COMPRESSIBLE BOUNDARY LAYERS WITH SHARP PRESSURE GRADIENTS

Michael John Reader-Harris

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ABSTRACT

The work of this thesis was undertaken as a C.A.S.E. award project in collaboration with Rolls-Royce to examine compressible laminar boundary layers with sharp adverse pressure gradients. Much of the work is devoted to the solution of two important particular problems. The first flow considered is that along a semi-infinite flat plate with uniform pressure when $X < X_0$ and with the pressure for $X > X_0$ being so chosen that the boundary layer is just on the point of separation for all $X > X_0$. Immediately downstream of X_0 there is a sharp pressure rise to which the flow reacts mainly in a thin inner sublayer; so inner and outer asymptotic expansions are derived and matched for the stream function and a function of the temperature. Throughout the thesis the ratio of the viscosity to the absolute temperature is taken to be a function of x, the distance along the wall, alone, and the Illingworth-Stewartson transformation is applied. The Prandtl number, σ , is taken to be of order unity and detailed results are presented for $\sigma = 1$ and 0.72. The second flow considered is that along a finite flat plate where the transformed external velocity $U_1(X)$ is chosen such that

$$U_1(X) = u_0(-X/L)^{\epsilon}, \text{ where } 0<\epsilon\ll1,$$

L is the transformed length of the plate and X represents transformed distance downstream from the trailing edge. The skin friction, position of separation and heat transfer right up to separation are determined.

On the basis of these two solutions, another solution which is not presented in detail, and a solution (due to Curle) to a fourth sharp pressure gradient problem, a general Stratford-type method for computing compressible boundary layers is derived, which may be used to predict the position of separation, skin friction, heat transfer, displacement and momentum thicknesses for a compressible boundary layer with an unfavourable pressure gradient.

In all this work techniques of series analysis are used to good effect. This led us to look at another boundary-layer problem in which such techniques could be used, one in which two parallel infinite disks are initially rotating with angular velocity Ω about a common axis in incompressible fluid, the appropriate Reynolds number being very large. Suddenly the angular velocity of one of the disks is reversed. A new examination of this problem is presented in the appendix to the thesis.

DECLARATION

I declare that the following Thesis is a record of research work carried out by me, that the Thesis is my own composition, and that it has not been previously presented in application for a higher degree.

POSTGRADUATE CAREER

I was admitted into the University of St. Andrews as a research student under Ordinance General No. 12 in October 1978 to work on Compressible Boundary Layers with Sharp Pressure Gradients under the supervision of Professor S. N. Curle. I was admitted under the above resolution as a candidate for the degree of Ph.D. in October 1979.

CERTIFICATE

I certify that Michael J. Reader-Harris has satisfied the conditions of the Ordinance and Regulations and is thus qualified to submit the accompanying application for the degree of Doctor of Philosophy.

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CONTENTS

NOTATION :

In chapters 1, 4 and 5 lower case letters are used for physical co-ordinates and velocity components; upper case letters are used for co-ordinates and velocity components in compressible flow transformed with the Illingworth-Stewartson transformation. In those chapters in which exact solutions of the boundary-layer equations are computed the notation of chapters 1, 4 and 5 is inconvenient. In chapter 2, following Curle (1978 and 1979b), lower case letters are used for transformed co-ordinates and velocity components. In chapter 3, lower case letters are used for transformed and non-dimensionalized co-ordinates and velocity components; other notations are defined in that chapter as required; the notation is similar to that of Riley and Stewartson (1969). The appendix to the thesis uses unstarred lower case letters for non-dimensionalized variables.

NOMENCLATURE:

x	distance measured parallel to the plate (measured from the leading edge of the plate unless otherwise stated)
y	distance measured normal to the plate $(y = 0)$
p	pressure
ρ	density
T	temperature
μ	viscosity
ν	kinematic viscosity, $= \mu/\rho$
k	thermal conductivity
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
γ	ratio of the specific heats, $= c_p/c_v$
a	speed of sound
M	Mach number
τ_w	skin friction, $= \mu_w \, (\partial u/\partial y)_w$
Q_w	heat transfer rate, $= k_w \, (\partial T/\partial y)_w$
δ_1	displacement thickness, $= \int_0^{\infty} \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy$
δ_2	momentum thickness, $= \int_0^{\infty} \frac{\rho u}{\rho_1 u_1} \left(1 - \frac{u}{u_1}\right) dy$

Suffices

0	conditions at a specific reference position upstream of the plate
1	local conditions at the edge of the boundary layer
w	conditions at the wall

1. INTRODUCTION

1.1. Motivation

The aim of this thesis is twofold, to gain new insight into compressible boundary layers with sharp pressure gradients by computing solutions of significant problems and, using the new solutions, to devise a simple but accurate procedure which may be used to predict the position of separation, the skin friction, the heat transfer, the displacement thickness and the momentum thickness for a compressible laminar boundary layer with an unfavourable pressure gradient. The method used is a generalization to compressible flow of the boundary-layer calculation methods of Stratford (1954) and Curle (1977). The work has been undertaken as a C.A.S.E. award project in collaboration with Rolls-Royce Ltd., Aero Division, in Derby.

It is clearly not unreasonable to ask why another method for computing compressible boundary layers approximately is required, when there are already several in use, including those of Cohen and Reshotko (1956b) and Poots (1960). Part of the answer lies in the fact that the method proposed here is at its most accurate in regions of sharp adverse pressure gradient where other methods are at their least accurate. Moreover in regions of sharp adverse pressure gradient at any rate this method is well based physically and does not involve either an arbitrary choice of velocity and temperature profiles or disregarding the thermal-energy integral equation. It is also simple to use.

It is also not unfair to ask whether there is a continued need for approximate methods of computing boundary layers as the development of computing makes the direct solution of the partial differential equations of the compressible boundary layer more tractable. The fact that this work has been undertaken as a C.A.S.E. award project shows that Rolls-Royce finds a continuing need for approximate methods which provide reliable answers quickly. Moreover the particular solutions of the compressible boundary-layer equations here computed and the approximate method derived from these solutions indicate which are the important parameters, show the type of dependence on the parameters which is to be expected and thus enable us to interpret better the results of solving the partial differential equations directly.

1.2 Background to the problem

Many of the first approximate methods for solving the incompressible boundary-layer equations were based on an idea due to K. Pohlhausen (1921). It consisted of making some plausible assumption about the shape of the velocity profile in the boundary layer. In the simplest form of the method the velocity profile was taken to be a quartic polynomial in the non-dimensional co-ordinate normal to the wall $\eta = y/\delta$, where δ is a length characteristic of the thickness of the layer:

$$u/u_1 = f(\eta) = a_o + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4. \quad (1.1)$$

This profile was then made to satisfy some of the same conditions as the true velocity profile did at the wall and at the edge of the boundary layer :

$$\left.\begin{aligned} u = 0, \frac{\partial^2 u}{\partial y^2} &= -\frac{u_1}{\nu}\frac{du_1}{dx}, \text{ when } y = 0, \\ \text{and} \quad u = u_1, \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} &= 0, \text{ when } y = \delta. \end{aligned}\right\} \quad (1.2)$$

When these boundary conditions are used to determine the coefficients in (1.1),

$$\left.\begin{aligned} u/u_1 &= 2\eta - 2\eta^3 + \eta^4 + 1/6\,\Lambda\eta(1-\eta)^3, \\ \text{where} \quad \Lambda &= \frac{\delta^2}{\nu}\frac{du_1}{dx}. \end{aligned}\right\} \quad (1.3)$$

Hence the skin friction, displacement and momentum thicknesses can be obtained in terms of δ , and δ itself can be determined from the momentum integral equation which becomes:

$$\frac{d\Lambda}{dx} = g(\Lambda)\frac{u_1'}{u_1} + h(\Lambda)\frac{u_1''}{u_1'} , \qquad (1.4)$$

Where primes denote derivatives with respect to x.

This method appears in general to give good results in regions of favourable pressure gradient; on the other hand it becomes rapidly less accurate when the pressure gradient is unfavourable. In the case of Howarth's problem (1938), for example, the error in the distance to separation is 30 per cent. Moreover the degree of the polynomial taken and exactly which boundary conditions are satisfied are arbitrary; if a quintic instead of a quartic polynomial is used in (1.1), errors near separation are much reduced but the method fails completely near to a stagnation point.

The Pohlhausen method can be rewritten so that u_1'' does not appear explicitly in the formulation. Introducing the parameters

$$\lambda_\tau = u_1' \;\frac{\delta_2^{\,2}}{\nu} = \left(\frac{\delta_2}{\delta}\right)^2 \Lambda\,, \quad l = \frac{\tau_w \delta_2}{\mu u_1} \text{ and } H = \frac{\delta_1}{\delta_2}\,, \tag{1.5}$$

the momentum integral equation becomes

$$u_1 \frac{d}{dx}\left(\frac{\delta_2^{\,2}}{\nu}\right) = u_1 \frac{d}{dx}\left(\frac{\lambda_\tau}{u_1'}\right) = 2\left\{ l - \lambda_\tau(H+2)\right\} = L\,. \tag{1.6}$$

Then using (1.3), l, H and L may be found as functions of λ_τ and equation (1.6) solved numerically for λ_τ, whence δ_2, δ_1 and τ_w follow easily. This alternative approach merely improves the convenience of the method; it can still be fairly criticized for arbitrariness and inaccuracy.

Thwaites (1949), however, pointed out that if what was required was the calculation of δ_1, δ_2 and τ_w then a detailed knowledge of the velocity profile within the layer was not necessary, but rather all that was required was a suitable correlation between the boundary layer properties H, l, L and λ_τ. He found that whereas there was some variation in the curves of H and l against λ_τ from solution to solution, especially for negative λ_τ (that is, in regions of unfavourable pressure gradient) the variations of L (λ_τ) were less pronounced and that $L(\lambda_\tau)$ could be taken as roughly linear for all solutions. He found that choosing $L(\lambda_\tau) = 0.45 - 6\lambda_\tau$ gave good agreement with all the known solutions and with this form for L (λ_τ) equation (1.6) could be integrated to yield

$$\delta_2^{\,2} = 0.45\,\nu\, u_1^{-6} \int_0^x u_1^5 \, dx. \tag{1.7}$$

From the known solutions, tables of H (λ_τ) and l (λ_τ) were constructed and so, using these, δ_1 and τ_w could be obtained too. This method reproduced the known 'exact' solutions to a reasonable accuracy and with its ease of application was widely accepted as one of the better practical methods. It was shown in an even better light when Truckenbrodt (1952) showed that by making simple approximations in the kinetic energy integral equation (Leibenson, 1935) Thwaites' fitting of L as a linear function of λ_τ could be justified. The method was improved by Curle (1967), who, following Tani (1949), introduced a second parameter $\mu_\tau = \lambda_\tau^2 \, u_1 \, u_1'' / (u_1'^2)$, so that $L = F_0(\lambda_\tau) - \mu_\tau G_0(\lambda_\tau)$, and the method was refined by Lister (1971), whose method produces errors which are typically only 5% of those given by Thwaites' method.

All these methods work least well when there is a sharp adverse pressure gradient. Stratford (1954), however, was concerned to produce a criterion for predicting boundary-layer separation which would give good results especially in the range of sharp pressure gradients. He developed exact solutions for two particular cases. He first considered a boundary layer for which the pressure is constant when $x < x_0$, with a large uniform adverse pressure gradient $dp/dx = \lambda \rho u_0^2 / x_0$ when $x > x_0$, and showed that, as $\lambda \to \infty$, the pressure coefficient $Cp = (p - p_0)/\frac{1}{2}\rho u_0^2$ satisfied the condition

$$\left\{ 10\, x_0 \frac{dCp}{dx} \right\}^2 Cp = 0.764 \qquad (1.8)$$

at separation. Stratford then considered a further problem in which the pressure is constant when $x < x_0$, but the pressure when $x > x_0$ is so chosen that the skin friction is everywhere zero when $x > x_0$. He found that the pressure coefficient immediately downstream of $x = x_0$ satisfied

$$\left\{10\, x_0 \frac{dCp}{dx}\right\}^2 \; Cp = 0.591 \,. \qquad (1.9)$$

Stratford then sought to extend the usefulness of these results. Since separation occurs precisely at $x = x_0$ in the two limiting cases cited he replaced x_0 by x in equations (1.8) and (1.9) and gave a physical argument why this change might improve these formulae for problems in which the pressure gradient is modest and separation occurs downstream of $x = x_0$. He also noted that

$$\left\{10\, x \frac{dCp}{dx}\right\}^2 \; Cp = 1.002 \qquad (1.10)$$

for Howarth's problem in which the external velocity is linearly retarded, a problem in which the pressure gradient is not sharp. Since, moreover, in a formula of the type

$$\left\{10\, x \frac{dCp}{dx}\right\}^2 \; Cp = k, \qquad (1.11)$$

the predicted distance to separation will depend roughly on $k^{\frac{1}{3}}$, he suggested the simple formula for predicting the position of separation

$$\left\{10\, x \frac{dCp}{dx}\right\}^2 \; Cp = 0.764 \qquad (1.12)$$

for use when errors of order 10 per cent are acceptable.

Stratford also produced a more accurate formula in which the constant on the right hand side of (1.12) is replaced by a function of two parameters, Δ and Γ, defined by

$$\Delta = \frac{Cp}{x\,dCp/dx} \tag{1.13}$$

and

$$\Gamma = Cp\,\frac{d^2Cp}{dx^2} \Big/ \left[\frac{dCp}{dx}\right]^2 . \tag{1.14}$$

Δ is a measure of the sharpness of the pressure gradient; Γ measures the change in the pressure gradient with x. $\Delta = 0$ for each of Stratford's two solutions; $\Delta = 1.068$ at separation for Howarth's problem. For the large uniform adverse pressure gradient problem, $\Gamma \equiv 0$, while $\Gamma = -\frac{1}{2}$ for the continuous incipient separation problem and $\Gamma = -0.1451$ at separation for Howarth's problem. Stratford proposed that the function to replace the constant k in (1.11) should be such as to fit his continuous incipient separation solution both at x_0 and downstream of x_0, as it tends to the Falkner-Skan (1930) continuous incipient separation solution, as well as Howarth's solution; accordingly he predicted that separation would occur when

$$\left\{10x\,\frac{dCp}{dx}\right\}^2 Cp = 0.764\,(1+0.35\Delta)\left(1+0.46\,\frac{1+0.14\Delta}{1+0.80\Delta}\Gamma\right). \tag{1.15}$$

Stratford also increased the usefulness of his approximate method by proposing that in the situation where the pressure gradient is initially favourable and later adverse the part of the flow in which the pressure gradient is favourable is replaced in the calculation of the flow downstream of the pressure minimum by an equivalent distance with a mainstream velocity constant and equal to the peak mainstream velocity in the actual flow. The momentum thickness at the point of peak mainstream velocity is taken as the criterion of equivalence; hence the Thwaites' method gives

$$x_0 = \int_0^{x_0'} \left(\frac{u_1}{u_m}\right)^5 dx' , \qquad (1.16)$$

where x is the equivalent distance, x' the actual distance, u_m the peak mainstream velocity and u_1 the actual mainstream velocity. Stratford also tested his method on an example with an initially favourable pressure gradient and obtained good results.

Stratford's method, though soundly based physically, fitted exactly only a small number of solutions and was questioned by Riley and Stewartson (1969), who studied a boundary layer with external velocity.

$$u_1 (x) = u_0 \left[1 - (x/c)\right]^{\alpha} \qquad (1.17)$$

in the limit as $\alpha \to 0$. Their solution (on correction of a numerical coefficient) gives the position of separation as

$$x/c = 1 - 40.8\,\alpha^{3/2} . \qquad (1.18)$$

They claimed that the value of 1 - x/c at separation is not even predicted by Stratford's method to be a multiple of $\alpha^{3/2}$. This criticism is not entirely fair, however, since their remarks are based upon Stratford's simplified formula, valid only if Δ and Γ are small, whereas Γ is infinite for their problem. Curle (1977) pointed out that the full Stratford formula (1.15) correctly gives a solution of the form (1.18) with the constant equal to 47.7, an error of 15 per cent, and set out to show that Stratford's method was essentially sound and to adjust the details to achieve good agreement with the increased number of accurate solutions now available.

Curle (1976a) first reconsidered the boundary layer developing with uniform pressure when $x < x_0$ and with dp/dx large and constant when $x > x_0$. For large values of

$$\lambda = \frac{x_0}{\rho u_0^2}\frac{dp}{dx},$$ he solved the problem by deriving and

matching inner and outer asymptotic expansions, whereas Stratford had considered only the inner solution, his outer boundary condition being determined by physical arguments. Curle was able to obtain not only the distribution of skin friction but also the displacement and momentum thicknesses. His results justify Stratford's analysis completely, giving rigorous justification to the outer boundary condition assumed by Stratford on physical grounds, and extending the analysis from the case of infinite λ to large finite λ. Moreover he obtained very accurate numerical results which agree well with Straford's values. In particular, as $\lambda \to \infty$,

$$\left\{10x\frac{dCp}{dx}\right\}^2 Cp = 0.745(14) \qquad (1.19)$$

at separation, which agrees well with (1.8).

Curle (1976b) went on to reconsider Stratford's continuous incipient separation problem, again using matched asymptotic expansions and obtaining boundary-layer thicknesses as well as the pressure distribution. He was able to justify Stratford's replacement of x_0 by x in equation (1.8) and (1.9) by showing that, for non-zero values of $x - x_0$, the formula (1.9) becomes

$$\left\{10x_0\frac{dCp}{dx}\right\}^2 Cp = 0.59077\left\{1 - 2.00431\left(\frac{x}{x_0} - 1\right) + 2.99223\left(\frac{x}{x_0} - 1\right)^2 + \dots\right\}$$

and thus

$$\left\{10x\,\frac{dCp}{dx}\right\}^2 Cp = 0.59077\left\{1-0.00431\left(\frac{x}{x_0}-1\right) - 0.01640\left(\frac{x}{x_0}-1\right)^2 + \ldots\right\}, \qquad (1.20)$$

which varies with $x-x_0$ only very slowly.

Curle (1977) then produced a paper which extended Stratford's method, based essentially on seven solutions, the two solutions of Stratford reconsidered by Curle, the solution of Howarth for which a more accurate solution was subsequently given by Leigh (1955), the Riley-Stewartson solution, two numerical solutions due to Williams (1976) for compressible flows with zero heat transfer, transformed into incompressible form by the transformation of Illingworth (1949) and Stewartson (1949) (see equations (1.24) to (1.32)), and an additional solution derived in appendix A of Curle's (1977) paper for a problem in which $dp/dx = 0$ when $x < x_0$, and

$$\frac{dp}{dx} = \lambda\frac{\rho u_0^2}{x_0}\left(\frac{x}{x_0} - 1\right) \qquad (1.21)$$

when $x > x_0$ and $\lambda \to \infty$. He showed how $(10x\ dCp/dx)^2 Cp$ varies at separation with Δ and Γ and thus devised a highly accurate procedure for predicting the position of boundary-layer separation. The method was designed to predict exactly the position of separation for each of the seven problems listed above and also gave extremely good results when tested on Tani's (1949) and Banks's (1967) problems.

Curle not only refined Stratford's method but also extended it: he found that when a non-dimensional form of the skin friction T_c is plotted against z , a function of the pressure coefficient, points from the various exact solutions collapsed accurately on to a single curve, enabling the skin friction to be accurately predicted for an arbitrary problem. Moreover examination of the displacement and momentum thicknesses for his recalculation (Curle, 1976a and b) of Stratford's two problems suggested that non-dimensional forms of δ_1 and δ_2 might be expanded in series of powers of $\Delta^{\frac{1}{3}}$. In particular, he suggested that

$$\left(\frac{u_0}{2\nu x}\right)^{\frac{1}{2}} \delta_1 = 1.216783 + \Delta^{\frac{1}{3}}\, p_1(z) + \Delta^{\frac{2}{3}}\, p_2(z) + \Delta^{4/3} \log \Delta\, p_{4L}(z) + \dots , \qquad (1.22)$$

and that

$$\left(\frac{u_0}{2\nu x}\right)^{\frac{1}{2}} \delta_2 = 0.469600 + 0.232246 \left[F(X)\right]^{\frac{1}{3}} z \Delta^{\frac{2}{3}} + \dots , \qquad (1.23)$$

where p_1, p_2, p_{4L} and F(X) are defined in his paper. He showed that, even in the cases (Williams, 1976) for which Δ is not at all small, the later (omitted) terms in the expansions make only modest contributions, and proposed formulae consisting of the above terms plus correction terms chosen to give good agreement with the exact solutions. The accuracy achieved is most impressive: T_c, taking values between 0 and 1, is predicted to within $\pm$ 0.0057, δ_1 to within 0.9%, and δ_2 to within 0.6%. In the case of sharp pressure gradient problems the errors would be much less.

While approximate methods for computing incompressible boundary layers have been developed, approximate methods for compressible boundary layers have been produced too. In solving the compressible boundary-layer equations one major simplification is provided by the Illingworth (1949) - Stewartson (1949) transformation, which is used in many approximate methods and throughout this thesis. The basic equations of compressible boundary-layer flow are (Curle and Davies, 1971, p. 275) in standard notation :

$$\left.\begin{aligned} &\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \\ &\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \\ \text{and } &\rho u \frac{\partial}{\partial x}(c_p T) + \rho v \frac{\partial}{\partial y}(c_p T) = u\frac{dp}{dx} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2. \end{aligned}\right\} \quad (1.24)$$

The assumption is made that the ratio of the viscosity μ to the absolute temperature T is a function of x alone, so that

$$\frac{\mu}{\mu_0} = C(x)\,\frac{T}{T_0}\,, \quad (1.25)$$

where suffix zero refers to values outside the boundary layer at x = 0, and the variables are changed so that

$$\left.\begin{aligned} X &= \int_0^x C(x)\left(\frac{a_1}{a_0}\right)^{(3\gamma - 1)/(\gamma - 1)} dx, \\ \text{and } \quad Y &= \frac{a_1}{a_0}\int_0^y \frac{\rho}{\rho_0}\, dy. \end{aligned}\right\} \quad (1.26)$$

On defining ψ, U, and V such that

$$\left.\begin{aligned} &\rho u = \rho_0 \frac{\partial \psi}{\partial y}\,,\ \rho v = -\rho_0 \frac{\partial \psi}{\partial x}\,, \\ &U = \frac{\partial \psi}{\partial Y}\,,\ V = -\frac{\partial \psi}{\partial X}\,, \end{aligned}\right\} \quad (1.27)$$

the equations of motion (1.24) reduce to

$$\left.\begin{aligned}
&\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \\
&U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = U_1\frac{dU_1}{dx}(1+S) + \nu_0\frac{\partial^2 U}{\partial Y^2}, \\
&\text{and} \\
&U\frac{\partial S}{\partial X} + V\frac{\partial S}{\partial Y} = \frac{\nu_0}{\sigma}\frac{\partial^2 S}{\partial Y^2} - \nu_0\frac{(1-\sigma)}{\sigma}\beta\left\{1-\beta+\beta(U_1/u_0)^2\right\}^{-1} \\
&\qquad\frac{\partial^2}{\partial Y^2}\left(\frac{U^2}{u_0^2}\right),
\end{aligned}\right\} \quad (1.28)$$

where

$$\beta = \frac{\gamma-1}{2}M_0^2\left(1+\frac{\gamma-1}{2}M_0^2\right)^{-1}, \quad (1.29)$$

M_0 is the upstream Mach number and S is related to the temperature by

$$S\left(1+\frac{\gamma-1}{2}M_1^2\right) = \frac{T}{T_1} - \frac{\gamma-1}{2a_1^2}(u_1^2-u^2) - 1 \quad (1.30)$$

The boundary conditions on the equations are

$$\left.\begin{aligned}
&U = V = 0,\ S = S_w \text{ when } Y = 0 \\
&U \to U_1,\quad S \to 0 \quad \text{when } Y \to \infty,
\end{aligned}\right\} \quad (1.31)$$

where

$$\left.\begin{aligned}
&S_w = \frac{T_w}{T_s} - 1 \\
&\text{and}\quad T_s = T_1\left\{1+\frac{\gamma-1}{2}M_1^2\right\}
\end{aligned}\right\} \quad (1.32)$$

is the stagnation temperature.

In the case where $\sigma = 1$ Cohen and Řeshotko both found similar solutions to these equations (1956a) and used them as the basis for an approximate method of calculating the development of compressible boundary layers (1956b). To do this they introduced boundary-layer thicknesses defined by

$$\left.\begin{aligned} \delta_1^* &= \int_0^\infty \left(1 + S - \frac{U}{U_1}\right) dY, \\ \delta_2^* &= \int_0^\infty \frac{U}{U_1}\left(1 - \frac{U}{U_1}\right) dY, \\ \text{and} \quad \delta_4^* &= \int_0^\infty S \frac{U}{U_1} dY. \end{aligned}\right\} \quad (1.33)$$

These appear in the momentum integral and thermal-energy integral equations:

$$\frac{d\delta_2^*}{dX} + \frac{\delta_1^* + 2\delta_2^*}{U_1} \frac{dU_1}{dX} = \frac{\nu_0}{U_1^2} \left(\frac{\partial U}{\partial Y}\right)_w , \quad (1.34)$$

and

$$\frac{d\delta_4^*}{dX} + \frac{1}{U_1} \frac{dU_1}{dX} \delta_4^* = - \frac{\nu_0}{U_1}\left(\frac{\partial S}{\partial Y}\right)_w . \quad (1.35)$$

Following Thwaites, Cohen and Reshotko introduced non-dimensional parameters

$$\left.\begin{aligned} l &= \frac{\delta_2^*}{U_1}\left(\frac{\partial U}{\partial Y}\right)_w , \\ \lambda_c &= - \frac{\delta_2^{*2}}{(1+S_w)U_1}\left(\frac{\partial^2 U}{\partial Y^2}\right)_w = \frac{\delta_2^{*2}}{\nu_0} \frac{dU_1}{dX} , \\ r_c &= \frac{\delta_2^{*3}}{U_1}\left(\frac{\partial^3 U}{\partial Y^3}\right)_w = -\lambda_c \delta_2^* \left(\frac{\partial S}{\partial Y}\right)_w , \\ \text{and} \quad H^* &= \delta_1^*/\delta_2^* . \end{aligned}\right\} \quad (1.36)$$

On substituting into (1.34) it is found that

$$\frac{U_1}{\nu_0} \frac{d}{dX}\left(\delta_2^{*2}\right) = 2\left\{1 - \lambda_c(H^* + 2)\right\} = L . \qquad (1.37)$$

The assumption is then made that l, r_c, H* and L are functions of λ_c and S_w alone. This means that it is not possible to satisfy both (1.34) and (1.35). Cohen and Reshotko focused their attention primarily on the velocity rather than the thermal characteristics and satisfied (1.34) rather than (1.35). The assumed relationships L (λ_c, Sw) are determined from the similar solutions and (1.37) becomes an ordinary differential equation for $\delta_2^*(X)$. After solving this equation, the other boundary-layer characteristics follow immediately. Curle and Davies (1971, p. 279) point out that examination of Cohen and Reshotko's results suggests that

$$L \doteq 0.45 - (6+3S_w)\lambda_c, \qquad (1.38)$$

so that (1.37) integrates to give

$$\delta_2^{*2} = 0.45\nu_0 \, U_1^{-(6+3Sw)} \int_0^X U_1^{(5+3Sw)} \, dX. \qquad (1.39)$$

This method was refined by Monaghan (1960). One obvious defect of the Cohen and Reshotko method is that no attempt is made to satisfy the thermal-energy integral equation, so that the method is unlikely to be of any great accuracy in predicting heat-transfer rates. On the other hand it does involve solution of the momentum integral equation and is therefore more likely to give good predictions of the velocity profile. One possibility is to use Cohen and

Reshotko's method to determine the velocity profile and then feed that velocity profile into Curle's method to determine the heat transfer (Curle, 1962, and Curle and Davies, 1971, p. 282).

There are several other approximate methods of calculating compressible boundary layers in use, one of the most theoretically acceptable of which is due to Poots (1960), which is an extension of a method due to Tani (1954) for calculating incompressible boundary layers. Poots began by taking (1.34), (1.35) and the kinetic-energy integral equation

$$U_1 \frac{d\delta_3^*}{dX} + 2\frac{dU_1}{dX}\left(3\delta_3^* + 2\delta_4^*\right) = \frac{2\nu_0}{U_1^2}\int_0^\infty \left(\frac{\partial U}{\partial Y}\right)^2 dY, \quad (1.40)$$

where

$$\delta_3^* = \int_0^\infty \frac{U}{U_1}\left(1 - \frac{U^2}{U_1^2}\right) dY \quad (1.41)$$

and other quantities are defined earlier.

The approximation was then made that U/U_1 and S may be expressed as quartics in Y/δ^*, the coefficients being chosen to satisfy the conditions

$$\left.\begin{aligned} &U = 0,\ S = S_w,\ \text{when } Y = 0,\\ &\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} = S = \frac{\partial S}{\partial Y} = \frac{\partial^2 S}{\partial Y^2} = 0,\ \frac{U}{U_1} = 1,\ \text{when } Y = \delta^*. \end{aligned}\right\} \quad (1.42)$$

This yielded

$$\left.\begin{aligned} U/U_1 &= a\eta^*(1-\eta^*)^3 + \eta^{*2}(6-8\eta^*+3\eta^{*2}),\\ \text{and}\quad S &= S_w(1-\eta^*)^3(1+3\eta^*) + b\eta^*(1-\eta^*)^3,\\ \text{where}\quad \eta^* &= Y/\delta^*. \end{aligned}\right\} \quad (1.43)$$

Upon substituting into (1.33) and (1.41), taking the upper limit of integration as δ^*, and then into the three equations (1.34), (1.35) and (1.40), three simultaneous ordinary differential equations for δ^*, a and b were obtained.

This method thus provides simultaneously solutions for skin friction, heat transfer, and indeed any desired boundary-layer property. It does, however, require the solution of three simultaneous ordinary differential equations and is unlikely to be very accurate if there is a sharp adverse pressure gradient. It still does not meet the need to find a method which will be physically sound, simple to use and reasonably accurate even in the case of a sharp adverse pressure gradient.

1.3. Aims of this thesis

In this opening chapter we have given a brief review of some of the approximate methods which have been and are being used to calculate both incompressible and compressible boundary layers together with the history of the particular type of method which was originally devised by Stratford, was developed by Curle and is generalized to compressible flow in this thesis.

In the second and third chapters of this thesis two new solutions of the compressible boundary-layer equations are derived both for flows where there is a sharp adverse pressure gradient. The second chapter considers a compressible boundary layer on a semi-infinite flat plate with uniform pressure when $X < X_0$ and with the pressure for $X > X_0$ being so chosen that the boundary layer is just on the point of separation for all $X > X_0$. The third chapter considers a compressible boundary layer on a finite flat plate where the transformed external velocity U_1 is chosen such that

$$U_1 (X) = u_0(-X/L)^{\epsilon} ,$$

where $0 < \epsilon << 1$,

$$L = \int_{-1}^{0} C(x) \left(\frac{a_1}{a_0} \right)^{\frac{3\gamma-1}{\gamma-1}} dx,$$

l is the physical length of the plate and x and X represent physical and transformed distances measured downstream from the trailing edge. In both these problems the Prandtl number, σ, is taken to be of order unity and detailed results are presented for $\sigma = 1$ and 0.72. The fourth chapter both derives and presents the approximate method for computing compressible boundary layers and the fifth chapter presents our conclusions.

Throughout the thesis techniques of series analysis are used to good effect. This led us to look at another boundary-layer problem in which these techniques could be used, a problem in which two parallel infinite disks are rotating with the same constant angular velocity about a common axis in incompressible fluid, the appropriate Reynolds number being very large, until at a certain time the angular velocity of one of the disks is suddenly reversed. This problem, originally studied by Bodonyi and Stewartson (1977), is examined in the appendix to the thesis.

2. DEVELOPMENT OF A COMPRESSIBLE LAMINAR BOUNDARY LAYER UNDER CONDITIONS OF CONTINUOUS INCIPIENT SEPARATION

2.1. Introduction

The problem considered in this chapter is that of a compressible laminar boundary layer with uniform pressure when the distance x along the wall satisfies $x < x_0$ and with the pressure when $x > x_0$ being so chosen that the boundary layer is just on the point of separation for all $x > x_0$. This problem is thus a generalization to compressible flow of the problem studied first by Stratford (1954) and later by Curle (1976b). This problem was proposed as part of the C.A.S.E. award project and is of crucial importance for the construction of the approximate method in chapter 4.

It is assumed that the ratio of the viscosity μ to the absolute temperature T is a function of x alone, so that

$$\mu = C(x)\mu_0 \frac{T}{T_0}, \qquad (2.1)$$

where suffix zero refers to values outside the boundary layer at x = 0. The theory holds for all Prandtl numbers σ of order unity; detailed numerical calculations are presented for $\sigma = 1$ and $\sigma = 0.72$ (appropriate to air).

The transformation of Illingworth (1949) and Stewartson (1949) partially reduces the equations to incompressible form and yields (equation (1.28))

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad (2.2)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1 \frac{du_1}{dx}(1+S) + \nu_0 \frac{\partial^2 u}{\partial y^2}, \qquad (2.3)$$

$$u \frac{\partial S}{\partial x} + v \frac{\partial S}{\partial y} = \frac{\nu_0}{\sigma} \frac{\partial^2 S}{\partial y^2} - \frac{\nu_0 (1 - \sigma)}{\sigma} \beta \left\{ 1 - \beta + \beta (u_1/u_0)^2 \right\}^{-1}$$

$$\frac{\partial^2}{\partial y^2} \left(\frac{u^2}{u_0^2} \right), \qquad (2.4)$$

where

$$\beta = \frac{\gamma - 1}{2} M_0^2 \left\{ 1 + \frac{\gamma - 1}{2} M_0^2 \right\}^{-1}, \qquad (2.5)$$

M_0 is the upstream Mach number and S is related to the temperature by

$$S \left\{ 1 + \frac{\gamma - 1}{2} M_1^2 \right\} = \frac{T}{T_1} - 1 - \frac{\gamma - 1}{2} M_1^2 \left\{ 1 - \frac{u^2}{u_1^2} \right\}. \qquad (2.6)$$

Except in (2.1) where x is measured in the physical plane, x, y, u and v represent distances and velocities in the transformed plane.

It is clear that $S \to 0$ at the edge of the boundary layer.

The value of S when y=0 is

$$S_w = \frac{T_w}{T_s} - 1, \qquad (2.7)$$

where $T_s = T_1 \left\{ 1 + \frac{\gamma - 1}{2} M_1^2 \right\}$

is the stagnation temperature.

If $\sigma = 1$, the solution is found to depend only on S_w. If $\sigma \neq 1$, the solution depends also on β, i.e. upon the Mach number M_0.

When $x < x_0$ the pressure gradient is zero, u_1 takes the constant value u_0, and the solution of (2.2) - (2.4) was given by Blasius (1908) and E. Pohlhausen (1921).

Downstream of $x = x_0$, it is assumed following Stratford (1954) and Curle (1976b) that the pressure distribution for continuous incipient separation is

$$\frac{1}{2}\left(1 - \frac{u_1^2}{u_0^2}\right) = K_2\xi^2 + K_3\xi^3 + K_4\xi^4 + K_5\xi^5 + \dots , \qquad (2.8)$$

$$\text{where } \xi = \left(\frac{x}{x_0} - 1\right)^{1/3}. \qquad (2.9)$$

The flow reacts to the sharp pressure rise primarily within a thin inner sublayer; so inner and outer asymptotic expansions are obtained for S and for a stream function ψ, and matched.

It may be anticipated that as ξ increases, and the inner sub-layer becomes thicker, the velocity profile will ultimately tend to that given by the zero skin-friction solution derived by Cohen and Reshotko (1956a), if $\sigma = 1$, and derived in section 2.6, if $\sigma \neq 1$.

2.2. The Outer Solution

When $x < x_0$, u_1 (x) takes the constant value u_0, and equations (2.2)-(2.4) are satisfied by the Blasius-Pohlhausen solution. Thus, introducing a stream function ψ, such that $u = \psi_y$ and $v = -\psi_x$, we write

$$\psi = (2u_0\nu_0 x)^{1/2} f_0(\eta), \; S = S_0(\eta),$$

where

$$\eta = \left(\frac{u_0}{2\nu_0 x}\right)^{1/2} y.$$

It follows (Blasius, 1908) that

$$f_0''' + f_0 f_0'' = 0,$$

where $f_0(0) = f_0'(0) = 0$, $f_0'(\eta) \to 1$ as $\eta \to \infty$,
and(E. Pohlhausen (1921)) that

$$S_0'' + \sigma f_0 S_0' = 2\beta(1-\sigma)\left\{ f_0' f_0''' + (f_0'')^2 \right\},$$

where $S_0(0) = S_w$, $S_0(\eta) \to 0$, as $\eta \to \infty$.

The relevant properties of $f_0(\eta)$, $S_0(\eta)$ are well known.

Where the pressure gradient is discontinuous at $x = x_0$, the boundary layer approximation fails, but equations (2.2)-(2.4) hold when $x > x_0$. The outer solution is a perturbation of the Blasius-Pohlhausen solution; so write

$$\left.\begin{aligned} \psi &= (2u_0\nu_0 x)^{1/2} F(\xi,\eta), \\ S &= S(\xi,\eta). \end{aligned}\right\} \quad (2.10)$$

Substituting into (2.2)-(2.4), using (2.8), gives

$$\xi^2 (F_{\eta\eta\eta} + FF_{\eta\eta}) = {}^2/_3\,(1+\xi^3)(F_\eta F_{\xi\eta} - F_\xi F_{\eta\eta}) + {}^2/_3\,(1+S)(1+\xi^3)\frac{dK}{d\xi} \quad (2.11)$$

and $$\xi^2(S_{\eta\eta} + \sigma F S_\eta) = {}^2/_3\sigma(1+\xi^3)(F_\eta S_\xi - F_\xi S_\eta) + 2\beta(1-\sigma)\,\xi^2(F_\eta F_{\eta\eta\eta} + F_{\eta\eta}^{\;2})\,(1 - 2\beta K(\xi))^{-1}, \quad (2.12)$$

where $K(\xi) = K_2\xi^2 + K_3\xi^3 + K_4\xi^4 + K_5\xi^5 + \ldots$. (2.13)

A solution is sought in the form

$$F(\xi,\eta) = f_0(\eta) + \xi f_1(\eta) + \xi^2\log\xi\, f_{2L}(\eta) + \xi^2 f_2(\eta) + \xi^3\log\xi\, f_{3L}(\eta) + \xi^3 f_3(\eta) + \xi^4(\log\xi)^2 f_{4LL}(\eta) + \xi^4\log\xi\, f_{4L}(\eta) + \xi^4 f_4(\eta) + \xi^5(\log\xi)^2 f_{5LL}(\eta) + \xi^5\log\xi\, f_{5L}(\eta) + \xi^5 f_5(\eta) + \ldots \quad (2.14)$$

$$S(\xi,\eta) = S_0(\eta) + \xi S_1(\eta) + \xi^2\log\xi\, S_{2L}(\eta) + \xi^2 S_2(\eta)$$

$$+ \xi^3 \log\xi\, S_{3L}(\eta) + \xi^3 S_3(\eta) + \xi^4(\log\xi)^2 S_{4LL}(\eta) + \xi^4 \log\xi\, S_{4L}(\eta) + \xi^4 S_4(\eta) + \dots \quad (2.15)$$

The equations for f_1, S_1, f_{2L}, S_{2L}, etc., are

$$f_0'f_1' - f_0''f_1 = 0, \quad (2.16)$$

$$f_0'S_1 = f_1S_0', \quad (2.17)$$

$$f_0'f_{2L}' - f_0''f_{2L} = 0, \quad (2.18)$$

$$f_0'f_2' - f_0''f_2 = -K_2(1+S_0) + {}^1/_2(f_1f_1'' - f_1'^2 + f_{2L}f_0'' - f_{2L}'f_0'), \quad (2.19)$$

$$f_0'S_{2L} = f_{2L}S_0', \quad (2.20)$$

$$f_0'S_2 = f_2S_0' + {}^1/_2(f_1S_1' - f_1'S_1 + f_{2L}S_0' - f_0'S_{2L}), \quad (2.21)$$

$$f_0'f_{3L}' - f_0''f_{3L} = {}^2/_3\ (f_1''f_{2L} - f_1'f_{2L}') + {}^1/_3(f_1f_{2L}'' - f_1'f_{2L}'), \quad (2.22)$$

$$f_0'f_3' - f_0''f_3 = -K_3(1+S_0) - {}^2/_3K_2S_1 + {}^2/_3(f_1''f_2 - f_1'f_2') + {}^1/_3(f_2''f_1 - f_2'f_1' + f_1''f_{2L} - f_1'f_{2L}' + f_0''f_{3L} - f_0'f_{3L}') + {}^1/_2\ (f_0''' + f_0f_0''), \quad (2.23)$$

$$f_0'S_{3L} = f_{3L}S_0' + {}^2/_3(f_{2L}S_1' - S_{2L}f_1') + {}^1/_3(f_1S_{2L}' - S_1f_{2L}'), \quad (2.24)$$

$$f_0'S_3 = f_3S_0' + {}^2/_3(f_2S_1' - f_1'S_2) + {}^1/_3(f_1S_2' - f_2'S_1$$

$$+ f_{2L}S_1' - f_1'S_{2L} + f_{3L}S_0' - f_0'S_{3L}) + \frac{1}{2\sigma}(S_0'' + \sigma f_0 S_0'$$

$$- 2\beta(1-\sigma)(f_0'f_0''' + (f_0'')^2)), \qquad (2.25)$$

$$f_0'f_{4LL}' - f_0''f_{4LL} = {}^1/_2(f_{2L}f_{2L}'' - f_{2L}'^2), \qquad (2.26)$$

$$f_0'f_{4L}' - f_0''f_{4L} = -{}^1/_2K_2S_{2L} + {}^3/_4(f_1''f_{3L} - f_1'f_{3L}')$$

$$+ {}^1/_2(f_0''f_{4LL} - f_0'f_{4LL}' + f_{2L}''f_2 - f_{2L}'f_2' + f_2''f_{2L}$$

$$- f_2'f_{2L}') + {}^1/_4(f_{2L}''f_{2L} - f_{2L}'^2 + f_{3L}''f_1 - f_{3L}'f_1'), \qquad (2.27)$$

$$f_0'f_4' - f_0''f_4 = -K_4(1+S_0) - {}^3/_4K_3S_1 - {}^1/_2K_2S_2 + {}^3/_4(f_1''f_3 -$$

$$f_1'f_3') + {}^1/_2(f_2''f_2 - f_2'^2) + {}^1/_4(f_0''f_{4L} - f_0'f_{4L}' + f_1''f_{3L}$$

$$- f_1'f_{3L}' + f_2''f_{2L} - f_2'f_{2L}' + f_3''f_1 - f_3'f_1' + f_0''f_1 - f_0'f_1')$$

$$+ {}^3/_8(f_1''' + f_0f_1'' + f_0''f_1), \qquad (2.28)$$

$$f_0'S_{4LL} = f_{4LL}S_0' + {}^1/_2(f_{2L}S_{2L}' - f_{2L}'S_{2L}), \qquad (2.29)$$

$$f_0'S_{4L} = f_{4L}S_0' + {}^3/_4(f_{3L}S_1' - f_1'S_{3L}) + {}^1/_2(f_2S_{2L}'$$

$$- f_{2L}'S_2 + f_{2L}S_2' - f_2'S_{2L} + f_{4LL}S_0' - f_0'S_{4LL}) + {}^1/_4(f_{2L}S_{2L}'$$

$$- f_{2L}'S_{2L} + f_1S_{3L}' - f_{3L}'S_1), \qquad (2.30)$$

$$f_0'S_4 = f_4S_0' + {}^3/_4(f_3S_1' - f_1'S_3) + {}^1/_2(f_2S_2' - f_2'S_2)$$

$$+ {}^1/_4(f_1S_3' - f_3'S_1 + f_{2L}S_2' - f_2'S_{2L} + f_{3L}S_1' - f_1'S_{3L}$$

$$+ f_{4L}S_0' - f_0'S_{4L} + f_1S_0' - f_0'S_1) + \frac{3}{8\sigma}(S_1'' + \sigma(f_0S_1'$$

$$+ f_1S_0') - 2\beta(1-\sigma)(f_0'f_1''' + f_1'f_0''' + 2f_0''f_1'')), \qquad (2.31)$$

$$f_0'f_{5LL}' - f_0''f_{5LL} = {}^4/_5(f_1''f_{4LL} - f_1'f_{4LL}') + {}^3/_5(f_{2L}''f_{3L}$$

$$- f_{2L}'f_{3L}') + {}^2/_5(f_{3L}''f_{2L} - f_{3L}'f_{2L}') + {}^1/_5(f_{4LL}''f_1$$

$$- f_{4LL}'f_1'), \tag{2.32}$$

$$f_0'f_{5L}' - f_0''f_{5L} = -{}^3/_5K_3S_{2L} - {}^2/_5K_2S_{3L} + {}^4/_5(f_1''f_{4L} - f_1'f_{4L}')$$
$$+ {}^3/_5(f_{2L}''f_3 - f_{2L}'f_3' + f_2''f_{3L} - f_2'f_{3L}') + {}^2/_5(f_0''f_{5LL}$$
$$- f_0'f_{5LL}' + f_1''f_{4LL} - f_1'f_{4LL}' + f_{3L}''f_2 - f_{3L}'f_2' + f_3''f_{2L}$$
$$- f_3'f_{2L}' + f_0''f_{2L} - f_0'f_{2L}') + {}^1/_5(f_{2L}''f_{3L} - f_{2L}'f_{3L}'$$
$$+ f_{3L}''f_{2L} - f_{3L}'f_{2L}' + f_{4L}''f_1 - f_{4L}'f_1') + {}^3/_{10}(f_{2L}'''$$
$$+ f_0f_{2L}'' + f_0''f_{2L}), \tag{2.33}$$

$$f_0'f_5' - f_0''f_5 = - K_5(1+S_0) - {}^4/_5K_4S_1 - {}^3/_5K_3S_2 - {}^2/_5K_2S_3$$
$$- {}^2/_5K_2(1+S_0) + {}^4/_5(f_1''f_4 - f_1'f_4') + {}^3/_5(f_2''f_3 - f_2'f_3')$$
$$+ {}^2/_5(f_3''f_2 - f_3'f_2' + f_0''f_2 - f_0'f_2') + {}^1/_5(f_0''f_{5L} - f_0'f_{5L}'$$
$$+ f_1''f_{4L} - f_1'f_{4L}' + f_2''f_{3L} - f_2'f_{3L}' + f_3''f_{2L} - f_3'f_{2L}'$$
$$+ f_4''f_1 - f_4'f_1' + f_0''f_{2L} - f_0'f_{2L}' + f_1f_1'' - f_1'^2)$$
$$+ {}^3/_{10}(f_2''' + f_0f_2'' + f_1f_1'' + f_2f_0''). \tag{2.34}$$

The boundary conditions as $\eta \to \infty$ are readily determined from (2.7) and (2.8), and are

$$f_1' \to 0,\ S_1 \to 0,\ f_{2L}' \to 0,\ f_2' \to -K_2,\ S_{2L} \to 0,$$
$$S_2 \to 0,\ f_{3L}' \to 0,\ f_3' \to -K_3,\ S_{3L} \to 0,\ S_3 \to 0,\ f_{4LL}' \to 0,$$
$$f_{4L}' \to 0,\ f_4' \to -K_4 - {}^1/_2K_2^2,\ S_{4LL} \to 0,\ S_{4L} \to 0,\ S_4 \to 0,$$
$$f_{5LL}' \to 0,\ f_{5L}' \to 0,\ f_5' \to -K_5 - K_2K_3. \tag{2.35}$$

Equations (2.16) to (2.34) may be solved in turn:

$$f_1 = a_1f_0', \tag{2.36}$$

$$S_1 = a_1 S_0', \qquad (2.37)$$

$$f_{2L} = a_{2L} f_0', \qquad (2.38)$$

$$f_2 = a_2 f_0' + {}^1/_2 a_1{}^2 f_0'' - K_2 \phi_1 , \qquad (2.39)$$

$$S_{2L} = a_{2L} S_0', \qquad (2.40)$$

$$S_2 = {}^1/_2 a_1{}^2 S_0'' + a_2 S_0' - K_2 \frac{\phi_1 S_0'}{f_0'} , \qquad (2.41)$$

$$f_{3L} = a_{3L} f_0' + a_1 a_{2L} f_0'', \qquad (2.42)$$

$$f_3 = a_3 f_0' + a_1 a_2 f_0'' + {}^1/_6 a_1{}^3 f_0''' - a_1 K_2 \phi_1' - K_3 \phi_1, \qquad (2.43)$$

$$S_{3L} = a_{3L} S_0' + a_1 a_{2L} S_0'', \qquad (2.44)$$

$$S_3 = a_3 S_0' + a_1 a_2 S_0'' + {}^1/_6 a_1{}^3 S_0''' - a_1 K_2 \left(\frac{\phi_1 S_0'}{f_0'}\right)'$$

$$- K_3 \frac{\phi_1 S_0'}{f_0'}, \qquad (2.45)$$

$$f_{4LL} = a_{4LL} f_0' + {}^1/_2 a_{2L}{}^2 f_0'', \qquad (2.46)$$

$$f_{4L} = a_{4L} f_0' + (a_1 a_{3L} + a_2 a_{2L}) f_0'' + {}^1/_2 a_1{}^2 a_{2L} f_0'''$$

$$- a_{2L} K_2 \phi_1', \qquad (2.47)$$

$$f_4 = a_4 f_0' + (a_1 a_3 + {}^1/_2 a_2{}^2) f_0'' + {}^1/_2 a_1{}^2 a_2 f_0'''$$

$$+ {}^1/_{24} a_1{}^4 f_0^{IV} - {}^1/_2 a_1{}^2 K_2 \phi_1'' - (a_2 K_2 + a_1 K_3) \phi_1' - K_4 \phi_1$$

$$+ {}^1/_2 K_2{}^2 \phi_2, \qquad (2.48)$$

$$S_{4LL} = a_{4LL} S_0' + {}^1/_2 a_{2L}{}^2 S_0'', \qquad (2.49)$$

$$S_{4L} = a_{4L}S_0' + (a_1a_{3L} + a_2a_{2L})\, S_0'' + {}^1/_2 a_1{}^2 a_{2L} S_0''' - a_{2L}K_2 \left(\frac{\phi_1 S_0'}{f_0'}\right)', \qquad (2.50)$$

$$S_4 = a_4 S_0' + (a_1a_3 + {}^1/_2 a_2{}^2) S_0'' + {}^1/_2 a_1{}^2 a_2 S_0''' + {}^1/_{24} a_1{}^4 S_0^{IV} - {}^1/_2 a_1{}^2 K_2 \left(\frac{\phi_1 S_0'}{f_0'}\right)'' - (a_2K_2 + a_1K_3)\left(\frac{\phi_1 S_0'}{f_0'}\right)' - K_4 \frac{\phi_1 S_0'}{f_0'} + {}^1/_2 K_2{}^2 \left\{ \frac{\phi_2 S_0'}{f_0'} + \frac{\phi_1}{f_0'}\left(\frac{\phi_1 S_0'}{f_0'}\right)' - \frac{\phi_1'}{f_0'}\,\frac{\phi_1 S_0'}{f_0'} \right\}, \qquad (2.51)$$

$$f_{5LL} = a_{5LL} f_0' + (a_1 a_{4LL} + a_{2L} a_{3L}) f_0'' + {}^1/_2 a_1 a_{2L}{}^2 f_0''', \qquad (2.52)$$

$$f_{5L} = a_{5L} f_0' + (a_3 a_{2L} + a_2 a_{3L} + a_1 a_{4L})\, f_0'' + a_1(a_2 a_{2L} + {}^1/_2 a_1 a_{3L}) f_0''' + {}^1/_6 a_1{}^3 a_{2L} f_0^{IV} - (K_2 a_{3L} + a_{2L} K_3)\phi_1' - a_1 a_{2L} K_2 \phi_1'', \qquad (2.53)$$

$$f_5 = a_5 f_0' + (a_1 a_4 + a_2 a_3) f_0'' + {}^1/_2 a_1 (a_1 a_3 + a_2{}^2) f_0''' + {}^1/_6 a_1{}^3 a_2 f_0^{IV} + {}^1/_{120} a_1{}^5 f_0^{V} - K_5\phi_1 - (a_1K_4 + a_2K_3 + a_3K_2)\phi_1' - a_1(a_2K_2 + {}^1/_2 a_1 K_3)\phi_1'' - {}^1/_6 K_2 a_1{}^3 \phi_1''' + {}^1/_2 a_1 K_2{}^2 \phi_2' + K_2K_3\phi_2 - {}^3/_{10} K_2 \phi_3, \qquad (2.54)$$

where a_1, a_{2L}, a_2, a_{3L}, a_3, a_{4LL}, a_{4L}, a_4, a_{5LL}, a_{5L}, and a_5 are arbitrary constants and ϕ_1, ϕ_2, and ϕ_3 are the solutions of the following equations:

$$f_0'\phi_1' - f_0''\phi_1 = 1 + S_0\,, \qquad (2.55)$$

$$f_0'\phi_2' - f_0''\phi_2 = \frac{\phi_1 S_0'}{f_0'} + \phi_1\phi_1'' - \phi_1'^2, \qquad (2.56)$$

$$f_0'\phi_3' - f_0''\phi_3 = \phi_1''' + f_0\phi_1'' + f_0''\phi_1, \qquad (2.57)$$

where in each case the solution of the equation for which the coefficient of η in the series expansion for small η is zero is chosen.

It is easily shown that, for small η,

$$\phi_1(1+S_w)^{-1} = (-\alpha^{-1} + {}^1/_{12}\eta^3 + O(\eta^6)) + B_1(\alpha^{-1}\eta\log\eta$$

$$+ O(\eta^4\log\eta)) + B_2(1-\sigma)(\alpha\eta^2 + O(\eta^5)), \qquad (2.58)$$

$$\phi_2(1+S_w)^{-2} = (-{}^1/_2\alpha^{-2}\eta\log\eta + O(\eta^4)) + B_1(\alpha^{-3}\eta^{-1} + O(\eta^2\log\eta))$$

$$+ B_2(1-\sigma)(4\alpha^{-1} + O(\eta^3)) + B_1{}^2(\alpha^{-3}((\log\eta)^2 + 2\log\eta + 3)$$

$$+ O(\eta^3(\log\eta)^2)) + B_1B_2(1-\sigma)(-2\alpha^{-1}\eta\log\eta + O(\eta^4\log\eta))$$
$$+B_2{}^2(1-\sigma)^2(O(\eta^2)), \qquad (2.59)$$

$$\phi_3(1+S_w)^{-1} = ({}^1/_2\alpha^{-1} + O(\eta^3)) + B_1({}^1/_3\alpha^{-2}\eta^{-2} + O(\eta\log\eta))$$

$$+ B_2(1-\sigma)(O(\eta^2)), \qquad (2.60)$$

where $\alpha = f_0''(0) = 0.4696000$, $\Theta = S_0'(0)$, $B_1 = \Theta(1+S_w)^{-1}$, and $B_2 = \beta(1+S_w)^{-1}$.

If $\sigma = 1$, $S_0'(0) = -\alpha S_w$.

If $\sigma \neq 1$, $S_0'(0)$ must be determined numerically: if $\sigma = 0.72$, $\Theta = -0.4180913\ S_w - 0.0636704\beta$.

The arbitrary constants are derived from the matching with the inner profile, whose scale is determined by writing

$$z = \xi^{-1}\eta.$$

The outer boundary conditions on the inner solution are obtained by expanding the outer solution for small values of η and rewriting in terms of z:

$$F \sim \{{}^1/_2\alpha(z+a_1)^2 + \alpha^{-1}K_2(1+S_w) + \ldots\}\xi^2 + \{\alpha(a_{2L} - K_2\alpha^{-2}\theta)$$

$$(z+a_1) + \ldots\}\xi^3 \log\xi + \{-K_2\theta\alpha^{-1}(z+a_1)\log(z+a_1) + \alpha a_2(z+a_1)$$

$$+ \alpha^{-1}K_3(1+S_w) + {}^1/_2\alpha^{-3}K_2^2(1+S_w)\theta(z+a_1)^{-1} - {}^1/_{10}\alpha^{-2}K_2\theta(z+a_1)^{-2}$$

$$+ \ldots\}\xi^3 + \{{}^1/_2\alpha(a_{2L} - K_2\alpha^{-2}\theta)^2 + \ldots\}\xi^4(\log\xi)^2 + \{\alpha(a_{3L}$$

$$- K_3\alpha^{-2}\theta)(z+a_1) - \alpha^{-1}K_2\theta(a_{2L} - K_2\alpha^{-2}\theta)(\log(z+a_1) + 1)$$

$$-\alpha a_2(a_{2L} - K_2\alpha^{-2}\theta) + \ldots\}\xi^4\log\xi + \{-K_2\beta(1-\sigma)\alpha(z+a_1)^2$$

$$-\alpha^{-1}K_3\theta(z+a_1)\log(z+a_1) + \alpha a_3(z+a_1) + {}^1/_2 K_2^2\alpha^{-3}\theta^2((\log(z+a_1))^2$$

$$+ 2\log(z+a_1) + 3) - \alpha^{-1}a_2K_2\theta(\log(z+a_1) + 1) + {}^1/_2 a_2^2\alpha + \alpha^{-1}$$

$$K_4(1+S_w) + 2\alpha^{-1}K_2^2(1+S_w)\beta(1-\sigma) + \alpha^{-3}K_2K_3(1+S_w)\theta(z+a_1)^{-1}$$

$$+ \ldots\}\xi^4 + \{a_{4LL}\alpha(z+a_1) + \alpha(a_{2L} - K_2\alpha^{-2}\theta)(a_{3L} - K_3\alpha^{-2}\theta) + \ldots\}\xi^5$$

$$(\log\xi)^2 + \{\alpha(a_{4L} - {}^1/_4\alpha^{-3}K_2^2(1+S_w)^2 - \alpha^{-2}K_4\theta - \alpha^{-2}K_2\beta(1-\sigma)$$

$$(K_2\theta + 2\alpha^2a_{2L}))(z+a_1) - \theta\alpha^{-1}(K_2(a_{3L} - K_3\alpha^{-2}\theta) + K_3(a_{2L}$$

$$- K_2\alpha^{-2}\theta))(\log(z+a_1) + 1) + \alpha(a_2(a_{3L} - K_3\alpha^{-2}\theta) + a_3(a_{2L}$$

$$- K_2\alpha^{-2}\theta)) + \ldots\}\xi^5\log\xi + \{-\frac{1}{5!}\alpha^2(z+a_1)^5 - \frac{1}{12}K_2(1+S_w)$$

$$(z+a_1)^3 - \alpha K_3\beta(1-\sigma)(z+a_1)^2 - (\alpha^{-1}K_4\theta + \alpha^{-1}K_2^2\theta\beta(1-\sigma)$$

$$+ \tfrac{1}{4}\alpha^{-2} K_2^{\,2}(1+S_w)^2)\ (z+a_1) \log (z+a_1) + (a_4\alpha - 2\alpha a_2K_2$$

$$\beta(1-\sigma))\ (z+a_1) + \alpha^{-3}K_2K_3\theta^2((\log(z+a_1))^2 + 2\log(z+a_1)$$

$$+ 3) - \alpha^{-1}\theta(a_2K_3 + a_3K_2)\ (\log(z+a_1) + 1) + a_2a_3\alpha + 4\alpha^{-1}K_2K_3$$

$$(1+S_w)\beta(1-\sigma) + \alpha^{-1}K_5\ (1+S_w) - \tfrac{3}{20}\alpha^{-1}K_2(1+S_w) + \ldots\}\,\xi^5$$

$$+ \ldots \tag{2.61}$$

Since the inner expansion is found to contain no $\log\xi$ terms, it is clear that $a_{2L} = \alpha^{-2}K_2\theta,$ (2.62)

$$a_{3L} = \alpha^{-2}K_3\theta, \tag{2.63}$$

$$a_{4LL} = 0, \tag{2.64}$$

and $a_{4L} = \tfrac{1}{4}\alpha^{-3}K_2^{\,2}(1+S_w)^2 + \alpha^{-2}K_4\theta + 3\alpha^{-2}K_2^{\,2}\theta\beta(1-\sigma).$ (2.65)

Using (2.62)-(2.65),

$$S \sim S_w + \theta\,\{(z+a_1) + \alpha^{-2}K_2(1+S_w)(z+a_1)^{-1} - \tfrac{1}{2}\alpha^{-4}K_2^{\,2}(1+S_w)^2$$

$$(z+a_1)^{-3} + \ldots\}\,\xi + \{\alpha^2\beta(1-\sigma)\ (z+a_1)^2 - \alpha^{-2}K_2\theta^2\log(z+a_1)$$

$$+ 2K_2(1+S_w)\beta(1-\sigma) + a_2\theta + \alpha^{-2}K_3(1+S_w)\theta(z+a_1)^{-1} + \alpha^{-4}K_2^{\,2}\theta^2$$

$$(1+S_w)(\log(z+a_1) + \tfrac{1}{2})(z+a_1)^{-2} - \alpha^{-2}a_2K_2\theta(1+S_w)(z+a_1)^{-2}$$

$$+ \ldots\}\,\xi^2 + \{-K_2\theta\beta(1-\sigma)(2\log(z+a_1) + 1)(z+a_1) + 2\alpha^2a_2\beta(1-\sigma)$$

$$(z+a_1) - \alpha^{-2} K_3\theta^2\log(z+a_1) + a_3\theta + 2K_3(1+S_w)\ \beta(1-\sigma)$$

$$+ \alpha^{-4}K_2^{\,2}\theta^3(\log(z+a_1) + \tfrac{3}{2})\ (z+a_1)^{-1} + \alpha^{-2}(K_2\theta(1+S_w)$$

$$- a_2K_2\theta^2 + 4K_2^{\,2}(1+S_w)\theta\beta(1-\sigma))\ (z+a_1)^{-1} + \ldots\}\,\xi^3 + \ldots \tag{2.66}$$

2.3 The Inner Solution

In the region just downstream of $x=x_0$, the boundary layer reacts to the sharp adverse pressure gradient, and there is an inner sub-layer in which the appropriate coordinate normal to the wall is

$$z = \xi^{-1}\eta. \qquad (2.67)$$

Upon changing from variables (η,ξ) to (z,ξ), it may be shown that the equations (2.11) and (2.12) now take the form

$$^3/_2\xi^2(1+\xi^3)^{-1}(F_{zzz} + \xi FF_{zz}) = \xi F_z F_{\xi z} - \xi F_\xi F_{zz} - F_z{}^2 + \xi^3(1+S)$$

$$\frac{d}{d\xi} K\ (\xi) \qquad (2.68)$$

and

$$\xi^2(S_{zz} + \sigma FS_z \xi) = {}^2/_3\sigma(1+\xi^3)\xi(F_z S_\xi - F_\xi S_z) + 2\beta(1-\sigma)$$

$$(F_z F_{zzz} + F_{zz}{}^2)\ (1 - 2\beta K\ (\xi))^{-1}. \qquad (2.69)$$

The solution to (2.68) and (2.69) is sought in the form

$$F = \xi^2 F_0(z) + \xi^3 F_1(z) + \xi^4 F_2(z) + \xi^5 F_3(z) + \ldots \quad (2.70)$$

and

$$S = S_w + \xi S_1(z) + \xi^2 S_2(z) + \xi^3 S_3(z) + \ldots \quad . \quad (2.71)$$

Then

$$F_0''' + {}^4/_3 F_0 F_0'' - {}^2/_3 F_0'^2 = {}^4/_3(1+S_w)K_2, \qquad (2.72)$$

$$S_1'' + {}^4/_3\sigma F_0 S_1' - {}^2/_3\sigma F_0' S_1 = 0, \qquad (2.73)$$

$$F_1''' + {}^4/_3 F_0 F_1'' - 2F_0' F_1' + 2F_0'' F_1 = 2(1+S_w)\ K_3$$

$$+{}^4/_3K_2S_1, \qquad (2.74)$$

$$S_2'' + {}^4/_3\sigma F_0S_2' - {}^4/_3\sigma F_0'S_2 = {}^2/_3\sigma F_1'S_1 - 2\sigma F_1S_1' + 2\beta(1-\sigma)$$

$$(F_0'F_0''' + F_0''^2), \qquad (2.75)$$

$$F_2''' + {}^4/_3F_0F_2'' - {}^8/_3F_0'F_2' + {}^8/_3F_0''F_2 = {}^4/_3F_1'^2 - 2F_1F_1''$$

$$+ {}^8/_3K_4(1+S_w) + 2K_3S_1 + {}^4/_3K_2\,S_2, \qquad (2.76)$$

$$S_3'' + {}^4/_3\sigma F_0S_3' - 2\sigma F_0'S_3 = {}^4/_3\sigma F_1'\,S_2 + {}^2/_3\sigma F_2'S_1$$

$$- 2\sigma F_1S_2' - {}^8/_3\sigma F_2S_1' + 2\beta(1-\sigma)(F_0'F_1''' + F_1'F_0'''$$

$$+ 2F_0''F_1''), \qquad (2.77)$$

and

$$F_3''' + {}^4/_3F_0F_3'' - {}^{10}/_3F_0'F_3' + {}^{10}/_3F_0''F_3 = {}^{10}/_3F_1'F_2'$$

$$- 2F_1F_2'' - {}^8/_3F_2F_1'' + {}^2/_3F_0'^2 - {}^7/_3F_0F_0'' + (1+S_w)$$

$$({}^4/_3K_2 + {}^{10}/_3K_5) + {}^8/_3K_4S_1 + 2K_3S_2 + {}^4/_3K_2S_3. \qquad (2.78)$$

The inner boundary conditions are $F_n(0) = F_n'(0) = F_n''(0) = 0$ $(n= 0,1,2,3)$ and $S_n(0) = 0$ $(n= 1,2,3)$. (2.79)

The outer boundary conditions are given by (2.61) and (2.66).

The equations (2.72) to (2.78) are solved successively.

Write $K_2 = \lambda_0(1+S_w)^{-1}$. (2.80)

Then (2.72) becomes

$$F_0''' + {}^4/_3F_0F_0'' - {}^2/_3F_0'^2 = {}^4/_3\lambda_0, \qquad (2.81a)$$

with boundary conditions

$$\left.\begin{aligned} &F_0(0) = F_0'(0) = F_0''(0) = 0, \\ \text{and}\quad &F_0 \sim {}^1/_2\,(z+a_1)^2 + \lambda_0\alpha^{-1}, \text{ as } z \to \infty. \end{aligned}\right\} \qquad (2.81b)$$

The numerical solution yields

$$K_2(1+S_w) = \lambda_0 = 0.118441628, \qquad (2.82)$$

and

$$a_1 = -1.6062864. \qquad (2.83)$$

Write $S_1 = \Theta M_1$. (2.84)

Then (2.73) becomes

$$\left.\begin{aligned} &M_1'' + {}^4/_3\sigma F_0 M_1' - {}^2/_3\sigma F_0' M_1 = 0, \\ &\text{with boundary conditions} \\ &M_1(0) = 0, \\ &\text{and } M_1 \sim (z+a_1) + \lambda_0\alpha^{-2}(z+a_1)^{-1} - {}^1/_2\lambda_0^2\alpha^{-4}(z+a_1)^{-3} \\ &\quad + \ldots \text{ as } z \to \infty. \end{aligned}\right\} (2.85)$$

The numerical solution yields

$$M_1'(0) = \begin{cases} 0.5751939 & \text{if } \sigma = 1 \\ 0.5953621 & \text{if } \sigma = 0.72 \end{cases} \qquad (2.86)$$

Write $K_3 = \lambda_1 B_1 (1+S_w)^{-1}$, $F_1 = B_1 J_1$, and $a_2 = a_2' B_1$. (2.87)

Then (2.74) becomes

$$J_1''' + {}^4/_3 F_0 J_1'' - 2F_0' J_1' + 2F_0'' J_1 = 2\lambda_1 + {}^4/_3\lambda_0 M_1, \qquad (2.88a)$$

with boundary conditions

$$J_1(0) = J_1'(0) = J_1''(0) = 0, \qquad (2.88b)$$

and

$$J_1 \sim -\lambda_0\alpha^{-1}\ (z+a_1)\log(z+a_1) + \alpha\, a_2'(z+a_1) + \lambda_1\alpha^{-1} + {}^1/_2\lambda_0{}^2\alpha^{-3}$$

$$(z+a_1)^{-1} - \frac{1}{10}\lambda_0\alpha^{-2}\ (z+a_1)^{-2} + \dots\ ,\ \text{as } z \to \infty. \quad (2.88c)$$

The numerical solution yields

$$\lambda_1 = \begin{cases} -0.1067636 & \text{if } \sigma = 1 \\ -0.1087672 & \text{if } \sigma = 0.72 \end{cases} \quad (2.89)$$

and

$$a_2' = \begin{cases} -0.4650589 & \text{if } \sigma = 1. \\ -0.4719941 & \text{if } \sigma = 0.72. \end{cases}$$

Write $S_2 = (1+S_w)(B_1{}^2O_2 + B_2Q_2).$ (2.90)

Then (2.75) gives

$$O_2'' + {}^4/_3\sigma F_0O_2' - {}^4/_3\sigma F_0'O_2 = {}^2/_3\sigma J_1'M_1 - 2\sigma J_1M_1',$$

with boundary conditions

$$O_2(0) = 0, \quad (2.91)$$

and $O_2 \sim -\lambda_0\alpha^{-2}\log(z+a_1) + a_2' + \lambda_1\alpha^{-2}(z+a_1)^{-1}$

$+\lambda_0{}^2\alpha^{-4}(\log(z+a_1) - a_2'\lambda_0{}^{-1}\alpha^2 + {}^1/_2)\ (z+a_1)^{-2} + \dots$

as $z \to \infty$,

and

$$Q_2'' + {}^4/_3\sigma F_0Q_2' - {}^4/_3\sigma F_0'Q_2 = 2(1-\sigma)(F_0'F_0''' + F_0''^2),$$

with boundary conditions

$$Q_2(0) = 0, \quad (2.92)$$

$$Q_2 \sim \alpha^2(1-\sigma)(z+a_1)^2 + 2\lambda_0(1-\sigma) \quad \text{as } z \to \infty.$$

The numerical solution yields

$$\left.\begin{aligned} O_2'(0) &= \begin{cases} -0.1937759 & \text{if } \sigma = 1, \\ -0.1968201 & \text{if } \sigma = 0.72, \end{cases} \\ Q_2'(0) &= \begin{cases} 0 & \text{if } \sigma = 1. \\ 0.0143476 & \text{if } \sigma = 0.72. \end{cases} \end{aligned}\right\} \quad (2.93)$$

Write $K_4 = (1+S_w)^{-1}(\lambda_{21}\, B_1^{\,2} + \lambda_{22}\, B_2)$, $F_2 = B_1^{\,2}J_2$ $+ B_2L_2$, and $a_3 = a_{31}\, B_1^{\,2} + a_{32}\, B_2$. (2.94)

Then (2.76) gives

$$J_2''' + {}^4/_3F_0J_2'' - {}^8/_3F_0'J_2' + {}^8/_3F_0''J_2 = {}^4/_3J_1'^2 - 2J_1J_1''$$
$$+ 2\lambda_1M_1 + {}^4/_3\lambda_0O_2 + {}^8/_3\lambda_{21},$$

with boundary conditions

$$J_2(0) = J_2'(0) = J_2''(0) = 0,$$

and $J_2 \sim -\lambda_1\alpha^{-1}(z+a_1)\log(z+a_1) + \alpha\, a_{31}(z+a_1) + {}^1/_2\lambda_0^{\,2}\alpha^{-3}$
$((\log(z+a_1))^2 + 2\log(z+a_1) + 3) - \alpha^{-1}a_2'\lambda_0(\log(z+a_1) + 1)$
$+ {}^1/_2a_2'^{\,2} + \alpha^{-1}\lambda_{21} + \alpha^{-3}\lambda_0\lambda_1(z+a_1)^{-1} + \dots$ as $z \to \infty$, (2.95)

and

$$L_2''' + {}^4/_3F_0L_2'' - {}^8/_3F_0'\, L_2' + {}^8/_3F_0''L_2 = {}^4/_3\lambda_0Q_2 + {}^8/_3\lambda_{22},$$

with boundary conditions

$$L_2(0) = L_2'(0) = L_2''(0) = 0,$$

and $L_2 \sim -\lambda_0\alpha(1-\sigma)(z+a_1)^2 + \alpha\, a_{32}(z+a_1) + \alpha^{-1}\lambda_{22}$
$+ 2\alpha^{-1}\lambda_0^{\,2}(1-\sigma)$, as $z \to \infty$. (2.96)

The numerical solution yields

$$\left.\begin{aligned}
\lambda_{21} &= \begin{cases} 0.0929310 & \text{if } \sigma = 1 \\ 0.0962037 & \text{if } \sigma = 0.72 \end{cases} \\
a_{31} &= \begin{cases} -0.16635 & \text{if } \sigma = 1 \\ -0.16016 & \text{if } \sigma = 0.72 \end{cases} \\
\lambda_{22} &= \begin{cases} 0 & \text{if } \sigma = 1 \\ -0.0071220 & \text{if } \sigma = 0.72 \end{cases} \\
a_{32} &= \begin{cases} 0 & \text{if } \sigma = 1. \\ -0.0664444 & \text{if } \sigma = 0.72. \end{cases}
\end{aligned}\right\} \quad (2.97)$$

Write $S_3 = (1+S_w)(B_1{}^3O_3 + B_1B_2Q_3)$. (2.98)

Then (2.77) gives

$$O_3'' + {}^4/_3\sigma F_0O_3' - 2\sigma F_0'O_3 = {}^4/_3\sigma J_1'O_2 + {}^2/_3\sigma J_2'M_1$$

$$- 2\sigma J_1O_2' - {}^8/_3\sigma J_2M_1',$$

with boundary conditions (2.99)

$$O_3(0) = 0,$$

$$O_3 \sim - \alpha^{-2}\lambda_1\log(z+a_1) + a_{31} + \alpha^{-4}\lambda_0{}^2(\log(z+a_1)$$

$$+ {}^3/_2)(z+a_1)^{-1} + \alpha^{-2}(\lambda_{21} - a_2'\lambda_0)(z+a_1)^{-1} + \dots ,$$

as $z \to \infty$.

and

$$Q_3'' + {}^4/_3\sigma F_0Q_3' - 2\sigma F_0'Q_3 = {}^4/_3\sigma J_1'Q_2 + {}^2/_3\sigma L_2'M_1$$

$$- 2\sigma J_1Q_2' - {}^8/_3\sigma L_2M_1' + 2(1-\sigma)(F_0'\ J_1''' + J_1'F_0'''$$

$$+ 2F_0''J_1''), \tag{2.100a}$$

with boundary conditions

$$\left.\begin{aligned} &Q_3(0) = 0 , \\ &Q_3 \sim -\lambda_0(1-\sigma)(2\log(z+a_1) + 1)(z+a_1) + 2\alpha^2 a_2'\,(1-\sigma) \\ &(z+a_1) + a_{32} + 2\lambda_1\,(1-\sigma) + (\lambda_{22}\alpha^{-2} + 4\alpha^{-2}\lambda_0{}^2(1-\sigma)) \\ &(z+a_1)^{-1} + \dots \quad , \text{ as } z \to \infty . \end{aligned}\right\} \tag{2.100b}$$

The numerical solution yields

$$\left.\begin{aligned} &O_3'(0) = \begin{cases} 0.1471404 & \text{if } \sigma = 1, \\ 0.1523021 & \text{if } \sigma = 0.72, \end{cases} \\ &Q_3'(0) = \begin{cases} 0 & \text{if } \sigma = 1. \\ -0.0116558 & \text{if } \sigma = 0.72. \end{cases} \end{aligned}\right\} \tag{2.101}$$

Write $K_5 = (1+S_w)^{-1}(\lambda_{31}B_1{}^3 + \lambda_{32}B_1B_2 + \lambda_{33})$, $F_5 =$

$B_1{}^3J_3 + B_1B_2L_3 + T_3$, and $a_4 = a_{41}B_1{}^3 + a_{42}B_1B_2 + a_{43}$. (2.102)

Then (2.78) gives

$$J_3''' + {}^4/_3F_0J_3'' - {}^{10}/_3F_0'J_3' + {}^{10}/_3F_0''J_3 = {}^1/_3(10J_1'J_2' - 6J_1J_2'' - 8J_1''J_2 + 8\lambda_{21}M_1 + 6\lambda_1O_2 + 4\lambda_0O_3 + 10\lambda_{31}), \tag{2.103a}$$

with boundary conditions

$$J_3(0) = J_3'(0) = J_3''(0) = 0, \tag{2.103b}$$

$$J_3 \sim -\lambda_{21}\alpha^{-1}\,(z+a_1)\log(z+a_1) + a_{41}\alpha(z+a_1) + \lambda_0\lambda_1\alpha^{-3}$$

$$((\log(z+a_1))^2 + 2\log(z+a_1) + 3) - \alpha^{-1}(a_2'\lambda_1 + a_{31}\lambda_0)$$

$(\log(z+a_1) + 1) + \alpha a_2{}' a_{31} + \alpha^{-1}\lambda_{31} + \dots$, as $z \to \infty$, (2.103c)

and

$$\left.\begin{aligned}
&L_3''' + {}^4/_3 F_0 L_3'' - {}^{10}/_3 F_0' L_3' + {}^{10}/_3 F_0'' L_3 = {}^1/_3 (10 J_1' \\
&L_2' - 6 J_1 L_2'' - 8 J_1'' L_2 + 8\lambda_{22} M_1 + 6\lambda_1 Q_2 + 4\lambda_0 Q_3 + 10\lambda_{32}), \\
&\text{with boundary conditions} \\
&\qquad L_3(0) = L_3'(0) = L_3''(0) = 0, \\
&L_3 \sim -\lambda_1 \alpha (1-\sigma)(z+a_1)^2 - (\alpha^{-1}\lambda_{22} + \alpha^{-1}\lambda_0^2 (1-\sigma))\ (z+a_1)\log \\
&(z+a_1) + (a_{42}\alpha - 2\alpha a_2{}' \lambda_0 (1-\sigma))(z+a_1) - \alpha^{-1}\lambda_0 a_{32} (\log \\
&(z+a_1) + 1) + a_2{}' a_{32}\alpha + 4\alpha^{-1}\lambda_0\lambda_1(1-\sigma) + \alpha^{-1}\lambda_{32} + \dots ,
\end{aligned}\right\} (2.104)$$

as $z \to \infty$,

and

$$\left.\begin{aligned}
&T_3''' + {}^4/_3 F_0 T_3'' - {}^{10}/_3 F_0' T_3' + {}^{10}/_3 F_0'' T_3 = {}^1/_3 (2F_0'^2 \\
&- 7F_0 F_0'' + 4\lambda_0 + 10\lambda_{33}), \\
&\text{with boundary conditions} \\
&\qquad T_3(0) = T_3'(0) = T_3''(0) = 0, \\
&T_3 \sim -\frac{1}{5!}\alpha^2 (z+a_1)^5 - \frac{\lambda_0}{12}(z+a_1)^3 - \frac{\lambda_0^2}{4}\alpha^{-2}(z+a_1)\log(z+a_1) \\
&+ a_{43}\alpha(z+a_1) + \alpha^{-1}\lambda_{33} - {}^3/_{20}\alpha^{-1}\lambda_0 + \dots , \text{ as } z \to \infty .
\end{aligned}\right\} (2.105)$$

The numerical solution of these equations yields

$$\lambda_{31} = \left\{\begin{array}{ll} -0.0790458 & \text{if } \sigma = 1 \\ -0.0830081 & \text{if } \sigma = 0.72 \end{array}\right\} \qquad (2.106a)$$

$$\left.\begin{aligned}
a_{41} &= \begin{cases} 0.458 & \text{if } \sigma = 1 \\ 0.468 & \text{if } \sigma = 0.72 \end{cases} \\
\lambda_{32} &= \begin{cases} 0 & \text{if } \sigma = 1 \\ 0.0119208 & \text{if } \sigma = 0.72 \end{cases} \\
a_{42} &= \begin{cases} 0 & \text{if } \sigma = 1 \\ 0.06047 & \text{if } \sigma = 0.72 \end{cases} \\
\lambda_{33} &= -0.0394201 \quad \text{for all } \sigma \\
a_{43} &= 0.3254475 \quad \text{for all } \sigma .
\end{aligned}\right\} \quad (2.106b)$$

Using the values of K_2, K_3, K_4 and K_5 given by (2.82), (2.89), (2.97) and (2.106), the external velocity distribution is given by

$$\tfrac{1}{2}\left(1 - \frac{u_1^2}{u_0^2}\right)(1+S_w) = 0.118441628\,\xi^2 - 0.1067636\, B_1\xi^3$$

$$+ 0.0929310\, B_1^2\xi^4 + (-0.0394201 - 0.0790458\, B_1^3)\xi^5 + \dots ,$$

$$\text{if } \sigma = 1, \qquad (2.107)$$

and $\tfrac{1}{2}\left(1 - \frac{u_1^2}{u_0^2}\right)(1+S_w) = 0.118441628\,\xi^2 - 0.1087672$

$B_1\xi^3 + (0.0962037\, B_1^2 - 0.0071220\, B_2)\xi^4 + (-0.0394201 -$

$0.0830081\, B_1^3 + 0.0119208\, B_1B_2)\xi^5 + \dots$, if $\sigma = 0.72$. (2.108)

Likewise, using the values of $S_1'(0)$, $S_2'(0)$, and $S_3'(0)$ given by (2.86), (2.93) and (2.101), the heat transfer is given by

$$\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}} \left(\frac{\partial T}{\partial y}\right)_w \frac{1}{T_w} = 0.5751939\ B_1 - 0.1937759\ B_1^{\,2}\xi$$

$$+\ 0.1471404\ B_1^{\,3}\xi^2 + \ldots, \text{ if } \sigma = 1, \qquad (2.109)$$

and $$\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}} \left(\frac{\partial T}{\partial y}\right)_w \frac{1}{T_w} = 0.5953621\ B_1 + (-0.1968201\ B_1^{\,2}$$

$$+\ 0.0143476\ B_2)\xi + (0.1523021\ B_1^{\,3} - 0.0116558\ B_1 B_2)\xi^2$$

$$+ \ldots, \text{ if } \sigma = 0.72. \qquad (2.110)$$

It is noticeable how little each of these coefficients changes with σ.

2.4. Calculation of Displacement Thickness and Momentum Thickness

The quantities $\delta_1{}^*$ and $\delta_2{}^*$ are defined by

$$\delta_1{}^* = \int_0^\infty \left(1 - \frac{u}{u_1} + S\right) dy, \qquad (2.111)$$

and

$$\delta_2{}^* = \int_0^\infty \frac{u}{u_1}\left(1 - \frac{u}{u_1}\right) dy. \qquad (2.112)$$

Using (2.8) and (2.10),

$$\left(\frac{u_0}{2\nu_0 x}\right)^{\frac{1}{2}} \delta_1{}^* = \int_0^\infty \left\{1 + S - F_\eta (1 - 2K(\xi))^{-\frac{1}{2}}\right\} d\eta$$

$$= \lim_{\eta\to\infty}\left\{\eta - F(1 - 2K(\xi))^{-\frac{1}{2}}\right\} + \int_0^\infty S\ d\eta. \qquad (2.113)$$

To evaluate the above integral it is necessary to deal separately with the inner and outer regions and to write

$$\int_0^\infty S\,d\eta = \xi\int_0^{\epsilon\xi^{-1}} S\,dz + \int_\epsilon^\infty S\,d\eta. \qquad (2.114)$$

$$\text{Now} \int_0^z S\,dz = S_w\,z + (1+S_w)\int_0^z (\xi B_1 M_1(z) + \xi^2(B_1{}^2 O_2(z)$$

$$+ B_2 Q_2(z)) + \xi^3(B_1{}^3 O_3(z) + B_1 B_2 Q_3(z)) + \dots)dz. \quad (2.115)$$

From (2.85), using the equation to derive an extra term in the expansion,

$$\int_0^z M_1(z)dz = {}^1/_2(z+a_1)^2 + \lambda_0\alpha^{-2}\log(z+a_1) + \beta_1 + {}^1/_4\lambda_0{}^2\alpha^{-4}$$

$$(z+a_1)^{-2} - {}^1/_5\sigma^{-1}\lambda_0\alpha^{-3}(z+a_1)^{-3} + \dots \quad . \qquad (2.116)$$

$$\text{From (2.91)} \int_0^z O_2(z)dz = \lambda_0\alpha^{-2}(-\log(z+a_1) + \alpha^2 a_2{}'\lambda_0{}^{-1} + 1)$$

$$(z+a_1) + \lambda_1\alpha^{-2}\log(z+a_1) + \beta_2 + \lambda_0{}^2\alpha^{-4}(-\log(z+a_1)$$
$$+ \alpha^2 a_2{}'\lambda_0{}^{-1} - {}^3/_2)(z+a_1)^{-1} + \lambda_0\alpha^{-3}({}^1/_2\lambda_1\alpha^{-1} - {}^1/_{20}$$
$$(3\sigma^{-1} - 1))(z+a_1)^{-2} + \dots \quad . \qquad (2.117)$$

$$\text{From (2.92)} \int_0^z Q_2(z)dz = (1-\sigma)\alpha^2\,({}^1/_3(z+a_1)^3 + 2\lambda_0\alpha^{-2}$$

$$(z+a_1)) + \beta_3 + \dots \quad . \qquad (2.118)$$

From (2.99)
$$\int_0^z O_3(z)dz = \lambda_1\alpha^{-2}(-(\log(z+a_1) - a_{31}\alpha^2\lambda_1^{-1})$$
$$+ 1)(z+a_1) + {}^1/_2\lambda_0^2\alpha^{-4}(\log(z+a_1))^2 + (\lambda_{21}\alpha^{-2} - a_2'\lambda_0\alpha^{-2}$$
$$+ {}^3/_2\lambda_0^2\alpha^{-4})\log(z+a_1) + \beta_4 + \lambda_0\lambda_1\alpha^{-4}(-(\log(z+a_1) - a_2'$$
$$\alpha^2\lambda_0^{-1}) - (\log(z+a_1) - a_{31}\alpha^2\lambda_1^{-1}) - 3)(z+a_1)^{-1} + \dots \qquad (2.119)$$

From (2.100)
$$\int_0^z Q_3(z)dz = -(1-\sigma)\lambda_0(z+a_1)^2(\log(z+a_1)$$
$$- a_2'\alpha^2\lambda_0^{-1}) + (a_{32} + 2\lambda_1(1-\sigma))(z+a_1) + (4\lambda_0^2\alpha^{-2}(1-\sigma)$$
$$+ \lambda_{22}\alpha^{-2})\log(z+a_1) + \beta_5 + \lambda_0\alpha^{-1}({}^2/_{25}(1-\sigma) + a_{32}\alpha^{-1})$$
$$(z+a_1)^{-1} + \dots \qquad (2.120)$$

From the numerical solutions,

$$\text{if } \sigma = 1,\ \beta_1 = 1.5393398,\ \beta_2 = 0.058227,\ \beta_3 = 0,\ \beta_4 = 0.29,$$
$$\beta_5 = 0, \qquad (2.121)$$
$$\text{if } \sigma = 0.72,\ \beta_1 = 1.6573493,\ \beta_2 = 0.12971,\ \beta_3 = -0.008264,$$
$$\beta_4 = 0.26,\ \beta_5 = 0.0730. \qquad (2.122)$$

Upon making the appropriate substitutions and after considerable further numerical integration and algebra the integrals in (2.114) are evaluated and (2.113) finally yields

$$\left(\frac{u_0}{2\nu_0 x_0}\right)^{\frac{1}{2}} (1+S_w)^{-1}\delta_1^* = 1.216781 + 1.606286\xi - 1.074183$$

$$B_1\xi^2\log\xi + (-0.289701 + 1.92998(7)\ B_1 + 0.653522\ B_1^{\ 2})\xi^2 + 0.968272\ B_1^{\ 2}\xi^3\log\xi + (0.608390 + 0.261137\ B_1 + 0.29165\ B_1^{\ 2} - 0.58909\ B_1^{\ 3})\xi^3 + (-0.033866 - 1.275520\ B_1^{\ 3})\xi^4\log\xi + (0.460774 - 0.0587\ B_1 - 0.927\ B_1^{\ 2} - 0.58\ B_1^{\ 3} + 0.5127\ B_1^{\ 4})\xi^4 + \ldots, \text{ if } \sigma = 1, \qquad (2.123)$$

$$\left(\frac{u_0}{2\nu_0 x_0}\right)^{\frac{1}{2}} (1+S_w)^{-1}\delta_1^*, = 1.216781 - 0.368143\ B_1 + 0.067902\ B_2 + 1.606286\xi - 1.074183\ B_1\xi^2\log\xi + (-0.289701 + 1.98299(5)\ B_1 + 0.123069\ B_2 + 0.466483\ B_1^{\ 2} + 0.086764\ B_1B_2 + 0.005036\ B_2^{\ 2})\xi^2 + 0.986445\ B_1^{\ 2}\xi^3\log\xi + (0.608390 + 0.081966\ B_1 + 0.092131\ B_2 + 0.42427\ B_1^{\ 2} - 0.113016\ B_1B_2 - 0.428380\ B_1^{\ 3} - 0.079677\ B_1^{\ 2}B_2 - 0.004624\ B_1B_2^{\ 2})\xi^3 + (-0.033866 - 0.095716\ B_1B_2 - 1.305201\ B_1^{\ 3})\xi^4\log\xi + (0.460774 - 0.0646\ B_1 - 0.017177\ B_2 - 1.008\ B_1^{\ 2} + 0.118\ B_1B_2 + 0.002960\ B_2^{\ 2} - 0.79\ B_1^{\ 3} + 0.21081\ B_1^{\ 2}B_2 - 0.066505\ B_1B_2^{\ 2} - 0.000007\ B_2^{\ 3} + 0.378899\ B_1^{\ 4} + 0.070474\ B_1^{\ 3}B_2 + 0.004090\ B_1^{\ 2}\ B_2^{\ 2})\xi^4 + \ldots, \text{ if } \sigma = 0.72. \qquad (2.124)$$

Next the momentum thickness, δ_2^*, is calculated, using the momentum integral equation which in the transformed plane takes the form (Curle and Davies 1971, p. 278)

$$\frac{d}{dx}\,(u_1^{\ 2}\delta_2^*) = \nu_0\left(\frac{\partial u}{\partial y}\right)_w - u_1\,\frac{du_1}{dx}\,\delta_1^*.$$

Appropriate substitutions for $u_1\frac{du_1}{dx}$ and $\delta_1{*}$ are made from (2.8), (2.123) and (2.124), the variable is changed from x to ξ using (2.9), and after more algebra it emerges that

$$\left(\frac{u_0}{2\nu_0 x_0}\right)^{\frac{1}{2}} \delta_2{*} = 0.469600 + (0.255358 + 0.236883\ B_1)\xi^2 + (0.126834 - 0.230180\ B_1 - 0.213527\ B_1^2)\xi^3 - 0.063614\ B_1\xi^4\log\xi + (0.043334 + 0.186505\ B_1 + 0.358552\ B_1^2 + 0.185862\ B_1^3)\xi^4 + \ldots, \text{ if } \sigma = 1, \qquad (2.125)$$

$$\left(\frac{u_0}{2\nu_0 x_0}\right)^{\frac{1}{2}} \delta_2{*} = 0.469600 + (0.255358 + 0.222464\ B_1 + 0.024983\ B_2)\xi^2 + (0.126834 - 0.234500\ B_1 - 0.204293\ B_1^2 - 0.022942\ B_1B_2)\xi^3 - 0.063614\ B_1\xi^4\log\xi + (0.043334 + 0.199684 B_1 + 0.007063\ B_2 + 0.361083\ B_1^2 + 0.013942\ B_1B_2 - 0.000303\ B_2^2 + 0.180695\ B_1^3 + 0.020292\ B_1^2\ B_2)\xi^4 + \ldots$$

$$\text{if } \sigma = 0.72. \qquad (2.126)$$

Finally the non-dimensional shape factor $H{*} = \delta_1{*}/\delta_2{*}$ is calculated:

$$H{*}(1+S_w)^{-1}\lambda^{-1} = 1 + 1.320112\xi - 0.882807\ B_1\xi^2\log\xi + (-0.781865 + 1.08170(6)\ B_1 + 0.537091\ B_1^2)\xi^2 + 0.795765\ B_1^2\xi^3\log\xi + (-0.487937 + 0.038864\ B_1 + 0.69439\ B_1^2 - 0.484136\ B_1^3)\xi^3 + (-0.027833 + 0.615515\ B_1 + 0.445320\ B_1^2 - 1.048274\ B_1^3)\xi^4\log\xi + (0.355017 + 0.0078((5))\ B_1$$

$$- 1.763\, B_1^{\,2} - 1.14\, B_1^{\,3} + 0.4214\, B_1^{\,4})\xi^4 + \ldots, \text{ if } \sigma = 1, \qquad (2.127)$$

and $H^*(1+S_w)^{-1}\lambda^{-1} = 1 - 0.302555\, B_1 + 0.055805\, B_2$

$$+ 1.320112\xi - 0.882807\, B_1\xi^2\log\xi + (-0.781865 + 1.32049(8)\, B_1$$

$$+ 0.017597\, B_2 + 0.526704\, B_1^{\,2} + 0.060966\, B_1B_2 + 0.001170\, B_2^{\,2})\,\xi^2$$

$$+ 0.810701\, B_1^{\,2}\xi^3\log\xi + (-0.487937 + 0.023065\, B_1 - 0.009586\, B_2$$

$$+ 0.63263\, B_1^{\,2} - 0.016160\, B_1B_2 - 0.483682\, B_1^{\,3} - 0.055986\, B_1^{\,2}B_2$$

$$- 0.001074\, B_1B_2^{\,2})\xi^3 + (-0.027833 + 0.615515\, B_1 + 0.377227\, B_1^{\,2}$$

$$- 0.024138\, B_1B_2 - 1.072668\, B_1^{\,3})\xi^4\log\xi + (0.355017$$

$$- 0.1388((5))\, B_1 - 0.002281\, B_2 - 1.806\, B_1^{\,2} + 0.000(9)\, B_1B_2$$

$$+ 0.000666\, B_2^{\,2} - 1.05\, B_1^{\,3} + 0.03921\, B_1^{\,2}B_2 - 0.060306\, B_1B_2^{\,2}$$

$$- 0.000032\, B_2^{\,3} + 0.427813\, B_1^{\,4} + 0.049520\, B_1^{\,3}B_2 + 0.000950$$

$$B_1^{\,2}B_2^{\,2})\xi^4 + \ldots, \text{ if } \sigma = 0.72, \qquad (2.128)$$

where $\lambda = 2.591100$.

2.5. Solution for $\xi = O(1)$

Immediately downstream of $x=x_0$, where the boundary layer reacts to the sharp adverse pressure gradient, there is an inner sublayer whose thickness is $O(\xi)$. As ξ increases the inner layer thickens and for larger ξ the concept of an inner sub-layer becomes blurred and the velocity tends with increasing ξ to that given by the separating profile derived by Falkner and Skan (1930) for the incompressible case, by

Cohen and Reshotko (1956a) for the compressible case where $\sigma = 1$, and in this paper (Section 2.6) for the compressible case where $\sigma \neq 1$.

To improve the convergence of the series for the pressure coefficient (2.107) and (2.108) and for the heat transfer rate (2.109) and (2.110) it is necessary first to derive more terms in the series. This is easily done in the incompressible case and in the compressible case is possible if B_1 and B_2 are not too large. From examination of the inner solution it is clear that

$$\begin{aligned} K(\xi)(1+S_w) = {} & P_2\xi^2 + P_5\xi^5 + P_8\xi^8 + \dots \\ & + B_1(P_3\xi^3 + P_6\xi^6 + \dots) \\ & + B_1^2(P_4\xi^4 + P_7\xi^7 + \dots) \\ & + B_2(R_4\xi^4 + R_7\xi^7 + \dots) \\ & + \dots \end{aligned} \tag{2.129}$$

$$\begin{aligned} F = {} & \xi^2F_0 + \xi^5F_3 + \xi^8F_6 + \dots \\ & + B_1(\xi^3J_1 + \xi^6J_4 + \xi^9J_7 + \dots) \\ & + B_1^2(\xi^4J_2 + \xi^7J_5 + \xi^{10}J_8 + \dots) \\ & + B_2(\xi^4L_2 + \xi^7L_5 + \xi^{10}L_8 + \dots) \\ & + \dots \end{aligned} \tag{2.130}$$

$$\begin{aligned} S = S_w + (1+S_w)\Big\{ & B_1(\xi M_1 + \xi^4M_4 + \xi^7M_7 + \dots) \\ & + B_1^2(\xi^2O_2 + \xi^5O_5 + \xi^8O_8 + \dots) \\ & + B_2(\xi^2Q_2 + \xi^5Q_5 + \xi^8Q_8 + \dots) \\ & + \dots\Big\} . \end{aligned} \tag{2.131}$$

From (2.68) and (2.69) the equations which these functions satisfy can be derived:

$$F_{3n}''' = {}^1/_3\Big\{(6n+4)P_{3n+2} + (6n-2)P_{3n-1}\,\pi_{n0} + \sum_{r=0}^{n}\Big\{F_{3r}'\big((3n+2)$$

$$F_{3(n-r)}' + (3n-1)F_{3(n-r-1)}'\,\pi_{rn}\big) - F_{3r}\Big((6r+4)F_{3(n-r)}''$$

$$+ (6r+7)F_{3(n-r-1)}''\,\pi_{rn}\Big)\Big\}\Big\}, \qquad (2.132)$$

where $\pi_{rn} = 1$, if $r \neq n$, and $= 0$, if $r = n$. (2.133)

$$M_{3n+1}'' + \sigma\pi_{n0}\sum_{r=0}^{n-1} F_{3r}M'_{3(n-r)-2} = {}^2/_3\sigma\sum_{r=0}^{n}\Big\{(3r+1)M_{3r+1}\Big(F'_{3(n-r)}$$

$$+ F'_{3(n-r-1)}\pi_{rn}\Big) - (3r+2)F_{3r}\Big(M'_{3(n-r)+1} + M'_{3(n-r)-2}\pi_{rn}\Big)\Big\},$$

(2.134)

$$Q''_{3n+2} + \sigma\pi_{n0}\sum_{r=0}^{n-1} F_{3r}Q'_{3(n-r)-1} = {}^2/_3\sigma\sum_{r=0}^{n}(3r+2)\Big\{Q_{3r+2}$$

$$\Big(F'_{3(n-r)} + F'_{3(n-r-1)}\pi_{rn}\Big) - F_{3r}\Big(Q'_{3(n-r)+2} + Q'_{3(n-r)-1}\pi_{rn}\Big)\Big\}$$

$$+ 2(1-\sigma)\sum_{r=0}^{n}\Big(F_{3r}'F'''_{3(n-r)} + F_{3r}''F''_{3(n-r)}\Big). \qquad (2.135)$$

$${}^3/_2\Big(J'''_{3n+1} + \pi_{n0}\sum_{r=0}^{n-1}\Big(F_{3r}J''_{3(n-r)-2} + J_{3r+1}F''_{3(n-r-1)}\Big)\Big)$$

$$= 3(n+1)P_{3(n+1)} + 3nP_{3n}\pi_{n0} + \sum_{r=0}^{n}\Big\{F'_{3r}\Big(3(n+1)J'_{3(n-r)+1}$$

$$+ 3nJ'_{3(n-r)-2}\pi_{rn}\Big) - (3r+2)F_{3r}\Big(J''_{3(n-r)+1} + J''_{3(n-r)-2}\pi_{rn}\Big)$$

$$- 3(r+1)J_{3r+1}\Big(F''_{3(n-r)} + F''_{3(n-r-1)}\pi_{rn}\Big) + M_{3r+1}\Big((3(n-r)+2)$$

$$P_{3(n-r)+2} + (3(n-r)-1)\,P_{3(n-r)-1}\pi_{rn}\Big)\Big\}. \qquad (2.136)$$

$$3/_2\Big(L'''_{3n+2} + \pi_{n0}\sum_{r=0}^{n-1}\Big(F_{3r}L''_{3(n-r)-1} + L_{3r+2}F''_{3(n-r)}\Big)\Big)$$

$$= (3n+4)R_{3n+4} + (3n+1)R_{3n+1}\pi_{n0} + \sum_{r=0}^{n}\Big\{F'_{3r}\Big((3n+4)L'_{3(n-r)+2}$$

$$+ (3n+1)L'_{3(n-r)-1}\pi_{rn}\Big) - (3r+2)F_{3r}\Big(L''_{3(n-r)+2} + L''_{3(n-r)-1}\pi_{rn}\Big)$$

$$- (3r+4)L_{3r+2}\Big(F''_{3(n-r)} + F''_{3(n-r-1)}\pi_{rn}\Big) + Q_{3r+2}\Big((3(n-r)+2)P_{3(n-r)+2}$$

$$+ (3(n-r)-1)P_{3(n-r)-1}\pi_{rn}\Big)\Big\}. \qquad (2.137)$$

$$O''_{3n+2} + \sigma\pi_{n0}\sum_{r=0}^{n-1}\Big(F_{3r}O'_{3(n-r)-1} + J_{3r+1}M'_{3(n-r)-2}\Big)$$

$$= {}^{2}/_{3}\sigma\sum_{r=0}^{n}\Big\{(3r+1)M_{3r+1}\Big(J'_{3(n-r)+1} + J'_{3(n-r)-2}\pi_{rn}\Big) + (3r+2)$$

$$\Big(O_{3r+2}\Big(F'_{3(n-r)} + F'_{3(n-r-1)}\pi_{rn}\Big) - F_{3r}\Big(O'_{3(n-r)+2} + O'_{3(n-r)-1}\pi_{rn}\Big)\Big)$$

$$- 3(r+1)J_{3r+1}\Big(M'_{3(n-r)+1} + M'_{3(n-r)-2}\pi_{rn}\Big)\Big\}. \qquad (2.138)$$

$$
{}^3/_2\left(J'''_{3n+2} + \pi_{n0}\sum_{r=0}^{n-1}\left(F_{3r}J''_{3(n-r)-1} + J_{3r+1}J''_{3(n-r)-2} + J_{3r+2}F''_{3(n-r-1)}\right)\right) = (3n+4)P_{3n+4} + (3n+1)P_{3n+1}\pi_{n0}
$$

$$
+ \sum_{r=0}^{n}\Big\{F'_{3r}\left((3n+4)J'_{3(n-r)+2} + (3n+1)J'_{3(n-r)-1}\pi_{rn}\right) + {}^1/_2 J'_{3r+1}\left((3n+4)J'_{3(n-r)+1} + (3n+1)J'_{3(n-r)-2}\pi_{rn}\right) - (3r+2)F_{3r}\left(J''_{3(n-r)+2}\right.
$$

$$
\left.+J''_{3(n-r)-1}\pi_{rn}\right) - 3(r+1)J_{3r+1}\left(J''_{3(n-r)+1} + J''_{3(n-r)-2}\pi_{rn}\right)
$$

$$
- (3r+4)J_{3r+2}\left(F''_{3(n-r)} + F''_{3(n-r-1)}\pi_{rn}\right) + M_{3r+1}
$$

$$
\left(3(n-r+1)P_{3(n-r+1)} + 3(n-r)P_{3(n-r)}\pi_{rn}\right) + O_{3r+2}\left((3(n-r)+2)\right.
$$

$$
\left.P_{3(n-r)+2} + (3(n-r)-1)\,P_{3(n-r)-1}\pi_{rn}\right)\Big\}. \tag{2.139}
$$

The inner boundary conditions on these equations are

$$
\left.\begin{aligned}
&F_{3n}(0) = F_{3n}'(0) = F_{3n}''(0) = 0,\\
&J_{3n+1}(0) = J'_{3n+1}(0) = J''_{3n+1}(0) = 0,\\
&J_{3n+2}(0) = J'_{3n+2}(0) = J''_{3n+2}(0) = 0,\\
&L_{3n+2}(0) = L'_{3n+2}(0) = L''_{3n+2}(0) = 0,\\
&\text{and}\\
&M_{3n+1}(0) = O_{3n+2}(0) = Q_{3n+2}(0) = 0.
\end{aligned}\right\} \tag{2.140}
$$

The outer boundary conditions on the equations are derived from the series expansions in powers of η, of f_0, s_0, ϕ_1, and $\frac{\phi_1 S_0'}{f_0'}$:

$$f_0(\eta) = \alpha\eta^2 \sum_{n=0}^{\infty} a_n \alpha^n \eta^{3n},$$

where $a_0 = {}^1/_2$, $a_1 = -{}^1/_{5!}$, $a_2 = \frac{11}{8!}$, $a_3 = \frac{-375}{11!}$, $a_4 = \frac{27897}{14!}$,

$a_5 = \frac{-3817137}{17!}$, $a_6 = \frac{865874115}{20!}$, $a_7 = \frac{-303083960103}{23!}$,

$$a_8 = \frac{155172279780289}{26!},\ a_9 = \frac{-111431991012221729}{29!},\ \ldots \quad (2.141)$$

$$S_0(\eta) = S_w + \theta\eta \sum_{n=0}^{\infty} b_n \alpha^n \eta^{3n} + \beta(1-\sigma)\eta^2\alpha^2 \sum_{n=0}^{\infty} c_n \alpha^n \eta^{3n},$$

where $b_0 = 1$, $b_1 = \frac{-\sigma}{4!}$, $b_2 = \frac{\sigma(10\sigma+1)}{7!}$, $b_3 = \frac{-\sigma(280\sigma^2+84\sigma+11)}{10!}$, ...

and $c_0 = 1$, $c_1 = \frac{-1}{20}(\sigma + {}^5/_3)$, $c_2 = \frac{90\sigma^2+162\sigma+246}{8!}$,

$$c_3 = \frac{-(3240\sigma^3+6588\sigma^2+10314\sigma+15510)}{11!},\ \ldots . \quad (2.142)$$

$$\phi_1(1+S_w)^{-1} = \alpha^{-1} \sum_{n=0}^{\infty} d_n \alpha^n \eta^{3n} + B_1\alpha^{-1}\eta \sum_{n=0}^{\infty} \alpha^n \eta^{3n}((3n+2)$$

$$a_n \log\eta + e_n) + B_2(1-\sigma)\alpha\eta^2 \sum_{n=0}^{\infty} \alpha^n \eta^{3n} f_n,$$

where $d_0 = -1$, $d_1 = \frac{1}{12}$, $d_2 = \frac{-27}{10.6!}$, $d_3 = \frac{657}{10!}$, $d_4 = \frac{-142083}{440.11!}$,

$d_5 = \frac{3230037}{110.14!}$, ... , $e_0 = 0$, $e_1 = \frac{-(\sigma-2)}{72}$, $e_2 = \frac{40\sigma^2+4\sigma-123}{3.8!}$,

$$e_3 = \frac{-(1120\sigma^3-264\sigma^2+1304\sigma-6435)}{36.10\,!}, \ \ldots\ , f_0 = 1,\ f_1 = -\frac{1}{4!}\left(1+\frac{3\sigma}{10}\right),$$

$$f_2 = \frac{90\sigma^2+141\sigma+616}{7.8!},\quad f_3 = \frac{-(22680\sigma^3+31266\sigma^2+64179\sigma+288750)}{70.11!}, \ \ldots$$

(2.143)

$$\frac{\phi_1 S_0'}{f_0'}(1+S_w)^{-2} = B_1\alpha^{-2}\eta^{-1}\sum_{n=0}^{\infty} g_n\alpha^n\eta^{3n} + B_2(1-\sigma)\sum_{n=0}^{\infty}$$

$$h_n\alpha^n\eta^{3n} + B_1^2\alpha^{-2}\sum_{n=0}^{\infty}\alpha^n\eta^{3n}((3n+1)b_n\log\eta + i_n) + B_1B_2(1-\sigma)\eta$$

$$\sum_{n=0}^{\infty}\alpha^n\eta^{3n}((3n+2)\ c_n\log\eta + j_n) + B_2^2(1-\sigma)^2\alpha^2\eta^2\sum_{n=0}^{\infty}\alpha^n\eta^{3n}k_n,$$

where $g_0 = -1,\ g_1 = \frac{4\sigma+1}{4!},\ g_2 = \frac{-(2800\sigma^2+1680\sigma-34)}{5.8!}$,

$$g_3 = \frac{(5600\sigma^3+5880\sigma^2+436\sigma-45)}{2.10!}, \ \ldots\ , h_0 = -2,\ h_1 = \frac{\sigma+2}{4},$$

$$h_2 = \frac{-(900\sigma^2+2145\sigma+3318)}{10.7!},\ h_3 = \frac{3240\sigma^3+9288\sigma^2+15021\sigma+22590}{10!}, \ \ldots$$

$$i_0 = 0,\ i_1 = \frac{-(\sigma-2)}{72}, \ \ldots\ , j_0 = 1,\ j_1 = \frac{-(149\sigma-40)}{720}, \ \ldots$$

$$k_0 = 2,\ k_1 = \frac{-(33\sigma+50)}{5!}, \ \ldots \qquad (2.144)$$

From these the outer boundary conditions are

$$F_{3n} \sim \alpha^{n+1}a_n(z+a_1)^{3n+2} - \lambda_0\alpha^{n-1}d_n(z+a_1)^{3n} + \ldots \qquad (2.145)$$

$$M_{3n+1} \sim \alpha^n b_n(z+a_1)^{3n+1} - \lambda_0\alpha^{n-2}g_n(z+a_1)^{3n-1} + \ldots \qquad (2.146)$$

$$Q_{3n+2} \sim \alpha^{n+2}c_n(1-\sigma)(z+a_1)^{3n+2} - \lambda_0\alpha^n h_n(1-\sigma)(z+a_1)^{3n} + \ldots$$

(2.147)

$$J_{3n+1} \sim -\lambda_0 \alpha^{n-1}((3n+2)(\log(z+a_1) - a_2'\alpha^2\lambda_0^{-1})a_n + e_n)$$

$$(z+a_1)^{3n+1} - \lambda_1 \alpha^{n-1} d_n (z+a_1)^{3n} + \ldots \qquad (2.148)$$

$$L_{3n+2} \sim -\lambda_0 \alpha^{n+1} f_n (1-\sigma)(z+a_1)^{3n+2} + a_{32}\alpha^{n+1}(3n+2)a_n$$

$$(z+a_1)^{3n+1} + \ldots \qquad (2.149)$$

$$O_{3n+2} \sim -\lambda_0 \alpha^{n-2}((3n+1)(\log(z+a_1) - a_2'\alpha^2\lambda_0^{-1})b_n + i_n)$$

$$(z+a_1)^{3n} - \lambda_1 \alpha^{n-2} g_n (z+a_1)^{3n-1} + \ldots \qquad (2.150)$$

$$J_{3n+2} \sim -\lambda_1 \alpha^{n-1}((3n+2)(\log(z+a_1) - a_{31}\alpha^2\lambda_1^{-1})a_n + e_n)$$

$$(z+a_1)^{3n+1} + \ldots \qquad (2.151)$$

Solving these sets of equations gives

$$P(\xi) = P_2\xi^2 + P_5\xi^5 + P_8\xi^8 + \ldots = \xi^2(0.118441628 - 0.03942014\,\xi^3$$
$$+ 0.02253761\,\xi^6 - 0.0154805\,\xi^9 + 0.0116713\,\xi^{12}$$
$$- 0.0093072(0)\,\xi^{15} + 0.0077042\,\xi^{18} - 0.006545\,\xi^{21}$$
$$+ 0.005683\,\xi^{24} - 0.005005\,\xi^{27} + \ldots), \qquad (2.152)$$

$$K(\xi)(1+S_w) = P(\xi)$$
$$+ B_1\xi^3(-0.1067636 + 0.0552033\,\xi^3 - 0.0372567\,\xi^6 + 0.02816(6)\,\xi^9$$
$$- \ldots)$$
$$+ B_1^2\xi^4(0.0929310 - 0.065856\,\xi^3 + \ldots)$$
$$+ B_1^3\xi^5(-0.0790458 + \ldots)$$
$$+ \ldots, \qquad \text{if } \sigma = 1, \qquad (2.153)$$

$$K(\xi)(1+S_w) = P(\xi)$$

$$+ B_1\xi^3(-\ 0.1087672 + 0.0546505\,\xi^3 - 0.0362980\,\xi^6$$

$$+ 0.02714(2)\,\xi^9 - \ldots)$$

$$+ B_2\xi^4(-\ 0.0071220 + 0.0059420\,\xi^3 - 0.005161(4)\,\xi^6$$

$$+ 0.0046(4)\,\xi^9 - \ldots)$$

$$+ B_1{}^2\xi^4(0.0962037 - 0.065235\,\xi^3 + \ldots)$$

$$+ B_1B_2\xi^5(0.0119208 - \ldots)$$

$$+ B_1{}^3\xi^5(-\ 0.0830081 + \ldots)$$

$$+ \ldots\ ,\ \text{if } \sigma = 0.72\ . \qquad (2.154)$$

Also $$\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}}\left(\frac{\partial T}{\partial y}\right)_w \frac{1}{T_w} = B_1(0.5751939 + 0.0027410\,\xi^3$$

$$- 0.0055113\,\xi^6$$

$$+ 0.0047(8)\,\xi^9 - \ldots)$$

$$+ B_1{}^2\xi(-\ 0.1937759 + 0.05280\,\xi^3 - \ldots)$$

$$+ B_1{}^3\xi^2(0.1471404 - \ldots)$$

$$+ \ldots\ ,\ \text{if } \sigma = 1. \qquad (2.155)$$

$$\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}}\left(\frac{\partial T}{\partial y}\right)_w \frac{1}{T_w} = B_1(0.5953621 + 0.0004433\,\xi^3 - 0.0039735\,\xi^6$$

$$+ 0.0036(2)\,\xi^9 - \ldots)$$

$$+ B_2\xi(0.0143476 - 0.0019877\,\xi^3 + 0.0009128\,\xi^6$$

$$- 0.000621\,\xi^9 + \ldots)$$

$$+ B_1{}^2\xi(-\ 0.1968201 + 0.04357\,\xi^3 - \ldots)$$

$$+ B_1B_2\xi^2(-0.0116558 + \ldots)$$

$$+ B_1^3\xi^2(0.1523021 + \ldots)$$

$$+ \ldots, \text{ if } \sigma = 0.72. \qquad (2.156)$$

It is interesting to note that if B_1 is small the expression in (2.155) and (2.156) varies very slowly with ξ. This result is similar to that obtained by Stratford (1954) and Curle (1977) who showed that, in the incompressible case, F_p varies very slowly with ξ, where

$$F_p = \left(10x\frac{dC_p}{dx}\right)^2 \quad C_p = 0.59077\left\{1 - 0.00431\xi^3 - 0.01639\xi^6 - \ldots\right\} \text{ and } C_p = {}^1/_2\left(1 - \frac{u_1^2}{u_0^2}\right).$$

The series for the pressure coefficient $P(\xi)$ clearly has a singularity at $\xi = -1$, corresponding to $x=0$, the leading edge of the flat plate. This can be confirmed by considering ratios of successive coefficients of the series and extrapolating to the limit using Neville tables in the manner suggested by Gaunt and Guttman (1974, p.187-194).

The Euler transformation is therefore made:

$$r = \frac{\xi^3}{\xi^3+1}. \qquad (2.157)$$

This gives

$$P(\xi) = P^*(r) = \left(\frac{r}{1-r}\right)^{\frac{2}{3}}(0.118441628 - 0.03942014\,r - 0.01688253\,r^2 - 0.0098250\,r^3 - 0.0065774\,r^4 - 0.0047746\,r^5 - 0.003655((6))\,r^6 - 0.002904\,r^7 - 0.00235\,r^8 - 0.0018(5)\,r^9 - \ldots). \qquad (2.158)$$

$$K(\xi)(1+S_w) = K^*(r)(1+S_w) = P^*(r)$$

$$+ B_1\left(\frac{r}{1-r}\right)(-0.1067636 + 0.0552033\,r + 0.0179466\,r^2$$

$$+ 0.00885(6)\,r^3 + \dots)$$

$$+ B_1^2\left(\frac{r}{1-r}\right)^{4/3}(0.0929310 - 0.065856\,r - \dots)$$

$$+ B_1^3\left(\frac{r}{1-r}\right)^{5/3}(-0.0790458 + \dots)$$

$$+ \dots, \quad \text{if } \sigma = 1. \qquad (2.159)$$

$$K^*(r)(1+S_w) = P^*(r)$$

$$+ B_1\left(\frac{r}{1-r}\right)(-0.1087672 + 0.0546505\,r + 0.0183525\,r^2 + 0.00919(7)\,r^3$$

$$+ \dots)$$

$$+ B_2\left(\frac{r}{1-r}\right)^{4/3}(-0.0071220 + 0.0059420\,r + 0.000780(6)\,r^2$$

$$+ 0.0002(6)\,r^3 + \dots)$$

$$+ B_1^2\left(\frac{r}{1-r}\right)^{4/3}(0.0962037 - 0.065235\,r - \dots)$$

$$+ B_1B_2\left(\frac{r}{1-r}\right)^{5/3}(0.0119208 - \dots)$$

$$+ B_1^3\left(\frac{r}{1-r}\right)^{5/3}(-0.0830081 + \dots)$$

$$+ \dots, \quad \text{if } \sigma = 0.72. \qquad (2.160)$$

The heat transfer rate (2.155) and (2.156) can be transformed in the same way.

It is these values of the pressure coefficient and the heat transfer rate which will be used to determine the constants in the solution for large ξ.

2.6. Solution for large ξ

Following Cohen and Reshotko (1956a) and Curle (1976b) (appendix) a solution to (2.2)-(2.4) is sought in which the external velocity

$$u_1 = \chi u_0 z^m V(z), \tag{2.161}$$

where $z = \left({}^x/_{x_0} - a\right)$ and χ and a are constants.

Making a change of variables similar to that made by Falkner and Skan, write

$$\left.\begin{aligned} \psi &= \left(\frac{\chi 2 u_0 \nu_0 x_0}{m+1}\right)^{\frac{1}{2}} F(z,\eta) z^{\frac{1}{2}(m+1)}, \\ S &= S(z,\eta), \\ \text{and} \quad \eta &= \left(\frac{\chi(m+1)u_0}{2\nu_0 x_0}\right)^{\frac{1}{2}} z^{\frac{m-1}{2}} y. \end{aligned}\right\} \tag{2.162}$$

Substituting into (2.2)-(2.4) gives

$$\frac{2z}{m+1}\left\{F_\eta F_{z\eta} - F_z F_{\eta\eta}\right\} = F_{\eta\eta\eta} + FF_{\eta\eta} - \frac{2m}{m+1} F_\eta^{\ 2} + \frac{2}{m+1}\,(zVV' + mV^2)(1+S) \tag{2.163}$$

and

$$\frac{1}{\sigma} S_{\eta\eta} + FS_\eta = \frac{2}{m+1}\, z(F_\eta S_z - F_z S_\eta) + 2\frac{(1-\sigma)}{\sigma}\chi^2 \beta z^{2m}(F_{\eta\eta}^{\ 2} + F_\eta F_{\eta\eta\eta})\left\{1 - \beta(1 - z^{2m}[V(z)]^2\chi^2)\right\}^{-1} \tag{2.164}$$

Write

$$\begin{aligned} &F(z,\eta,\sigma,S_w,\beta) \\ &= F_0(\eta,\sigma,S_w,\beta) + z^{-\lambda_1(\sigma,S_w,\beta)} F_1(\eta,\sigma,S_w,\beta) \\ &+ z^{-2\lambda_1(\sigma,S_w,\beta)} F_2(\eta,\sigma,S_w,\beta) + \dots + z^{-\lambda_2(\sigma,S_w,\beta)} F_1{}^*(\eta,\sigma,S_w,\beta) \end{aligned}$$

$$+ z^{-(\lambda_2(\sigma,S_w,\beta)+\lambda_1(\sigma,S_w,\beta))}F^{**}(\eta,\sigma,S_w,\beta) + \ldots , \quad (2.165)$$

$$S(z,\eta,\sigma,S_w,\beta)$$

$$= S_0(\eta,\sigma,S_w,\beta) + z^{-\lambda_1(\sigma,S_w,\beta)}S_1(\eta,\sigma,S_w,\beta)$$

$$+ z^{-2\lambda_1(\sigma,S_w,\beta)}S_2(\eta,\sigma,S_w,\beta) + \ldots + z^{-\lambda_2(\sigma,S_w,\beta)}S_1{}^*(\eta,\sigma,S_w,\beta)$$

$$+ z^{-(\lambda_2(\sigma,S_w,\beta)+\lambda_1(\sigma,S_w,\beta))}S^{**}(\eta,\sigma,S_w,\beta) + \ldots , \quad (2.166)$$

and

$$V(z,\sigma,S_w,\beta)$$

$$= 1 + z^{-\lambda_1(\sigma,S_w,\beta)}K_1(\sigma,S_w,\beta)$$

$$+ z^{-2\lambda_1(\sigma,S_w,\beta)}K_2(\sigma,S_w,\beta) + \ldots + z^{-\lambda_2(\sigma,S_w,\beta)}K_1{}^*(\sigma,S_w,\beta)$$

$$+ z^{-(\lambda_2(\sigma,S_w,\beta)+\lambda_1(\sigma,S_w,\beta))}K^{**}(\sigma,S_w,\beta) + \ldots , \quad (2.167)$$

where $\lambda_1 = -2m$ and λ_2 is the smallest value of λ which satisfies the appropriate eigenvalue problem, the compressible analogue of the incompressible eigenvalue problem in Curle (1976b) (appendix).

The equations satisfied by F_0 etc., together with their associated boundary conditions are readily derived.

Then it is clear that further expansion is possible:

Write $F_0(\eta,\sigma,S_w,\beta) = F_{00}(\eta,S_w) + (1-\sigma)F_{01}(\eta,S_w) + (1-\sigma)^2$

$F_{02}(\eta,S_w) + \ldots$

$F_1(\eta,\sigma,S_w,\beta) = (1-\sigma)F_{11}(\eta,S_w,\beta) + (1-\sigma)^2F_{12}(\eta,S_w,\beta) + \ldots$

$$F_2(\eta,\sigma,S_w,\beta) = (1-\sigma)F_{21}(\eta,S_w,\beta) + (1-\sigma)^2F_{22}(\eta,S_w,\beta) + \ldots$$

$$F_1{}^*(\eta,\sigma,S_w,\beta) = F_{10}{}^*(\eta,S_w) + (1-\sigma)F_{11}{}^*(\eta,S_w) + (1-\sigma)^2F_{12}{}^*(\eta,S_w) + \ldots$$

$$F^{**}(\eta,\sigma,S_w,\beta) = (1-\sigma)F_1{}^{**}(\eta,S_w,\beta) + (1-\sigma)^2F_2{}^{**}(\eta,S_w,\beta) + \ldots \quad (2.168)$$

Expand S and V similarly, and write

$$\left.\begin{aligned} \frac{2m}{m+1} &= r_0(S_w) + (1-\sigma)r_1(S_w) + (1-\sigma)^2r_2(S_w) + \ldots \\ \frac{2\lambda_2}{m+1} &= \mu_{20}(S_w) + (1-\sigma)\mu_{21}(S_w) + (1-\sigma)^2\mu_{22}(S_w) + \ldots \,. \end{aligned}\right\} (2.169)$$

If, as in section 2.5 , it is assumed that S_w and β are not large and, if $\sigma = 1$, only coefficients of 1, S_w, and $S_w{}^2$ are retained, whereas if $\sigma \neq 1$ only coefficients of 1, S_w and β are retained, then still further expansion is possible and sets of ordinary differential equations with associated boundary conditions can be derived. These derivations are straight forward, but tedious. When the equations are solved in turn, V(z) and the heat transfer rate are determined.

Given that
$$\begin{aligned} \chi(\sigma,S_w;\beta) = \chi_{00} &+ \chi_{01}S_w + \chi_{02}S_w{}^2 + \ldots \\ &+ (1-\sigma)(\chi_{110}S_w + \chi_{101}\beta + \ldots) \\ &+ (1-\sigma)^2(\chi_{210}S_w + \chi_{201}\beta + \ldots) \\ &+ \ldots \end{aligned} \quad (2.170)$$

and
$$\begin{aligned} K_1{}^*(\sigma,S_w,\beta) = K_{100} &+ K_{101}S_w + K_{102}S_w{}^2 + \ldots \\ &+ (1-\sigma)(K_{111}S_w + \ldots) \end{aligned}$$

$$+ (1-\sigma)^2(K_{121}S_w + \ldots)$$

$$+ \ldots , \qquad (2.171)$$

$$V(z) = 1 + (K_{100} + K_{101}\, S_w + K_{102}S_w^{\,2} + \ldots)z^{-\lambda_2} + \ldots$$
$$+ (1-\sigma)(z^{-\lambda_1}(-\,0.0182455\beta\chi_{00}^{\,2} + \ldots) + z^{-\lambda_2}(K_{111}S_w + \ldots)$$
$$+ z^{-(\lambda_1+\lambda_2)}(-\,1.898788\; K_{010}\chi_{00}^{\,2}\beta + \ldots) + \ldots) + (1-\sigma)^2$$
$$(z^{-\lambda_1}(-\,0.0404083\beta\chi_{00}^{\,2} + \ldots) + z^{-\lambda_2}(K_{121}S_w + \ldots) + z^{-(\lambda_1+\lambda_2)}$$
$$(-\,4.177194\,\beta K_{010}\chi_{00}^{\,2} + \ldots) \;+ \ldots) + (1-\sigma)^3(z^{-\lambda_1}(-\,0.0397190$$
$$\beta\chi_{00}^{\,2} + \ldots) + z^{-\lambda_2}(K_{131}S_w + \ldots) + z^{-(\lambda_1+\lambda_2)}(-\,7.515846$$
$$\beta K_{010}\chi_{00}^{\,2} + \ldots) + \ldots) + (1-\sigma)^4(z^{-\lambda_1}(-\,0.0368369\beta\chi_{00}^{\,2} + \ldots)$$
$$+ z^{-\lambda_2}\,(K_{141}S_w + \ldots) + z^{-(\lambda_1+\lambda_2)}(-\,10.99227\,\beta K_{010}\chi_{00}^{\,2} + \ldots)$$
$$+ \ldots) + \ldots . \qquad (2.172)$$

$$\frac{\partial S}{\partial \eta}\,(z,0) = -0.3258111\; S_w - 0.0230575\; S_w^{\,2} + \ldots$$
$$+ z^{-\lambda_2}(-\,3.0683987\;(K_{100}S_w + K_{101}S_w^{\,2} + \ldots) - 1.7400635$$
$$K_{100}S_w^{\,2} + \ldots)$$
$$+ (1-\sigma)(0.0878766\; S_w + \ldots + (-\,0.1302331\beta\chi_{00}^{\,2} + \ldots)z^{-\lambda_1}$$
$$+ (0.4558656\; K_{100}S_w + \ldots)z^{-\lambda_2} + (-\,1.54038\; K_{010}\chi_{00}^{\,2}\beta + \ldots)$$
$$z^{-(\lambda_1+\lambda_2)} + \ldots)$$
$$+ (1-\sigma)^2(0.0348091\; S_w + \ldots + (0.0018505\,\beta\chi_{00}^{\,2} + \ldots)z^{-\lambda_1}$$
$$+ (0.1698977\; K_{100}S_w + \ldots)z^{-\lambda_2} + (-\,0.8381918\; K_{010}\chi_{00}^{\,2}\beta \;+ \ldots)$$

$$z^{-(\lambda_1+\lambda_2)} + \ldots)$$

$$+ (1-\sigma)^3(0.0211265\ S_w + \ldots + (0.0077149\beta\chi_{00}{}^2 + \ldots)\ z^{-\lambda_1}$$

$$+ (0.1461291\ K_{100}S_w + \ldots)z^{-\lambda_2} + (1.091895\ K_{010}\chi_{00}{}^2\beta + \ldots)$$

$$z^{-(\lambda_1+\lambda_2)} + \ldots)$$

$$+ (1-\sigma)^4(0.0149472\ S_w + \ldots + (0.0074122\beta\chi_{00}{}^2 + \ldots)z^{-\lambda_1}$$

$$+ (0.1458170\ K_{100}S_w + \ldots)z^{-\lambda_2} + (-\ 0.1700329\ K_{010}\chi_{00}{}^2\beta + \ldots)$$

$$z^{-(\lambda_1+\lambda_2)} + \ldots)$$

$$+ \ldots\ . \qquad (2.173)$$

$$\frac{2m}{m+1} = -\ 0.19883774 + 0.1020737\ S_w - 0.0448122\ S_w{}^2 + \ldots$$
$$+ (1-\sigma)(0.0343396\ S_w + \ldots)$$
$$+ (1-\sigma)^2(0.0218330\ S_w + \ldots)$$
$$+ (1-\sigma)^3(0.0164886\ S_w + \ldots)$$
$$+ (1-\sigma)^4(0.0134646\ S_w + \ldots)$$
$$+ \ldots\ . \qquad (2.174)$$

$$\frac{2\lambda_2}{m+1} = 3.7615197 + 0.0110326\ S_w + 0.0033665\ S_w{}^2 + \ldots$$
$$+ (1-\sigma)(-\ 0.2123742\ S_w + \ldots)$$
$$+ (1-\sigma)^2(-\ 0.7357164\ S_w + \ldots)$$
$$+ (1-\sigma)^3(-\ 1.463326\ S_w + \ldots)$$
$$+ (1-\sigma)^4(-\ 2.345605\ S_w + \ldots)$$
$$+ \ldots\ . \qquad (2.175)$$

If $\sigma = 1$, $m = -\ 0.09042856 + 0.0422238\ S_w - 0.0165769\ S_w{}^2 + \ldots$,

$$(2.176)$$

$$\lambda_2 = 1.7106854 + 0.0844303\ S_w - 0.0294133\ S_w^{\ 2} + \dots\ . \tag{2.177}$$

$$\text{If } \sigma = 0.72,\ V(z) = 1 - 0.00944\,\chi_{00}^{\ 2}\beta z^{-\lambda_1} + z^{-\lambda_2}(K_{100} + K_{1*1}S_w + \dots) - z^{-(\lambda_1+\lambda_2)}K_{010}\chi_{00}^{\ 2}\beta\,1.12(9) + \dots \tag{2.178}$$

$$\frac{\partial S}{\partial \eta}(z,0) = (-\ 0.2978(9) - K_{100}z^{-\lambda_2}\ 2.922(9) + \dots)\ S_w + z^{-\lambda_1}(-\ 0.03608 - K_{100}z^{-\lambda_2}\ 0.47(5) + \dots)\chi_{00}^{\ 2}\beta + \dots\ . \tag{2.179}$$

$$\left.\begin{aligned} \frac{2m}{m+1} &= -\ 0.198837784 + 0.11387\ S_w + \dots \\ m &= -\ 0.09042856 + 0.04710\ S_w + \dots \end{aligned}\right\} \tag{2.180}$$

$$\left.\begin{aligned} \frac{2\lambda_2}{m+1} &= 3.7615197 - 0.1616(8)\ S_w + \dots \\ \lambda_2 &= 1.7106854 + 0.0301\ S_w + \dots\ . \end{aligned}\right\} \tag{2.181}$$

As in the solution for small ξ the displacement and momentum thicknesses can be calculated and finally the shape factor $H^* = \delta_1^*/\delta_2^*$ evaluated:

$$H^*/4.0292280 = 1 + 0.6407587\ S_w + 0.0476295\ S_w^{\ 2} - 0.0342470\ S_w^{\ 3} + \dots$$
$$+ z^{-\lambda_2}(-\ 2.975636\ K_{100} + (-\ 3.577145\ K_{100} - 2.975636\ K_{101})S_w + (-\ 1.401113\ K_{100} - 3.577145\ K_{101} - 2.975636\ K_{102})S_w^{\ 2} + \dots)$$
$$+ \dots$$

$$+ (1-\sigma)\big(0.2155640\ S_w + \ldots + (0.0016386\,\chi_{00}{}^2\beta + \ldots)z^{-\lambda_1} + \ldots$$

$$+ z^{-\lambda_2}((-\ 0.864357\ K_{100} - 2.975636\ K_{111})S_w + \ldots)$$

$$+ z^{-(\lambda_1\ +\lambda_2)}(7.213116\ K_{100}\chi_{00}{}^2\beta + \ldots) + \ldots\big)$$

$$+ (1-\sigma)^2\big(0.1370547\ S_w + \ldots + (0.0746578\,\chi_{00}{}^2\beta + \ldots)z^{-\lambda_1} + \ldots$$

$$+ z^{-\lambda_2}((0.537498\ K_{100} - 2.975636\ K_{121})S_w + \ldots)$$

$$+ z^{-(\lambda_1+\lambda_2)}(14.37487\ K_{100}\chi_{00}{}^2\beta + \ldots) + \ldots\big)$$

$$+ (1-\sigma)^3\big(\ 0.1035057\ S_w + \ldots + (0.0745488\,\lambda_{00}{}^2\beta + \ldots)z^{-\lambda_1} + \ldots$$

$$+ z^{-\lambda_2}((1.987217\ K_{100} - 2.975636\ K_{131})S_w + \ldots)$$

$$+ z^{-(\lambda_1+\lambda_2)}(27.55408\ K_{100}\lambda_{00}{}^2\beta\ + \ldots) + \ldots\big)$$

$$+ (1-\sigma)^4\big(0.0844809\ S_w + \ldots + (0.0673214\,\chi_{00}{}^2\beta + \ldots)z^{-\lambda_1} + \ldots$$

$$+ z^{-\lambda_2}((3.619247\ K_{100} - 2.975636\ K_{141})\ S_w + \ldots)$$

$$+ z^{-(\lambda_1+\lambda_2)}\ (39.69103\ K_{100}\chi_{00}{}^2\beta\ + \ldots) + \ldots\big)$$

$$+ \ldots\ . \qquad (2.182)$$

If $\sigma = 0.72$,

$$H^* = 4.0292280\ (1 + 0.7148(2)\ S_w + \ldots + z^{-\lambda_1}(0.0085\chi_{00}{}^2\beta + \ldots)$$

$$+ \ldots$$

$$+ z^{-\lambda_2}\big(-\ 2.9756362\ (K_{100} + K_{1*1}\ S_w) - 3.69\ K_{100}S_w + \ldots\big)$$

$$+ z^{-(\lambda_1+\lambda_2)}(4.1\ K_{100}\lambda_{00}{}^2\beta + \ldots) + \ldots)\ . \qquad (2.183)$$

It remains to determine χ, a, K_{100} and K_{1*1}. This is done by writing the external velocity and the heat transfer rate for both small and large ξ in powers of S_w and β and comparing the coefficients of S_w and β over suitable intervals of ξ. It is not possible to derive accurate values of a, K_{100} or K_{1*1} but it is clear that

$$\chi = \begin{cases} 0.95 + 0.06(3)\ S_w - 0.04(3)\ S_w^{\ 2} + \ldots\ , \text{ if } \sigma = 1, \\ 0.95 + 0.06(6)\ S_w + 0.003\beta + \ldots, \text{ if } \sigma = 0.72. \end{cases} \quad (2.184)$$

Even with poor values of a, K_{100}, and K_{1*1} it is nevertheless possible to draw graphs showing the external velocity and the heat transfer rate.

$$\text{Let } {}^{u_1}/_{u_0} = \begin{cases} V_{00} + V_{01}S_w + V_{02}S_w^{\ 2} + \ldots\ , \text{ if } \sigma = 1, \\ V_{00} + V_{01}S_w + V_{10}\beta + \ldots\ , \text{ if } \sigma = 0.72. \end{cases} \quad (2.185)$$

Then V_{00} is the incompressible term and thus does not depend on σ ; V_{01} is a function of σ and is up to 5% larger when $\sigma = 0.72$ than when $\sigma = 1$, but on the scale of the graph the two curves (for $\sigma = 1$ and $\sigma = 0.72$) are indistinguishable;

$|V_{10}| < 0.001$; V_{02} was only calculated for $\sigma = 1$. V_{00}, V_{01}, and V_{02} are plotted in figure 2.1.

Let

$$\left(\frac{2\nu_0 x_0}{u_0}\right)^{\frac{1}{2}} \left(\frac{\partial T}{\partial y}\right)_w \frac{1}{T_w} = \begin{cases} W_{01}\ S_w + W_{02}\ S_w^{\ 2} + \ldots\ , \text{ if } \sigma = 1, \\ W_{01}S_w + W_{10}\beta + \ldots\ , \text{ if } \sigma = 0.72. \end{cases} \quad (2.186)$$

W_{01} and W_{02}, for $\sigma = 1$, are plotted in figure 2.2 and W_{01} and W_{10}, for $\sigma = 0.72$, are plotted in figure 2.3.

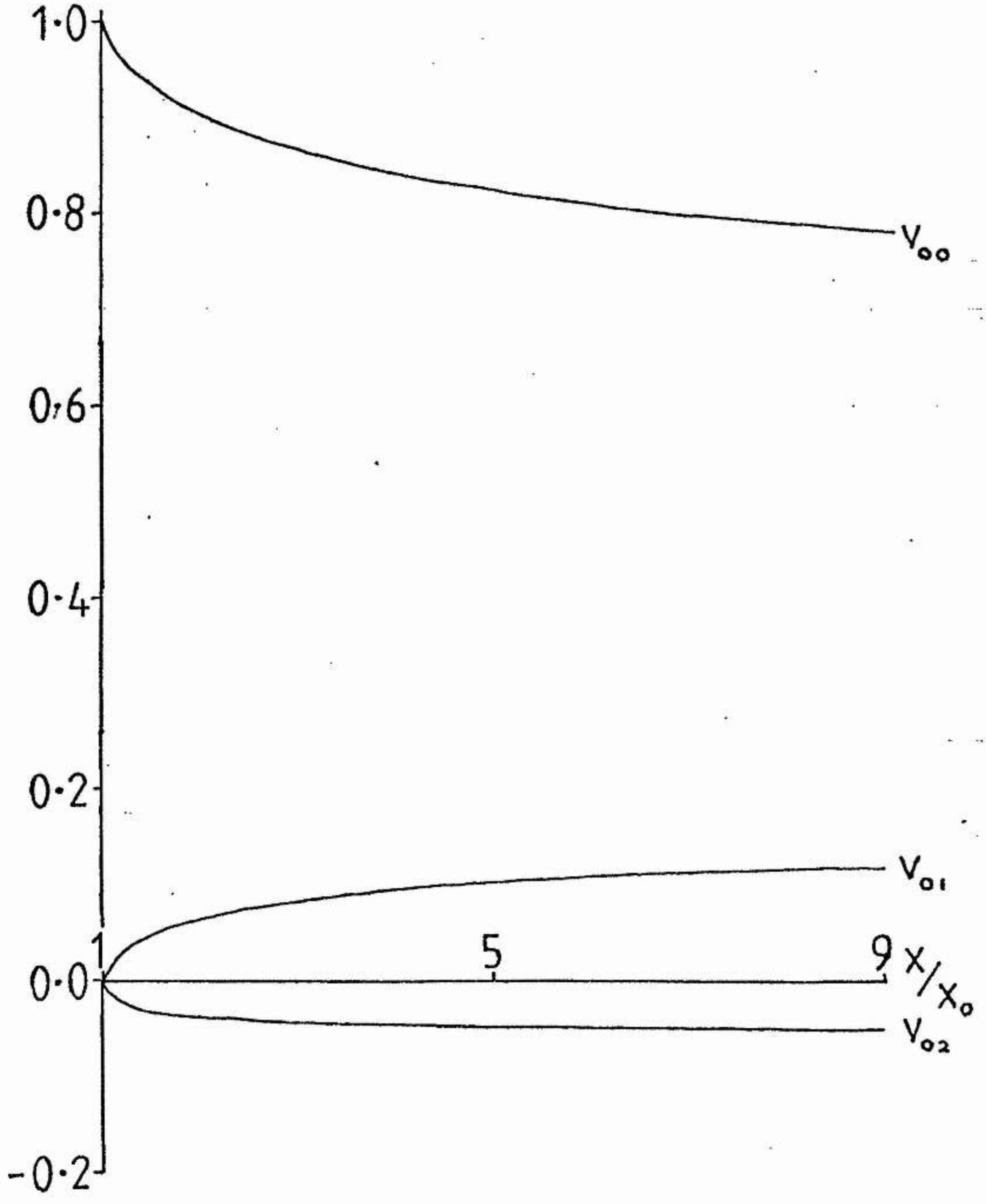

Figure 2.1 External velocity

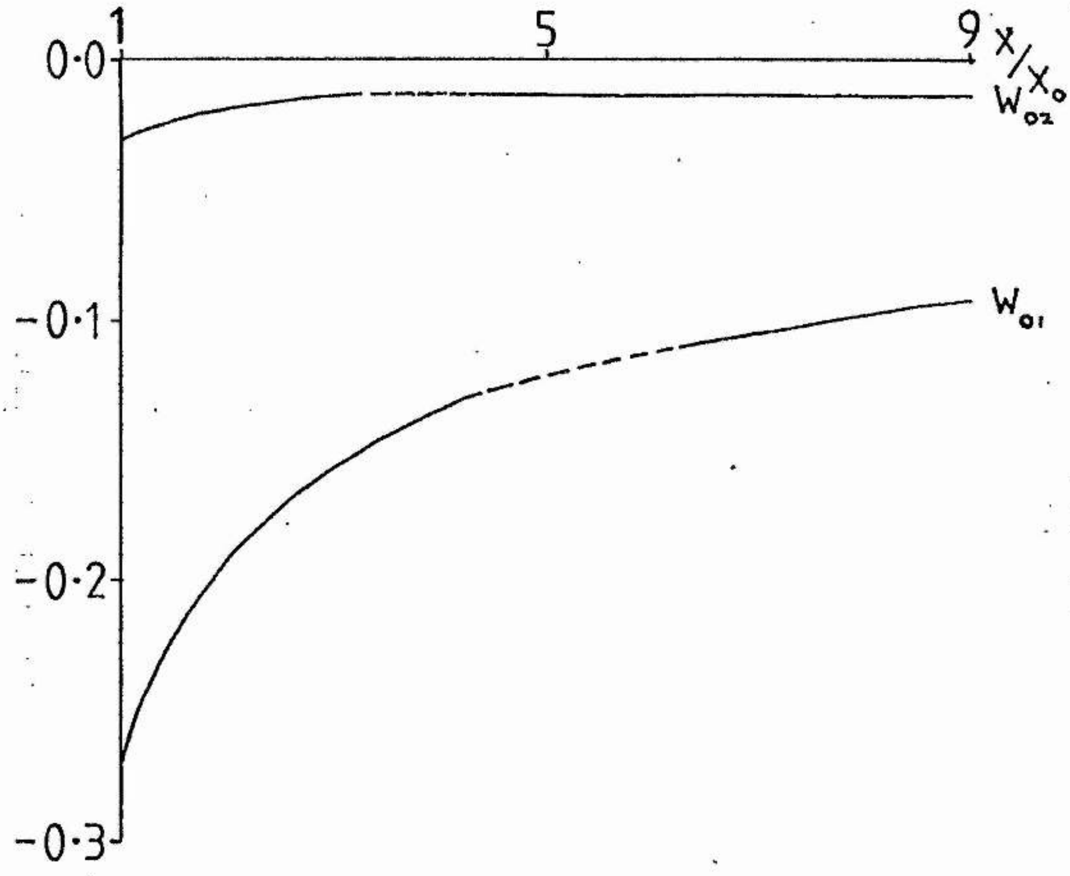

Figure 2.2 Heat transfer rate (σ = 1)

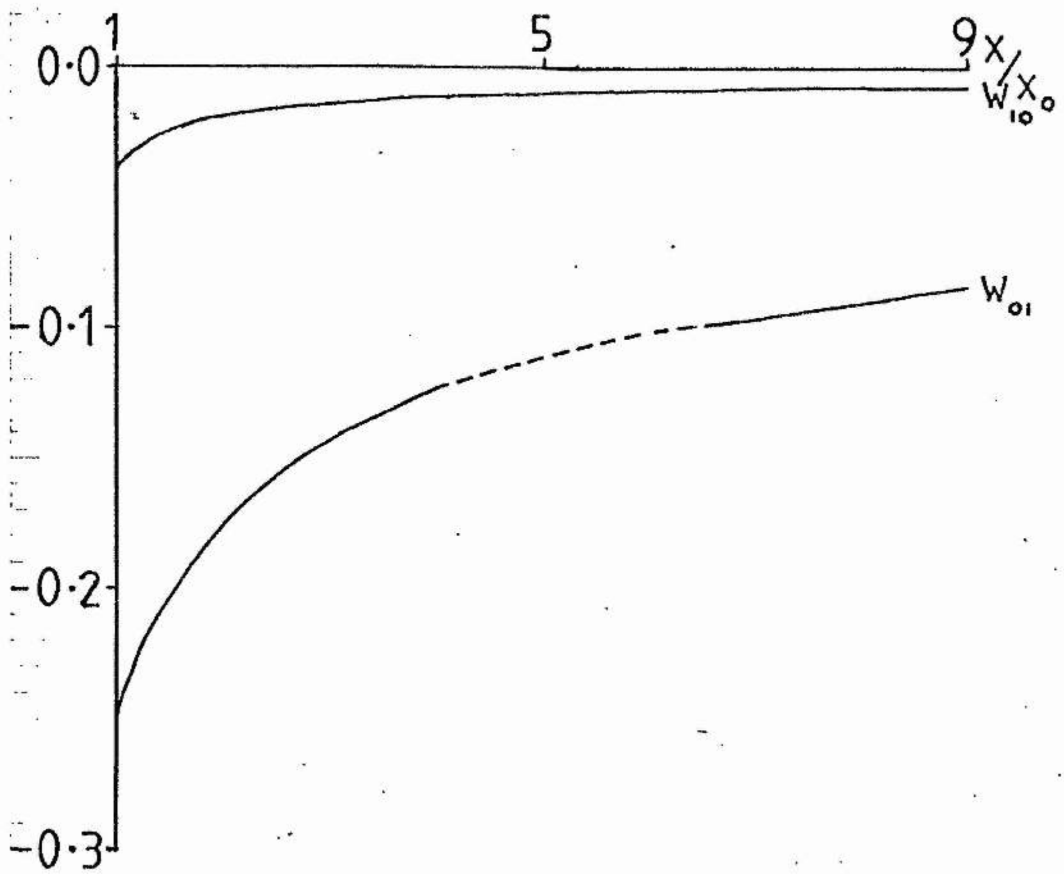

Figure 2.3 Heat transfer rate (σ = 0.72)

3. DEVELOPMENT AND SEPARATION OF A COMPRESSIBLE LAMINAR BOUNDARY LAYER UNDER THE ACTION OF A VERY SHARPLY INCREASING ADVERSE PRESSURE GRADIENT

3.1. Introduction

This chapter considers a compressible laminar boundary layer on a finite flat plate with an adverse pressure gradient which, although small near the leading edge, becomes increasingly sharp towards the trailing edge. The assumption is made that the ratio of the viscosity μ to the absolute temperature T is a function of x^* alone, where x^* measures the distance downstream from the trailing edge in the direction parallel to the plate. Thus

$$\mu = C(x^*)\mu_0 T/T_0 \tag{3.1}$$

where μ_0 and T_0 are values at a suitable reference position.

The theory holds for all Prandtl numbers σ of order unity; detailed numerical calculations are presented for $\sigma=1$ and $\sigma=0.72$ (appropriate to air).

Accordingly a transformation of variables due to Illingworth (1949) and Stewartson (1949) is made, which partially reduces the equations to incompressible form. After transformation, the equations of motion become (equation (1.28))

$$\frac{\partial U^*}{\partial X^*} + \frac{\partial V^*}{\partial Y^*} = 0, \tag{3.2a}$$

$$U^* \frac{\partial U^*}{\partial X^*} + V^* \frac{\partial U^*}{\partial Y^*} = U_1^* \frac{dU_1^*}{dX^*} (1+S) + \nu_0 \frac{\partial^2 U^*}{\partial Y^{*2}}, \tag{3.2b}$$

$$U^* \frac{\partial S}{\partial X^*} + V^* \frac{\partial S}{\partial Y^*} = \frac{\nu_0}{\sigma} \frac{\partial^2 S}{\partial Y^{*2}} - \frac{\nu_0 \, (1-\sigma)}{\sigma} \beta \left\{ 1 - \beta + \beta \left(\frac{U_1^*}{u_0} \right)^2 \right\}^{-1}$$

$$\frac{\partial^2}{\partial Y^{*2}} \left(\frac{U^{*2}}{u_0^{\,2}} \right) , \qquad (3.2c)$$

where $\beta = \frac{\gamma-1}{2} M_0^{\,2} \left(1 + \frac{\gamma-1}{2} M_0^{\,2} \right)^{-1}$,

M_0 is the upstream Mach number and S is related to the temperature by

$$S \left(1 + \frac{\gamma-1}{2} M_1^{\,2} \right) = \frac{T}{T_1} - 1 - \frac{\gamma-1}{2} M_1^{\,2} \left(1 - \frac{U^{*2}}{U_1^{\,*2}} \right) . \qquad (3.3)$$

In these equations Y^* represents distance measured normal to the plate and X^* represents distance measured <u>downstream</u> from the trailing edge, both in the transformed plane, with associated transformed velocity components U^* and V^*. The suffix 1 refers to values at the edge of the boundary layer.

From (3.3) it is clear the $S \to 0$ at the edge of the boundary layer, where $U^* \to U^*_1$ and $T \to T_1$. Likewise at the wall

$$S_w = T_w / T_s - 1 , \qquad (3.4)$$

where

$$T_s = T_1 \left[1 + \tfrac{1}{2} (\gamma - 1) M_1^{\,2} \right] .$$

The equations are non-dimensionalized by writing

$$U^* = u_0 u ,$$

$$X^* = Lx, \text{ where } L = \int_{-1}^{0} C(x^*) \left(\frac{a_1}{a_0} \right)^{\frac{3\gamma-1}{\gamma-1}} dx^*$$

and l is the length of the untransformed plate,

$Y^* = LR^{-\frac{1}{2}}y$, where $R = Lu_0/\nu_0$,

and $V^* = u_0\, R^{-\frac{1}{2}}v$. (3.5)

$$\left.\begin{aligned} &\text{Then } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1 \frac{du_1}{dx}(1+S) + \frac{\partial^2 u}{\partial y^2}, \\ &u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 S}{\partial y^2} - \frac{(1-\sigma)}{\sigma}\beta\left(1-\beta+\beta u_1{}^2\right)^{-1}\frac{\partial^2}{\partial y^2}(u^2). \end{aligned}\right\} \quad (3.6)$$

The external velocity u_1 is selected to be

$$u_1 = (-x)^{\epsilon}, \text{ where } 0 < \epsilon << 1. \quad (3.7)$$

This problem is thus a generalization to compressible flow of the problem first studied by Riley and Stewartson (1969) and for which a more accurate numerical solution was computed by Williams (1976). Its solution must therefore be obtained in each of the four regions which they consider, which are indicated in figure 3.1 (which is not drawn to scale). The assumption that σ is of order unity removes the need for further regions, as there will not be separate thermal layers.

It is necessary to examine (a) region I in which the flow is irrotational (b) region II, the classical Prandtl boundary layer of thickness $O(R^{-\frac{1}{2}})$ between I and the body. Now it is found that near the trailing edge and specifically within a distance $O(\epsilon^{3/2})$ of it, region II splits into (c) region III, an inner boundary layer of thickness $O(\epsilon^{\frac{1}{2}}R^{-\frac{1}{2}})$,

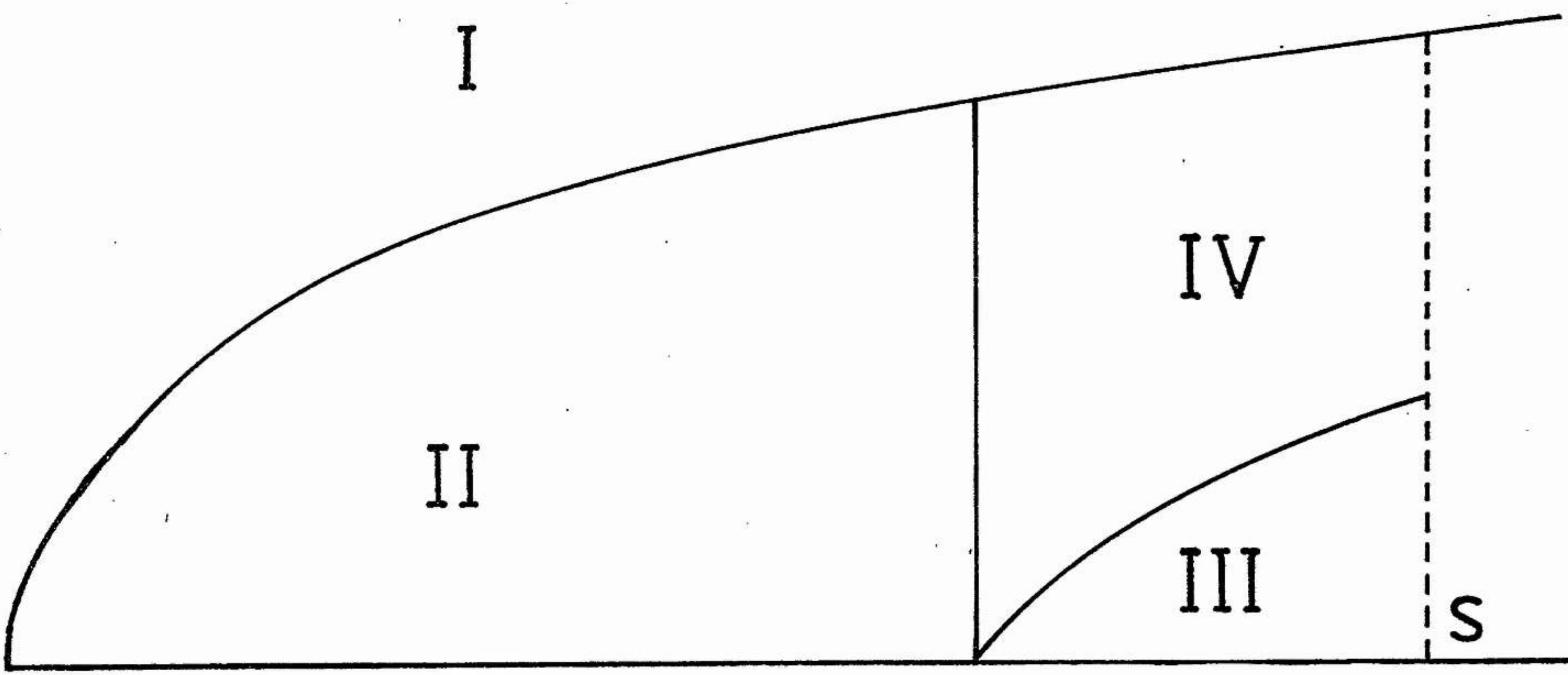

FIGURE 3.1

and (d) region IV. This is an essentially inviscid region of thickness $O(R^{-\frac{1}{2}})$ into which vorticity from region II is convected. The structure of the solution in regions II, III and IV is analysed in detail using perturbation techniques. The response of the flow in III to the rapidly varying pressure leads to separation taking place before the trailing edge is reached and within a distance $O(\epsilon^{3/2})$ of it.

3.2. Solution in Region II

In this region write $\xi = 1 + x$.

Then

$$u_1 \frac{du_1}{dx} = -\epsilon(-x)^{2\epsilon-1} = -\epsilon \sum_{n=0}^{\infty} \xi^n + O(\epsilon^2). \quad (3.8)$$

The governing equations (3.6) become

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial y} = 0 , \quad (3.9a)$$

$$u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial y} = -\epsilon \sum_{n=0}^{\infty} \xi^n(1+S) + \frac{\partial^2 u}{\partial y^2} + O(\epsilon^2), \quad (3.9b)$$

$$u\frac{\partial S}{\partial \xi} + v\frac{\partial S}{\partial y} = \frac{1}{\sigma}\frac{\partial^2 S}{\partial y^2} - \frac{(1-\sigma)}{\sigma}\beta\left\{1 + 2\beta\epsilon \sum_{n=0}^{\infty} \frac{\xi^{n+1}}{(n+1)} + O(\epsilon^2)\right\}\frac{\partial^2}{\partial y^2}(u^2), \quad (3.9c)$$

with boundary conditions

$$\left.\begin{aligned} &u = v = 0, \ S = S_w \text{ on } y = 0 , \\ &u \sim 1 - \epsilon \sum_{n=1}^{\infty} \frac{\xi^n}{n} + O(\epsilon^2), \ S \to 0 \text{ as } y \to \infty . \end{aligned}\right\} \quad (3.10)$$

The stream function ψ is introduced, such that $u = \psi_y$ and $v = -\psi_\xi$; then ψ is written as

$$\psi = (2\xi)^{\frac{1}{2}}\left\{ f_B(\tilde{\eta}) + \epsilon \sum_{n=0}^{\infty} \xi^{n+1} f_n(\tilde{\eta}) + O(\epsilon^2)\right\}, \qquad (3.11)$$

where $\tilde{\eta} = \frac{y}{(2\xi)^{\frac{1}{2}}}$.

$$S = S_B(\tilde{\eta}) + \epsilon \sum_{n=0}^{\infty} \xi^{n+1} S_n(\tilde{\eta}) + O(\epsilon^2). \qquad (3.12)$$

$$\left.\begin{aligned} &\text{Then } f_B''' + f_B f_B'' = 0, \\ &\text{where } f_B(0) = f_B'(0) = 0,\ f_B'(\tilde{\eta}) \to 1 \text{ as } \tilde{\eta} \to \infty, \end{aligned}\right\} (3.13)$$

$$\left.\begin{aligned} &\text{and } S_B'' + \sigma f_B S_B' = 2\beta(1-\sigma)\left\{ f_B' f_B''' + (f_B'')^2 \right\}, \\ &\text{where } S_B(0) = S_w,\ S_B(\tilde{\eta}) \to 0 \text{ as } \tilde{\eta} \to \infty. \end{aligned}\right\} (3.14)$$

Considering terms of $O(\epsilon)$ and equating coefficients of ξ^n give

$$\left.\begin{aligned} &f_n''' + f_B f_n'' - 2(n+1) f_B' f_n' + (2n+3) f_B'' f_n = 2(1+S_B), \\ &\text{where} \\ &f_n(0) = f_n'(0) = 0,\ f_n' \to -(n+1)^{-1} \text{ as } \tilde{\eta} \to \infty, \end{aligned}\right\} (3.15)$$

$$\left.\begin{aligned} &\text{and } S_n'' + \sigma f_B S_n' - 2\sigma(n+1) f_B' S_n = -\sigma(2n+3) S_B' f_n \\ &+ 2(1-\sigma)\beta\left\{ f_B''' f_n' + 2 f_B'' f_n'' + f_B' f_n''' + \frac{2\beta}{(n+1)} \right. \\ &\qquad \left. (f_B' f_B''' + (f_B'')^2)\right\}, \\ &\text{where} \\ &\qquad S_n(0) = 0,\ S_n \to 0 \text{ as } \tilde{\eta} \to \infty. \end{aligned}\right\} (3.16)$$

Although these equations cannot be solved in closed form for arbitrary n, the solutions for large n are important since they lead to the terms which are singular as $\xi \to 1$.

Since $f_n = f_B'$ satisfies the homogeneous version of (3.15) and is such that $f_n' \to 0$ as $\tilde{\eta} \to \infty$, for $n >> 1$ f_n can be written as

$$f_n = \frac{Q}{n+1} + \frac{R}{(n+1)^2} + O\left(\frac{1}{(n+1)^3}\right) + D_n \ f_B', \quad (3.17)$$

where D_n is a constant.

Then $f_B'Q' - f_B''Q + 1 + S_B = 0; \quad Q' \to 1$ as $\tilde{\eta} \to \infty$; (3.18)

$$f_B'R' - f_B''R = f_B''Q + \frac{f_BQ''}{2} + \frac{Q'''}{2}; \quad R' \to 0 \text{ as } \tilde{\eta} \to \infty. \quad (3.19)$$

These have particular integrals

$$Q = f_B'(\tilde{\eta}) \int_{\tilde{\eta}}^{\infty} \left\{ \frac{1 + S_B(t)}{f_B'^2(t)} - 1 \right\} dt - \tilde{\eta} f_B'(\tilde{\eta}) \quad (3.20)$$
$$= \Lambda_1(\tilde{\eta}).$$

$$R = - f_B'(\tilde{\eta}) \int_{\tilde{\eta}}^{\infty} \frac{1}{f_B'^2(s)} \left\{ f_B''(s) \ Q(s) + \frac{f_B(s)Q''(s)}{2} + \frac{Q'''(s)}{2} \right\} ds. \quad (3.21)$$

For small $\tilde{\eta}$

$$\left.\begin{aligned} Q &= \frac{1+S_w}{\alpha} - \frac{\theta}{\alpha}\tilde{\eta}\log\tilde{\eta} + \alpha A^*\tilde{\eta} + O(\tilde{\eta}^2), \\ \text{and} \quad R &= -\frac{\theta}{6\alpha^2\tilde{\eta}^2} + O(1), \end{aligned}\right\} (3.22)$$

where $\alpha = f_B''(0) = 0.4696000$,

and $\theta = S_B'(0)$,

and A^* is a constant. (3.23)

If $\sigma = 1$, $\theta = -\alpha S_w$.

If $\sigma \neq 1$, θ must be evaluated numerically:

if $\sigma = 0.72$, $\theta = -0.4180913\ S_w - 0.0636704\ \beta$.

For no constants D_n can the inner boundary conditions be satisfied. It is clear that there is a thin inner region in which the scale perpendicular to the plate is different. To discuss this region it is necessary to write

$$\tilde{\eta} = n^{-\frac{1}{3}}\xi, \tag{3.24}$$

$$f_n = \frac{F(\xi)}{n} + \frac{G(\xi)}{n^{4/3}} + O(n^{-5/3}). \tag{3.25}$$

Then
$$\left.\begin{aligned} &F''' - 2\alpha\xi F' + 2\alpha F = 2(1+S_w), \\ &F(0) = F'(0) = 0. \end{aligned}\right\} \tag{3.26}$$

$$\left.\begin{aligned} &G''' - 2\alpha\xi G' + 2\alpha G = 2\theta\xi, \\ &G(0) = G'(0) = 0 . \end{aligned}\right\} \tag{3.27}$$

Differentiation gives $F^{IV} - 2\alpha\xi F'' = 0$ (3.28)

and $G^{IV} - 2\alpha\xi G'' = 2\theta.$ (3.29)

Excluding the complementary function which is exponentially large as $\xi \to \infty$,

$$F'' = a\ Ai((2\alpha)^{1/3}\ \xi). \tag{3.30}$$

Moreover $F'''(0) = 2(1+S_w) = (2\alpha)^{1/3}\ a\ Ai'(0)$

implies that $a = -\Gamma(1/3)3^{1/3}\ \alpha^{-1/3}\ 2^{2/3}\ (1+S_w)$. (3.31)

Integrating twice with respect to ξ gives

$$F(\xi) - a(2\alpha)^{-1/3}\xi \int_0^{(2\alpha)^{1/3}\xi} Ai(t)\,dt$$

$$= a(2\alpha)^{-2/3}\left\{ Ai'(0) - Ai'((2\alpha)^{1/3}\xi)\right\} . \qquad (3.32)$$

As $\xi \to \infty$,

$$F(\xi) = -(1/3)!\,\alpha^{-2/3}\,2^{1/3}\,3^{1/3}\,(1+S_w)\xi$$

$$+ \frac{1+S_w}{\alpha} + \text{terms which are exponentially small.} \qquad (3.33)$$

Matching with the outer solution gives

$$D_n = \frac{C}{(n+1)^{2/3}} + o((n+1)^{-2/3}) \qquad (3.34)$$

and $C = -\alpha^{-5/3}\,2^{1/3}\,3^{1/3}\,(1/3)!\,(1+S_w)$. (3.35)

Solving (3.27) gives (after excluding the complementary function which is exponentially large as $\tilde{\eta} \to \infty$)

$$G'' = b\,Ai((2\alpha)^{1/3}\xi) - 2^{1/3}\frac{\theta\pi}{a^{2/3}}\,Gi((2\alpha)^{1/3}\xi), \qquad (3.36)$$

where Gi (ξ) is as defined in Abramowitz and Stegun (1965) (p.448).

$G'''(0) = 0$ implies that $b = -\dfrac{2^{1/3}\theta\pi}{\alpha^{2/3}\sqrt{3}}$. (3.37)

Integrating twice with respect to ξ gives

$$G(\xi) + \frac{\theta\pi}{\alpha}\xi\left\{\int_0^{(2\alpha)^{1/3}\xi}\left(\frac{Ai(s)}{\sqrt{3}} + Gi(s)\right)ds - \pi^{-1}\right\}$$

$$= \frac{\theta\pi}{\alpha}(2\alpha)^{-1/3}\left\{\frac{Ai'((2\alpha)^{1/3}\xi)}{\sqrt{3}} + Gi'((2\alpha)^{1/3}\xi)\right\} . \qquad (3.38)$$

As $\xi \to \infty$

$$G(\xi) = -\frac{\theta}{\alpha}\left(\xi\log\xi + B\xi + \frac{1}{6\alpha\xi^2} + O({}^1/_{\xi^5})\right), \qquad (3.39)$$

$$\text{where } B = \frac{\pi}{3\sqrt{3}} - 1 + {}^2/_3\gamma + {}^1/_3 \log(6\alpha), \qquad (3.40)$$

where $\gamma = 0.57721566$ is Euler's Constant.

If Q and R are expanded for small $\tilde{\eta}$ it is found that matching occurs provided

$$D_n = \frac{C}{(n+1)^{2/3}} + \frac{1}{\alpha(n+1)}\left(-{}^1/_3 \frac{\theta}{\alpha} \log n - \frac{\theta}{\alpha} B - \alpha A^*\right)$$

$$+ O((n+1)^{-4/3}). \qquad (3.41)$$

$$\text{So } f_n = \frac{\Lambda_1(\tilde{\eta})}{(n+1)} + O((n+1)^{-2})$$

$$+ \left\{ \frac{C}{(n+1)^{2/3}} - {}^1/_3 \frac{\theta \log n}{\alpha^2(n+1)} + \frac{-\theta B - \alpha^2 A^*}{\alpha^2(n+1)} + O((n+1)^{-4/3}) \right\}$$

$$f_B'(\tilde{\eta}), \qquad (3.42)$$

and in the outer region

$$\psi = (2\xi)^{\frac{1}{2}}\left(f_B(\tilde{\eta}) + \epsilon\left\{f_B'(\tilde{\eta}) \sum_{n=0}^{\infty} \xi^{n+1}\left(\frac{C}{(n+1)^{2/3}} - \right.\right.\right.$$

$$- {}^1/_3 \frac{\theta \log n}{\alpha^2(n+1)}$$

$$\left. - \frac{\theta B + \alpha^2 A^*}{\alpha^2(n+1)}\right) + \Lambda_1(\tilde{\eta}) \sum_{n=0}^{\infty} \frac{\xi^{n+1}}{(n+1)} + \text{terms which remain finite as}$$

$$\left.\left. \xi \to 1 \right\} + O(\epsilon^2)\right). \qquad (3.43)$$

As $\xi \to 1$

$$\psi = 2^{\frac{1}{2}} f_B\left({}^y/_{2^{\frac{1}{2}}}\right) + \epsilon\left\{ \overline{C} f_B'\left({}^y/_{2^{\frac{1}{2}}}\right) (-x)^{-1/3} + \text{terms which tend to}\right.$$

infinity more slowly as $\xi \to 1 \Big\} + O(\epsilon^2)$, (3.44)

where $\overline{C} = - \dfrac{2\pi\, 2^{5/6}\, (1/3)\,!}{\alpha^{5/3}\, 3^{1/6}\, (-1/3)\,!}\,(1 + S_w)$. (3.45)

Similarly the wall skin friction

$$\left(\frac{\partial^2\psi}{\partial y^2}\right)_{y=0} = \frac{\alpha}{(2\xi)^{1/2}} + \epsilon\left\{\sum_{n=1}^{\infty}\left(\frac{F''(0)}{n^{1/3}} + \frac{G''(0)}{n^{2/3}}\right)\frac{\xi^{n+1/2}}{2^{1/2}}\right.$$

$$\left. + \text{ terms which tend to infinity more slowly as } \xi \to 1\right\} + O(\epsilon^2). \tag{3.46}$$

As $\xi \to 1$

$$\left(\frac{\partial^2\psi}{\partial y^2}\right)_{y=0} = \frac{\alpha}{2^{1/2}} + \epsilon\left\{\frac{-2\pi\, 2^{1/6}}{\alpha^{1/3}\,(-1/3)!\; 3^{5/6}}\,(-x)^{-2/3}\,(1 + S_w)\right.$$

$$- \frac{(2\pi)^2\,(-x)^{-1/3}}{2^{1/6}\, 3^{5/3}\left[(-1/3)!\right]^2 \alpha^{2/3}}\,\theta$$

$$\left. + \text{ terms which tend to infinity more slowly as } \xi \to 1\right\} + O(\epsilon^2). \tag{3.47}$$

The energy equation (3.16) has solution

$$S_n = S_B'\, D(n) + \frac{S_B'}{f_B'}\,\frac{\Lambda_1(\tilde{\eta})}{n+1} + O((n+1)^{-2}) \tag{3.48}$$

outside the thin inner region near the wall.

In the inner region in which $\tilde{\eta} = n^{-1/3}\,\xi$

$$S_n = \frac{A}{n^{2/3}} + O(1/n). \tag{3.49}$$

Then (3.16) implies that

$$\frac{A''}{\sigma} - 2\alpha\xi A = -2\theta F, \tag{3.50}$$

where A(0) = 0 and A does not become exponentially large as

$\xi \to \infty$.

If $\sigma=1$, this equation can be solved analytically:

$$A = \frac{-2^{1/3}\theta(1+S_w)}{\alpha^{5/3}}\left\{\frac{\pi}{\sqrt{3}}\,Ai((2\alpha)^{1/3}\xi) - \pi\,Gi\,((2\alpha)^{1/3}\xi) + 3^{1/3}\,\Gamma(1/3)\int_0^{(2\alpha)^{1/3}\xi} Ai(t)dt\right\}, \tag{3.51}$$

which gives

$$A'(0) = -\frac{2^{2/3}\theta(1+S_W)}{\alpha^{4/3}\,3^{1/3}}\left\{\frac{\Gamma(1/3)}{\Gamma(2/3)} - \frac{2\pi}{\sqrt{3}\,\Gamma(1/3)}\right\}. \tag{3.52}$$

If $\sigma\neq 1$, this equation must be solved numerically:

if $\sigma = 0.72$, $A'(0) = -1.7954672\theta(1+S_w)$. (3.53)

So, in the outer region,

$$S = S_B(\tilde{\eta}) + \epsilon\left\{S_B'(\tilde{\eta})\; C\sum_{n=0}^{\infty}\frac{\xi^{n+1}}{(n+1)^{2/3}} + \text{terms which tend to infinity more slowly as } \xi \to 1\right\} + O(\epsilon^2). \tag{3.54}$$

As $\xi \to 1$,

$$S = S_B\left(\frac{y}{2^{1/2}}\right) + \epsilon\left\{S_B'\left(\frac{y}{2^{1/2}}\right)\frac{\bar{C}}{2^{1/2}}(-x)^{-1/3} + \text{terms which tend to infinity more slowly as } \xi \to 1\right\} + O(\epsilon^2). \tag{3.55}$$

Similarly the heat transfer rate

$$\left(\frac{\partial S}{\partial y}\right)_{y=0} = \frac{\theta}{(2\xi)^{1/2}} + \epsilon\left\{\sum_{n=1}^{\infty} \frac{\xi^{n+1}}{(2\xi)^{1/2}} \frac{A'(0)}{n^{1/3}} + \text{terms which tend}\right.$$

$$\left.\text{to infinity more slowly as } \xi \to 1\right\} + O(\epsilon^2). \qquad (3.56)$$

As $\xi \to 1$,

$$\left(\frac{\partial S}{\partial y}\right)_{y=0} = \frac{\theta}{2^{1/2}} + \epsilon\left\{\frac{\sqrt{2}\,\pi\, A'(0)(-x)^{-2/3}}{\sqrt{3}\,\Gamma(1/3)} + O((-x)^{-1/3})\right\}$$

$$+ O(\epsilon^2), \qquad (3.57)$$

where $A'(0)$ is given by (3.52) and (3.53).

One of the conditions (3.83) to be imposed on the solution in region IV is that it must match asymptotically as $\epsilon \to 0$ with (3.44) and (3.55).

Likewise the skin friction (3.47) and heat transfer (3.57) in region III must match asymptotically with the skin friction (3.78) and heat transfer (3.79) in region III.

3.3. Solution in Region III

The variables appropriate to region III are given by

$$x = 2\alpha^{-2}\epsilon^{3/2} X, \; y = 2^{1/2}\alpha^{-1}\epsilon^{1/2} Y, \; u = \epsilon^{1/2} U, \; v = 2^{-1/2}\alpha\epsilon^{-1/2} V. \qquad (3.58)$$

The governing equations become

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \qquad (3.59a)$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -(-X)^{-1}\ (1 + O(\epsilon\log\epsilon))\ (1 + S) + \frac{\partial^2 U}{\partial Y^2}\ , \quad (3.59b)$$

$$U\frac{\partial S}{\partial X} + V\frac{\partial S}{\partial Y} = \frac{1}{\sigma}\ \frac{\partial^2 S}{\partial Y^2} + O(\epsilon). \quad (3.59c)$$

$$\left.\begin{array}{l} \text{At } Y=0\ U=V=0;\quad S=S_w. \\ \text{As } X \to -\infty,\ U \to Y + O(\epsilon^{1/2}); \\ \qquad S \to S_w + \epsilon^{1/2}\frac{\theta Y}{\alpha} + O(\epsilon). \end{array}\right\} (3.60)$$

Moreover the skin friction must match with region II.

Introduce a stream function $\psi = 2^{1/2}\alpha^{-1}\epsilon\Psi$ so that

$$U = \frac{\partial\Psi}{\partial Y}, \quad V = -\frac{\partial\Psi}{\partial X}\ . \quad (3.61)$$

The solution is given by

$$\left.\begin{array}{l} \Psi = \Psi_0 + \epsilon^{1/2}\ \Psi_1 + \ldots \\ S = S_w + \epsilon^{1/2}\ \Sigma_1 + \ldots \end{array}\right\} (3.62)$$

$$O(1) : \ \Psi_{0_Y}\ \Psi_{0_{XY}} - \Psi_{0_X}\Psi_{0_{YY}} = -(-X)^{-1}\ (1+S_w) + \Psi_{0_{YYY}}. \quad (3.63)$$

The solution of this equation can be written in the form

$$\Psi_0 = \sum_{m=0}^{\infty} (-X)^{2/3 - 2/3m}\ (1+S_w)^m\ \tilde{f}_m(\eta),\ \text{where}\ \eta = \frac{Y}{(-X)^{1/3}}\ , \quad (3.64)$$

and $\tilde{f}_m(0) = \tilde{f}_m'(0) = 0 \ \ \forall m$,

$$\tilde{f}_0'(\eta) \sim \eta \quad \text{as} \quad \eta \to \infty,$$

$\tilde{f}_m'(\eta)$ is bounded as $\eta \to \infty$ $m \geq 1$ (to make matching with region IV possible),

and $\tilde{f}_m$ is independent of S_w $\forall m$.

Then $\tilde{f}_0(\eta) = {}^1/_2\eta^2$,

$$3\tilde{f}_1''' - \eta^2 \tilde{f}_1'' = 3,$$

$$3\tilde{f}_2''' - \eta^2 \tilde{f}_2'' - 2\eta \tilde{f}_2' + 2\tilde{f}_2 = \tilde{f}_1'^2,$$

$$3\tilde{f}_3''' - \eta^2 \tilde{f}_3'' - 4\eta \tilde{f}_3' + 4\tilde{f}_3 = 4\tilde{f}_1'\tilde{f}_2' - 2\tilde{f}_1'' \tilde{f}_2,$$

$$3\tilde{f}_4''' - \eta^2 \tilde{f}_4'' - 6\eta \tilde{f}_4' + 6\tilde{f}_4 = 6\tilde{f}_1'\tilde{f}_3' + 3\tilde{f}_2'^2 - 4\tilde{f}_1''\tilde{f}_3 - 2\tilde{f}_2''\tilde{f}_2,$$

$$3\tilde{f}_5''' - \eta^2 \tilde{f}_5'' - 8\eta\tilde{f}_5' + 8\tilde{f}_5 = 8(\tilde{f}_1'\tilde{f}_4'+\tilde{f}_2'\tilde{f}_3') - 6\tilde{f}_1''\tilde{f}_4 - 4\tilde{f}_2''\tilde{f}_3 - 2\tilde{f}_3''\tilde{f}_2, \qquad (3.65)$$

$$3\tilde{f}_6''' - \eta^2\tilde{f}_6'' - 10\eta \tilde{f}_6' + 10\tilde{f}_6 = 10(\tilde{f}_1'\tilde{f}_5'+\tilde{f}_2'\tilde{f}_4') + 5\tilde{f}_3'^2 - 8\tilde{f}_1''\tilde{f}_5 - 6\tilde{f}_2''\tilde{f}_4 - 4\tilde{f}_3''\tilde{f}_3 - 2\tilde{f}_4''\tilde{f}_2,$$

etc..

The linear equations (3.65) can be solved successively without great difficulty. Applying the condition that $\tilde{f}_m'(\eta)$ should not be exponentially large as $\eta \to \infty$ gives that $\tilde{f}_m'$ tends to a constant for $m \geq 1$.

As $\eta \to \infty$ $\dfrac{\partial \Psi_0}{\partial Y} \sim Y - g(X)$,

$$\text{where } g(X) = - \sum_{m=1}^{\infty} (-X)^{{}^1/_3 - {}^2/_3 m} \tilde{f}_m'(\infty) (1+S_w)^m. \qquad (3.66)$$

Hence $\Psi_0 \sim {}^1/_2 Y^2 - g(X)Y$, as $Y \to \infty$ for finite X. (3.67)

$O(\epsilon^{1/2})$: Energy equation:

$$\Psi_{0_Y}\,\Sigma_{1_X} - \Psi_{0_X}\,\Sigma_{1_Y} = \frac{1}{\sigma}\Sigma_{1_{YY}}. \qquad (3.68)$$

At $Y=0$, $\Sigma_1=0$.

As $X \to -\infty$, $\Sigma_1 \sim \frac{\theta}{\alpha} Y$.

The solution can be written in the form

$$\Sigma_1 = -\frac{\theta}{\alpha} \sum_{m=0}^{\infty} (-X)^{1/3 - 2/3 m}\, \tilde{S}_m(\eta)\, (1+S_w)^m, \qquad (3.69)$$

where $\tilde{S}_m(0) = 0 \ \forall m$,

$\tilde{S}_0 \sim -\eta$ as $\eta \to \infty$,

$\tilde{S}_m$ is bounded as $\eta \to \infty$ for $m \geq 1$

(to make matching with region IV possible),

and $\tilde{S}_m$ is independent of $S_w \ \forall m$.

Thus $\tilde{S}_0(\eta) = -\eta$,

$$\tilde{S}_1''/\sigma - 1/3\,\eta^2\tilde{S}_1' - 1/3\,\eta\tilde{S}_1 = 1/3\,\eta\tilde{f}_1',$$

$$\tilde{S}_2''/\sigma - 1/3\,\eta^2\tilde{S}_2' - \eta\tilde{S}_2 = 1/3\,\eta\tilde{f}_2' + 2/3\,\tilde{f}_2 + 1/3\,\tilde{f}_1'\tilde{S}_1,$$

$$\tilde{S}_3''/\sigma - 1/3\,\eta^2\tilde{S}_3' - 5/3\,\eta\tilde{S}_3 = 1/3\,\eta\tilde{f}_3' + 4/3\,\tilde{f}_3 + \tilde{f}_1'\tilde{S}_2 + 1/3\,\tilde{f}_2'\tilde{S}_1 - 2/3\,\tilde{f}_2\tilde{S}_1',$$

$$\tilde{S}_4''/\sigma - 1/3\,\eta^2\tilde{S}_4' - 7/3\,\eta\tilde{S}_4 = 1/3\,\eta\tilde{f}_4' + 2\tilde{f}_4 + 5/3\,\tilde{f}_1'\tilde{S}_3 + \tilde{f}_2'\tilde{S}_2 + 1/3\,\tilde{f}_3'\tilde{S}_1 - 4/3\,\tilde{S}_1'\tilde{f}_3 - 2/3\,\tilde{S}_2'\tilde{f}_2, \qquad (3.70)$$

etc..

These equations are also solved successively and for $m \geq 1$ in each case $\tilde{S}_m \to$ a constant as $\eta \to \infty$.

So write

$$\Sigma_1 = \frac{\theta}{\alpha} Y + g^*(X) + \dots, \text{ as } Y \to \infty \text{ for finite } X. \quad (3.71)$$

Then (3.69) implies that

$$g^{*\prime} + \frac{\theta}{\alpha} g'(X) = 0.$$

Since $g^*(X) \to 0$, when $X \to -\infty$,

$$g^*(X) = -\frac{\theta}{\alpha} g(X). \quad (3.72)$$

So $\Sigma_1 \sim \frac{\theta}{\alpha}(Y - g(X))$, as $Y \to \infty$, for finite X. (3.73)

$O(\epsilon^{1/2})$: Momentum equation

$$\Psi_{0_Y} \Psi_{1_{YX}} + \Psi_{1_Y} \Psi_{0_{YX}} - \Psi_{0_X} \Psi_{1_{YY}} - \Psi_{1_X} \Psi_{0_{YY}} =$$

$$-(-X)^{-1}\Sigma_1 + \Psi_{1_{YYY}}. \quad (3.74)$$

At Y=0, $U_1=V_1=0$.

As $X \to -\infty$ the skin friction must match with region II.

The solution of these equations can be written in the form

$$\Psi_1 = -\frac{\theta}{\alpha} \sum_{m=0}^{\infty} (-X)^{1/3-2/3\,m} \tilde{h}_m(\eta)\,(1+S_w)^m, \quad (3.75)$$

where $\tilde{h}_m(0) = \tilde{h}_m'(0) = 0 \;\forall m$,

$\tilde{h}_m$ does not grow exponentially as $\eta \to \infty$,

and $\tilde{h}_m$ is independent of $S_w \;\forall m$.

Thus $\tilde{h}_0''' - \frac{1}{3}\eta^2\tilde{h}_0'' + \frac{1}{3}\eta\tilde{h}_0' - \frac{1}{3}\tilde{h}_0 = \tilde{S}_0$, (3.76)

$$\tilde{h}_1''' - \frac{1}{3}\eta^2\tilde{h}_1'' - \frac{1}{3}\eta\tilde{h}_1' + \frac{1}{3}\tilde{h}_1 = \tilde{S}_1 + \frac{1}{3}\tilde{h}_0'\tilde{f}_1'$$
$$+ \frac{1}{3}\tilde{f}_1''\tilde{h}_0,$$

$$\tilde{h}_2''' - {}^1/_3\eta^2\tilde{h}_2'' - \eta\tilde{h}_2' + \tilde{h}_2 = \tilde{S}_2 + \tilde{h}_0'\tilde{f}_2' + \tilde{h}_1'\tilde{f}_1'$$
$$- {}^2/_3\tilde{f}_2\tilde{h}_0'' - {}^1/_3\tilde{h}_1\tilde{f}_1'' + {}^1/_3\tilde{h}_0\tilde{f}_2'',$$

$$\tilde{h}_3''' - {}^1/_3\eta^2\tilde{h}_3'' - {}^5/_3\eta\tilde{h}_3' + {}^5/_3\tilde{h}_3 = \tilde{S}_3 + {}^5/_3\,(\tilde{h}_0'\tilde{f}_3' +$$
$$\tilde{h}_1'\tilde{f}_2' + \tilde{h}_2'\tilde{f}_1') - {}^4/_3\tilde{f}_3\tilde{h}_0'' - {}^2/_3\tilde{f}_2\tilde{h}_1'' - \tilde{h}_2\tilde{f}_1'' -$$
$${}^1/_3\tilde{h}_1\tilde{f}_2'' + {}^1/_3\tilde{h}_0\tilde{f}_3'',$$

etc..

These equations are solved successively as before.

It remains to check the matching of the skin friction and heat transfer with region II:

As $X \to -\infty$,

$$\left(\frac{\partial^2\Psi}{\partial Y^2}\right)_{Y=0} = \tilde{f}_0''(0) + (-X)^{-2/3}\,\tilde{f}_1''(0)\,(1+S_w) + O((-X)^{-4/3})$$
$$+ \epsilon^{1/2}\left\{-\frac{\theta}{\alpha}\,(-X)^{-1/3}\,\tilde{h}_0''(0)\,(1+S_w) + O((-X)^{-1})\right\} + O\,(\epsilon\log\epsilon), \tag{3.77}$$

which, from (3.A3) and (3.A14),

$$= 1 - \frac{2\pi(1+S_w)}{3^{5/6}\,(-1/3)!}\,(-X)^{-2/3} + O((-X)^{-4/3})$$
$$+ \epsilon^{1/2}\left\{-\frac{(2\pi)^2\,\theta\,(-X)^{-1/3}}{3^{5/3}\,\alpha\left[(-1/3)!\right]^2} + O((-X)^{-1})\right\} + O(\epsilon\log\epsilon), \tag{3.78}$$

which matches with (3.47) as required.

As $X \to -\infty$,

$$\left(\frac{\partial S}{\partial Y}\right)_{Y=0} = \epsilon^{1/2}\,(-S_w - \frac{\theta}{\alpha}\,(-X)^{-2/3}\,\tilde{S}_1'(0)(1+S_w) + O((-X)^{-4/3}))$$
$$+ O(\epsilon). \tag{3.79}$$

If $\sigma = 1$, $\tilde{S}_1'(0)$ can be obtained analytically, and from (3.A22)

$$\tilde{S}_1'(0) = \frac{2\pi}{3^{5/6}}\left\{\frac{1}{\Gamma(2/3)} - \frac{2\pi}{\sqrt{3}\,[\Gamma(1/3)]^2}\right\}.$$

When $\sigma \neq 1$, $\tilde{S}_1'(0)$ must be obtained numerically: if $\sigma = 0.72$, $\tilde{S}_1'(0) = 0.5590510$.

So (3.79) matches with (3.57) as required.

3.4. Solution in Region IV

The variables appropriate to region IV are given by

$$x = 2\alpha^{-2}\,\epsilon^{3/2}\,\bar{X},\; y = \bar{Y},\; \psi = \bar{\psi}. \tag{3.80}$$

The governing equations become

$$\left.\begin{aligned} &\bar{\psi}_{\bar{Y}}\bar{\psi}_{\bar{X}\bar{Y}} - \bar{\psi}_{\bar{X}}\bar{\psi}_{\bar{Y}\bar{Y}} = \epsilon\frac{(1+S)}{\bar{X}} + 2\alpha^{-2}\,\epsilon^{3/2}\,\bar{\psi}_{\bar{Y}\bar{Y}\bar{Y}} + O(\epsilon^2\log\epsilon), \\ &\bar{\psi}_{\bar{Y}}\,S_{\bar{X}} - \bar{\psi}_{\bar{X}}\,S_{\bar{Y}} = O(\epsilon^{3/2}). \end{aligned}\right\} \tag{3.81}$$

As $\bar{Y} \to 0$ from (3.67) and (3.73)

$$\left.\begin{aligned} &\bar{\psi} \sim \frac{1}{2^{3/2}}\alpha\bar{Y}^2 - \epsilon^{1/2}\,\bar{Y}g(\bar{X}) + O(\epsilon(\log\epsilon)^2), \\ &S \sim S_w + 2^{-1/2}\,\theta\,\bar{Y} - \epsilon^{1/2}\frac{\theta}{\alpha}\,g(\bar{X}) + O(\epsilon(\log\epsilon)^2). \end{aligned}\right\} \tag{3.82}$$

As $\bar{X} \to -\infty$, from (3.44) and (3.55)

$$\bar{\psi} \sim 2^{1/2}\,f_B\left(\frac{\bar{Y}}{2^{1/2}}\right) + \epsilon^{1/2}\frac{\alpha^{2/3}\,\bar{C}}{2^{1/3}}\,f_B'\left(\frac{\bar{Y}}{2^{1/2}}\right)(-\bar{X})^{-1/3}$$

$$+ O(\epsilon(\log\epsilon)^2), \tag{3.83a}$$

$$S \sim S_B\left(\frac{\bar{Y}}{2^{1/2}}\right) + \epsilon^{1/2}\,\frac{\alpha^{2/3}\bar{C}}{2^{5/6}}\;S_B'\left(\frac{\bar{Y}}{2^{1/2}}\right)(-X)^{-1/3}$$

$$+\ O(\epsilon(\log\epsilon)^2). \qquad (3.83b)$$

As $\bar{Y} \to \infty$, from (3.7) and (3.4)

$$\frac{\partial\bar{\psi}}{\partial\bar{Y}} \sim 1 + O(\epsilon\log\epsilon) \text{ and } S \to 0\,. \qquad (3.84)$$

So a solution to (3.81) is sought in the form

$$\left.\begin{aligned} \bar{\psi} &= \bar{\psi}_0 + \epsilon^{1/2}\,\bar{\psi}_1 + O(\epsilon(\log\epsilon)^2), \\ S &= \bar{S}_0 + \epsilon^{1/2}\,\bar{S}_1 + O(\epsilon(\log\epsilon)^2). \end{aligned}\right\} \qquad (3.85)$$

$$\left.\begin{aligned} O(1) :\ & \bar{\psi}_{0_{\bar{Y}}}\,\bar{\psi}_{0_{\bar{X}\bar{Y}}} - \bar{\psi}_{0_{\bar{X}}}\,\bar{\psi}_{0_{\bar{Y}\bar{Y}}} = 0, \\ & \bar{\psi}_{0_{\bar{Y}}}\,S_{0_{\bar{X}}} - \bar{\psi}_{0_{\bar{X}}}\,S_{0_{\bar{Y}}} = 0, \end{aligned}\right\} \qquad (3.86)$$

with boundary conditions

$$\text{as } \bar{Y} \to 0,\ \bar{\psi}_0 \sim \frac{1}{2^{3/2}}\,\alpha\,\bar{Y}^2,\ S \to S_w, \text{ from (3.82)}, \qquad (3.87)$$

$$\text{as } \bar{X} \to -\infty,\ \bar{\psi}_0 \sim 2^{1/2}\,f_B\left(\frac{\bar{Y}}{2^{1/2}}\right),\ S \sim S_B\left(\frac{\bar{Y}}{2^{1/2}}\right), \text{ from (3.83)}, \qquad (3.88)$$

$$\text{as } \bar{Y} \to \infty,\ \bar{\psi}_0 \sim \bar{Y},\ S \to 0, \text{ from (3.84)}. \qquad (3.89)$$

(3.86) have solution

$$\left.\begin{aligned} \bar{\psi}_0 &= 2^{1/2}\,f_B\left(\frac{\bar{Y}}{2^{1/2}}\right), \\ S_0 &= S_B\left(\frac{\bar{Y}}{2^{1/2}}\right). \end{aligned}\right\} \qquad (3.90)$$

$$O(\epsilon^{1/2}) \; : \; \bar{\psi}_{0_{\bar{Y}}} \bar{\psi}_{1_{\bar{X}\bar{Y}}} - \bar{\psi}_{1_{\bar{X}}} \bar{\psi}_{0_{\bar{Y}\bar{Y}}} = 0, \qquad \bar{\psi}_{0_{\bar{Y}}} S_{1_{\bar{X}}} - \bar{\psi}_{1_{\bar{X}}} S_{0_{\bar{Y}}} = 0, \tag{3.91}$$

with boundary conditions

$$\text{as } \bar{Y} \to 0, \; \bar{\psi}_1 \sim -\bar{Y} g(\bar{X}), \; S_1 \sim -\frac{\theta}{\alpha} g(\bar{X}), \text{ from (3.82)}, \tag{3.92}$$

$$\text{as } \bar{X} \to -\infty, \; \bar{\psi}_1 \sim \frac{\alpha^{2/3}\bar{C}}{2^{1/3}} f_B'\left(\frac{\bar{Y}}{2^{1/2}}\right)(-\bar{X})^{-1/3}, \qquad S_1 \sim \frac{\alpha^{2/3}\bar{C}}{2^{5/6}} S_B'\left(\frac{\bar{Y}}{2^{1/2}}\right)(-\bar{X})^{-1/3}, \text{ from (3.83)}, \tag{3.93}$$

$$\text{as } \bar{Y} \to \infty, \; \frac{\partial \bar{\psi}_1}{\partial \bar{Y}} \to 0, \; S_1 \to 0, \text{ from (3.84)}. \tag{3.94}$$

$$(3.91) \Rightarrow \bar{\psi}_{0_{\bar{Y}}}^{-2} \frac{\partial}{\partial \bar{Y}} \left[\bar{\psi}_{1_{\bar{X}}} / \bar{\psi}_{0_{\bar{Y}}} \right] = 0 \tag{3.95}$$

$$\Rightarrow \bar{\psi}_1 = F_1(\bar{X}) \, f_B'\left(\frac{\bar{Y}}{2^{1/2}}\right) + \Gamma_1(\bar{Y}). \tag{3.96}$$

$$(3.92) \Rightarrow F_1(\bar{X}) = -\frac{g(\bar{X}) 2^{1/2}}{\alpha}, \; \Gamma_1(\bar{Y}) = o(\bar{Y}), \tag{3.97}$$

and (3.93) is satisfied provided that $\Gamma_1(\bar{Y}) \equiv 0$
(using (3.A5)); (3.98)

(3.94) is then satisfied.

$$\text{Then } (3.91) \Rightarrow S_1 = -\frac{g(\bar{X})}{\alpha} S_B'\left(\frac{\bar{Y}}{2^{1/2}}\right) + \Gamma_2(\bar{Y}); \tag{3.99}$$

$$(3.93) \Rightarrow \Gamma_2(\bar{Y}) \equiv 0; \tag{3.100}$$

(3.92) and (3.94) are then satisfied.

The matching is complete !

3.5 Position of separation

From (3.64) & (3.75) the following series for $\tau = \left(\frac{\partial U}{\partial Y}\right)_{Y=0}$ was obtained :

$$\tau = 1 + \tilde{f}_1''(0)\chi + \tilde{f}_2''(0)\chi^2 + \tilde{f}_3''(0)\chi^3 + \tilde{f}_4''(0)\chi^4 + \ldots$$

$$- \epsilon^{1/2} \frac{\theta}{\alpha(1+S_w)^{1/2}} \chi^{1/2} (\tilde{h}_0''(0) + \tilde{h}_1''(0)\chi + \tilde{h}_2''(0)\chi^2$$

$$+ \tilde{h}_3''(0)\chi^3 + \ldots) + O(\epsilon(\log\epsilon)^2),$$

where $\chi = (-X)^{-2/3}(1+S_w)$. (3.101)

Write $\tau = \tau_0 + \epsilon^{1/2}\tau_1 + O(\epsilon(\log\epsilon)^2)$. (3.102)

Substituting numerical values gives

$$\tau_0 = 1 - 1.85747\chi - 0.73139\chi^2 - 0.91578\chi^3 - 1.51241\chi^4$$

$$- 2.84792\chi^5 - 5.79375\chi^6 - \ldots$$

Writing $\tau_0 = \sum_{n=0}^{\infty} a_n \chi^n$ gives $a_0 = 1$ and the coefficients a_n $(n \geqslant 1)$ as in table 3.1. (3.103)

TABLE 3.1

n	a_n	n	a_n
1	-1.857472236	11	-347.2687130
2	-0.731394491	12	-834.6808880
3	-0.915777155	13	-2029.499242
4	-1.512413003	14	-4983.104139
5	-2.847920469	15	-12338.09366
6	-5.793747622	16	-30771.50532
7	-12.40519071	17	-77233.97433
8	-27.54582352	18	-194940.8295
9	-62.85547009	19	-494494.9315
10	-146.4964754	20	-1259963.433

The boundary layer separates at the first zero, κ , of this function (3.103); the value of κ can be estimated by various methods, of which the following proved the most convergent. Using the fact that the radius of convergence of the series (3.103) is κ, consider the ratios of successive terms of it and then extrapolate using Neville tables (Gaunt & Guttman, 1974, p.187-192).

Table 3.2 gives three series of estimates of $1/\kappa$: the second column gives the value of a_n/a_{n-1}, the third column gives the first Neville extrapolation and the fourth column the second extrapolation.

TABLE 3.2

n	a_n/a_{n-1}		
12	2.40355914		
13	2.43146725	2.76636458	
14	2.45533678	2.76564074	2.76129771
15	2.47598551	2.76506775	2.76134332
16	2.49402429	2.76460596	2.76137339
17	2.50991863	2.76422796	2.76139299
18	2.52402950	2.76391434	2.76140540
19	2.53664116	2.76365103	2.76141283
20	2.54798048	2.76342760	2.76141674

The common limit of these sequences is $1/\kappa = 2.76142$, and thus $\kappa = 0.362132((5))$. (3.104)

Including terms of $O(\epsilon^{1/2})$ it is clear that $\tau = 0$ when

$$\chi = \chi_{sep} = \kappa + \epsilon^{1/2} \frac{\theta}{(1+S_w)^{1/2}} \lambda + \dots \tag{3.105}$$

Write $\bar{\chi} = \chi/\chi_{sep}$. Then, from (3.101),

$$\lambda = \lim_{\bar{\chi}\to 1} \left\{ \frac{\kappa^{1/2}}{\alpha \bar{\chi}^{1/2}} (\tilde{h}_0''(0) + \kappa \tilde{h}_1''(0)\bar{\chi} + \kappa^2 \tilde{h}_2''(0)\bar{\chi}^2 + \dots) \right.$$
$$(\tilde{f}_1''(0) + 2\kappa \tilde{f}_2''(0)\bar{\chi} + 3\kappa^2 \tilde{f}_3''(0)\bar{\chi}^2 + 4\kappa^3 \tilde{f}_4''(0)\bar{\chi}^3$$
$$\left. + \dots)^{-1} \right\}. \tag{3.106}$$

If $\sigma = 1$,

$$\lambda = -3.9554349 \lim_{\kappa_* \to \kappa} \left\{ \frac{\kappa}{\kappa_*^{1/2}} (1 + 0.7000507\kappa_* \right.$$

$$+ 1.3381367\kappa_*^2 + 2.9797876\kappa_*^3 + 7.0632692\kappa_*^4$$

$$+ 17.324866\kappa_*^5 + 43.423143\kappa_*^6 + 110.47241\kappa_*^7 + 284.14078\kappa_*^8$$

$$+ 736.95635\kappa_*^9 + 1924.0401\kappa_*^{10} + 5050.1560\kappa_*^{11} + \ldots)$$

$$(1 + 0.7875159\kappa_* + 1.4790700\kappa_*^2 + 3.2569273\kappa_*^3 + 7.6661185\kappa_*^4$$

$$+ 18.714942\kappa_*^5 + 46.749735\kappa_*^6 + 118.63789\kappa_*^7 + 304.55326\kappa_*^8$$

$$+ 788.68730\kappa_*^9 + 2056.5346\kappa_*^{10} + 5392.3663\kappa_*^{11} + \ldots)^{-1}\Big\}$$

$$= -3.9554349 \lim_{\kappa_* \to \kappa} \Big\{ \frac{\kappa}{\kappa_*^{1/2}} (1 - 0.08746521\kappa_*$$

$$- 0.07205302\kappa_*^2 - 0.09102963\kappa_*^3 - 0.1397227\kappa_*^4 - 0.2402128\kappa_*^5$$

$$- 0.4450109\kappa_*^6 - 0.8693851\kappa_*^7 - 1.767377\kappa_*^8 - 3.705812\kappa_*^9$$

$$- 7.965175\kappa_*^{10} - 17.47098\kappa_*^{11} + \ldots)\Big\} .$$

Repeated Aitken extrapolation gives

$$\lambda = -2.2518(2). \qquad (3.107)$$

If $\sigma = 0.72$,

$$\lambda = -3.9554349 \lim_{\kappa_* \to \kappa} \Big\{ \frac{\kappa}{\kappa_*^{1/2}} (1 + 0.7091940\kappa_* +$$

$$+ 1.3529351\kappa_*^2 + 3.0095415\kappa_*^3 + 7.1292645\kappa_*^4 + 17.47937\kappa_*^5$$

$$+ 43.797284\kappa_*^6 + 111.39924\kappa_*^7 + 286.47457\kappa_*^8 + 742.90520\kappa_*^9$$

$$+ 1939.3482\kappa_*^{10} + 5089.8504\kappa_*^{11} + \ldots) (1 + 0.7875159\kappa_*$$

$$+ 1.4790700\kappa_*^2 + 3.2569273\kappa_*^3 + 7.6661185\kappa_*^4 + 18.714942\kappa_*^5$$

$$+ 46.749735\kappa_*^6 + 118.63789\kappa_*^7 + 304.55326\kappa_*^8 + 788.68730\kappa_*^9$$

$$+ 2056.5346\kappa_*^{10} + 5392.3663\kappa_*^{11} + \ldots)^{-1}\Big\}$$

$$= -3.9554349 \lim_{\kappa_* \to \kappa} \Big\{ \frac{\kappa}{\kappa_*^{1/2}} (1 - 0.07832197\kappa_*$$

$$- 0.06445506\kappa_*^2 - 0.08078268\kappa_*^3 - 0.1228139\kappa_*^4 - 0.2090123\kappa_*^5$$
$$- 0.3831158\kappa_*^6 - 0.7406530\kappa_*^7 - 1.489366\kappa_*^8 - 3.088347\kappa_*^9$$
$$- 6.562952\kappa_*^{10} - 14.22857\kappa_*^{11} - \ldots)\}.$$

Repeated Aitken extrapolation gives

$$\lambda = -2.2665(4). \qquad (3.108)$$

3.6. Analysis of flow close to the position of separation

To analyse flow close to the position of separation, following Goldstein (1948) and Stewartson (1962) we introduce a length scale l^*, defined by

$$l^* = \frac{-U_1^*(X^*_s)}{\left[\frac{dU_1^*}{dX^*}\right]_s (1+S_w)}$$

and a Reynolds number R, defined by

$$R = \frac{U_1^*(X_s^*)l^*}{\nu_0} .$$

Buckmaster (1970) showed that in the neighbourhood of separation, if $\sigma = 1$,

$$\frac{l^* R^{-1/2}}{U_1^*(X_s^*)}\left(\frac{\partial U^*}{\partial Y^*}\right)_w = 2\sqrt{2}\left\{\xi^{*2}(\alpha_{10}\log\xi^* + \alpha_{12}\log\left|\log\xi^*\right| + \alpha_{11}\right.$$
$$\left. + \alpha_{13}\log\left|\log\xi^*\right|/\log\xi^* + \ldots) + O\left(\xi^{*3}\left(\log\xi^*\right)^2\right)\right\} \qquad (3.109)$$

$$\text{and} \quad \frac{l^{*}R^{-1/2}}{(1+S_w)}\left(\frac{\partial S}{\partial Y^{*}}\right)_w = \sqrt{2}\, B^{*}\Big\{ {}^{1}/_{2} - \bar{g}_2'(0)\,\xi^{*}\,(\alpha_{10}\log\xi^{*}$$

$$+ \alpha_{12}\log\left|\log\xi^{*}\right| + \alpha_{11} + \alpha_{13}\log\left|\log\xi^{*}\right|/\log\xi^{*} + \ldots)$$

$$+ O\left(\xi^{*2}\left(\log\xi^{*}\right)^{2}\right)\Big\}, \qquad (3.110)$$

$$\left.\begin{aligned} &\text{where } \xi^{*} = \left(\frac{X_s^{*} - X^{*}}{l^{*}}\right)^{1/4},\ \alpha_{10} = -\,0.091148\,B^{*},\ \alpha_{12} \\ &= 0.035210B^{*},\ \alpha_{13} = -0.013601B^{*},\ \text{and } \bar{g}_2'(0) = -\,1.111552. \end{aligned}\right\} (3.111)$$

In incompressible flow, however, $B^{*} = 0$, and it is possible to take the series for the skin friction much further as Goldstein (1948), Stewartson (1958) and Terrill (1960) showed :

$$\frac{l^{*}R^{-1/2}}{U_1^{*}(X_s^{*})}\left(\frac{\partial U^{*}}{\partial Y^{*}}\right)_w = 2\sqrt{2}\,\Big\{\alpha_{11}\,\xi^{*2} + 1.77848\alpha_{11}^{2}\,\xi^{*3}$$

$$+ 3.31101\alpha_{11}^{3}\,\xi^{*4} + 7.15731\alpha_{11}^{4}\,\xi^{*5} + \ldots\Big\}, \qquad (3.112)$$

In this particular problem

$$\left.\begin{aligned} &U_1^{*}(X_s^{*}) = u_0(1 + O(\epsilon\log\epsilon)), \\ &l^{*} = \frac{2\alpha^{-2}\,\epsilon^{1/2}\,(-X_s)L}{(1 + S_w)}, \\ &\text{and } R = \frac{2\alpha^{-2}\epsilon^{1/2}\,(-X_s)R_0\,(1 + O(\epsilon\log\epsilon)),}{(1 + S_w)} \\ &\text{where } R_0 = \frac{Lu_0}{\nu_0}\,. \end{aligned}\right\} (3.113)$$

Then $$\frac{l^* R^{-1/2}}{U_1^*(X_s^*)}\left(\frac{\partial U^*}{\partial Y^*}\right)_w = \frac{(-X_s)^{1/2}\,\epsilon^{1/4}}{(1+S_w)^{1/2}}\,(1 + O(\epsilon\log\epsilon))\left(\frac{\partial U}{\partial Y}\right)_w \tag{3.114}$$

and $$\frac{l^* R^{-1/2}}{(1+S_w)}\left(\frac{\partial S}{\partial Y^*}\right)_w = \frac{(-X_s)^{1/2}\,\epsilon^{-1/4}}{(1 + S_w)^{3/2}}\,(1 + O(\epsilon\log\epsilon))\left(\frac{\partial S}{\partial Y}\right)_w . \tag{3.115}$$

Now $$\xi^* = \epsilon^{1/4}\,(1+S_w)^{1/4}\left[\left(\frac{-X}{-X_s}\right)-1\right]^{1/4} . \tag{3.116}$$

Since $(\partial S/\partial Y)_w = O(\epsilon^{1/2})$, it is clear that $B^* = O(\epsilon^{1/4})$. However, since $(\partial U/\partial Y)_w = O(1)$ it is clear that $\alpha_{11} = O(\epsilon^{-1/4})$. So, although this flow is compressible, in the neighbourhood of separation it behaves to first order like an incompressible flow. This argument, of course, only holds for $\sigma = 1$, but it is clear from (3.101) that if $\sigma \neq 1$ but is of order unity the same result holds.

We write $$\left.\begin{aligned} \alpha_{11} &= \phi\,\epsilon^{-1/4}(1+S_w)^{-1/4} + O(\epsilon^{1/4}\log\epsilon) , \\ B^* &= \sqrt{2}\,\kappa^{-3/4}(1+S_w)^{-3/4}\epsilon^{1/4}\,\theta\beta_1 + O(\epsilon^{3/4}\log\epsilon), \\ \text{and}\quad \omega &= \left[\frac{(-X)}{(-X_s)}\right]^{-2/3} . \end{aligned}\right\} \tag{3.117}$$

Then $$\left(\frac{\partial U}{\partial Y}\right)_w = 2\sqrt{2}\,\kappa^{3/4}\Big\{(3/2)^{1/2}\,\phi(1-\omega)^{1/2} + 1.77848\,(3/2)^{3/4}\,\phi^2(1-\omega)^{3/4} + 3.31101\,(3/2)\,\phi^3(1-\omega) + 7.15731\,(3/2)^{5/4}\,\phi^4(1-\omega)^{5/4} + \dots\Big\} , \tag{3.118}$$

Following Curle (1979a) rewrite (3.103) in terms of ω; subtract the leading term of (3.118) (expanded in powers of ω); and equate the nth partial sums (evaluated at $\omega=1$) to zero. This gives a convergent sequence of estimates for ϕ. A better sequence of estimates is obtained by retaining the ϕ^2 terms and an even better one by retaining ϕ^4 terms as well (the ϕ^3 terms have no effect on the result).

The estimates of ϕ from the quartic equations are shown in Table 3.3.

TABLE 3.3

Estimates of ϕ

n	Quartic
12	0.3065 523
13	0.3063 738
14	0.3062 256
15	0.3061 014
16	0.3059 967
17	0.3059 079
18	0.3058 323
19	0.3057 677
20	0.3057 125

Extrapolation gives

$$\phi = 0.30538 \qquad (3.119)$$

and thus $\alpha_{11} = \epsilon^{-1/4}\, 0.30538(1+S_w)^{-1/4} + \ldots \qquad (3.120)$

The terms of $O(\xi^{*6} \log \xi^{*})$ and $O(\xi^{*6})$ in the series for the skin friction (3.112) were determined by Terrill (1960), who showed that the coefficient of $\overset{*}{\xi}{}^{6} \log \overset{*}{\xi}$ is a function of α_{11} but that the coefficient of $\overset{*}{\xi}{}^{6}$, α_{55}, is independent of α_{11}. Accordingly we made an attempt to obtain the value of α_{55}, but its results were inconclusive.

We next consider the heat transfer: from (3.70)

$$t = \left(\frac{\partial \Sigma_1}{\partial Y}\right)_{Y=0} = -\frac{\theta}{\alpha}\left\{-1 + \tilde{S}_1'(0)\chi + \tilde{S}_2'(0)\chi^2 + \tilde{S}_3'(0)\chi^3 + \dots\right\}. \tag{3.121}$$

If $\sigma = 1$,

$$t = -\frac{\theta}{\alpha}\left\{-1 + 0.5861005\chi + 0.5096957\chi^2 + 0.7546612\chi^3 + 1.3575590\chi^4 + 2.7045185\chi^5 + 5.7378054\chi^6 + 12.703829\chi^7 + 29.007993\chi^8 + 67.800243\chi^9 + 161.38880\chi^{10} + 389.84201\chi^{11} + 953.09801\chi^{12} + \dots\right\}. \tag{3.122}$$

If $\sigma = 0.72$,

$$t = -\frac{\theta}{\alpha}\left\{-1 + 0.5590510\chi + 0.4880867\chi^2 + 0.7232773\chi^3 + 1.3017150\chi^4 + 2.5941815\chi^5 + 5.5053291\chi^6 + 12.192204\chi^7 + 27.846009\chi^8 + 65.097537\chi^9 + 154.98401\chi^{10} + 374.43446\chi^{11} + 915.57277\chi^{12} + \dots\right\}. \tag{3.123}$$

As $\chi \to \kappa$, these series (3.122) and (3.123) converge increasingly slowly so that they only give extremely rough estimates of the heat transfer at separation. However, since (even though the heat transfer is not small) B^* is small, more terms in the series for the heat transfer in the neighbourhood of separation (3.110) can be derived for general σ (Akinrelere 1977) and it can be shown that

$$t = \theta\beta_1\left\{1 + 2.22310\sigma^{1/4}\left(\frac{3}{2}\right)^{1/4}\phi(1-\omega)^{1/4}\right.$$

$$\left. + (2.62098\sigma^{1/2} + 2.09679\sigma^{1/4})\left(\frac{3}{2}\right)^{1/2}\phi^2(1-\omega)^{1/2} + \ldots\right\}. \tag{3.124}$$

If $\sigma = 1$, the next term in the series can also be derived,

$$10.69395\left(\frac{3}{2}\right)^{3/4}\phi^3(1-\omega)^{3/4}. \tag{3.125}$$

Expanding Akinrelere's series ((3.124) and (3.125)) and dividing into (3.122) (rewritten in terms of ω) give

$$t/\left\{1 + 0.751316(1-\omega)^{1/4} + 0.538845(1-\omega)^{1/2} + 0.411245(1-\omega)^{3/4}\right\}$$

$$= 0.7882829\ \theta(1 + 0.0711935\omega + 0.0186168\omega^2 + 0.0077084\omega^3$$

$$+ 0.0040404\omega^4 + 0.0024393\omega^5 + 0.0016167\omega^6 + 0.0011443\omega^7$$

$$+ 0.0008505\omega^8 + 0.0006562\omega^9 + 0.0005216\omega^{10} + 0.0004247\omega^{11}$$

$$+ 0.0003528\omega^{12} + \ldots)\ ,\ \text{if } \sigma = 1. \tag{3.126}$$

It is useful also for purposes of comparison to derive

$$t/\left\{1 + 0.751316(1-\omega)^{1/4} + 0.538845(1-\omega)^{1/2}\right\}$$

$$= 0.9298352\theta(1 - 0.0125869\omega - 0.0091878\omega^2 - 0.0057843\omega^3$$

$$- 0.0039402\omega^4 - 0.0028645\omega^5 - 0.0021848\omega^6 - 0.0017272\omega^7$$

$$- 0.0014036\omega^8 - 0.0011659\omega^9 - 0.0009856\omega^{10} - 0.0008455\omega^{11}$$

$$- 0.0007342\omega^{12} - \ldots),\ \text{if } \sigma = 1. \tag{3.127}$$

$$t/\left\{1 + 0.6920789(1-\omega)^{1/4} + 0.4746184(1-\omega)^{1/2}\right\}$$

$$= 0.9828193\theta(1 - 0.0130706\omega - 0.0091562\omega^2 - 0.0056730\omega^3$$
$$- 0.0038270\omega^4 - 0.0027635\omega^5 - 0.0020972\omega^6 - 0.0016515\omega^7$$
$$- 0.0013379\omega^8 - 0.0011083\omega^9 - 0.0009349\omega^{10} - 0.0008004\omega^{11}$$
$$- 0.0006938\omega^{12} - \ldots) \text{ , if } \sigma = 0.72. \qquad (3.128)$$

At separation repeated extrapolation gives

$$\left.\begin{aligned} t_s &= 0.8778\,\theta && \text{if } \sigma = 1. \\ t_s &= 0.930\,\theta && \text{if } \sigma = 0.72. \end{aligned}\right\}(3.129)$$

So $$\left(\frac{\partial T}{\partial y}\right)_{y=0}\Bigg|_{sep} = T_s \frac{\alpha}{\sqrt{2}} \epsilon^{-1/2} \left(\frac{\partial S}{\partial Y}\right)_{Y=0}\Bigg|_{sep}$$

$$= \frac{T_s \alpha t_s}{\sqrt{2}} + O(\epsilon^{1/2})$$

$$= \theta T_s t_s^*(\sigma) + O(\epsilon^{1/2}),$$

$$\left.\begin{aligned} \text{where } t_s^*(1) &= 0.2915 \text{ ,} \\ t_s^*(0.72) &= 0.309 \text{ .} \end{aligned}\right\} \quad (3.130)$$

APPENDIX TO CHAPTER 3

EVALUATION OF $\tilde{f}_1''(0)$, $\tilde{f}_1'(\infty)$, $\tilde{h}_0''(0)$ and, when $\sigma = 1$, $\tilde{S}_1'(0)$.

From (3.65) $3\tilde{f}_1''' - \eta^2 \tilde{f}_1'' = 3,$

where $\tilde{f}_1(0) = \tilde{f}_1'(0) = 0$, $\tilde{f}_1' \to$ a constant as $\eta \to \infty$ } (3.A1)

$$\Rightarrow \tilde{f}_1''(\eta) = -e^{\eta^3/9} \int_{\eta}^{\infty} e^{-t^3/9}\, dt \tag{3.A2}$$

$$\Rightarrow \tilde{f}_1''(0) = -\int_0^{\infty} e^{-t^3/9}\, dt = \frac{-2\pi}{3^{5/6}(-1/3)!} . \tag{3.A3}$$

$$\tilde{f}_1'(\infty) = -\int_0^{\infty} e^{\eta^3/9} \int_{\eta}^{\infty} e^{-t^3/9}\, dt\, d\eta \tag{3.A4}$$

$$= -\Gamma(2/3)\, 3^{-2/3}\, B(1/3, 1/3)$$

$$= \frac{-2\pi\ \Gamma(4/3)}{3^{1/6}\ \Gamma(2/3)} . \tag{3.A5}$$

From (3.76) $\tilde{h}_0''' - 1/3\,\eta^2\tilde{h}_0'' + 1/3\,\eta\tilde{h}_0' - 1/3\,\tilde{h}_0 = -\eta,$

where $\tilde{h}_0(0) = \tilde{h}_0'(0) = 0$ and $\tilde{h}_0$ does not grow exponentially as $\eta \to \infty$. (3.A6)

$$(3.A6) \Rightarrow \tilde{h}_0^{IV} - 1/3\,\eta^2\tilde{h}_0''' - 1/3\,\eta\tilde{h}_0'' = -1. \tag{3.A7}$$

Write $z = \frac{1}{9}\eta^3$, $\tilde{h}_0'' = h^*$: (3.A8)

$$z\frac{d^2h^*}{dz^2} + (2/3 - z)\frac{dh^*}{dz} - 1/3\, h^* = \frac{1}{z^{1/3}\, 3^{2/3}} . \tag{3.A9}$$

Following Erdélyi et al. (1953, p.265),

set $h^* = (^1/_2\xi)^{1/6} e^{\xi} J(\xi), \quad \xi = {}^1/_2 z$: (3.A10)

Then $J''(\xi) + {}^1/_\xi J'(\xi) - \left(1 + \frac{1}{36\xi^2}\right) J(\xi) = -\frac{e^{-\xi} 2^{5/6}}{\xi^{3/2} 3^{2/3}}$.

(3.A11)

The solutions which do not grow exponentially as $\eta \to \infty$ are

$$J(\xi) = \frac{2^{5/6}}{3^{2/3}} \left\{ I_{1/6}(\xi) \int_\xi^\infty \frac{e^{-s}}{s^{1/2}} K_{1/6}(s)ds + K_{1/6}(\xi) \left(\int_0^\xi \frac{e^{-s}}{s^{1/2}} I_{1/6}(s)ds + E \right) \right\}, \tag{3.A12}$$

where E is a constant.

Now, since $\frac{d^3\tilde{h}_0}{d\eta^3}(0) = 0$,

$$E = \frac{\sqrt{3}}{\pi} \int_0^\infty \frac{e^{-s}}{s^{1/2}} K_{1/6}(s)ds$$

$$= \sqrt{\frac{2\pi}{3}} \text{ (using Luke (1962, p.107)).} \tag{3.A13}$$

So $$\frac{d^2\tilde{h}_0}{d\eta^2}(0) = \frac{(2\pi)^2}{3^{5/3}\left(\Gamma(^2/_3)\right)^2}. \tag{3.A14}$$

If $\sigma = 1$, $\tilde{S}_1'' - {}^1/_3\eta^2\tilde{S}_1' - {}^1/_3\eta\tilde{S}_1 = {}^1/_3\eta\tilde{f}_1'$ (from (3.70)),

where $\tilde{S}_1(0) = 0$, $\tilde{S}_1 \to$ a constant as $\eta \to \infty$. (3.A15)

Write $\tilde{S}_1 = -\tilde{f}_1{}' + S^*$. (3.A16)

Then $S^{*''} - {}^1/_3\eta^2 S^{*'} - {}^1/_3\eta S^* = 1,$

where $S^*(0) = 0$, $S^* \to$ a constant as $\eta \to \infty$. (3.A17)

Set $S^* = ({}^1/_2\xi)^{1/6}\, e^{\xi}\, I(\xi)$, $\xi = \frac{1}{18}\eta^3$. (3.A18)

Then $I''(\xi) + {}^1/_\xi\, I'(\xi) - \left(1 + \frac{1}{36\xi^2}\right) I(\xi) = \frac{e^{-\xi}\, 2^{5/6}}{\xi^{3/2}\; 3^{2/3}}$.

(3.A19)

The solution of (3.A19) which satisfies the boundary conditions is

$$I(\xi) = -\frac{2^{5/6}}{3^{2/3}}\left\{ I_{1/6}(\xi) \int_\xi^\infty \frac{e^{-s}}{s^{1/2}} K_{1/6}(s)\,ds + K_{1/6}(\xi) \int_0^\xi \frac{e^{-s}}{s^{1/2}}\, I_{1/6}(s)\,ds \right\}. \quad (3.A20)$$

So $$\frac{ds^*}{d\eta}(0) = -\frac{2^{1/6}}{3^{4/3}\,\Gamma({}^7/_6)} \int_0^\infty \frac{e^{-s}}{s^{1/2}} K_{1/6}(s)\,ds =$$

$$-\frac{(2\pi)^2}{3^{4/3}\left(\Gamma({}^1/_3)\right)^2}. \quad (3.A21)$$

So $$\tilde{S}_1{}'(0) = \frac{2\pi}{3^{5/6}}\left\{ \frac{1}{\Gamma({}^2/_3)} - \frac{2\pi}{\sqrt{3}\,\left(\Gamma({}^1/_3)\right)^2} \right\}. \quad (3.A22)$$

4. AN ACCURATE CALCULATION METHOD FOR TWO-DIMENSIONAL COMPRESSIBLE BOUNDARY LAYERS WITH AN UNFAVOURABLE PRESSURE GRADIENT.

4.1. Introduction

This chapter provides a method of calculating compressible boundary layers, predicting positions of separation, skin friction, heat transfer, displacement thickness and momentum thickness. The method is a generalization to compressible flow of the work of Stratford (1954) and Curle (1977), which is summarized in chapter 1 ((1.8) to (1.23)). Sections 4.2 to 4.6 present the method and its derivation for the case $\sigma = 1$ making use of the results of Davies and Walker (1977) for a compressible flow with a linearly retarded external velocity (untransformed); section 4.7 gives the generalization to the case $\sigma = O(1)$. The whole procedure makes use of the Illingworth (1949) - Stewartson (1949) transformation.

4.2. A Criterion for Predicting Boundary-Layer Separation

Following Stratford (1954) the following definitions are made:

$$Cp = 1 - \frac{U_1^2}{u_0^2}, \qquad (4.1)$$

$$\Delta = \frac{Cp}{XdCp/dX}, \qquad (4.2)$$

$$\Gamma = Cp\,\frac{d^2Cp}{dX^2} \Big/ \left[\frac{dCp}{dX}\right]^2 , \qquad (4.3)$$

$$\text{and} \quad F = \left\{ 10X\frac{dCp}{dX} \right\}^2 Cp , \qquad (4.4)$$

where X is the transformed co-ordinate parallel to the plate and U_1 the transformed external velocity.

We begin by examining the compressible boundary layer for which the pressure is constant when $X < X_0$ and the transformed external velocity U_1 is chosen such that

$$\lambda = -\frac{X_0}{u_0^2}\,U_1\,\frac{dU_1}{dX}\,\frac{T_w}{T_s}$$

is constant, where T_w is the wall temperature and T_s the stagnation temperature. This will be referred to as the sharp quasi-uniform pressure gradient problem; the solution was obtained by Curle (1978).

From his paper it is clear that at separation

$$F = 0.74514(1+S_w)^{-3}\left\{ 1 + 0.041269\sigma_w\,\lambda^{-1} + 0.003594\sigma_w^{\,2}\lambda^{-2} + \ldots\right\}^3 ,$$

where $S_w = T_w/T_s - 1$

and $\sigma_w = S_w/(1+S_w)$.

It is easy to show that at separation

$$\lambda^{-1}\,\xi_s = \Delta^{1/3} + O(\Delta). \qquad (4.5)$$

Eliminating λ gives that at separation

$$(1+S_w)F^{1/3} = 0.90659 + 0.3831\sigma_w \Delta^{1/3} + 0.1797\sigma_w^2 \Delta^{2/3} + \dots \qquad (4.6)$$

The compressible continuous incipient separation problem (chapter 2) gives

$$(1+S_w)F^{1/3} = 0.839087\,(1 + 0.001020r - 0.010695r^2 - 0.011794r^3 - \dots) + 0.413710\sigma_w \Delta^{1/3}(1 - 0.029665r - 0.021212r^2 - 0.017527r^3 - \dots) + 0.206971\sigma_w^2 \Delta^{2/3}(1 - 0.069787r - \dots) + 0.105360\sigma_w^3 \Delta\,(1 - \dots) + \dots, \qquad (4.7)$$

where $r = 1 - X_0/X$.

It is clear that the coefficients of powers of $\sigma_w \Delta^{1/3}$ in the expansion of $(1 + S_w)F^{1/3}$ vary very slowly with X, just as F varies very slowly with x in the incompressible case (Curle, 1976b).

The compressible analogue of the Riley and Stewartson problem (chapter 3) gives

$$(1+S_w)F^{1/3} = 0.77399\,\Gamma^{1/3} + 0.65215\sigma_w \Delta^{1/3} + \dots, \qquad (4.8)$$

as $\Gamma \to \infty$.

A new solution for a problem in which the pressure is uniform when $X < X_0$ and the transformed external velocity U_1 is chosen so that for $X > X_0$

$$\frac{X_0}{u_0^2}\, U_1 \frac{dU_1}{dX} (1+S_w) = -\lambda\left(\frac{X}{X_0} - 1\right),$$

where λ is large and constant, the compressible analogue of the solution derived in appendix A of Curle (1977), gives

$$(1+S_w)F^{1/3} = 0.96743 + 0.3736\sigma_w \Delta^{1/3} + 0.1501\sigma_w^2 \Delta^{2/3} + \ldots \quad (4.9)$$

The number 0.96743 is a more accurate value than the number 0.9704 quoted by Curle. The details of the solution are not presented here.

From these results it is clear that, if $\Delta << 1$

$$(1+S_w)F^{1/3} = a_0(\Gamma) + a_1(\Gamma)\,\sigma_w\,\Delta^{1/3} + a_2(\Gamma)\sigma_w^2\,\Delta^{2/3} + \ldots, \quad (4.10)$$

where a_0, a_1, and a_2 are shown in table 4.1.

TABLE 4.1

Γ	a_0	a_1	a_2
$-1/2$	0.8391	0.4137	0.2070
0	0.9066	0.3831	0.1797
$1/2$	0.9674	0.3736	0.1501
$\to \infty$	$0.7740\Gamma^{1/3}$	0.6521	unknown

Moreover, examining the results from the continuous incipient separation problem (4.7), we see that, if Δ is not small, a_1 changes only very slowly as X changes and is roughly constant along the line along which a_0 is constant. Now a_0 may be closely approximated by $F_c^{1/3}$, where F_c is an approximation to the value which F would have at separation for an incompressible problem with the same Δ and Γ as the compressible problem. The approximation used is that derived by Curle (1977), that is,

$$F_c = 0.46367\, X_c + (0.10161\, X_c^2 + 0.36224\, X_c + 0.74514)e^{-2X_c/3}, \tag{4.11}$$

where X_c is the greater root of the equation

$$X_c^2 + \left(\frac{14.1783}{\Delta} + 1.2462\right) X_c = 13.3794 + \frac{14.1783}{\Delta}\Gamma \; . \tag{4.12}$$

Furthermore, extrapolation is logical; so the above suggests that to a good approximation at separation

$$(1+S_w)F^{1/3} = F_c^{1/3} + a_1(X_c)\,\sigma_w\,\Delta^{1/3} + a_2(X_c)\sigma_w^2\,\Delta^{2/3}/(1 - a_3(X_c)\sigma_w\,\Delta^{1/3}), \tag{4.13}$$

where $a_3 \doteqdot a_2/a_1$.

It might appear best to choose a_1 and a_2 so that the formula (4.13) gives exact results as $\Delta \to 0$. However, closer examination of the continuous incipient separation problem reveals that as X increases the curve defined parametrically by

$$\Delta = \Delta(X)$$

and $\Gamma = \Gamma(X)$,

along which $(1+S_w)F^{1/3}$ is a slowly varying function of X, depends significantly on the value of σ_w. In fact in the case of this problem much better results are obtained by taking the value of a_1 when X_c is $-{}^1/_2$ as 0.366 rather than 0.414.

In the case of the linearly retarded external velocity solutions (Davies and Walker, 1977) it is found that better results are obtained by taking the value of a_1 when X_c is ${}^1/_2$ as 0.405 rather than 0.374. So we take as our approximation

$$a_1(X_c) = 0.6521 - (0.0231X_c^2 + 0.1398\,X_c + 0.2690)e^{-2X_c/3}. \tag{4.14}$$

This is exact at 0 and as $X_c \to \infty$ and takes values 0.3662 when $X_c = -{}^1/_2$ and 0.4051 when $X_c = {}^1/_2$.

a_2 is approximated by

$$a_2(X_c) = 1.11\;a_1^2/F_c^{\,1/3},$$

which is exact at 0 and probably has the right form as $X_c \to \infty$.

In the case of both the linearly retarded external velocity and the continuous incipient separation solutions better results are obtained by taking a_3 slightly larger than a_2/a_1; it seems best therefore to predict that separation occurs when

$$(1+S_w)F^{1/3} = F_c^{\,1/3} + a_1\,\sigma_w\,\Delta^{1/3} + 1.11(a_1^2/F_c^{\,1/3})\sigma_w^{\,2}\,\Delta^{2/3}/$$
$$(1-1.3\,(a_1/F_c^{\,1/3})\,\sigma_w\,\Delta^{1/3}). \tag{4.15}$$

We write

$$\Phi = (1+S_w)\;(F/F_c)^{1/3}, \tag{4.16}$$

$$\Phi_o = 1 + (a_1/F_c^{\,1/3})\sigma_w\,\Delta^{1/3} + 1.11\;(a_1^2/F_c^{\,2/3})\sigma_w^{\,2}\,\Delta^{2/3}/$$

$$(1 - 1.3(a_1/F_c{}^{1/3})\sigma_w\Delta^{1/3}), \tag{4.17}$$

and $z = \Phi / \Phi_0$. (4.18)

Then at separation z should equal 1.

In the continuous incipient separation case some values of z are shown in table 4.2. In this table the values of Δ, Γ and z may not necessarily be accurate in the last decimal place.

TABLE 4.2

S_w	r	ξ	Δ	Γ	z
$-1/2$	0.1	0.4808	0.173	-0.851	1.004
$-1/2$	0.2	0.6300	0.374	-1.095	1.007
$-1/2$	0.4	0.8736	0.901	-1.667	1.010
1	0.1	0.4808	0.150	-0.546	1.000
1	0.2	0.6300	0.312	-0.680	1.001
1	0.4	0.8736	0.696	-1.035	1.000

To use the method ((4.16) - (4.18)) to solve a particular problem it is necessary to make the Illingworth-Stewartson transformation and to calculate F, Δ and Γ and hence X_c, F_c, a_1 and z for each of a number of values of x. The approximate separation position is given by the criterion

$$z = 1.$$

The results of applying the method to five of the linearly retarded external velocity solutions calculated by Davies and Walker are shown in table 4.3.

TABLE 4.3

Mach Number	S_w	Exact Separation position	Predicted Separation position
1	$-1/2$	1.3305	1.3343
1	1	0.5240	0.5252
1	$-1/6$	0.9934	0.9944
$1/2$	$-1/2$	1.3833	1.3886
$1/2$	1	0.5711	0.5720

4.3. A procedure for predicting the distribution of skin friction.

We begin by examining the sharp quasi-uniform pressure gradient problem in which

$$\frac{2\nu_0 X}{u_0^3}\left(\frac{\partial U}{\partial Y}\right)^2 = F_0(\bar{\xi}) + \sigma_w \lambda^{-1} F_1(\bar{\xi}) + \sigma_w^2 \lambda^{-2} F_2(\bar{\xi}) + \ldots, \tag{4.19}$$

where F_0, F_1 and F_2 are given by Curle (1978) as

$$F_0(\bar{\xi})/(1-\bar{\xi}) = 0.22052 - 0.06722\bar{\xi} - 0.01731\bar{\xi}^2 - 0.00904\bar{\xi}^3 - 0.00590\bar{\xi}^4 - 0.00431\bar{\xi}^5 - \ldots,$$

$$100F_1(\bar{\xi})/(1-\bar{\xi}) = -0.1860\log(1-\bar{\xi}) - 1.37350\bar{\xi} + 0.38298\bar{\xi}^2 + 0.04506\bar{\xi}^3 + 0.01157\bar{\xi}^4 + 0.00317\bar{\xi}^5 + \ldots,$$

and $$1000F_2(\bar{\xi})/(1-\bar{\xi}) = -0.09175\log(1-\bar{\xi}) - 1.12594\bar{\xi} + 0.39670\bar{\xi}^2 - 0.04009\bar{\xi}^3 + 0.00354\bar{\xi}^4 + \ldots .$$

The definition of z (4.18) gives

$$z = \bar{\xi}\,\frac{(1 + 0.4226\sigma_w \Delta_{sep}^{1/3} + 0.1983\,\sigma_w^2\, \Delta_{sep}^{2/3} + \ldots)}{(1 + 0.4226\sigma_w \Delta^{1/3} + 0.1983\,\sigma_w^2\, \Delta^{2/3} + \ldots)}\ .$$

Since $\bar{\xi} = (\Delta/\Delta_{sep})^{1/3} + O(\Delta^{4/3})$, eliminating Δ_{sep} and writing $\bar{\xi}$ as a function of z give

$$\bar{\xi} = z - 0.4226\ (1-z)\sigma_w \Delta^{1/3} - 0.\ 1983(1-z)(1 + z^{-1})\sigma_w^2\, \Delta^{2/3} + \ldots\ . \tag{4.20}$$

Substituting into (4.19) gives

$$\frac{2\nu_0 X}{u_0^3}\left(\frac{\partial U}{\partial Y}\right)^2 = F_0(z) + \sigma_w\, \Delta^{1/3}\ F_1^*(z) + \sigma_w^2\ \Delta^{2/3}\ F_2^*(z) + \ldots , \tag{4.21}$$

where $F_1^*(z)/(1-z) = -0.01905(z^{-1}\log(1-z) + 1) - 0.00297z - 0.02696z^2 - 0.01138z^3 - 0.00305z^4 - \ldots ,$

and $F_2^*(z)/(1-z) = -0.0029z - 0.0010z^2 - \ldots$.

We note that, unless z is very near 1, $F_1^*(z)$ and $F_2^*(z)$ are small compared with $F_0(z)$, even if Δ is not small; so it seems reasonable to suppose that the same function of z which gave a good approximation in the incompressible case may give good results in the compressible case too. So we take as our approximation

$$\frac{1}{\alpha}\left(\frac{2\nu_0 X}{u_0^3}\right)^{1/2}\left(\frac{\partial U}{\partial Y}\right)_w = T_c(z) \tag{4.22}$$

where $\alpha = 0.4696000$ (4.23)

and $T_c(z) = 0.48213(1-z)^{1/2} + 0.37428(1-z)^{3/4} + 0.14359(1-z)$, (4.24)

following Curle (1977).

Some values of τ_w (their notation) for the two solutions tabulated by Davies and Walker in their paper are shown in table 4.4 together with the predictions of the formula (4.22).

TABLE 4.4

$S_w = 1$

x	z	τ_w (exact)	τ_w (predicted)
0.1	0.2113	0.9132	0.9132
0.2	0.4122	0.5417	0.5420
0.3	0.6033	0.3488	0.3494
0.4	0.7850	0.2082	0.2092
0.5	0.9578	0.0694	0.0714

$S_w = -1/2$

x	z	τ_w (exact)	τ_w (predicted)
0.2	0.1872	0.6679	0.6699
0.5	0.4403	0.3459	0.3487
0.8	0.6638	0.2052	0.2083
1.1014	0.8624	0.1040	0.1065
1.3039	0.9828	0.0276	0.0297

4.4. A procedure for predicting the heat transfer

In the compressible continuous incipient separation problem the heat transfer can be shown (from (2.155)) to be given by

$$G^* = -\frac{1}{\alpha}\left(\frac{2\nu_0 X}{u_0}\right)^{1/2}\frac{(\partial T/\partial Y)_w}{T_w - T_s} \qquad (4.25)$$

$$= 0.575194 + 0.002741r - 0.002770r^2 - 0.003502r^3 - \dots$$
$$+ 0.079493\sigma_w \ \Delta^{1/3} (1 - 0.105558r - \dots)$$
$$+ 0.029662\sigma_w^2 \ \Delta^{2/3} (1 - \dots) + \dots \qquad (4.26)$$

It is noticeable how slowly the coefficients of powers of $\sigma_w \ \Delta^{1/3}$ vary with r.

In the sharp quasi-uniform pressure gradient problem

$$G^* = G_0^*(z) + \sigma_w \ \Delta^{1/3} \ G_1^*(z) + \dots , \qquad (4.27)$$

where $G_0^*(z) / \left\{ 1 + 0.628505(1-z)^{1/4} + 0.377082(1-z)^{1/2} + 0.240746(1-z)^{3/4} \right\} = 0.445170 \ (1 + 0.028400z + 0.004497z^2 + 0.001224z^3 + 0.000382z^4 + 0.000076z^5 + \dots)$

$$(4.28)$$

and $G_1^*(z) = 0.039093z + 0.010159z^2 + 0.005873z^3 + 0.004185z^4 + \dots$. (4.29)

In the sharp increasing pressure gradient problem (see(4.9))

$$G^* = G_0^*(z) + \sigma_w \ \Delta^{1/3} \ G_1^* \ (z) + \dots , \qquad (4.30)$$

where $G_0^*(z) / \left\{ 1 + 0.681751 \ (1-z)^{1/4} + 0.443681 \ (1-z)^{1/2} + 0.308418 \ (1-z)^{3/4} \right\}$
$= 0.410872 \ (1 + 0.048637z + 0.011696z^2 + 0.004718z^3 + 0.002456z^4 + 0.001484z^5 + \dots)$ (4.31)

and $G_1^*(z) = 0.047976z + 0.014521z^2 + 0.009172z^3 + 0.007002z^4 + \dots$. (4.32)

In the compressible version of the Riley-Stewartson problem (3.126)

$$G^* = G_0^*(z) + O(\Delta^{1/3}), \qquad (4.33)$$

where $G_0^*(z)/\{1 + 0.751316\,(1-z)^{1/4} + 0.538845\,(1-z)^{1/2} + 0.411245(1-z)^{3/4}\}$

$$= 0.370178\,(1+ 0.071194\,z + 0.018617z^2 + 0.007708z^3 + 0.004040z^4 + 0.002439z^5 + \ldots). \qquad (4.34)$$

If $\Delta = 0$ the values of G^* at separation, G^*_{sep}, are shown in table 4.5.

TABLE 4.5

Γ	G^*_{sep}
$-1/2$	0.5752
0	0.4606
$1/2$	0.442
$\to\infty$	0.4122

It is not possible to fit these values of G^*_{sep} to a function of the form $a + (b\Gamma^2 + c\Gamma + d)\,e^{-n\Gamma}$ where $n < 1.5$ which does not increase rapidly for $1/2 < \Gamma < 2$; if, however, $n = 2$, the function of that form which fits the data will decrease continuously. So if $\Delta = 0$ we approximate G^*_{sep} by

$$G_c = 0.4122 + (0.0883\Gamma^2 + 0.0210\Gamma + 0.0484)\,e^{-2\Gamma}.$$

Equation (4.26) suggests that if σ_w is very small G^*_{sep} may well be constant along the same lines in the Γ, Δ plane as F_c is; so, if σ_w is very small, G^*_{sep} may be approximated by

$$G_c = 0.4122 + (0.0883X_c^2 + 0.0210X_c + 0.0484)e^{-2X_c}, \qquad (4.35)$$

where X_c was defined in (4.12).

If σ_w is not small, following (4.26) we look for an approximation to the heat transfer at separation of the form

$$G^*_{sep} = G_c + b_1(X_c)\sigma_w \Delta^{1/3}/(1 - b_2(X_c)\sigma_w \Delta^{1/3}), \quad (4.36)$$

where $b_2 \triangleq b_1/G_c$.

It is not obvious exactly what the correct form of $b_1(X_c)$ is. After some experiment we take

$$b_1 = \frac{0.0707 + (0.0429 - 0.1222X_c + 0.11X_c^2)e^{-2X_c}}{\left(X_c + (1.1 - 0.6X_c)e^{-2X_c}\right)^{1/3}}, \quad (4.37)$$

which decays like $\lambda/X_c^{1/3}$ for large X_c and has roughly the right values at $X_c = 1/2$, 0, - 0.4 and - 0.8. Then we take

$$b_2 = 1.15b_1/G_c.$$

This gives good results in the case of the tabulated solutions of Davies and Walker (given in table 4.7) and in the case of the continuous incipient separation problem. In table 4.6 some values of the heat transfer, Q, for the latter problem together with the predictions of the formula (4.36) are given, where

$$Q = \left(\frac{2\nu_0 X}{u_0}\right)^{1/2} \frac{(\partial T/\partial Y)_w}{T_w} = -\alpha\sigma_w G^* .$$

TABLE 4.6

S_w	r	ξ	Q (exact)	Q(predicted)
$-^1/_2$	0.1	0.4808	0.253	0.250
$-^1/_2$	0.2	0.6300	0.250	0.248
$-^1/_2$	0.4	0.8736	0.246	0.246
1	0.1	0.4808	-0.141	-0.142
1	0.2	0.6300	-0.142	-0.144
1	0.4	0.8736	-0.145	-0.150

It is possible to derive an approximate formula for the heat transfer at points other than the position of separation. From (4.27) to (4.34) it is clear what form of approximation we should try: the following formula gives an exceedingly good fit for those three problems and we take it as our approximation to the heat transfer:

$$G^* = G_c\left\{(1 + (0.7513 - 0.1228e^{-X_c})(1-z)^{1/4} + (0.5388 - 0.1618e^{-X_c})(1-z)^{1/2} + (0.4112 - 0.1705e^{-X_c})(1-z)^{3/4})(1 + (0.0712 - 0.0415e^{-X_c})z + (0.0187 - 0.0133e^{-X_c})z^2 + (0.0077 - 0.0059e^{-X_c})z^3/(1 - (0.6734 + 0.2223e^{-X_c} - 0.2856e^{-2X} z))/(1.1135 - 0.0568e^{-X_c} - 0.0220e^{-2X_c})\right\} + b_1(X_c)\sigma_w \Delta^{1/3}/(1 - 1.15b_1\sigma_w \Delta^{1/3}/G_c)L(z), \qquad (4.38)$$

where
$$L(z) = \begin{cases} z(0.4467 + 0.1162z + 0.0671z^2 + 0.0478z^3/(1-z)), & \text{if } z \leq 0.5. \\ z(1 - 0.55(1-z)^{1/4}), & \text{if } z > 0.5. \end{cases} \qquad (4.39)$$

Some values of the heat transfer, Q_s (their notation), for the two solutions tabulated by Davies and Walker (1977) in their paper are shown in table 4.7 together with the predictions of the formula (4.38).

TABLE 4.7

$S_w = 1$

x	z	Q_s (exact)	Q_s (predicted)
0.1	0.2113	0.5103	0.5130
0.2	0.4122	0.3480	0.3493
0.3	0.6033	0.2704	0.2714
0.4	0.7850	0.2174	0.2182
0.5	0.9578	0.1647	0.1659
sep	1	0.1209	0.1191

$S_w = -1/2$

x	z	Q_s (exact)	Q_s (predicted)
0.2	0.1872	-0.7302	-0.7328
0.5	0.4403	-0.4448	-0.4441
0.8	0.6638	-0.3306	-0.3284
1.1014	0.8624	-0.2504	-0.2481
1.3039	0.9828	-0.1809	-0.1812
sep	1	-0.1272	-0.1268

4.5. Estimating the displacement thickness

In the case of the sharp quasi-uniform pressure gradient problem (Curle (1978))

$$\left(\frac{u_0}{2\nu_0 X}\right)^{1/2} \delta_1^* (1+S_w)^{-1} = \Delta_1^* = 1.21678 + \lambda^{-1} B_1(\xi)$$

$$- 0.00397\sigma_w \lambda^{-2} \log\lambda \xi^3 - 0.00228\lambda^{-2} \xi^3 + \sigma_w \lambda^{-2} Q_1(\xi)$$

$$+ 0.00113\sigma_w^{\,2}\,\lambda^{-2}\,\bar{\xi}^{\,3} + \ldots\,, \qquad (4.40)$$

where $10B_1(\bar{\xi}) = 0.60425\,\bar{\xi}^{\,2} + 0.17897\,\bar{\xi}^{\,3} + 0.09479\,\bar{\xi}^{\,4} + 0.06084\,\bar{\xi}^{\,5} + 0.04424\,\bar{\xi}^{\,6} + \ldots$

and $100\,Q_1(\bar{\xi}) = 0.49874\,\bar{\xi}^{\,2} + 0.39669\,\bar{\xi}^{\,3}\log\bar{\xi} - 1.13129\,\bar{\xi}^{\,3} - 0.14865\,\bar{\xi}^{\,4} - 0.06143\,\bar{\xi}^{\,5} - 0.03302\,\bar{\xi}^{\,6} - \ldots\,.$

Using (4.5) and (4.20),

$$\Delta_1^{*} = 1.21678 + \Delta^{1/3}\,P_1(z) + \sigma_w\,\Delta^{2/3}\log\Delta\,P_{2LS}(z) + \Delta^{2/3}\,P_2(z) + \sigma_w\,\Delta^{2/3}\,P_{2S}(z) + \sigma_w^{\,2}\,\Delta^{2/3}\,P_{2SS}(z) + O\,(\Delta\log\Delta), \qquad (4.41)$$

Where $P_1(z) = 0.61872z + 0.18326z^2 + 0.09706z^3 + 0.06230z^4 + 0.04530z^5 + \ldots\,, \qquad (4.42)$

$$P_{2LS}(z) = 0.13865z\,, \qquad (4.43)$$

$$P_2(z) = -0.23887z + \ldots\,, \qquad (4.44)$$

$$P_{2S}(z) = -0.18943z - 0.16504z^2 - 0.07300z^3 - 0.04418z^4 - \ldots \qquad (4.45)$$

and $P_{2SS}(z) = 0.11879z. \qquad (4.46)$

Following Curle (1977) the singularity in P_1 can be extracted and P_1 rewritten as

$$P_1(z) \doteq -1.33115\,(1-z)^{1/2} - 0.66931\,(1-z)^{3/4} + 2.00046 - 0.54884z - 0.04588z^2 - 0.01228z^3 - 0.00440z^4/(1 - 0.38892z). \qquad (4.47)$$

At separation

$$\Delta_1^* = 1.21678 + 1.38625\,\Delta^{1/3} + 0.13865\sigma_w\,\Delta^{2/3}\log\Delta$$
$$- 0.23885\,\Delta^{2/3} + P_{2S}(1)\sigma_w\,\Delta^{2/3} + 0.11879\sigma_w^2\,\Delta^{2/3}$$
$$+ O\,(\Delta\log\Delta). \qquad (4.48)$$

The exact value of $P_{2S}(1)$ is unknown but is clearly approximately -0.6.

In the case of the continuous incipient separation problem (2.123) substitution for ξ in terms of Δ gives

$$\Delta_1^* = 1.21678 + 1.40322\,\Delta^{1/3} + 0.12832\sigma_w\,\Delta^{2/3}\log\Delta$$
$$- 0.22108\,\Delta^{2/3} - 0.65720\sigma_w\,\Delta^{2/3} + 0.10998\sigma_w^{\,2}\,\Delta^{2/3}$$
$$+ 0.06327\sigma_w^{\,2}\,\Delta\log\Delta - 0.10900\sigma_w\,\Delta - 0.01452\sigma_w^{\,2}\Delta + 0.05423\sigma_w^{\,3}\Delta$$
$$+ O\,(\Delta^{4/3}\log\Delta). \qquad (4.49)$$

We note that the coefficients in the series (4.48) and (4.49) are approximately the same and also that the ratios 0.12832 : 0.06327, - 0.22108 : -0.10900, 0.10998 : 0.05423 are in each case 1 : 0.49305, which is the same ratio as 0.83909 : 0.41371 in (4.7).

These suggest that a good approximation to Δ_1^* is likely to be of the form

$$\Delta_1^* = 1.21678 + \Delta^{1/3}\,P_1(z) + \sigma_w\,\Delta^{2/3}\log\Delta\,P_{2LS}(z)\Phi_0 +$$
$$\Delta^{2/3}\,P_2(z)\Phi_0 + \sigma_w\,\Delta^{2/3}\,(1 + 0.0256\sigma_w\,\Delta^{1/3})\,P_{2S}(z)$$
$$+ \sigma_w^{\,2}\,\Delta^{2/3}\,P_{2SS}(z)\Phi_0 + O\,(\Delta^{4/3}\log\Delta), \qquad (4.50)$$

where, following the incompressible problem,

$$P_2(z) = -\,0.23887z + 0.01779z^6 \qquad (4.51)$$

and, in order to give roughly the right answer at $z = 1$,

$$P_{2S}(z) = -0.18943z - 0.16504z^2 - 0.07300z^3$$

$$- 0.04418z^4/(1 - 0.8077z). \qquad (4.52)$$

The accurate calculation of the remainder term would need a larger number of available solutions. For the present the best that can be obtained is

$$\Phi_0^{-3} \left(\Delta^{4/3} \log\Delta \, P_{4L}(z) + D \right), \qquad (4.53)$$

where, following the incompressible problem,

$$P_{4L}(z) = - 0.00768z^2 + 0.00110z^3 \qquad (4.54)$$

and $D = - (0.3522z + 0.1554z^2) \Delta^{4/3} + (0.0310z - 0.0485z^2)$

$$\Gamma \Delta^{4/3} + (0.3911 + 0.0283\Gamma) z \Delta^{5/3}. \qquad (4.55)$$

So we take as our approximation to Δ_1^*

$$\Delta_1^* = 1.21678 + \Delta^{1/3} P_1(z) + \sigma_w \Delta^{2/3} \log\Delta P_{2LS}(z) \Phi_0$$

$$+ \Delta^{2/3} P_2(z)\Phi_0 + \sigma_w \Delta^{2/3} (1 + 0.0256\sigma_w \Delta^{1/3}) P_{2S}(z)$$

$$+ \sigma_w^2 \Delta^{2/3} P_{2SS}(z)\Phi_0 + \Delta^{4/3}(\log\Delta \, P_{4L}(z) + D)\Phi_0^{-3}, \quad (4.56)$$

where P_1, P_{2LS}, P_2, P_{2S}, P_{2SS}, P_{4L}, and D are defined in (4.47), (4.43), (4.51), (4.52), (4.46), (4.54), and (4.55) respectively.

Some values of δ_1 for the two solutions tabulated by Davies and Walker (1977) in their paper are shown in table 4.8 together with the predictions of the formula (4.56). It should be noted that to obtain the predicted value of δ_1 requires the calculation of the predicted values of both δ_1* and δ_2* .

TABLE 4.8

x	z	δ_1 (exact)	δ_1 (predicted)	δ_2 (exact)	δ_2 (predicted)
		$S_w = 1$			
0.1	0.2113	1.3922	1.3916	0.2152	0.2130
0.2	0.4112	2.0497	2.0343	0.3121	0.3073
0.3	0.6033	2.6377	2.5914	0.3925	0.3860
0.4	0.7850	3.2573	3.1631	0.4661	0.4598
0.5	0.9578	4.1051	3.9601	0.5368	0.5332
sep	1	4.6342	4.6241	0.5537	0.5520
		$S_w = -1/2$			
0.2	0.1872	0.5429	0.5635	0.3038	0.3069
0.5	0.4403	0.9261	0.9795	0.4996	0.5084
0.8	0.6638	1.2960	1.3602	0.6616	0.6719
1.1014	0.8624	1.7644	1.8216	0.8196	0.8220
1.3039	0.9828	2.3236	2.3587	0.9304	0.9191
sep	1	2.5661	2.5814	0.9456	0.9335

4.6. Estimating the Momentum Thickness

Following the procedure adopted for the displacement thickness, we first calculate the momentum thickness for the sharp quasi-uniform pressure gradient problem, where it is found that

$$\left(\frac{u_0}{2\nu_0 X}\right)^{1/2} \delta_2^{\,*} = \Delta_2^{\,*} = 0.46960 + (0.21055 - 0.09172\sigma_w)\, z\, \Delta^{2/3}$$

$$+ \left\{ (0.08896\sigma_w - 0.03876\sigma_w^2) + q_2(z) \right\} \Delta + O(\Delta^{4/3}\log\Delta), \tag{4.57}$$

where $q_2(z) = -0.11489z + 0.02222z^2 + 0.00352z^3$

$$+ 0.00124z^4 + 0.00058z^5 + \dots . \tag{4.58}$$

At separation

$$\Delta_2^* = 0.46960 + (0.21055 - 0.09172\sigma_w)\Delta^{2/3}$$

$$+ (-0.08682 + 0.08896\sigma_w - 0.03876\sigma_w^2)\Delta + O(\Delta^{4/3}\log\Delta). \tag{4.59}$$

Likewise for the continuous incipient separation problem, from (2.125),

$$\Delta_2^* = 0.46960 + (0.19488 - 0.08489\sigma_w)\Delta^{2/3}$$

$$+ (-0.07198 + 0.09608\sigma_w - 0.04186\sigma_w^2)\Delta + O(\Delta^{4/3}\log\Delta). \tag{4.60}$$

The reason for the difference between the coefficients of $\Delta^{2/3}$ can be derived from the momentum integral equation, which in the transformed plane takes the form (Curle and Davies, 1971, p. 278)

$$\frac{d}{dX}(U_1^2\delta_2^*) = \nu_0\left(\frac{\partial U}{\partial Y}\right)_w - U_1\frac{dU_1}{dX}\delta_1^* .$$

Upon setting

$$U_1^2 = u_0^2(1-Cp) \text{ and} \left(\frac{\partial U}{\partial Y}\right)_w = 0.46960u_0\left(\frac{u_0}{2\nu_0X}\right)^{1/2}T,$$

this becomes

$$\frac{d}{dX}\left[(1-Cp)\delta_2^*\right] = 0.46960\left(\frac{\nu_0}{2u_0X}\right)^{1/2}T + {}^1/_2\frac{dCp}{dX}\delta_1^* .$$

When the pressure gradient is sharp, this may be written as

$$\frac{d}{dX}\left[(1 - Cp)\delta_2^*\right] = 0.60839\ (1+S_w)\left(\frac{2\nu_0 X}{u_0}\right)^{1/2}\frac{dCp}{dX} + \text{smaller terms}$$

immediately downstream of the commencement of the pressure gradient, from which it follows that

$$\frac{d}{dX}\left[(1 - Cp)\Delta_2^*\right] = 0.60839\ (1+S_w)\frac{dCp}{dX} + \dots ,$$

and hence

$$(1 - Cp)\Delta_2^* = 0.46960 + 0.60839(1+S_w)Cp + \dots .$$

Since Cp is small; it follows that

$$\Delta_2^* = 0.46960 + (1.07799 + 0.60839\ S_w)\ Cp + \dots . \quad (4.61)$$

From the definitions of Δ, F_c, Φ_0, and z in equations (4.2), (4.11), (4.17) and (4.18) respectively, we deduce that

$$Cp = \left(\frac{F_c}{100}\right)^{1/3}\frac{\Phi_0}{1+S_w}\, z\, \Delta^{2/3},$$

using which (4.61) becomes

$$\Delta_2^* = 0.46960 + (0.23225 - 0.10117\sigma_w)\ F_c^{1/3}\,\Phi_0\, z\, \Delta^{2/3} + \dots , \quad (4.62)$$

which is readily shown to be consistent with (4.57) and (4.60). The accurate calculation of the remainder term would need a larger number of available solutions. For the present the best that can be obtained is

$$Mq_2(z)\Delta , \quad (4.63)$$

where $M = 0.0925 + 1.1800F_c\,\Phi_0$. (4.64)

So we take as our approximation to Δ_2^*

$$\Delta_2^* = 0.46960 + (0.23225 - 0.10117\sigma_w)F_c^{1/3}\,\Phi_0\,\dot{z}\,\Delta^{2/3} + Mq_2(z)\Delta\ , \qquad (4.65)$$

where M and q_2 are defined in (4.64) and (4.58) respectively. Some values of δ_2 for the two solutions tabulated by Davies and Walker in their paper are shown in table 4.8 together with the predictions of the formula (4.65).

4.7. Generalization of the method to $\sigma = 0\ (1)$

If $\sigma = 0\ (1)$ the dependence of the position of separation, heat transfer, etc., on σ_w is replaced by a dependence on the two parameters B_1 and B_2 defined in chapter 2 by

$$\left.\begin{aligned} B_1 &= S'(0)/(1+S_w) \\ \text{and } B_2 &= \beta/(1+S_w)\ , \end{aligned}\right\}\ (4.66)$$

where $S(\eta)$ is the solution of the Pohlhausen equation given below (4.68) and

$$\left.\begin{aligned} &\beta = \frac{\gamma-1}{2} M_0^2 \left(1 + \frac{\gamma-1}{2} M_0^2\right)^{-1} \\ &\text{and } M_0 \text{ is the upstream Mach number.} \end{aligned}\right\}\ (4.67)$$

The Pohlhausen equation is the energy equation for compressible flow in a boundary layer with no pressure gradient:

$$S'' + \sigma f S' = 2\beta(1-\sigma)\left\{f'f''' + f''^2\right\}, \qquad (4.68)$$

where $S(0) = S_w$, $S(\eta) \to 0$ as $\eta \to \infty$,

and f is the solution of the Blasius equation

$$f''' + ff'' = 0\ ,$$

where $f(0) = f'(0) = 0$, $f'(\eta) \to 1$ as $\eta \to \infty$.

In the case of the sharp quasi-uniform pressure gradient problem (Curle 1979b) it is observed that when the separation position is written as a function of B_1 and B_2 the coefficient of B_1 varies little with σ and the coefficient of B_2 is small. In the case of the compressible version of the Riley-Stewartson problem ((3.105) - (3.108)) the same is true. The external flow for the compressible continuous incipient separation problem ((2.159) and (2.160)) when written as a function of B_1 and B_2 has coefficients which vary little with σ . So defining Δ and Φ as in (4.2) and (4.16) respectively we write

$$\Phi_0 = 1 - \alpha^{-1} (a_1/F_c^{1/3}) B_1 \Delta^{1/3} + 1.11 \alpha^{-2} (a_1^2/F_c^{2/3}) B_1^2 \Delta^{2/3} / (1 + 1.3 \alpha^{-1} (a_1/F_c^{1/3}) B_1 \Delta^{1/3}) \qquad (4.69)$$

$$\text{and } z = \Phi / \Phi_0 , \qquad (4.70)$$

where α , a_1 and F_c are defined in (4.23), (4.14) and (4.11) respectively. Then the approximate position of separation is again given by the criterion

$$z = 1 . \qquad (4.71)$$

As in section 4.3 the skin friction is given by

$$\frac{1}{\alpha} \left(\frac{2\nu_0 X}{u_0^3}\right)^{1/2} \left(\frac{\partial U}{\partial Y}\right)_w = T_c(z), \qquad (4.72)$$

where T_c is defined in (4.24).

To predict the heat transfer we define

$$G^{**} = \left(\frac{2\nu_0 X}{u_0}\right)^{1/2} \frac{(\partial T/\partial Y)_w}{T_w B_1} . \qquad (4.73)$$

If $\sigma = 1$,

$$G^{**} = G^{*} .$$

Examination of the heat transfer rate in the compressible version of the Riley-Stewartson problem ((3.124)-(3.129)) shows that when $\Gamma \to \infty$

$$G^{**} \doteq 0.4122 \left\{ \frac{2.2901}{1 + 0.9908\,\sigma^{1/4} + 0.2993\,\sigma^{1/2}} \right\}$$

$$\left(1 + 0.7513\,\sigma^{1/4}(1-z)^{1/4} + 0.5388 \left(\frac{5\sigma^{1/2} + 4\sigma^{1/4}}{9} \right) (1-z)^{1/2}\right)$$

$$(1 - 0.0126\,z - 0.0092\,z^2 - 0.0058\,z^3/(1 - 0.83z))/0.9441$$
$$+ \, 0\,(\Delta^{1/3}).$$

This gives excellent results at separation and retains the correct dependence on σ in the coefficients of $(1-z)^{1/4}$ while omitting it in the coefficients of z where it has only a small effect.

Taking into consideration also the sharp quasi-uniform pressure gradient and the sharp increasing pressure gradient (see (4.9)) problems, we take as our general approximation

$$G^{**} = G_c \left\{ \frac{2.2901 - 0.2846 e^{-X_c}}{1 + (0.9908 - 0.1947 e^{-X_c})\sigma^{1/4} + (0.2993 - 0.0899 e^{-X_c})\sigma} \right.$$

$$\left\{ 1 + (0.7513 - 0.1228 e^{-X_c})\sigma^{1/4}(1-z)^{1/4} + (0.5388 - 0.1618 e^{-X_c}) \right.$$

$$\left. \left(\frac{5\sigma^{1/2} + 4\sigma^{1/4}}{9} \right) (1-z)^{1/2} \right\} \left\{ 1 - (0.0126 + 0.0199 e^{-X_c})\,z \right.$$

$$- (0.0092 + 0.0025e^{-X_c})z^2 - 0.0058z^3/(1 - 0.83z)\Big\}/$$
$$(0.9441 - 0.0224e^{-X_c}) - \alpha^{-1} b_1(X_c)B_1 \Delta^{1/3}/(1 + 1.15\alpha^{-1}$$
$$b_1 B_1 \Delta^{1/3}/G_c)L(z) + 0.05\ (1-\sigma)\ \frac{B_2}{B_1}\ \Delta^{1/3}\ i(z), \qquad (4.74)$$

where $i(z) = z(1.5399 - 0.2613z - 0.0867z^2 - 0.0473z^3/$
$(1 - 0.7534z))$ (4.75)

and G_c, X_c, b_1 and $L(z)$ are as defined in (4.35), (4.12), (4.37), and (4.39) respectively.

If $|B_1| << 1$, this approximation breaks down; in this case the heat transfer is small anyway. If $B_1 = 0$ a rough approximation would be

$$\left(\frac{2\nu_0 X}{u_0}\right)^{1/2} \frac{(\partial T/\partial Y)_w}{B_2\ T_w} = 0.05(1-\sigma)\ \Delta^{1/3}\ i(z). \qquad (4.76)$$

Obtaining an approximation to the displacement thickness does not prove easy; there is not as good a match between the displacement thicknesses for the sharp quasi-uniform pressure gradient problem at separation and the continuous incipient separation problem (written out in powers of B_1 and B_2 and $\Delta^{1/3}$) when $\sigma = 0.72$ as there is when $\sigma = 1$ ((4.48) and (4.49)). Probably the best approximation that we can give is

$$\left(\frac{u_0}{2\nu_0 X}\right)^{1/2} \delta_1^* \ (1 + S_w)^{-1} = \psi + \Delta^{1/3} p_1(z) -$$
$$\alpha^{-1} B_1 \Delta^{2/3} \log \Delta\ P_{2LS}(z)\Phi_0 + \Delta^{2/3}\ P_2(z)\ \Phi_0 - \alpha^{-1}$$
$$B_1 \Delta^{2/3}\ (1 - 0.0256\,\alpha^{-1}\ B_1 \Delta^{1/3})\ P_{2S}(z)$$

$$+ \alpha^{-2} B_1^2 \Delta^{2/3} P_{2SS}(z)\Phi_0 + \Phi_0^{-3}\Delta^{4/3}(\log \Delta P_{4L}(z) + D), \quad (4.77)$$

where

$$\psi = \frac{1.21678 + \int_0^{\infty} S d\eta}{1 + S_w}, \quad (4.78)$$

S is defined in (4.68),

and P_1 etc. are defined in section 4.5.

Following the same method which was used to derive the approximation to the momentum thickness in section 4.6 we obtain

$$\left(\frac{u_0}{2\nu_0 X}\right)^{1/2} \delta_2^* = 0.46960 + F_c^{1/3} \Phi_0 z \Delta^{2/3} 100^{-1/3}$$

$$\left(\frac{\alpha}{1 + S_w} + \frac{\psi}{2}\right) + Mq_2(z), \quad (4.79)$$

when F_c, M and q_2 are defined in (4.11), (4.64) and (4.58) respectively.

5. CONCLUSIONS

The aim of the thesis as agreed with Rolls-Royce and as stated in the first chapter has been fulfilled. In order to make this possible two new solutions of the compressible boundary layer equations have been derived for flows where there is a sharp adverse pressure gradient. The second chapter considers a compressible boundary layer on a semi-infinite flat plate with uniform pressure when $X < X_0$ and with the pressure for $X > X_0$ being so chosen that the boundary layer is just on the point of separation for all $X > X_0$. Immediately downstream of X_0 there is a sharp pressure rise, to which the flow reacts mainly in a thin inner sublayer of thickness $O(\xi)$, where $\xi = (X/X_0 - 1)^{1/3}$; so inner and outer asymptotic expansions are derived and matched for functions F and S which determine the stream function and the temperature. The external velocity, heat transfer rate, displacement thickness and momentum thickness are determined as series in powers of ξ (and $\log\xi$) and involve two parameters B_1 and B_2 which depend upon the wall temperature and the Mach number. Detailed calculations are presented for $\sigma = 1$ and 0.72 (appropriate to air). These series have radius of convergence 1 because of a singularity at $\xi = -1$ (the leading edge of the plate), but the series for the external velocity and heat transfer rate are so well behaved that it is possible to make an Euler transformation and thus derive values for them for $\xi = O(1)$. When ξ is large the velocity and heat transfer tend with increasing ξ to those given by the separating profile derived by Cohen and Reshotko (1956a), if $\sigma = 1$, and derived in section 2.6 of chapter 2 if $\sigma \neq 1$.

Just as in the incompressible continuous incipient separation problem Stratford (1954) suggested and Curle (1976b) proved that F varies very slowly with ξ, so in the compressible problem it is found that the coefficients of powers of $\sigma_w \Delta^{1/3}$ in the expansions for both $(1+S_w)F^{1/3}$ and the heat transfer change little with distance downstream: when $\sigma = 1$,

$$(1+S_w)F^{1/3} = 0.839087\ (1 + 0.001020r - 0.010695r^2 - 0.011794r^3 - \ldots) + 0.413710\sigma_w \Delta^{1/3}\ (1 - 0.029665r - 0.021212r^2 - 0.017527r^3 - \ldots) + 0.206971\sigma_w^2\ \Delta^{2/3}$$

$$(1 - 0.069787r - \ldots) + 0.105360\sigma_w^3 \Delta\ (1 - \ldots) + \ldots$$

$$\text{and } -\alpha^{-1}\left(\frac{2\nu_0 X}{u_0}\right)^{1/2}\ \frac{(\partial T/\partial Y)_w}{T_w - T_s} = 0.575194 + 0.002741r - 0.002770r^2 - 0.003502r^3 - \ldots + 0.079493\sigma_w\ \Delta^{1/3}$$

$$(1 - 0.105558r - \ldots) + 0.029662\sigma_w^2\ \Delta^{2/3}\ (1 - \ldots) + \ldots ,$$

where $F = \left\{10X\frac{dCp}{dX}\right\}^2 Cp$, $\Delta = \dfrac{Cp}{XdCp/dX}$, $Cp = 1 - \dfrac{U_1^2}{u_0^2}$,

$r = 1 - X_0/X$ and $\alpha = 0.469600$.

If $\sigma \neq 1$ but is of order 1 it is shown that the dependence of the pressure coefficient, heat transfer, etc., on σ_w is replaced by a dependence on the two parameters B_1 and B_2 and that when the pressure coefficient is written as a function of B_1 and B_2 the coefficient of B_1 varies little with σ and the coefficient of B_2 is small.

The third chapter considers a compressible boundary layer on a finite flat plate with an adverse pressure gradient which although small near the leading edge becomes increasingly sharp towards the trailing edge. This is in contrast to the second chapter where the pressure gradient downstream of X_0 is very sharp but becoming less so. In chapter 3 the transformed external velocity $U_1(X)$ is chosen such that

$$U_1(X) = u_0 \, (-X/L)^{\epsilon} ,$$

where $0 < \epsilon << 1$,

$$L = \int_{-1}^{0} C(x) \left(\frac{a_1}{a_0}\right)^{\frac{3\gamma-1}{\gamma-1}} dx,$$

l is the physical length of the plate and x and X represent physical and transformed distances measured downstream from the trailing edge. This problem is thus a generalization to compressible flow of the problem first studied by Riley and Stewartson (1969). The boundary-layer equations are solved using the method of matched asymptotic expansions. In particular it is shown that separation occurs when

$$(-\bar{X}/L)^{-2/3} (T_w/T_s) = 0.362132((5)) + \epsilon^{1/2} \lambda B_1 (T_w/T_s)^{1/2} + \ldots ,$$

where $X = 2\alpha^{-2} \epsilon^{3/2} \bar{X}$ $(\alpha = 0.4696000)$,

and $\lambda(\sigma)$ is a function of the Prandtl number σ such that

$\lambda(1) = -2.2518(2)$ and $\lambda(0.72) = -2.2665(4)$.

The flow near separation is studied and it is found that, although the flow is compressible, it behaves to first order like an incompressible flow. Thus the constant α_{11} which arises in an expansion about the singularity (as shown by Goldstein (1948), Stewartson (1958), Terrill (1960) and Buckmaster (1970)) is determined :

$$\alpha_{11} = \epsilon^{-1/4} (1+S_w)^{-1/4}\, 0.30538 + \ldots .$$

Using this and the unpublished work of Akinrelere (1977) the heat transfer at separation is determined :

$$\left(\frac{L\nu_0}{u_0}\right)^{1/2} \left(\frac{\partial T}{\partial Y}\right)_{Y=0}\Bigg|_{sep} = B_1 T_w t_s^*(\sigma) + O(\epsilon^{1/2}),$$

where $t_s^*(1) = 0.2915$ and $t_s^*(0.72) = 0.309$.

The fourth chapter examines the solutions of chapters 2 and 3, the sharp quasi-uniform pressure gradient problem (Curle, 1978) and another problem whose solution is not presented in detail in which the pressure is uniform when $X < X_0$ and the transformed external velocity U_1 is chosen so that when $X > X_0$

$$\frac{X_0}{u_0^2}\, U_1 \frac{dU_1}{dX} (1 + S_w) = -\lambda\left(\frac{X}{X_0} - 1\right),$$

where λ is large and constant, the compressible analogue of the solution derived in appendix A of Curle (1977). The approximate method of computing compressible boundary layers is derived for the case where $\sigma = 1$ in sections 4.2

to 4.6, starting from the fact that for each sharp pressure gradient problem at separation

$$(1 + S_w)F^{1/3} = a_0(\Gamma) + a_1(\Gamma)\sigma_w \Delta^{1/3} + a_2(\Gamma)\sigma_w^2 \Delta^{2/3}$$

$$+ \ldots, \text{ if } \sigma = 1,$$

where $\Gamma = Cp \dfrac{d^2Cp}{dX^2} \Big/ \left[\dfrac{dCp}{dX}\right]^2$, $\Delta = \dfrac{Cp}{XdCp/dX}$ and $Cp = 1 - \dfrac{U_1^2}{u_0^2}$.

Then to provide a general method of predicting the position of separation if $\sigma = 1$ even in cases where the pressure gradient is not sharp the following definitions are made:

$$\Phi = (1 + S_w)(F/F_c)^{1/3},$$

$$\Phi_0 = 1 + (a_1/F_c)^{1/3}\sigma_w \Delta^{1/3} + 1.11(a_1^2/F_c^{2/3})\sigma_w^2 \Delta^{2/3}$$

$$/ (1 - 1.3(a_1/F_c^{1/3})\sigma_w \Delta^{1/3}),$$

and $z = \Phi / \Phi_0$,

where F_c and a_1 are both functions of X_c and X_c is a function of Δ and Γ, both F_c and X_c being the same as functions defined in the incompressible approximate method. Then separation is predicted to occur when $z = 1$.

The appropriate non-dimensional form of the skin friction is found to be a function of z to a good approximation. Examination of the sharp adverse pressure gradient problems gives a form for the heat transfer at separation analogous to the form for $(1 + S_w)F^{1/3}$ at separation, and a form for the heat transfer at all points follows from examining how the heat transfer changes as a function of z and X_c. Estimates of the displacement thickness and momentum thickness are also provided. In each case the results of Davies and Walker (1977) are invaluable both for the determination of correction terms and for the checking of the approximate method. The method is easy to use and remarkably accurate. A generalization to the case where $\sigma \neq 1$ but is of order 1, which makes use of the results of chapters 2 and 3, is provided in the last section of the chapter.

Throughout the thesis techniques of series analysis, Euler transformations, Aitken extrapolation, Neville tables and the ratio method (Gaunt and Guttman, 1974, p.187-199), are used with good results. This led us to look at another boundary-layer problem in which we were able to use series expansions and both the standard methods of series analysis, the ratio method and Padé approximants. This problem is one in which two parallel infinite disks are initially rotating with angular velocity Ω about a common axis in incompressible fluid, the appropriate Reynolds number being very large. At a certain time the angular velocity of one of the disks is suddenly reversed. This problem was originally studied by Bodonyi and Stewartson (1977), who integrated the partial differential equations numerically, and is here studied using series expansions in the appendix to the thesis. Bodonyi and Stewartson found that the boundary layer which is growing near the disturbed disk breaks down when $\Omega t \simeq 2.36$, as all the velocity components become infinite. They also constructed an asymptotic expansion in the neighbourhood of the breakdown which contained the singularities of the numerical solution in a moderately but not entirely satisfactory way. This work both confirms their numerical solution and exhibits the same breakdown of the solution but does not resolve the difficulties which they found in fitting it to the asymptotic expansion and in fact shows that their asymptotic expansion must at least be incomplete.

APPENDIX: THE UNSTEADY LAMINAR BOUNDARY LAYER ON A ROTATING DISK IN A COUNTER-ROTATING INCOMPRESSIBLE FLUID

The problem considered in this appendix is one in which two parallel infinite disks are initially rotating with angular velocity Ω about a common axis in incompressible fluid, the appropriate Reynolds number being very large. At time $t^* = 0$ the angular velocity of one of the disks is suddenly reversed to become $-\Omega$. This problem was studied by Bodonyi and Stewartson (1977), who integrated the partial differential equations numerically and found that the boundary layer which is growing near the disturbed disk breaks down when $\Omega t^* = t_E \doteq 2.36$, as all the velocity components become infinite. They also constructed an asymptotic expansion in the neighbourhood of the breakdown which contained the singularities of the numerical solution in a moderately but not entirely satisfactory way. In this work the partial differential equations are solved by expanding in powers of one variable, with coefficients which are functions of a second variable. This work both confirms the numerical solution of Bodonyi and Stewartson and exhibits the same breakdown of the solution when $t = t_E$ but does not resolve the difficulties which they found in fitting it to the asymptotic expansion.

The equations themselves are derived as follows: consider an incompressible fluid with kinematic viscosity ν confined between the two parallel planes $z^* = 0$ and $z^* = d$. At first the fluid and the disks are rotating with angular velocity Ω ; then at time $t^* = 0$ the angular velocity of the disk $z^* = 0$ is instantaneously reversed. Then if $R \gg 1$, where $R = \Omega d^2/\nu$ is the Reynolds number of the flow, the principal disturbance to the flow occurs in a thin boundary layer near $z^* = 0$ for a finite range of values of Ωt^*. Relative to cylindrical polar coordinates (r^*, θ, z^*), where the axis is the common axis of rotation of the disks, the velocity components of the fluid can be written as

$$(\Omega r^* \partial F/\partial z, \Omega r^* G, -2(\nu\Omega)^{1/2} F), \tag{A1}$$

where $z = z^*(\Omega/\nu)^{1/2}$, $t = \Omega t^*$, and F and G are functions of z and t only. Further, since $R \gg 1$, the governing equations reduce to boundary layer form

$$F_{zt} = F_{zzz} + 2FF_{zz} - F_z^{\ 2} + G^2 - 1, \tag{A2}$$

$$G_t = G_{zz} + 2FG_z - 2GF_z, \tag{A3}$$

since the continuity equation is identically satisfied by (A1). The boundary conditions are

$$F \equiv 0,\ G \equiv 1,\ \text{for all } t \leq 0; \tag{A4}$$

$$G = -1,\ F = F_z = 0 \text{ at } z = 0,\ \text{for all } t > 0; \tag{A5}$$

$$G \to 1,\ F_z \to 0,\ \text{as } z \to \infty,\ \text{for all } t. \tag{A6}$$

We introduce the variable η,

where
$$\eta = z/\sqrt{t} \tag{A7}$$

and we look for a solution of the form

$$G = g_0(\eta) + t^2 g_1(\eta) + t^4 g_2(\eta) + \ldots, \tag{A8}$$

where
$$\left.\begin{aligned} &g_0(0) = -1,\ g_0(\infty) = 1, \\ &g_n(0) = g_n(\infty) = 0 \text{ for } n \geq 1, \end{aligned}\right\} \tag{A9}$$

and

$$F = -t^{3/2}\{ f_0(\eta) + t^2 f_1(\eta) + t^4 f_2(\eta) + \ldots \}, \tag{A10}$$

where
$$f_n(0) = f_n'(0) = f_n'(\infty) = 0 \text{ for all } n. \tag{A11}$$

Substituting into (A2) and (A3) gives

$$g_0'' + {}^1/_2\eta g_0' = 0,$$
$$f_0''' + {}^1/_2\eta f_0'' - f_0' = -1 + g_0^2,$$
$$g_1'' + {}^1/_2\eta g_1' - 2g_1 = 2(f_0 g_0' - f_0' g_0).$$
$$f_1''' + {}^1/_2\eta f_1'' - 3f_1' = 2f_0 f_0'' - f_0'^2 + 2g_0 g_1,$$
$$g_2'' + {}^1/_2\eta g_2' - 4g_2 = 2(f_0 g_1' + f_1 g_0' - f_0' g_1 - f_1' g_0),$$
$$f_2''' + {}^1/_2\eta f_2'' - 5f_2' = 2(f_0 f_1'' + f_1 f_0'') - 2f_0' f_1'$$
$$+ 2g_0 g_2 + g_1^2,$$
$$g_3'' + {}^1/_2\eta g_3' - 6g_3 = 2(f_0 g_2' + f_1 g_1' + f_2 g_0' - f_0' g_2$$
$$-f_1' g_1 - f_2' g_0),$$
$$f_3''' + {}^1/_2\eta f_3'' - 7f_3' = 2(f_0 f_2'' + f_1 f_1'' + f_2 f_0'') - 2f_0' f_2'$$
$$-f_1'^2 + 2g_0 g_3 + 2g_1 g_2, \text{ etc.} \qquad \text{(A12)}$$

The boundary conditions are given in (A9) and (A11).

These equations are linear and are solved successively. The wall derivatives can be obtained accurately, but the accuracy of the values of $f_n(\infty)$ tails off rapidly because one of the complementary functions of the equation for f_n is a polynomial of degree $4n + 3$. The values of $g_n'(0)$ and $f_n''(0)$ are given in table A1. $-t^{-3/2}F(\infty, t)$ is given by

$$- t^{-3/2} F(\infty,t) = H(u) = 0.8352827988195 + 0.230123851184\ u$$
$$+ 0.059040047051\ u^2 + 0.01395760982\ u^3 + 0.00311923162\ u^4$$
$$+ 0.00067068781\ u^5 + 0.0001403001\ u^6 + 0.00002875783\ u^7$$
$$+ 0.00000580326\ u^8 + 0.00000115672\ u^9 + 0.00000022828\ u^{10}$$
$$+ 0.00000004468\ u^{11} + \ldots , \qquad \text{(A13)}$$

where $u = t^2$.

TABLE A1

n	$g_n'(0) \times 10^n$	$f_n''(0) \times 10^n$
0	1.128 379 167 096	0.616 635 619 812
1	-1.962 357 309 89	0.218 126 572 18
2	-2.352 744 992 97	-0.174 892 132 65
3	-2.718 507 180 57	-0.855 709 582 46
4	-2.147 013 957 6	-1.621 607 488 9
5	0.158 822 778 7	-2.013 256 765 1
6	4.401 763 495 0	-1.502 576 731 7
7	9.419 680 233	0.160 736 620
8	12.367 008 930	2.610 276 199
9	9.638 888 540	4.813 475 461
10	-0.678 598 28	5.507 080 60
11	-15.847 334 32	3.969 239 31
12	-27.533 599 5	0.556 432 3
13	-25.673 740 9	-3.638 719 1
14	-7.292 054 7	-7.966 270 5
15	15.268 446 6	-12.919 579 2
16	17.519 069 2	-18.379 687 6
17	-14.099 592 4	-19.175 848 6
18	-47.325 878	-2.488 453
19	6.924 908	43.855 445
20	229.932 747	106.675 378

These coefficients are interesting because, whereas the coefficients of the series for H are of constant sign and fall into a very regular pattern, the wall derivative series have coefficients of variable sign which do not fall into an obvious pattern. The reason for this is that the former series is dominated by a singularity on the real axis while each of the latter is dominated by singularities in the complex plane which do not lie on the real axis.

We begin by examining the series for $H(u)$. Since it has a singularity on the real axis, we use the ratio method to estimate $\mu = 1/u_E$ where $u = u_E$ is the value of u at the singularity. The second and third columns of table A2 give two sequences of estimates of μ : the second column gives the values of $\mu_n = a_n/a_{n-1}$ where a_n is the coefficient of u^n and the third column the first Neville extrapolants μ_n^+. The latter sequence clearly tends to a limit μ' where

$$\mu' = 0.17945 \pm 0.00003 . \qquad \text{(A14)}$$

If we invert the series for $H(u)$ the inverted series has coefficients of variable sign and no singularity at $u = u_E$; so it seems reasonable to suppose that the singularity in H is a simple or multiple pole at $u = u_E$. This idea is supported by the fact that if we assume that

$$H(u) = A(1-\mu u)^{-(1+g)} \left\{1 + O(1-\mu u)\right\}, \quad u \to u_E \qquad \text{(A15)}$$

then we can form a sequence of estimates of g (Domb and Sykes 1957)

$$g_n \equiv n\left(\frac{\mu_n}{\mu'} - 1\right) \doteq g\left\{1 + O(1/n)\right\}, \text{ as } n \to \infty. \qquad \text{(A16)}$$

These estimates, taking $\mu' = 0.17945$, are shown in the fourth column of table A2 and appear to be tending to 1. This result is in accordance with the results of Bodonyi and Stewartson (1977, p. 674); so we use this value of g to form a more rapidly convergent sequence of estimates of μ from the sequence (Domb and Sykes 1961)

$$\mu_n^* \equiv \frac{n\mu_n}{n+g} \doteq \mu\left\{1 + 0\left(\frac{1}{n^2}\right)\right\}, \text{ as } n \to \infty. \qquad (A17)$$

These estimates are shown in the fifth column of table A2 and have a limit μ'' where

$$\mu'' = 0.179435 \pm 0.00001. \qquad (A18)$$

Then

$$\left.\begin{aligned} u_E &= 5.5730 \pm 0.0003\,; \\ t_E &= u_E^{1/2} = 2.36072 \pm 0.00005\,. \end{aligned}\right\} \qquad (A19)$$

Finally the amplitude of the singularity, A, may be estimated from the sequence (Sykes and Fisher 1962)

$$A_n = \frac{a_n}{\binom{g+n}{n}\mu''^{\,n}}, \qquad (A20)$$

which in this problem becomes simply

$$A_n = \frac{a_n}{(n+1)\mu''^{\,n}}\,. \qquad (A21)$$

These estimates, taking $\mu'' = 0.179435$, are shown in the sixth column of table A2. The limit is sensitive to the value of μ'' but can be given as

$$A' = 0.599 \pm 0.001\,. \qquad (A22)$$

Thus near $t = t_E$,

$$F(\infty, t) \doteq -\pi/(\alpha(t_E - t)^2), \qquad (A23)$$

where $\alpha = 1.038 \pm 0.002$.

TABLE A2

n	μ_n	μ_n^+	g_n	μ_n^*	A_n
5	0.2150170	-	0.9910	0.1791808	0.60094
6	0.2091885	0.180046	0.9943	0.1793044	0.60051
7	0.2049736	0.179684	0.9956	0.1793519	0.60023
8	0.2017976	0.179566	0.9963	0.179376	0.60003
9	0.199322	0.17952	0.9967	0.179390	0.59988

Having determined the nature of the singularity in $F(\infty, t)$, we need also to evaluate $F_{zz}(0, t)$ and $G_z(0, t)$ up to breakdown. Since just adding up the series gives poor results as breakdown is approached especially when evaluating $\frac{d}{dt}(F_{zz}(0, t))$ and $\frac{d}{dt}(G_z(0, t))$ we use the method of Padé approximants (Padé, 1892; Wall, 1948) which enables several singularities lying anywhere in the complex plane to be studied simultaneously and provides a method of approximately analytically continuing a function beyond its radius of convergence. The $[L, M]$ Padé approximant to a function $F(z)$ is the ratio of a polynomial $P_L(z)$ of degree L to a polynomial $Q_M(z)$ of degree M,

$$[L, M] = \frac{P_L(z)}{Q_M(z)} \equiv \frac{P_0+P_1z+P_2z^2+\ldots+P_Lz^L}{1+q_1z+q_2z^2+\ldots+q_Mz^M},$$

where the coefficients $P_0, P_1, \ldots, P_L$ and $q_1, q_2, \ldots, q_M$ are chosen so that the expansion of $[L,M]$ agrees with the expansion of $F(z)$ up to terms of order L+M, i.e.

$$F(z) = [L,M] + O(z^{L+M+1}).$$

The coefficients are unique and can be obtained directly by solving the sets of linear equations, although the set of equations is often ill-conditioned. It is usual to look at diagonal or near-diagonal approximants $[n + j, n]$. Much information is provided by Gaunt and Guttman (1974, p. 202-210), who give a very full list of references.

To determine $F_{zz}(0, t)$ and $G_z(0, t)$ we make the definitions

$$\left.\begin{aligned} - t^{-1/2} F_{zz}(0, t) &= I(u) = f_0''(0) + f_1''(0)\,u + f_2''(0)\,u^2 + \ldots , \\ \text{and} \quad t^{1/2} G_z(0, t) &= J(u) = g_0'(0) + g_1'(0)u + g_2'(0)u^2 + \ldots , \\ \text{where } u &= t^2. \end{aligned}\right\} \text{(A2}$$

The Padé approximants to $I(u)$ and $J(u)$ converge well until breakdown is approached and the results obtained for $I(u)$ and $J(u)$ and hence for $F_{zz}(0, t)$ and $G_z(0, t)$ are shown in table A3 for $t = {}^1/_2, 1, 1{}^1/_2, 2$ and $2{}^1/_4$ together with the comparable results of Bodonyi and Stewartson.

TABLE A3

t	I(u)	$-F_{zz}(0, t)$	
		present analysis	Bodonyi and Stewartson
0.5	0.621965453	0.439796	0.4450
1.0	0.635659898	0.635660	0.6373
1.5	0.641632281	0.785836	0.7858
2.0	0.556775	0.787399	0.7883
2.25	0.4022	0.6033	0.6062

t	J(u)	$G_z(0, t)$ present analysis	$G_z(0, t)$ Bodonyi and Stewartson
0.5	1.077806456	1.524249	1.527
1.0	0.905689843	0.905690	0.9065
1.5	0.532304187	0.434625	0.4381
2.0	-0.217597	-0.153864	-0.1509
2.25	-0.7483(7)	-0.4989	-0.4956

As breakdown is approached and H becomes singular, I and J remain finite; so in tables A4 and A5 the [n-1, n], [n, n] and [n+1, n] approximants to $I(u_E)$ and $I'(u_E)/I(u_E)$ and $J(u_E)$ and $J'(u_E)/J(u_E)$ are shown. Attempts were made to find the location and nature of the most dominant singularities of I(u) by analysing $I'(u)/I(u)$ (Baker, 1961), but the results were not sufficiently accurate to enable a better estimate of $I(u_E)$ to be obtained.

TABLE A4

n	$I(u_E)$ [n-1, n]	$I(u_E)$ [n, n]	$I(u_E)$ [n+1, n]
6	0.30159	0.29221	0.29309
7	0.29315	0.29423	0.29390
8	0.29391	0.29291	0.29350
9	0.29350	0.29357	0.29428
10	0.29455	0.29369	-

continued over

n	$I'(u_E)/I(u_E)$ [n-1, n]	[n, n]	[n+1, n]
6	-0.7692	-0.7778	-0.7800
7	-0.7815	-0.7241	-0.7665
8	-0.7674	-0.7545	-0.7805
9	-0.7850	-0.7879	-0.7939
10	-0.7643	-	-

Hence $I(u_E) \doteq 0.294$; $I'(u_E)/I(u_E) \doteq -0.78$. (A25)

TABLE A5

n	$J(u_E)$ [n-1, n]	[n, n]	[n+1, n]
6	-0.97356	-0.97744	-0.97611
7	-0.97607	-0.98472	-0.97283
8	-0.97277	-0.97312	-0.97321
9	-0.97320	-0.97321	-0.97321
10	-0.97320	-0.97308	-

n	$J'(u_E)/J(u_E)$ [n-1, n]	[n, n]	[n+1, n]
6	0.3668	0.4148	0.4154
7	0.4260	0.4220	0.4190
8	0.4361	0.3976	0.4051
9	0.4064	0.4063	0.4067
10	0.4064	-	-

Hence $J(u_E) \doteq -0.9732$; $J'(u_E)/J(u_E) \doteq 0.406(4)$. (A26)

So in the neighbourhood of t_E

$$\left.\begin{aligned} F_{zz}(0,\ t) &= -0.452\ -1.5(7)\ (t_E-t) + \ldots \\ \text{and}\quad G_z\ (0,\ t) &= -0.6334 + 1.081\ (t_E-t) + \ldots . \end{aligned}\right\} \quad (A27)$$

It is thus not possible to choose values of α and β which fit Bodonyi and Stewartson's asymptotic analysis, which gives in the neighbourhood of t_E (p.678):

$$F_{zz}(0,\ t) = {}^1/_2\alpha\beta - \alpha(t_E-t) + \ldots$$

and $G_z(0,\ t) = -{}^1/_2\alpha - \alpha\beta(t_E-t) + \ldots$.

The asymptotic analysis must at least be incomplete.

Thus this work indeed both confirms the numerical solution of Bodonyi and Stewartson and exhibits the same breakdown of the solution when $t = t_E$, but does not resolve the difficulties which they found in fitting it to the asymptotic expansion .

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