $n\geq 1$, $j=1-2\omega$, and for $l<0$ and $n\geq 0$ or for $l\geq 0$ and $n>l$, the associated polynomials $A_n^{(l)}(z)$ satisfy*

$$\frac{z^{n-1}A_n^{(l)}(\beta^2/z)}{A_n^{(l)}(0)}=\frac{z^j}{\mu_j}\left[A_n^{(j-l)}(z)-M^{(j)}(z)B_n^{(j-l)}(z)\right]. \tag{5.3.7}$$

Proof: From (5.3.6), then

$$\frac{z^{n-1}A_n^{(l)}(\beta^2/z)}{A_n^{(l)}(0)}=-\frac{B_n^{(l)}(0)}{A_n^{(l)}(0)}\frac{z^j}{\beta^{j+1}}\left[A_n^{(j-l)}(z)-M^{(j)}(z)B_n^{(j-l)}(z)\right].$$

Since (5.2.14) holds for $l<0$ and $n\geq 0$ or for $l\geq 0$ and $n>l$, using the relations (3.1.2) the result follows. □

We now see the symmetric behaviour of the associated polynomials $C_n^{(r)}(z)$.

Theorem 5.3.3 *Let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $0<\beta<b\leq\infty$, $a=\beta^2/b$ and $2\omega\in\boldsymbol{Z}$. Then for $n\geq 1$ and $j=1-2\omega$, the associated polynomials $C_n^{(l)}(z)$ satisfy*

$$\frac{z^{n-1}C_n^{(l)}(\beta^2/z)}{C_n^{(l)}(0)}=\frac{C_n^{(j-l)}(z)}{\mu_{j-l}},\quad \text{for } l=0,\pm1,\pm2,\dots. \tag{5.3.8}$$

Proof: From (5.3.3), with $j=1-2\omega$, we can write

$$\frac{z^{n-1}C_n^{(l)}(\beta^2/z)}{C_n^{(l)}(0)}=-\beta^{2l-1-j}\frac{B_n^{(l)}(0)}{C_n^{(l)}(0)}C_n^{(j-l)}(z),\quad l=0,\pm1,\pm2,\dots.$$

The result holds since the relations (5.2.13) and the relations (3.1.2) with $m=l-1$ imply that $\mu_{j-l}=\mu_{l-1}/\beta^{2l-1-j}$. □

The following corollary gives some relations satisfied by the associated polynomials $C_n^{(r)}(z)$ when (i) 2ω is odd and (ii) 2ω is even.

Corollary 5.3.3.1 *Let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $0 < \beta < b \leq \infty$, $a = \beta^2/b$ and $2\omega \in \boldsymbol{Z}$. Then for $n \geq 1$, the associated polynomials $C_n^{(l)}(z)$ satisfy*
(i) for 2ω odd and $j = \frac{1}{2} - \omega$,

$$\frac{z^{n-1}C_n^{(j+l)}(\beta^2/z)}{C_n^{(j+l)}(0)} = \frac{C_n^{(j-l)}(z)}{\mu_{j-l}}, \quad \text{for } l = 0, \pm 1, \pm 2, ... , \tag{5.3.9}$$

(ii) for 2ω even and $j = -\omega$,

$$\frac{z^{n-1}C_n^{(j+l)}(\beta^2/z)}{C_n^{(j+l)}(0)} = \frac{C_n^{(j+1-l)}(z)}{\mu_{j+1-l}}, \quad \text{for } l = 0, \pm 1, \pm 2, ... \tag{5.3.10}$$

5.4 Quadrature formulae using the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$

In this section we consider the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$, $n \geq 0$, for an integer $r \geq 1$, that are defined in section 2.5. We recall that the zeros of these polynomials are denoted by $z_{n,i}^{\lambda^{(r)}}$, $i = 1, 2, ..., n$. Let the coefficients $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}$ be such that the polynomial $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ has zeros that are all real and distinct. Then we can construct the quadrature formula

$$\int_a^b f(t)d\psi(t) = \sum_{i=1}^{n} w_{n,i}^{\lambda^{(r)}} f(z_{n,i}^{\lambda^{(r)}}) + \mathbb{E}_n(f). \tag{5.4.1}$$

By a similar method to that used in section 5.1 we find that the weights $w_{n,i}^{\lambda^{(r)}}$ are given by

$$w_{n,i}^{\lambda^{(r)}} = \frac{(z_{n,i}^{\lambda^{(r)}})^{n-r}}{B_n'(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z_{n,i}^{\lambda^{(r)}})} \int_a^b \frac{t^{-n+r}B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; t)}{t - z_{n,i}^{\lambda^{(r)}}} d\psi(t), \tag{5.4.2}$$

for $i = 1, 2..., n,$. The remainder is given by

$$\mathbb{E}_n(f) = \int_a^b t^{-n+r}B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)F[z_{n,1}^{\lambda^{(r)}}, ..., z_{n,n}^{\lambda^{(r)}}, z]d\psi(t). \tag{5.4.3}$$

Theorem 5.4.1 *For the quadrature rule (5.4.1),*

$$\mathbb{E}_n(f) = 0 \quad \textit{whenever} \quad z^{n-r}f(z) \in \mathbb{P}_{2n-1-r}.$$

Proof: Since $F(z) = z^{n-r}f(z) \in \mathbb{P}_{2n-1-r}$, then $F[z_{n,1}^{\lambda^{(r)}}, ..., z_{n,n}^{\lambda^{(r)}}, z] \in \mathbb{P}_{n-1-r}$ and hence (5.4.3) becomes

$$\mathbb{E}_n(f) = \int_a^b t^{-n+r} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; t) \sum_{s=0}^{n-1-r} v_s t^s d\psi(t)$$

or

$$\mathbb{E}_n(f) = \sum_{s=0}^{n-1-r} v_s \int_a^b t^{-n+s+r} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; t) d\psi(t).$$

From (2.5.2) we know that the above integral is zero for $0 \leq s \leq n-1-r$. This completes the proof. □

Setting $f(t) = 1$ in (5.4.1), we can see that $\sum_{i=1}^{n} w_{n,i}^{\lambda^{(r)}} = \mu_0$.

These quadrature formulae for the case when $r = 1$ were studied by Sri Ranga, de Andrade and McCabe in [48].

For $r \geq 1$, we define the associated polynomials $A_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$, $n \geq 1$, of degree $n-1$ by

$$A_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z) = \int_a^b z^{n-r} \frac{z^{-n+r} B_n(\lambda_{n,1}^{(r)}, .., \lambda_{n,r}^{(r)}; z) - t^{-n+r} B_n(\lambda_{n,1}^{(r)}, .., \lambda_{n,r}^{(r)}; t)}{z-t} d\psi(t). \tag{5.4.4}$$

We now give some properties involving these polynomials.

Theorem 5.4.2 *For $n \geq r$, the polynomials $A_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ satisfy*

$$A_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z) = A_n^{(0)}(z) + \lambda_{n,1}^{(r)} A_{n-1}^{(0)}(z) + \cdots + \lambda_{n,r}^{(r)} A_{n-r}^{(0)}(z). \tag{5.4.5}$$

Proof: From the definitions (2.5.1) and (5.4.4) we obtain

$$A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)=\int_a^b z^{n-r}\frac{z^{-n+r}\sum_{i=0}^{r}\lambda_{n,i}^{(r)}B_{n-i}^{(0)}(z)-t^{-n+r}\sum_{i=0}^{r}\lambda_{n,i}^{(r)}B_{n-i}^{(0)}(t)}{z-t}d\psi(t),$$

where $\lambda_{n,0}^{(r)}=1$, hence,

$$A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)=\sum_{i=0}^{r}\lambda_{n,i}^{(r)}\int_a^b z^{n-r}\frac{z^{-n+r}B_{n-i}^{(0)}(z)-t^{-n+r}B_{n-i}^{(0)}(t)}{z-t}d\psi(t)\,.$$

Setting $r=0$ in (5.2.4), we obtain

$$A_{n-i}^{(0)}(z)=\int_a^b z^p\,\frac{z^{-p}B_{n-i}^{(0)}(z)-t^{-p}B_{n-i}^{(0)}(t)}{z-t}d\psi(t),\quad \text{for}\quad 0\le p\le n-i.$$

Since $i=0,1,...,r$ and $n-r\le n-i$, the result holds with $p=n-r$ in (5.2.4).□

Property 5.4.1 *The polynomials $A_n(\lambda_{n,1}^{(r)},..,\lambda_{n,r}^{(r)};z)$ for $n\ge r$ can be given by, for $p=0,1,...,n-r$,*

$$A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)=$$
$$\int_a^b z^{n-r-p}\frac{z^{-n+r+p}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)-t^{-n+r+p}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)}{z-t}d\psi(t),\quad (5.4.6)$$

Proof: From the definition (5.4.4), we can write

$$A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)=$$
$$\int_a^b z^{n-r-p}\frac{z^{-n+r+p}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)-z^pt^{-n+r}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)}{z-t}d\psi(t)\,.$$

Adding and then subtracting the term $t^{-n+r+p}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)$ to the numerator of the integrand we obtain

$$A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)=$$
$$\int_a^b z^{n-r-p}\frac{z^{-n+r+p}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)-t^{-n+r+p}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)}{z-t}d\psi(t)$$

$$-\int_a^b z^{n-r-p}t^{-n+r}\frac{z^p-t^p}{z-t}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)d\psi(t).$$

The second integral above is equal to

$$-\int_a^b z^{n-r-p}t^{-n+r}\sum_{j=0}^{p-1} z^{p-1-j}t^j B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)d\psi(t)$$

or

$$-\sum_{j=0}^{p-1} z^{n-r-1-j}\int_a^b t^{-n+r+j} B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)d\psi(t).$$

According to the conditions (2.5.2) the integral above vanishes for $0 \leq j \leq n-1-r$. Hence,

$$\sum_{j=0}^{p-1} z^{n-r-1-j}\int_a^b t^{-n+r+j} B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)d\psi(t) = 0, \quad \text{for} \quad 1 \leq p \leq n-r\,.$$

The result also holds when $p = 0$. □

We can also write (5.4.6) in the equivalent form

$$A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z) = \int_a^b z^p \frac{z^{-p}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z) - t^{-p}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)}{z-t} d\psi(t),$$

for $p = 0, 1, ..., n-r$. Setting $p = 0$, for example, we obtain

$$A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z) = \int_a^b \frac{B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z) - B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)}{z-t} d\psi(t). \quad (5.4.7)$$

Property 5.4.2 *The associated polynomials $A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)$, $n \geq 1$, defined by (5.4.4) satisfy*

$$A_n(\hat{\lambda}_{n,1}^{(r)},...,\hat{\lambda}_{n,r}^{(r)};z) = A_n^{(r)}(z), \quad (5.4.8)$$

where $\hat{\lambda}_{n,1}^{(r)},...,\hat{\lambda}_{n,r}^{(r)}$ are defined in (2.5.7).

Proof: The result follows from substituting $B_n(\hat{\lambda}_{n,1}^{(r)},...,\hat{\lambda}_{n,r}^{(r)};z) = B_n^{(r)}(z)$, $n \geq 1$, in the definition (5.4.4) and using the relation (5.2.4) with $p = n$. □

We now can give some results concerning the weights and nodes of the quadrature formula of the form (5.4.1).

Theorem 5.4.3 *The weights* $w_{n,i}^{\lambda^{(r)}}$, $i=1,2...,n$, $n\geq 1$, *can be given by*

$$w_{n,i}^{\lambda^{(r)}}=\frac{A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}{B_n'(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})},\quad i=1,2...,n\,. \tag{5.4.9}$$

Proof: From the definition (5.4.4) with $z=z_{n,i}^{\lambda^{(r)}}$, $i=1,2...,n$

$$\frac{A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}{B_n'(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}=$$

$$\int_a^b (z_{n,i}^{\lambda^{(r)}})^{n-r}\frac{(z_{n,i}^{\lambda^{(r)}})^{-n+r}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})-t^{-n+r}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)}{B_n'(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})(z_{n,i}^{\lambda^{(r)}}-t)}d\psi(t).$$

Since $B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})=0$, it follows that

$$\frac{A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}{B_n'(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}=\frac{(z_{n,i}^{\lambda^{(r)}})^{n-r}}{B_n'(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}\int_a^b\frac{t^{-n+r}B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};t)}{t-z_{n,i}^{\lambda^{(r)}}}d\psi(t),$$

hence, from (5.4.2), the result follows. □

Theorem 5.4.4 *For* $n\geq 1$, *we can write*

$$\frac{A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)}{B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)}=\sum_{i=1}^{n}\frac{w_{n,i}^{\lambda^{(r)}}}{z-z_{n,i}^{\lambda^{(r)}}}. \tag{5.4.10}$$

Proof: The polynomial $A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)$ has degree $n-1$. Hence it coincides with its own interpolating polynomial on the n roots of $B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)$. Thus,

$$A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)=\sum_{i=1}^{n}\frac{B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)}{(z-z_{n,i}^{\lambda^{(r)}})B_n'(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}}),$$

and then

$$\frac{A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)}{B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)}=\sum_{i=1}^{n}\frac{A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}{(z-z_{n,i}^{\lambda^{(r)}})B_n'(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z_{n,i}^{\lambda^{(r)}})}=\sum_{i=1}^{n}\frac{w_{n,i}^{\lambda^{(r)}}}{z-z_{n,i}^{\lambda^{(r)}}}.\quad\square$$

We now consider $S^3[\omega,\beta,b]$ distributions and we show some symmetric relations satisfied by the weights and the nodes of the quadrature rule (5.4.1).

Theorem 5.4.5 *Given an integer $r \geq 1$, let $\psi(t)$ be an $S^3[\omega, \beta, b]$ distribution with $\omega = (1-r)/2$, $0 < \beta < b \leq \infty$ and $a = \beta^2/b$. Let the real parameters $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}$ and $\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}$ be such that*

$$\frac{z^n B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/z)}{B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0)} = B_n(\eta_{n,1}^{(r)} ..., \eta_{n,r}^{(r)}; z), \quad n \geq r. \tag{5.4.11}$$

Let $z_{n,1}^{\lambda^{(r)}} < z_{n,2}^{\lambda^{(r)}} < \cdots < z_{n,m}^{\lambda^{(r)}}$, where $1 \leq m \leq n$, be the positive distinct zeros of the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ and $z_{n,1}^{\eta^{(r)}} < z_{n,2}^{\eta^{(r)}} < \cdots < z_{n,m}^{\eta^{(r)}}$ be the positive distinct zeros of the polynomials $B_n(\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}; z)$. Then the weights $w_{n,i}^{\lambda^{(r)}}$ and $w_{n,i}^{\eta^{(r)}}$ and the positive zeros $z_{n,i}^{\lambda^{(r)}}$ and $z_{n,i}^{\eta^{(r)}}$ are related by

$$\frac{w_{n,i}^{\lambda^{(r)}}}{(z_{n,i}^{\lambda^{(r)}})^{\omega}} = \frac{w_{n,m+1-i}^{\eta^{(r)}}}{(z_{n,m+1-i}^{\eta^{(r)}})^{\omega}}, \quad \text{for } i = 1, ..., m.$$

Proof: From (5.4.11) we obtain

$$\begin{aligned} B_n'(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/z) = & \frac{nz}{\beta^2} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/z) \\ & - \frac{z^{-n+2}}{\beta^2} B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0) B_n'(\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}; z). \end{aligned}$$

Since $z_{n,m+1-i}^{\lambda^{(r)}} = \beta^2/z_{n,i}^{\eta^{(r)}}$, for $i = 1, 2, ..., m$, then

$$B_n'(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/z_{n,i}^{\eta^{(r)}}) = \frac{-B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0)(z_{n,i}^{\eta^{(r)}})^{-n+2}}{\beta^2} B_n'(\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}; z_{n,i}^{\eta^{(r)}}). \tag{5.4.12}$$

From (5.4.7) replacing z by β^2/z and t by β^2/t, we obtain

$$A_n(\lambda_{n,1}^{(r)}, .., \lambda_{n,r}^{(r)}; \beta^2/z) = \int_b^a \frac{B_n(\lambda_{n,1}^{(r)}, .., \lambda_{n,r}^{(r)}; \beta^2/z) - B_n(\lambda_{n,1}^{(r)}, .., \lambda_{n,r}^{(r)}; \beta^2/t)}{\beta^2/z - \beta^2/t} d\psi(\beta^2/t).$$

Using the property (3.1.1) and the relation (5.4.11), we obtain

$$A_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/z) =$$

$$B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0) \int_b^a \frac{zt}{\beta^2} \frac{z^{-n} B_n(\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}; z) - t^{-n} B_n(\eta_{n,1}^{(r)}, ..., \eta_{n,r}^{(r)}; t)}{t - z} \frac{\beta^{2\omega} d\psi(t)}{t^{2\omega}}.$$

Substituting $z = z_{n,i}^{\eta^{(r)}}$ for $i = 1,2,...,m$, and since $r = 1-2\omega$, we obtain

$$A_n(\lambda_{n,1}^{(r)},..,\lambda_{n,r}^{(r)};\beta^2/z_{n,i}^{\eta^{(r)}}) = \frac{-B_n(\lambda_{n,1}^{(r)},..,\lambda_{n,r}^{(r)};0)z_{n,i}^{\eta^{(r)}}}{(\beta^2)^{1-\omega}}\int_b^a t^{-n+r}\frac{B_n(\eta_{n,1}^{(r)},..,\eta_{n,r}^{(r)};t)}{t-z_{n,i}^{\eta^{(r)}}}d\psi(t). \tag{5.4.13}$$

For $i = 1,2,...m$,

$$w_{n,m+1-i}^{\lambda^{(r)}} = \frac{A_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};\beta^2/z_{n,i}^{\eta^{(r)}})}{B_n'(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};\beta^2/z_{n,i}^{\eta^{(r)}})},$$

then from (5.4.12) and (5.4.13)

$$w_{n,m+1-i}^{\lambda^{(r)}} = \frac{\beta^{2\omega}(z_{n,i}^{\eta^{(r)}})^{n-r-2\omega}}{B_n'(\eta_{n,1}^{(r)},...,\eta_{n,r}^{(r)};z_{n,i}^{\eta^{(r)}})}\int_b^a t^{-n+r}\frac{B_n(\eta_{n,1}^{(r)},...,\eta_{n,r}^{(r)};t)}{t-z_{n,i}^{\eta^{(r)}}}d\psi(t).$$

From the definition (5.4.2),

$$w_{n,m+1-i}^{\lambda^{(r)}} = \beta^{2\omega}(z_{n,i}^{\eta^{(r)}})^{-2\omega}w_{n,i}^{\eta^{(r)}}, \quad i = 1,...,m$$

or

$$w_{n,m+1-i}^{\lambda^{(r)}} = \left(\frac{\beta^2}{z_{n,i}^{\eta^{(r)}}}\right)^{\omega}\frac{w_{n,i}^{\eta^{(r)}}}{(z_{n,i}^{\eta^{(r)}})^{\omega}}, \quad i = 1,...,m.$$

Since $z_{n,m+1-i}^{\lambda^{(r)}} = \beta^2/z_{n,i}^{\eta^{(r)}}$, for $i = 1,2,...,m$, the result holds. □

As a consequence the following corollary holds.

Corollary 5.4.5.1 *Given an integer $r \geq 1$, let $\psi(t)$ be an $S^3[\omega,\beta,b]$ distribution with $\omega = (1-r)/2$, $0 < \beta < b \leq \infty$ and $a = \beta^2/b$. Let the real parameters $\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)}$ be such that*

$$\frac{z^n B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};\beta^2/z)}{B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};0)} = B_n(\lambda_{n,1}^{(r)}...,\lambda_{n,r}^{(r)};z), \quad n \geq r.$$

Let $z_{n,1}^{\lambda^{(r)}} < z_{n,2}^{\lambda^{(r)}} < \cdots < z_{n,m}^{\lambda^{(r)}}$, $m \leq n$, be the positive distinct zeros of the polynomials $B_n(\lambda_{n,1}^{(r)},...,\lambda_{n,r}^{(r)};z)$. Then the weights $w_{n,i}^{\lambda^{(r)}}$ and the zeros $z_{n,i}^{\lambda^{(r)}}$ are related by

$$\frac{w_{n,i}^{\lambda^{(r)}}}{(z_{n,i}^{\lambda^{(r)}})^{\omega}} = \frac{w_{n,m+1-i}^{\lambda^{(r)}}}{(z_{n,m+1-i}^{\lambda^{(r)}})^{\omega}}, \quad \text{for } i = 1,...,m.$$

We now make the following conjecture about the complex zeros or negative zeros of the polynomials $B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; z)$.

Conjecture 5.1 *For $r = 2$, let $\psi(t)$ be an $S^3[-1/2, \beta, b]$ distribution. If the related polynomial $B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; z)$ has two negative zeros or two complex conjugate zeros, $z_{n,1}^{\lambda^{(2)}}$ and $z_{n,2}^{\lambda^{(2)}}$. Then the weights $w_{n,1}^{\lambda^{(2)}}$ and $w_{n,2}^{\lambda^{(2)}}$ and the zeros $z_{n,1}^{\lambda^{(2)}}$ and $z_{n,2}^{\lambda^{(2)}}$ satisfy*

$$\frac{w_{n,1}^{\lambda^{(2)}}}{(z_{n,1}^{\lambda^{(2)}})^{\omega}} = \overline{\left(\frac{w_{n,2}^{\lambda^{(2)}}}{(z_{n,2}^{\lambda^{(2)}})^{\omega}}\right)},$$

where $\overline{z}$ is the conjugate complex of $z \in \mathbb{C}$.

We can make remarks about the sign of the weights $w_{n,i}^{\lambda^{(r)}}$, $i = 1, ..., n$.

For the case $r = 1$, if we choose

$$f(t) = t^{-n+1} \left\{ \frac{B_n(\lambda_{n,1}^{(1)}; t)}{t - z_{n,i}^{\lambda^{(1)}}} \right\}^2, \quad i = 1, 2, ..., n,$$

in the quadrature formula (5.4.1), then

$$\int_a^b f(t) d\psi(t) > 0.$$

Since $t^{n-1} f(t) \in \mathbb{P}_{2n-2}$, then

$$\int_a^b f(t) d\psi(t) = w_{n,i}^{\lambda^{(1)}} (z_{n,i}^{\lambda^{(1)}})^{-n+1} \left\{ B_n'(\lambda_{n,1}^{(1)}; z_{n,i}^{\lambda^{(1)}}) \right\}^2, \quad i = 1, 2, ..., n.$$

We can conclude that, since $z_{n,i}^{\lambda^{(1)}} > 0$, for $i = 2, 3, ..., n$ then $w_{n,i}^{\lambda^{(1)}} > 0$, for $i = 2, 3, ..., n$. If n is odd then also $w_{n,1}^{\lambda^{(1)}} > 0$. If n is even then $w_{n,1}^{\lambda^{(1)}}$ has the same sign that $z_{n,1}^{\lambda^{(1)}}$.

We now consider $r = 2$. Hence for $S^3[-1/2, \beta, b]$ distributions there exist real parameters $\lambda_{n,1}^{(r)}$ and $\lambda_{n,2}^{(2)}$ such that

$$\frac{z^n B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; \beta^2/z)}{B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; 0)} = B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; z), \quad n \geq 2.$$

Let $n-2 \le m \le n$ and $z_{n,1}^{\lambda^{(2)}} < z_{n,2}^{\lambda^{(2)}} < \cdots < z_{n,m}^{\lambda^{(2)}}$ be the distinct zeros of the polynomial $B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; z)$ located inside the interval (a,b).

We set

$$f(t) = t^{-n+2}\left\{\frac{B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; t)}{(t - z_{n,i}^{\lambda^{(2)}})(t - \beta^2/z_{n,i}^{\lambda^{(2)}})}\right\}^2, \quad i = 1, 2, ..., m.$$

Since $t^{n-2}f(t) \in \mathbb{P}_{2n-3}$, from (5.4.1) we obtain

$$\int_a^b f(t)d\psi(t) = w_{n,i}^{\lambda^{(2)}} \frac{(B_n'(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; z_{n,i}^{\lambda^{(2)}}))^2}{(z_{n,i}^{\lambda^{(2)}} - \beta^2/z_{n,i}^{\lambda^{(2)}})^2} + w_{n,m+1-i}^{\lambda^{(2)}} \frac{(B_n'(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; \beta^2/z_{n,i}^{\lambda^{(2)}}))^2}{(\beta^2/z_{n,i}^{\lambda^{(2)}} - z_{n,i}^{\lambda^{(2)}})^2},$$

for $i = 1, 2, ..., m$.

Since

$$B_n'(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; \beta^2/z_{n,i}^{\lambda^{(2)}}) = \frac{-B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; 0)(z_{n,i}^{\lambda^{(2)}})^{-n+2}}{\beta^2} B_n'(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; z_{n,i}^{\lambda^{(2)}}),$$

and $(B_n(\lambda_{n,1}^{(2)}, \lambda_{n,2}^{(2)}; 0))^2 = \beta^{2n}$, we then obtain

$$\frac{(z_{n,i}^{\lambda^{(2)}} - \beta^2/z_{n,i}^{\lambda^{(2)}})^2 (z_{n,i}^{\lambda^{(2)}})^{n-2}}{(B_n'(\lambda_{n,1}^{(2)}, ..., \lambda_{n,2}^{(2)}; z_{n,i}^{\lambda^{(2)}}))^2} \int_a^b f(t)d\psi(t) = w_{n,i}^{\lambda^{(2)}} (z_{n,i}^{\lambda^{(2)}})^{n-2} + w_{n,m+1-i}^{\lambda^{(2)}} \left(\frac{\beta^2}{z_{n,i}^{\lambda^{(2)}}}\right)^{n-2}.$$

Since the left hand side is positive and

$$z_{n,m+1-i}^{\lambda^{(2)}} = \beta^2/z_{n,i}^{\lambda^{(2)}}, \quad \text{for } i = 1, 2, ..., m,$$

we obtain

$$w_{n,i}^{\lambda^{(2)}} (z_{n,i}^{\lambda^{(2)}})^{n-2} + w_{n,m+1-i}^{\lambda^{(2)}} (z_{n,m+1-i}^{\lambda^{(2)}})^{n-2} > 0.$$

Concluding remarks

This study of the distributions $S^3[\omega, \beta, b]$ has led us to find interesting results involving the orthogonal L-polynomials and the associated polynomials. For example, some of the results in Chapter 3, namely Theorem 3.2.1 and Theorem 3.2.2, and in Chapter 5, namely Theorem 5.3.2 and Theorem 5.3.3. In addition we have found interesting symmetric properties involving the weights and nodes of the related quadrature formulae, Theorem 5.3.1 for example.

From the study of the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ related to the distributions $S^3[\omega, \beta, b]$ with $\omega = (1-r)/2$, we could find parameters $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)} \in \mathbb{R}$ such that the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ satisfy the inversive symmetric property

$$\frac{z^n B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; \beta^2/z)}{B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; 0)} = B_n(\lambda_{n,1}^{(r)} ..., \lambda_{n,r}^{(r)}; z), \quad n \geq r. \tag{5.4.14}$$

In section 4.2 and section 4.3 we gave the values of $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}$ for $r = 1$ and $r = 2$ respectively.

A future objective could be to find the values of the parameters $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}$, for $r = 3, 4, ...$, such that (5.4.14) holds. Another objective is to find the exact location of the zeros of the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ and the behaviour of the weights of the quadrature formulae related with these polynomials for $r = 2, 3, ...$. In particular, to find the values of the parameters $\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}$, for $r = 2, 3, ...$, such that the zeros of the polynomials $B_n(\lambda_{n,1}^{(r)}, ..., \lambda_{n,r}^{(r)}; z)$ are all real, distinct, and inside the interval (a, b).

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