PARAMETERISATION-INVARIANT VERSIONS OF WALD TESTS

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Parameterisation-invariant Versions of Wald Tests

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Declarations

I, *Pia Veldt Larsen*, hereby certify that this thesis, approximately 35,000 words in length, has been composed by me, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree or professional qualification.

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I was admitted to the Faculty of Science of the University of St Andrews as a research student in January 1996 as a candidate for the degree of Ph.D. in January 1997; the higher study for which this is a record was carried out in the University of St Andrews between 1996 and 1999.

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Abstract

Although Wald tests form one of the major classes of hypothesis tests, they suffer from the well-known major drawback that they are not invariant under reparameterisation. This thesis uses the differential-geometric concept of a yoke to introduce one-parameter families of geometric Wald statistics, which are parameterisation-invariant statistics in the spirit of the traditional Wald statistics. Both the geometric Wald statistics based on the expected likelihood yokes and those based on the observed likelihood yokes are investigated. Bartlett-type adjustments of the geometric Wald statistics are obtained, in order to improve the accuracy of the chi-squared approximations to their distributions under the null hypothesis.

Contents

Chapter 1

Introduction

1.1 Outline

One of the key test statistics for testing hypotheses in parametric models is the Wald statistic. It was suggested by Abraham Wald [33] in 1943, and is used widely today as an alternative to the better known likelihood ratio test statistic and the score statistic. It has the advantage of often being very simple to compute, as it involves only the maximum likelihood estimates of the parameters and the expected information matrix. In particular, in econometrics and finance the Wald statistic is used extensively. However, it does have a major drawback in that the value of the Wald statistic depends on how the hypotheses are formulated - or equivalently, it depends on the parameterisation used to describe the model. One can end up with two completely different values of what should have been the same test statistic. As it is customary to compare the value of the Wald statistic with its limiting large-sample null distribution (which is always is the same), it can easily happen that different parameterisations lead to contradictory conclusions. In contrast to the Wald statistics, the likelihood ratio statistic and the score statistics are invariant under changes of the parameterisation.

In this thesis we look into the problem from a differential geometric point of view, and try to explain geometrically why the Wald statistic suffers from this drawback. We remark that the intention of the Wald statistic was that it should provide a measure of the difference between two values of the parameter. However, the concept of *difference* is defined only when a parameterisation is chosen, and it depends on the choice of parameterisation.

We define a wide range of parameterisation-invariant *geometric Wald statistics* in the spirit of Wald but not suffering from the above drawback. We concentrate on two very important geometries in statistics: expected geometries, using moments of derivatives of the log-likelihood function; and observed geometries, using mixed derivatives of the log likelihood function.

We show that the geometric Wald test statistics defined using expected or observed geometries have a known asymptotic distribution with error of order $O(n^{-\frac{1}{2}})$ under the null hypothesis. We improve the distributional results to order $O(n^{-\frac{3}{2}})$ by obtaining Bartlett-type adjustments of the type suggested by Cordeiro & Ferrari [14].

The thesis is divided into five chapters. Chapter 1 introduces terminology and notation used throughout the thesis. It also contains a brief summary of the requisite differential geometry and reminders on moments, cumulants and Hermite polynomials. In Chapter 2 we have a closer look at the traditional Wald statistic, illuminating the lack of parameterisation-invariance both analytically and from a differential-geometric point of view. We define a family of new geometric Wald test statistics for a general geometry, and emphasise the importance of the geometric Wald statistics based on expected geometry and observed geometry. In Chapter 3 we obtain Bartlett-type adjustments of the expected geometric Wald statistics, and in Chapter 4 we introduce a corresponding Bartlett-type adjustment (using observed geometry) of the observed geometric Wald statistic. We look closer at the case of testing simple null-hypotheses in Chapter 5, and we include a few simple examples of the adequacy of the new test statistics. Finally, in Chapter 6 there is a short discussion on the results obtained in this thesis and on the use of differential geometry in statistics. Since the notation can appear complicated (as terminology from both analysis and differential geometry is used), an index of the most important notation is included on pages 100–101.

1.2 Notation and Assumptions

Consider a parametric statistical model $\mathcal{M}$ with probability density function $p(x;\theta)$ with respect to some dominating measure. The parameter θ runs through the parameter space Θ, which is assumed to be a differentiable manifold (a 'smooth' surface in $\mathbb{R}^r$, see Section 1.3 on Differential Geometry for a formal definition), so that $\theta = (\theta^1, \ldots, \theta^r)$ in some local coordinate system on Θ. All vectors are assumed to be row-vectors. Let ψ be a p-dimensional interest parameter and let χ be a q-dimensional nuisance parameter, such that $r = p + q$ and $\theta = (\psi^1, \dots, \psi^p, \chi^1, \dots, \chi^q)$. We shall consider only interest-respecting re-parameterisations of Θ, that is under re-parameterisation of (ψ, χ) to (ϕ, ξ), the new interest parameter ϕ depends only on ψ and not on χ.

Denote the components of θ corresponding to the general parameter by θ^i, θ^j, etc., the components corresponding to the interest parameter ψ by θ^a, θ^b, etc., and the components corresponding to the nuisance parameter χ by θ^α, θ^β, etc.

Let $x_1, \dots, x_n$ be observations from n independent, identically distributed random variables $\mathbf{X}_1, \dots, \mathbf{X}_n$ with distribution depending on the parameter θ. Let $l^{(1)}(\theta; \mathbf{X}_1)$ denote the log-likelihood function based on one observation, and let $l(\theta; \mathbf{X})$ denote the log-likelihood function based on n observations. We shall usually refer to the log-likelihood functions simply as $l^{(1)}(\theta)$ and $l(\theta)$, respectively. We are assuming that the log-likelihood functions are differentiable at least four times with respect to θ, and that all relevant moments of the derivatives of the log-likelihood functions exist. The derivatives of the log-likelihood function are denoted by $l_i(\theta)$, $l_{ij}(\theta)$, etc.

Suppose we wish to test the composite null hypothesis

$$H_0 : \psi = \psi_0$$

against the general alternative hypothesis

$$H_1 : \theta \in \Theta.$$

Denote the maximum likelihood estimate of θ under the full hypothesis by $\hat{\theta}$, and the maximum likelihood estimate under the null hypothesis by $\tilde{\theta} = (\psi_0^1, \dots, \psi_0^p, \tilde{\chi}^1, \dots, \tilde{\chi}^q)$. A hat above an expression denotes evaluation of the expression at $\hat{\theta}$ and a tilde denotes evaluation at $\tilde{\theta}$. We shall assume appropriate regularity conditions, so that the maximum likelihood estimators are consistent. See Cox & Hinkley [15, Section 9] for an outline.

The Einstein summation convention, i.e. automatically summing with respect to a letter represented as both a subscript and as a superscript, is used. Furthermore, for any set of indices $I = \{I_1, \dots, I_m\}$ where each of the subsets I_j is of the form $I_j = \{i_{j1}, \dots, i_{jm_j}\}$, the following summation convention will be used extensively

$$\zeta_{I_1,\dots,I_m}\{\cdot\}_{I_1\dots I_m} = \sum\nolimits_{I'/|I_1|\dots|I_m|} \zeta_{I'_1,\dots,I'_m}, \tag{1.2.1}$$

summing over all partitions I' of I into m subsets of 'sizes' $|I_1| \dots |I_m|$ such that the order of the indices in each of these subsets is the same as their order in I. The number in the braces is the number of terms to be added. E.g.

$$\zeta_{i,j,kl}\,\{3\}_{jkl} = \zeta_{i,j,kl} + \zeta_{i,k,jl} + \zeta_{i,l,jk}.$$

The subscripts of the braces are omitted when we are summing over all indices of ζ.

Remark 1:

Note that this definition differs from the $[\cdot]$-notation of Barndorff-Nielsen & Cox [5, 6], Blæsild [11] and others, as it is *essential* here that $|I_1|, \dots, |I_m|$ stay fixed.

Let h be a real-valued function on the parameter space Θ. We denote the derivatives of h by $h_{/I}$ or simply h_I, where $I = i_1 \dots i_r$ is a multi-index of length $r \geq 0$. More precisely,

$$h_{/I}(\theta) = h_I(\theta) = \frac{\partial^r}{\partial\theta^{i_1} \cdots \partial\theta^{i_r}} h(\theta).$$

Also, we shall need to consider real-valued functions on $\Theta \times \Theta$. Let g be a function on $\Theta \times \Theta$. Then the derivatives of g are denoted by $g_{I;J}$, where $I = (i_1 \cdots i_r)$ and $J = (j_1 \cdots j_s)$ are multi-indices of lengths $r \geq 0$ and $s \geq 0$, respectively. That is,

$$g_{I;J}(\theta;\theta') = \frac{\partial^{r+s}}{\partial\theta^{i_1} \dots \partial\theta^{i_r} \partial\theta'^{j_1} \dots \partial\theta'^{j_s}} g(\theta;\theta'). \tag{1.2.2}$$

If $s = 0$, i.e. J is empty, we write $g_I(\theta;\theta')$ rather than $g_{I;}(\theta;\theta')$.

We use a diagonal 'slash' through a function to indicate the restriction of the function to the diagonal, e.g.

$$\not{g}_{I;J}(\theta) = g_{I;J}(\theta;\theta')|_{\theta'=\theta}. \tag{1.2.3}$$

Unless otherwise stated, all such 'diagonalised' functions are evaluated at the true value θ of the parameter, and we shall omit the argument and write e.g. $\not{g}_{I;J}$ for $\not{g}_{I;J}(\theta)$.

Finally, the derivative with respect to θ^k of $\not{g}_{I;J}(\theta)$ is denoted by $\not{g}_{I;J/k}(\theta)$, i.e.

$$\begin{aligned} \not{g}_{I;J/k}(\theta) &= \frac{\partial}{\partial\theta^k} g_{I;J}(\theta;\theta) \\ &= \not{g}_{Ik;J}(\theta) + \not{g}_{I;Jk}(\theta). \end{aligned} \quad (1.2.4)$$

1.3 Differential Geometry

Much of the theory of statistical inference, e.g. the large-sample behaviour of maximum likelihood estimators, comes from results based on differential calculus. However, care is required when using higher-order derivatives, since operations such as differentiating or Taylor-expanding functions with respect to the parameters depend on the parameterisation, and thus any inferences made may depend on the parameterisation.

A very efficient way of ensuring that inferences do not depend on parameterisation is to use some concepts and language from differential geometry.

In this thesis we are concentrating on parameterisation-invariance of a specific statistical object, the Wald test statistic. We shall introduce only enough differential-geometric theory to support the definition in Chapter 2 of a family of new test statistics, *geometric Wald statistics* and to obtain Bartlett-type adjustments of these new statistics.

We start with a few fundamental definitions. For more details see Barndorff-Nielsen & Cox [5, 6], Barndorff-Nielsen & Jupp [8], Lauritzen [24] and Murray & Rice [29].

A topological space Ω is an *r-dimensional smooth differentiable manifold* if around each point ω of Ω there exist an open subset U of Ω and a map $h_U : U \to h_U(U) \subseteq \mathbb{R}^r$ such that h_U is invertible, $h_U(U)$ is open in $\mathbb{R}^r$, and both h_U and h_U^{-1} are infinitely differentiable. The mapping h_U is called a *parameterisation* or a *local coordinate system* of Ω. Note that Ω need not be a linear space, nor need it be given a metric.

We define a *parameterised path* on the manifold to be a smooth map $\gamma : (-\epsilon, \epsilon) \to \Omega$, where $\epsilon > 0$. The *tangent vector* to a path γ at the point $\omega = \gamma(0)$ in a manifold is the vector $\frac{\partial\gamma}{\partial t}(0)$. The tangent vectors do not depend on the parameterisation. The set of all tangent vectors through a point ω in an r-dimensional manifold is an r-dimensional real vector space referred to as the *tangent space* to Ω at ω and denoted by $T_\omega\Omega$. Let h be a parameterisation of Ω at ω and define the paths $\gamma_1^h, \dots \gamma_r^h$ by

$$h^j(\gamma_i^h(t)) = h^j(\omega) + \delta_i^j t,$$

where δ_i^j is the Kronecker delta (i.e. $\delta_i^j = 1$ if $i = j$ and $\delta_i^j = 0$ if $i \neq j$) for $i, j = 1, \dots, r$. Then the tangent space at ω is the span of the vectors $\frac{\partial}{\partial t}\gamma_1^h(0), \dots, \frac{\partial}{\partial t}\gamma_r^h(0)$. The *cotangent space* $T_\omega^*\Omega$ is the dual of $T_\omega\Omega$.

A function g which assigns to each ω in Ω a symmetric, bilinear, positive definite function g_ω from $T_\omega\Omega \times T_\omega\Omega$ into $\mathbb{R}$ is called a *Riemannian metric.* In statistical inference, the most important Riemannian metrics of interest are the *expected information metric* (the Fisher information metric), $i(\omega)$, represented by the matrix

$$\begin{aligned} i_{i,j}(\omega) &= n^{-1}\mathbb{E}\left[l_i(\omega)l_j(\omega)\right] \\ &= \mathbb{E}\left[l_i^{(1)}(\omega)l_j^{(1)}(\omega)\right], \end{aligned} \tag{1.3.5}$$

and the (per observation) *observed information metric*, $j(\omega)$, represented by the matrix

$$j_{i,j}(\omega) = n^{-1}l_{i;j}(\omega;\omega,a), \tag{1.3.6}$$

where a is an auxiliary statistic such that the statistic $(\hat{\omega}, a)$ is minimal sufficient for ω.

1.3.1 Yokes

Most of the differential geometry used in statistics comes from yokes. Let g be a real-valued function on $\Omega \times \Omega$ with derivatives $g_{I;J}(\omega;\omega')$. Then g is a *yoke* if everywhere on Ω

$$\begin{aligned} &\text{(i)} \quad \not{g}_i(\omega) = 0 \\ &\text{(ii)} \quad \text{the matrix } [\not{g}_{i;j}(\omega)] \text{ is non-singular.} \end{aligned} \tag{1.3.7}$$

Furthermore, g is a *normalised yoke* if g is a yoke and everywhere on Ω

$$\text{(iii)} \qquad \not{g}(\omega) = 0. \tag{1.3.8}$$

For any yoke g, $\not{g}_{i;j}(\omega)$ is a Riemannian metric, and by differentiating (1.3.7) we find the following *balance relations*

$$\not{g}_{ij}(\omega) + \not{g}_{i;j}(\omega) = 0 \tag{1.3.9}$$

$$\not{g}_{ijk}(\omega) + \not{g}_{i;jk}(\omega) + \not{g}_{ij;k}(\omega)\,\{2\}_{jk} = 0 \tag{1.3.10}$$

$$\not{g}_{ijkl}(\omega) + \not{g}_{i;jkl}(\omega) + \not{g}_{ij;kl}(\omega)\,\{3\}_{jkl} + \not{g}_{ijk;l}(\omega)\,\{3\}_{jkl} = 0 \tag{1.3.11}$$

etc.

Moreover, if g is a normalised yoke, differentiating (1.3.8) gives us the following relations

$$\not{g}_{;i}(\omega) = 0 \tag{1.3.12}$$

$$\not{g}_{ij}(\omega) + \not{g}_{i;j}(\omega)\,\{2\} + \not{g}_{;ij}(\omega) = 0 \tag{1.3.13}$$

$$\not{g}_{ijk}(\omega) + \not{g}_{i;jk}(\omega)\,\{3\} + \not{g}_{ij,k}(\omega)\,\{3\} + \not{g}_{;ijk}(\omega) = 0 \tag{1.3.14}$$

$$\not{g}_{ijkl}(\omega) + \not{g}_{i;jkl}(\omega)\,\{4\} + \not{g}_{ij;kl}(\omega)\,\{6\}_{jkl} + \not{g}_{ijk;l}(\omega)\,\{4\} + \not{g}_{;ijkl}(\omega) = 0 \tag{1.3.15}$$

etc.

See Barndorff-Nielsen & Cox [6, Section 5.6] for more information on yokes.

1.3.2 Expected and Observed Geometries

In the statistical context there are two fundamental types of geometries, both closely connected to the behaviour of the log-likelihood function on the parameter space Θ. We shall talk about

(i) expected geometries,

based on moments of derivatives of the log-likelihood function, and

(ii) observed geometries,

based on mixed derivatives of the log-likelihood function with respect to the parameter θ, and with respect to the maximum likelihood estimate $\hat{\theta}$ of θ. In observed geometry we regard l as depending on the data through $(\hat{\theta}, a)$ where a is an auxiliary statistic such that the statistic $(\hat{\theta}, a)$ is minimal sufficient for θ.

Expected geometries use moments, cumulants, skewness of the score, the Fisher information matrix, etc., all of which are obtained by taking expectations of the log-likelihood function or its derivatives. We can regard this as putting a geometric structure (an 'expected geometry') on the manifold of statistical models. A natural way of comparing two elements in this manifold is by using the Kullback-Leibler divergence (Barndorff-Nielsen & Cox [6]), or – changing the sign – the *expected likelihood yoke*

$$\begin{aligned} f(\theta;\theta') &= \mathbb{E}_{\theta'}\left[l^{(1)}(\theta) - l^{(1)}(\theta')\right] \\ &= n^{-1}\mathbb{E}_{\theta'}\left[l(\theta) - l(\theta')\right]. \end{aligned} \tag{1.3.16}$$

By differentiating the expected likelihood yoke appropriately, we get all the above moments, cumulants, etc. Moreover, two important parameterisations arise naturallly from this yoke; in full exponential models these parameterisations correspond to the canonical and the expectation parameterisations, supporting the general perception that these two parameterisations in some sense are 'natural' (see Efron [20]). The expected likelihood yoke suggests a generalisation of the canonical and the expectation parameterisations to all statistical models.

Observed geometries were introduced by Barndorff-Nielsen [1] as an alternative to expected geometries. He (and others) suggested ways of replacing the (expected) moments, cumulants, skewnesses, Fisher information matrix, etc. with observed versions based on mixed derivatives of the log-likelihood functions. In a sense, observed geometries are 'closer to the data' than expected geometries, in that they do not involve integration (for taking expectations). Observed geometries usually require the specification of an auxiliary statistic. We can use the *observed likelihood yoke* as an alternative to the expected likelihood yoke,

$$g(\theta;\theta') = n^{-1}\left\{l(\theta;\theta',a) - l(\theta';\theta',a)\right\}. \tag{1.3.17}$$

By differentiating the observed likelihood yoke in the same way as the expected likelihood yoke, we obtain the observed versions of the expected objects. The coordinate systems based on the observed likelihood yoke correspond to the expected coordinate systems. In full exponential models, the two geometries coincide, and thus the canonical and expectation parameterisations stand out as the key parameterisations in observed geometry.

Remark 2:

Note that the two likelihood yokes are normalised yokes. If g is the expected or observed likelihood yoke then the Riemannian metric $[\not{g}_{i;j}(\theta)]$ is the expected information metric

$i(\theta)$, or the observed information metric $j(\theta)$, respectively. The expected and the observed likelihood yokes agree to order $O(n^{-\frac{1}{2}})$. Hence, the derivatives $f_{I;J}$ and $g_{I;J}$ agree to order $O(n^{-\frac{1}{2}})$, I and J being any sets of indices. It is straightforward to verify that the two likelihood yokes coincide for full exponential families.

1.3.3 Tensors

Many of the parameterisation-invariant objects which arise in higher-order asymptotic statistics can be expressed neatly in terms of tensors. Basically, a tensor is a function on a manifold which transforms in a linear way when we change the parameterisation of the manifold, the coefficients being derivatives of the new parameterisation with respect to the old one. More precisely, let $\xi = (\xi^1, \ldots, \xi^r)$ and $\tau = (\tau^1, \ldots, \tau^r)$ be parameterisations on an r-dimensional differentiable manifold, and put

$$\begin{aligned} \xi^i_{/a} &= \frac{\partial \xi^i}{\partial \tau^a} \\ \tau^a_{/i} &= \frac{\partial \tau^a}{\partial \xi^i}. \end{aligned}$$

Let

$$T^{i_1,\ldots,i_u}_{j_1,\ldots,j_v}(\xi), \tag{1.3.18}$$

where each of the indices $i_1, \ldots, i_u, j_1, \ldots, j_v$ takes values 1 to r, be a real-valued function of the parameter ξ with the indices each representing components of the parameter. The function (1.3.18) is an (u, v) *tensor* if, under re-parameterisation, it satisfies

$$T^{a_1,\ldots,a_u}_{b_1,\ldots,b_v}(\tau) = T^{i_1,\ldots,i_u}_{j_1,\ldots,j_v}(\xi)\tau^{a_1}_{/i_1}\cdots\tau^{a_u}_{/i_u}\xi^{j_1}_{/b_1}\cdots\xi^{j_v}_{/b_v}. \tag{1.3.19}$$

A $(u, 0)$ tensor is called a *contravariant tensor* and a $(0, v)$ tensor is called a *covariant tensor*. Note that a $(0, 0)$ tensor is a scalar and that $(0, 1)$ and $(1, 0)$ tensors are vectors. In particular, observe that the expected and observed information matrices (1.3.5) and (1.3.6) are $(0, 2)$-tensors.

We shall use the terms *to lift* and *to lower* (indices of) tensors by a Riemannian metric. Let $T^{i_1\cdots i_u}_{j_1\cdots j_v}$ be a (u, v) tensor and let $\not{g}_{i;j}$ be a Riemannian metric with inverse $\not{g}^{i;j}$. Then we call the $(u+v, 0)$ tensor $T^{i_1\cdots i_u i_{u+1}\cdots i_{u+v}} = T^{i_1\cdots i_u}_{j_1\cdots j_v}\not{g}^{i_{u+1},j_1}\cdots\not{g}^{i_{u+v},j_v}$ the *lifted version* of $T^{i_1\cdots i_u}_{j_1\cdots j_v}$. Similarly, we call the $(0, u+v)$ tensor $T_{j_{v+1}\cdots j_{v+u}j_1\cdots j_v} = T^{i_1\cdots i_u}_{j_1\cdots j_v}\not{g}_{i_1,j_{v+1}}\cdots\not{g}_{j_{u+v},j_v}$ the *lowered version* of $T^{i_1\cdots i_u}_{j_1\cdots j_v}$.

1.4 Reminders

We shall be using various properties of cumulants and of Hermite polynomials. Brief reminders on these two topics are included here, together with a result on differentiating inverses of matrices. For more detailed information and for proofs of the properties of cumulants and Hermite polynomials, see e.g. Barndorff-Nielsen *et al.* [2] or Barndorff-Nielsen & Cox [5].

1.4.1 Moments and Cumulants

Let $\mathbf{X} = (X_1, \ldots, X_m)$ be a random vector. Then, for $k = 1, 2, \ldots$, and $r_i \in \{1, \ldots, m\}$ with $i = 1, \ldots, k$, we write

$$\begin{aligned} \kappa_{r_1,\ldots,r_k} &= \mathbb{E}\left[X_{r_1} \cdots X_{r_k}\right] \\ \nu_{r_1,\ldots,r_k} &= \mathbb{E}\left[(X_{r_1} - \kappa_{r_1}) \cdots (X_{r_k} - \kappa_{r_k})\right] \end{aligned}$$

for the *moments* and the *central moments*, respectively, of $\mathbf{X}$. The *moment generating function* of $\mathbf{X}$ is defined by

$$M_{\mathbf{X}}(t) = \mathbb{E}\left[e^{t\mathbf{X}^T}\right].$$

The $(r_1, \ldots, r_k)^{th}$ moment can be found by

$$\kappa_{r_1,\ldots,r_k} = \frac{\partial^k}{\partial t_{r_1} \cdots \partial t_{r_k}} M_{\mathbf{X}}(0). \tag{1.4.20}$$

Likewise, the $(r_1, \ldots, r_k)^{th}$ central moment can be found by

$$\nu_{r_1,\ldots,r_k} = \frac{\partial^k}{\partial t_{r_1} \cdots \partial t_{r_k}} M_{\mathbf{X}-(\kappa_1,\ldots,\kappa_m)}(0). \tag{1.4.21}$$

Define the *cumulant generating function* by

$$K_{\mathbf{X}}(t) = \log M_{\mathbf{X}}(t).$$

The $(r_1, \ldots, r_k)^{th}$ *cumulant* of $\mathbf{X}$, denoted by $\lambda_{r_1,\ldots,r_k}$, is defined as

$$\lambda_{r_1,\ldots,r_k} = \frac{\partial^k}{\partial t_{r_1} \cdots \partial t_{r_k}} K_{\mathbf{X}}(0). \tag{1.4.22}$$

We shall use the cumulants up to order 4. In terms of the (non-central) moments, they are

$$\lambda_r = \kappa_r \tag{1.4.23}$$

$$\lambda_{r,s} = \kappa_{r,s} - \kappa_r\kappa_s \tag{1.4.24}$$

$$\lambda_{r,s,t} = \kappa_{r,s,t} - \kappa_{r,s}\kappa_t\{3\} + 2\kappa_r\kappa_s\kappa_t \tag{1.4.25}$$

$$\begin{aligned}\lambda_{r,s,t,u} &= \kappa_{r,s,t,u} - \kappa_{r,s,t}\kappa_u\{4\} + 2\kappa_{r,s}\kappa_t\kappa_u\{6\} \\ &\quad -6\kappa_r\kappa_s\kappa_t\kappa_u - \kappa_{r,s}\kappa_{t,u}\{3\}.\end{aligned} \tag{1.4.26}$$

In particular, if the κ_r vanish (and hence the moments coincide with the central moments) then (1.4.23)–(1.4.26) simplify to

$$\lambda_r = 0 \tag{1.4.27}$$

$$\lambda_{r,s} = \kappa_{r,s} \tag{1.4.28}$$

$$\lambda_{r,s,t} = \kappa_{r,s,t} \tag{1.4.29}$$

$$\lambda_{r,s,t,u} = \kappa_{r,s,t,u} - \kappa_{r,s}\kappa_{t,u}\{3\}. \tag{1.4.30}$$

One of the most useful properties of cumulants is the following, which is a simple consequence of the definition.

Property:

Let $\mathbf{X_1}, \ldots, \mathbf{X_n}$ be n independent, identically distributed copies of the random vector $\mathbf{X}$. Then the cumulants $\lambda_R^{(n)}$ of the sum $\mathbf{S}^{(n)} = \sum_{i=1}^n \mathbf{X}_i$ satisfy

$$\lambda_R^{(n)} = n\lambda_R, \tag{1.4.31}$$

where λ_R are the cumulants based on one *observation only.*

Note that this property does not hold for moments.

1.4.2 Hermite Polynomials

Let $\mathbf{X} = (X_1, \ldots, X_m)$ be a random vector having the m-dimensional normal distribution with zero mean and covariance matrix $V = [v_{i,j}]$. We write $\mathbf{X} \sim N_m(0, V)$ and denote the corresponding probability density function by $\varphi(x; V)$. The *covariant Hermite polynomial* with covariance matrix V and index $I_r = i_1 \ldots i_r$, denoted by $h_{I_r}(x;V)$, is defined by

$$\varphi(x;V)h_{I_r}(x;V) = (-1)^r \varphi_{/I_r}(x;V), \tag{1.4.32}$$

where $\varphi_{/I_r}(x;V)$ is the rth derivative of $\varphi(x;V)$ with respect to $x_{i_1} \ldots x_{i_r}$. That is, $\varphi_{/I_r}(x;V) = \frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}} \varphi(x;V)$. The covariant Hermite polynomials used here are

$$\begin{aligned}
h_{ij}(x;V) &= x_i x_j - v_{i,j}, & (1.4.33)\\
h_{ijk}(x;V) &= x_i x_j x_k - v_{i,j} x_k \{3\} & (1.4.34)\\
h_{ijkl}(x;V) &= x_i x_j x_k x_l - v_{i,j} x_k x_l \{6\} + v_{i,j} v_{k,l} \{3\} & (1.4.35)\\
h_{ijklmn}(x;V) &= x_i x_j x_k x_l x_m x_n - v_{i,j} x_k x_l x_m x_n \{15\} & \\
&\quad + v_{i,j} v_{k,l} x_m x_n \{45\} - v_{i,j} v_{k,l} v_{m,n} \{15\}. & (1.4.36)
\end{aligned}$$

Re-arranging (1.4.33)-(1.4.36) gives

$$\begin{aligned}
x_i x_j &= h_{ij}(x;V) + v_{i,j} & (1.4.37)\\
x_i x_j x_k &= h_{ijk}(x;V) + v_{i,j} x_k \{3\} & (1.4.38)\\
x_i x_j x_k x_l &= h_{ijkl}(x;V) + v_{i,j} h_{kl}(x;V) \{6\} + v_{i,j} v_{k,l} \{3\} & (1.4.39)\\
x_i x_j x_k x_l x_m x_n &= h_{ijklmn}(x;V) + v_{i,j} h_{lkmn}(x;V) \{15\} & \\
&\quad + v_{i,j} v_{k,l} h_{mn}(x;V) \{45\} + v_{i,j} v_{k,l} v_{m,n} \{15\}. & (1.4.40)
\end{aligned}$$

From the definitions of the Hermite polynomials, we find that

$$\begin{aligned}
x_i x_j x_k h_{lmn}(x;V) &= h_{ijklmn}(x;V) + v_{i,j} h_{klmn}(x;V) \{15\} \\
&\quad + v_{i,j} v_{k,l} h_{mn}(x;V) \{45\} + v_{i,j} v_{k,l} v_{m,n} \{15\} \\
&\quad - (h_{ijkl}(x;V) + v_{i,j} h_{kl}(x;V) \{6\} + v_{i,j} v_{k,l} \{3\}) v_{m,n} \\
&\quad - (h_{ijkm}(x;V) + v_{i,j} h_{km}(x;V) \{6\} + v_{i,j} v_{k,m} \{3\}) v_{l,n} \\
&\quad - (h_{ijkn}(x;V) + v_{i,j} h_{kn}(x;V) \{6\} + v_{i,j} v_{k,n} \{3\}) v_{l,m}. \qquad (1.4.41)
\end{aligned}$$

We shall use the following properties of Hermite polynomials.

Property 1:

Let $\mathbf{X} \sim N_m(0,V)$, and let I_r be an arbitrary non-empty set of indices of length r, then

$$\mathbb{E}\left[h_{I_r}(\mathbf{X};V)\right] = 0. \tag{1.4.42}$$

Property 2:

Let $\mathbf{X} \sim N_m(0, V)$*, and let* I_r *and* J_s *be two arbitrary sets of indices of lengths* r *and* s *respectively, then*

$$\mathbb{E}\left[h_{I_r}(\mathbf{X}; V) h_{J_s}(\mathbf{X}; V)\right] = \begin{cases} v_{i_1,j_1} \cdots v_{i_r,j_r} \{r!\}_{j_1,\dots,j_r} & \text{if} s = r \\ 0 & \text{otherwise.} \end{cases} \quad (1.4.43)$$

Property 3:

Let $\mathbf{X} \sim N_m(0, V + dT)$*, where* d *is a scalar and* T *is a* $m \times m$ *matrix, then*

$$\mathbb{E}\left[h_{i_1 \dots i_{2r}}(\mathbf{X}; V)\right] = (T_{i_1 i_2}) \dots (T_{i_{2r-1} i_{2r}}) \left\{\frac{(2r)!}{2^r r!}\right\} d^r. \quad (1.4.44)$$

Note that (1.4.44) does not depend on the covariance matrix V*. Furthermore,*

$$\mathbb{E}\left[h_{i_1 \dots i_{2r+1}}(\mathbf{X}; V)\right] = 0. \quad (1.4.45)$$

For proofs of Property 1 and Property 2, see Barndorff-Nielsen *et al.* [2] or Barndorff-Nielsen & Cox [5, pp. 152–153]. Property 3 follows from Exercise 5.10 in McCullagh [27].

1.4.3 Differentiating Inverses of Matrices

We shall make use of the following result on differentiating inverses of matrices.

Let A *be an invertible* $r \times r$ *matrix which is a differentiable function of a parameter* θ*. Then*

$$AA^{-1} = \mathbb{I}_r, \quad (1.4.46)$$

where $\mathbb{I}_r$ *is the* $r \times r$ *identity matrix. Differentiation of (1.4.46) shows that*

$$\left(A^{-1}\right)_{/i} = -A^{-1} A_{/i} A^{-1} \quad (1.4.47)$$

and

$$\left(A^{-1}\right)_{/ij} = -A^{-1} A_{/ij} A^{-1} + A^{-1} A_{/i} A^{-1} A_{/j} A^{-1} \{2\}_{ij}. \quad (1.4.48)$$

Chapter 2

The Wald Test

There are three major types of standard parametric tests: the likelihood ratio tests, which reject the null hypothesis H_0 for large values of

$$w = 2\left\{l(\hat{\theta}) - l(\tilde{\theta})\right\};$$

the score tests, which reject H_0 for large values of

$$S = n^{-1} l_i(\tilde{\theta}) i^{i,j}(\tilde{\theta}) l_j(\tilde{\theta});$$

and the Wald tests, which reject H_0 for large values of

$$W = n\left(\hat{\theta}^i - \tilde{\theta}^i\right) i_{i,j}(\hat{\theta}) \left(\hat{\theta}^j - \tilde{\theta}^j\right).$$

The likelihood ratio test was proposed by Neyman & Pearson [30], the score test by Rao [32], and the Wald test by Wald [33]. As the likelihood ratio and score tests are well known, we do not review their properties here.

The Wald test has a severe drawback, which will be discussed in Section 2.2. The main purpose of Chapter 2 is to define versions of Wald tests which do not suffer from this drawback.

2.1 The Traditional Wald Test

The idea behind the Wald test is that the distance between the maximum likelihood estimator $\hat{\theta}$ under the full hypothesis and the maximum likelihood estimator $\tilde{\theta}$ under the null hypothesis should be close to zero if the null hypothesis is true (see e.g. Buse [13]).

The Wald test statistic is defined as the squared distance between the two maximum likelihood estimates, as measured by the information metric, that is

$$W = n\left(\hat{\theta} - \tilde{\theta}\right) i(\hat{\theta}) \left(\hat{\theta} - \tilde{\theta}\right)^T, \tag{2.1.1}$$

where $i(\theta)$ is the expected information matrix (1.3.5). Under H_0, W is asymptotically χ^2-distributed with p degrees of freedom with error of order $O(n^{-\frac{1}{2}})$.

Remark 3:
Definition (2.1.1) of the Wald statistic is slightly different from the traditional Wald statistic, usually defined as

$$W = n\left(\hat{\psi} - \psi_0\right) i_{\psi\psi}(\hat{\psi}) \left(\hat{\psi} - \psi_0\right)^T, \tag{2.1.2}$$

where $i_{\psi\psi}(\psi)$ denotes the interest part of the expected information matrix. The reason we use definition (2.1.1) rather than definition (2.1.2) is partly that it is simpler to understand geometrically, and partly that definition (2.1.1) takes into account the information contained in the nuisance parameters about the statistical model we are investigating, whereas definition (2.1.2) treats the nuisance parameters as fixed and equal to the unrestricted maximum likelihood estimate $\chi = \hat{\chi}$ (see Critchley *et al.* [18]). Note that the two definitions coincide when we are testing simple null hypotheses, and when the parameter space Θ splits as $\Theta = \Psi \times X$ and $\tilde{\chi} = \hat{\chi}$. For example, in parametric families with cuts the parameter space splits in this way and the two Wald statistics are the same.

2.2 Drawback of the Wald Test

For a given set of data, the value of the Wald statistic varies depending on how the null hypothesis is formulated. That is, the test statistics for testing two algebraically equivalent hypotheses can be very different. Breusch & Schmidt [12] show that for a given set of data it is possible to make the Wald statistic attain *any* positive value - just by rewriting the null hypothesis in an algebraically equivalent way.

Thus, changing the parameterisation of the statistical model may result in a different value of the test statistic. We now indicate why this occurs.

2.2.1 Obtaining any Value of W

Suppose for simplicity that we are testing the simple hypothesis

$$H_0 : \theta = 0$$

against

$$H_1 : \theta \in \Theta.$$

Then the Wald statistic (2.1.1) is

$$W = n\hat{\theta}\ i(\hat{\theta})\ \hat{\theta}^T.$$

Now, let $\xi = \xi(\theta)$ be an alternative parameterisation of Θ, such that $\xi(\cdot)$ is a one-to-one, differentiable function, mapping Θ into $\Xi = \xi(\Theta)$. For simplicity we assume that $\xi(0)$=0 and that the matrix $\frac{\partial \xi}{\partial \theta^T}$ is diagonal, i.e. $\xi^i(\theta) = \xi^i(\theta^i)$ for $i = 1, ..., r$. We can rewrite the null hypothesis as

$$H_0^\dagger : \xi(\theta) = 0,$$

and we obtain the Wald statistic given by the ξ-parameterisation as

$$\begin{aligned} W^\dagger &= n\hat{\xi}\ i(\hat{\xi})\ \hat{\xi}^T \\ &= n\xi(\hat{\theta}) \left(\frac{\partial}{\partial \theta^T}\xi(\hat{\theta})\right)^{-1} i(\hat{\theta}) \left(\frac{\partial}{\partial \theta}\xi^T(\hat{\theta})\right)^{-1} \xi^T(\hat{\theta}). \end{aligned}$$

For any number $K > 0$ and any value of θ there exists a function ξ such that for all i=1,..., r, $\frac{\xi^i(\hat{\theta})}{\xi^i_i(\hat{\theta})} = K\hat{\theta}^i$, where $\xi^i_i(\theta) = \frac{\partial}{\partial \theta^i}\xi^i(\theta)$. Applying this function we get

$$\begin{aligned} \xi(\hat{\theta}) \left(\frac{\partial \xi}{\partial \theta^T}(\hat{\theta})\right)^{-1} &= \left(\frac{\xi^1(\hat{\theta})}{\xi^1_1(\hat{\theta})}, ..., \frac{\xi^r(\hat{\theta})}{\xi^r_r(\hat{\theta})}\right) \\ &= K\hat{\theta}, \end{aligned}$$

and hence $W^\dagger = K^2 W$. By choosing K appropriately, it is possible to obtain any positive value of $W^\dagger$.

Remark 4:

The following explanation of the problem caused by the lack of parameterisation-invariance of the Wald test was given by Phillips & Park [31]. The *true* distributions of W and $W^\dagger$ are (in general) different. If we knew these distributions then we could use them to assess the significance of observed values of W and $W^\dagger$, respectively, and the observed significance levels would be the same. However, since the distributions are unknown, it is usual to compare the observed values of W and $W^\dagger$ with their asymptotic χ^2 distribution. This asymptotic distribution is determined by the distribution of the first order Taylor expansion of the test statistic. Different parameterisations lead to different higher-order terms of the Taylor-expansions, and so the test statistic changes with the parameterisation, while the asymptotic distribution remains unchanged. This means that, for the same set of data, we might accept H_0 using the θ-parameterisation and reject it using the ξ-parameterisation.

2.2.2 Differential Geometric Considerations

Let $\hat{\theta}$ and $\tilde{\theta}$ denote the maximum likelihood estimates on Θ under H_1 and H_0 respectively. Note that the maximum likelihood estimates are points on the manifold, and do not depend on the parameterisation.

The Wald test attempts to measure the distance between the points $\hat{\theta}$ and $\tilde{\theta}$ on the manifold Θ, using the inner product from the expected information matrix $i(\hat{\theta})$. However, the inner product defined by $i(\hat{\theta})$ is an inner product on the *tangent space* $T_{\hat{\theta}}\Theta$ of θ at $\hat{\theta}$, whereas $\hat{\theta}$ and $\tilde{\theta}$ are points on the *manifold* Θ. There is no inner product given on Θ, and thus no such thing as 'the distance' between two points (see e.g. Critchley *et al.* [18] or Murray & Rice [29]).

When the Wald statistic is computed, a parameterisation, h is implicitly *chosen* to get points $h(\hat{\theta})$ and $h(\tilde{\theta})$ in a parameter space which is then identified with the tangent space $T_{\hat{\theta}}\Theta$. Once we have points in the tangent space we can use the inner product based on $i(\hat{\theta})$ to obtain the Wald statistic. However, as the points $h(\hat{\theta})$ and $h(\tilde{\theta})$ obviously depend on the parameterisation chosen, the Wald statistic depends on the parameterisation. In other words, the squared distance used in the Wald statistic is the distance between $h(\hat{\theta})$ and $h(\tilde{\theta})$ for some (not unique) parameterisation h (see e.g. Critchley *et al.* [17, 18] or

Le Cam [26]).

Remark 5:

The likelihood ratio statistic

$$w = 2\left\{l(\hat{\theta}) - l(\tilde{\theta})\right\} \tag{2.2.3}$$

is simply a comparison of values taken by the likelihood function on the manifold and does not depend on the parameterisation. Also, the score statistic

$$S = n^{-1} l_i(\tilde{\theta}) i^{i,j}(\tilde{\theta}) l_j(\tilde{\theta}) \tag{2.2.4}$$

is parameterisation-invariant, as it measures the length of a vector $l_*(\theta) = \frac{\partial l}{\partial \theta}(\theta; x)$ in the cotangent space, with a metric $i(\theta)^{-1}$ on the cotangent space.

Remark 6:

Various authors (e.g. Critchley *et al.* [17, 18], Le Cam [26] and Dagenais & Dufour [19]) have come up with suggestions for re-defining the Wald statistic in ways avoiding the dependence on the parameterisation.

Critchley *et al.* [17, 18] discuss defining a Wald-type test statistic entirely on the manifold Θ. Their *Fisher geodesic statistic* is defined as the squared geodesic distance between $\hat{\theta}$ and the nearest point in Θ_0, where Θ_0 is the (sub)manifold corresponding to the points on Θ for which the null hypothesis is satisfied. They use the geodesic based on the expected information matrix on Ω. Calculation of the Fisher geodesic statistic involves solving second order differential equations (to obtain geodesics) and then minimising (to obtain the geodesic distance). In general, this can be very complicated.

Le Cam [26] considers the confidence ellipsoids of the Wald statistic. Naturally these confidence ellipsoids depend on the parameterisation in the same way as the Wald statistic. As an alternative, he suggests using a function of the Hellinger distance to measure the distance between $\hat{\theta}$ and $\tilde{\theta}$ on the manifold. He then bases confidence ellipsoids on these (parameterisation-invariant) distances. Unfortunately, the Hellinger distances, and thus the new confidence ellipsoids, can be difficult to compute, unless the model is very simple.

Dagenais & Dufour suggest using special formulations of the generalised Neyman's $C(\alpha)$-statistic which are invariant and can be chosen such that they are easily computed in the same situations where the Wald statistic is easy to compute. The Neyman's $C(\alpha)$-tests are not directly linked to the ideas of the Wald test.

2.3 Geometric Wald Tests

A family of new Wald statistics, *geometric Wald statistics*, is formulated here. The main ideas are closely related to the arguments of Critchley *et al* [17, 18], the fundamental difference being that while they construct a new statistic (the Fisher geodesic test statistic) using the manifold itself (see Remark 6 in Section 2.2.2), here a family of parameterisation-invariant statistics is defined using the tangent spaces to the manifold. We shall use the differential-geometric concept of a yoke to form a coordinate system on the manifold Θ, taking values in the tangent space $T_{\hat{\theta}}\Theta$ such that $\hat{\theta}$ is mapped to the origin and $\tilde{\theta}$ is mapped to a 'relevant' point of $T_{\hat{\theta}}\Theta$. A new Wald-type test statistic can now be defined as the squared distance (given by a suitable Riemannian metric) between this 'relevant' point and the origin.

Recall that a yoke g induces a Riemannian metric $\not{g}_{i;j}(\theta)$ on Ω. Furthermore, for a fixed point θ, define

$$\overset{\alpha}{\Gamma}_* (\theta;\theta') = n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} \not{g}^{i;j} g_{;j}(\theta';\theta) + \frac{1-\alpha}{2} \not{g}^{i;j} g_{j;}(\theta;\theta') \right\} \frac{\partial}{\partial \theta^i},$$

where α is an arbitrary real number and $\not{g}^{i;j}$ is the $(i,j)^{th}$ element of the inverse of the matrix $[g_{i;j}(\theta;\theta)]$. The functions $\overset{\alpha}{\Gamma}_* (\theta;\cdot)$ provide a family of (normal) coordinate systems, around $\theta \in \Theta$, taking values in the tangent space to Θ at θ (see Blæsild [10]). It is straightforward to see that $\overset{\alpha}{\Gamma}_* (\theta;\theta')$ is a vector in the correct tangent space $T_\theta\Theta$. An intuitive argument for looking at these particular coordinates hinges on the importance of the yoke involved. The yoke $g(\theta;\theta')$ contains useful information about the manifold of statistical models and about the difference between θ and θ'. By definition of a yoke, we know that $g(\theta;\theta')$ is smooth and concave locally around the diagonal $\not{g}(\theta)$. Thus, for fixed θ the steepness of the slope at $(\theta;\theta')$ indicates how 'far away' we are from the diagonal i.e. how far θ' is from θ. Likewise, for fixed θ' the steepness of $g(\theta;\theta')$ tells us how far θ is from θ'. The constant α is used as a weight between the two ways of expressing the discrepancy between θ and θ'. These coordinate systems are the key to defining our statistics.

We use the inner product obtained from the Riemannian metric $\not{g}_{i,j}$, based on the yoke used in the $\overset{\alpha}{\Gamma}_*$-coordinate systems, to measure the squared length of the vector $\overset{\alpha}{\Gamma}_* (\hat{\theta};\tilde{\theta})$ in $T_{\hat{\theta}}\Theta$. More precisely, for each $\alpha \in \mathbb{R}$, we can define a parameterisation-invariant test statistic on the *tangent space* $T_{\hat{\theta}}\Theta$ of Θ by

$$\overset{\alpha}{\mathcal{W}} \;=\; \overset{\alpha}{\Gamma}_* (\hat{\theta};\tilde{\theta}) \not{g}(\hat{\theta}) \, \overset{\alpha}{\Gamma}_*{}^T(\hat{\theta};\tilde{\theta})$$

$$= \overset{\alpha}{\Gamma}_*{}^i(\hat{\theta};\tilde{\theta}) g_{i;j}(\hat{\theta}) \overset{\alpha}{\Gamma}_*{}^j(\hat{\theta};\tilde{\theta}). \tag{2.3.5}$$

We call this family of test statistics the *geometric Wald statistics* based on the yoke g.

Observe that we can write the geometric Wald statistics $\overset{\alpha}{\mathcal{W}}$ as

$$\overset{\alpha}{\mathcal{W}} = \overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta}) g^{i;j}(\hat{\theta}) \overset{\alpha}{\Gamma}_j(\hat{\theta};\tilde{\theta}), \tag{2.3.6}$$

where

$$\overset{\alpha}{\Gamma}_i(\theta;\theta') = n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} g_{;i}(\theta';\theta) + \frac{1-\alpha}{2} g_i(\theta;\theta') \right\} \tag{2.3.7}$$

is the lowered version of $\overset{\alpha}{\Gamma}_*(\theta;\theta')$.

In particular, geometric Wald tests based on the expected and observed likelihood yokes, respectively, will be considered here. We call the geometric Wald statistic based on the expected likelihood yoke (1.3.16) the *expected geometric Wald statistic*, and the geometric Wald statistic based on the observed likelihood yoke (1.3.17) the *observed geometric Wald statistic*. In comparison to other suggestions for re-defining the Wald statistic (see Remark 6 in Section 2.2.2), the geometric Wald statistic can be calculated fairly easily from the yoke and the two maximum likelihood estimates $\hat{\theta}$ and $\tilde{\theta}$.

In Remark 11 and Remark 14 in Section 3.5.3 and Section 4.4.3, respectively, we find that, under H_0, the geometric Wald statistics based on either of the two likelihood yokes are χ^2-distributed with error of order $O(n^{-\frac{1}{2}})$.

Remark 7:

We mentioned in Remark 3 that the traditional Wald test is slightly different from the definition used here. Ideas for defining geometric Wald statistics closer to the original statistic (i.e. not depending on the nuisance parameters) include (i) using the profile likelihood for the interest parameter ψ, (ii) using a marginal distribution only depending on ψ, or (iii) treating the nuisance parameters as fixed and equal to the unrestricted maximum likelihood estimates $\hat{\chi}$.

2.3.1 Modified Wald Tests

Hayakawa & Puri [23] suggest a modification of the Wald test when testing simple hypotheses

$$H_0 : \theta = \theta_0.$$

They suggest replacing the expected information matrix $i(\hat{\theta})$ at $\hat{\theta}$ in (2.1.1) by $i(\theta_0)$, obtaining the test statistic

$$\widetilde{W} = \left(\hat{\theta} - \theta_0\right) i(\theta_0) \left(\hat{\theta} - \theta_0\right)^T.$$

The motivation behind the modification is that sometimes it is easier to calculate the expected information matrix at θ_0 than at $\hat{\theta}$. We can generalise the modification to composite null hypotheses as

$$\widetilde{W} = \left(\hat{\theta} - \tilde{\theta}\right) i(\tilde{\theta}) \left(\hat{\theta} - \tilde{\theta}\right)^T. \tag{2.3.8}$$

Note that this modified statistic depends on the parameterisation.

2.3.2 Modified Geometric Wald Tests

The geometric Wald test statistics can be modified in a way similar to that used by Hayakawa & Puri to modify the traditional Wald statistic. In the geometric Wald statistic, we mapped $\tilde{\theta}$ to a point in the tangent space $T_{\hat{\theta}}\Theta$ at $\hat{\theta}$, in order to use the inner product from $i(\hat{\theta})$. We can modify this method by instead mapping $\hat{\theta}$ to a point in the tangent space $T_{\tilde{\theta}}\Theta$ at $\tilde{\theta}$ and using the inner product from $i(\tilde{\theta})$ to form a *modified geometric Wald statistic.* We define

$$\begin{aligned} \overset{\alpha}{\widetilde{\mathcal{W}}} &= \overset{\alpha}{\Gamma}_* (\tilde{\theta}; \hat{\theta}) \not{g}(\tilde{\theta}) \overset{\alpha T}{\Gamma}_* (\tilde{\theta}; \hat{\theta}) \\ &= \overset{\alpha}{\Gamma}_i (\tilde{\theta}; \hat{\theta}) \not{g}^{i,j}(\tilde{\theta}) \overset{\alpha}{\Gamma}_j (\tilde{\theta}; \hat{\theta}), \end{aligned} \tag{2.3.9}$$

where $\not{g}^{i,j}$ and $\overset{\alpha}{\Gamma}_i (\theta; \theta')$ are based on either the expected likelihood yoke (1.3.16) or the observed likelihood yoke (1.3.17).

The modified geometric Wald statistics are parameterisation-invariant. Like the unmodified statistics, the modified geometric Wald statistics all have χ^2_p distributions under

H_0 with error of order $O(n^{-\frac{1}{2}})$.

Remark 8:

Definitions (2.3.6) and (2.3.9) cover a whole range of test statistics, when different yokes and different values of α are considered. It is interesting to observe that the score test statistic (2.2.4),

$$S = n^{-1} l_i(\tilde{\theta}; \hat{\theta}, a) i^{i,j}(\tilde{\theta}) l_j(\tilde{\theta}; \hat{\theta}, a),$$

is very nearly the same as the modified observed geometric Wald statistic with $\alpha = -1$,

$$\overset{-1}{\widetilde{W}} = n^{-1} l_i(\tilde{\theta}; \hat{\theta}, a) j^{i,j}(\tilde{\theta}) l_j(\tilde{\theta}; \hat{\theta}, a).$$

The only difference is that the geometric Wald statistic always uses the Riemannian metric corresponding to the yoke itself – in this case the observed information metric – whereas the score test statistic involves both the expected and the observed likelihood yokes. Note also that when we consider models with cuts, the geometric Wald statistics based on either of the likelihood yokes simplify to the geometric Wald statistics for the marginal model of the interest parameter.

Example 2.1:

Consider a linear regression model

$$Y_i = a + bx_i + \epsilon_i, \qquad i = 1 \dots n$$

where, for $i = 1 \dots n$, Y_i are dependent variables, x_i are controlled variables, a is a constant, b is a slope parameter and ϵ_i are independent, identically normally distributed random variables with zero mean and unknown variance σ^2. For convenience, we write the parameters as $\theta = (a, b, \sigma^2)$. Suppose we wish to test that the parameter θ belongs to some subspace Θ_0 of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$.

The log-likelihood function of θ is

$$l(\theta) = -\frac{n}{2}\left(\log 2\pi + \log \sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - a - bx_i)^2 .$$

The expected likelihood yoke (1.3.16) is then

$$f(\theta; \theta') = n^{-1}\mathbb{E}_{\theta'}\left[l(\theta) - l(\theta')\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\log\sigma'^2 - \log\sigma^2 + 1 - \frac{\sigma'^2}{\sigma^2}\right)\\
&\quad -\frac{1}{2\sigma^2}\left((a'-a)^2 + 2(a'-a)(b'-b)\bar{x}. + (b'-b)^2 SS_x\right),
\end{aligned}
$$

where $\bar{x}.$ denotes the mean value $\frac{1}{n}\sum_{i=1}^{n} x_i$ and where SS_x is the mean sum of squares $SS_x = \frac{1}{n}\sum_{i=1}^{n} x_i^2$. Thus,

$$
\begin{aligned}
f_a(\theta;\theta') &= \frac{1}{\sigma^2}\left(a' - a + (b'-b)\bar{x}.\right)\\
f_b(\theta;\theta') &= \frac{1}{\sigma^2}\left((a'-a)\bar{x}. + (b'-b)SS_x\right)\\
f_{\sigma^2}(\theta;\theta') &= -\frac{1}{2\sigma^2}\left(1 - \frac{\sigma'^2}{\sigma^2}\right) + \frac{1}{2n\sigma^4}\sum_{i=1}^{n}\left(a' - a + (b'-b)x_i\right)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
f_{;a}(\theta';\theta) &= \frac{1}{\sigma'^2}\left(a' - a + (b'-b)\bar{x}.\right)\\
f_{;b}(\theta';\theta) &= \frac{1}{\sigma'^2}\left((a'-a)\bar{x}. + (b'-b)SS_x\right)\\
f_{;\sigma^2}(\theta';\theta) &= \frac{1}{2}\left(\frac{1}{\sigma^2} - \frac{1}{\sigma'^2}\right).
\end{aligned}
$$

Furthermore, we find that the expected information matrix is

$$
i(\theta) = \begin{bmatrix} \frac{1}{\sigma^2} & \frac{1}{\sigma^2}\bar{x}. & 0 \\ \frac{1}{\sigma^2}\bar{x}. & \frac{1}{\sigma^2}SS_x & 0 \\ 0 & 0 & \frac{1}{2\sigma^4} \end{bmatrix},
$$

and thus the expected formation matrix is

$$
i^{-1}(\theta) = \begin{bmatrix} \frac{\sigma^2}{SSD_x}SS_x & -\frac{\sigma^2}{SSD_x}\bar{x}. & 0 \\ -\frac{\sigma^2}{SSD_x}\bar{x}. & \frac{\sigma^2}{SSD_x} & 0 \\ 0 & 0 & 2\sigma^4 \end{bmatrix},
$$

where SSD_x is the mean sum of squares of deviations $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}.)^2$. We find $\overset{\alpha}{\Gamma}(\hat{\theta};\tilde{\theta})$ by

$$
\overset{\alpha}{\Gamma}_\theta(\hat{\theta};\tilde{\theta}) = n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}f_{;\theta}(\tilde{\theta};\hat{\theta}) + \frac{1-\alpha}{2}f_\theta(\hat{\theta};\tilde{\theta})\right\},
$$

that is

$$
\overset{\alpha}{\Gamma}_a(\hat{\theta};\tilde{\theta}) = n^{\frac{1}{2}}\left(\frac{1+\alpha}{2\tilde{\sigma}^2} + \frac{1-\alpha}{2\hat{\sigma}^2}\right)\left(\tilde{a} - \hat{a} + (\tilde{b} - \hat{b})\bar{x}.\right)
$$

$$\begin{aligned}
\overset{\alpha}{\Gamma}_b\,(\hat\theta;\tilde\theta) &= n^{\frac{1}{2}}\left(\frac{1+\alpha}{2\tilde\sigma^2}+\frac{1-\alpha}{2\hat\sigma^2}\right)\left((\tilde a-\hat a)\bar x. + (\tilde b-\hat b)SS_x\right)\\
\overset{\alpha}{\Gamma}_{\sigma^2}\,(\hat\theta;\tilde\theta) &= n^{\frac{1}{2}}\frac{\tilde\sigma^4-\hat\sigma^4-\alpha\left(\tilde\sigma^2-\hat\sigma^2\right)^2}{4\tilde\sigma^2\hat\sigma^4}+\frac{1-\alpha}{4\sqrt{n}\hat\sigma^4}\sum_{i=1}^n\left(\tilde a-\hat a+(\tilde b-\hat b)x_i\right)^2.
\end{aligned}$$

Finally, we get the expected geometric Wald statistic as

$$\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= \hat\sigma^2\left(\frac{1+\alpha}{2\tilde\sigma^2}+\frac{1-\alpha}{2\hat\sigma^2}\right)^2\sum_{i=1}^n\left(\tilde a-\hat a+(\tilde b-\hat b)x_i\right)^2\\
&\quad+n\frac{1}{2}\left(\frac{\tilde\sigma^4-\hat\sigma^4-\alpha\left(\tilde\sigma^2-\hat\sigma^2\right)^2}{2\tilde\sigma^2\hat\sigma^2}+\frac{1-\alpha}{2n\hat\sigma^2}\sum_{i=1}^n\left(\tilde a-\hat a+(\tilde b-\hat b)x_i\right)^2\right)^2.
\end{aligned}$$

The modified geometric Wald statistic in expected geometry follows by swapping $\hat\theta$ and $\tilde\theta$ in the above formula. We find that

$$\begin{aligned}
\widetilde{\overset{\alpha}{\mathcal{W}}} &= \tilde\sigma^2\left(\frac{1+\alpha}{2\hat\sigma^2}+\frac{1-\alpha}{2\tilde\sigma^2}\right)^2\sum_{i=1}^n\left(\tilde a-\hat a+(\tilde b-\hat b)x_i\right)^2\\
&\quad+n\frac{1}{2}\left(\frac{\hat\sigma^4-\tilde\sigma^4-\alpha\left(\tilde\sigma^2-\hat\sigma^2\right)^2}{2\tilde\sigma^2\hat\sigma^2}+\frac{1-\alpha}{2n\tilde\sigma^2}\sum_{i=1}^n\left(\tilde a-\hat a+(\tilde b-\hat b)x_i\right)^2\right)^2.
\end{aligned}$$

The likelihood ratio test statistic (2.2.3) for testing the same hypothesis is

$$\begin{aligned}
w &= 2\left\{l(\hat\theta)-l(\tilde\theta)\right\}\\
&= n\left(\log\tilde\sigma^2-\log\hat\sigma^2\right)-\frac{1}{\hat\sigma^2}\sum_{i=1}^n\left(Y_i-\hat a-\hat bx_i\right)^2+\frac{1}{\tilde\sigma^2}\sum_{i=1}^n\left(Y_i-\tilde a-\tilde bx_i\right)^2\\
&= n\log\frac{\tilde\sigma^2}{\hat\sigma^2}+n\left(\frac{\hat\sigma^2}{\tilde\sigma^2}-1\right)+\frac{1}{\tilde\sigma^2}\sum_{i=1}^n\left(\hat a-\tilde a+(\hat b-\tilde b)x_i\right)^2,
\end{aligned}$$

where we have used that the maximum likelihood estimate of the variance is $\hat\sigma^2 = \frac{1}{n}\sum_{i=1}^n\left(Y_i-\hat a-\hat bx_i\right)^2$ and that $\sum_{i=1}^n\left(Y_i-\tilde a-\tilde bx_i\right)^2 = n\hat\sigma^2+\sum_{i=1}^n\left(\hat a-\tilde a+(\hat b-\tilde b)x_i\right)^2$. The score statistic (2.2.4) becomes

$$\begin{aligned}
S &= n^{-1}l_\theta(\tilde\theta)i^{-1}(\tilde\theta)l_\theta^T(\tilde\theta)\\
&= \frac{1}{\tilde\sigma^2}\sum_{i=1}^n\left(\tilde a-\hat a+(\tilde b-\hat b)x_i\right)^2\\
&\quad+n\frac{1}{2}\left(\frac{\hat\sigma^2-\tilde\sigma^2}{\tilde\sigma^2}+\frac{1}{n\tilde\sigma^2}\sum_{i=1}^n\left(\tilde a-\hat a+(\tilde b-\hat b)x_i\right)^2\right)^2.
\end{aligned}$$

Observe that the score statistic is exactly the modified geometric Wald statistic with $\alpha = -1$. This happens because the linear regression model is a full exponential model and, as we shall soon see, for all full exponential models we have that $S = \widetilde{\overset{-1}{\mathcal{W}}}$.

2.3.3 Full Exponential Models

Suppose we have a full exponential model (see e.g. Barndorff-Nielsen & Cox [5] or Efron [20]) with density function

$$f(\mathbf{x};\theta) = e^{\theta t(\mathbf{x})^T - a(\theta)},$$

and hence log likelihood function

$$l(\theta) = \theta t(\mathbf{x})^T - a(\theta).$$

Then θ is the canonical parameter. Let η be the expectation parameter, that is

$$\eta = \eta(\theta) = \mathbb{E}_\theta\left[t(\mathbf{X})\right],$$

where $\mathbb{E}_\theta[\cdot]$ denotes the mean value with respect to θ. Observe that $\mathbb{E}_\theta\left[l_i(\theta)\right] = 0$ implies that $\eta = \frac{\partial}{\partial\theta}a(\theta)$. Let $i(\theta)$ be the expected information matrix in the θ-parameterisation. Then the expected information matrix in the η-parameterisation is

$$\begin{aligned} i(\eta) &= \left(\frac{\partial \eta}{\partial \theta^T}\right)^{-1} i(\theta) \left(\frac{\partial \eta^T}{\partial \theta}\right)^{-1} \\ &= \left(\frac{\partial^2}{\partial \theta^T \partial \theta} a(\theta)\right)^{-1} i(\theta) \left(\frac{\partial^2}{\partial \theta \partial \theta^T} a(\theta)\right)^{-1} \\ &= i^{-1}(\theta), \end{aligned}$$

where we have used that $i(\theta) = -\frac{\partial^2}{\partial\theta\partial\theta^T}a(\theta)$. The two likelihood yokes coincide for full exponential models, and they are equal to

$$\begin{aligned} g(\theta_1;\theta_2) &= n^{-1}\mathbb{E}_{\theta_2}\left[l(\theta_1) - l(\theta_2)\right] \\ &= \left(\theta_1^i - \theta_2^i\right) a_i(\theta_2)^T - a(\theta_1) + a(\theta_2), \end{aligned}$$

where $a_i = \frac{\partial a}{\partial \theta^i}$. Thus

$$g_i(\theta_1;\theta_2) = a_i(\theta_2) - a_i(\theta_1)$$

$$\begin{aligned} g_{;i}(\theta_2;\theta_1) &= \left(\theta_2^j - \theta_1^j\right) a_{ij}(\theta_1) - a_i(\theta_1) + a_i(\theta_1) \\ &= \left(\theta_1^j - \theta_2^j\right) i_{i,j}(\theta_1). \end{aligned}$$

The geometric Wald statistics are

$$\begin{aligned}\overset{\alpha}{\mathcal{W}} &= n\left(\frac{1+\alpha}{2}(\hat{\theta}-\tilde{\theta})i(\hat{\theta})+\frac{1-\alpha}{2}(\tilde{\eta}-\hat{\eta})\right)i^{-1}(\hat{\theta}) \\ &\quad\times\left(\frac{1+\alpha}{2}(\hat{\theta}-\tilde{\theta})i(\hat{\theta})+\frac{1-\alpha}{2}(\tilde{\eta}-\hat{\eta})\right)^T, \end{aligned} \tag{2.3.10}$$

and the modified geometric Wald statistics are

$$\begin{aligned}\overset{\alpha}{\widetilde{\mathcal{W}}} &= n\left(\frac{1+\alpha}{2}(\tilde{\theta}-\hat{\theta})i(\tilde{\theta})+\frac{1-\alpha}{2}(\hat{\eta}-\tilde{\eta})\right)i^{-1}(\tilde{\theta}) \\ &\quad\times\left(\frac{1+\alpha}{2}(\tilde{\theta}-\hat{\theta})i(\tilde{\theta})+\frac{1-\alpha}{2}(\hat{\eta}-\tilde{\eta})\right)^T. \end{aligned} \tag{2.3.11}$$

In particular, observe that

$$\begin{aligned}\overset{1}{\mathcal{W}} &= n\left(\hat{\theta}-\tilde{\theta}\right)i(\hat{\theta})\left(\hat{\theta}-\tilde{\theta}\right)^T \\ \overset{-1}{\mathcal{W}} &= n\,(\hat{\eta}-\tilde{\eta})\,i(\hat{\eta})\,(\hat{\eta}-\tilde{\eta})^T\end{aligned}$$

are the traditional Wald test statistics (2.1.1) for the canonical and expectation parameterisation, respectively. Furthermore,

$$\begin{aligned}\overset{1}{\widetilde{\mathcal{W}}} &= n\left(\hat{\theta}-\tilde{\theta}\right)i(\tilde{\theta})\left(\hat{\theta}-\tilde{\theta}\right)^T \\ \overset{-1}{\widetilde{\mathcal{W}}} &= n\,(\hat{\eta}-\tilde{\eta})\,i(\tilde{\eta})\,(\hat{\eta}-\tilde{\eta})^T\end{aligned}$$

are the modified traditional Wald statistics (2.3.8) for the the canonical and expectation parameterisation, respectively. Finally, note that, as the two geometries coincide for full exponential models, the score test statistic (2.2.4) is equivalent to $\overset{-1}{\widetilde{\mathcal{W}}}$. Moreover, for multivariate normal distributions with known variance, it is straightforward to see that, for any α, $\overset{\alpha}{\mathcal{W}} = \overset{\alpha}{\widetilde{\mathcal{W}}} = S = w$.

Example 2.2:

A very simple example of the geometric Wald statistics is given by looking at the 1-dimensional exponential model consisting of the exponential distributions, with density function

$$f(\mathbf{x};\theta) = \theta^n e^{-n\theta\bar{x}},$$

where $\bar{x}$ denotes the sample mean. We have a full exponential model with canonical parameter θ and expectation parameter $\eta = \theta^{-1}$. The expected information matrix in the canonical parameterisation is $i(\theta) = \theta^{-2}$. Thus the geometric Wald statistic (2.3.10) becomes

$$
\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= n\left(\frac{1+\alpha}{2}(\hat{\theta}-\tilde{\theta})\hat{\theta}^{-2}+\frac{1-\alpha}{2}(\tilde{\theta}^{-1}-\hat{\theta}^{-1})\right)^2\hat{\theta}^2 \\
&= n\frac{1}{4}\left((1+\alpha)\left(1-\frac{\tilde{\theta}}{\bar{\hat{\theta}}}\right)+(1-\alpha)\left(\frac{\hat{\theta}}{\bar{\tilde{\theta}}}-1\right)\right)^2 .
\end{aligned}
$$

Likewise, we find the modified geometric Wald statistic (2.3.11) to be

$$
\begin{aligned}
\widetilde{\overset{\alpha}{\mathcal{W}}} &= n\frac{1}{4}\left((1+\alpha)\left(1-\frac{\hat{\theta}}{\bar{\tilde{\theta}}}\right)+(1-\alpha)\left(\frac{\tilde{\theta}}{\bar{\hat{\theta}}}-1\right)\right)^2 \\
&= \overset{-\alpha}{\mathcal{W}} .
\end{aligned}
$$

To assess the adequacy of the large-sample χ^2 approximation to the null distribution of the geometric Wald statistics, 1000 simulations were run for the exponential distribution, testing the null hypothesis

$$H_0 : \theta = 1,$$

where θ is the canonical parameter. For sample sizes $n = 12$ and $n = 5$, the following results were obtained:

	n=12				n=5			
	nominal cumulative probability				nominal cumulative probability			
α	0.900	0.950	0.975	0.990	0.900	0.950	0.975	0.990
0	0.893	0.937	0.957	0.976	0.868	0.912	0.938	0.964
0.33	0.889	0.931	0.947	0.968	0.862	0.899	0.923	0.949
-0.33	0.897	0.938	0.966	0.984	0.878	0.925	0.959	0.970
0.5	0.887	0.926	0.948	0.966	0.859	0.899	0.920	0.942
-0.5	0.902	0.942	0.970	0.986	0.884	0.933	0.960	0.979
1	0.883	0.922	0.941	0.952	0.852	0.882	0.904	0.919
-1	0.916	0.959	0.975	0.988	0.928	0.960	0.967	0.983
10	0.702	0.742	0.777	0.796	0.622	0.654	0.682	0.716
-10	0.733	0.768	0.796	0.826	0.623	0.662	0.691	0.720

Table 1: $P\left(\overset{\alpha}{\mathcal{W}} \leq \chi^2_{1;c}\right)$ for $c = 0.90, 0.95, 0.975, 0.99$ based on 1000 simulations.

In this example, the nominal cumulative probabilities (from the asymptotic χ^2_1 distribution) of the geometric Wald statistic with $\alpha = -1$ are consistently better than those for any of the other values of α used here. The χ^2-approximation is very poor for $\alpha = \pm 10$, and not particularly good for most of the other values, not even for the score statistic which is the geometric Wald statistic for the value $\alpha = 1$. The approximation clearly worsens as the sample size is reduced. In the next two chapters we shall try and improve the approximations to the limiting distributions for small sample sizes. We do this by obtaining Bartlett-type adjustments of the geometric Wald statistics.

Chapter 3

Bartlett-type Adjustment for $\overset{\alpha}{\mathcal{W}}$

3.1 Bartlett and Bartlett-type Adjustments

By multipying the likelihood ratio statistic by a suitable factor, the χ^2 approximation to the asymptotic null distribution can be improved remarkably. This factor (the Bartlett adjustment) does not depend on the observations but is purely a function of the cumulants of the log-likelihood derivatives (see e.g. Barndorff-Nielsen & Cox [4, 6, Section 6.3] or Cribari-Neto & Cordeiro [16]).

A result corresponding to the Bartlett adjustment of the likelihood ratio statistic does not in general hold for other test statistics such as the score statistic (2.2.4) and the Wald statistic (2.1.1). In particular, it does not hold for the geometric Wald statistics (2.3.6) and (2.3.9). There are, however, various other ways of adjusting a wide range of test statistics to improve the approximation to the asymptotic distribution. Here we shall use the moment generating function to find a *Bartlett-type adjustment* of the geometric Wald statistics. We start out with a brief reminder of the Bartlett adjustment of the likelihood ratio statistic and an outline of the argument behind the Bartlett-type adjustment.

3.1.1 Likelihood Ratio Test

Recall that the likelihood ratio statistic (2.2.3) for testing H_0 against H_1 is given by

$$w = 2\left\{l(\hat{\theta}) - l(\tilde{\theta})\right\}. \tag{3.1.1}$$

The statistic w is invariant under re-parameterisation and is χ^2-distributed under H_0, with error of order $O(n^{-\frac{1}{2}})$.

It can be shown (e.g. Barndorff-Nielsen & Cox [4]) that the mean value of w can be written as

$$\mathbb{E}[w] = \left(1 + \frac{B}{pn}\right) p + O(n^{-\frac{3}{2}}), \tag{3.1.2}$$

where B is a function of the parameter $\theta = (\theta^1 \ldots \theta^r)$. Define

$$w' = \left\{1 + \frac{B}{pn}\right\}^{-1} w. \tag{3.1.3}$$

Then, trivially,

$$\mathbb{E}[w'] = p + O(n^{-\frac{3}{2}}). \tag{3.1.4}$$

Furthermore, all the cumulants of w are corrected to order $O(n^{-\frac{3}{2}})$. Thus w' has a χ^2 distribution with error of order $O(n^{-\frac{3}{2}})$. The correction term B is a function of the joint cumulants of the log-likelihood derivatives up to order 4. Barndorff-Nielsen & Hall [7] show that the error in approximating the distribution of the Bartlett adjusted likelihood ratio statistic by its limiting χ^2 distribution is of order $O(n^{-2})$ rather than order $O(n^{-\frac{3}{2}})$.

3.1.2 Bartlett-type Adjustments

Cordeiro & Ferrari [14] provide a general method of using the moment generating functions to obtain Bartlett-type corrections of test statistics which are asymptotically χ^2-distributed with error of order $O(n^{-\frac{1}{2}})$. Their corrections improve the approximation to the asymptotic distribution to order $O(n^{-\frac{3}{2}})$. An outline of their method is as follows.

Let S be a test statistic with a χ^2 distribution with error of order $O(n^{-\frac{1}{2}})$. Then we can find an adjusted statistic S', defined by

$$S' = \left\{1 - \frac{1}{n}(c + bS + aS^2)\right\} S, \tag{3.1.5}$$

having a χ^2 distribution with error of order $O(n^{-\frac{3}{2}})$.

Denote the moment generating function of the null distribution of S by $M_S(t)$. The asymptotic expansion to order $O(n^{-1})$ of $M_S(t)$ can be written as

$$\begin{aligned} M_S(t) &= \mathbb{E}\left[e^{St}\right] \\ &= (1-2t)^{-\frac{p}{2}}\left(1+(24n)^{-1}\left\{A_1 d + A_2 d^2 + A_3 d^3 + O(d^4)\right\}\right) \\ &\quad + O(n^{-\frac{3}{2}}), \end{aligned} \tag{3.1.6}$$

where

$$d = \frac{2t}{1-2t} \tag{3.1.7}$$

and A_1, A_2 and A_3 are constants.

We can rewrite (3.1.6) as

$$\begin{aligned} M_S(t) &= (1-2t)^{-\frac{p}{2}} \\ &\quad +(24n)^{-1}\left\{A_3(1-2t)^{-\frac{p+6}{2}} + (A_2-3A_3)(1-2t)^{-\frac{p+4}{2}}\right. \\ &\quad +(A_1-A_2+3A_3)(1-2t)^{-\frac{p+2}{2}} \\ &\quad \left.+(A_2-A_1-A_3)(1-2t)^{-\frac{p}{2}} + O\left((1-2t)^{-\frac{p+8}{2}}\right)\right\} \\ &\quad +O(n^{-\frac{3}{2}}), \end{aligned} \tag{3.1.8}$$

from which it follows that

$$\begin{aligned} f_S(x) &= g_p(x) + (24n)^{-1}\{A_3 g_{p+6}(x) + (A_2-3A_3)g_{p+4}(x) \\ &\quad +(A_1-A_2+3A_3)g_{p+2}(x) + (A_2-A_1-A_3)g_p(x) + O(g_{p+8}(x))\} \\ &\quad +O(n^{-\frac{3}{2}}), \end{aligned} \tag{3.1.9}$$

where $f_S(x)$ is the probability density function of S and $g_p(x)$ is the probability density function of a $\chi^2(p)$ random variable. Now, from the recurrence relation $g_{p+2}(x) = xp^{-1}g_p(x)$ we have that

$$f_S(x) = g_p(x)\left(1 + B_0 + B_1 x + B_2 x^2 + B_3 x^3 + O(x^4)\right) + O(n^{-\frac{3}{2}}), \tag{3.1.10}$$

where

$$B_0 = \frac{A_2 - A_1 - A_3}{24n}$$

$$B_1 = \frac{3A_3 - 2A_2 + A_1}{24pn}$$
$$B_2 = \frac{A_2 - 3A_3}{24p(p+2)n}$$
$$B_3 = \frac{A_3}{24p(p+2)(p+4)n}.$$

This can be used to obtain the following result by Cordeiro & Ferrari [14].

Proposition 1:

If a statistic S has a moment generating function of the form (3.1.6) then the statistic S' given by (3.1.5), where

$$a = \frac{A_3}{12p(p+2)(p+4)}$$
$$b = \frac{A_2 - 2A_3}{12p(p+2)}$$
$$c = \frac{A_1 - A_2 + A_3}{12p}$$

has a $\chi^2(p)$ distribution with error of order $O(n^{-\frac{3}{2}})$.

3.2 Notation

Before we start Taylor-expanding the geometric Wald statistics, we introduce some more notation that will simplify the expansions. Let $f = f(\hat{\theta};\theta)$ denote the expected likelihood yoke (1.3.16). Recall from Remark 2 in Section 1.3.2 that $[f_{i;j}(\theta)] = i(\theta)$, where $i(\theta)$ is the expected information matrix of order $O(1)$. We define the (normalised) derivatives of the scores by

$$Z_i = n^{-\frac{1}{2}} l_i(\theta;\mathbf{X}) \tag{3.2.11}$$
$$Z_{ij} = n^{-\frac{1}{2}}[l_{ij}(\theta;\mathbf{X}) - nf_{ij}] \tag{3.2.12}$$
$$Z_{ijk} = n^{-\frac{1}{2}}[l_{ijk}(\theta;\mathbf{X}) - nf_{ijk}], \tag{3.2.13}$$

and

$$Z^i = i^{i,j}(\theta)Z_j. \tag{3.2.14}$$

We shall use subscripts $\alpha, \beta, \gamma, \ldots$ to denote the nuisance parts of the derivatives of the scores, that is

$$
\begin{aligned}
Z_\alpha &= n^{-\frac{1}{2}} l_\alpha(\theta;\mathbf{X}) \\
Z_{i\alpha} &= n^{-\frac{1}{2}} [l_{i\alpha}(\theta;\mathbf{X}) - n \not{f}_{i\alpha}] \\
&\mathit{etc.}
\end{aligned}
$$

Letting $[\nu^{\alpha,\beta}]$ denote the inverse of the nuisance part $[i_{\alpha,\beta}]$ of the information matrix i, we define

$$
Z^\alpha = \nu^{\alpha,\beta} Z_\beta. \tag{3.2.15}
$$

Note that all the (normalised) derivatives (Z_i, Z_{ij}, *etc.*) of the score are of order $O(1)$.

3.2.1 Tensorial Versions of Derivatives of the Score

In order to simplify the calculations in the expansion of the geometric Wald statistics and to give them an invariant form, we use the following tensorial versions of the derivatives of the score vector. Let

$$
Y_i = Z_i \tag{3.2.16}
$$

$$
Y_{ij} = Z_{ij} - \not{f}_{ij;k} Z^k \tag{3.2.17}
$$

$$
Y_{ijk} = Z_{ijk} - \not{f}_{ij;l} Z_{km} \{3\}_{ijk} i^{l,m} + \left(\not{f}_{ij;m} \not{f}_{kn;l} \{3\}_{ijk} i^{m,n} - \not{f}_{ijk;l} \right) Z^l \tag{3.2.18}
$$

Subscripts $\alpha, \beta, \gamma, \ldots$ denote the nuisance parts of the Ys, that is

$$
\begin{aligned}
Y_\alpha &= Z_\alpha \\
Y_{i\alpha} &= Z_{i\alpha} - \not{f}_{i\alpha;k} Z^k \\
&\mathit{etc.}
\end{aligned}
$$

Note that the Ys are all of order $O(1)$.

3.2.2 Some Useful Tensors

To simplify the expansions of the geometric Wald statistics we introduce the following tensors (see Barndorff-Nielsen *et al.* [2] or Blæsild [11])

$$\begin{aligned}
\not{t}_{ijk} &= -\ell_{ij;k}\{3\}_{ijk} - \ell_{ijk} && (3.2.19)\\
&= \ell_{i;jk} - \ell_{jk;i} (= \ell_{j;ik} - \ell_{ik;j} = \ell_{k;ij} - \ell_{ij;k}) &&\\
\not{t}_{ij;kl} &= -\ell_{ij;m}\ell_{n;kl} i^{m,n} + \ell_{ij;kl} && (3.2.20)\\
\not{t}_{i;jkl} &= -\not{t}_{ijm}\ell_{kl;n}\{3\}_{jkl} i^{m,n} + \ell_{i;jkl} - \ell_{jkl;i} && (3.2.21)\\
\not{t}_{ijkl;} &= -\not{t}_{i;jkl} - \not{t}_{ij;kl}\{3\}_{jkl}. && (3.2.22)
\end{aligned}$$

The geometrical interpretation of the tensor $\not{t}_{ijk}$, is the *skewness tensor* (see Lauritzen [24]).

Subscripts $\alpha, \beta, \gamma, \ldots$ denote the nuisance parts of the $\not{t}$-tensors, that is

$$\begin{aligned}
\not{t}_{ij\alpha} &= \ell_{i;j\alpha} - \ell_{j\alpha;i}\\
\not{t}_{\alpha\beta\gamma} &= \ell_{\alpha;\beta\gamma} - \ell_{\beta\gamma;\alpha}\\
\not{t}_{ij;\alpha\beta} &= -\ell_{ij;m}\ell_{n;\alpha\beta} i^{m,n} + \ell_{ij;\alpha\beta}\\
& etc.
\end{aligned}$$

Observe that all $\not{t}$s are tensors and are of order $O(1)$.

Furthermore, because we are considering only interest-respecting re-parameterisations, it is easy to see that the following expressions are tensors

$$\begin{aligned}
\overset{1}{H}{}^{i}_{\alpha,\beta} &= \ell_{\alpha\beta;j} i^{i,j} - \delta^{i}_{\gamma}\ell_{\alpha\beta;\delta}\nu^{\gamma,\delta} && (3.2.23)\\
\overset{-1}{H}{}^{i}_{\alpha,\beta} &= \left(\ell_{\alpha\beta;j} + \not{t}_{j\alpha\beta}\right) i^{i,j} - \delta^{i}_{\gamma}\left(\ell_{\alpha\beta;\delta} + \not{t}_{\alpha\beta\gamma}\right)\nu^{\gamma,\delta}. && (3.2.24)
\end{aligned}$$

The geometrical interpretations of $\overset{1}{H}{}^{i}_{\alpha,\beta}$ and $\overset{-1}{H}{}^{i}_{\alpha,\beta}$ are the expected 1-embedding curvature and the expected (-1)-embedding curvature, respectively, of Θ_0 in Θ (see e.g. Barndorff-Nielsen & Blæsild [3]). We also define the following lowered versions of the embedding curvatures,

$$\begin{aligned}
\overset{1}{H}_{i;\alpha,\beta} &= i_{i,j}\,\overset{1}{H}{}^{j}_{\alpha,\beta} && (3.2.25)\\
\overset{-1}{H}_{i;\alpha,\beta} &= i_{i,j}\,\overset{-1}{H}{}^{j}_{\alpha,\beta}. && (3.2.26)
\end{aligned}$$

Finally, we define the following tensors

$$
\begin{aligned}
\tau_{ijkl} &= \not{t}_{ijm} \not{t}_{kln} i^{m,n} && (3.2.27)\\
\overset{1}{\tau}_{i;\alpha,jk} &= \overset{1}{H}_{i;\alpha,\beta} \not{t}_{jk\gamma} \nu^{\beta,\gamma} && (3.2.28)\\
\overset{1}{\tau}_{i;j;\alpha,\beta} &= \overset{1}{H}_{i;\alpha,\gamma} \overset{1}{H}_{j;\beta,\delta} \nu^{\gamma,\delta} && (3.2.29)\\
\overset{1}{\tau}_{i;\alpha\beta\gamma} &= -\overset{1}{H}_{i;\alpha,\delta} \not{f}_{\beta\epsilon;\gamma} \nu^{\delta,\epsilon} \{3\}_{\alpha\beta\gamma} + \not{f}_{\alpha\beta\gamma;i} - i_{i,\delta} \nu^{\delta,\epsilon} \not{f}_{\alpha\beta\gamma;\epsilon} && (3.2.30)\\
\overset{-1}{\tau}_{\alpha,\beta,ij} &= \overset{-1}{H}_{k;\alpha,\beta} \not{t}_{ijl} i^{k,l} && (3.2.31)\\
\overset{-1}{\tau}_{\alpha,\beta,\gamma,\delta} &= \overset{-1}{H}_{i;\alpha,\beta} \overset{-1}{H}_{j;\gamma,\delta} i^{i,j}. && (3.2.32)
\end{aligned}
$$

As before, we use subscripts $\alpha, \beta, \ldots$ for the nuisance parts of the tensors, e.g. $\tau_{ijk\alpha} = \not{t}_{ijl} \not{t}_{km\alpha} i^{l,m}$, etc.

3.2.3 Sub- and Superscripts

In this section we shall differentiate between single and multiple indices. Let lower-case letters such as $i, j, k, \ldots$ run through $\{1, \ldots, r\}$, and let the upper-case letters $I, J, K, \ldots$ run through $\{1, \ldots, r, 11, \ldots, rr, 111, \ldots, rrr\}$. We shall use upper case letters $R, S, T \ldots$ to denote sets of multi-indices $\{(I), (I, J), (I, J, K), \ldots\}$, e.g. the cumulant λ_R means the set of all cumulants $\{\lambda_i, \lambda_{ij}, \ldots, \lambda_{i,j}, \ldots\}$.

3.3 Expected Geometric Wald Tests

The family of geometric Wald statistics (2.3.6) in expected geometry is defined as

$$
\overset{\alpha}{\mathcal{W}} = \overset{\alpha}{\Gamma}_i(\hat{\theta}; \tilde{\theta}) i^{i,j}(\hat{\theta}) \overset{\alpha}{\Gamma}_j(\hat{\theta}; \tilde{\theta}), \qquad (3.3.33)
$$

where

$$
\overset{\alpha}{\Gamma}_i(\theta; \theta') = n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} f_{;i}(\theta'; \theta) + \frac{1-\alpha}{2} f_i(\theta; \theta') \right\},
$$

with $f(\theta; \theta')$ as the expected likelihood yoke (1.3.16),

$$
f(\theta; \theta') = n^{-1} \mathbb{E}_{\theta'} \left[l(\theta) - l(\theta') \right]. \qquad (3.3.34)
$$

Note that $\overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta}')$, $\overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta}')$ and $\overset{\alpha}{\mathcal{W}}$ have order $O(1)$.

Modified Exoected Geometric Wald Test

When the expected likelihood yoke is used, the modified geometric Wald test (2.3.9) becomes

$$\overset{\alpha}{\widetilde{\mathcal{W}}} = \overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta})i^{i;j}(\tilde{\theta})\overset{\alpha}{\Gamma}_j(\tilde{\theta};\hat{\theta}). \tag{3.3.35}$$

3.4 Bartlett-type Adjustments

In order to compute the moment generating functions of the geometric Wald statistics and the modified geometric Wald statistics, and hence their Bartlett-type corrections, we need to find the asymptotic distributions of $\overset{\alpha}{\mathcal{W}}$ and $\overset{\alpha}{\widetilde{\mathcal{W}}}$ up to order $O(n^{-\frac{3}{2}})$. This is done by Taylor-expanding $\overset{\alpha}{\mathcal{W}}$ and $\overset{\alpha}{\widetilde{\mathcal{W}}}$ as polynomials in the (normalised) score vector $Y_* = (Y_1, \ldots, Y_r, Y_{11}, \ldots, Y_{rr}, Y_{111}, \ldots, Y_{rrr})$ and using a result by Barndorff-Nielsen & Cox [5] on the distribution of Y_*. Letting λ_R denote the cumulants of Y_*, observe that $\mathbb{E}[Y_*] = (0, \ldots, 0)$ and the variance of Y_* is $\lambda = [\lambda_{I,J}]$. Let $\lambda^{-1} = [\lambda^{I,J}]$ denote the inverse of λ.

From Barndorff-Nielsen & Cox [5, Section 6.3] we find the probability density function of Y_* to be

$$p(Y_*;\theta) = \varphi(Y_*;\lambda)\left\{1 + n^{-\frac{1}{2}}Q_1 + n^{-1}Q_2 + O(n^{-\frac{3}{2}})\right\}, \tag{3.4.36}$$

where $\varphi(\cdot;\lambda)$ denotes the probability density function of the $(r+r^2+r^3)$-dimensional normal distribution with mean zero and variance matrix λ, and

$$\begin{aligned} Q_1 &= \frac{1}{6}\lambda^{I,J,K}h_{IJK}(Y_*;\lambda) \\ Q_2 &= \frac{1}{24}\lambda^{I,J,K,L}h_{IJKL}(Y_*;\lambda) + \frac{1}{72}\lambda^{I,J,K}\lambda^{L,M,N}h_{IJKLMN}(Y_*;\lambda), \end{aligned}$$

where $h_{IJ}(Y_*;\lambda)$, $h_{IJK}(Y_*;\lambda)$, etc. are the covariant Hermite polynomials (1.4.33)–(1.4.36), defined by

$$h_{IJ}(Y_*;\lambda) = Y_IY_J - \lambda_{I,J}, \tag{3.4.37}$$

$$h_{IJK}(Y_*;\lambda) = Y_IY_JY_K - \lambda_{I,J}Y_K\,\{3\}, \tag{3.4.38}$$

$$h_{IJKL}(Y_*;\lambda) = Y_IY_JY_KY_L - \lambda_{I,J}Y_KY_L\{6\} + \lambda_{I,J}\lambda_{K,L}\{3\}, \qquad (3.4.39)$$

$$h_{IJKLMN}(Y_*;\lambda) = Y_IY_JY_KY_LY_MY_N - \lambda_{I,J}Y_KY_LY_MY_N\{15\} + \lambda_{I,J}\lambda_{K,L}Y_MY_N\{45\} - \lambda_{I,J}\lambda_{K,L}\lambda_{M,N}\{15\}, \qquad (3.4.40)$$

and where $\lambda^{I,J,K}$ and $\lambda^{I,J,K,L}$ are lifted by λ^{-1}, e.g. $\lambda^{I,J,K} = \lambda^{I,L}\lambda^{J,M}\lambda^{K,N}\lambda_{L,M,N}$.

The λs and hs are all of order $O(1)$. Thus the terms Q_1 and Q_2 are both of order $O(1)$.

Remark 9:

Note that

$$\begin{aligned} \lambda_{i,j} &= \mathbb{E}[Y_iY_j] = i_{i,j} \\ \lambda_{i,jk} &= \mathbb{E}[Y_iY_{jk}] = 0 \\ \lambda_{i,jkl} &= \mathbb{E}[Y_iY_{jkl}] = 0. \end{aligned}$$

Thus,

$$\lambda^{i,J} = i^{i,j}\delta_j^J. \qquad (3.4.41)$$

Remark 10:

Let κ_R denote the moments of the derivatives of the log-likelihood function for one observation. We can write $f_{I;J}(\theta)$ in terms of the moments as

$$f_I = \kappa_I \qquad (3.4.42)$$

$$f_{I;j} = \kappa_{I,j} \qquad (3.4.43)$$

$$f_{I;jk} = \kappa_{I,jk} + \kappa_{I,j,k} \qquad (3.4.44)$$

$$f_{I;jkl} = \kappa_{I,jkl} + \kappa_{I,j,kl}\{3\}_{jkl} + \kappa_{I,j,k,l} \qquad (3.4.45)$$

etc.

Note from (1.3.12)–(1.3.15) that we can write $f_{;J}(\theta)$ as a sum of functions $f_{I;K}(\theta)$, such that $I \cup K = J$ and I is non-empty. Thus we can write all derivatives of $f(\theta)$ in terms of the moments. It is not (necessarily) possible to write an isolated moment as a function of the $f_{I;J}(\theta)$s.

3.5 Taylor Expansions

Taylor expansion (in any coordinate system on the full parameter space Θ) of $\hat{Z}_i = Z_i(\hat{\theta})$ around the true value θ of the parameter gives

$$\begin{aligned}
0 &= \hat{Z}_i \\
&= n^{-\frac{1}{2}} l_i(\hat{\theta}; \mathbf{X}) \\
&= n^{-\frac{1}{2}} l_i(\theta; \mathbf{X}) + n^{-\frac{1}{2}} l_{ij}(\theta; \mathbf{X}) \left(\hat{\theta}^j - \theta^j\right) + n^{-\frac{1}{2}} \frac{1}{2} l_{ijk}(\theta; \mathbf{X}) \left(\hat{\theta}^j - \theta^j\right)\left(\hat{\theta}^k - \theta^k\right) \\
&\quad + n^{-\frac{1}{2}} \frac{1}{6} l_{ijkl}(\theta; \mathbf{X}) \left(\hat{\theta}^j - \theta^j\right)\left(\hat{\theta}^k - \theta^k\right)\left(\hat{\theta}^l - \theta^l\right) + \ldots \\
&= Z_i + \left[Z_{ij} + n^{\frac{1}{2}} \nu_{ij}\right]\left(\hat{\theta}^j - \theta^j\right) + \frac{1}{2}\left[Z_{ijk} + n^{\frac{1}{2}} \nu_{ijk}\right]\left(\hat{\theta}^j - \theta^j\right)\left(\hat{\theta}^k - \theta^k\right) \\
&\quad + \frac{1}{6}\left[Z_{ijkl} + n^{\frac{1}{2}} \nu_{ijkl}\right]\left(\hat{\theta}^j - \theta^j\right)\left(\hat{\theta}^k - \theta^k\right)\left(\hat{\theta}^l - \theta^l\right) + \ldots .
\end{aligned}$$

By substituting $\hat{\delta} = n^{\frac{1}{2}}\left(\hat{\theta} - \theta\right)$ and re-arranging, we find that

$$\begin{aligned}
-Z_i &= -i_{i,j}\hat{\delta}^j + n^{-\frac{1}{2}}\left\{Z_{ij}\hat{\delta}^j + \frac{1}{2}\nu_{ijk}\hat{\delta}^j\hat{\delta}^k\right\} + n^{-1}\left\{\frac{1}{2}Z_{ijk}\hat{\delta}^j\hat{\delta}^k + \frac{1}{6}\nu_{ijkl}\hat{\delta}^j\hat{\delta}^k\hat{\delta}^l\right\} \\
&\quad + \left(n^{-\frac{3}{2}}\right).
\end{aligned}$$

Solving for $\hat{\delta}^i$ gives

$$\begin{aligned}
\hat{\delta}^i &= Z^i + n^{-\frac{1}{2}}\left\{i^{i,j} Z_{jk} Z^k + \frac{1}{2} i^{i,j} \nu_{jkl} Z^k Z^l\right\} \\
&\quad + n^{-1}\left\{i^{i,j} i^{k,l} Z_{jk} Z_{lm} Z^m + \frac{1}{2} i^{i,j} Z_{jkl} Z^k Z^l + i^{i,j} \nu_{jkl} i^{k,m} Z_{mn} Z^l Z^n\right. \\
&\quad + \frac{1}{2} i^{i,j} i^{k,l} \nu_{lmn} Z_{jk} Z^m Z^n + \frac{1}{6} i^{i,j} \nu_{jklm} Z^k Z^l Z^m \\
&\quad \left. + \frac{1}{2} i^{i,j} \nu_{jkl} i^{l,m} \nu_{mno} Z^k Z^n Z^o\right\} \\
&\quad + O\left(n^{-\frac{3}{2}}\right). \qquad (3.5.46)
\end{aligned}$$

Let $\tilde{\delta} = n^{\frac{1}{2}}\left(\tilde{\theta} - \theta\right)$. Clearly, $\tilde{\delta}^a = n^{\frac{1}{2}}\left(\tilde{\theta}^a - \theta^a\right)$ vanishes. Furthermore, we observe that $\widetilde{Z}_\alpha = Z_\alpha(\tilde{\theta}) = 0$. Thus expanding $\widetilde{Z}_\alpha$ around the true value θ, substituting $\tilde{\delta}^\alpha$, and solving for $\tilde{\delta}^\alpha$, we obtain

$$\begin{aligned}
\tilde{\delta}^\alpha &= Z^\alpha + n^{-\frac{1}{2}}\left\{\nu^{\alpha,\beta} Z_{\beta\gamma} Z^\gamma + \frac{1}{2}\nu^{\alpha,\beta}\nu_{\beta\gamma\delta} Z^\gamma Z^\delta\right\} \\
&\quad + n^{-1}\left\{\nu^{\alpha,\beta}\nu^{\gamma,\delta} Z_{\beta\gamma} Z_{\delta\epsilon} Z^\epsilon + \frac{1}{2}\nu^{\alpha,\beta} Z_{\beta\gamma\delta} Z^\gamma Z^\delta + \nu^{\alpha,\beta}\nu_{\beta\gamma\delta}\nu^{\gamma,\epsilon} Z_{\epsilon\psi} Z^\delta Z^\psi\right.
\end{aligned}$$

$$+\frac{1}{2}\nu^{\alpha,\beta}\nu^{\gamma,\delta}f_{\delta\epsilon\psi}Z_{\beta\gamma}Z^{\epsilon}Z^{\psi}+\frac{1}{6}\nu^{\alpha,\beta}f_{\beta\gamma\delta\epsilon}Z^{\gamma}Z^{\delta}Z^{\epsilon}+\frac{1}{2}\nu^{\alpha,\beta}f_{\beta\gamma\delta}\nu^{\delta,\epsilon}f_{\epsilon\psi\phi}Z^{\gamma}Z^{\psi}Z^{\phi}\Big\}$$
$$+O\left(n^{-\frac{3}{2}}\right). \tag{3.5.47}$$

3.5.1 Taylor-expanding i^{-1}

Observe that $\frac{\partial}{\partial\theta^k}i_{i,j}(\theta)=f_{i;j/k}(\theta)$ and $\frac{\partial^2}{\partial\theta^k\partial\theta^l}i_{i,j}(\theta)=f_{i;j/kl}(\theta)$. Using (1.4.47) and (1.4.48), we find that

$$\frac{\partial}{\partial\theta^k}i^{i,j}(\theta) = -i^{i,l}i^{j,m}f_{l;m/k} \tag{3.5.48}$$
$$\frac{\partial^2}{\partial\theta^k\partial\theta^l}i^{i,j}(\theta) = i^{i,m}i^{j,n}(-f_{m;n/kl}+f_{m;p/k}i^{p,q}f_{n;p/l}\,\{2\}_{kl}). \tag{3.5.49}$$

Thus

$$\begin{aligned}
i^{i,j}(\hat\theta) &= i^{i,j}(\theta)+n^{-\frac{1}{2}}\frac{\partial}{\partial\theta^k}i^{i,j}(\theta)\hat\delta^k+n^{-1}\frac{1}{2}\frac{\partial^2}{\partial\theta^k\partial\theta^l}i^{i,j}(\theta)\hat\delta^k\hat\delta^l+\dots\\
&= i^{i,j}+n^{-\frac{1}{2}}\left\{-i^{i,l}i^{j,m}f_{l;m/k}Z^k\right\}\\
&\quad +n^{-1}\Big\{-i^{i,m}i^{j,n}f_{m;n/p}(i^{p,q}Z_{qk}Z^k+\frac{1}{2}i^{p,q}f_{klq}Z^kZ^l)\\
&\quad +\frac{1}{2}i^{i,m}i^{j,n}\left(-f_{m;n/kl}+2f_{m;p/k}i^{p,q}f_{n;q/l}\right)Z^kZ^l\Big\}+O(n^{-\frac{3}{2}})\\
\\
&= i^{i,j}+n^{-\frac{1}{2}}\left\{-i^{i,l}i^{j,m}f_{l;m/k}Z^k\right\}\\
&\quad +n^{-1}\Big\{-i^{i,l}i^{j,m}f_{l;m/n}i^{n,p}Z_{kp}Z^k\\
&\quad +\frac{1}{2}i^{i,m}i^{j,n}\left[-f_{m;n/p}i^{p,q}f_{klq}+2f_{m;p/k}i^{p,q}f_{n;l/q}-f_{m;n/kl}\right]Z^kZ^l\Big\}\\
&\quad +O(n^{-\frac{3}{2}}).
\end{aligned} \tag{3.5.50}$$

Likewise, we find that

$$\begin{aligned}
i^{i,j}(\tilde\theta) &= i^{i,j}(\theta)+n^{-\frac{1}{2}}\frac{\partial}{\partial\theta^k}i^{i,j}(\theta)\tilde\delta^k+n^{-1}\frac{1}{2}\frac{\partial^2}{\partial\theta^k\partial\theta^l}i^{i,j}(\theta)\tilde\delta^k\tilde\delta^l+\dots\\
&= i^{i,j}+n^{-\frac{1}{2}}\left\{-i^{i,k}i^{j,l}f_{k;l/\alpha}Z^\alpha\right\}\\
&\quad +n^{-1}\Big\{-i^{i,l}i^{j,m}f_{m;n/\gamma}(\nu^{\gamma,\delta}Z_{\alpha\delta}Z^\alpha+\frac{1}{2}\nu^{\gamma,\delta}f_{\alpha\beta\delta}Z^\alpha Z^\beta)\\
&\quad +\frac{1}{2}i^{i,k}i^{j,l}\left(-f_{k;l/\alpha\beta}+2f_{k;m/\alpha}i^{m,n}f_{k;n/\beta}\right)Z^\alpha Z^\beta\Big\}+O(n^{-\frac{3}{2}})
\end{aligned}$$

$$
\begin{aligned}
&= i^{i,j} + n^{-\frac{1}{2}} \left\{ -i^{i,k} i^{j,l} f_{k;l/\alpha} Z^{\alpha} \right\} \\
&\quad + n^{-1} \left\{ -i^{i,l} i^{j,m} f_{m;n/\beta} \nu^{\beta,\gamma} Z_{\alpha\gamma} Z^{\alpha} \right. \\
&\quad \left. + \frac{1}{2} i^{i,k} i^{j,l} \left[-f_{k;l/\gamma} \nu^{\gamma,\delta} f_{\alpha\beta\delta} + 2 f_{k;m/\alpha} i^{m,n} f_{l,n/\beta} - f_{k;l/\alpha\beta} \right] Z^{\alpha} Z^{\beta} \right\} \\
&\quad + O(n^{-\frac{3}{2}}).
\end{aligned} \tag{3.5.51}
$$

3.5.2 Taylor Expanding $\overset{\alpha}{\Gamma}_i(\theta;\theta')$

Let $\overset{\alpha}{\Gamma}_{iI;J}(\theta;\theta')$ denote the derivatives of $\overset{\alpha}{\Gamma}_i(\theta;\theta')$, e.g. $\overset{\alpha}{\Gamma}_{i;j}(\theta;\theta') = \frac{\partial}{\partial \theta'^j} \overset{\alpha}{\Gamma}_i(\theta;\theta')$, $\overset{\alpha}{\Gamma}_{ij}(\theta;\theta') = \frac{\partial}{\partial \theta^j} \overset{\alpha}{\Gamma}_i(\theta;\theta')$, etc. We find from (1.3.9)–(1.3.15) that

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_i &= n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} f_{;i} + \frac{1-\alpha}{2} f_i \right\} \\
&= 0
\end{aligned} \tag{3.5.52}
$$

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{ij} &= n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} f_{;ij} + \frac{1-\alpha}{2} f_{ij} \right\} \\
&= -n^{\frac{1}{2}} i_{i,j}
\end{aligned} \tag{3.5.53}
$$

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{ijk} &= n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} (-f_{ijk} - f_{i;jk}\{3\} - f_{ij;k}\{3\}) + \frac{1-\alpha}{2} f_{ijk} \right\} \\
&= n^{\frac{1}{2}} \left\{ -\frac{1}{2}(3+\alpha) f_{ijk} - f_{ij;k}\{3\} \right\}
\end{aligned} \tag{3.5.54}
$$

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{ijkl} &= n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} (-f_{ijkl} - f_{i;jkl}\{4\} - f_{ijk;l}\{4\} - f_{ij;kl}\{6\}) + \frac{1-\alpha}{2} f_{ijkl} \right\} \\
&= n^{\frac{1}{2}} \left\{ \frac{1}{2} (-f_{i;jkl}\{4\} - f_{ijk;l}\{4\} - f_{ij;kl}\{6\}) \right. \\
&\quad \left. + \frac{\alpha}{2} \left(f_{i;jkl} - f_{jkl;i} + (f_{ikl;j} - f_{j;ikl} + f_{ij;kl} - f_{kl;ij})\{3\}_{jkl} \right) \right\},
\end{aligned} \tag{3.5.55}
$$

and that

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{i;j} &= n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} f_{j;i} + \frac{1-\alpha}{2} f_{i;j} \right\} \\
&= n^{\frac{1}{2}} i_{i,j}
\end{aligned} \tag{3.5.56}
$$

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{i;jk} &= n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} f_{jk;i} + \frac{1-\alpha}{2} f_{i;jk} \right\} \\
&= n^{\frac{1}{2}} \left\{ \frac{1}{2} (f_{jk;i} + f_{i;jk}) - \frac{\alpha}{2} f_{ijk} \right\}
\end{aligned} \tag{3.5.57}
$$

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{ij;k} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\ell_{k;ij}+\frac{1-\alpha}{2}\ell_{ij;k}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\ell_{k;ij}+\ell_{ij;k})+\frac{\alpha}{2}t_{ijk}\right\}
\end{aligned} \tag{3.5.58}
$$

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{i;jkl} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\ell_{jkl;i}+\frac{1-\alpha}{2}\ell_{i;jkl}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\ell_{i;jkl}+\ell_{jkl;i})+\frac{\alpha}{2}(\ell_{jkl;i}-\ell_{i;jkl})\right\}
\end{aligned} \tag{3.5.59}
$$

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{ij;kl} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\ell_{kl;ij}+\frac{1-\alpha}{2}\ell_{ij;kl}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\ell_{ij;kl}+\ell_{kl;ij})+\frac{\alpha}{2}(\ell_{kl;ij}-\ell_{ij;kl})\right\}
\end{aligned} \tag{3.5.60}
$$

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_{ijk;l} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\ell_{l;ijk}+\frac{1-\alpha}{2}\ell_{ijk;l}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\ell_{ijk;l}+\ell_{l;ijk})+\frac{\alpha}{2}(\ell_{l;ijk}-\ell_{ijk;l})\right\}.
\end{aligned} \tag{3.5.61}
$$

Thus, we find the Taylor expansion of $\overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta})$ around $(\theta;\theta)$ as

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta}) &= \overset{\alpha}{\Gamma}_i + n^{-\frac{1}{2}}\left\{\overset{\alpha}{\Gamma}_{ij}\hat{\delta}^j+\overset{\alpha}{\Gamma}_{i;j}\tilde{\delta}^j\right\} \\
&\quad + n^{-1}\left\{\frac{1}{2}\overset{\alpha}{\Gamma}_{ijk}\hat{\delta}^j\hat{\delta}^k+\overset{\alpha}{\Gamma}_{ij;k}\hat{\delta}^j\tilde{\delta}^k+\frac{1}{2}\overset{\alpha}{\Gamma}_{i;jk}\tilde{\delta}^j\tilde{\delta}^k\right\} \\
&\quad + n^{-\frac{3}{2}}\left\{\frac{1}{6}\overset{\alpha}{\Gamma}_{ijkl}\hat{\delta}^j\hat{\delta}^k\hat{\delta}^l+\frac{1}{2}\overset{\alpha}{\Gamma}_{ijk;l}\hat{\delta}^j\hat{\delta}^k\tilde{\delta}^l+\frac{1}{2}\overset{\alpha}{\Gamma}_{ij;kl}\hat{\delta}^j\tilde{\delta}^k\tilde{\delta}^l+\frac{1}{6}\overset{\alpha}{\Gamma}_{i;jkl}\tilde{\delta}^j\tilde{\delta}^k\tilde{\delta}^l\right\}+\dots \\
&= n^{-\frac{1}{2}}\left\{\overset{\alpha}{\Gamma}_{ij}Z^j+\overset{\alpha}{\Gamma}_{i;\alpha}Z^\alpha\right\} \\
&\quad + n^{-1}\left\{\overset{\alpha}{\Gamma}_{ij}(i^{j,k}Z_{kl}Z^l+\frac{1}{2}i^{j,k}\ell_{klm}Z^lZ^m)+\overset{\alpha}{\Gamma}_{i;\alpha}(\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma+\frac{1}{2}\nu^{\alpha,\beta}\ell_{\beta\gamma\delta}Z^\gamma Z^\delta)\right. \\
&\quad \left.+\frac{1}{2}\overset{\alpha}{\Gamma}_{ijk}Z^jZ^k+\overset{\alpha}{\Gamma}_{ij;\alpha}Z^jZ^\alpha+\frac{1}{2}\overset{\alpha}{\Gamma}_{i;\alpha\beta}Z^\alpha Z^\beta\right\} \\
&\quad + n^{-\frac{3}{2}}\left\{\overset{\alpha}{\Gamma}_{ij}\left(i^{j,k}i^{l,m}Z_{kl}Z_{mn}Z^n+\frac{1}{2}i^{j,k}Z_{klm}Z^lZ^m\right.\right. \\
&\quad +i^{j,k}\ell_{klm}i^{m,n}Z_{np}Z^lZ^p+\frac{1}{2}i^{j,k}i^{l,m}\ell_{mnp}Z_{kl}Z^nZ^p+\frac{1}{6}i^{j,k}\ell_{klmn}Z^lZ^mZ^n \\
&\quad \left.+\frac{1}{2}i^{j,k}\ell_{klm}i^{m,n}\ell_{npq}Z^lZ^pZ^q\right) \\
&\quad +\overset{\alpha}{\Gamma}_{i;\alpha}\nu^{\alpha,\beta}\left(\nu^{\gamma,\delta}Z_{\beta\gamma}Z_{\delta\epsilon}Z^\epsilon+\frac{1}{2}Z_{\beta\gamma\delta}Z^\gamma Z^\delta+\ell_{\beta\gamma\delta}\nu^{\gamma,\epsilon}Z_{\epsilon\psi}Z^\delta Z^\psi\right. \\
&\quad \left.+\frac{1}{2}\nu^{\gamma,\delta}\ell_{\delta\epsilon\psi}Z_{\beta\gamma}Z^\epsilon Z^\psi+\frac{1}{6}\ell_{\beta\gamma\delta\epsilon}Z^\gamma Z^\delta Z^\epsilon+\frac{1}{2}\ell_{\beta\gamma\delta}\nu^{\delta,\epsilon}\ell_{\epsilon\psi\phi}Z^\gamma Z^\psi Z^\phi\right) \\
&\quad +\overset{\alpha}{\Gamma}_{ijk}Z^j\left(i^{k,l}Z_{lm}Z^m+\frac{1}{2}i^{k,l}\ell_{lmn}Z^mZ^n\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\overset{\alpha}{\not{\Gamma}}_{ij;\alpha}\left(i^{j,k}Z_{kl}Z^{l}Z^{\alpha}+\frac{1}{2}i^{j,k}\not{\ell}_{klm}Z^{l}Z^{m}Z^{\alpha}+\nu^{\alpha,\beta}Z_{\beta\gamma}Z^{j}Z^{\gamma}+\frac{1}{2}\nu^{\alpha,\beta}\not{\ell}_{\beta\gamma\delta}Z^{j}Z^{\gamma}Z^{\delta}\right)\\
& +\overset{\alpha}{\not{\Gamma}}_{i;\alpha\beta}Z^{\alpha}\left(\nu^{\beta\gamma}Z_{\gamma\delta}Z^{\delta}+\frac{1}{2}\nu^{\beta,\gamma}\not{\ell}_{\gamma\delta\epsilon}Z^{\delta}Z^{\epsilon}\right)\\
& +\frac{1}{6}\overset{\alpha}{\not{\Gamma}}_{ijkl}Z^{j}Z^{k}Z^{l}+\frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ijk;\alpha}Z^{j}Z^{k}Z^{\alpha}+\frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ij;\alpha\beta}Z^{j}Z^{\alpha}Z^{\beta}+\frac{1}{6}\overset{\alpha}{\not{\Gamma}}_{i;\alpha\beta\gamma}Z^{\alpha}Z^{\beta}Z^{\gamma}\Big\}\\
& +O(n^{-2})\\
= \; & -Z_{i}+i_{i,\alpha}Z^{\alpha}\\
& +n^{-\frac{1}{2}}\Big\{-Z_{ij}Z^{j}+i_{i,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^{\gamma}-\frac{1}{4}(1+\alpha)\not{t}_{ijk}Z^{j}Z^{k}\\
& +\left[\not{\ell}_{ij;\alpha}+\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha}\right]Z^{j}Z^{\alpha}+\left[\frac{1}{4}(1-\alpha)\not{t}_{i\alpha\beta}+\frac{1}{2}\not{\ell}_{\alpha\beta;i}+\frac{1}{2}i_{i,\gamma}\nu^{\gamma,\delta}\not{\ell}_{\alpha\beta\delta}\right]Z^{\alpha}Z^{\beta}\Big\}\\
& +n^{-1}\Big\{-\frac{1}{2}Z_{ijk}Z^{j}Z^{k}-i^{j,k}Z_{ij}Z_{kl}Z^{l}-\frac{1}{2}(1+\alpha)\not{t}_{ijk}i^{k,l}Z_{lm}Z^{j}Z^{m}\\
& -\frac{1}{2}i^{j,k}\not{\ell}_{klm}Z_{ij}Z^{l}Z^{m}+\left[\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha}+\not{\ell}_{ij;\alpha}\right]i^{j,k}Z_{kl}Z^{l}Z^{\alpha}\\
& +\left[\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha}+\not{\ell}_{ij;\alpha}\right]\nu^{\alpha,\beta}Z_{\beta\gamma}Z^{j}Z^{\gamma}+\left[\frac{1}{2}(1-\alpha)\not{t}_{i\alpha\beta}+\not{\ell}_{\alpha\beta;i}\right]\nu^{\beta,\gamma}Z_{\gamma\delta}Z^{\alpha}Z^{\delta}\\
& +\Big[-\frac{1}{4}(1+\alpha)\not{t}_{ijm}i^{m,n}\not{\ell}_{kln}\\
& -\frac{1}{12}(1+\alpha)\left(2\not{\ell}_{ijkl}+\not{\ell}_{i;jkl}\{4\}+\not{\ell}_{ijk;l}\{4\}+\not{\ell}_{ij;kl}\{6\}\right)\Big]Z^{j}Z^{k}Z^{l}\\
& +\Big[\frac{1}{4}(1+\alpha)\not{t}_{il\alpha}i^{l,m}\not{\ell}_{jkm}+\frac{1}{2}\not{\ell}_{il;\alpha}i^{l,m}\not{\ell}_{jkm}\\
& +\frac{1}{4}\left(\not{\ell}_{ijk;\alpha}+\not{\ell}_{\alpha;ijk}+\alpha(\not{\ell}_{\alpha;ijk}-\not{\ell}_{ijk;\alpha})\right)\Big]Z^{j}Z^{k}Z^{\alpha}\\
& +\Big[\frac{1}{4}(1+\alpha)\not{t}_{ij\gamma}\nu^{\gamma,\delta}\not{\ell}_{\alpha\beta\delta}+\frac{1}{2}\not{\ell}_{ij;\gamma}\nu^{\gamma,\delta}\not{\ell}_{\alpha\beta\delta}\\
& +\frac{1}{4}\left(\not{\ell}_{ij;\alpha\beta}+\not{\ell}_{\alpha\beta;ij}+\alpha(\not{\ell}_{\alpha\beta;ij}-\not{\ell}_{ij;\alpha\beta})\right)\Big]Z^{j}Z^{\alpha}Z^{\beta}\\
& +\Big[\frac{1}{4}(1+\alpha)\not{t}_{i\alpha\delta}\nu^{\delta,\epsilon}\not{\ell}_{\beta\gamma\epsilon}+\frac{1}{2}\not{\ell}_{\alpha\delta;i}\nu^{\delta,\epsilon}\not{\ell}_{\beta\gamma\epsilon}\\
& +\frac{1}{12}\left(\not{\ell}_{i;\alpha\beta\gamma}+\not{\ell}_{\alpha\beta\gamma;i}+\alpha(\not{\ell}_{\alpha\beta\gamma;i}-\not{\ell}_{i;\alpha\beta\gamma})\right)\Big]Z^{\alpha}Z^{\beta}Z^{\gamma}\\
& +i_{i,\alpha}\nu^{\alpha,\beta}\Big(\nu^{\gamma,\delta}Z_{\beta\gamma}Z_{\delta\epsilon}Z^{\epsilon}+\frac{1}{2}Z_{\beta\gamma\delta}Z^{\gamma}Z^{\delta}+\not{\ell}_{\beta\gamma\delta}\nu^{\gamma,\epsilon}Z_{\epsilon\psi}Z^{\delta}Z^{\psi}\\
& +\frac{1}{2}\nu^{\gamma,\delta}\not{\ell}_{\delta\epsilon\psi}Z_{\beta\gamma}Z^{\epsilon}Z^{\psi}+\frac{1}{6}\not{\ell}_{\beta\gamma\delta\epsilon}Z^{\gamma}Z^{\delta}Z^{\epsilon}\\
& +\frac{1}{2}\not{\ell}_{\beta\gamma\delta}\nu^{\delta,\epsilon}\not{\ell}_{\epsilon\psi\phi}Z^{\gamma}Z^{\psi}Z^{\phi}\Big)\Big\}+O(n^{-\frac{3}{2}}),
\end{aligned}
\tag{3.5.62}
$$

and we find the Taylor expansion of $\overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta})$ as

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta}) \;=\; & \overset{\alpha}{\not{\Gamma}}_{i}+n^{-\frac{1}{2}}\left\{\overset{\alpha}{\not{\Gamma}}_{ij}\tilde{\delta}^{j}+\overset{\alpha}{\not{\Gamma}}_{i;j}\hat{\delta}^{j}\right\}\\
& +n^{-1}\left\{\frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ijk}\tilde{\delta}^{j}\tilde{\delta}^{k}+\overset{\alpha}{\not{\Gamma}}_{ij;k}\tilde{\delta}^{j}\hat{\delta}^{k}+\frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{i;jk}\hat{\delta}^{j}\hat{\delta}^{k}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+n^{-\frac{3}{2}}\left\{\frac{1}{6}\overset{\alpha}{\Gamma}_{ijkl}\tilde{\delta}^j\tilde{\delta}^k\tilde{\delta}^l+\frac{1}{2}\overset{\alpha}{\Gamma}_{ijk;l}\tilde{\delta}^j\tilde{\delta}^k\hat{\delta}^l+\frac{1}{2}\overset{\alpha}{\Gamma}_{ij;kl}\tilde{\delta}^j\hat{\delta}^k\hat{\delta}^l\frac{1}{6}\overset{\alpha}{\Gamma}_{i;jkl}\hat{\delta}^j\hat{\delta}^k\hat{\delta}^l\right\}+\ldots \\
=\;& n^{-\frac{1}{2}}\left\{\overset{\alpha}{\Gamma}_{i;j}Z^j+\overset{\alpha}{\Gamma}_{i\alpha}Z^\alpha\right\} \\
&+n^{-1}\left\{\overset{\alpha}{\Gamma}_{i;j}(i^{j,k}Z_{kl}Z^l+\frac{1}{2}i^{j,k}\ell_{klm}Z^lZ^m)+\overset{\alpha}{\Gamma}_{i\alpha}(\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma+\frac{1}{2}\nu^{\alpha,\beta}\ell_{\beta\gamma\delta}Z^\gamma Z^\delta)\right. \\
&\left.+\frac{1}{2}\overset{\alpha}{\Gamma}_{i;jk}Z^jZ^k+\overset{\alpha}{\Gamma}_{i\alpha;j}Z^jZ^\alpha+\frac{1}{2}\overset{\alpha}{\Gamma}_{i\alpha\beta}Z^\alpha Z^\beta\right\} \\
&+n^{-\frac{3}{2}}\left\{\overset{\alpha}{\Gamma}_{i;j}\left(i^{j,k}i^{l,m}Z_{kl}Z_{mn}Z^n+\frac{1}{2}i^{j,k}Z_{klm}Z^lZ^m\right.\right. \\
&+i^{j,k}\ell_{klm}i^{m,n}Z_{np}Z^lZ^p+\frac{1}{2}i^{j,k}i^{l,m}\ell_{mnp}Z_{kl}Z^nZ^p+\frac{1}{6}i^{j,k}\ell_{klmn}Z^lZ^mZ^n \\
&\left.+\frac{1}{2}i^{j,k}\ell_{klm}i^{m,n}\ell_{npq}Z^lZ^pZ^q\right) \\
&+\overset{\alpha}{\Gamma}_{i\alpha}\nu^{\alpha,\beta}\left(\nu^{\gamma,\delta}Z_{\beta\gamma}Z_{\delta\epsilon}Z^\epsilon+\frac{1}{2}Z_{\beta\gamma\delta}Z^\gamma Z^\delta+\ell_{\beta\gamma\delta}\nu^{\gamma,\epsilon}Z_{\epsilon\psi}Z^\delta Z^\psi\right. \\
&\left.+\frac{1}{2}\nu^{\gamma,\delta}\ell_{\delta\epsilon\psi}Z_{\beta\gamma}Z^\epsilon Z^\psi+\frac{1}{6}\ell_{\beta\gamma\delta\epsilon}Z^\gamma Z^\delta Z^\epsilon+\frac{1}{2}\ell_{\beta\gamma\delta}\nu^{\delta,\epsilon}\ell_{\epsilon\psi\phi}Z^\gamma Z^\psi Z^\phi\right) \\
&+\overset{\alpha}{\Gamma}_{i;jk}Z^j\left(i^{k,l}Z_{lm}Z^m+\frac{1}{2}i^{k,l}\ell_{lmn}Z^mZ^n\right) \\
&+\overset{\alpha}{\Gamma}_{i\alpha;j}\left(i^{j,k}Z_{kl}Z^lZ^\alpha+\frac{1}{2}i^{j,k}\ell_{klm}Z^lZ^mZ^\alpha+\nu^{\alpha,\beta}Z_{\beta\gamma}Z^jZ^\gamma+\frac{1}{2}\nu^{\alpha,\beta}\ell_{\beta\gamma\delta}Z^jZ^\gamma Z^\delta\right) \\
&+\overset{\alpha}{\Gamma}_{i\alpha\beta}Z^\alpha\left(\nu^{\beta\gamma}Z_{\gamma\delta}Z^\delta+\frac{1}{2}\nu^{\beta,\gamma}\ell_{\gamma\delta\epsilon}Z^\delta Z^\epsilon\right) \\
&\left.+\frac{1}{6}\overset{\alpha}{\Gamma}_{i;jkl}Z^jZ^kZ^l+\frac{1}{2}\overset{\alpha}{\Gamma}_{i\alpha;jk}Z^jZ^kZ^\alpha+\frac{1}{2}\overset{\alpha}{\Gamma}_{i\alpha\beta;j}Z^jZ^\alpha Z^\beta+\frac{1}{6}\overset{\alpha}{\Gamma}_{i\alpha\beta\gamma}Z^\alpha Z^\beta Z^\gamma\right\} \\
&+O(n^{-2}) \\
=\;& Z_i-i_{i,\alpha}Z^\alpha \\
&+n^{-\frac{1}{2}}\left\{Z_{ij}Z^j-i_{i,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma+\left[-\frac{1}{4}(1+\alpha)t_{ijk}-\ell_{ij;k}\right]Z^jZ^k\right. \\
&+\left[\ell_{i\alpha;j}+\frac{1}{2}(1+\alpha)t_{ij\alpha}\right]Z^jZ^\alpha \\
&\left.+\left[-\frac{1}{4}(3+\alpha)t_{i\alpha\beta}-\frac{1}{2}\ell_{i\alpha;\beta}\{3\}-\frac{1}{2}i_{i,\gamma}\nu^{\gamma,\delta}\ell_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\right\} \\
&+n^{-1}\left\{\frac{1}{2}Z_{ijk}Z^jZ^k+i^{j,k}Z_{ij}Z_{kl}Z^l+\frac{1}{2}i^{j,k}\ell_{klm}Z_{ij}Z^lZ^m\right. \\
&+\left[\left(-\frac{1}{2}(1+\alpha)t_{ijk}-\ell_{ij;k}-\ell_{ik;j}\right)Z^j+\left(\frac{1}{2}(1+\alpha)t_{ik\alpha}+\ell_{i\alpha;k}\right)Z^\alpha\right]i^{k,l}Z_{lm}Z^m \\
&+\left[\frac{1}{2}(1+\alpha)t_{ij\alpha}+\ell_{i\alpha;j}\right]\nu^{\alpha,\beta}Z_{\beta\gamma}Z^jZ^\gamma+\left[-\frac{1}{2}(1+\alpha)t_{i\alpha\beta}+\ell_{i\alpha\beta}\right]\nu^{\beta,\gamma}Z_{\gamma\delta}Z^\alpha Z^\delta \\
&+\left[-\frac{1}{4}\left((1+\alpha)t_{ijm}+2\ell_{ij;m}+2\ell_{im;j}\right)i^{m,n}\ell_{kln}\right. \\
&\left.+\frac{1}{12}\left(2\ell_{ijkl}+\ell_{i;jkl}+\ell_{jkl;i}+\alpha(\ell_{jkl;i}-\ell_{i;jkl})\right)\right]Z^jZ^kZ^l \\
&+\left[\frac{1}{4}(1+\alpha)t_{il\alpha}i^{l,m}\ell_{jkm}+\frac{1}{2}\ell_{i\alpha;l}i^{l,m}\ell_{jkm}\right. \\
&\left.+\frac{1}{4}\left(\ell_{i\alpha;jk}+\ell_{jk;i\alpha}+\alpha(\ell_{jk;i\alpha}-\ell_{i\alpha;jk})\right)\right]Z^jZ^kZ^\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{1}{4}(1+\alpha)\not{t}_{ij\gamma}\nu^{\gamma,\delta}f_{\alpha\beta\delta}+\frac{1}{2}f_{i\gamma;j}\nu^{\gamma,\delta}f_{\alpha\beta\delta}\right.\\
&\left.+\frac{1}{4}\left(f_{j;i\alpha\beta}+f_{i\alpha\beta;j}+\alpha(f_{j;i\alpha\beta}-f_{i\alpha\beta;j}\right)\right]Z^jZ^\alpha Z^\beta\\
&+\left[-\frac{1}{4}(1+\alpha)\not{t}_{i\alpha\delta}\nu^{\delta,\epsilon}f_{\beta\gamma\epsilon}+\frac{1}{2}f_{i\alpha\delta}\nu^{\delta,\epsilon}f_{\beta\gamma\epsilon}\right.\\
&+\frac{1}{12}\left(-f_{i;\alpha\beta\gamma}\{4\}-f_{\alpha\beta\gamma;i}\{4\}-f_{i\alpha;\beta\gamma}\{6\}\right.\\
&\left.\left.+\ \alpha(f_{i;\alpha\beta\gamma}-f_{\alpha\beta\gamma;i}+3f_{i\alpha\beta;\gamma}-3f_{\gamma;i\alpha\beta}+3f_{i\alpha;\beta\gamma}-3f_{\beta\gamma;i\alpha})\right)\right]Z^\alpha Z^\beta Z^\gamma\\
&-i_{i,\alpha}\nu^{\alpha,\beta}\left(\nu^{\gamma,\delta}Z_{\beta\gamma}Z_{\delta\epsilon}Z^\epsilon+\frac{1}{2}Z_{\beta\gamma\delta}Z^\gamma Z^\delta+f_{\beta\gamma\delta}\nu^{\gamma,\epsilon}Z_{\epsilon\psi}Z^\delta Z^\psi\right.\\
&+\frac{1}{2}\nu^{\gamma,\delta}f_{\delta\epsilon\psi}Z_{\beta\gamma}Z^\epsilon Z^\psi+\frac{1}{6}f_{\beta\gamma\delta\epsilon}Z^\gamma Z^\delta Z^\epsilon\\
&\left.\left.+\frac{1}{2}f_{\beta\gamma\delta}\nu^{\delta,\epsilon}f_{\epsilon\psi\phi}Z^\gamma Z^\psi Z^\phi\right)\right\}+O(n^{-\frac{3}{2}}).
\end{aligned}
\tag{3.5.63}
$$

3.5.3 Expansion for the Geometric Wald Statistic

Now, expanding the geometric Wald test statistic is simply a matter of putting together the expansions (3.5.50) and (3.5.62) of $i^{i,j}(\hat{\theta})$, and $\overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta})$ and $\overset{\alpha}{\Gamma}_j(\hat{\theta};\tilde{\theta})$,

$$
\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= \overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta})i^{i,j}(\hat{\theta})\overset{\alpha}{\Gamma}_j(\hat{\theta};\tilde{\theta})\\
&= (-Z_i+i_{i,\alpha}Z^\alpha)i^{i,j}(-Z_j+i_{j,\alpha}Z^\alpha)\\
&\quad+n^{-\frac{1}{2}}\left\{2(-Z_i+i_{i,\alpha}Z^\alpha)i^{i,j}\right.\\
&\quad\times\left(-Z_{jk}Z^k+i_{j,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma-\frac{1}{4}(1+\alpha)\not{t}_{jkl}Z^kZ^l\right.\\
&\quad\left.+\left[f_{jk;\alpha}+\frac{1}{2}(1+\alpha)\not{t}_{jk\alpha}\right]Z^kZ^\alpha+\left[\frac{1}{4}(1-\alpha)\not{t}_{j\alpha\beta}+\frac{1}{2}f_{\alpha\beta;j}+\frac{1}{2}i_{j,\gamma}\nu^{\gamma,\delta}f_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\right)\\
&\quad\left.+(-Z_i+i_{i,\alpha}Z^\alpha)(-Z_j+i_{j,\alpha}Z^\alpha)\left(-i^{i,l}i^{j,m}(f_{kl;m}+f_{l;km})Z^k\right)\right\}\\
&\quad+n^{-1}\left\{2(-Z_j+i_{j,\alpha}Z^\alpha)(i^{i,j})\right.\\
&\quad\times\left(-\frac{1}{2}Z_{ijk}Z^jZ^k-i^{j,k}Z_{ij}Z_{kl}Z^l-\frac{1}{2}(1+\alpha)\not{t}_{ijk}i^{k,l}Z_{lm}Z^jZ^m\right.\\
&\quad-\frac{1}{2}i^{j,k}f_{klm}Z_{ij}Z^lZ^m+\left[\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha}+f_{ij;\alpha}\right]i^{j,k}Z_{kl}Z^lZ^\alpha\\
&\quad+\left[\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha}+f_{ij;\alpha}\right]\nu^{\alpha,\beta}Z_{\beta\gamma}Z^jZ^\gamma+\left[\frac{1}{2}(1-\alpha)\not{t}_{i\alpha\beta}+f_{\alpha\beta;i}\right]\nu^{\beta,\gamma}Z_{\gamma\delta}Z^\alpha Z^\delta\\
&\quad+\left[-\frac{1}{4}(1+\alpha)\not{t}_{ijm}i^{m,n}f_{kln}\right.\\
&\quad\left.-\frac{1}{12}(1+\alpha)\left(2f_{ijkl}+f_{i;jkl}\{4\}+f_{ijk;l}\{4\}+f_{ij;kl}\{6\}\right)\right]Z^jZ^kZ^l\\
&\quad+\left[\frac{1}{4}(1+\alpha)\not{t}_{il\alpha}i^{l,m}f_{jkm}+\frac{1}{2}f_{il;\alpha}i^{l,m}f_{jkm}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}\left(\ell_{ijk;\alpha}+\ell_{\alpha;ijk}+\alpha(\ell_{\alpha;ijk}-\ell_{ijk;\alpha})\right)\Big] Z^j Z^k Z^\alpha \\
&+\Big[\frac{1}{4}(1+\alpha)\not{\ell}_{ij\gamma}\nu^{\gamma,\delta}\ell_{\alpha\beta\delta}+\frac{1}{2}\ell_{il;\gamma}\nu^{\gamma,\delta}\ell_{\alpha\beta\delta} \\
&+\frac{1}{4}\left(\ell_{ij;\alpha\beta}+\ell_{\alpha\beta;ij}+\alpha(\ell_{\alpha\beta;ij}-\ell_{ij;\alpha\beta}\right)\Big] Z^j Z^\alpha Z^\beta \\
&+\Big[\frac{1}{4}(1+\alpha)\not{\ell}_{i\alpha\delta}\nu^{\delta,\epsilon}\ell_{\beta\gamma\epsilon}+\frac{1}{2}\ell_{\alpha\delta;i}\nu^{\delta,\epsilon}\ell_{\beta\gamma\epsilon} \\
&+\frac{1}{12}\left(\ell_{i;\alpha\beta\gamma}+\ell_{\alpha\beta\gamma;i}+\alpha(\ell_{\alpha\beta\gamma;i}-\ell_{i;\alpha\beta\gamma})\right)\Big] Z^\alpha Z^\beta Z^\gamma \\
&+i_{i,\alpha}\nu^{\alpha,\beta}\Big(\nu^{\gamma,\delta}Z_{\beta\gamma}Z_{\delta\epsilon}Z^\epsilon+\frac{1}{2}Z_{\beta\gamma\delta}Z^\gamma Z^\delta+\ell_{\beta\gamma\delta}\nu^{\gamma,\epsilon}Z_{\epsilon\psi}Z^\delta Z^\psi \\
&+\frac{1}{2}\nu^{\gamma,\delta}\ell_{\delta\epsilon\psi}Z_{\beta\gamma}Z^\epsilon Z^\psi+\frac{1}{6}\ell_{\beta\gamma\delta\epsilon}Z^\gamma Z^\delta Z^\epsilon \\
&+\frac{1}{2}\ell_{\beta\gamma\delta}\nu^{\delta,\epsilon}\ell_{\epsilon\psi\phi}Z^\gamma Z^\psi Z^\phi\Big)\Big) \\
&+(-Z_i+i_{i,\alpha}Z^\alpha)\left(-i^{i,l}i^{j,m}(\ell_{lk;m}+\ell_{l;km}+\ell_{mk;l}+\ell_{m;kl})Z^k\right) \\
&\times\Big(-Z_{jk}Z^k+i_{j,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma-\frac{1}{4}(1+\alpha)\not{\ell}_{jkl}Z^k Z^l \\
&+\left[\ell_{jk;\alpha}+\frac{1}{2}(1+\alpha)\not{\ell}_{jk\alpha}\right]Z^k Z^\alpha+\left[\frac{1}{4}(1-\alpha)\not{\ell}_{j\alpha\beta}+\frac{1}{2}\ell_{\alpha\beta;j}+\frac{1}{2}i_{j,\gamma}\nu^{\gamma,\delta}\ell_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\Big) \\
&+i^{i,j}\Big(-Z_{ij}Z^j+i_{i,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma-\frac{1}{4}(1+\alpha)\not{\ell}_{ijk}Z^j Z^k \\
&+\left[\ell_{ij;\alpha}+\frac{1}{2}(1+\alpha)\not{\ell}_{ij\alpha}\right]Z^j Z^\alpha+\left[\frac{1}{4}(1-\alpha)\not{\ell}_{i\alpha\beta}+\frac{1}{2}\ell_{\alpha\beta;i}+\frac{1}{2}i_{i,\gamma}\nu^{\gamma,\delta}\ell_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\Big) \\
&\times\Big(-Z_{jk}Z^k+i_{j,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma-\frac{1}{4}(1+\alpha)\not{\ell}_{jkl}Z^k Z^l \\
&+\left[\ell_{jk;\alpha}+\frac{1}{2}(1+\alpha)\not{\ell}_{jk\alpha}\right]Z^k Z^\alpha+\left[\frac{1}{4}(1-\alpha)\not{\ell}_{j\alpha\beta}+\frac{1}{2}\ell_{\alpha\beta;j}+\frac{1}{2}i_{j,\gamma}\nu^{\gamma,\delta}\ell_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\Big) \\
&+(-Z_i+i_{i,\alpha}Z^\alpha)(-Z_j+i_{j,\alpha}Z^\alpha)\Big(-i^{i,l}i^{j,m}\ell_{l,m/n}i^{n,p}Z_{kp}Z^k \\
&+\frac{1}{2}i^{i,m}i^{j,n}\left[-\ell_{m;n/p}i^{p,q}\ell_{klq}+2\ell_{m;p/k}i^{p,q}\ell_{n;q/l}-\ell_{m;n/kl}\right]Z^k Z^l\Big)\Big\} \\
&+O(n^{-\frac{3}{2}}) \\
=\;& i_{i,j}Z^i Z^j-i_{\alpha,\beta}Z^\alpha Z^\beta \\
&+n^{-\frac{1}{2}}\Big\{2\left[Z_{ij}-\ell_{ij;k}Z^k\right]Z^i Z^j-2\left[Z_{i\alpha}-\ell_{i\alpha;k}Z^k\right]Z^i Z^\alpha-\frac{1}{2}(1-\alpha)\not{\ell}_{ijk}Z^i Z^j Z^k \\
&+\frac{1}{2}(1-3\alpha)\not{\ell}_{ij\alpha}Z^i Z^j Z^\alpha-\left[\ell_{\alpha\beta;i}+\frac{1}{2}(1-3\alpha)\not{\ell}_{i\alpha\beta}\right]Z^i Z^\alpha Z^\beta \\
&+\left[\ell_{\alpha\beta;\gamma}+\frac{1}{2}(1-\alpha)\not{\ell}_{\alpha\beta\gamma}\right]Z^\alpha Z^\beta Z^\gamma\Big\} \\
&n^{-1}\Big\{\left[Z_{ijk}-3\ell_{ij;l}i^{l,m}Z_{km}+(3\ell_{ij;m}i^{m,n}\not{\ell}_{kln}-\ell_{ijk;l})Z^l\right]Z^i Z^j Z^k \\
&-\left[Z_{ij\alpha}-3\ell_{ij;l}i^{l,m}Z_{m\alpha}+(3\ell_{ij;m}i^{m,n}\not{\ell}_{ln\alpha}-\ell_{ij\alpha;l})Z^l\right]Z^i Z^j Z^\alpha \\
&+3i^{j,l}\left[Z_{ij}-\ell_{ij;k}Z^k\right]\left[Z_{lm}-\ell_{lm;n}Z^n\right]Z^i Z^m \\
&-2i^{j,l}\left[Z_{ij}-\ell_{ij;k}Z^k\right]\left[Z_{l\alpha}-\ell_{l\alpha;n}Z^n\right]Z^i Z^\alpha
\end{aligned}
$$

$$
\begin{aligned}
& -2\nu^{\beta,\gamma}\left[Z_{i\beta}-\not{\ell}_{i\beta;k}Z^k\right]\left[Z_{\alpha\gamma}-\not{\ell}_{\alpha\gamma;n}Z^n\right]Z^iZ^\alpha \\
& +\nu^{\gamma,\delta}\left[Z_{\alpha\gamma}-\not{\ell}_{\alpha\gamma;i}Z^i\right]\left[Z_{\beta\delta}-\not{\ell}_{\beta\delta;j}Z^j\right]Z^\alpha Z^\beta \\
& +\frac{1}{2}(-5+3\alpha)\not{t}_{ijl}i^{l,m}\left[Z_{km}-\not{\ell}_{km;n}Z^n\right]Z^iZ^jZ^k \\
& +(1-3\alpha)\not{t}_{ik\alpha}i^{k,l}\left[Z_{jl}-\not{\ell}_{jl;m}Z^m\right]Z^iZ^jZ^\alpha \\
& -\left(\frac{1}{2}(1-3\alpha)\not{t}_{j\alpha\beta}+\not{\ell}_{\alpha\beta;j}\right)i^{j,k}\left[Z_{ik}-\not{\ell}_{ik;l}Z^l\right]Z^iZ^\alpha Z^\beta \\
& +i^{k,l}\not{t}_{ijk}\left[Z_{l\alpha}-\not{\ell}_{l\alpha;m}Z^m\right]Z^iZ^jZ^\alpha-2\nu^{\beta,\gamma}\not{\ell}_{\alpha\beta;i}\left[Z_{j\gamma}-\not{\ell}_{j\gamma;k}Z^k\right]Z^iZ^jZ^\alpha \\
& +\nu^{\gamma,\delta}\left(\not{t}_{\alpha\beta\gamma}+\not{\ell}_{\alpha\beta;\gamma}+2\not{\ell}_{\alpha\gamma;\beta}\right)\left[Z_{i\delta}-\not{\ell}_{i\delta;j}Z^j\right]Z^iZ^\alpha Z^\beta \\
& +\frac{1}{2}(1-3\alpha)\nu^{\beta,\gamma}\not{t}_{ij\beta}\left[Z_{\alpha\gamma}-\not{\ell}_{\alpha\gamma;k}Z^k\right]Z^iZ^jZ^\alpha \\
& -(1-3\alpha)\nu^{\gamma,\delta}\not{t}_{i\alpha\gamma}\left[Z_{\beta\delta}-\not{\ell}_{\beta\delta;j}Z^j\right]Z^iZ^\alpha Z^\beta \\
& +\frac{1}{2}(1-3\alpha)\nu^{\delta,\epsilon}\not{t}_{\alpha\beta\delta}\left[Z_{\gamma\epsilon}-\not{\ell}_{\gamma\epsilon;i}Z^i\right]Z^\alpha Z^\beta Z^\gamma \\
& +\Big[(\frac{9}{16}-\frac{7}{8}\alpha+\frac{1}{16}\alpha^2)\not{t}_{ijm}\not{t}_{kln}i^{m,n}+i^{m,n}\not{\ell}_{ij;m}\not{\ell}_{kl;n}-\not{\ell}_{ij;kl} \\
& +(\frac{1}{2}-\alpha)i^{m,n}\not{\ell}_{ij;m}\not{t}_{kln}+\frac{1}{6}(1-2\alpha)\left(\not{\ell}_{ijk;l}-\not{\ell}_{l;ijk}\right)\Big]Z^iZ^jZ^kZ^l \\
& +\Big[(-\frac{3}{4}+2\alpha-\frac{1}{4}\alpha^2)i^{l,m}\not{t}_{ijl}\not{t}_{km\alpha}+\frac{1}{4}(1+\alpha)\left(i^{l,m}\not{t}_{ijl}\not{\ell}_{k\alpha;m}\{3\}_{jk\alpha}-\not{\ell}_{i;jk\alpha}+\not{\ell}_{ij\alpha;k}\right) \\
& -\frac{1}{12}(5-7\alpha)\left(3i^{l,m}\not{t}_{il\alpha}\not{\ell}_{jk;m}-\not{\ell}_{\alpha;ijk}+\not{\ell}_{ijk;\alpha}\right)-2i^{l,m}\not{\ell}_{i\alpha;l}\not{\ell}_{m;jk}+2\not{\ell}_{i\alpha;jk} \\
& +\frac{1}{2}(1-3\alpha)\nu^{\beta,\gamma}\overset{1}{H}_{i;\alpha,\beta}\not{t}_{jk\gamma}\Big]Z^iZ^jZ^kZ^\alpha \\
& +\Big[\left(\frac{1}{8}-\frac{1}{4}\alpha+\frac{1}{8}\alpha^2\right)i^{k,l}\not{t}_{ijk}\not{t}_{l\alpha\beta}+\left(\frac{1}{4}-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2\right)i^{k,l}\not{t}_{ik\alpha}\not{t}_{jl\beta} \\
& +\frac{1}{4}(1-3\alpha)i^{k,l}\not{t}_{ijk}\overset{-1}{H}_{l;\alpha,\beta}+\frac{1}{2}i^{k,l}\not{\ell}_{ij;k}\not{\ell}_{l;\alpha\beta}-\frac{1}{2}\not{\ell}_{ij;\alpha\beta}+\frac{1}{2}i^{k,l}\not{\ell}_{\alpha\beta;k}\not{\ell}_{l;ij}-\frac{1}{2}\not{\ell}_{\alpha\beta;ij} \\
& +i^{k,l}\not{\ell}_{i\alpha;k}\not{\ell}_{l;j\beta}-\not{\ell}_{i\alpha;j\beta}-\alpha\left(i^{k,l}\not{t}_{ik\alpha}\not{\ell}_{j\beta;l}\{3\}_{ij\beta}-\not{\ell}_{\alpha;ij\beta}+\not{\ell}_{ij\beta;\alpha}\right) \\
& +\frac{1}{2}\alpha\left(i^{k,l}\not{t}_{ijk}\not{\ell}_{\alpha\beta;l}\{3\}_{j\alpha\beta}-\not{\ell}_{i;j\alpha\beta}+\not{\ell}_{j\alpha\beta;i}\right)-(1-3\alpha)\nu^{\gamma,\delta}\overset{1}{H}_{i;\alpha,\gamma}\not{t}_{j\beta\delta} \\
& -\nu^{\gamma,\delta}\overset{1}{H}_{i;\alpha,\gamma}\overset{1}{H}_{j;\beta,\delta}\Big]Z^iZ^jZ^\alpha Z^\beta \\
& +\Big[\left(\frac{1}{4}-\alpha-\frac{1}{4}\alpha^2\right)i^{j,k}\not{t}_{ij\alpha}\not{t}_{k\beta\gamma}-\frac{1}{2}(1-3\alpha)i^{j,k}\not{t}_{ij\alpha}\overset{1}{H}_{k;\beta,\gamma} \\
& -i^{j,k}\not{\ell}_{i\alpha;j}\not{\ell}_{k;\beta\gamma}+\not{\ell}_{i\alpha;\beta\gamma}+\frac{1}{4}(1+\alpha)\left(i^{j,k}\not{t}_{ij\alpha}\not{\ell}_{\beta\gamma;k}\{3\}_{i\beta\gamma}-\not{\ell}_{\alpha;i\beta\gamma}+\not{\ell}_{i\alpha\beta;\gamma}\right) \\
& -\frac{1}{12}(1+5\alpha)\left(3i^{j,k}\not{t}_{ij\alpha}\not{\ell}_{\beta\gamma;k}-\not{\ell}_{i;\alpha\beta\gamma}+\not{\ell}_{\alpha\beta\gamma;i}\right)+\frac{3}{2}(1-\alpha)\nu^{\delta,\epsilon}\overset{1}{H}_{i;\alpha,\delta}\not{t}_{\beta\gamma\epsilon} \\
& +3\nu^{\delta,\epsilon}\overset{1}{H}_{i;\alpha,\delta}\not{\ell}_{\beta\epsilon;\gamma}-\not{\ell}_{\alpha\beta\gamma;i}+i_{i,\delta}\nu^{\delta,\epsilon}\not{\ell}_{\alpha\beta\gamma;\epsilon}\Big]Z^iZ^\alpha Z^\beta Z^\gamma \\
& +\Big[\left(-\frac{7}{16}+\frac{5}{8}\alpha+\frac{1}{16}\alpha^2\right)i^{i,j}\not{t}_{i\alpha\beta}\not{t}_{j\gamma\delta}-\frac{1}{6}(1-\alpha)(3i^{i,j}\not{t}_{i\alpha\beta}\not{\ell}_{\gamma\delta;j}-\not{\ell}_{\alpha;\beta\gamma\delta}+\not{\ell}_{\alpha\beta\gamma;\delta}) \\
& +\frac{1}{4}(1-3\alpha)i^{i,j}\overset{-1}{H}_{i;\alpha,\beta}\not{t}_{j\gamma\delta}+\frac{1}{4}i^{i,j}\overset{-1}{H}_{i;\alpha,\beta}\overset{-1}{H}_{j;\gamma,\delta}\Big]Z^\alpha Z^\beta Z^\gamma Z^\delta\Big\}+O(n^{-\frac{3}{2}}) \\
= \;& \overset{\alpha}{C}{}^{i,j}Y_iY_j+n^{-\frac{1}{2}}\overset{\alpha}{C}{}^{i,j,K}Y_iY_jY_K+n^{-1}\overset{\alpha}{C}{}^{i,j,K,L}Y_iY_jY_KY_L+O(n^{-\frac{3}{2}}), \qquad (3.5.64)
\end{aligned}
$$

where

$$\overset{\alpha}{C}{}^{i,j} = i^{i,j} - \boldsymbol{\delta}^{i,j}_{\alpha,\beta}\, \nu^{\alpha,\beta} \tag{3.5.65}$$

$$\overset{\alpha}{C}{}^{i,j,kl} = 2i^{i,k}i^{j,l} - 2\boldsymbol{\delta}^{j,l}_{\alpha,\beta}\, i^{i,k}\nu^{\alpha,\beta} \tag{3.5.66}$$

$$\begin{aligned}
\overset{\alpha}{C}{}^{i,j,k} = {} & i^{i,l}i^{j,m}i^{k,n}\left[-\frac{1}{2}(1-\alpha)\,\not{t}_{lmn}\right] + \boldsymbol{\delta}^{k}_{\alpha}\, i^{i,l}i^{j,m}\nu^{\alpha,\beta}\left[\frac{1}{2}(1-3\alpha)\,\not{t}_{lm\beta}\right] \\
& + \boldsymbol{\delta}^{j,k}_{\alpha,\beta}\, i^{i,l}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[-\frac{1}{2}(1-3\alpha)\,\not{t}_{l\gamma\delta} - \overset{1}{H}_{l;\gamma,\delta}\right] \\
& + \boldsymbol{\delta}^{i,j,k}_{\alpha,\beta,\gamma}\, \nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\frac{1}{2}(1-\alpha)\,\not{t}_{\delta\epsilon\psi}\right]
\end{aligned} \tag{3.5.67}$$

$$\overset{\alpha}{C}{}^{i,j,k,lmn} = i^{i,l}i^{j,m}i^{k,n} - \boldsymbol{\delta}^{k,n}_{\alpha,\beta}\, i^{i,l}i^{j,m}\nu^{\alpha,\beta} \tag{3.5.68}$$

$$\begin{aligned}
\overset{\alpha}{C}{}^{i,j,kl,mn} = {} & 3i^{i,k}i^{j,m}i^{l,n} - 2\boldsymbol{\delta}^{j,n}_{\alpha,\beta}\, i^{ik}i^{l,m}\nu^{\alpha,\beta} - 2\boldsymbol{\delta}^{j,l,m,n}_{\alpha,\beta,\gamma,\delta}\, i^{i,k}\nu^{\alpha,\gamma}\nu^{\beta,\delta} \\
& + \boldsymbol{\delta}^{i,j,k,l,m,n}_{\alpha,\beta,\gamma,\delta,\epsilon,\psi}\, \nu^{\alpha,\gamma}\nu^{\beta,\epsilon}\nu^{\delta,\psi}
\end{aligned} \tag{3.5.69}$$

$$\begin{aligned}
\overset{\alpha}{C}{}^{i,j,k,lm} = {} & i^{i,n}i^{j,p}i^{k,l}i^{m,q}\left[\frac{1}{2}(-5+3\alpha)\,\not{t}_{npq}\right] \\
& + \boldsymbol{\delta}^{k}_{\alpha}\, i^{i,n}i^{j,l}i^{m,p}\nu^{\alpha,\beta}\left[(1-3\alpha)\,\not{t}_{np\beta}\right] \\
& + \boldsymbol{\delta}^{j,k}_{\alpha,\beta}\, i^{i,l}i^{m,n}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[-\frac{1}{2}(1-3\alpha)\,\not{t}_{n\gamma\delta} - \overset{1}{H}_{n;\gamma,\delta}\right] \\
& + \boldsymbol{\delta}^{k,m}_{\alpha,\beta}\, i^{i,n}\left[i^{j,p}i^{l,q}\nu^{\alpha,\beta}\,\not{t}_{npq} - 2i^{j,l}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\,\overset{1}{H}_{n;\gamma,\delta}\right] \\
& + \boldsymbol{\delta}^{j,k,m}_{\alpha,\beta,\gamma}\, i^{i,l}\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\not{t}_{\delta\epsilon\psi}\right] \\
& + \boldsymbol{\delta}^{k,l,m}_{\alpha,\beta,\gamma}\, i^{i,n}i^{j,p}\nu^{\alpha,\beta}\nu^{\gamma,\delta}\left[\frac{1}{2}(1-3\alpha)\,\not{t}_{np\delta}\right] \\
& + \boldsymbol{\delta}^{j,k,l,m}_{\alpha,\beta,\gamma,\delta}\, i^{i,n}\nu^{\alpha,\gamma}\nu^{\beta,\epsilon}\nu^{\delta,\psi}\left[-(1-3\alpha)\,\not{t}_{n\epsilon\psi}\right] \\
& + \boldsymbol{\delta}^{i,j,k,l,m}_{\alpha,\beta,\gamma,\delta,\epsilon}\, \nu^{\alpha,\delta}\nu^{\beta,\psi}\nu^{\gamma,\phi}\nu^{\epsilon,\chi}\left[\frac{1}{2}(1-3\alpha)\,\not{t}_{\psi\phi\chi}\right]
\end{aligned} \tag{3.5.70}$$

$$\begin{aligned}
\overset{\alpha}{C}{}^{i,j,k,l} = {} & i^{i,m}i^{j,n}i^{k,p}i^{l,q}\left[\left(\frac{9}{16} - \frac{7}{8}\alpha + \frac{1}{16}\alpha^2\right)\tau_{mnpq} - \frac{1}{6}(1-2\alpha)\,\not{t}_{m;npq} - \not{t}_{mn;pq}\right] \\
& + \boldsymbol{\delta}^{l}_{\alpha}\, i^{i,m}i^{j,n}i^{k,p}\nu^{\alpha,\beta}\left[\left(-\frac{3}{4} + 2\alpha - \frac{1}{4}\alpha^2\right)\tau_{mnp\beta} - \frac{1}{4}(1+\alpha)\,\not{t}_{m;np\beta}\right. \\
& \left. + \frac{1}{12}(5-7\alpha)\,\not{t}_{\beta;mnp} + 2\,\not{t}_{m\beta;np} + \frac{1}{2}(1-3\alpha)\,\overset{1}{\tau}_{m;\beta,np}\right] \\
& + \boldsymbol{\delta}^{k,l}_{\alpha,\beta}\, i^{i,m}i^{j,n}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[\left(\frac{1}{8} - \frac{1}{4}\alpha + \frac{1}{8}\alpha^2\right)\tau_{mn\gamma\delta}\right. \\
& + \left(\frac{1}{4} - \frac{1}{2}\alpha + \frac{1}{4}\alpha^2\right)\tau_{m\gamma n\delta} + \left(\frac{1}{4} - \frac{3}{4}\alpha\right)\overset{-1}{\tau}_{\gamma,\delta,mn} - \frac{1}{2}\,\not{t}_{mn;\gamma\delta} - \frac{1}{2}\,\not{t}_{\gamma\delta;mn}
\end{aligned}$$

$$
\begin{aligned}
&-\not{t}_{m\gamma;n\delta} + \alpha\,\not{t}_{\gamma;mn\delta} - \frac{1}{2}\alpha\,\not{t}_{m;n\gamma\delta} - (1-3\alpha)\,\overset{1}{\tau}_{m;\gamma,n\delta} - \overset{1}{\tau}_{m;n;\gamma,\delta}\Big] \\
&+\delta^{j,k,l}_{\alpha,\beta,\gamma}\, i^{i,m}\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\left(\frac{1}{4} - \alpha - \frac{1}{4}\alpha^2\right)\tau_{m\delta\epsilon\psi}\right. \\
&-\frac{1}{2}(1-3\alpha)\,\overset{-1}{\tau}_{\delta,\epsilon,m\psi} + \not{t}_{m\delta;\epsilon\psi} - \frac{1}{4}(1+\alpha)\,\not{t}_{\delta;m\epsilon\psi} \\
&\left.+\frac{1}{12}(1+5\alpha)\,\not{t}_{m;\delta\epsilon\psi} + \frac{3}{2}(1-\alpha)\,\overset{1}{\tau}_{m;\delta,\epsilon\psi} - \overset{1}{\tau}_{m;\delta\epsilon\psi}\right] \\
&+\delta^{i,j,k,l}_{\alpha,\beta,\gamma,\delta}\,\nu^{\alpha,\epsilon}\nu^{\beta,\psi}\nu^{\gamma,\phi}\nu^{\delta,\chi}\left[\left(-\frac{7}{16} + \frac{5}{8}\alpha + \frac{1}{16}\alpha^2\right)\tau_{\epsilon\psi\phi\chi}\right. \\
&\left.+\frac{1}{4}(1-3\alpha)\,\overset{-1}{\tau}_{\epsilon,\psi,\phi\chi} + \frac{1}{4}\,\overset{-1}{\tau}_{\epsilon,\psi,\phi,\chi} + \frac{1}{6}(1-\alpha)\,\not{t}_{\epsilon;\psi\phi\chi}\right],
\end{aligned}
\qquad (3.5.71)
$$

and all other $\overset{\alpha}{C}$s vanish.

Remark 11:

We see from (3.5.64) that with error of order $O(n^{-\frac{1}{2}})$ the geometric Wald statistics have the same asymptotic expansion as the likelihood ratio statistic (and the score statistic). Thus, with error of order $O(n^{-\frac{1}{2}})$ the geometric Wald statistics have the same asymptotic χ^2 distribution with p degrees of freedom as the likelihood ratio statistic.

Alternatively, observe that the expansion of the expected geometric Wald statistics with error of order $O(n^{-\frac{1}{2}})$ can be re-written as

$$
\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= i^{i,j}Y_iY_j - \nu^{\alpha,\beta}Y_\alpha Y_\beta + O(n^{-\frac{1}{2}}) \\
&= i^{a,b}\left(Y_a - i_{a,\alpha}\nu^{\alpha,\beta}Y_\beta\right)\left(Y_b - i_{b,\alpha}\nu^{\alpha,\beta}Y_\beta\right) + O(n^{-\frac{1}{2}}),
\end{aligned}
\qquad (3.5.72)
$$

where $[i^{a,b}]$ is the interest-part of the expected formation matrix $i^{-1}(\theta)$. From general results (e.g. Barndorff-Nielsen & Cox [5]), we know that with error of order $O(n^{-\frac{1}{2}})$, the score vector $(Y_1, \ldots, Y_r)$ has a central r-dimensional normal distribution with covariance matrix $i(\theta)$. Straightforward linear transformations of multi-dimensional normal distributions give us that the vector $(Y_1 - i_{1,\alpha}\nu^{\alpha,\beta}Y_\beta, \ldots, Y_p - i_{p,\alpha}\nu^{\alpha,\beta}Y_\beta)$ has a central p-dimensional normal distribution with variance matrix $M(\theta)$, where $M(\theta)$ is the inverse of the matrix $[i^{a,b}(\theta)]$. Thus, with error of order $O(n^{-\frac{1}{2}})$, the expected geometric Wald statistics (3.5.72) have a χ^2 distribution with p degrees of freedom.

Remark 12:

Observe that the coefficients $\overset{\alpha}{C}{}^{i,j}$, $\overset{\alpha}{C}{}^{i,j,kl}$, $\overset{\alpha}{C}{}^{i,j,k,lmn}$ and $\overset{\alpha}{C}{}^{i,j,kl,mn}$ in (3.5.65), (3.5.66), (3.5.68) and (3.5.69) do not depend on the value of α.

If we have a model with vanishing skewness tensor $\not{t}_{ijk}$, most of the coefficients of the form $\overset{\alpha}{C}{}^{i,j,k}$ and $\overset{\alpha}{C}{}^{i,j,k,lm}$ vanish, the remaining ones simplifying to

$$\begin{aligned}
\overset{\alpha}{C}{}^{i,\alpha,\beta} &= -\nu^{\alpha,\gamma}\nu^{\beta,\delta}\overset{1}{H}{}^{i}_{\gamma,\delta} \\
\overset{\alpha}{C}{}^{i,\alpha,\beta,jk} &= -i^{i,j}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\overset{1}{H}{}^{k}_{\gamma,\delta} \\
\overset{\alpha}{C}{}^{i,j,\alpha,k\beta} &= -2i^{j,k}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\overset{1}{H}{}^{i}_{\gamma,\delta}.
\end{aligned}$$

Note that the remaining coefficients do not depend on α. Furthermore, the following $\not{t}$-tensor and τ-tensors occurring in $\overset{\alpha}{C}{}^{i,j,k,l}$ simplify to

$$\begin{aligned}
\not{t}_{i;jkl} &= f_{i;jkl} - f_{jkl;i} \\
\tau_{ijkl} &= 0 \\
\overset{1}{\tau}{}_{i;\alpha,jk} &= 0 \\
\overset{-1}{\tau}{}_{ij,\alpha,\beta} &= 0 \\
\overset{-1}{\tau}{}_{\alpha,\beta,\gamma,\delta} &= i^{i,j}\overset{1}{H}{}_{i;\alpha,\beta}\overset{1}{H}{}_{j;\gamma,\delta}.
\end{aligned}$$

When both the embedding curvature $\overset{1}{H}{}^{i}_{\alpha,\beta}$ and the skewness tensor $\not{t}_{ijk}$ vanish, the expansion of the geometric Wald statistics simplifies even more. All coefficients except $\overset{\alpha}{C}{}^{i,j}$ and $\overset{\alpha}{C}{}^{i,j,k,l}$ vanish. Also, most of the $\not{t}$-tensors and τ-tensors in $\overset{\alpha}{C}{}^{i,j,k,l}$ vanish, leaving only

$$\begin{aligned}
\not{t}_{ij;kl} &= -f_{ij;m}f_{n;kl}i^{m,n} + f_{ij;kl} \\
\not{t}_{i;jkl} &= f_{i;jkl} - f_{jkl;i} \\
\overset{-1}{\tau}{}_{i;\alpha\beta\gamma} &= f_{\alpha\beta\gamma;i} - i_{i,\delta}\nu^{\delta,\epsilon}f_{\alpha\beta\gamma;\epsilon}.
\end{aligned}$$

3.5.4 Expansion for the Modified Geometric Wald Statistic

To find the expansion of the modified geometric Wald statistic, we use the expansions (3.5.63) and (3.5.51) of $\overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta})$, $\overset{\alpha}{\Gamma}_j(\tilde{\theta};\hat{\theta})$ and $i^{-1}(\tilde{\theta})$ to get

$$\begin{aligned}
\overset{\alpha}{\widetilde{\mathcal{W}}} &= \overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta})i^{i,j}(\tilde{\theta})\overset{\alpha}{\Gamma}_j(\tilde{\theta};\hat{\theta}) \\
&= (Z_i - i_{i,\alpha}Z^\alpha)i^{i,j}(Z_j - i_{j,\alpha}Z^\alpha) \\
&\quad + n^{-\frac{1}{2}}\Big\{2(Z_i - i_{i,\alpha}Z^\alpha)i^{i,j}\Big(Z_{jk}Z^k - i_{j,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma \\
&\quad + \left[-\frac{1}{4}(1+\alpha)\not{t}_{jkl} - f_{jk;l}\right]Z^kZ^l + \left[f_{j\alpha;k} + \frac{1}{2}(1+\alpha)\not{t}_{jk\alpha}\right]Z^kZ^\alpha
\end{aligned}$$

$$+\left[-\frac{1}{4}(3+\alpha)\not{t}_{j\alpha\beta}-\frac{1}{2}\not{f}_{j\alpha;\beta}\{3\}-\frac{1}{2}i_{j,\gamma}\nu^{\gamma,\delta}\not{f}_{\alpha\beta\delta}\right]Z^{\alpha}Z^{\beta}\Big)$$

$$+(Z_i-i_{i,\alpha}Z^{\alpha})(Z_j-i_{j,\alpha}Z^{\alpha})\left(-i^{i,k}i^{j,l}\not{f}_{k;l/\alpha}Z^{\alpha}\right)\Big\}$$

$$+n^{-1}\{2(Z_j-i_{j,\alpha}Z^{\alpha})(i^{i,j})$$

$$\times\Big(\frac{1}{2}Z_{ijk}Z^jZ^k+i^{j,k}Z_{ij}Z_{kl}Z^l+\frac{1}{2}i^{j,k}\not{f}_{klm}Z_{ij}Z^lZ^m$$

$$+\left[\left(-\frac{1}{2}(1+\alpha)\not{t}_{ijk}-\not{f}_{ij;k}-\not{f}_{ik;j}\right)Z^j+\left(\frac{1}{2}(1+\alpha)\not{t}_{ik\alpha}+\not{f}_{i\alpha;k}\right)Z^{\alpha}\right]i^{k,l}Z_{lm}Z^m$$

$$+\left[\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha}+\not{f}_{i\alpha;j}\right]\nu^{\alpha,\beta}Z_{\beta\gamma}Z^jZ^{\gamma}+\left[-\frac{1}{2}(1+\alpha)\not{t}_{i\alpha\beta}+\not{f}_{i\alpha\beta}\right]\nu^{\beta,\gamma}Z_{\gamma\delta}Z^{\alpha}Z^{\delta}$$

$$+\Big[-\frac{1}{4}\left((1+\alpha)\not{t}_{ijm}+2\not{f}_{ij;m}+2\not{f}_{im;j}\right)i^{m,n}\not{f}_{kln}$$

$$+\frac{1}{12}\left(2\not{f}_{ijkl}+\not{f}_{i;jkl}+\not{f}_{jkl;i}+\alpha(\not{f}_{jkl;i}-\not{f}_{i;jkl})\right)\Big]Z^jZ^kZ^l$$

$$+\Big[\frac{1}{4}(1+\alpha)\not{t}_{il\alpha}i^{l,m}\not{f}_{jkm}+\frac{1}{2}\not{f}_{i\alpha;l}i^{l,m}\not{f}_{jkm}$$

$$+\frac{1}{4}\left(\not{f}_{i\alpha;jk}+\not{f}_{jk;i\alpha}+\alpha(\not{f}_{jk;i\alpha}-\not{f}_{i\alpha;jk})\right)\Big]Z^jZ^kZ^{\alpha}$$

$$+\Big[\frac{1}{4}(1+\alpha)\not{t}_{ij\gamma}\nu^{\gamma,\delta}\not{f}_{\alpha\beta\delta}+\frac{1}{2}\not{f}_{i\gamma;j}\nu^{\gamma,\delta}\not{f}_{\alpha\beta\delta}$$

$$+\frac{1}{4}\left(\not{f}_{j;i\alpha\beta}+\not{f}_{i\alpha\beta;j}+\alpha(\not{f}_{j;i\alpha\beta}-\not{f}_{i\alpha\beta;j})\right)\Big]Z^jZ^{\alpha}Z^{\beta}$$

$$+\Big[-\frac{1}{4}(1+\alpha)\not{t}_{i\alpha\delta}\nu^{\delta,\epsilon}\not{f}_{\beta\gamma\epsilon}+\frac{1}{2}\not{f}_{i\alpha\delta}\nu^{\delta,\epsilon}\not{f}_{\beta\gamma\epsilon}$$

$$+\frac{1}{12}\Big(-\not{f}_{i;\alpha\beta\gamma}\{4\}-\not{f}_{\alpha\beta\gamma;i}\{4\}-\not{f}_{i\alpha;\beta\gamma}\{6\}$$

$$+\alpha(\not{f}_{i;\alpha\beta\gamma}-\not{f}_{\alpha\beta\gamma;i}+3\not{f}_{i\alpha\beta;\gamma}-3\not{f}_{\gamma;i\alpha\beta}+3\not{f}_{i\alpha;\beta\gamma}-3\not{f}_{\beta\gamma;i\alpha})\Big)\Big]Z^{\alpha}Z^{\beta}Z^{\gamma}$$

$$-i_{i,\alpha}\nu^{\alpha,\beta}\Big(\nu^{\gamma,\delta}Z_{\beta\gamma}Z_{\delta\epsilon}Z^{\epsilon}+\frac{1}{2}Z_{\beta\gamma\delta}Z^{\gamma}Z^{\delta}+\not{f}_{\beta\gamma\delta}\nu^{\gamma,\epsilon}Z_{\epsilon\psi}Z^{\delta}Z^{\psi}$$

$$+\frac{1}{2}\nu^{\gamma,\delta}\not{f}_{\delta\epsilon\psi}Z_{\beta\gamma}Z^{\epsilon}Z^{\psi}+\frac{1}{6}\not{f}_{\beta\gamma\delta\epsilon}Z^{\gamma}Z^{\delta}Z^{\epsilon}$$

$$+\frac{1}{2}\not{f}_{\beta\gamma\delta}\nu^{\delta,\epsilon}\not{f}_{\epsilon\psi\phi}Z^{\gamma}Z^{\psi}Z^{\phi}\Big)\Big)$$

$$+(Z_i-i_{i,\alpha}Z^{\alpha})\left(-i^{i,k}i^{j,l}\left(\not{f}_{k;l/\alpha}+\not{f}_{l;k/\alpha}\right)Z^{\alpha}\right)$$

$$\times\Big(Z_{jk}Z^k-i_{j,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^{\gamma}+\left[-\frac{1}{4}(1+\alpha)\not{t}_{jkl}-\not{f}_{jk;l}\right]Z^kZ^l$$

$$+\left[\not{f}_{j\alpha;k}+\frac{1}{2}(1+\alpha)\not{t}_{jk\alpha}\right]Z^kZ^{\alpha}$$

$$+\left[-\frac{1}{4}(3+\alpha)\not{t}_{j\alpha\beta}-\frac{1}{2}\not{f}_{j\alpha;\beta}\{3\}-\frac{1}{2}i_{j,\gamma}\nu^{\gamma,\delta}\not{f}_{\alpha\beta\delta}\right]Z^{\alpha}Z^{\beta}\Big)$$

$$+i^{i,j}\Big(Z_{ij}Z^j-i_{i,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^{\gamma}+\left[-\frac{1}{4}(1+\alpha)\not{t}_{ijk}-\not{f}_{ij;k}\right]Z^jZ^k$$

$$+\left[\not{f}_{i\alpha;j}+\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha}\right]Z^jZ^{\alpha}$$

$$+\left[-\frac{1}{4}(3+\alpha)\not{t}_{i\alpha\beta}-\frac{1}{2}\not{f}_{i\alpha;\beta}\{3\}-\frac{1}{2}i_{i,\gamma}\nu^{\gamma,\delta}\not{f}_{\alpha\beta\delta}\right]Z^{\alpha}Z^{\beta}\Big)$$

$$\times\left(Z_{jk}Z^k - i_{j,\alpha}\nu^{\alpha,\beta}Z_{\beta\gamma}Z^\gamma + \left[-\frac{1}{4}(1+\alpha)\not{t}_{jkl} - \not{f}_{jk;l}\right]Z^kZ^l\right.$$

$$+\left[\not{f}_{j\alpha;k} + \frac{1}{2}(1+\alpha)\not{t}_{jk\alpha}\right]Z^kZ^\alpha$$

$$\left.+\left[-\frac{1}{4}(3+\alpha)\not{t}_{j\alpha\beta} - \frac{1}{2}\not{f}_{j\alpha;\beta}\{3\} - \frac{1}{2}i_{j,\gamma}\nu^{\gamma,\delta}\not{f}_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\right)$$

$$+(Z_i - i_{i,\alpha}Z^\alpha)(Z_j - i_{j,\alpha}Z^\alpha)\left(-i^{i,l}i^{j,m}\not{f}_{m;n/\beta}\nu^{\beta,\gamma}Z_{\alpha\gamma}Z^\alpha\right.$$

$$\left.\left.+\frac{1}{2}i^{i,k}i^{j,l}\left[-\not{f}_{k;l/\gamma}\nu^{\gamma,\delta}\not{f}_{\alpha\beta\delta} + 2\not{f}_{k;m/\alpha}i^{m,n}\not{f}_{l,n/\beta} - \not{f}_{k;l/\alpha\beta}\right]Z^\alpha Z^\beta\right)\right\}$$

$$+O(n^{-\frac{3}{2}})$$

$$= \quad i_{i,j}Z^iZ^j - i_{\alpha,\beta}Z^\alpha Z^\beta$$

$$+n^{-\frac{1}{2}}\left\{2\left[Z_{ij} - \not{f}_{ij;k}Z^k\right]Z^iZ^j - 2\left[Z_{i\alpha} - \not{f}_{i\alpha;k}Z^k\right]Z^iZ^\alpha - \frac{1}{2}(1+\alpha)\not{t}_{ijk}Z^iZ^jZ^k\right.$$

$$+\frac{1}{2}(1+3\alpha)\not{t}_{ij\alpha}Z^iZ^jZ^\alpha - \left[\not{f}_{\alpha\beta;i} + \frac{1}{2}(1+3\alpha)\not{t}_{i\alpha\beta}\right]Z^iZ^\alpha Z^\beta$$

$$\left.+\left[\not{f}_{\alpha\beta;\gamma} + \frac{1}{2}(1+\alpha)\not{t}_{\alpha\beta\gamma}\right]Z^\alpha Z^\beta Z^\gamma\right\}$$

$$n^{-1}\left\{\left[Z_{ijk} - 3\not{f}_{ij;l}i^{l,m}Z_{km} + (3\not{f}_{ij;m}i^{m,n}\not{t}_{kln} - \not{f}_{ijk;l})Z^l\right]Z^iZ^jZ^k\right.$$

$$-\left[Z_{ij\alpha} - 3\not{f}_{ij;l}i^{l,m}Z_{m\alpha} + (3\not{f}_{ij;m}i^{m,n}\not{t}_{ln\alpha} - \not{f}_{ij\alpha;l})Z^l\right]Z^iZ^jZ^\alpha$$

$$+3i^{j,l}\left[Z_{ij} - \not{f}_{ij;k}Z^k\right]\left[Z_{lm} - \not{f}_{lm;n}Z^n\right]Z^iZ^m$$

$$-2i^{j,l}\left[Z_{ij} - \not{f}_{ij;k}Z^k\right]\left[Z_{l\alpha} - \not{f}_{l\alpha;n}Z^n\right]Z^iZ^\alpha$$

$$-2\nu^{\beta,\gamma}\left[Z_{i\beta} - \not{f}_{i\beta;k}Z^k\right]\left[Z_{\alpha\gamma} - \not{f}_{\alpha\gamma;n}Z^n\right]Z^iZ^\alpha$$

$$+\nu^{\gamma,\delta}\left[Z_{\alpha\gamma} - \not{f}_{\alpha\gamma;i}Z^i\right]\left[Z_{\beta\delta} - \not{f}_{\beta\delta;j}Z^j\right]Z^\alpha Z^\beta$$

$$-\frac{1}{2}(5+3\alpha)\not{t}_{ijl}i^{l,m}\left[Z_{km} - \not{f}_{km;n}Z^n\right]Z^iZ^jZ^k$$

$$+(1+3\alpha)\not{t}_{ik\alpha}i^{k,l}\left[Z_{jl} - \not{f}_{jl;m}Z^m\right]Z^iZ^jZ^\alpha$$

$$-\left(\frac{1}{2}(1+3\alpha)\not{t}_{j\alpha\beta} + \not{f}_{\alpha\beta;j}\right)i^{j,k}\left[Z_{ik} - \not{f}_{ik;l}Z^l\right]Z^iZ^\alpha Z^\beta$$

$$+i^{k,l}\not{t}_{ijk}\left[Z_{l\alpha} - \not{f}_{l\alpha;m}Z^m\right]Z^iZ^jZ^\alpha - 2\nu^{\beta,\gamma}\not{f}_{\alpha\beta;i}\left[Z_{j\gamma} - \not{f}_{j\gamma;k}Z^k\right]Z^iZ^jZ^\alpha$$

$$+\nu^{\gamma,\delta}\left(\not{t}_{\alpha\beta\gamma} + \not{f}_{\alpha\beta;\gamma} + 2\not{f}_{\alpha\gamma;\beta}\right)\left[Z_{i\delta} - \not{f}_{i\delta;j}Z^j\right]Z^iZ^\alpha Z^\beta$$

$$+\frac{1}{2}(1+3\alpha)\nu^{\beta,\gamma}\not{t}_{ij\beta}\left[Z_{\alpha\gamma} - \not{f}_{\alpha\gamma;k}Z^k\right]Z^iZ^jZ^\alpha$$

$$-(1+3\alpha)\nu^{\gamma,\delta}\not{t}_{i\alpha\gamma}\left[Z_{\beta\delta} - \not{f}_{\beta\delta;j}Z^j\right]Z^iZ^\alpha Z^\beta$$

$$+\frac{1}{2}(1+3\alpha)\nu^{\delta,\epsilon}\not{t}_{\alpha\beta\delta}\left[Z_{\gamma\epsilon} - \not{f}_{\gamma\epsilon;i}Z^i\right]Z^\alpha Z^\beta Z^\gamma$$

$$+\left[(\frac{9}{16} + \frac{5}{8}\alpha + \frac{1}{16}\alpha^2)\not{t}_{ijm}\not{t}_{kln}i^{m,n} + i^{m,n}\not{f}_{ij;m}\not{f}_{kl;n} - \not{f}_{ij;kl}\right.$$

$$\left.+\frac{1}{2}(1+\alpha)i^{m,n}\not{f}_{ij;m}\not{t}_{kln} + \frac{1}{6}(1+\alpha)\left(\not{f}_{ijk;l} - \not{f}_{l;ijk}\right)\right]Z^iZ^jZ^kZ^l$$

$$+\left[(-\frac{3}{4} - \alpha - \frac{1}{4}\alpha^2)i^{l,m}\not{t}_{ijl}\not{t}_{km\alpha} + \frac{1}{4}(1+\alpha)\left(i^{l,m}\not{t}_{ijl}\not{f}_{k\alpha;m}\{3\}_{jk\alpha} - \not{f}_{i;jk\alpha} + \not{f}_{ij\alpha;k}\right)\right.$$

$$
\begin{aligned}
& -\frac{5}{12}(1+\alpha)\left(3i^{l,m}\,\not{t}_{il\alpha}\not{f}_{jk;m}-\not{f}_{\alpha;ijk}+\not{f}_{ijk;\alpha}\right)-2i^{l,m}\not{f}_{i\alpha;l}\not{f}_{m;jk}+2\not{f}_{i\alpha;jk} \\
& +\frac{1}{2}(1+3\alpha)\nu^{\beta,\gamma}\overset{1}{H}_{i;\alpha,\beta}\,\not{t}_{jk\gamma}\Big]\,Z^iZ^jZ^kZ^\alpha \\
& +\Big[\left(\frac{1}{8}-\frac{1}{4}\alpha+\frac{1}{8}\alpha^2\right)i^{k,l}\,\not{t}_{ijk}\,\not{t}_{l\alpha\beta}+\left(\frac{1}{4}-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2\right)i^{k,l}\,\not{t}_{ik\alpha}\,\not{t}_{jl\beta} \\
& +\frac{1}{4}(1+3\alpha)i^{k,l}\,\not{t}_{ijk}\overset{-1}{H}_{l;\alpha,\beta}+\frac{1}{2}i^{k,l}\not{f}_{ij;k}\not{f}_{l;\alpha\beta}-\frac{1}{2}\not{f}_{ij;\alpha\beta}+\frac{1}{2}i^{k,l}\not{f}_{\alpha\beta;k}\not{f}_{l;ij}-\frac{1}{2}\not{f}_{\alpha\beta;ij} \\
& +i^{k,l}\not{f}_{i\alpha;k}\not{f}_{l;j\beta}-\not{f}_{i\alpha;j\beta}+\frac{1}{2}\alpha\left(i^{k,l}\,\not{t}_{ik\alpha}\not{f}_{j\beta;l}\{3\}_{ij\beta}-\not{f}_{\alpha;ij\beta}+\not{f}_{ij\beta;\alpha}\right) \\
& -\alpha\left(i^{k,l}\,\not{t}_{ijk}\not{f}_{\alpha\beta;l}\{3\}_{j\alpha\beta}-\not{f}_{i;j\alpha\beta}+\not{f}_{j\alpha\beta;i}\right)-(1+3\alpha)\nu^{\gamma,\delta}\overset{1}{H}_{i;\alpha,\gamma}\,\not{t}_{j\beta\delta} \\
& -\nu^{\gamma,\delta}\overset{1}{H}_{i;\alpha,\gamma}\overset{1}{H}_{j;\beta,\delta}\Big]\,Z^iZ^jZ^\alpha Z^\beta \\
& +\Big[\left(\frac{1}{4}+2\alpha-\frac{1}{4}\alpha^2\right)i^{j,k}\,\not{t}_{ij\alpha}\,\not{t}_{k\beta\gamma}-\frac{1}{2}(1+3\alpha)i^{j,k}\,\not{t}_{ij\alpha}\overset{1}{H}_{k;\beta,\gamma} \\
& -i^{j,k}\not{f}_{i\alpha;j}\not{f}_{k;\beta\gamma}+\not{f}_{i\alpha;\beta\gamma}+\frac{1}{4}(1+\alpha)\left(i^{j,k}\,\not{t}_{ij\alpha}\not{f}_{\beta\gamma;k}\{3\}_{i\beta\gamma}-\not{f}_{\alpha;i\beta\gamma}+\not{f}_{i\alpha\beta;\gamma}\right) \\
& -\frac{1}{12}(1-7\alpha)\left(3i^{j,k}\,\not{t}_{ij\alpha}\not{f}_{\beta\gamma;k}-\not{f}_{i;\alpha\beta\gamma}+\not{f}_{\alpha\beta\gamma;i}\right)+\frac{3}{2}(1+\alpha)\nu^{\delta,\epsilon}\overset{1}{H}_{i;\alpha,\delta}\,\not{t}_{\beta\gamma\epsilon} \\
& +3\nu^{\delta,\epsilon}\overset{1}{H}_{i;\alpha,\delta}\not{f}_{\beta\epsilon;\gamma}-\not{f}_{\alpha\beta\gamma;i}+i_{i,\delta}\nu^{\delta,\epsilon}\not{f}_{\alpha\beta\gamma;\epsilon}\Big]\,Z^iZ^\alpha Z^\beta Z^\gamma \\
& +\Big[\left(-\frac{7}{16}-\frac{7}{8}\alpha+\frac{1}{16}\alpha^2\right)i^{i,j}\,\not{t}_{i\alpha\beta}\,\not{t}_{j\gamma\delta}-\frac{1}{6}(1+2\alpha)(3i^{i,j}\,\not{t}_{i\alpha\beta}\not{f}_{\gamma\delta;j}-\not{f}_{\alpha;\beta\gamma\delta}+\not{f}_{\alpha\beta\gamma;\delta}) \\
& +\frac{1}{4}(1+3\alpha)i^{i,j}\overset{-1}{H}_{i;\alpha,\beta}\,\not{t}_{j\gamma\delta}+\frac{1}{4}i^{i,j}\overset{-1}{H}_{i;\alpha,\beta}\overset{-1}{H}_{j;\gamma,\delta}\Big]\,Z^\alpha Z^\beta Z^\gamma Z^\delta\Big\}+O(n^{-\frac{3}{2}}) \\
= \; & \overset{\alpha}{\tilde{C}}{}^{i,j}Y_iY_j+n^{-\frac{1}{2}}\overset{\alpha}{\tilde{C}}{}^{i,j,K}Y_iY_jY_K+n^{-1}\overset{\alpha}{\tilde{C}}{}^{i,j,K,L}Y_iY_jY_KY_L+O(n^{-\frac{3}{2}}), \qquad (3.5.73)
\end{aligned}
$$

where

$$\overset{\alpha}{\tilde{\tilde{C}}}{}^{i,j} = i^{i,j}-\delta^{i,j}_{\alpha,\beta}\,\nu^{\alpha,\beta} \qquad (3.5.74)$$

$$\overset{\alpha}{\tilde{\tilde{C}}}{}^{i,j,kl} = 2i^{i,k}i^{j,l}-2\delta^{j,l}_{\alpha,\beta}\,i^{i,k}\nu^{\alpha,\beta} \qquad (3.5.75)$$

$$
\begin{aligned}
\overset{\alpha}{\tilde{\tilde{C}}}{}^{i,j,k} = \; & i^{i,l}i^{j,m}i^{k,n}\left[-\frac{1}{2}(1+\alpha)\,\not{t}_{lmn}\right]+\delta^{k}_{\alpha}\,i^{i,l}i^{j,m}\nu^{\alpha,\beta}\left[\frac{1}{2}(1+3\alpha)\,\not{t}_{lm\beta}\right] \\
& +\delta^{j,k}_{\alpha,\beta}\,i^{i,l}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[-\frac{1}{2}(1+3\alpha)\,\not{t}_{l\gamma\delta}-\overset{1}{H}_{l;\gamma,\delta}\right] \\
& +\delta^{i,j,k}_{\alpha,\beta,\gamma}\,\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\frac{1}{2}(1+\alpha)\,\not{t}_{\delta\epsilon\psi}\right] \qquad (3.5.76)
\end{aligned}
$$

$$\overset{\alpha}{\tilde{\tilde{C}}}{}^{i,j,k,lmn} = i^{i,l}i^{j,m}i^{k,n}-\delta^{k,n}_{\alpha,\beta}\,i^{i,l}i^{j,m}\nu^{\alpha,\beta} \qquad (3.5.77)$$

$$\overset{\alpha}{\tilde{\tilde{C}}}{}^{i,j,kl,mn} = 3i^{i,k}i^{j,m}i^{l,n}-2\delta^{j,n}_{\alpha,\beta}\,i^{ik}i^{l,m}\nu^{\alpha,\beta}-2\delta^{j,l,m,n}_{\alpha,\beta,\gamma,\delta}\,i^{i,k}\nu^{\alpha,\gamma}\nu^{\beta,\delta}$$

$$+\delta^{i,j,k,l,m,n}_{\alpha,\beta,\gamma,\delta,\epsilon,\psi}\, \nu^{\alpha,\gamma}\nu^{\beta,\epsilon}\nu^{\delta,\psi} \tag{3.5.78}$$

$$\begin{aligned}
\overset{\alpha}{\tilde{C}}{}^{i,j,k,lm} = {} & i^{i,n}i^{j,p}i^{k,l}i^{m,q}\left[-\frac{1}{2}(5+3\alpha)\,\not{t}_{npq}\right] \\
& +\delta^{k}_{\alpha}\, i^{i,n}i^{j,l}i^{m,p}\nu^{\alpha,\beta}\left[(1+3\alpha)\,\not{t}_{np\beta}\right] \\
& +\delta^{j,k}_{\alpha,\beta} i^{i,l}i^{m,n}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[-\frac{1}{2}(1+3\alpha)\,\not{t}_{n\gamma\delta} - \overset{1}{H}_{n;\gamma,\delta}\right] \\
& +\delta^{k,m}_{\alpha,\beta}\, i^{i,n}\left[i^{j,p}i^{l,q}\nu^{\alpha,\beta}\,\not{t}_{npq} - 2i^{j,l}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\,\overset{1}{H}_{n;\gamma,\delta}\right] \\
& +\delta^{j,k,m}_{\alpha,\beta,\gamma}\, i^{i,l}\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\not{t}_{\delta\epsilon\psi}\right] \\
& +\delta^{k,l,m}_{\alpha,\beta,\gamma}\, i^{i,n}i^{j,p}\nu^{\alpha,\beta}\nu^{\gamma,\delta}\left[\frac{1}{2}(1+3\alpha)\,\not{t}_{np\delta}\right] \\
& +\delta^{j,k,l,m}_{\alpha,\beta,\gamma,\delta}\, i^{i,n}\nu^{\alpha,\gamma}\nu^{\beta,\epsilon}\nu^{\delta,\psi}\left[-(1+3\alpha)\,\not{t}_{n\epsilon\psi}\right] \\
& +\delta^{i,j,k,l,m}_{\alpha,\beta,\gamma,\delta,\epsilon}\, \nu^{\alpha,\delta}\nu^{\beta,\psi}\nu^{\gamma,\phi}\nu^{\epsilon,\chi}\left[\frac{1}{2}(1+3\alpha)\,\not{t}_{\psi\phi\chi}\right]
\end{aligned} \tag{3.5.79}$$

$$\begin{aligned}
\overset{\alpha}{\tilde{C}}{}^{i,j,k,l} = {} & i^{i,m}i^{j,n}i^{k,p}i^{l,q}\left[\left(\frac{9}{16}+\frac{5}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{mnpq} - \frac{1}{6}(1+\alpha)\,\not{t}_{m;npq} - \not{t}_{mn;pq}\right] \\
& +\delta^{l}_{\alpha}\, i^{i,m}i^{j,n}i^{k,p}\nu^{\alpha,\beta}\left[\left(-\frac{3}{4}-\alpha-\frac{1}{4}\alpha^2\right)\tau_{mnp\beta} - \frac{1}{4}(1+\alpha)\,\not{t}_{m;np\beta}\right. \\
& \left. +\frac{5}{12}(1+\alpha)\,\not{t}_{\beta;mnp} + 2\,\not{t}_{m\beta;np} + \frac{1}{2}(1+3\alpha)\,\overset{1}{\tau}_{m;\beta,np}\right] \\
& +\delta^{k,l}_{\alpha,\beta}\, i^{i,m}i^{j,n}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[\left(\frac{1}{8}-\frac{1}{4}\alpha+\frac{1}{8}\alpha^2\right)\tau_{mn\gamma\delta}\right. \\
& + \left(\frac{1}{4}-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2\right)\tau_{m\gamma n\delta} + \left(\frac{1}{4}+\frac{3}{4}\alpha\right)\overset{-1}{\tau}_{\gamma,\delta,mn} - \frac{1}{2}\,\not{t}_{mn;\gamma\delta} - \frac{1}{2}\,\not{t}_{\gamma\delta;mn} \\
& \left. - \not{t}_{m\gamma;n\delta} - \frac{1}{2}\alpha\,\not{t}_{\gamma;mn\delta} + \alpha\,\not{t}_{m;n\gamma\delta} - (1+3\alpha)\,\overset{1}{\tau}_{m;\gamma,n\delta} - \overset{1}{\tau}_{m;n;\gamma,\delta}\right] \\
& +\delta^{j,k,l}_{\alpha,\beta,\gamma}\, i^{i,m}\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\left(\frac{1}{4}+2\alpha-\frac{1}{4}\alpha^2\right)\tau_{m\delta\epsilon\psi} - \frac{1}{2}(1+3\alpha)\,\overset{-1}{\tau}_{\delta,\epsilon,m\psi}\right. \\
& + \not{t}_{m\delta;\epsilon\psi} - \frac{1}{4}(1+\alpha)\,\not{t}_{\delta;m\epsilon\psi} + \frac{1}{12}(1-7\alpha)\,\not{t}_{m;\delta\epsilon\psi} \\
& \left. +\frac{3}{2}(1+\alpha)\,\overset{1}{\tau}_{m;\delta,\epsilon\psi} - \overset{1}{\tau}_{m;\delta\epsilon\psi}\right] \\
& +\delta^{i,j,k,l}_{\alpha,\beta,\gamma,\delta}\, \nu^{\alpha,\epsilon}\nu^{\beta,\psi}\nu^{\gamma,\phi}\nu^{\delta,\chi}\left[\left(-\frac{7}{16}-\frac{7}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{\epsilon\psi\phi\chi}\right. \\
& \left. +\frac{1}{4}(1+3\alpha)\,\overset{-1}{\tau}_{\epsilon,\psi,\phi\chi} + \frac{1}{4}\,\overset{-1}{\tau}_{\epsilon,\psi,\phi,\chi} + \frac{1}{6}(1+2\alpha)\,\not{t}_{\epsilon;\psi\phi\chi}\right].
\end{aligned} \tag{3.5.80}$$

By an argument similar to Remark 11 in Section 3.5.3 we observe that, under the null hypothesis, $\overset{\alpha}{\widetilde{\mathcal{W}}}$ has a χ^2 distribution with p degrees of freedom with error of order $O(n^{-\frac{1}{2}})$. Also, the coefficients $\overset{\alpha}{\tilde{C}}{}^{i,j,K}$ and $\overset{\alpha}{\tilde{C}}{}^{i,j,K,L}$ in the expansion of the modified Wald statistic simplify in the same way as $\overset{\alpha}{C}{}^{i,j,K}$ and $\overset{\alpha}{C}{}^{i,j,K,L}$ in Remark 12 in Section 3.5.3.

3.5.5 Bartlett-type Adjustment for $\overset{\alpha}{\mathcal{W}}$

To produce the Bartlett-type adjustment (3.1.5), we need to find A_1, A_2 and A_3 in (3.1.6). We use (3.5.64), (3.4.36) and the approximation $e^x \simeq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ when x is close to zero, and write $\overset{\alpha}{w}{}^0(Y_*)$, $\overset{\alpha}{w}{}^1(Y_*)$ and $\overset{\alpha}{w}{}^2(Y_*)$ for $\overset{\alpha}{C}{}^{i,j}Y_iY_j$, $\overset{\alpha}{C}{}^{i,j,K}Y_iY_jY_K$ and $\overset{\alpha}{C}{}^{i,j,K,L}Y_iY_jY_KY_L$, respectively, where $\overset{\alpha}{C}{}^{i,j}$, $\overset{\alpha}{C}{}^{i,j,K}$ and $\overset{\alpha}{C}{}^{i,j,K,L}$ are defined in (3.5.65)–(3.5.71). We find that

$$
\begin{aligned}
M_{\overset{\alpha}{\mathcal{W}}}(t) &= \int e^{\left(\overset{\alpha}{w}{}^0(y_*)+n^{-\frac{1}{2}}\left\{\overset{\alpha}{w}{}^1(y_*)\right\}+n^{-1}\left\{\overset{\alpha}{w}{}^2(y_*)\right\}\right)t}\varphi(y_*;\lambda) \\
&\quad \times\left(1+n^{-\frac{1}{2}}\left\{Q_1(y_*)\right\}+n^{-1}\left\{Q_2(y_*)\right\}\right)dy_*+O(n^{-\frac{3}{2}}) \\
&= \int |2\pi\lambda|^{-\frac{1}{2}}e^{i^{i,j}y_iy_jt-\frac{1}{2}y_*\lambda^{-1}y_*^T}e^{\left(n^{-\frac{1}{2}}\left\{\overset{\alpha}{w}{}^1(y_*)\right\}+n^{-1}\left\{\overset{\alpha}{w}{}^2(y_*)\right\}\right)t} \\
&\quad \times\left(1+n^{-\frac{1}{2}}\left\{Q_1(y_*)\right\}+n^{-1}\left\{Q_2(y_*)\right\}\right)dy_*+O(n^{-\frac{3}{2}}) \\
&= \int |2\pi\lambda|^{-\frac{1}{2}}e^{-\frac{1}{2}y_*(\lambda^{-1}-2tK^{-1})y_*^T} \\
&\quad \times\left(1+n^{-\frac{1}{2}}\left\{\overset{\alpha}{w}{}^1(y_*)t\right\}+n^{-1}\left\{\frac{1}{2}\left(\overset{\alpha}{w}{}^1(y_*)t\right)^2+\overset{\alpha}{w}{}^2(y_*)t\right\}\right) \\
&\quad \times\left(1+n^{-\frac{1}{2}}\left\{Q_1(y_*)\right\}+n^{-1}\left\{Q_2(y_*)\right\}\right)dy_*+O(n^{-\frac{3}{2}}) \\
&= \left(\frac{|V|}{|\lambda|}\right)^{\frac{1}{2}}\int |2\pi V|^{-\frac{1}{2}}e^{-\frac{1}{2}y_*V^{-1}y_*^T}\left(1+n^{-\frac{1}{2}}\left\{\overset{\alpha}{w}{}^1(y_*)t+Q_1(y_*)\right\}\right. \\
&\quad \left.+n^{-1}\left\{\frac{1}{2}\left(\overset{\alpha}{w}{}^1(y_*)t\right)^2+\overset{\alpha}{w}{}^1(y_*)t+Q_1(y_*)\overset{\alpha}{w}{}^1(y_*)t+Q_2(y_*)\right\}\right)dy_*+O(n^{-\frac{3}{2}}) \\
&= \left(\frac{|V|}{|\lambda|}\right)^{\frac{1}{2}}\left(1+n^{-\frac{1}{2}}\left\{\mathbb{E}\left[\overset{\alpha}{w}{}^1(Y_*)\right]t+\mathbb{E}\left[Q_1(Y_*)\right]\right\}\right. \\
&\quad \left.+n^{-1}\left\{\frac{1}{2}\mathbb{E}\left[\left(\overset{\alpha}{w}{}^1(Y_*)t\right)^2\right]+\mathbb{E}\left[\overset{\alpha}{w}{}^2(Y_*)\right]t+\mathbb{E}\left[Q_1(Y_*)\overset{\alpha}{w}{}^1(Y_*)\right]t+\mathbb{E}\left[Q_2(Y_*)\right]\right\}\right) \\
&\quad +O(n^{-\frac{3}{2}}), \qquad (3.5.81)
\end{aligned}
$$

where

$$
K^{-1} = \begin{bmatrix} i^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad (3.5.82)
$$

and $V^{-1} = \lambda^{-1} - 2tK^{-1}$. Denote the generalised inverse of K^{-1} by K, that is

$$
K = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix},
$$

and let

$$V = \lambda + dK, \tag{3.5.83}$$

where d is defined in (3.1.7). Observe that by Remark 9 in Section 3.4

$$\begin{aligned} VV^{-1} &= \left(\lambda + \frac{2t}{1-2t}K\right)\left(\lambda^{-1} - 2tK^{-1}\right) \\ &= \begin{bmatrix} \left(1 + \frac{2t}{1-2t} - 2t - \frac{4t^2}{1-2t}\right)\mathbb{I}_r & 0 \\ 0 & \mathbb{I}_{r^2+r^3} \end{bmatrix} \\ &= \mathbb{I}_{r+r^2+r^3}, \end{aligned}$$

where $\mathbb{I}_s$ denotes the $s \times s$-identity matrix. Thus, V is the inverse of V^{-1}. Note that $|V| = (1-2t)^{-p}|\lambda|$, and hence (3.5.81) is of the form of (3.1.6).

In (3.5.81) we are taking the expectation with respect to a normal distribution with zero mean and variance $V = \lambda + dK$. Recall that the notation suppresses dependence of the Q_1 and Q_2 on λ, i.e. $Q_1 = Q_1(Y_*;\lambda)$ and $Q_2 = Q_2(Y_*;\lambda)$, where $Y_* \sim N_{p+p^2+p^3}(0, \lambda + dK)$. Using Property 3 of Hermite polynomials given in (1.4.45), (1.4.37)–(1.4.41), (3.4.41) and symmetry when appropriate, and recalling that all odd moments of a central normal distribution vanish, we find that

$$\begin{aligned} \mathbb{E}\left[\overset{\alpha}{w}{}^1(Y_*)\right]t &= \overset{\alpha}{C}{}^{ijK}\mathbb{E}\left[Y_iY_jY_K\right]t \\ &= 0, \end{aligned} \tag{3.5.84}$$

and

$$\begin{aligned} \mathbb{E}\left[Q_1(Y_*;\lambda)\right] &= \frac{1}{6}\lambda^{I,J,K}\mathbb{E}\left[h_{IJK}(Y_*;\lambda)\right] \\ &= 0. \end{aligned} \tag{3.5.85}$$

Thus, the $O(n^{-\frac{1}{2}})$-term of (3.5.81) vanishes. Furthermore, we find that

$$\begin{aligned} \mathbb{E}\left[Q_2(Y_*;\lambda)\right] &= \frac{1}{24}\lambda^{I,J,K,L}\mathbb{E}\left[h_{IJKL}(Y_*;\lambda)\right] + \frac{1}{72}\lambda^{I,J,K}\lambda^{L,M,N}\mathbb{E}\left[h_{IJKLMN}(Y_*;\lambda)\right] \\ &= \frac{1}{8}\lambda^{i,j,k,l}i^{i,j}i^{k,l}d^2 + \frac{1}{24}\left(3\not{t}_{ijk}\not{t}_{lmn} + 2\not{t}_{ijm}\not{t}_{kln}\right)i^{i,j}i^{k,l}i^{m,n}d^3 \\ &= \frac{1}{24}\left[D_{11}d + D_{12}d^2 + D_{13}d^3\right], \end{aligned} \tag{3.5.86}$$

where

$$\begin{aligned}
D_{11} &= 0 \\
D_{12} &= 3\lambda_{i,j,k,l} i^{i,j} i^{k,l} \\
D_{13} &= (3\tau_{ijkl} + 2\tau_{ikjl})\, i^{i,j} i^{k,l}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbb{E}\left[\overset{\alpha}{w}{}^{1}(Y_*) Q_1(Y_*;\lambda)\right] t &= \overset{\alpha}{C}{}^{i,j,K} \frac{1}{6} \lambda^{L,M,N} \mathbb{E}\left[Y_i Y_j Y_K h_{LMN}\right] \left(\frac{1}{2}(d - d^2 + d^3) + O(d^4)\right) \\
&= \frac{1}{6} \overset{\alpha}{C}{}^{i,j,K} \lambda^{L,M,N} \left[K_{i,j} K_{K,L} K_{M,N}\{15\} + K_{i,j} K_{K,L} \lambda_{M,N}\{45\}\right. \\
&\quad + K_{i,j} \lambda_{K,L} \lambda_{M,N}\{45\} + \lambda_{i,j} \lambda_{K,L} \lambda_{M,N}\{15\} \\
&\quad \left. - \left(K_{i,j} K_{K,L}\{3\} + K_{i,j} \lambda_{K,L}\{6\} + \lambda_{i,j} \lambda_{K,L}\{3\}\right) \lambda_{M,N}\{3\}_{L,M,N}\right] \\
&\quad \times \left(\frac{1}{2}(d - d^2 + d^3) + O(d^4)\right) \\
&= \frac{1}{24} \left\{ 12 \left[\overset{\alpha}{C}{}^{i,j,kl} \lambda^{m,n,P} \lambda_{kl,P} i_{i,m} i_{j,n} + \overset{\alpha}{C}{}^{i,j,k} \lambda^{l,m,n} i_{i,l} i_{j,m} i_{k,n}\right] d \right. \\
&\quad + 6 \left[\overset{\alpha}{C}{}^{i,j,kl} \lambda^{m,n,P} \lambda_{kl,P} \left(i_{i,m} i_{j,n}\{3\}\right)\right. \\
&\quad \left. + \overset{\alpha}{C}{}^{i,j,k} \lambda^{l,m,n} i_{i,l} \left(3 i_{j,k} i_{m,n} + 4 i_{j,m} i_{k,n}\right)\right] d^2 \\
&\quad \left. + 6 \left[\overset{\alpha}{C}{}^{i,j,k} \lambda^{l,m,n} i_{i,l} \left(3 i_{j,k} i_{m,n} + 2 i_{j,m} i_{k,n}\right)\right] d^3 + O(d^4) \right\} \\
&\quad + O(n^{-\frac{1}{2}}) \\
&= \frac{1}{24} \left[D_{21} d + D_{22} d^2 + D_{23} d^3 + O(d^4)\right] + O(n^{-\frac{1}{2}}), \qquad (3.5.87)
\end{aligned}$$

where

$$\begin{aligned}
D_{21} &= 12 \overset{\alpha}{C}{}^{i,j,kl} \lambda_{i,jkl} + 12 \overset{\alpha}{C}{}^{i,j,k} \not{t}_{ijk} \\
D_{22} &= 12 \overset{\alpha}{C}{}^{i,j,kl} \lambda_{i,jkl} + 6 \overset{\alpha}{C}{}^{i,j,kl} i^{m,n} \lambda_{kl,m,n} i_{i,j} \\
&\quad + 24 \overset{\alpha}{C}{}^{i,j,k} \not{t}_{ijk} + 18 \overset{\alpha}{C}{}^{i,j,k} i^{l,m} \not{t}_{ilm} i_{j,k} \\
D_{23} &= 12 \overset{\alpha}{C}{}^{i,j,k} \not{t}_{ijk} + 18 \overset{\alpha}{C}{}^{i,j,k} i^{l,m} \not{t}_{ilm} i_{j,k}.
\end{aligned}$$

Further,

$$\begin{aligned}
\mathbb{E}\left[\overset{\alpha}{w}{}^{2}(Y_*)\right] t &= \mathbb{E}\left[\overset{\alpha}{C}{}^{i,j,K,L} \mathbf{Y}_i \mathbf{Y}_j \mathbf{Y}_K \mathbf{Y}_L\right] \\
&\quad \times \left(\frac{1}{2}(d - d^2 + d^3) + O(d^4)\right) + O(n^{-\frac{1}{2}}) \\
&= \overset{\alpha}{C}{}^{i,j,K,L} \left[K_{i,j} K_{K,L}\{3\}\, d^2 - \lambda_{i,j} K_{K,L}\{6\}\, d + \lambda_{i,j} \lambda_{K,L}\{3\}\right]
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{2}(d - d^2 + d^3) + O(d^4)\right) + O(n^{-\frac{1}{2}}) \\
&= \frac{1}{24}\left\{12\left[\overset{\alpha}{C}{}^{i,j,kl,mn}\lambda_{kl,mn}i_{i,j} + \overset{\alpha}{C}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right)\right] d \right.\\
&\quad \left. + 12\overset{\alpha}{C}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right) d^2\right\} + O(d^4) \\
&= \frac{1}{24}\left[D_{31}d + D_{32}d^2 + D_{33}d^3 + O(d^4)\right],
\end{aligned}
\tag{3.5.88}
$$

where

$$
\begin{aligned}
D_{31} &= 12\overset{\alpha}{C}{}^{i,j,kl,mn}\lambda_{kl,mn}i_{i,j} + 12\overset{\alpha}{C}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right) \\
D_{32} &= 12\overset{\alpha}{C}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right) \\
D_{33} &= 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\left(\overset{\alpha}{w}{}^1(Y_*)\right)^2\right]t^2 &= \frac{1}{2}\mathbb{E}[\overset{\alpha}{C}{}^{i,j,K}\overset{\alpha}{C}{}^{l,m,N}\mathbf{Y}_i\mathbf{Y}_j\mathbf{Y}_K\mathbf{Y}_l\mathbf{Y}_m\mathbf{Y}_N]\left(\frac{1}{4}(d^2 - 2d^3) + O(d^4)\right) \\
&\quad + O(n^{-\frac{1}{2}}) \\
&= \frac{1}{2}\overset{\alpha}{C}{}^{i,j,K}\overset{\alpha}{C}{}^{l,m,N}\left[K_{i,j}K_{K,l}K_{m,N}\{15\}d^3 + K_{i,j}K_{K,l}\lambda_{m,N}\{45\}d^2\right. \\
&\quad \left. + K_{i,j}\lambda_{K,l}\lambda_{m,N}\{45\}d + \lambda_{i,j}\lambda_{K,l}\lambda_{m,N}\{15\}\right] \\
&\quad \times \left(\frac{1}{4}(d^2 - 2d^3) + O(d^4)\right) + O(n^{-\frac{1}{2}}) \\
&= \frac{1}{24}\left\{3\left[\overset{\alpha}{C}{}^{i,j,kl}\overset{\alpha}{C}{}^{m,n,pq}\lambda_{kl,pq}\left(i_{i,j}i_{m,n}\{3\}\right)\right.\right. \\
&\quad \left. + \overset{\alpha}{C}{}^{i,j,k}\overset{\alpha}{C}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right)\right] d^2 \\
&\quad \left. + 3\,\overset{\alpha}{C}{}^{i,j,k}\overset{\alpha}{C}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right)d^3 + O(d^4)\right\} \\
&= \frac{1}{24}\left[D_{41}d + D_{42}d^2 + D_{43}d^3 + O(d^4)\right],
\end{aligned}
\tag{3.5.89}
$$

where

$$
\begin{aligned}
D_{41} &= 0 \\
D_{42} &= 3\,\overset{\alpha}{C}{}^{i,j,kl}\overset{\alpha}{C}{}^{m,n,pq}\lambda_{kl,pq}\left(i_{i,j}i_{m,n}\{3\}\right) + 3\,\overset{\alpha}{C}{}^{i,j,k}\overset{\alpha}{C}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right) \\
D_{43} &= 3\,\overset{\alpha}{C}{}^{i,j,k}\overset{\alpha}{C}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right).
\end{aligned}
$$

Then, substituting (3.5.86)–(3.5.89) into (3.5.81) yields

$$
M_{\overset{\alpha}{\mathcal{W}}}(t) = \mathbb{E}\left[e^{\overset{\alpha}{\mathcal{W}}t}\right]
$$

$$= (1-2t)^{-\frac{p}{2}}\left(1+(24n)^{-1}\left\{A_1 d + A_2 d^2 + A_3 d^3 + O(d^4)\right\}\right)$$
$$+O(n^{-\frac{3}{2}}), \tag{3.5.90}$$

as in (3.1.6), where

$$\begin{aligned}
A_1 &= D_{11} + D_{21} + D_{31} + D_{41} \\
&= 12\overset{\alpha}{C}{}^{i,j,kl}\lambda_{i,j,kl} + 12\overset{\alpha}{C}{}^{i,j,k}\not{t}_{ijk} \\
&\quad +12\overset{\alpha}{C}{}^{i,j,kl,mn}\lambda_{kl,mn}i_{i,j} + 12\overset{\alpha}{C}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right),
\end{aligned}$$

$$\begin{aligned}
A_2 &= 3\lambda_{i,j,k,l}i^{i,j}i^{k,l} + 12\overset{\alpha}{C}{}^{i,j,kl}\lambda_{i,j,kl} + 6\overset{\alpha}{C}{}^{i,j,kl}i^{m,n}\lambda_{m,n,kl}i_{i,j} \\
&\quad +24\overset{\alpha}{C}{}^{i,j,k}\not{t}_{ijk} + 18\overset{\alpha}{C}{}^{i,j,k}i^{l,m}\not{t}_{ilm}i_{j,k} + 12\overset{\alpha}{C}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right) \\
&\quad +3\overset{\alpha}{C}{}^{i,j,kl}\overset{\alpha}{C}{}^{m,n,pq}\lambda_{kl,pq}\left(i_{i,j}i_{m,n}\{3\}\right) \\
&\quad +3\overset{\alpha}{C}{}^{i,j,k}\overset{\alpha}{C}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right)
\end{aligned}$$

$$\begin{aligned}
A_3 &= \left(3\tau_{ijkl} + 2\tau_{ikjl}\right)i^{i,j}i^{k,l} + 12\overset{\alpha}{C}{}^{i,j,k}\not{t}_{ijk} + 18\overset{\alpha}{C}{}^{i,j,k}i^{l,m}\not{t}_{ilm}i_{j,k} \\
&\quad +3\overset{\alpha}{C}{}^{i,j,k}\overset{\alpha}{C}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right).
\end{aligned}$$

From Proposition 1 in Section 3.1.2 and Remark 11 in Section 3.5.3 we find that the Bartlett-type adjusted geometric Wald statistic $\overset{\alpha}{\mathcal{W}}{}'$ has a χ^2_p distribution with error of order $O(n^{-\frac{3}{2}})$, where

$$\overset{\alpha}{\mathcal{W}}{}' = \left\{1 - \frac{1}{n}\left(c + b\,\overset{\alpha}{\mathcal{W}} + a\overset{\alpha}{\mathcal{W}}{}^2\right)\right\}\overset{\alpha}{\mathcal{W}}, \tag{3.5.91}$$

with

$$\begin{aligned}
a &= \frac{A_3}{12p(p+2)(p+4)} \\
&= \frac{1}{12p(p+2)(p+4)}\left\{\left(3\tau_{ijkl} + 2\tau_{ikjl}\right)i^{i,j}i^{k,l} + 12\overset{\alpha}{C}{}^{i,j,k}\not{t}_{ijk}\right. \\
&\quad \left.+18\overset{\alpha}{C}{}^{i,j,k}i^{l,m}\not{t}_{ilm}i_{j,k} + 3\overset{\alpha}{C}{}^{i,j,k}\overset{\alpha}{C}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right)\right\}
\end{aligned} \tag{3.5.92}$$

$$\begin{aligned}
b &= \frac{A_2 - 2A_3}{12p(p+2)} \\
&= \frac{1}{12p(p+2)}\left\{3\lambda_{i,j,k,l}i^{i,j}i^{k,l} - \left(6\tau_{ijkl} + 4\tau_{ikjl}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+12\overset{\alpha}{C}^{i,j,kl}\lambda_{i,j,kl} + 6\overset{\alpha}{C}^{i,j,kl}i^{m,n}\lambda_{m,n,kl}i_{i,j} \\
&-18\overset{\alpha}{C}{}^{i,j,k}i^{l,m}\not{t}_{ilm}i_{j,k} + 12\overset{\alpha}{C}{}^{i,j,k,l}(i_{i,j}i_{k,l}\{3\}) \\
&+3\overset{\alpha}{C}{}^{i,j,kl}\overset{\alpha}{C}{}^{m,n,pq}\lambda_{kl,pq}(i_{i,j}i_{m,n}\{3\}) \\
&-3\overset{\alpha}{C}{}^{i,j,k}\overset{\alpha}{C}{}^{l,m,n}(i_{i,j}i_{k,l}i_{m,n}\{15\})\Big\}
\end{aligned} \tag{3.5.93}
$$

$$
\begin{aligned}
c &= \frac{A_1 - A_2 + A_3}{12p} \\
&= \frac{1}{12p}\Big\{(-3\lambda_{i,j,k,l} + 3\tau_{ijkl} + 2\tau_{ikjl})i^{i,j}i^{k,l} \\
&\quad -6\overset{\alpha}{C}{}^{i,j,kl}i^{m,n}\lambda_{m,n,kl}i_{i,j} + 12\overset{\alpha}{C}{}^{i,j,kl,mn}\lambda_{kl,mn}i_{i,j} \\
&\quad -3\overset{\alpha}{C}{}^{i,j,kl}\overset{\alpha}{C}{}^{m,n,pq}\lambda_{kl,pq}(i_{i,j}i_{m,n}\{3\})\Big\}.
\end{aligned} \tag{3.5.94}
$$

Remark 13:

By Remark 12 in Section 3.5.3 the coefficients $\overset{\alpha}{C}{}^{i,j,kl}$ and $\overset{\alpha}{C}{}^{i,j,kl,mn}$ do not depend on the value of α. Thus c in (3.5.94) does not depend on α.

3.5.6 Bartlett-type Adjustment for $\overset{\alpha}{\widetilde{\mathcal{W}}}$

Using the same procedure as in the un-modified case and letting $\overset{\alpha}{\widetilde{w}}{}^0(Z_*)$, $\overset{\alpha}{\widetilde{w}}{}^1(Z_*)$ and $\overset{\alpha}{\widetilde{w}}{}^2(Z_*)$ be short for $\overset{\alpha}{\widetilde{C}}{}^{i,j}Z_iZ_j$, $\overset{\alpha}{\widetilde{C}}{}^{i,j,K}Z_iZ_jZ_K$ and $\overset{\alpha}{\widetilde{C}}{}^{i,j,K,L}Z_iZ_jZ_KZ_L$, respectively, where $\overset{\alpha}{\widetilde{C}}{}^{i,j}$, $\overset{\alpha}{\widetilde{C}}{}^{i,j,K}$ and $\overset{\alpha}{\widetilde{C}}{}^{i,j,K,L}$ are defined in (3.5.74)–(3.5.80), we find $\widetilde{A}_1$, $\widetilde{A}_2$ and $\widetilde{A}_3$ in the moment generating functions (3.1.6) for the modified geometric Wald statistics, that is

$$
\begin{aligned}
M_{\overset{\alpha}{\widetilde{\mathcal{W}}}}(t) &= \mathbb{E}\left[e^{\overset{\alpha}{\widetilde{\mathcal{W}}}t}\right] \\
&= (1-2t)^{-\frac{p}{2}}\left(1 + (24n)^{-1}\left\{\widetilde{A}_1 d + \widetilde{A}_2 d^2 + \widetilde{A}_3 d^3 + O(d^4)\right\}\right) \\
&\quad + O(n^{-\frac{3}{2}}),
\end{aligned} \tag{3.5.95}
$$

by determining coefficients $\widetilde{D}_{11}\ldots\widetilde{D}_{43}$ analogous to $D_{11}\ldots D_{43}$. Trivially,

$$
\begin{aligned}
\widetilde{D}_{11} &= D_{11} = 0 \\
\widetilde{D}_{12} &= D_{12} = 3\lambda_{i,j,k,l}i^{i,j}i^{k,l} \\
\widetilde{D}_{13} &= D_{13} = (3\tau_{ijkl} + 2\tau_{ikjl})i^{i,j}i^{k,l}.
\end{aligned}
$$

Further, we find that

$$
\begin{aligned}
\mathbb{E}\left[\overset{\alpha}{\widetilde{w}}{}^{1}(Y_*)Q_1(Y_*;\lambda)\right]t &= \overset{\alpha}{\widetilde{C}}{}^{i,j,K}\frac{1}{6}\lambda^{L,M,N}\mathbb{E}\left[Y_iY_jY_Kh_{LMN}\right]\left(\frac{1}{2}(d-d^2+d^3)+O(d^4)\right)\\
&= \frac{1}{6}\overset{\alpha}{\widetilde{C}}{}^{i,j,K}\lambda^{L,M,N}\left[K_{i,j}K_{K,L}K_{M,N}\{15\}+K_{i,j}K_{K,L}\lambda_{M,N}\{45\}\right.\\
&\quad +K_{i,j}\lambda_{K,L}\lambda_{M,N}\{45\}+\lambda_{i,j}\lambda_{K,L}\lambda_{M,N}\{15\}\\
&\quad \left.-\left(K_{i,j}K_{K,L}\{3\}+K_{i,j}\lambda_{K,L}\{6\}+\lambda_{i,j}\lambda_{K,L}\{3\}\right)\lambda_{M,N}\{3\}_{L,M,N}\right]\\
&\quad \times\left(\frac{1}{2}(d-d^2+d^3)+O(d^4)\right)\\
&= \frac{1}{24}\left\{12\left[\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\lambda^{m,n,P}\lambda_{kl,P}i_{i,m}i_{j,n}+\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\lambda^{l,m,n}i_{i,l}i_{j,m}i_{k,n}\right]d\right.\\
&\quad +6\left[\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\lambda^{m,n,P}\lambda_{kl,P}\left(i_{i,m}i_{j,n}\{3\}\right)\right.\\
&\quad \left.+\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\lambda^{l,m,n}i_{i,l}\left(3i_{j,k}i_{m,n}+4i_{j,m}i_{k,n}\right)\right]d^2\\
&\quad \left.+6\left[\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\lambda^{l,m,n}i_{i,l}\left(3i_{j,k}i_{m,n}+2i_{j,m}i_{k,n}\right)\right]d^3+O(d^4)\right\}\\
&\quad +O(n^{-\frac{1}{2}})\\
&= \frac{1}{24}\left[\widetilde{D}_{21}d+\widetilde{D}_{22}d^2+\widetilde{D}_{23}d^3+O(d^4)\right]+O(n^{-\frac{1}{2}}),
\end{aligned}
\tag{3.5.96}
$$

where

$$
\begin{aligned}
\widetilde{D}_{21} &= 12\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\lambda_{i,j,kl}+12\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{l}_{ijk}\\
\widetilde{D}_{22} &= 12\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{l}_{ijk}+6\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}i^{m,n}\lambda_{m,n,kl}i_{i,j}\\
&\quad +24\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{l}_{ijk}i_{i,l}+18\overset{\alpha}{\widetilde{C}}{}^{i,j,k}i^{l,m}\not{l}_{ilm}i_{j,k}\\
\widetilde{D}_{23} &= 12\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{l}_{ijk}i_{i,l}+18\overset{\alpha}{\widetilde{C}}{}^{i,j,k}i^{l,m}\not{l}_{ilm}i_{j,k}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathbb{E}\left[\overset{\alpha}{\widetilde{w}}{}^{2}(Y_*)\right]t &= \mathbb{E}[\overset{\alpha}{\widetilde{C}}{}^{i,j,K,L}Y_iY_jY_KY_L]\\
&\quad \times\left(\frac{1}{2}(d-d^2+d^3)+O(d^4)\right)+O(n^{-\frac{1}{2}})\\
&= \overset{\alpha}{\widetilde{C}}{}^{i,j,K,L}\left[K_{i,j}K_{K,L}\{3\}d^2-\lambda_{i,j}K_{K,L}\{6\}d+\lambda_{i,j}\lambda_{K,L}\{3\}\right]\\
&\quad \times\left(\frac{1}{2}(d-d^2+d^3)+O(d^4)\right)+O(n^{-\frac{1}{2}})\\
&= \frac{1}{24}\left\{12\left[\overset{\alpha}{\widetilde{C}}{}^{i,j,kl,mn}\lambda_{kl,mn}i_{i,j}+\overset{\alpha}{\widetilde{C}}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right)\right]d\right.
\end{aligned}
$$

$$
\begin{aligned}
& + 12\overset{\alpha}{\widetilde{C}}^{i,j,k,l}\left(i_{i,j}i_{k,l}\left\{3\right\}\right)d^2\Big\} + O(d^4) \\
=\; & \frac{1}{24}\left[\widetilde{D}_{31}d + \widetilde{D}_{32}d^2 + \widetilde{D}_{33}d^3 + O(d^4)\right],
\end{aligned}
\qquad (3.5.97)
$$

where

$$
\begin{aligned}
\widetilde{D}_{31} &= 12\overset{\alpha}{\widetilde{C}}^{i,j,kl,mn}\lambda_{kl,mn}i_{i,j} + 12\overset{\alpha}{\widetilde{C}}^{i,j,k,l}\left(i_{i,j}i_{k,l}\left\{3\right\}\right) \\
\widetilde{D}_{32} &= 12\overset{\alpha}{\widetilde{C}}^{i,j,k,l}\left(i_{i,j}i_{k,l}\left\{3\right\}\right) \\
\widetilde{D}_{33} &= 0.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\left(\overset{\alpha}{\widetilde{w}}^1(Y_*)\right)^2\right]t^2 &= \frac{1}{2}\mathbb{E}[\overset{\alpha}{\widetilde{C}}^{i,j,K}\overset{\alpha}{\widetilde{C}}^{l,m,N}Y_iY_jY_KY_lY_mY_N]\left(\frac{1}{4}(d^2-2d^3)+O(d^4)\right) \\
&\quad +O(n^{-\frac{1}{2}}) \\
&= \frac{1}{2}\overset{\alpha}{\widetilde{C}}^{i,j,K}\overset{\alpha}{\widetilde{C}}^{l,m,N}\Big[K_{i,j}K_{K,l}K_{m,N}\left\{15\right\}d^3 + K_{i,j}K_{K,l}\lambda_{m,N}\left\{45\right\}d^2 \\
&\quad + K_{i,j}\lambda_{K,l}\lambda_{m,N}\left\{45\right\}d + \lambda_{i,j}\lambda_{K,l}\lambda_{m,N}\left\{15\right\}\Big] \\
&\quad \times\left(\frac{1}{4}(d^2-2d^3)+O(d^4)\right) + O(n^{-\frac{1}{2}}) \\
&= \frac{1}{24}\Big\{3\Big[\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\overset{\alpha}{\widetilde{C}}^{m,n,pq}\lambda_{kl,pq}\left(i_{i,j}i_{m,n}\left\{3\right\}\right) \\
&\quad + \overset{\alpha}{\widetilde{C}}{}^{i,j,k}\overset{\alpha}{\widetilde{C}}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\left\{15\right\}\right)\Big]d^2 \\
&\quad + 3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\overset{\alpha}{\widetilde{C}}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\left\{15\right\}\right)d^3 + O(d^4)\Big\} \\
&= \frac{1}{24}\left[\widetilde{D}_{41}d + \widetilde{D}_{42}d^2 + \widetilde{D}_{43}d^3 + O(d^4)\right],
\end{aligned}
\qquad (3.5.98)
$$

where

$$
\begin{aligned}
\widetilde{D}_{41} &= 0 \\
\widetilde{D}_{42} &= 3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\overset{\alpha}{\widetilde{C}}^{m,n,pq}\lambda_{kl,pq}\left(i_{i,j}i_{m,n}\left\{3\right\}\right) + 3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\overset{\alpha}{\widetilde{C}}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\left\{15\right\}\right) \\
\widetilde{D}_{43} &= 3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\overset{\alpha}{\widetilde{C}}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\left\{15\right\}\right).
\end{aligned}
$$

Substituting (3.5.86)–(3.5.98) into the analogue for $\overset{\alpha}{\widetilde{\mathcal{W}}}$ of (3.5.81) yields

$$
\begin{aligned}
M_{\overset{\alpha}{\widetilde{\mathcal{W}}}}(t) &= \mathbb{E}\left[e^{\overset{\alpha}{\widetilde{\mathcal{W}}}t}\right] \\
&= (1-2t)^{-\frac{p}{2}}\left(1+(24n)^{-1}\left\{A_1d + A_2d^2 + A_3d^3 + O(d^4)\right\}\right)
\end{aligned}
$$

$$+O(n^{-\frac{3}{2}}), \tag{3.5.99}$$

as in (3.1.6), where

$$\begin{aligned}
\widetilde{A}_1 &= \widetilde{D}_{11} + \widetilde{D}_{21} + \widetilde{D}_{31} + \widetilde{D}_{41} \\
&= 12\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\lambda_{i,j,kl} + 12\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{t}_{ijk} \\
&\quad +12\overset{\alpha}{\widetilde{C}}{}^{i,j,kl,mn}\lambda_{kl,mn}i_{i,j} + 12\overset{\alpha}{\widetilde{C}}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right),
\end{aligned}$$

$$\begin{aligned}
\widetilde{A}_2 &= \widetilde{D}_{12} + \widetilde{D}_{22} + \widetilde{D}_{32} + \widetilde{D}_{42} \\
&= 3\lambda_{i,j,k,l}i^{i,j}i^{k,l} + 12\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{t}_{ijk} + 6\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}i^{m,n}\lambda_{m,n,kl}i_{i,j} \\
&\quad +24\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{t}_{ijk} + 18\overset{\alpha}{\widetilde{C}}{}^{i,j,k}i^{l,m}\not{t}_{ilm}i_{j,k} + 12\overset{\alpha}{\widetilde{C}}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right) \\
&\quad +3\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\overset{\alpha}{\widetilde{C}}{}^{m,n,pq}\lambda_{kl,pq}\left(i_{i,j}i_{m,n}\{3\}\right) \\
&\quad +3\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\overset{\alpha}{\widetilde{C}}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right)
\end{aligned}$$

$$\begin{aligned}
\widetilde{A}_3 &= \widetilde{D}_{13} + \widetilde{D}_{23} + \widetilde{D}_{33} + \widetilde{D}_{43} \\
&= \left(3\tau_{ijkl} + 2\tau_{ikjl}\right)i^{i,j}i^{k,l} + 12\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{t}_{ijk} + 18\overset{\alpha}{\widetilde{C}}{}^{i,j,k}i^{l,m}\not{t}_{ilm}i_{j,k} \\
&\quad +3\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\overset{\alpha}{\widetilde{C}}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right).
\end{aligned}$$

By Remark 11 in Section 3.5.3 and Proposition 1 in Section 3.1.2, we find a Bartlett-type adjusted modified geometric Wald statistic $\overset{\alpha}{\widetilde{\mathcal{W}}}{}'$, with χ^2 distribution under the null hypothesis with p degrees of freedom, and error of order $O(n^{-\frac{3}{2}})$, where

$$\overset{\alpha}{\widetilde{\mathcal{W}}}{}' = \left\{1 - \frac{1}{n}\left(\tilde{c} + \tilde{b}\overset{\alpha}{\widetilde{\mathcal{W}}} + \tilde{a}\overset{\alpha}{\widetilde{\mathcal{W}}}{}^2\right)\right\}\overset{\alpha}{\widetilde{\mathcal{W}}}, \tag{3.5.100}$$

with

$$\begin{aligned}
\tilde{a} &= \frac{\widetilde{A}_3}{12p(p+2)(p+4)} \\
&= \frac{1}{12p(p+2)(p+4)}\left\{\left(3\tau_{ijkl} + 2\tau_{ikjl}\right)i^{i,j}i^{k,l} + 12\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{t}_{ijk}\right. \\
&\quad \left. +18\overset{\alpha}{\widetilde{C}}{}^{i,j,k}i^{l,m}\not{t}_{ilm}i_{j,k} + 3\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\overset{\alpha}{\widetilde{C}}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right)\right\}
\end{aligned} \tag{3.5.101}$$

$$
\begin{aligned}
\tilde{b} &= \frac{\widetilde{A}_2 - 2\widetilde{A}_3}{12p(p+2)} \\
&= \frac{1}{12p(p+2)}\Big\{3\lambda_{i,j,k,l}i^{i,j}i^{k,l} - (6\tau_{ijkl} + 4\tau_{ikjl}) \\
&\quad +12\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\not{t}_{ijk} + 6\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}i^{m,n}\lambda_{m,n,kl}i_{i,j} \\
&\quad -18\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}i^{l,m}\not{t}_{ilm}i_{j,k} + 12\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k,l}\left(i_{i,j}i_{k,l}\{3\}\right) \\
&\quad +3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\overset{\alpha}{\widetilde{C}}{}^{m,n,pq}\lambda_{kl,pq}\left(i_{i,j}i_{m,n}\{3\}\right) \\
&\quad -3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\overset{\alpha}{\widetilde{C}}{}^{l,m,n}\left(i_{i,j}i_{k,l}i_{m,n}\{15\}\right)\Big\} \qquad (3.5.102)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{c} &= \frac{\widetilde{A}_1 - \widetilde{A}_2 + \widetilde{A}_3}{12p} \\
&= \frac{1}{12p}\Big\{-3\lambda_{i,j,k,l}i^{i,j}i^{k,l} + (3\tau_{ijkl} + 2\tau_{ikjl}) \\
&\quad -6\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}i^{m,n}\lambda_{m,n,kl}i_{i,j} + 12\,\overset{\alpha}{\widetilde{C}}{}^{i,j,kl,mn}\lambda_{kl,mn}i_{i,j} \\
&\quad -3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,kl}\overset{\alpha}{\widetilde{C}}{}^{m,n,pq}\lambda_{kl,pq}\left(i_{i,j}i_{m,n}\{3\}\right)\Big\}. \qquad (3.5.103)
\end{aligned}
$$

Note that, by an argument analogous to Remark 13 in Section 3.5.5, $\tilde{c}$ does not depend on α. Moreover, $\tilde{c}$ is equal to c in (3.5.94).

Chapter 4

Observed Bartlett-type Adjustment for $\overset{\alpha}{\mathcal{W}}$

Observed geometries are natural alternatives to expected geometries. They are 'closer to the data' than expected geometries, in that they do not involve taking expectations. However, they require the specification of a suitable auxiliary statistic. Since the introduction of observed geometries by Barndorff-Nielsen [1], observed equivalents of most geometric objects based on expected geometry have been developed (see e.g. Barndorff-Nielsen & Cox [5, 6], Mora [28] or Murray & Rice [29]). In the spirit of the observed Bartlett adjustments discussed in Barndorff-Nielsen & Cox [5, Section 5.4], we introduce in this Chapter an observed version of the Bartlett-type adjustment considered in Chapter 3.

4.1 Notation

Let the log-likelihood function based on n observations be denoted by $l(\theta; \hat{\theta}, a)$ where a is an auxiliary statistic such that $(\hat{\theta}, a)$ is a one-to-one function of a minimal sufficient statistic. Note that a can depend on the number of observations, n. Let $g = g(\theta; \theta')$ denote the observed likelihood yoke (1.3.17) of order $O(1)$, that is

$$g(\theta; \theta') = n^{-1} \{l(\theta; \theta', a) - l(\theta'; \theta', a)\}. \tag{4.1.1}$$

Recall from Remark 2 in Section 1.3.2 that $[\not{l}_{i,j}(\theta)] = j(\theta)$, where $j(\theta)$ is the observed information matrix of order $O(1)$. Define the (scaled) score by

$$Z_i = n^{-\frac{1}{2}} l_i \left(\theta; \hat{\theta}, a\right), \tag{4.1.2}$$

and let

$$Z^i = j^{i,j}(\theta) Z_j. \tag{4.1.3}$$

We shall use subscripts $\alpha, \beta, \gamma, \ldots$ to denote the nuisance part of the derivative of the score, that is

$$Z_\alpha = n^{-\frac{1}{2}} l_\alpha (\theta; \mathbf{X}).$$

Letting $[\nu^{\alpha,\beta}]$ denote the inverse of the nuisance part $[j_{\alpha,\beta}]$ of the information matrix j, we define

$$Z^\alpha = \nu^{\alpha,\beta} Z_\beta. \tag{4.1.4}$$

Note that the derivatives of the score are of order $O(1)$.

4.1.1 Some Useful Tensors

Define the following useful tensors analogous to (3.2.19)–(3.2.22):

$$\begin{aligned} \not{t}_{ijk} &= -\not{l}_{ij;k}\{3\}_{ijk} - \not{l}_{ijk} && (4.1.5) \\ &= \not{l}_{i;jk} - \not{l}_{jk;i} \; (= \not{l}_{j;ik} - \not{l}_{ik;j} = \not{l}_{k;ij} - \not{l}_{ij;k}) \\ \not{t}_{ij;kl} &= -\not{l}_{ij;m} j^{m,n} \not{l}_{n;kl} + \not{l}_{ij;kl} && (4.1.6) \\ \not{t}_{i;jkl} &= -\not{t}_{ijm} j^{m,n} \not{l}_{kl;n}\{3\}_{jkl} + \not{l}_{i;jkl} - \not{l}_{jkl;i} && (4.1.7) \\ \not{t}_{ijkl;} &= -\not{t}_{i;jkl} - \not{t}_{ij;kl}\{3\}_{jkl}. && (4.1.8) \end{aligned}$$

The tensor $\not{t}_{ijk}$ has a geometrical interpretation as an observed version of the skewness tensor (3.2.19).

We shall use subscripts $\alpha, \beta, \gamma, \ldots$ to denote the nuisance parts of the $\not{t}$-tensors, that is

$$
\begin{aligned}
\not{t}_{ij\alpha} &= \not{\ell}_{i;j\alpha} - \not{\ell}_{j\alpha;i} \\
\not{t}_{\alpha\beta\gamma} &= \not{\ell}_{\alpha;\beta\gamma} - \not{\ell}_{\beta\gamma;\alpha} \\
\not{t}_{ij;\alpha\beta} &= -\not{\ell}_{ij;m}\not{\ell}_{n;\alpha\beta}\jmath^{m,n} + \not{\ell}_{ij;\alpha\beta} \\
& etc.
\end{aligned}
$$

Observe that all $\not{t}$s are tensors and are of order $O(1)$.

We find that the following expressions (which are analogues in observed geometry of (3.2.23)–(3.2.24) in expected geometry) are tensors

$$
\begin{aligned}
\overset{1}{H}{}^{i}_{\alpha,\beta} &= \not{\ell}_{\alpha\beta;j}\jmath^{i,j} - \delta^{i}_{\gamma}\not{\ell}_{\alpha\beta;\delta}\nu^{\gamma,\delta} && (4.1.9) \\
\overset{-1}{H}{}^{i}_{\alpha,\beta} &= \left(\not{\ell}_{\alpha\beta;j} + \not{t}_{j\alpha\beta}\right)\jmath^{i,j} - \delta^{i}_{\gamma}\left(\not{\ell}_{\alpha\beta;\delta} + \not{t}_{\alpha\beta\gamma}\right)\nu^{\gamma,\delta}. && (4.1.10)
\end{aligned}
$$

The geometrical interpretations of $\overset{1}{H}{}^{i}_{\alpha,\beta}$ and $\overset{-1}{H}{}^{i}_{\alpha,\beta}$ are the observed 1-embedding curvature and the observed (-1)-embedding curvature, respectively, of Θ_0 in Θ, corresponding to the expected embedding curvatures (3.2.23) and (3.2.24). Define the lowered versions of the embedding curvatures by

$$
\begin{aligned}
\overset{1}{H}_{i;\alpha,\beta} &= \jmath_{i,j}\,\overset{1}{H}{}^{j}_{\alpha,\beta} && (4.1.11) \\
\overset{-1}{H}_{i;\alpha,\beta} &= \jmath_{i,j}\,\overset{-1}{H}{}^{j}_{\alpha,\beta}. && (4.1.12)
\end{aligned}
$$

Finally, we define the following tensors

$$
\begin{aligned}
\tau_{ijkl} &= \not{t}_{ijm}\not{t}_{kln}\imath^{m,n} && (4.1.13) \\
\overset{1}{\tau}_{i;\alpha,jk} &= \overset{1}{H}_{i;\alpha,\beta}\,\not{t}_{jk\gamma}\nu^{\beta,\gamma} && (4.1.14) \\
\overset{1}{\tau}_{i;j;\alpha,\beta} &= \overset{1}{H}_{i;\alpha,\gamma}\overset{1}{H}_{j;\beta,\delta}\,\nu^{\gamma,\delta} && (4.1.15) \\
\overset{1}{\tau}_{i;\alpha\beta\gamma} &= -\overset{1}{H}_{i;\alpha,\delta}\,\not{\ell}_{\beta\epsilon;\gamma}\nu^{\delta,\epsilon}\,\{3\}_{\alpha\beta\gamma} + \not{\ell}_{\alpha\beta\gamma;i} - \jmath_{i,\delta}\nu^{\delta,\epsilon}\not{\ell}_{\alpha\beta\gamma;\epsilon} && (4.1.16) \\
\overset{-1}{\tau}_{\alpha,\beta,ij} &= \overset{-1}{H}_{k;\alpha,\beta}\,\not{t}_{ijl}\jmath^{k,l} && (4.1.17) \\
\overset{-1}{\tau}_{\alpha,\beta,\gamma,\delta} &= \overset{-1}{H}_{i;\alpha,\beta}\overset{-1}{H}_{j;\gamma,\delta}\,\jmath^{i,j}. && (4.1.18)
\end{aligned}
$$

As before, we use subscripts $\alpha, \beta, \ldots$ for the nuisance parts of the tensors, e.g. $\tau_{ijk\alpha} = \not{t}_{ijl}\not{t}_{km\alpha}\jmath^{l,m}$, etc.

4.2 Observed Geometric Wald Tests

In observed geometry we use the observed likelihood yoke (4.1.1) to define the family of geometric Wald test statistics (2.3.6) by

$$\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= \overset{\alpha}{\Gamma}(\hat{\theta};\tilde{\theta})j^{-1}(\hat{\theta})\overset{\alpha}{\Gamma}{}^T(\hat{\theta};\tilde{\theta}) \\
&= \overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta})j^{i,j}(\hat{\theta})\overset{\alpha}{\Gamma}_j(\hat{\theta};\tilde{\theta}), \qquad (4.2.19)
\end{aligned}$$

where

$$\overset{\alpha}{\Gamma}_i(\theta;\theta') = n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}g_{;i}(\theta';\theta) + \frac{1-\alpha}{2}g_i(\theta;\theta')\right\}.$$

Note that $\overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta})$, $\overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta})$ and $\overset{\alpha}{\mathcal{W}}$ have order $O(1)$.

Modified Observed Geometric Wald Test

When the observed likelihood yoke is used, the modified observed Wald test statistic (2.3.9) becomes

$$\begin{aligned}
\overset{\alpha}{\widetilde{\mathcal{W}}} &= \overset{\alpha}{\Gamma}(\tilde{\theta};\hat{\theta})j^{-1}(\tilde{\theta})\overset{\alpha}{\Gamma}{}^T(\tilde{\theta};\hat{\theta}) \\
&= \overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta})j^{i,j}(\tilde{\theta})\overset{\alpha}{\Gamma}_j(\tilde{\theta};\hat{\theta}). \qquad (4.2.20)
\end{aligned}$$

4.3 Observed Bartlett-type Adjustments

An asymptotic expansion, in observed geometry, for the conditional distribution for the conditional distribution of the score vector $l_* = (l_1, \ldots, l_r)$ given the auxiliary a is given by Mora [28]. Using her result, we find the probability density function of $Z_* = n^{-\frac{1}{2}}l_*$ to be

$$\begin{aligned}
p(Z_*;\theta|a) &= p(n^{-\frac{1}{2}}l_*;\theta|a) \\
&= \varphi(Z_*;j)\left\{1 + n^{-\frac{1}{2}}R_1 + n^{-1}R_2 + O(n^{-\frac{3}{2}})\right\} \qquad (4.3.21)
\end{aligned}$$

where $\varphi(\cdot;j)$ denotes the probability density function of the r-dimensional normal distribution with mean zero and variance matrix $j(\theta)$,

$$R_1 = \frac{1}{6}j^{i,l}j^{j,m}j^{k,n}\not{t}_{ijk}h_{lmn}(Z_*;j) \qquad (4.3.22)$$

$$
\begin{aligned}
R_2 &= -\frac{1}{4}\jmath^{k,l}\jmath^{i,m}\jmath^{j,n}\not{l}_{ij;kl}h_{mn}(Z_*;\jmath) \\
&\quad +\frac{1}{24}\jmath^{i,m}\jmath^{j,n}\jmath^{k,p}\jmath^{l,q}(\not{l}_{i;jkl}-\not{l}_{kl;ij}\{3\}_{jkl})h_{mnpq}(Z_*;\jmath) \\
&\quad +\frac{1}{72}\jmath^{i,p}\jmath^{j,q}\jmath^{k,r}\jmath^{l,s}\jmath^{m,t}\jmath^{n,u}\not{l}_{ijk}\not{l}_{lmn}h_{pqrstu}(Z_*;\jmath), \qquad (4.3.23)
\end{aligned}
$$

and $h_{ij}(Z_*;\jmath)$, $h_{ijk}(Z_*;\jmath)$, $h_{ijkl}(Z_*;\jmath)$ and $h_{ijkl}(Z_*;\jmath)$ are the covariant Hermite polynomials defined by

$$
\begin{aligned}
h_{ij}(Z_*;\jmath) &= Z_iZ_j - \jmath_{i,j}; \qquad (4.3.24)\\
h_{ijk}(Z_*;\jmath) &= Z_iZ_jZ_k - \jmath_{i,j}Z_k\{3\}; \qquad (4.3.25)\\
h_{ijkl}(Z_*;\jmath) &= Z_iZ_jZ_kZ_l - \jmath_{i,j}Z_kZ_l\{6\} + \jmath_{i,j}\jmath_{k,l}\{3\}; \qquad (4.3.26)\\
h_{ijklmn}(Z_*;\jmath) &= Z_iZ_jZ_kZ_lZ_mZ_n - \jmath_{i,j}Z_kZ_lZ_mZ_n\{15\} \\
&\quad +\jmath_{i,j}\jmath_{k,l}Z_mZ_n\{45\} - \jmath_{i,j}\jmath_{k,l}\jmath_{m,n}\{15\}. \qquad (4.3.27)
\end{aligned}
$$

The hs are all of order $O(1)$, and thus the terms R_1 and R_2 are both of order $O(1)$ with error of order $O(n^{-1})$.

4.4 Taylor Expansions

By Taylor expanding $Z_i = Z_i(\theta)$ (in any coordinate system on Θ) around the true value of the parameter θ, and letting $\hat{\delta} = n^{\frac{1}{2}}(\hat{\theta}-\theta)$, we have that

$$
\begin{aligned}
Z_i &= n^{-\frac{1}{2}}l_i(\theta;\hat{\theta},a) \\
&= n^{\frac{1}{2}}g_i(\theta;\hat{\theta}) \\
&= n^{\frac{1}{2}}\not{g}_i(\theta) + n^{\frac{1}{2}}\not{g}_{i;j}(\theta)\left(\hat{\theta}^j-\theta^j\right) + n^{\frac{1}{2}}\frac{1}{2}\not{g}_{i;jk}(\theta)\left(\hat{\theta}^j-\theta^j\right)\left(\hat{\theta}^k-\theta^k\right) \\
&\quad +n^{\frac{1}{2}}\frac{1}{6}\not{g}_{i;jkl}(\theta;\theta,a)\left(\hat{\theta}^j-\theta^j\right)\left(\hat{\theta}^k-\theta^k\right)\left(\hat{\theta}^l-\theta^l\right)+\dots \\
&= 0 + \jmath_{i,j}\hat{\delta}^j + n^{-\frac{1}{2}}\left\{\frac{1}{2}\not{g}_{i;jk}\hat{\delta}^j\hat{\delta}^k\right\} + n^{-1}\left\{\frac{1}{6}\not{g}_{i;jkl}\hat{\delta}^j\hat{\delta}^k\hat{\delta}^l\right\} + O(n^{-\frac{3}{2}}).
\end{aligned}
$$

Solving for $\hat{\delta}^i$ gives

$$
\begin{aligned}
\hat{\delta}^i &= Z^i - n^{-\frac{1}{2}}\left\{\frac{1}{2}\jmath^{i,j}\not{g}_{j;kl}Z^kZ^l\right\} \\
&\quad +n^{-1}\left\{\frac{1}{2}\jmath^{i,j}\not{g}_{j;kl}\jmath^{l,m}\not{g}_{m;np}Z^kZ^nZ^p - \frac{1}{6}\jmath^{i,j}\not{g}_{j;klm}Z^kZ^lZ^m\right\} \\
&\quad +O\left(n^{-\frac{3}{2}}\right). \qquad (4.4.28)
\end{aligned}
$$

Similarly, let $\tilde{\delta} = n^{\frac{1}{2}}\left(\tilde{\theta} - \theta\right)$. Then $\tilde{\delta}^a = n^{\frac{1}{2}}\left(\tilde{\theta}^a - \theta^a\right)$ vanishes. Observe that $\widetilde{Z}_\alpha = n^{-\frac{1}{2}} l_\alpha(\tilde{\theta}; \hat{\theta}, a) = 0$ by the definition of $\tilde{\theta}$. Thus expanding around the true value θ we find that

$$
\begin{aligned}
0 &= \widetilde{Z}_\alpha \\
&= n^{-\frac{1}{2}} l_\alpha(\tilde{\theta}; \hat{\theta}, a) \\
&= n^{\frac{1}{2}} g_\alpha(\tilde{\theta}; \hat{\theta}) \\
&= n^{\frac{1}{2}}\left\{\not{g}_\alpha + \not{g}_{\alpha\beta}(\theta)\left(\tilde{\theta}^\beta - \theta^\beta\right) + \not{g}_{\alpha;i}(\theta)\left(\hat{\theta}^i - \theta^i\right)\right. \\
&\quad + \frac{1}{2}\not{g}_{\alpha\beta\gamma}(\theta)\left(\tilde{\theta}^\beta - \theta^\beta\right)\left(\tilde{\theta}^\gamma - \theta^\gamma\right) + \not{g}_{\alpha\beta;i}(\theta)\left(\tilde{\theta}^\beta - \theta^\beta\right)\left(\hat{\theta}^i - \theta^i\right) \\
&\quad + \frac{1}{2}\not{g}_{\alpha;ij}(\theta)\left(\hat{\theta}^i - \theta^i\right)\left(\hat{\theta}^j - \theta^j\right) + \frac{1}{6}\not{g}_{\alpha\beta\gamma\delta}(\theta)\left(\tilde{\theta}^\beta - \theta^\beta\right)\left(\tilde{\theta}^\gamma - \theta^\gamma\right)\left(\tilde{\theta}^\delta - \theta^\delta\right) \\
&\quad + \frac{1}{2}\not{g}_{\alpha\beta\gamma;i}(\theta)\left(\tilde{\theta}^\beta - \theta^\beta\right)\left(\tilde{\theta}^\gamma - \theta^\gamma\right)\left(\hat{\theta}^i - \theta^i\right) + \frac{1}{2}\not{g}_{\alpha\beta;ij}(\theta)\left(\tilde{\theta}^\beta - \theta^\beta\right)\left(\hat{\theta}^i - \theta^i\right)\left(\hat{\theta}^j - \theta^j\right) \\
&\quad \left. + \frac{1}{6}\not{g}_{\alpha;ijk}(\theta)\left(\hat{\theta}^i - \theta^i\right)\left(\hat{\theta}^j - \theta^j\right)\left(\hat{\theta}^k - \theta^k\right) + \ldots\right\} \\
&= j_{\alpha;i}\hat{\delta}^i - j_{\alpha;\beta}\tilde{\delta}^\beta + n^{-\frac{1}{2}}\left\{\frac{1}{2}\not{g}_{\alpha\beta\gamma}\tilde{\delta}^\beta\tilde{\delta}^\gamma + \not{g}_{\alpha\beta;i}\tilde{\delta}^\beta\hat{\delta}^i + \frac{1}{2}\not{g}_{\alpha;ij}\hat{\delta}^i\hat{\delta}^j\right\} \\
&\quad + n^{-1}\left\{\frac{1}{6}\not{g}_{\alpha\beta\gamma\delta}\tilde{\delta}^\beta\tilde{\delta}^\gamma\tilde{\delta}^\delta + \frac{1}{2}\not{g}_{\alpha\beta\gamma;i}\tilde{\delta}^\beta\tilde{\delta}^\gamma\hat{\delta}^i + \frac{1}{2}\not{g}_{\alpha\beta;ij}\tilde{\delta}^\beta\hat{\delta}^i\hat{\delta}^j\right. \\
&\quad \left. + \frac{1}{6}\not{g}_{\alpha;ijk}\hat{\delta}^i\hat{\delta}^j\hat{\delta}^k\right\} + O(n^{-\frac{3}{2}}).
\end{aligned}
$$

Substituting (4.4.28) for $\hat{\delta}^i$, and solving for $\tilde{\delta}^\alpha$, we obtain

$$
\begin{aligned}
\tilde{\delta}^\alpha &= Z^\alpha + n^{-\frac{1}{2}}\left\{\nu^{\alpha,\beta}\not{g}_{\beta\gamma;i}Z^iZ^\gamma + \frac{1}{2}\nu^{\alpha,\beta}\not{g}_{\beta\gamma\delta}Z^\gamma Z^\delta\right\} \\
&\quad + n^{-1}\left\{\nu^{\alpha,\beta}\left[\not{g}_{\beta\delta;i}\nu^{\delta,\epsilon}\not{g}_{\gamma\epsilon;j} + \frac{1}{2}\not{g}_{\beta\gamma;ij} - \frac{1}{2}\not{g}_{\beta\gamma;k}j^{k,l}\not{g}_{l;ij}\right]Z^iZ^jZ^\gamma\right. \\
&\quad + \nu^{\alpha,\beta}\left[\not{g}_{\beta\gamma\epsilon}\nu^{\epsilon,\psi}\not{g}_{\delta\psi;i} + \frac{1}{2}\not{g}_{\beta\gamma\delta;i} + \frac{1}{2}\not{g}_{\beta\epsilon;i}\nu^{\epsilon,\psi}\not{g}_{\gamma\delta\psi}\right]Z^iZ^\gamma Z^\delta \\
&\quad \left. + \nu^{\alpha,\beta}\left[\frac{1}{6}\not{g}_{\beta\gamma\delta\epsilon} + \frac{1}{2}\not{g}_{\beta\gamma\psi}\nu^{\psi,\phi}\not{g}_{\delta\epsilon\phi}\right]Z^\gamma Z^\delta Z^\epsilon\right\} + O(n^{-\frac{3}{2}}). \qquad (4.4.29)
\end{aligned}
$$

4.4.1 Taylor Expanding j^{-1}

Observe that $\frac{\partial}{\partial\theta^k}j_{i,j}(\theta) = \not{g}_{i;j/k}(\theta)$ and $\frac{\partial^2}{\partial\theta^k\partial\theta^l}j_{i,j}(\theta) = \not{g}_{i;j/kl}(\theta)$. Then from (1.4.47) and (1.4.48) we find that

$$
\begin{aligned}
\frac{\partial}{\partial\theta^k}j^{i,j}(\theta) &= -j^{i,l}j^{j,m}\not{g}_{l;m/k} \qquad (4.4.30) \\
\frac{\partial^2}{\partial\theta^k\partial\theta^l}j^{i,j}(\theta) &= j^{i,m}j^{j,n}(-\not{g}_{m;n/kl} + \not{g}_{m;p/k}j^{p,q}\not{g}_{n;p/l}\{2\}_{kl}). \qquad (4.4.31)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
j^{i,j}(\hat{\theta}) &= j^{i,j}(\theta) + n^{-\frac{1}{2}} \frac{\partial}{\partial \theta^k} j_{i,j}(\theta) \hat{\delta}^k + n^{-1} \frac{1}{2} \frac{\partial^2}{\partial \theta^k \partial \theta^l} j_{i,j}(\theta) \hat{\delta}^k \hat{\delta}^l + \ldots \\
&= j^{i,j} + n^{-\frac{1}{2}} \left\{ -j^{i,l} j^{j,m} \not{\ell}_{l;m/k} Z^k \right\} \\
&\quad + n^{-1} \left\{ (-j^{i,l} j^{j,m} \not{\ell}_{l;m/k})(-\frac{1}{2} j^{k,n} \not{\ell}_{n;pq} Z^p Z^q) \right. \\
&\quad \left. + \frac{1}{2} j^{i,m} j^{j,n} \left(-\not{\ell}_{m;n/kl} + 2 \not{\ell}_{m;p/k} j^{p,q} \not{\ell}_{n;q/l} \right) Z^k Z^l \right\} + O(n^{-\frac{3}{2}}) \\
&= j^{i,j} + n^{-\frac{1}{2}} \left\{ -j^{i,l} j^{j,m} \not{\ell}_{l;m/k} Z^k \right\} \\
&\quad + n^{-1} \left\{ \frac{1}{2} j^{i,m} j^{j,n} \left[(\not{\ell}_{m;np} + \not{\ell}_{mp;n}) j^{p,q} \not{\ell}_{q;kl} \right. \right. \\
&\quad \left. \left. + 2(\not{\ell}_{m;pk} + \not{\ell}_{km;p}) j^{p,q} (\not{\ell}_{n;lq} + \not{\ell}_{nl;q}) - \not{\ell}_{m;n/kl} \right] Z^k Z^l \right\} \\
&\quad + O(n^{-\frac{3}{2}}). \qquad (4.4.32)
\end{aligned}
$$

Likewise, we find that

$$
\begin{aligned}
j^{i,j}(\tilde{\theta}) &= j^{i,j}(\theta) + n^{-\frac{1}{2}} \frac{\partial}{\partial \theta^k} j^{i,j}(\theta) \tilde{\delta}^k + n^{-1} \frac{1}{2} \frac{\partial^2}{\partial \theta^k \partial \theta^l} j^{i,j}(\theta) \tilde{\delta}^k \tilde{\delta}^l + \ldots \\
&= j^{i,j} + n^{-\frac{1}{2}} \left\{ -j^{i,k} j^{j,l} \not{\ell}_{k;l/\alpha} Z^\alpha \right\} \\
&\quad + n^{-1} \left\{ -j^{i,l} j^{j,m} \not{\ell}_{m;n/\gamma} \nu^{\gamma,\delta} \left(\not{\ell}_{\alpha\delta;i} Z^i Z^\alpha + \frac{1}{2} \not{\ell}_{\alpha\beta\delta} Z^\alpha Z^\beta \right) \right. \\
&\quad \left. + \frac{1}{2} j^{i,k} j^{j,l} \left(-\not{\ell}_{k;l/\alpha\beta} + 2 \not{\ell}_{k;m/\alpha} j^{m,n} \not{\ell}_{k;n/\beta} \right) Z^\alpha Z^\beta \right\} + O(n^{-\frac{3}{2}}) \\
&= j^{i,j} + n^{-\frac{1}{2}} \left\{ -j^{i,k} j^{j,l} \not{\ell}_{k;l/\alpha} Z^\alpha \right\} \\
&\quad + n^{-1} \left\{ -j^{i,l} j^{j,m} \not{\ell}_{m;n/\beta} \nu^{\beta,\gamma} \not{\ell}_{\alpha\gamma;i} Z^i Z^\alpha \right. \\
&\quad \left. + \frac{1}{2} j^{i,k} j^{j,l} \left[-\not{\ell}_{k;l/\gamma} \nu^{\gamma,\delta} \not{\ell}_{\alpha\beta\delta} + 2 \not{\ell}_{k;m/\alpha} j^{m,n} \not{\ell}_{l;n/\beta} - \not{\ell}_{k;l/\alpha\beta} \right] Z^\alpha Z^\beta \right\} \\
&\quad + O(n^{-\frac{3}{2}}). \qquad (4.4.33)
\end{aligned}
$$

4.4.2 Taylor Expanding $\overset{\alpha}{\Gamma}_i(\theta; \theta')$

From (1.3.9)-(1.3.15) we get that

$$
\begin{aligned}
\overset{\alpha}{\not{\Gamma}}_i &= n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} \not{\ell}_{;i} + \frac{1-\alpha}{2} \not{\ell}_i \right\} \\
&= 0 \qquad (4.4.34)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{ij} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\not{g}_{;ij}+\frac{1-\alpha}{2}\not{g}_{ij}\right\} \\
&= -n^{\frac{1}{2}}j_{i,j} \qquad (4.4.35)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{ijk} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}(-\not{g}_{ijk}-\not{g}_{i;jk}\,\{3\}-\not{g}_{ij;k}\,\{3\})+\frac{1-\alpha}{2}\not{g}_{ijk}\right\} \\
&= n^{\frac{1}{2}}\left\{-\frac{1}{2}(3+\alpha)\not{t}_{ijk}-\not{g}_{ij;k}\,\{3\}\right\} \qquad (4.4.36)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{ijkl} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}(-\not{g}_{ijkl}-\not{g}_{i;jkl}\,\{4\}-\not{g}_{ijk;l}\,\{4\}-\not{g}_{ij;kl}\,\{6\})+\frac{1-\alpha}{2}\not{g}_{ijkl}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(-\not{g}_{i;jkl}\,\{4\}-\not{g}_{ijk;l}\,\{4\}-\not{g}_{ij;kl}\,\{6\})\right. \\
&\quad \left.+\frac{\alpha}{2}\left(\not{g}_{i;jkl}-\not{g}_{jkl;i}+(\not{g}_{ikl;j}-\not{g}_{j;ikl}+\not{g}_{ij;kl}-\not{g}_{kl;ij})\,\{3\}_{jkl}\right)\right\}, \qquad (4.4.37)
\end{aligned}
$$

and that

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{i;j} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\not{g}_{j;i}+\frac{1-\alpha}{2}\not{g}_{i;j}\right\} \\
&= n^{\frac{1}{2}}j_{i,j} \qquad (4.4.38)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{i;jk} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\not{g}_{jk;i}+\frac{1-\alpha}{2}\not{g}_{i;jk}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\not{g}_{jk;i}+\not{g}_{i;jk})-\frac{\alpha}{2}\not{t}_{ijk}\right\} \qquad (4.4.39)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{ij;k} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\not{g}_{k;ij}+\frac{1-\alpha}{2}\not{g}_{ij;k}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\not{g}_{k;ij}+\not{g}_{ij;k})+\frac{\alpha}{2}\not{t}_{ijk}\right\} \qquad (4.4.40)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{i;jkl} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\not{g}_{jkl;i}+\frac{1-\alpha}{2}\not{g}_{i;jkl}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\not{g}_{i;jkl}+\not{g}_{jkl;i})+\frac{\alpha}{2}(\not{g}_{jkl;i}-\not{g}_{i;jkl})\right\} \qquad (4.4.41)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{ij;kl} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\not{g}_{kl;ij}+\frac{1-\alpha}{2}\not{g}_{ij;kl}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\not{g}_{ij;kl}+\not{g}_{kl;ij})+\frac{\alpha}{2}(\not{g}_{kl;ij}-\not{g}_{ij;kl})\right\} \qquad (4.4.42)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\not\Gamma}_{ijk;l} &= n^{\frac{1}{2}}\left\{\frac{1+\alpha}{2}\not{g}_{l;ijk}+\frac{1-\alpha}{2}\not{g}_{ijk;l}\right\} \\
&= n^{\frac{1}{2}}\left\{\frac{1}{2}(\not{g}_{ijk;l}+\not{g}_{l;ijk})+\frac{\alpha}{2}(\not{g}_{l;ijk}-\not{g}_{ijk;l})\right\}. \qquad (4.4.43)
\end{aligned}
$$

Thus, we find the Taylor expansion of $\overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta})$ as

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta}) &= \overset{\alpha}{\not{\Gamma}}_i + n^{-\frac{1}{2}}\left\{\overset{\alpha}{\not{\Gamma}}_{ij}\hat{\delta}^j + \overset{\alpha}{\not{\Gamma}}_{i;j}\tilde{\delta}^j\right\} \\
&+n^{-1}\left\{\frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ijk}\hat{\delta}^j\hat{\delta}^k + \overset{\alpha}{\not{\Gamma}}_{ij;k}\hat{\delta}^j\tilde{\delta}^k + \frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{i;jk}\tilde{\delta}^j\tilde{\delta}^k\right\} \\
&+n^{-\frac{3}{2}}\left\{\frac{1}{6}\overset{\alpha}{\not{\Gamma}}_{ijkl}\hat{\delta}^j\hat{\delta}^k\hat{\delta}^l + \frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ijk;l}\hat{\delta}^j\hat{\delta}^k\tilde{\delta}^l + \frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ij;kl}\hat{\delta}^j\tilde{\delta}^k\tilde{\delta}^l + \frac{1}{6}\overset{\alpha}{\not{\Gamma}}_{i;jkl}\tilde{\delta}^j\tilde{\delta}^k\tilde{\delta}^l\right\} + \dots \\
&= n^{-\frac{1}{2}}\left\{\overset{\alpha}{\not{\Gamma}}_{ij}Z^j + \overset{\alpha}{\not{\Gamma}}_{i;\alpha}Z^\alpha\right\} \\
&+n^{-1}\left\{\overset{\alpha}{\not{\Gamma}}_{ij}(-\frac{1}{2}j^{j,k}\not{g}_{k;lm}Z^lZ^m) + \overset{\alpha}{\not{\Gamma}}_{i;\alpha}\nu^{\alpha,\beta}(\not{g}_{\beta\gamma;j}Z^jZ^\gamma + \frac{1}{2}\not{g}_{\beta\gamma\delta}Z^\gamma Z^\delta)\right. \\
&\left.+\frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ijk}Z^jZ^k + \overset{\alpha}{\not{\Gamma}}_{ij;\alpha}Z^jZ^\alpha + \frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{i;\alpha\beta}Z^\alpha Z^\beta\right\} \\
&+n^{-\frac{3}{2}}\left\{\overset{\alpha}{\not{\Gamma}}_{ij}\left(-\frac{1}{6}j^{j,k}\not{g}_{k;lmn}Z^lZ^mZ^n + \frac{1}{2}j^{j,k}\not{g}_{k;lm}j^{m,n}\not{g}_{n;pq}Z^lZ^pZ^q\right)\right. \\
&+\overset{\alpha}{\not{\Gamma}}_{i;\alpha}\nu^{\alpha,\beta}\left(\left[\not{g}_{\beta\delta;j}\nu^{\delta,\epsilon}\not{g}_{\gamma\epsilon;k} + \frac{1}{2}\not{g}_{\beta\gamma;jk} - \frac{1}{2}\not{g}_{\beta\gamma;l}j^{l,m}\not{g}_{m;jk}\right]Z^jZ^kZ^\gamma\right. \\
&+\left[\not{g}_{\beta\gamma\epsilon}\nu^{\epsilon,\psi}\not{g}_{\delta\psi;j} + \frac{1}{2}\not{g}_{\beta\gamma\delta;j} + \frac{1}{2}\not{g}_{\beta\epsilon;j}\nu^{\epsilon,\psi}\not{g}_{\gamma\delta\psi}\right]Z^jZ^\gamma Z^\delta \\
&\left.+\left[\frac{1}{6}\not{g}_{\beta\gamma\delta\epsilon} + \frac{1}{2}\not{g}_{\beta\gamma\psi}\nu^{\psi,\phi}\not{g}_{\delta\epsilon\phi}\right]Z^\gamma Z^\delta Z^\epsilon\right) \\
&+\overset{\alpha}{\not{\Gamma}}_{ijk}Z^j\left(-\frac{1}{2}j^{k,l}\not{g}_{l;mn}Z^mZ^n\right) \\
&+\overset{\alpha}{\not{\Gamma}}_{ij;\alpha}\left(-\frac{1}{2}j^{j,k}\not{g}_{k;lm}Z^lZ^nZ^\alpha + \nu^{\alpha,\beta}\not{g}_{\beta\gamma;k}Z^jZ^kZ^\gamma + \frac{1}{2}\nu^{\alpha,\beta}\not{g}_{\beta\gamma\delta}Z^jZ^\gamma Z^\delta\right) \\
&+\overset{\alpha}{\not{\Gamma}}_{i;\alpha\beta}Z^\alpha\left(\nu^{\beta\gamma}\not{g}_{\gamma\delta;j}Z^jZ^\delta + \frac{1}{2}\nu^{\beta,\gamma}\not{g}_{\gamma\delta\epsilon}Z^\delta Z^\epsilon\right) \\
&\left.+\frac{1}{6}\overset{\alpha}{\not{\Gamma}}_{ijkl}Z^jZ^kZ^l + \frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ijk;\alpha}Z^jZ^kZ^\alpha + \frac{1}{2}\overset{\alpha}{\not{\Gamma}}_{ij;\alpha\beta}Z^jZ^\alpha Z^\beta + \frac{1}{6}\overset{\alpha}{\not{\Gamma}}_{i;\alpha\beta\gamma}Z^\alpha Z^\beta Z^\gamma\right\} \\
&+O(n^{-2}) \\
&= -Z_i + i_{i,\alpha}Z^\alpha \\
&+n^{-\frac{1}{2}}\left\{\left[-\frac{1}{4}(1+\alpha)\not{t}_{ijk} - \not{g}_{ij;k}\right]Z^jZ^k\right. \\
&+\left[\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha} + \not{g}_{ij;\alpha} + \jmath_{i,\beta}\nu^{\beta,\gamma}\not{g}_{\alpha\gamma;j}\right]Z^jZ^\alpha \\
&\left.+\left[\frac{1}{4}(1-\alpha)\not{t}_{i\alpha\beta} + \frac{1}{2}\not{g}_{\alpha\beta;i} + \frac{1}{2}\jmath_{i,\gamma}\nu^{\gamma,\delta}\not{g}_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\right\} \\
&+n^{-1}\left\{\left[\left(\frac{1}{4}(1+\alpha)\not{t}_{ijm} + \frac{1}{2}\not{g}_{im;j} + \frac{1}{2}\not{g}_{ij;m}\right)j^{m,n}\not{g}_{n;kl}\right.\right. \\
&+\frac{1}{12}\left(\not{g}_{i;jkl} - \not{g}_{jkl;i} - 3\not{g}_{ijk;l} - 3\not{g}_{l;ijk} - \not{g}_{ij;kl}\{6\}\right) \\
&\left.+\frac{\alpha}{12}\left(\not{g}_{i;jkl} - \not{g}_{jkl;i} + 3\not{g}_{ijk;l} - 3\not{g}_{l;ijk} + 3\not{g}_{ij;kl} - 3\not{g}_{kl;ij}\right)\right]Z^jZ^kZ^l \\
&+\left[\left(-\frac{1}{4}(1+\alpha)\not{t}_{il\alpha} - \frac{1}{2}\not{g}_{il;\alpha}\right)j^{l,m}\not{g}_{m;jk} + \left(\frac{1}{2}(1+\alpha)\not{t}_{ij\beta} + \not{g}_{ij;\beta}\right)\nu^{\beta,\gamma}\not{g}_{\alpha\gamma;k}\right. \\
&\left.+\frac{1}{4}\left(\not{g}_{ijk;\alpha} + \not{g}_{\alpha;ijk} + \alpha(\not{g}_{\alpha;ijk} - \not{g}_{ijk;\alpha})\right)\right]Z^jZ^kZ^\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\left(\frac{1}{4}(1+\alpha)\not{\ell}_{ij\gamma}+\frac{1}{2}\not{g}_{ij;\gamma}\right)\nu^{\gamma,\delta}\not{g}_{\alpha\beta\delta}+\left(\frac{1}{2}(1-\alpha)\not{\ell}_{i\alpha\gamma}+\not{g}_{\alpha\gamma;i}\right)\nu^{\gamma,\delta}\not{g}_{\beta\delta;j}\right.\\
&\left.+\frac{1}{4}\left(\not{g}_{ij;\alpha\beta}+\not{g}_{\alpha\beta;ij}+\alpha(\not{g}_{\alpha\beta;ij}-\not{g}_{ij;\alpha\beta})\right)\right]Z^jZ^\alpha Z^\beta\\
&+\left[\left(\frac{1}{4}(1-\alpha)\not{\ell}_{i\alpha\delta}+\frac{1}{2}\not{g}_{\alpha\delta;i}\right)\nu^{\delta,\epsilon}\not{g}_{\beta\gamma\epsilon}\right.\\
&\left.+\frac{1}{12}\left(\not{g}_{i;\alpha\beta\gamma}+\not{g}_{\alpha\beta\gamma;i}+\alpha(\not{g}_{\alpha\beta\gamma;i}-\not{g}_{i;\alpha\beta\gamma})\right)\right]Z^\alpha Z^\beta Z^\gamma\\
&+j_{i,\alpha}\nu^{\alpha,\beta}\left(\left[\not{g}_{\beta\delta;j}\nu^{\delta,\epsilon}\not{g}_{\gamma\epsilon;k}+\frac{1}{2}\not{g}_{\beta\gamma;jk}-\frac{1}{2}\not{g}_{\beta\gamma;l}j^{l,m}\not{g}_{m;jk}\right]Z^jZ^kZ^\gamma\right.\\
&+\left[\not{g}_{\beta\gamma\epsilon}\nu^{\epsilon,\psi}\not{g}_{\delta\psi;j}+\frac{1}{2}\not{g}_{\beta\gamma\delta;j}+\frac{1}{2}\not{g}_{\beta\epsilon;j}\nu^{\epsilon,\psi}\not{g}_{\gamma\delta\psi}\right]Z^jZ^\gamma Z^\delta\\
&\left.\left.+\left[\frac{1}{6}\not{g}_{\beta\gamma\delta\epsilon}+\frac{1}{2}\not{g}_{\beta\gamma\psi}\nu^{\psi,\phi}\not{g}_{\delta\epsilon\phi}\right]Z^\gamma Z^\delta Z^\epsilon\right)\right\}+O(n^{-\frac{3}{2}}),
\end{aligned}
\tag{4.4.44}
$$

and we find the Taylor expansion of $\overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta})$ as

$$
\begin{aligned}
\overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta}) &= \overset{\alpha}{\not\Gamma}_i+n^{-\frac{1}{2}}\left\{\overset{\alpha}{\not\Gamma}_{ij}\tilde{\delta}^j+\overset{\alpha}{\not\Gamma}_{i;j}\hat{\delta}^j\right\}\\
&\quad+n^{-1}\left\{\frac{1}{2}\overset{\alpha}{\not\Gamma}_{ijk}\tilde{\delta}^j\tilde{\delta}^k+\overset{\alpha}{\not\Gamma}_{ij;k}\tilde{\delta}^j\hat{\delta}^k+\frac{1}{2}\overset{\alpha}{\not\Gamma}_{i;jk}\hat{\delta}^j\hat{\delta}^k\right\}\\
&\quad+n^{-\frac{3}{2}}\left\{\frac{1}{6}\overset{\alpha}{\not\Gamma}_{ijkl}\tilde{\delta}^j\tilde{\delta}^k\tilde{\delta}^l+\frac{1}{2}\overset{\alpha}{\not\Gamma}_{ijk;l}\tilde{\delta}^j\tilde{\delta}^k\hat{\delta}^l+\frac{1}{2}\overset{\alpha}{\not\Gamma}_{ij;kl}\tilde{\delta}^j\hat{\delta}^k\hat{\delta}^l\frac{1}{6}\overset{\alpha}{\not\Gamma}_{i;jkl}\hat{\delta}^j\hat{\delta}^k\hat{\delta}^l\right\}+\dots\\
&= n^{-\frac{1}{2}}\left\{\overset{\alpha}{\not\Gamma}_{i;j}Z^j+\overset{\alpha}{\not\Gamma}_{i\alpha}Z^\alpha\right\}\\
&\quad+n^{-1}\left\{\overset{\alpha}{\not\Gamma}_{i;j}(-\frac{1}{2}j^{j,k}\not{g}_{k;lm}Z^lZ^m)+\overset{\alpha}{\not\Gamma}_{i\alpha}\nu^{\alpha,\beta}(\not{g}_{\beta\gamma;j}Z^jZ^\gamma+\frac{1}{2}\not{g}_{\beta\gamma\delta}Z^\gamma Z^\delta)\right.\\
&\quad\left.+\frac{1}{2}\overset{\alpha}{\not\Gamma}_{i;jk}Z^jZ^k+\overset{\alpha}{\not\Gamma}_{i\alpha;j}Z^jZ^\alpha+\frac{1}{2}\overset{\alpha}{\not\Gamma}_{i\alpha\beta}Z^\alpha Z^\beta\right\}\\
&\quad+n^{-\frac{3}{2}}\left\{\overset{\alpha}{\not\Gamma}_{i;j}\left(-\frac{1}{6}j^{j,k}\not{g}_{k;lmn}Z^lZ^mZ^n+\frac{1}{2}j^{j,k}\not{g}_{k;lm}j^{m,n}\not{g}_{n;pq}Z^lZ^pZ^q\right)\right.\\
&\quad+\overset{\alpha}{\not\Gamma}_{i\alpha}\nu^{\alpha,\beta}\left(\left[\not{g}_{\beta\delta;j}\nu^{\delta,\epsilon}\not{g}_{\gamma\epsilon;k}+\frac{1}{2}\not{g}_{\beta\gamma;jk}-\frac{1}{2}\not{g}_{\beta\gamma;l}j^{l,m}\not{g}_{m;jk}\right]Z^jZ^kZ^\gamma\right.\\
&\quad+\left[\not{g}_{\beta\gamma\epsilon}\nu^{\epsilon,\psi}\not{g}_{\delta\psi;j}+\frac{1}{2}\not{g}_{\beta\gamma\delta;j}+\frac{1}{2}\not{g}_{\beta\epsilon;j}\nu^{\epsilon,\psi}\not{g}_{\gamma\delta\psi}\right]Z^jZ^\gamma Z^\delta\\
&\quad\left.+\left[\frac{1}{6}\not{g}_{\beta\gamma\delta\epsilon}+\frac{1}{2}\not{g}_{\beta\gamma\psi}\nu^{\psi,\phi}\not{g}_{\delta\epsilon\phi}\right]Z^\gamma Z^\delta Z^\epsilon\right)\\
&\quad+\overset{\alpha}{\not\Gamma}_{i;jk}Z^j\left(-\frac{1}{2}j^{k,l}\not{g}_{l;mn}Z^mZ^n\right)\\
&\quad+\overset{\alpha}{\not\Gamma}_{i\alpha;j}\left(-\frac{1}{2}j^{j,k}\not{g}_{k;lm}Z^lZ^nZ^\alpha+\nu^{\alpha,\beta}\not{g}_{\beta\gamma;k}Z^jZ^kZ^\gamma+\frac{1}{2}\nu^{\alpha,\beta}\not{g}_{\beta\gamma\delta}Z^jZ^\gamma Z^\delta\right)\\
&\quad+\overset{\alpha}{\not\Gamma}_{i\alpha\beta}Z^\alpha\left(\nu^{\beta\gamma}\not{g}_{\gamma\delta;j}Z^jZ^\delta+\frac{1}{2}\nu^{\beta,\gamma}\not{g}_{\gamma\delta\epsilon}Z^\delta Z^\epsilon\right)\\
&\quad\left.+\frac{1}{6}\overset{\alpha}{\not\Gamma}_{i;jkl}Z^jZ^kZ^l+\frac{1}{2}\overset{\alpha}{\not\Gamma}_{i\alpha;jk}Z^jZ^kZ^\alpha+\frac{1}{2}\overset{\alpha}{\not\Gamma}_{i\alpha\beta;j}Z^jZ^\alpha Z^\beta+\frac{1}{6}\overset{\alpha}{\not\Gamma}_{i\alpha\beta\gamma}Z^\alpha Z^\beta Z^\gamma\right\}\\
&\quad+O(n^{-2})\\
&= Z_i-i_{i,\alpha}Z^\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+n^{-\frac{1}{2}}\left\{-\frac{1}{4}(1+\alpha)\not{t}_{ijk}Z^jZ^k+\left[\frac{1}{2}(1+\alpha)\not{t}_{ij\alpha}+\not{g}_{i\alpha;j}-j_{i,\beta}\nu^{\beta,\gamma}\not{g}_{\alpha\beta;j}\right]Z^jZ^\alpha\right.\\
&\left.+\left[-\frac{1}{4}(3+\alpha)\not{t}_{i\alpha\beta}-\frac{1}{2}\not{g}_{\alpha\beta;i}\{3\}-\frac{1}{2}j_{i,\gamma}\nu^{\gamma,\delta}\not{g}_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\right\}\\
&+n^{-1}\left\{\left[\frac{1}{4}(1+\alpha)\not{t}_{ijm}j^{m,n}\not{g}_{n;kl}+\frac{1+\alpha}{12}\left(\not{g}_{jkl;i}-\not{g}_{i;jkl}\right)\right]Z^jZ^kZ^l\right.\\
&+\left[\left(-\frac{1}{4}(1+\alpha)\not{t}_{il\alpha}-\frac{1}{2}\not{g}_{i\alpha;l}\right)j^{l,m}\not{g}_{m;jk}+\left(\frac{1}{2}(1+\alpha)\not{t}_{ij\beta}+\not{g}_{i\beta;j}\right)\nu^{\beta,\gamma}\not{g}_{\alpha\gamma;k}\right.\\
&\left.+\frac{1}{4}\left(\not{g}_{i\alpha;jk}+\not{g}_{jk;i\alpha}+\alpha(\not{g}_{jk;i\alpha}-\not{g}_{i\alpha;jk})\right)\right]Z^jZ^kZ^\alpha\\
&+\left[\left(\frac{1}{4}(1+\alpha)\not{t}_{ij\gamma}+\frac{1}{2}\not{g}_{i\gamma;j}\right)\nu^{\gamma,\delta}\not{g}_{\alpha\beta\delta}-\left(\frac{1}{2}(3+\alpha)\not{t}_{i\alpha\gamma}+\not{g}_{i\alpha;\gamma}\{3\}\right)\nu^{\gamma,\delta}\not{g}_{\beta\delta;j}\right.\\
&\left.+\frac{1}{4}\left(\not{g}_{i\alpha\beta;j}+\not{g}_{j;i\alpha\beta}+\alpha(\not{g}_{j;i\alpha\beta}-\not{g}_{i\alpha\beta;j})\right)\right]Z^jZ^\alpha Z^\beta\\
&+\left[\left(-\frac{1}{4}(3+\alpha)\not{t}_{i\alpha\delta}-\frac{1}{2}\not{g}_{i\alpha;\delta}\{3\}\right)\nu^{\delta,\epsilon}\not{g}_{\beta\gamma\epsilon}\right.\\
&-\frac{1}{12}\left(\not{g}_{i;\alpha\beta\gamma}\{4\}+\not{g}_{\alpha\beta\gamma;i}\{4\}+\not{g}_{i\alpha;\beta\gamma}\{6\}\right)\\
&\left.+\frac{\alpha}{12}\left(\not{g}_{i;\alpha\beta\gamma}-\not{g}_{\alpha\beta\gamma;i}+3\not{g}_{i\alpha\beta;\gamma}-3\not{g}_{\gamma;i\alpha\beta}+3\not{g}_{i\alpha;\beta\gamma}-3\not{g}_{\beta\gamma;i\alpha}\right)\right]Z^\alpha Z^\beta Z^\gamma\\
&-j_{i,\alpha}\nu^{\alpha,\beta}\left(\left[\not{g}_{\beta\delta;j}\nu^{\delta,\epsilon}\not{g}_{\gamma\epsilon;k}+\frac{1}{2}\not{g}_{\beta\gamma;jk}-\frac{1}{2}\not{g}_{\beta\gamma;l}j^{l,m}\not{g}_{m;jk}\right]Z^jZ^kZ^\gamma\right.\\
&+\left[\not{g}_{\beta\gamma\epsilon}\nu^{\epsilon,\psi}\not{g}_{\delta\psi;j}+\frac{1}{2}\not{g}_{\beta\gamma\delta;j}+\frac{1}{2}\not{g}_{\beta\epsilon;j}\nu^{\epsilon,\psi}\not{g}_{\gamma\delta\psi}\right]Z^jZ^\gamma Z^\delta\\
&\left.\left.+\left[\frac{1}{6}\not{g}_{\beta\gamma\delta\epsilon}+\frac{1}{2}\not{g}_{\beta\gamma\psi}\nu^{\psi,\phi}\not{g}_{\delta\epsilon\phi}\right]Z^\gamma Z^\delta Z^\epsilon\right)\right\}+O(n^{-\frac{3}{2}}). \qquad (4.4.45)
\end{aligned}
$$

4.4.3 Expansion for the Geometric Wald Statistic

We obtain a Taylor expansion of the observed geometric Wald statistic by combining the expansions (4.4.32) and (4.4.44) of $j^{i,j}(\hat{\theta})$ and $\overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta})$ and $\overset{\alpha}{\Gamma}_j(\hat{\theta};\tilde{\theta})$,

$$
\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= \overset{\alpha}{\Gamma}_i(\hat{\theta};\tilde{\theta})j^{i,j}(\hat{\theta})\overset{\alpha}{\Gamma}_j(\hat{\theta};\tilde{\theta})\\
&= (-Z_i+j_{i,\alpha}Z^\alpha)j^{i,j}(-Z_j+j_{j,\alpha}Z^\alpha)\\
&\quad+n^{-\frac{1}{2}}\left\{2(-Z_i+j_{i,\alpha}Z^\alpha)j^{i,j}\right.\\
&\quad\times\left(\left[-\frac{1}{4}(1+\alpha)\not{t}_{jkl}-\not{g}_{jk;l}\right]Z^kZ^l\right.\\
&\quad+\left[\frac{1}{2}(1+\alpha)\not{t}_{jk\alpha}+\not{g}_{jk;\alpha}+j_{j,\beta}\nu^{\beta,\gamma}\not{g}_{\alpha\beta;k}\right]Z^kZ^\alpha\\
&\quad\left.+\left[\frac{1}{4}(1-\alpha)\not{t}_{j\alpha\beta}+\frac{1}{2}\not{g}_{\alpha\beta;j}+\frac{1}{2}j_{j,\gamma}\nu^{\gamma,\delta}\not{g}_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\right)\\
&\quad\left.+(-Z_i+j_{i,\alpha}Z^\alpha)(-Z_j+j_{j,\alpha}Z^\alpha)\left(-j^{i,l}j^{j,m}(\not{g}_{kl;m}+\not{g}_{l;km})Z^k\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+n^{-1}\Big\{2(-Z_j+j_{j,\alpha}Z^{\alpha})(j^{i,j})\\
&\times\Bigg(\Bigg[\left(\frac{1}{4}(1+\alpha)\not{l}_{ijm}+\frac{1}{2}\not{\ell}_{im;j}+\frac{1}{2}\not{\ell}_{ij;m}\right)j^{m,n}\not{\ell}_{n;kl}\\
&+\frac{1}{12}\left(\not{\ell}_{i;jkl}-\not{\ell}_{jkl;i}-3\not{\ell}_{ijk;l}-3\not{\ell}_{l;ijk}-\not{\ell}_{ij;kl}\,\{6\}\right)\\
&+\frac{\alpha}{12}\left(\not{\ell}_{i;jkl}-\not{\ell}_{jkl;i}+3\not{\ell}_{ijk;l}-3\not{\ell}_{l;ijk}+3\not{\ell}_{ij;kl}-3\not{\ell}_{kl;ij}\right)\Bigg]Z^jZ^kZ^l\\
&+\Bigg[\left(-\frac{1}{4}(1+\alpha)\not{l}_{il\alpha}-\frac{1}{2}\not{\ell}_{il;\alpha}\right)j^{l,m}\not{\ell}_{m;jk}+\left(\frac{1}{2}(1+\alpha)\not{l}_{ij\beta}+\not{\ell}_{ij;\beta}\right)\nu^{\beta,\gamma}\not{\ell}_{\alpha\gamma;k}\\
&+\frac{1}{4}\left(\not{\ell}_{ijk;\alpha}+\not{\ell}_{\alpha;ijk}+\alpha(\not{\ell}_{\alpha;ijk}-\not{\ell}_{ijk;\alpha})\right)\Bigg]Z^jZ^kZ^{\alpha}\\
&+\Bigg[\left(\frac{1}{4}(1+\alpha)\not{l}_{ij\gamma}+\frac{1}{2}\not{\ell}_{ij;\gamma}\right)\nu^{\gamma,\delta}\not{\ell}_{\alpha\beta\delta}+\left(\frac{1}{2}(1-\alpha)\not{l}_{i\alpha\gamma}+\not{\ell}_{\alpha\gamma;i}\right)\nu^{\gamma,\delta}\not{\ell}_{\beta\delta;j}\\
&+\frac{1}{4}\left(\not{\ell}_{ij;\alpha\beta}+\not{\ell}_{\alpha\beta;ij}+\alpha(\not{\ell}_{\alpha\beta;ij}-\not{\ell}_{ij;\alpha\beta})\right)\Bigg]Z^jZ^{\alpha}Z^{\beta}\\
&+\Bigg[\left(\frac{1}{4}(1-\alpha)\not{l}_{i\alpha\delta}+\frac{1}{2}\not{\ell}_{\alpha\delta;i}\right)\nu^{\delta,\epsilon}\not{\ell}_{\beta\gamma\epsilon}\\
&+\frac{1}{12}\left(\not{\ell}_{i;\alpha\beta\gamma}+\not{\ell}_{\alpha\beta\gamma;i}+\alpha(\not{\ell}_{\alpha\beta\gamma;i}-\not{\ell}_{i;\alpha\beta\gamma})\right)\Bigg]Z^{\alpha}Z^{\beta}Z^{\gamma}\\
&+j_{i,\alpha}\nu^{\alpha,\beta}\Bigg(\left[\not{\ell}_{\beta\delta;j}\nu^{\delta,\epsilon}\not{\ell}_{\gamma\epsilon;k}+\frac{1}{2}\not{\ell}_{\beta\gamma;jk}-\frac{1}{2}\not{\ell}_{\beta\gamma;l}j^{l,m}\not{\ell}_{m;jk}\right]Z^jZ^kZ^{\gamma}\\
&+\left[\not{\ell}_{\beta\gamma\epsilon}\nu^{\epsilon,\psi}\not{\ell}_{\delta\psi;j}+\frac{1}{2}\not{\ell}_{\beta\gamma\delta;j}+\frac{1}{2}\not{\ell}_{\beta\epsilon;j}\nu^{\epsilon,\psi}\not{\ell}_{\gamma\delta\psi}\right]Z^jZ^{\gamma}Z^{\delta}\\
&+\left[\frac{1}{6}\not{\ell}_{\beta\gamma\delta\epsilon}+\frac{1}{2}\not{\ell}_{\beta\gamma\psi}\nu^{\psi,\phi}\not{\ell}_{\delta\epsilon\phi}\right]Z^{\gamma}Z^{\delta}Z^{\epsilon}\Bigg)\Bigg)\\
&+(-Z_i+j_{i,\alpha}Z^{\alpha})\left(-j^{i,l}j^{j,m}(\not{\ell}_{lk;m}+\not{\ell}_{l;km}+\not{\ell}_{km;l}+\not{\ell}_{m;kl})Z^k\right)\\
&\times\Bigg(\left[-\frac{1}{4}(1+\alpha)\not{l}_{jkl}-\not{\ell}_{jk;l}\right]Z^kZ^l\\
&+\left[\frac{1}{2}(1+\alpha)\not{l}_{jk\alpha}+\not{\ell}_{jk;\alpha}+j_{j,\beta}\nu^{\beta,\gamma}\not{\ell}_{\alpha\beta;k}\right]Z^kZ^{\alpha}\\
&+\left[\frac{1}{4}(1-\alpha)\not{l}_{j\alpha\beta}+\frac{1}{2}\not{\ell}_{\alpha\beta;j}+\frac{1}{2}j_{j,\gamma}\nu^{\gamma,\delta}\not{\ell}_{\alpha\beta\delta}\right]Z^{\alpha}Z^{\beta}\Bigg)\\
&+j^{i,j}\Bigg(\left[-\frac{1}{4}(1+\alpha)\not{l}_{ijk}-\not{\ell}_{ij;k}\right]Z^jZ^k\\
&+\left[\frac{1}{2}(1+\alpha)\not{l}_{ij\alpha}+\not{\ell}_{ij;\alpha}+j_{i,\beta}\nu^{\beta,\gamma}\not{\ell}_{\alpha\beta;j}\right]Z^jZ^{\alpha}\\
&+\left[\frac{1}{4}(1-\alpha)\not{l}_{i\alpha\beta}+\frac{1}{2}\not{\ell}_{\alpha\beta;i}+\frac{1}{2}j_{i,\gamma}\nu^{\gamma,\delta}\not{\ell}_{\alpha\beta\delta}\right]Z^{\alpha}Z^{\beta}\Bigg)\\
&\times\Bigg(\left[-\frac{1}{4}(1+\alpha)\not{l}_{jkl}-\not{\ell}_{jk;l}\right]Z^kZ^l\\
&+\left[\frac{1}{2}(1+\alpha)\not{l}_{jk\alpha}+\not{\ell}_{jk;\alpha}+j_{j,\beta}\nu^{\beta,\gamma}\not{\ell}_{\alpha\beta;k}\right]Z^kZ^{\alpha}\\
&+\left[\frac{1}{4}(1-\alpha)\not{l}_{j\alpha\beta}+\frac{1}{2}\not{\ell}_{\alpha\beta;j}+\frac{1}{2}j_{j,\gamma}\nu^{\gamma,\delta}\not{\ell}_{\alpha\beta\delta}\right]Z^{\alpha}Z^{\beta}\Bigg)\\
&+(-Z_i+j_{i,\alpha}Z^{\alpha})(-Z_j+j_{j,\alpha}Z^{\alpha})\Big(-j^{i,l}j^{j,m}\not{\ell}_{l,m/n}j^{n,p}Z_{kp}Z^k\\
&+\frac{1}{2}j^{i,m}j^{j,n}\left[-\not{\ell}_{m;n/p}j^{p,q}\not{\ell}_{klq}+2\not{\ell}_{m;p/k}j^{p,q}\not{\ell}_{n;q/l}-\not{\ell}_{m;n/kl}\right]Z^kZ^l\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
& +O(n^{-\frac{3}{2}}) \\
= \; & j_{i,j} Z^i Z^j - j_{\alpha,\beta} Z^\alpha Z^\beta \\
& + n^{-\frac{1}{2}} \left\{ -\frac{1}{2}(1-\alpha) \not{t}_{ijk} Z^i Z^j Z^k \right. \\
& + \frac{1}{2}(1-3\alpha) \not{t}_{ij\alpha} Z^i Z^j Z^\alpha - \left[\not{g}_{\alpha\beta;i} + \frac{1}{2}(1-3\alpha) \not{t}_{i\alpha\beta} \right] Z^i Z^\alpha Z^\beta \\
& \left. + \left[\not{g}_{\alpha\beta;\gamma} + \frac{1}{2}(1-\alpha) \not{t}_{\alpha\beta\gamma} \right] Z^\alpha Z^\beta Z^\gamma \right\} \\
& + n^{-1} \left\{ \left[\left(\frac{9}{16} - \frac{7}{8}\alpha + \frac{1}{16}\alpha^2\right) \not{t}_{ijm} \not{t}_{kln} j^{m,n} \right. \right. \\
& \left. + \left(\frac{1}{2} - \alpha\right) j^{m,n} \not{g}_{ij;m} \not{t}_{kln} + \frac{1}{6}(1-2\alpha) \left(\not{g}_{ijk;l} - \not{g}_{l;ijk} \right) \right] Z^i Z^j Z^k Z^l \\
& + \left[\left(-\frac{3}{4} + 2\alpha - \frac{1}{4}\alpha^2\right) j^{l,m} \not{t}_{ijl} \not{t}_{km\alpha} + \frac{1}{4}(1+\alpha) \left(j^{l,m} \not{t}_{ijl} \not{g}_{k\alpha;m} \{3\}_{jk\alpha} - \not{g}_{i;jk\alpha} + \not{g}_{ij\alpha;k} \right) \right. \\
& - \frac{1}{12}(5-7\alpha) \left(3 j^{l,m} \not{t}_{il\alpha} \not{g}_{jk;m} - \not{g}_{\alpha;ijk} + \not{g}_{ijk;\alpha} \right) - j^{l,m} \not{g}_{i\alpha;l} \not{g}_{m;jk} + \not{g}_{i\alpha;jk} \\
& \left. + \frac{1}{2}(1-3\alpha) \nu^{\beta,\gamma} \overset{1}{H}_{i;\alpha,\beta} \not{t}_{jk\gamma} \right] Z^i Z^j Z^k Z^\alpha \\
& + \left[\left(\frac{1}{8} - \frac{1}{4}\alpha + \frac{1}{8}\alpha^2 \right) j^{k,l} \not{t}_{ijk} \not{t}_{l\alpha\beta} + \left(\frac{1}{4} - \frac{1}{2}\alpha + \frac{1}{4}\alpha^2 \right) j^{k,l} \not{t}_{ik\alpha} \not{t}_{jl\beta} \right. \\
& + \frac{1}{4}(1-3\alpha) j^{k,l} \not{t}_{ijk} \overset{-1}{H}_{l;\alpha,\beta} + \frac{1}{2} j^{k,l} \not{g}_{ij;k} \not{g}_{l;\alpha\beta} - \frac{1}{2} \not{g}_{ij;\alpha\beta} + \frac{1}{2} j^{k,l} \not{g}_{\alpha\beta;k} \not{g}_{l;ij} - \frac{1}{2} \not{g}_{\alpha\beta;ij} \\
& + j^{k,l} \not{g}_{i\alpha;k} \not{g}_{l;j\beta} - \not{g}_{i\alpha;j\beta} - \alpha \left(j^{k,l} \not{t}_{ik\alpha} \not{g}_{j\beta;l} \{3\}_{ij\beta} - \not{g}_{\alpha;ij\beta} + \not{g}_{ij\beta;\alpha} \right) \\
& + \frac{1}{2}\alpha \left(j^{k,l} \not{t}_{ijk} \not{g}_{\alpha\beta;l} \{3\}_{j\alpha\beta} - \not{g}_{i;j\alpha\beta} + \not{g}_{j\alpha\beta;i} \right) - (1-3\alpha) \nu^{\gamma,\delta} \overset{1}{H}_{i;\alpha,\gamma} \not{t}_{j\beta\delta} \\
& \left. - \nu^{\gamma,\delta} \overset{1}{H}_{i;\alpha,\gamma} \overset{1}{H}_{j;\beta,\delta} \right] Z^i Z^j Z^\alpha Z^\beta \\
& + \left[\left(\frac{1}{4} - \alpha - \frac{1}{4}\alpha^2 \right) j^{j,k} \not{t}_{ij\alpha} \not{t}_{k\beta\gamma} - \frac{1}{2}(1-3\alpha) j^{j,k} \not{t}_{ij\alpha} \overset{1}{H}_{k;\beta,\gamma} \right. \\
& - j^{j,k} \not{g}_{i\alpha;j} \not{g}_{k;\beta\gamma} + \not{g}_{i\alpha;\beta\gamma} + \frac{1}{4}(1+\alpha) \left(j^{j,k} \not{t}_{ij\alpha} \not{g}_{\beta\gamma;k} \{3\}_{i\beta\gamma} - \not{g}_{\alpha;i\beta\gamma} + \not{g}_{i\alpha\beta;\gamma} \right) \\
& - \frac{1}{12}(1+5\alpha) \left(3 j^{j,k} \not{t}_{ij\alpha} \not{g}_{\beta\gamma;k} - \not{g}_{i;\alpha\beta\gamma} + \not{g}_{\alpha\beta\gamma;i} \right) + \frac{3}{2}(1-\alpha) \nu^{\delta,\epsilon} \overset{1}{H}_{i;\alpha,\delta} \not{t}_{\beta\gamma\epsilon} \\
& \left. + 3 \nu^{\delta,\epsilon} \overset{1}{H}_{i;\alpha,\delta} \not{g}_{\beta\epsilon;\gamma} - \not{g}_{\alpha\beta\gamma;i} + j_{i,\delta} \nu^{\delta,\epsilon} \not{g}_{\alpha\beta\gamma;\epsilon} \right] Z^i Z^\alpha Z^\beta Z^\gamma \\
& + \left[\left(-\frac{7}{16} + \frac{5}{8}\alpha + \frac{1}{16}\alpha^2 \right) j^{i,j} \not{t}_{i\alpha\beta} \not{t}_{j\gamma\delta} - \frac{1}{6}(1-\alpha)(3 j^{i,j} \not{t}_{i\alpha\beta} \not{g}_{\gamma\delta;j} - \not{g}_{\alpha;\beta\gamma\delta} + \not{g}_{\alpha\beta\gamma;\delta}) \right. \\
& \left. \left. + \frac{1}{4}(1-3\alpha) j^{i,j} \overset{-1}{H}_{i;\alpha,\beta} \not{t}_{j\gamma\delta} + \frac{1}{4} j^{i,j} \overset{-1}{H}_{i;\alpha,\beta} \overset{-1}{H}_{j;\gamma,\delta} \right] Z^\alpha Z^\beta Z^\gamma Z^\delta \right\} + O(n^{-\frac{3}{2}}) \\
= \; & \overset{\alpha}{C}^{i,j} Z_i Z_j + n^{-\frac{1}{2}} \overset{\alpha}{C}^{i,j,k} Z_i Z_j Z_k + n^{-1} \overset{\alpha}{C}^{i,j,k,l} Z_i Z_j Z_k Z_l + O(n^{-\frac{3}{2}}), \qquad (4.4.46)
\end{aligned}
$$

where

$$
\overset{\alpha}{C}^{i,j} \; = \; j^{i,j} - \delta^{i,j}_{\alpha,\beta} \nu^{\alpha,\beta} \qquad (4.4.47)
$$

$$\overset{\alpha}{C}{}^{i,j,k} = j^{i,l}j^{j,m}j^{k,n}\left[-\frac{1}{2}(1-\alpha)\not{t}_{lmn}\right] + \delta^k_\alpha\, j^{i,l}j^{j,m}\nu^{\alpha,\beta}\left[\frac{1}{2}(1-3\alpha)\not{t}_{lm\beta}\right]$$
$$+\delta^{j,k}_{\alpha,\beta}\, j^{i,l}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[-\frac{1}{2}(1-3\alpha)\not{t}_{l\gamma\delta} - \overset{1}{H}_{l;\gamma,\delta}\right]$$
$$+\delta^{i,j,k}_{\alpha,\beta,\gamma}\,\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\frac{1}{2}(1-\alpha)\not{t}_{\delta\epsilon\psi}\right] \tag{4.4.48}$$

$$\overset{\alpha}{C}{}^{i,j,k,l} = j^{i,m}j^{j,n}j^{k,p}j^{l,q}\left[\left(\frac{9}{16}-\frac{7}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{mnpq} - \frac{1}{6}(1-2\alpha)\not{t}_{m;npq}\right]$$
$$+\delta^l_\alpha\, j^{i,m}j^{j,n}j^{k,p}\nu^{\alpha,\beta}\left[\left(-\frac{3}{4}+2\alpha-\frac{1}{4}\alpha^2\right)\tau_{mnp\beta} - \frac{1}{4}(1+\alpha)\not{t}_{m;np\beta}\right.$$
$$\left.+\frac{1}{12}(5-7\alpha)\not{t}_{\beta;mnp} + \not{t}_{m\beta;np} + \frac{1}{2}(1-3\alpha)\overset{1}{\tau}_{m;\beta,np}\right]$$
$$+\delta^{k,l}_{\alpha,\beta}\, j^{i,m}j^{j,n}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[\left(\frac{1}{8}-\frac{1}{4}\alpha+\frac{1}{8}\alpha^2\right)\tau_{mn\gamma\delta}\right.$$
$$+\left(\frac{1}{4}-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2\right)\tau_{m\gamma n\delta} + \left(\frac{1}{4}-\frac{3}{4}\alpha\right)\overset{-1}{\tau}_{\gamma,\delta,mn} - \frac{1}{2}\not{t}_{mn;\gamma\delta} - \frac{1}{2}\not{t}_{\gamma\delta;mn}$$
$$\left.-\not{t}_{m\gamma;n\delta} + \alpha\not{t}_{\gamma;mn\delta} - \frac{1}{2}\alpha\not{t}_{m;n\gamma\delta} - (1-3\alpha)\overset{1}{\tau}_{m;\gamma,n\delta} - \overset{1}{\tau}_{m;n;\gamma,\delta}\right]$$
$$+\delta^{j,k,l}_{\alpha,\beta,\gamma}\, j^{i,m}\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\left(\frac{1}{4}-\alpha-\frac{1}{4}\alpha^2\right)\tau_{m\delta\epsilon\psi}\right.$$
$$-\frac{1}{2}(1-3\alpha)\overset{-1}{\tau}_{\delta,\epsilon,m\psi} + \not{t}_{m\delta;\epsilon\psi} - \frac{1}{4}(1+\alpha)\not{t}_{\delta;m\epsilon\psi}$$
$$\left.+\frac{1}{12}(1+5\alpha)\not{t}_{m;\delta\epsilon\psi} + \frac{3}{2}(1-\alpha)\overset{1}{\tau}_{m;\delta,\epsilon\psi} - \overset{1}{\tau}_{m;\delta\epsilon\psi}\right]$$
$$+\delta^{i,j,k,l}_{\alpha,\beta,\gamma,\delta}\,\nu^{\alpha,\epsilon}\nu^{\beta,\psi}\nu^{\gamma,\phi}\nu^{\delta,\chi}\left[\left(-\frac{7}{16}+\frac{5}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{\epsilon\psi\phi\chi}\right.$$
$$\left.+\frac{1}{4}(1-3\alpha)\overset{-1}{\tau}_{\epsilon,\psi,\phi\chi} + \frac{1}{4}\overset{-1}{\tau}_{\epsilon,\psi,\phi,\chi} + \frac{1}{6}(1-\alpha)\not{t}_{\epsilon;\psi\phi\chi}\right]. \tag{4.4.49}$$

Remark 14:

Recall that the observed and the expected information matrices coincide to order $O(n^{-\frac{1}{2}})$ (see e.g. Barndorff-Nielsen & Cox [6, Section 5.5]). Then expansion (4.4.46) shows that to order $O(n^{-\frac{1}{2}})$ the geometric Wald statistics based on observed geometry have the same asymptotic distribution as the likelihood ratio statistic. That is, the geometric Wald statistics are χ^2-distributed with p degrees of freedom, with error of order $O(n^{-\frac{1}{2}})$.

Remark 15:

For models with vanishing observed skewness, the coefficient $\overset{\alpha}{C}{}^{i,j,k}$ simplifies to

$$\overset{\alpha}{C}{}^{i,j,k} = -\delta^{j,k}_{\alpha,\beta}\, i^{i,l}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\,\overset{1}{H}_{l;\gamma,\delta}$$

Furthermore, the following $\not{t}$-tensor and τ-tensors occurring in $\overset{\alpha}{C}{}^{i,j,k,l}$ simplify to

$$
\begin{aligned}
\not{t}_{i;jkl} &= \not{g}_{i;jkl} - \not{g}_{jkl;i} \\
\tau_{ijkl} &= 0 \\
\overset{1}{\tau}{}_{i;\alpha,jk} &= 0 \\
\overset{-1}{\tau}{}_{ij,\alpha,\beta} &= 0 \\
\overset{-1}{\tau}{}_{\alpha,\beta,\gamma,\delta} &= i^{i,j}\, \overset{1}{H}{}_{i;\alpha,\beta}\, \overset{1}{H}{}_{j;\gamma,\delta}.
\end{aligned}
$$

For models for which the observed skewness $\not{t}_{ijk}$ and observed embedding curvature $\overset{1}{H}{}^{i}_{\alpha,\beta}$ both vanish, the expansion of $\overset{\alpha}{\mathcal{W}}$ simplifies even more to

$$
\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= j^{i,j} Z_i Z_j - \nu^{\alpha,\beta} Z_\alpha Z_\beta \\
&\quad + n^{-1} \left\{ -\frac{1}{6}(1-2\alpha) \left(\not{g}_{i;jkl} - \not{g}_{jkl;i} \right) Z^i Z^j Z^k Z^l \right. \\
&\quad + \left[-\frac{1}{4}(1+\alpha) \left(\not{g}_{i;jk\alpha} - \not{g}_{jk\alpha;i} \right) + \frac{1}{12}(5-7\alpha) \left(\not{g}_{\alpha;ijk} - \not{g}_{ijk;\alpha} \right) + \not{t}_{i\alpha;jk} \right] Z^i Z^j Z^k Z^\alpha \\
&\quad + \left[-\frac{1}{2} \not{t}_{ij;\alpha\beta} - \frac{1}{2} \not{t}_{\alpha\beta;ij} - \not{t}_{i\alpha;j\beta} \right. \\
&\quad \left. + \alpha \left(\not{g}_{\alpha;ij\beta} - \not{g}_{ij\beta;\alpha} - \frac{1}{2} \not{g}_{i;j\alpha\beta} + \frac{1}{2} \not{g}_{j\alpha\beta;i} \right) \right] Z^i Z^j Z^\alpha Z^\beta \\
&\quad + \left[\not{t}_{i\alpha;\beta\gamma} - \frac{1}{4}(1+\alpha) \left(\not{g}_{\alpha;i\beta\gamma} - \not{g}_{i\beta\gamma;\alpha} \right) \right. \\
&\quad \left. + \frac{1}{12}(1+5\alpha) \left(\not{g}_{i;\alpha\beta\gamma} - \not{g}_{\alpha\beta\gamma;i} \right) - \not{t}_{\alpha\beta\gamma;i} + i_{i,\delta} \nu^{\delta,\epsilon} \not{t}_{\alpha\beta\gamma;\epsilon} \right] Z^i Z^\alpha Z^\beta Z^\gamma \\
&\quad \left. + \frac{1}{6}(1-\alpha) \left(\not{g}_{\alpha;\beta\gamma\delta} - \not{g}_{\beta\gamma\delta;\alpha} \right) Z^\alpha Z^\beta Z^\gamma Z^\delta \right\} \\
&\quad + O(n^{-\frac{3}{2}}).
\end{aligned}
$$

4.4.4 Expansion for the Modified Geometric Wald Statistic

We find the expansion of the observed modified geometric Wald statistic from the expansions (4.4.45) and (4.4.33) of $\overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta})$, $\overset{\alpha}{\Gamma}_j(\tilde{\theta};\hat{\theta})$ and $j^{-1}(\tilde{\theta})$ to be

$$
\begin{aligned}
\overset{\alpha}{\widetilde{\mathcal{W}}} &= \overset{\alpha}{\Gamma}_i(\tilde{\theta};\hat{\theta}) j^{i,j}(\tilde{\theta}) \overset{\alpha}{\Gamma}_j(\tilde{\theta};\hat{\theta}) \\
&= (Z_i - j_{i,\alpha} Z^\alpha) j^{i,j} (Z_j - j_{j,\alpha} Z^\alpha) \\
&\quad + n^{-\frac{1}{2}} \left\{ 2 (Z_i - j_{i,\alpha} Z^\alpha) j^{i,j} \left(-\frac{1}{4}(1+\alpha) \not{t}_{jkl} Z^k Z^l \right. \right. \\
&\quad + \left[\frac{1}{2}(1+\alpha) \not{t}_{jk\alpha} + \not{g}_{j\alpha;k} - j_{j,\beta} \nu^{\beta,\gamma} \not{g}_{\alpha\beta;k} \right] Z^k Z^\alpha
\end{aligned}
$$

$$+\left[-\frac{1}{4}(3+\alpha)\not\ell_{j\alpha\beta}-\frac{1}{2}\not g_{\alpha\beta;j}\{3\}-\frac{1}{2}j_{j,\gamma}\nu^{\gamma,\delta}\not g_{\alpha\beta\delta}\right]Z^\alpha Z^\beta\Big)$$

$$+(Z_i-j_{i,\alpha}Z^\alpha)(Z_j-j_{j,\alpha}Z^\alpha)\left(-j^{i,k}j^{j,l}\not g_{k;l/\alpha}Z^\alpha\right)\Big\}$$

$$+n^{-1}\{2(Z_j-j_{j,\alpha}Z^\alpha)(j^{i,j})$$

$$\times\Big(\Big[\frac{1}{4}(1+\alpha)\not\ell_{ijm}j^{m,n}\not g_{n;kl}+\frac{1+\alpha}{12}\left(\not g_{jkl;i}-\not g_{i;jkl}\right)\Big]Z^jZ^kZ^l$$

$$+\Big[\left(-\frac{1}{4}(1+\alpha)\not\ell_{il\alpha}-\frac{1}{2}\not g_{i\alpha;l}\right)j^{l,m}\not g_{m;jk}-\left(\frac{1}{2}(1+\alpha)\not\ell_{ij\beta}+\not g_{i\beta;j}\right)\nu^{\beta,\gamma}\not g_{\alpha\gamma;k}$$

$$+\frac{1}{4}\left(\not g_{i\alpha;jk}+\not g_{jk;i\alpha}+\alpha(\not g_{jk;i\alpha}-\not g_{i\alpha;jk})\right)\Big]Z^jZ^kZ^\alpha$$

$$+\Big[\left(\frac{1}{4}(1+\alpha)\not\ell_{ij\gamma}+\frac{1}{2}\not g_{i\gamma;j}\right)\nu^{\gamma,\delta}\not g_{\alpha\beta\delta}-\left(\frac{1}{2}(3+\alpha)\not\ell_{i\alpha\gamma}+\not g_{i\alpha;\gamma}\{3\}\right)\nu^{\gamma,\delta}\not g_{\beta\delta;j}$$

$$+\frac{1}{4}\left(\not g_{i\alpha\beta;j}+\not g_{j;i\alpha\beta}+\alpha(\not g_{j;i\alpha\beta}-\not g_{i\alpha\beta;j})\right)\Big]Z^jZ^\alpha Z^\beta$$

$$+\Big[\left(-\frac{1}{4}(3+\alpha)\not\ell_{i\alpha\delta}-\frac{1}{2}\not g_{i\alpha;\delta}\{3\}\right)\nu^{\delta,\epsilon}\not g_{\beta\gamma\epsilon}$$

$$-\frac{1}{12}\left(\not g_{i;\alpha\beta\gamma}\{4\}+\not g_{\alpha\beta\gamma;i}\{4\}+\not g_{i\alpha;\beta\gamma}\{6\}\right)$$

$$+\frac{\alpha}{12}\left(\not g_{i;\alpha\beta\gamma}-\not g_{\alpha\beta\gamma;i}+3\not g_{i\alpha\beta;\gamma}-3\not g_{\gamma;i\alpha\beta}+3\not g_{i\alpha;\beta\gamma}-3\not g_{\beta\gamma;i\alpha}\right)\Big]Z^\alpha Z^\beta Z^\gamma$$

$$-j_{i,\alpha}\nu^{\alpha,\beta}\Big(\Big[\not g_{\beta\delta;j}\nu^{\delta,\epsilon}\not g_{\gamma\epsilon;k}+\frac{1}{2}\not g_{\beta\gamma;jk}-\frac{1}{2}\not g_{\beta\gamma;l}j^{l,m}\not g_{m;jk}\Big]Z^jZ^kZ^\gamma$$

$$+\Big[\not g_{\beta\gamma\epsilon}\nu^{\epsilon,\psi}\not g_{\delta\psi;j}+\frac{1}{2}\not g_{\beta\gamma\delta;j}+\frac{1}{2}\not g_{\beta\epsilon;j}\nu^{\epsilon,\psi}\not g_{\gamma\delta\psi}\Big]Z^jZ^\gamma Z^\delta$$

$$+\Big[\frac{1}{6}\not g_{\beta\gamma\delta\epsilon}+\frac{1}{2}\not g_{\beta\gamma\psi}\nu^{\psi,\phi}\not g_{\delta\epsilon\phi}\Big]Z^\gamma Z^\delta Z^\epsilon\Big)\Big)$$

$$+(Z_i-j_{i,\alpha}Z^\alpha)\left(-j^{i,k}j^{j,l}(\not g_{k;l/\alpha}\not g_{l;k/\alpha})Z^\alpha\right)$$

$$\times\Big(-\frac{1}{4}(1+\alpha)\not\ell_{jkl}Z^kZ^l+\Big[\frac{1}{2}(1+\alpha)\not\ell_{jk\alpha}+\not g_{j\alpha;k}-j_{j,\beta}\nu^{\beta,\gamma}\not g_{\alpha\beta;k}\Big]Z^kZ^\alpha$$

$$+\Big[-\frac{1}{4}(3+\alpha)\not\ell_{j\alpha\beta}-\frac{1}{2}\not g_{\alpha\beta;j}\{3\}-\frac{1}{2}j_{j,\gamma}\nu^{\gamma,\delta}\not g_{\alpha\beta\delta}\Big]Z^\alpha Z^\beta\Big)$$

$$+j^{i,j}\Big(-\frac{1}{4}(1+\alpha)\not\ell_{ijk}Z^jZ^k+\Big[\frac{1}{2}(1+\alpha)\not\ell_{ij\alpha}+\not g_{i\alpha;j}-j_{i,\beta}\nu^{\beta,\gamma}\not g_{\alpha\beta;j}\Big]Z^jZ^\alpha$$

$$+\Big[-\frac{1}{4}(3+\alpha)\not\ell_{i\alpha\beta}-\frac{1}{2}\not g_{\alpha\beta;i}\{3\}-\frac{1}{2}j_{i,\gamma}\nu^{\gamma,\delta}\not g_{\alpha\beta\delta}\Big]Z^\alpha Z^\beta\Big)$$

$$\times\Big(-\frac{1}{4}(1+\alpha)\not\ell_{jkl}Z^kZ^l+\Big[\frac{1}{2}(1+\alpha)\not\ell_{jk\alpha}+\not g_{j\alpha;k}-j_{j,\beta}\nu^{\beta,\gamma}\not g_{\alpha\beta;k}\Big]Z^kZ^\alpha$$

$$+\Big[-\frac{1}{4}(3+\alpha)\not\ell_{j\alpha\beta}-\frac{1}{2}\not g_{\alpha\beta;j}\{3\}-\frac{1}{2}j_{j,\gamma}\nu^{\gamma,\delta}\not g_{\alpha\beta\delta}\Big]Z^\alpha Z^\beta\Big)$$

$$+(Z_i-j_{i,\alpha}Z^\alpha)(Z_j-j_{j,\alpha}Z^\alpha)\Big(-j^{i,l}j^{j,m}\not g_{m;n/\beta}\nu^{\beta,\gamma}Z_{\alpha\gamma}Z^\alpha$$

$$+\frac{1}{2}j^{i,k}j^{j,l}\Big[-\not g_{k;l/\gamma}\nu^{\gamma,\delta}\not g_{\alpha\beta\delta}+2\not g_{k;m/\alpha}j^{m,n}\not g_{l,n/\beta}-\not g_{k;l/\alpha\beta}\Big]Z^\alpha Z^\beta\Big)\Big\}$$

$$+O(n^{-\frac{3}{2}})$$

$$=\ j_{i,j}Z^iZ^j-j_{\alpha,\beta}Z^\alpha Z^\beta$$

$$+n^{-\frac{1}{2}}\Big\{-\frac{1}{2}(1+\alpha)\not\ell_{ijk}Z^iZ^jZ^k$$

$$
\begin{aligned}
& +\frac{1}{2}(1+3\alpha)\not{t}_{ij\alpha}Z^iZ^jZ^\alpha - \left[\not{g}_{\alpha\beta;i} + \frac{1}{2}(1+3\alpha)\not{t}_{i\alpha\beta}\right] Z^iZ^\alpha Z^\beta \\
& + \left[\not{g}_{\alpha\beta;\gamma} + \frac{1}{2}(1+\alpha)\not{t}_{\alpha\beta\gamma}\right] Z^\alpha Z^\beta Z^\gamma \Big\} \\
& +n^{-1}\Big\{\Big[(\frac{9}{16} + \frac{5}{8}\alpha + \frac{1}{16}\alpha^2)\not{t}_{ijm}\not{t}_{kln}j^{m,n} \\
& +\frac{1}{2}(1+\alpha)j^{m,n}\not{g}_{ij;m}\not{t}_{kln} + \frac{1}{6}(1+\alpha)\left(\not{g}_{ijk;l} - \not{g}_{l;ijk}\right)\Big] Z^iZ^jZ^kZ^l \\
& + \Big[(-\frac{3}{4} - \alpha - \frac{1}{4}\alpha^2)j^{l,m}\not{t}_{ijl}\not{t}_{km\alpha} + \frac{1}{4}(1+\alpha)\left(j^{l,m}\not{t}_{ijl}\not{g}_{k\alpha;m}\{3\}_{jk\alpha} - \not{g}_{i;jk\alpha} + \not{g}_{ij\alpha;k}\right) \\
& -\frac{5}{12}(1+\alpha)\left(3j^{l,m}\not{t}_{il\alpha}\not{g}_{jk;m} - \not{g}_{\alpha;ijk} + \not{g}_{ijk;\alpha}\right) - j^{l,m}\not{g}_{i\alpha;l}\not{g}_{m;jk} + \not{g}_{i\alpha;jk} \\
& +\frac{1}{2}(1+3\alpha)\nu^{\beta,\gamma}\overset{1}{H}_{i;\alpha,\beta}\not{t}_{jk\gamma}\Big] Z^iZ^jZ^kZ^\alpha \\
& + \Big[\left(\frac{1}{8} - \frac{1}{4}\alpha + \frac{1}{8}\alpha^2\right)j^{k,l}\not{t}_{ijk}\not{t}_{l\alpha\beta} + \left(\frac{1}{4} - \frac{1}{2}\alpha + \frac{1}{4}\alpha^2\right)j^{k,l}\not{t}_{ik\alpha}\not{t}_{jl\beta} \\
& +\frac{1}{4}(1+3\alpha)j^{k,l}\not{t}_{ijk}\overset{-1}{H}_{l;\alpha,\beta} + \frac{1}{2}j^{k,l}\not{g}_{ij;k}\not{g}_{l;\alpha\beta} - \frac{1}{2}\not{g}_{ij;\alpha\beta} + \frac{1}{2}j^{k,l}\not{g}_{\alpha\beta;k}\not{g}_{l;ij} - \frac{1}{2}\not{g}_{\alpha\beta;ij} \\
& +j^{k,l}\not{g}_{i\alpha;k}\not{g}_{l;j\beta} - \not{g}_{i\alpha;j\beta} + \frac{1}{2}\alpha\left(j^{k,l}\not{t}_{ik\alpha}\not{g}_{j\beta;l}\{3\}_{ij\beta} - \not{g}_{\alpha;ij\beta} + \not{g}_{ij\beta;\alpha}\right) \\
& -\alpha\left(j^{k,l}\not{t}_{ijk}\not{g}_{\alpha\beta;l}\{3\}_{j\alpha\beta} - \not{g}_{i;j\alpha\beta} + \not{g}_{j\alpha\beta;i}\right) - (1+3\alpha)\nu^{\gamma,\delta}\overset{1}{H}_{i;\alpha,\gamma}\not{t}_{j\beta\delta} \\
& -\nu^{\gamma,\delta}\overset{1}{H}_{i;\alpha,\gamma}\overset{1}{H}_{j;\beta,\delta}\Big] Z^iZ^jZ^\alpha Z^\beta \\
& + \Big[\left(\frac{1}{4} + 2\alpha - \frac{1}{4}\alpha^2\right)j^{j,k}\not{t}_{ij\alpha}\not{t}_{k\beta\gamma} - \frac{1}{2}(1+3\alpha)j^{j,k}\not{t}_{ij\alpha}\overset{1}{H}_{k;\beta,\gamma} \\
& -j^{j,k}\not{g}_{i\alpha;j}\not{g}_{k;\beta\gamma} + \not{g}_{i\alpha;\beta\gamma} + \frac{1}{4}(1+\alpha)\left(j^{j,k}\not{t}_{ij\alpha}\not{g}_{\beta\gamma;k}\{3\}_{i\beta\gamma} - \not{g}_{\alpha;i\beta\gamma} + \not{g}_{i\alpha\beta;\gamma}\right) \\
& -\frac{1}{12}(1-7\alpha)\left(3j^{j,k}\not{t}_{ij\alpha}\not{g}_{\beta\gamma;k} - \not{g}_{i;\alpha\beta\gamma} + \not{g}_{\alpha\beta\gamma;i}\right) + \frac{3}{2}(1+\alpha)\nu^{\delta,\epsilon}\overset{1}{H}_{i;\alpha,\delta}\not{t}_{\beta\gamma\epsilon} \\
& +3\nu^{\delta,\epsilon}\overset{1}{H}_{i;\alpha,\delta}\not{g}_{\beta\epsilon;\gamma} - \not{g}_{\alpha\beta\gamma;i} + j_{i,\delta}\nu^{\delta,\epsilon}\not{g}_{\alpha\beta\gamma;\epsilon}\Big] Z^iZ^\alpha Z^\beta Z^\gamma \\
& + \Big[\left(-\frac{7}{16} - \frac{7}{8}\alpha + \frac{1}{16}\alpha^2\right)j^{i,j}\not{t}_{i\alpha\beta}\not{t}_{j\gamma\delta} - \frac{1}{6}(1+2\alpha)(3j^{i,j}\not{t}_{i\alpha\beta}\not{g}_{\gamma\delta;j} - \not{g}_{\alpha;\beta\gamma\delta} + \not{g}_{\alpha\beta\gamma;\delta}) \\
& +\frac{1}{4}(1+3\alpha)j^{i,j}\overset{-1}{H}_{i;\alpha,\beta}\not{t}_{j\gamma\delta} + \frac{1}{4}j^{i,j}\overset{-1}{H}_{i;\alpha,\beta}\overset{-1}{H}_{j;\gamma,\delta}\Big] Z^\alpha Z^\beta Z^\gamma Z^\delta\Big\} + O(n^{-\frac{3}{2}}) \\
= \;& \overset{\alpha}{\tilde{C}}{}^{i,j}Z_iZ_j + n^{-\frac{1}{2}}\overset{\alpha}{\tilde{C}}{}^{i,j,k}Z_iZ_jZ_k + n^{-1}\overset{\alpha}{\tilde{C}}{}^{i,j,k,l}Z_iZ_jZ_kZ_l + O(n^{-\frac{3}{2}}), \qquad (4.4.50)
\end{aligned}
$$

where

$$
\overset{\alpha}{\tilde{C}}{}^{i,j} = j^{i,j} - \delta^{i,j}_{\alpha,\beta}\,\nu^{\alpha,\beta} \qquad (4.4.51)
$$

$$
\begin{aligned}
\overset{\alpha}{\tilde{C}}{}^{i,j,k} = \;& j^{i,l}j^{j,m}j^{k,n}\left[-\frac{1}{2}(1+\alpha)\not{t}_{lmn}\right] + \delta^k_\alpha\, j^{i,l}j^{j,m}\nu^{\alpha,\beta}\left[\frac{1}{2}(1+3\alpha)\not{t}_{lm\beta}\right] \\
& +\delta^{j,k}_{\alpha,\beta}\, j^{i,l}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[-\frac{1}{2}(1+3\alpha)\not{t}_{l\gamma\delta} - \overset{1}{H}_{l;\gamma,\delta}\right] \\
& +\delta^{i,j,k}_{\alpha,\beta,\gamma}\,\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\frac{1}{2}(1+\alpha)\not{t}_{\delta\epsilon\psi}\right] \qquad (4.4.52)
\end{aligned}
$$

$$
\begin{aligned}
\overset{\alpha}{\tilde{C}}{}^{i,j,k,l} &= j^{i,m}j^{j,n}j^{k,p}j^{l,q}\left[\left(\frac{9}{16}+\frac{5}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{mnpq}-\frac{1}{6}(1+\alpha)\not\ell_{m;npq}\right] \\
&+\delta^{l}_{\alpha}\, j^{i,m}j^{j,n}j^{k,p}\nu^{\alpha,\beta}\left[\left(-\frac{3}{4}-\alpha-\frac{1}{4}\alpha^2\right)\tau_{mnp\beta}-\frac{1}{4}(1+\alpha)\not\ell_{m;np\beta}\right. \\
&\left.+\frac{5}{12}(1+\alpha)\not\ell_{\beta;mnp}+\not\ell_{m\beta;np}+\frac{1}{2}(1+3\alpha)\overset{1}{\tau}_{m;\beta,np}\right] \\
&+\delta^{k,l}_{\alpha,\beta}\, j^{i,m}j^{j,n}\nu^{\alpha,\gamma}\nu^{\beta,\delta}\left[\left(\frac{1}{8}-\frac{1}{4}\alpha+\frac{1}{8}\alpha^2\right)\tau_{mn\gamma\delta}\right. \\
&+\left(\frac{1}{4}-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2\right)\tau_{m\gamma n\delta}+\left(\frac{1}{4}+\frac{3}{4}\alpha\right)\overset{-1}{\tau}_{\gamma,\delta,mn}-\frac{1}{2}\not\ell_{mn;\gamma\delta}-\frac{1}{2}\not\ell_{\gamma\delta;mn} \\
&\left.-\not\ell_{m\gamma;n\delta}-\frac{1}{2}\alpha\not\ell_{\gamma;mn\delta}+\alpha\not\ell_{m;n\gamma\delta}-(1+3\alpha)\overset{1}{\tau}_{m;\gamma,n\delta}-\overset{1}{\tau}_{m;n;\gamma,\delta}\right] \\
&+\delta^{j,k,l}_{\alpha,\beta,\gamma}\, j^{i,m}\nu^{\alpha,\delta}\nu^{\beta,\epsilon}\nu^{\gamma,\psi}\left[\left(\frac{1}{4}+2\alpha-\frac{1}{4}\alpha^2\right)\tau_{m\delta\epsilon\psi}\right. \\
&-\frac{1}{2}(1+3\alpha)\overset{-1}{\tau}_{\delta,\epsilon,m\psi}+\not\ell_{m\delta;\epsilon\psi}-\frac{1}{4}(1+\alpha)\not\ell_{\delta;m\epsilon\psi} \\
&\left.+\frac{1}{12}(1-7\alpha)\not\ell_{m;\delta\epsilon\psi}+\frac{3}{2}(1+\alpha)\overset{1}{\tau}_{m;\delta,\epsilon\psi}-\overset{1}{\tau}_{m;\delta\epsilon\psi}\right] \\
&+\delta^{i,j,k,l}_{\alpha,\beta,\gamma,\delta}\,\nu^{\alpha,\epsilon}\nu^{\beta,\psi}\nu^{\gamma,\phi}\nu^{\delta,\chi}\left[\left(-\frac{7}{16}-\frac{7}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{\epsilon\psi\phi\chi}\right. \\
&\left.+\frac{1}{4}(1+3\alpha)\overset{-1}{\tau}_{\epsilon,\psi,\phi\chi}+\frac{1}{4}\overset{-1}{\tau}_{\epsilon,\psi,\phi,\chi}+\frac{1}{6}(1+2\alpha)\not\ell_{\epsilon;\psi\phi\chi}\right]. \qquad (4.4.53)
\end{aligned}
$$

The same arguments as we used in Remark 14 in Section 4.4.3 show that the asymptotic distribution to order $O(n^{-\frac{1}{2}})$ of $\overset{\alpha}{\tilde{\mathcal{W}}}$ is a χ^2 distribution with p degrees of freedom. The simplifications in Remark 15 in Section 4.4.3 apply to $\overset{\alpha}{\tilde{\mathcal{W}}}$ as well.

4.4.5 Observed Bartlett-type Adjustment for $\overset{\alpha}{\mathcal{W}}$

An observed version of the Bartlett-type adjustment (3.5.91) can be found by determining A_1, A_2 and A_3 corresponding to the conditional moment generating function given the auxiliary statistic a, that is

$$
\begin{aligned}
M_{\overset{\alpha}{\mathcal{W}}|a}(t) &= \mathbb{E}\left[e^{\overset{\alpha}{\mathcal{W}}t}|a\right] \\
&= (1-2t)^{-\frac{p}{2}}\left(1+(24n)^{-1}\left\{A_1d+A_2d^2+A_3d^3+O(d^4)\right\}\right) \\
&\quad +O(n^{-\frac{3}{2}}). \qquad (4.4.54)
\end{aligned}
$$

Let $\overset{\alpha}{\tilde{w}}{}^0(Z_*)$, $\overset{\alpha}{\tilde{w}}{}^1(Z_*)$ and $\overset{\alpha}{\tilde{w}}{}^2(Z_*)$ be short for $\overset{\alpha}{\tilde{C}}{}^{i,j}Z_iZ_j$, $\overset{\alpha}{\tilde{C}}{}^{i,j,k}Z_iZ_jZ_k$ and $\overset{\alpha}{\tilde{C}}{}^{i,j,k,l}Z_iZ_jZ_kZ_l$, respectively, where $\overset{\alpha}{\tilde{C}}{}^{i,j}$, $\overset{\alpha}{\tilde{C}}{}^{i,j,k}$ and $\overset{\alpha}{\tilde{C}}{}^{i,j,k,l}$ are defined in (4.4.47)–(4.4.49). Then by (4.4.46), (4.3.21) and the approximation $e^x \simeq 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$ when x is close to zero, we find that

$$
\begin{aligned}
M_{\overset{\alpha}{\mathcal{W}}|a}(t) &= \int e^{\left(\overset{\alpha}{w}^0(z_*)+n^{-\frac{1}{2}}\left\{\overset{\alpha}{w}^1(z_*)\right\}+n^{-1}\left\{\overset{\alpha}{w}^2(z_*)\right\}\right)t}\varphi(z_*;j) \\
&\quad\times\left(1+n^{-\frac{1}{2}}\{R_1(z_*)\}+n^{-1}\{R_2(z_*)\}\right)dz_*+O(n^{-\frac{3}{2}}) \\
&= \int |2\pi j|^{-\frac{1}{2}}e^{\overset{\alpha}{w}^0(z_*)t-\frac{1}{2}z_*j^{-1}z_*^T}e^{\left(n^{-\frac{1}{2}}\left\{\overset{\alpha}{w}^1(z_*)\right\}+n^{-1}\left\{\overset{\alpha}{w}^2(z_*)\right\}\right)t} \\
&\quad\times\left(1+n^{-\frac{1}{2}}\{R_1(z_*)\}+n^{-1}\{R_2(z_*)\}\right)dz_*+O(n^{-\frac{3}{2}}) \\
&= \int |2\pi j|^{-\frac{1}{2}}e^{-\frac{1}{2}(1-2t)z_*j^{-1}z_*^T}\left(1+n^{-\frac{1}{2}}\left\{\overset{\alpha}{w}^1(z_*)t\right\}+n^{-1}\left\{\frac{1}{2}\left(\overset{\alpha}{w}^1(z_*)t\right)^2+\overset{\alpha}{w}^2(z_*)t\right\}\right) \\
&\quad\times\left(1+n^{-\frac{1}{2}}\{R_1(z_*)\}+n^{-1}\{R_2(z_*)\}\right)dz_*+O(n^{-\frac{3}{2}}) \\
&= (1-2t)^{-\frac{p}{2}}\int |2\pi(1-2t)^{-1}j|^{-\frac{1}{2}}e^{-\frac{1}{2}z_*\left((1-2t)^{-1}j\right)^{-1}z_*^T}\left(1+n^{-\frac{1}{2}}\left\{\overset{\alpha}{w}^1(z_*)t+R_1(z_*)\right\}\right. \\
&\quad\left.+n^{-1}\left\{\frac{1}{2}\left(\overset{\alpha}{w}^1(z_*)t\right)^2+\overset{\alpha}{w}^2(z_*)t+R_1(z_*)\overset{\alpha}{w}^1(z_*)t+R_2(z_*)\right\}\right)dz_*+O(n^{-\frac{3}{2}}) \\
&= (1-2t)^{-\frac{p}{2}}\left(1+n^{-\frac{1}{2}}\left\{\mathbb{E}\left[\overset{\alpha}{w}^1(Z_*)|a\right]t+\mathbb{E}\left[R_1(Z_*)|a\right]\right\}\right. \\
&\quad+n^{-1}\left\{\frac{1}{2}\mathbb{E}\left[\left(\overset{\alpha}{w}^1(Z_*)t\right)^2|a\right]+\mathbb{E}\left[\overset{\alpha}{w}^2(Z_*)|a\right]t+\mathbb{E}\left[R_1(Z_*)\overset{\alpha}{w}^1(Z_*)|a\right]t\right. \\
&\quad\left.\left.+\ \mathbb{E}\left[R_2(Z_*)|a\right]\right\}\right)+O(n^{-\frac{3}{2}}). \qquad (4.4.55)
\end{aligned}
$$

We are taking the conditional expectation with respect to a normal distribution with zero mean and variance $V=(1-2t)^{-1}j=(1+d)j$. Note that R_1 and R_2 both depend on the observed information matrix j, i.e. $R_1=R_1(Z_*;j)$ and $R_2=R_2(Z_*;j)$, where $Z_*\sim N_r(0,(1+d)j)$.

In the following, let $\mathbb{E}[\cdot]$ be short for the conditional expectation with respect to a $N_r(0,(1+d)j)$ distribution. By Property 3 of Hermite polynomials given in (1.4.45), and symmetry, we find that

$$
\begin{aligned}
\mathbb{E}\left[\overset{\alpha}{w}^1(Z_*)\right]t &= \overset{\alpha}{C}^{i,j,k}\mathbb{E}\left[Z_iZ_jZ_k\right]t \\
&= 0, \qquad (4.4.56)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[R_1(Z_*;j)\right] &= \frac{1}{6}j^{i,l}j^{j,m}j^{k,n}\not{\ell}_{ijk}\mathbb{E}\left[h_{lmn}\right] \\
&= 0. \qquad (4.4.57)
\end{aligned}
$$

Thus, the $O(n^{-\frac{1}{2}})$-term of (4.4.55) vanishes. Moreover, we find that

$$\begin{aligned}
\mathbb{E}\left[R_2(Z_*;j)\right] &= -\frac{1}{4} j^{k,l} j^{i,m} j^{j,n} \not{\ell}_{ij;kl} \mathbb{E}\left[h_{mn}\right] \\
&\quad + \frac{1}{24} j^{i,m} j^{j,n} j^{k,p} j^{l,q} (\not{\ell}_{i;jkl} - \not{\ell}_{kl;ij}\{3\}_{jkl}) \mathbb{E}\left[h_{mnpq}\right] \\
&\quad + \frac{1}{72} j^{i,p} j^{j,q} j^{k,r} j^{l,s} j^{m,t} j^{n,u} \not{\ell}_{ijk} \not{\ell}_{lmn} \mathbb{E}\left[h_{pqrstu}\right] \\
&= -\frac{1}{4} \not{\ell}_{ij;kl} j^{k,l} j^{i,j} d + \frac{1}{24} (\not{\ell}_{i;jkl} - \not{\ell}_{kl;ij}\{3\}_{jkl}) \left(j^{i,j} j^{k,l}\{3\}\right) d^2 \\
&\quad + \frac{1}{72} \not{\ell}_{ijk} \not{\ell}_{lmn} \left(j^{i,j} j^{k,l} j^{m,n}\{15\}\right) d^3 \\
&= \frac{1}{24}\left[D_{11} d + D_{12} d^2 + D_{13} d^3 + O(d^4)\right], \qquad (4.4.58)
\end{aligned}$$

where

$$\begin{aligned}
D_{11} &= -6\,\not{\ell}_{ij;kl} j^{i,j} j^{k,l} \\
D_{12} &= 3\left(\not{\ell}_{i;jkl} - \not{\ell}_{ij;kl}\{3\}_{jkl}\right) j^{i,j} j^{k,l} \\
D_{13} &= (3\tau_{ijkl} + 2\tau_{ikjl})\, j^{i,j} j^{k,l}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbb{E}\left[\overset{\alpha}{w}{}^1(Z_*) R_1(Z_*;j)\right] t &= \frac{1}{6} \overset{\alpha}{C}{}^{i,j,k} j^{l,p} j^{m,q} j^{n,r} \not{\ell}_{pqr} \mathbb{E}\left[Z_i Z_j Z_k h_{lmn}\right] \left(\frac{1}{2}(d - d^2 + d^3) + O(d^4)\right) \\
&= \frac{1}{12} \overset{\alpha}{C}{}^{i,j,k} \not{\ell}_{pqr} j^{l,p} j^{m,q} j^{n,r} \mathbb{E}\left[h_{ijklmn} + \jmath_{i,j} h_{klmn}\{15\} + \jmath_{i,j}\jmath_{k,l} h_{mn}\{45\}\right. \\
&\quad + \jmath_{i,j}\jmath_{k,l}\jmath_{m,n}\{15\} - (h_{ijkl} + \jmath_{i,j} h_{kl}\{6\} \\
&\quad \left. + \jmath_{i,j}\jmath_{k,l}\{3\})\, \jmath_{m,n}\{3\}_{lmn}\right] (d - d^2 + d^3) + O(d^4) \\
&= \frac{1}{12} \overset{\alpha}{C}{}^{i,j,k} \not{\ell}_{pqr} j^{l,p} j^{m,q} j^{n,r} \left[6 \jmath_{i,l}\jmath_{j,m}\jmath_{k,n} d\right. \\
&\quad + (9\jmath_{i,j}\jmath_{k,l}\jmath_{m,n} + 12\jmath_{i,l}\jmath_{j,m}\jmath_{k,n}) d^2 \\
&\quad \left. + (9\jmath_{i,j}\jmath_{k,l}\jmath_{m,n} + 6\jmath_{i,l}\jmath_{j,m}\jmath_{k,n}) d^3\right] + O(d^4) \\
&= \frac{1}{24}\left[D_{21} d + D_{22} d^2 + D_{23} d^3 + O(d^4)\right], \qquad (4.4.59)
\end{aligned}$$

where

$$\begin{aligned}
D_{21} &= 12\, \overset{\alpha}{C}{}^{i,j,k} \not{\ell}_{ijk} \\
D_{22} &= 24\, \overset{\alpha}{C}{}^{i,j,k} \not{\ell}_{ijk} + 18\, \overset{\alpha}{C}{}^{i,j,k} j^{l,m} \not{\ell}_{klm} \jmath_{i,j} \\
D_{23} &= 12\, \overset{\alpha}{C}{}^{i,j,k} \not{\ell}_{ijk} + 18\, \overset{\alpha}{C}{}^{i,j,k} j^{l,m} \not{\ell}_{klm} \jmath_{i,j}.
\end{aligned}$$

Further,

$$
\begin{aligned}
\mathbb{E}\left[\overset{\alpha}{w}{}^2(Z_*)\right]t &= \overset{\alpha}{C}{}^{i,j,k,l}\mathbb{E}\left[Z_iZ_jZ_kZ_l\right]\left(\frac{1}{2}(d-d^2+d^3)+O(d^4)\right) \\
&= \frac{1}{2}\overset{\alpha}{C}{}^{i,j,k,l}\mathbb{E}\left[h_{ijkl}+j_{i,j}h_{kl}\left\{6\right\}+j_{i,j}j_{k,l}\left\{3\right\}\right] \\
&\quad \times\left(d-d^2+d^3\right)+O(d^4) \\
&= \frac{1}{2}\overset{\alpha}{C}{}^{i,j,k,l}\left[j_{i,j}j_{k,l}\left\{3\right\}d^2+2j_{i,j}j_{k,l}\left\{3\right\}d+j_{i,j}j_{k,l}\left\{3\right\}\right] \\
&\quad \times\left(d-d^2+d^3\right)+O(d^4) \\
&= \frac{1}{2}\overset{\alpha}{C}{}^{i,j,k,l}\left[j_{i,j}j_{k,l}\left\{3\right\}(d+d^2)\right]+O(d^4) \\
&= \frac{1}{24}\left[D_{31}d+D_{32}d^2+D_{33}d^3+O(d^4)\right], \qquad (4.4.60)
\end{aligned}
$$

where

$$
\begin{aligned}
D_{31} &= 12\,\overset{\alpha}{C}{}^{i,j,k,l}\left(j_{i,j}j_{k,l}\left\{3\right\}\right) \\
D_{32} &= 12\,\overset{\alpha}{C}{}^{i,j,k,l}\left(j_{i,j}j_{k,l}\left\{3\right\}\right) \\
D_{33} &= 0.
\end{aligned}
$$

Finally, we find that

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\left(\overset{\alpha}{w}{}^1(Z_*)\right)^2\right]t^2 &= \frac{1}{2}\overset{\alpha}{C}{}^{i,j,k}\,\overset{\alpha}{C}{}^{l,m,n}\mathbb{E}\left[Z_iZ_jZ_kZ_lZ_mZ_n\right]\left(\frac{1}{4}(d^2-2d^3)+O(d^4)\right) \\
&= \frac{1}{8}\overset{\alpha}{C}{}^{i,j,k}\,\overset{\alpha}{C}{}^{l,m,n}\mathbb{E}\left[h_{ijklmn}+j_{i,j}h_{klmn}\left\{15\right\}+j_{i,j}j_{k,l}h_{mn}\left\{45\right\}\right. \\
&\quad \left.+j_{i,j}j_{k,l}j_{m,n}\left\{15\right\}\right](d^2-2d^3)+O(d^4) \\
&= \frac{1}{8}\overset{\alpha}{C}{}^{i,j,k}\,\overset{\alpha}{C}{}^{l,m,n}\left[j_{i,l}j_{j,m}j_{k,n}\left\{15\right\}(d^2+d^3)\right]+O(d^4) \\
&= \frac{1}{24}\left[D_{41}d+D_{42}d^2+D_{43}d^3+O(d^4)\right], \qquad (4.4.61)
\end{aligned}
$$

where

$$
\begin{aligned}
D_{41} &= 0 \\
D_{42} &= 3\,\overset{\alpha}{C}{}^{i,j,k}\,\overset{\alpha}{C}{}^{l,m,n}\left(j_{i,l}j_{j,m}j_{k,n}\left\{15\right\}\right) \\
D_{43} &= 3\,\overset{\alpha}{C}{}^{i,j,k}\,\overset{\alpha}{C}{}^{l,m,n}\left(j_{i,l}j_{j,m}j_{k,n}\left\{15\right\}\right).
\end{aligned}
$$

Then, substituting (4.4.58)–(4.4.61) into (4.4.55) gives

$$
\begin{aligned}
M_{\overset{\alpha}{\mathcal{W}}|a}(t) &= \mathbb{E}\left[e^{\overset{\alpha}{\mathcal{W}}t}|a\right] \\
&= (1-2t)^{-\frac{p}{2}}\left(1+(24n)^{-1}\left\{A_1d+A_2d^2+A_3d^3+O(d^4)\right\}\right)
\end{aligned}
$$

$$+O(n^{-\frac{3}{2}}), \tag{4.4.62}$$

as in (4.4.54), where

$$\begin{aligned} A_1 &= D_{11} + D_{21} + D_{31} + D_{41} \\ &= -6\,\not{t}_{ij;kl} j^{i,j} j^{k,l} + 12 \overset{\alpha}{C}{}^{i,j,k} \not{t}_{ijk} + 12 \overset{\alpha}{C}{}^{i,j,k,l} (j_{i,j} j_{k,l} \{3\}), \end{aligned}$$

$$\begin{aligned} A_2 &= D_{12} + D_{22} + D_{32} + D_{42} \\ &= 3\left(\not{t}_{i;jkl} - \not{t}_{kl;ij} \{3\}_{jkl}\right) j^{i,j} j^{k,l} + 24 \overset{\alpha}{C}{}^{i,j,k} \not{t}_{ijk} + 18 \overset{\alpha}{C}{}^{i,j,k} j^{l,m} \not{t}_{klm} j_{i,j} \\ &\quad +12 \overset{\alpha}{C}{}^{i,j,k,l} (j_{i,j} j_{k,l} \{3\}) + 3 \overset{\alpha}{C}{}^{i,j,k} \overset{\alpha}{C}{}^{l,m,n} (j_{i,j} j_{k,l} j_{l,m} \{15\}), \end{aligned}$$

$$\begin{aligned} A_3 &= D_{13} + D_{23} + D_{33} + D_{43} \\ &= (3\tau_{ijkl} + 2\tau_{ikjl}) j^{i,j} j^{k,l} + 12 \overset{\alpha}{C}{}^{i,j,k} \not{t}_{ijk} \\ &\quad +18 \overset{\alpha}{C}{}^{i,j,k} j^{l,m} \not{t}_{klm} j_{i,j} + 3 \overset{\alpha}{C}{}^{i,j,k} \overset{\alpha}{C}{}^{l,m,n} (j_{i,j} j_{k,l} j_{l,m} \{15\}). \end{aligned}$$

By Proposition 1 in Section 3.1.2 , we obtain an observed Bartlett-adjusted observed geometric Wald statistic,

$$\overset{\alpha}{\mathcal{W}}{}' = \left\{1 - \frac{1}{n}(c + b\,\overset{\alpha}{\mathcal{W}} + a\overset{\alpha}{\mathcal{W}}{}^2)\right\} \overset{\alpha}{\mathcal{W}}, \tag{4.4.63}$$

where

$$\begin{aligned} a &= \frac{A_3}{12p(p+2)(p+4)} \\ &= \frac{1}{12p(p+2)(p+4)} (3\tau_{ijkl} + 2\tau_{ikjl}) j^{i,j} j^{k,l} + 12 \overset{\alpha}{C}{}^{i,j,k} \not{t}_{ijk} \\ &\quad +18 \overset{\alpha}{C}{}^{i,j,k} j^{l,m} \not{t}_{klm} j_{i,j} + 3 \overset{\alpha}{C}{}^{i,j,k} \overset{\alpha}{C}{}^{l,m,n} (j_{i,j} j_{k,l} j_{l,m} \{15\}), \end{aligned} \tag{4.4.64}$$

$$\begin{aligned} b &= \frac{A_2 - 2A_3}{12p(p+2)} \\ &= \frac{1}{12p(p+2)} \left(\left[3\,\not{t}_{i;jkl} - 3\,\not{t}_{ij;kl} \{3\}_{jkl} - 6\tau_{ijkl} - 4\tau_{ikjl}\right] j^{i,j} j^{k,l} \right. \\ &\quad +12 \overset{\alpha}{C}{}^{i,j,k,l} (j_{i,j} j_{k,l} \{3\}) - 18 \overset{\alpha}{C}{}^{i,j,k} j^{l,m} \not{t}_{klm} j_{i,j} \\ &\quad \left. - 3 \overset{\alpha}{C}{}^{i,j,k} \overset{\alpha}{C}{}^{l,m,n} (j_{i,j} j_{k,l} j_{l,m} \{15\})\right), \end{aligned} \tag{4.4.65}$$

$$c = \frac{A_1 - A_2 + A_3}{12p}$$

$$= \frac{1}{12p}\left[\left(3\not{t}_{ijkl;} + 12\not{t}_{ik;jl} + 3\tau_{ijkl} + 2\tau_{ikjl}\right) j^{i,j}j^{k,l}\right]. \qquad (4.4.66)$$

Observe that c does not depend on the value of α. Under the null hypothesis, $\overset{\alpha}{\widetilde{\mathcal{W}}}\,'$ has a χ^2 distribution with p degrees of freedom and error of order $O(n^{-\frac{3}{2}})$.

4.4.6 Observed Bartlett-type Adjustment for $\overset{\alpha}{\widetilde{W}}$

Let $\overset{\alpha}{\widetilde{w}}{}^0(Z_*)$, $\overset{\alpha}{\widetilde{w}}{}^1(Z_*)$ and $\overset{\alpha}{\widetilde{w}}{}^2(Z_*)$ be short for $\overset{\alpha}{\widetilde{C}}{}^{i,j}Z_iZ_j$, $\overset{\alpha}{\widetilde{C}}{}^{i,j,k}Z_iZ_jZ_k$ and $\overset{\alpha}{\widetilde{C}}{}^{i,j,k,l}Z_iZ_jZ_kZ_l$, respectively, where $\overset{\alpha}{\widetilde{C}}{}^{i,j}$, $\overset{\alpha}{\widetilde{C}}{}^{i,j,k}$ and $\overset{\alpha}{\widetilde{C}}{}^{i,j,k,l}$ are defined in (4.4.51)–(4.4.53). We can find $\widetilde{A}_1$, $\widetilde{A}_2$ and $\widetilde{A}_3$ in the moment generating functions (3.1.6) for the modified geometric Wald statistics, that is

$$\begin{aligned} M_{\overset{\alpha}{\widetilde{\mathcal{W}}}}(t) &= \mathbb{E}\left[e^{\overset{\alpha}{\widetilde{\mathcal{W}}}t}\right] \\ &= (1-2t)^{-\frac{p}{2}}\left(1 + (24n)^{-1}\left\{\widetilde{A}_1 d + \widetilde{A}_2 d^2 + \widetilde{A}_3 d^3 + O(d^4)\right\}\right) \\ &\quad + O(n^{-\frac{3}{2}}), \end{aligned} \qquad (4.4.67)$$

by calculating coefficients $\widetilde{D}_{11}\ldots\widetilde{D}_{43}$ analogous to $D_{11}\ldots D_{43}$. We find that

$$\begin{aligned} \widetilde{D}_{11} &= D_{11} = 0 \\ \widetilde{D}_{12} &= D_{12} = 3\left(\not{t}_{i;jkl} - \not{t}_{ij;kl}\{3\}_{jkl}\right) j^{i,j}j^{k,l} \\ \widetilde{D}_{13} &= D_{13} = (3\tau_{ijkl} + 2\tau_{ikjl})\, j^{i,j}j^{k,l}. \end{aligned}$$

We find that

$$\begin{aligned} \mathbb{E}\left[\overset{\alpha}{\widetilde{w}}{}^1(Z_*)R_1(Z_*;j)\right]t &= \frac{1}{6}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k} j^{l,p}j^{m,q}j^{n,r}\not{t}_{pqr}\mathbb{E}\left[Z_iZ_jZ_kh_{lmn}\right]\left(\frac{1}{2}(d-d^2+d^3)+O(d^4)\right) \\ &= \frac{1}{12}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\not{t}_{pqr}j^{l,p}j^{m,q}j^{n,r}\mathbb{E}\left[h_{ijklmn} + j_{i,j}h_{klmn}\{15\}\right. \\ &\quad + j_{i,j}j_{k,l}h_{mn}\{45\} + j_{i,j}j_{k,l}j_{m,n}\{15\} - (h_{ijkl} + j_{i,j}h_{kl}\{6\} \\ &\quad \left. + j_{i,j}j_{k,l}\{3\})\, j_{m,n}\{3\}_{lmn}\right](d-d^2+d^3) + O(d^4) \\ &= \frac{1}{12}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\not{t}_{pqr}j^{l,p}j^{m,q}j^{n,r}\left[6j_{i,l}j_{j,m}j_{k,n}d\right. \\ &\quad + (9j_{i,j}j_{k,l}j_{m,n} + 12j_{i,l}j_{j,m}j_{k,n})d^2 \\ &\quad \left. + (9j_{i,j}j_{k,l}j_{m,n} + 6j_{i,l}j_{j,m}j_{k,n})d^3\right] + O(d^4) \\ &= \frac{1}{24}\left[\widetilde{D}_{21}d + \widetilde{D}_{22}d^2 + \widetilde{D}_{23}d^3 + O(d^4)\right], \end{aligned} \qquad (4.4.68)$$

where

$$
\begin{aligned}
\widetilde{D}_{21} &= 12\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\,\not{t}_{ijk} \\
\widetilde{D}_{22} &= 24\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\,\not{t}_{ijk} + 18\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\jmath^{l,m}\,\not{t}_{klm}\jmath_{i,j} \\
\widetilde{D}_{23} &= 12\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\,\not{t}_{ijk} + 18\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\jmath^{l,m}\,\not{t}_{klm}\jmath_{i,j}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathbb{E}\left[\overset{\alpha}{\widetilde{w}}{}^{2}(Z_*)\right]t &= \overset{\alpha}{\widetilde{C}}{}^{\,i,j,k,l}\,\mathbb{E}\left[Z_iZ_jZ_kZ_l\right]\left(\frac{1}{2}(d-d^2+d^3)+O(d^4)\right) \\
&= \frac{1}{2}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k,l}\,\mathbb{E}\left[h_{ijkl}+\jmath_{i,j}h_{kl}\,\{6\}+\jmath_{i,j}\jmath_{k,l}\,\{3\}\right] \\
&\quad\times\left(d-d^2+d^3\right)+O(d^4) \\
&= \frac{1}{2}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k,l}\left[\jmath_{i,j}\jmath_{k,l}\,\{3\}\,d^2+2\jmath_{i,j}\jmath_{k,l}\,\{3\}\,d+\jmath_{i,j}\jmath_{k,l}\,\{3\}\right] \\
&\quad\times\left(d-d^2+d^3\right)+O(d^4) \\
&= \frac{1}{2}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k,l}\left[\jmath_{i,j}\jmath_{k,l}\,\{3\}\,(d+d^2)\right]+O(d^4) \\
&= \frac{1}{24}\left[\widetilde{D}_{31}d+\widetilde{D}_{32}d^2+\widetilde{D}_{33}d^3+O(d^4)\right], \qquad (4.4.69)
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{D}_{31} &= 12\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k,l}\left(\jmath_{i,j}\jmath_{k,l}\,\{3\}\right) \\
\widetilde{D}_{32} &= 12\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k,l}\left(\jmath_{i,j}\jmath_{k,l}\,\{3\}\right) \\
\widetilde{D}_{33} &= 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\left(\overset{\alpha}{\widetilde{w}}{}^{1}(Z_*)\right)^2\right]t^2 &= \frac{1}{2}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\,\overset{\alpha}{\widetilde{C}}{}^{\,l,m,n}\,\mathbb{E}\left[Z_iZ_jZ_kZ_lZ_mZ_n\right]\left(\frac{1}{4}(d^2-2d^3)+O(d^4)\right) \\
&= \frac{1}{8}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\,\overset{\alpha}{\widetilde{C}}{}^{\,l,m,n}\,\mathbb{E}\left[h_{ijklmn}+\jmath_{i,j}h_{klmn}\,\{15\}+\jmath_{i,j}\jmath_{k,l}h_{mn}\,\{45\}\right. \\
&\quad\left.+\jmath_{i,j}\jmath_{k,l}\jmath_{m,n}\,\{15\}\right](d^2-2d^3)+O(d^4) \\
&= \frac{1}{8}\,\overset{\alpha}{\widetilde{C}}{}^{\,i,j,k}\,\overset{\alpha}{\widetilde{C}}{}^{\,l,m,n}\left[\jmath_{i,l}\jmath_{j,m}\jmath_{k,n}\,\{15\}\,(d^2+d^3)\right]+O(d^4) \\
&= \frac{1}{24}\left[\widetilde{D}_{41}d+\widetilde{D}_{42}d^2+\widetilde{D}_{43}d^3+O(d^4)\right], \qquad (4.4.70)
\end{aligned}
$$

where

$$
\widetilde{D}_{41} = 0
$$

$$\begin{aligned}
\widetilde{D}_{42} &= 3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\,\overset{\alpha}{\widetilde{C}}{}^{l,m,n}\left(\jmath_{i,l}\jmath_{j,m}\jmath_{k,n}\{15\}\right)\\
\widetilde{D}_{43} &= 3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\,\overset{\alpha}{\widetilde{C}}{}^{l,m,n}\left(\jmath_{i,l}\jmath_{j,m}\jmath_{k,n}\{15\}\right).
\end{aligned}$$

We substitute (4.4.68)–(4.4.70) into the analogue for $\overset{\alpha}{\widetilde{\mathcal{W}}}$ of (4.4.55) to get

$$\begin{aligned}
M_{\overset{\alpha}{\widetilde{\mathcal{W}}}|a}(t) &= \mathbb{E}\left[e^{\overset{\alpha}{\widetilde{\mathcal{W}}}t}|a\right]\\
&= (1-2t)^{-\frac{p}{2}}\left(1+(24n)^{-1}\left\{\widetilde{A}_1 d+\widetilde{A}_2 d^2+\widetilde{A}_3 d^3+O(d^4)\right\}\right)\\
&\quad +O(n^{-\frac{3}{2}}),
\end{aligned}\tag{4.4.71}$$

as in (4.4.67), where

$$\begin{aligned}
\widetilde{A}_1 &= \widetilde{D}_{11}+\widetilde{D}_{21}+\widetilde{D}_{31}+\widetilde{D}_{41}\\
&= -6\,\not{t}_{ij;kl}\,j^{i,j}j^{k,l}+12\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\,\not{t}_{ijk}+12\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k,l}\left(\jmath_{i,j}\jmath_{k,l}\{3\}\right),
\end{aligned}$$

$$\begin{aligned}
\widetilde{A}_2 &= \widetilde{D}_{12}+\widetilde{D}_{22}+\widetilde{D}_{32}+\widetilde{D}_{42}\\
&= 3\left(\not{t}_{i;jkl}-\not{t}_{kl;ij}\{3\}_{jkl}\right)j^{i,j}j^{k,l}+24\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\,\not{t}_{ijk}+18\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}j^{l,m}\,\not{t}_{klm}\jmath_{i,j}\\
&\quad +12\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k,l}\left(\jmath_{i,j}\jmath_{k,l}\{3\}\right)+3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\,\overset{\alpha}{\widetilde{C}}{}^{l,m,n}\left(\jmath_{i,j}\jmath_{k,l}\jmath_{l,m}\{15\}\right),
\end{aligned}$$

$$\begin{aligned}
\widetilde{A}_3 &= \widetilde{D}_{13}+\widetilde{D}_{23}+\widetilde{D}_{33}+\widetilde{D}_{43}\\
&= \left(3\tau_{ijkl}+2\tau_{ikjl}\right)j^{i,j}j^{k,l}+12\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\,\not{t}_{ijk}\\
&\quad +18\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}j^{l,m}\,\not{t}_{klm}\jmath_{i,j}+3\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\,\overset{\alpha}{\widetilde{C}}{}^{l,m,n}\left(\jmath_{i,j}\jmath_{k,l}\jmath_{l,m}\{15\}\right).
\end{aligned}$$

We find a modified observed Bartlett adjusted geometric Wald statistic $\overset{\alpha}{\widetilde{\mathcal{W}}}{}'$ by Proposition 1 in Section 3.1.2. The adjusted statistic has a χ^2 distribution under the null hypothesis with p degrees of freedom and error of order $O(n^{-\frac{3}{2}})$. Let

$$\overset{\alpha}{\widetilde{\mathcal{W}}}{}'=\left\{1-\frac{1}{n}(\tilde{c}+\tilde{b}\,\overset{\alpha}{\widetilde{\mathcal{W}}}+\tilde{a}\overset{\alpha}{\widetilde{\mathcal{W}}}{}^2)\right\}\overset{\alpha}{\widetilde{\mathcal{W}}},\tag{4.4.72}$$

where

$$\begin{aligned}
\tilde{a} &= \frac{\widetilde{A}_3}{12p(p+2)(p+4)}\\
&= \frac{1}{12p(p+2)(p+4)}\left(\left(3\tau_{ijkl}+2\tau_{ikjl}\right)j^{i,j}j^{k,l}+12\,\overset{\alpha}{\widetilde{C}}{}^{i,j,k}\,\not{t}_{ijk}\right.
\end{aligned}$$

$$+18 \overset{\alpha}{\widetilde{C}}{}^{i,j,k} j^{l,m} \not{t}_{klm} j_{i,j} + 3 \overset{\alpha}{\widetilde{C}}{}^{i,j,k} \overset{\alpha}{\widetilde{C}}{}^{l,m,n} (j_{i,j} j_{k,l} j_{l,m} \{15\}) \Big), \qquad (4.4.73)$$

$$\begin{aligned} \tilde{b} &= \frac{\widetilde{A}_2 - 2\widetilde{A}_3}{12p(p+2)} \\ &= \frac{1}{12p(p+2)} \Big(\Big[3 \not{t}_{i;jkl} - 3 \not{t}_{ij;kl} \{3\}_{jkl} - 6\tau_{ijkl} - 4\tau_{ikjl} \Big] j^{i,j} j^{k,l} \\ &\quad +12 \overset{\alpha}{\widetilde{C}}{}^{i,j,k,l} (j_{i,j} j_{k,l} \{3\}) - 18 \overset{\alpha}{\widetilde{C}}{}^{i,j,k} j^{l,m} \not{t}_{klm} j_{i,j} \\ &\quad -3 \overset{\alpha}{\widetilde{C}}{}^{i,j,k} \overset{\alpha}{\widetilde{C}}{}^{l,m,n} (j_{i,j} j_{k,l} j_{l,m} \{15\}), \end{aligned} \qquad (4.4.74)$$

$$\begin{aligned} \tilde{c} &= \frac{\widetilde{A}_1 - \widetilde{A}_2 + \widetilde{A}_3}{12p} \\ &= \frac{1}{12p} \Big[\Big(3 \not{t}_{ijkl;} + 12 \not{t}_{ik;jl} + 3\tau_{ijkl} + 2\tau_{ikjl} \Big) j^{i,j} j^{k,l} \Big]. \end{aligned} \qquad (4.4.75)$$

Observe that $\tilde{c}$ does not depend on α. Moreover, $\tilde{c}$ coincides with c in (4.4.66).

Chapter 5

Simple Null Hypotheses

When we are testing a simple null hypothesis, the Taylor expansions of the geometric Wald statistics simplify remarkably in both expected and observed geometries. Hence, obtaining the Bartlett-type adjustments (3.5.91) and (3.5.100), and (4.4.63) and (4.4.72) becomes more manageable.

Suppose we wish to test the simple hypothesis

$$H_0 : \theta = \theta_0$$

against

$$H_1 : \theta \in \Theta.$$

As in the previous sections, we denote the maximum likelihood estimate of θ under the alternative hypothesis by $\hat{\theta}$.

5.1 Expected Geometry

Recall that the family of geometric Wald statistics (2.3.6) in expected geometry is defined as

$$\overset{\alpha}{\mathcal{W}} = \overset{\alpha}{\Gamma}_i(\hat{\theta};\theta_0) i^{i,j}(\hat{\theta}) \overset{\alpha}{\Gamma}_j(\hat{\theta};\theta_0), \tag{5.1.1}$$

with

$$\overset{\alpha}{\Gamma}_i(\theta;\theta') = n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} f_{;i}(\theta';\theta) + \frac{1-\alpha}{2} f_i(\theta;\theta') \right\},$$

where $f(\theta;\theta')$ is the expected likelihood yoke (1.3.16),

$$f(\theta;\theta') = n^{-1}\mathbb{E}_{\theta'}\left[l(\theta) - l(\theta')\right]. \tag{5.1.2}$$

We use definitions (3.2.16)–(3.2.22) and (3.2.27) of the relevant Y- $\not{t}$- and τ-tensors from Section 3.2. Again, we denote the cumulants of the vector $Y_* = (Y_1, \ldots, Y_p, Y_{11}, \ldots, Y_{pp}, Y_{111}, \ldots Y_{ppp})$ by λ_R. Observe from (1.4.25) and (1.4.26) that

$$\begin{aligned}
\lambda_{i,j,k} &= \kappa_{i,j,k} \\
\lambda_{ij,kl} &= -\kappa_{ij;m} i^{m,n} \kappa_{n,kl} + \kappa_{ij,kl} - \kappa_{ij}\kappa_{kl} \\
\lambda_{ij,k,l} &= -\kappa_{ij,m} i^{m,n} \kappa_{n,k,l} + \kappa_{ij,k,l} - \kappa_{ij} i_{k,l} \\
\lambda_{i,j,k,l} &= \kappa_{i,j,k,l} - i_{i,j} i_{k,l}\{3\},
\end{aligned} \tag{5.1.3}$$

where the κs are the moments of the normalised derivatives Z_i, Z_{ij} in (3.2.11)–(3.2.12) of the log-likelihood function given by (1.4.20). Thus, by the definitions of the $\not{t}$-tensors and τ-tensors we find that

$$\not{t}_{ijk} = \lambda_{i,j,k} \tag{5.1.4}$$

$$\not{t}_{ij;kl} = \lambda_{ij,kl} + \lambda_{ij,k,l} \tag{5.1.5}$$

$$\not{t}_{i;jkl} = \lambda_{i,j,k,l} + \lambda_{i,j,kl}\{3\}_{jkl} \tag{5.1.6}$$

$$\not{t}_{ijkl;} = -\lambda_{i,j,k,l} - \lambda_{ij,kl}\{3\}_{jkl} - \lambda_{i,j,kl}\{6\} \tag{5.1.7}$$

$$\tau_{ijkl} = \lambda_{i,j,m} i^{m,n} \lambda_{k,l,n}. \tag{5.1.8}$$

5.1.1 Bartlett Correction of $\overset{\alpha}{\mathcal{W}}$

From (3.5.64) we find that the Taylor expansion of the geometric Wald statistic for testing a simple null hypothesis is

$$\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= \overset{\alpha}{\Gamma}_i(\hat\theta;\theta_0) i^{i,j}(\hat\theta) \overset{\alpha}{\Gamma}_j(\hat\theta;\theta_0) \\
&= i^{i,j} Y_i Y_j + n^{-\frac{1}{2}} \left\{ 2 i^{i,k} i^{j,l} Y_{ij} Y_k Y_k - \frac{1}{2}(1-\alpha) \not{t}_{ijk} i^{i,l} i^{j,m} i^{k,n} Y_l Y_m Y_n \right\} \\
&\quad n^{-1} \left\{ i^{i,l} i^{j,m} i^{k,n} Y_{ijk} Y_l Y_m Y_n + 3 i^{j,k} i^{i,m} i^{l,n} Y_{ij} Y_{kl} Y_m Y_n \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(-5+3\alpha)\not{t}_{ijk}i^{k,l}i^{i,n}i^{j,p}i^{m,q}Y_{lm}Y_nY_pY_q \\
&+\left[\left(\frac{9}{16}-\frac{7}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{ijkl}-\not{t}_{ij;kl}-\left(\frac{1}{6}-\frac{1}{3}\alpha\right)\not{t}_{i;jkl}\right]i^{i,m}i^{j,n}i^{k,p}i^{l,q}Y_mY_nY_pY_q\Big\} \\
&+O(n^{-\frac{3}{2}}). \qquad (5.1.9)
\end{aligned}
$$

Thus, the Bartlett-type adjusted geometric Wald statistic (3.5.91) simplifies to

$$
\overset{\alpha}{\mathcal{W}}{}' = \left\{1-\frac{1}{n}(c+b\overset{\alpha}{\mathcal{W}}+a\overset{\alpha}{\mathcal{W}}{}^2)\right\}\overset{\alpha}{\mathcal{W}}, \qquad (5.1.10)
$$

where

$$
\begin{aligned}
a &= \frac{1}{12p(p+2)(p+4)}\left(\left(\frac{3}{4}-\frac{9}{2}\alpha+\frac{27}{4}\alpha^2\right)\tau_{ijkl}+\left(\frac{1}{2}-3\alpha+\frac{9}{2}\alpha^2\right)\tau_{ikjl}\right)i^{i,j}i^{k,l}, \\
b &= \frac{1}{12p(p+2)}\Big((-3+12\alpha)\lambda_{i,j,k,l}+(-6+12\alpha)\lambda_{ij,k,l}\{3\}_{jkl} \\
&\quad +(3-6\alpha-6\alpha^2)\tau_{ijkl}+(5-12\alpha-3\alpha^2)\tau_{ikjl}\Big)i^{i,j}i^{k,l}, \\
c &= \frac{1}{12p}\left(-3\lambda_{i,j,k,l}+12(\lambda_{ik;jl}-\lambda_{ij,k,l}-\lambda_{ij;kl})+3\tau_{ijkl}+2\tau_{ikjl}\right)i^{i,j}i^{k,l}.
\end{aligned}
$$

Bartlett Correction of $\overset{\alpha}{\widetilde{\mathcal{W}}}$

Likewise, from (3.5.73) we find that the Taylor expansion of the modified geometric Wald statistic for testing a simple null hypothesis is

$$
\begin{aligned}
\overset{\alpha}{\widetilde{\mathcal{W}}} &= \overset{\alpha}{\Gamma}_i(\theta_0;\hat{\theta})i^{i,j}(\hat{\theta})\overset{\alpha}{\Gamma}_j(\theta_0;\hat{\theta}) \\
&= i^{i,j}Y_iY_j+n^{-\frac{1}{2}}\left\{2i^{i,k}i^{j,l}Y_{ij}Y_kY_k-\frac{1}{2}(1+\alpha)\not{t}_{ijk}i^{i,l}i^{j,m}i^{k,n}Y_lY_mY_n\right\} \\
&\quad n^{-1}\Big\{i^{i,l}i^{j,m}i^{k,n}Y_{ijk}Y_lY_mY_n+3i^{j,k}i^{i,m}i^{l,n}Y_{ij}Y_{kl}Y_mY_n \\
&\quad -\frac{1}{2}(5+3\alpha)\not{t}_{ijk}i^{k,l}i^{i,n}i^{j,p}i^{m,q}Y_{lm}Y_nY_pY_q \\
&\quad +\left[\left(\frac{9}{16}+\frac{5}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{ijkl}-\not{t}_{ij;kl}-\frac{1}{6}(1+\alpha)\not{t}_{i;jkl}\right]i^{i,m}i^{j,n}i^{k,p}i^{l,q}Y_mY_nY_pY_q\Big\} \\
&\quad +O(n^{-\frac{3}{2}}). \qquad (5.1.11)
\end{aligned}
$$

We find that the Bartlett-type adjusted modified geometric Wald statistic (3.5.100) simplifies to

$$\overset{\alpha}{\widetilde{\mathcal{W}}}{}' = \left\{ 1 - \frac{1}{n}(\tilde{c} + \tilde{b}\,\overset{\alpha}{\widetilde{\mathcal{W}}} + \tilde{a}\overset{\alpha}{\widetilde{\mathcal{W}}}{}^2) \right\} \overset{\alpha}{\widetilde{\mathcal{W}}}, \tag{5.1.12}$$

where

$$\begin{aligned}
\tilde{a} &= \frac{1}{12p(p+2)(p+4)} \left(\left(\frac{3}{4} + \frac{9}{2}\alpha + \frac{27}{4}\alpha^2 \right) \tau_{ijkl} + \left(\frac{1}{2} + 3\alpha + \frac{9}{2}\alpha^2 \right) \tau_{ikjl} \right) i^{i,j} i^{k,l}, \\
\tilde{b} &= \frac{1}{12p(p+2)} \Big(-3(1+2\alpha)\lambda_{i,j,k,l} - 6(1+\alpha)\lambda_{ij,k,l}\,\{3\}_{jkl} \\
&\quad + (3 + 3\alpha - 6\alpha^2)\tau_{ijkl} + (5 + 6\alpha - 3\alpha^2)\tau_{ikjl} \Big) i^{i,j} i^{k,l}, \\
\tilde{c} &= \frac{1}{12p} \left(-3\lambda_{i,j,k,l} + 12(\lambda_{ik;jl} - \lambda_{ij,k,l} - \lambda_{ij;kl} + 3\tau_{ijkl} + 2\tau_{ikjl} \right) i^{i,j} i^{k,l}.
\end{aligned}$$

5.2 Observed Geometry

The family of geometric Wald statistics (2.3.6) in observed geometry for testing a simple null hypothesis is defined as

$$\overset{\alpha}{\mathcal{W}} = \overset{\alpha}{\Gamma}(\hat{\theta};\theta_0) j^{-1}(\hat{\theta}) \overset{\alpha}{\Gamma}{}^T(\hat{\theta};\theta_0), \tag{5.2.13}$$

where

$$\overset{\alpha}{\Gamma}_i(\theta;\theta') = n^{\frac{1}{2}} \left\{ \frac{1+\alpha}{2} g_{;i}(\theta';\theta) + \frac{1-\alpha}{2} g_i(\theta;\theta') \right\},$$

and where $g(\theta;\theta')$ is the observed likelihood yoke (4.1.1)

$$g(\theta;\theta') = n^{-1} \left\{ l(\theta;\theta',a) - l(\theta';\theta',a) \right\}.$$

Let Z_i and Z^i be as in (4.1.2) and (4.1.3), respectively, and use (4.1.5)-(4.1.8) and (4.1.13) to define the relevant $\not{t}$- and τ-tensors from Section 4.2.

The expansion of $\overset{\alpha}{\mathcal{W}}$ simplifies to

$$\begin{aligned}
\overset{\alpha}{\mathcal{W}} &= \overset{\alpha}{\Gamma}_i(\hat{\theta};\theta_0) j^{i,j}(\hat{\theta}) \overset{\alpha}{\Gamma}_j(\hat{\theta};\theta_0) \\
&= j_{i,j} Z^i Z^j + n^{-\frac{1}{2}} \left\{ -\frac{1}{2}(1-\alpha) \not{t}_{ijk} Z^i Z^j Z^k \right\}
\end{aligned}$$

$$+n^{-1}\left\{\left[\left(\frac{9}{16}-\frac{7}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{ijkl}-\frac{1}{6}(1-2\alpha)\not{t}_{i;jkl}\right]Z^iZ^jZ^kZ^l\right\}$$
$$+O(n^{-\frac{3}{2}}). \tag{5.2.14}$$

Then the observed Bartlett-corrected geometric Wald statistic (4.4.63) for testing a simple null hypothesis becomes

$$\overset{\alpha}{\hat{\mathcal{W}}}{}' = \left\{1-\frac{1}{n}(c+b\,\overset{\alpha}{\hat{\mathcal{W}}}+a\overset{\alpha}{\hat{\mathcal{W}}}{}^2)\right\}\overset{\alpha}{\hat{\mathcal{W}}}, \tag{5.2.15}$$

where

$$\begin{aligned}
a &= \frac{1}{12p(p+2)(p+4)}\left(\left(\frac{3}{4}-\frac{9}{2}\alpha+\frac{27}{4}\alpha^2\right)\tau_{ijkl}+\left(\frac{1}{2}-3\alpha+\frac{9}{2}\alpha^2\right)\tau_{ikjl}\right)j^{i,j}j^{k,l},\\
b &= \frac{1}{12p(p+2)}\left(3\not{t}_{ijkl;}+12\alpha\not{t}_{i;jkl}+(3-6\alpha-6\alpha^2)\tau_{ijkl}+(5-12\alpha-3\alpha^2)\tau_{ikjl}\right)j^{i,j}j^{k,l},\\
c &= \frac{1}{12p}\left(3\not{t}_{ijkl;}+12\not{t}_{ik;jl}+3\tau_{ijkl}+2\tau_{ikjl}\right)j^{i,j}j^{k,l}.
\end{aligned}$$

Observed Bartlett Correction for $\overset{\alpha}{\widetilde{\mathcal{W}}}$

We find that the Taylor expansion of $\overset{\alpha}{\widetilde{\mathcal{W}}}$ for a simple null hypothesis is

$$\begin{aligned}
\overset{\alpha}{\widetilde{\mathcal{W}}} &= \overset{\alpha}{\Gamma}_i(\theta_0;\hat{\theta})j^{i,j}(\theta_0)\overset{\alpha}{\Gamma}_j(\theta_0;\hat{\theta})\\
&= j_{i,j}Z^iZ^j+n^{-\frac{1}{2}}\left\{-\frac{1}{2}\not{t}_{ijk}Z^iZ^jZ^k\right\}\\
&\quad +n^{-1}\left\{\left[\left(\frac{9}{16}+\frac{5}{8}\alpha+\frac{1}{16}\alpha^2\right)\tau_{ijkl}-\frac{1}{6}(1+\alpha)\not{t}_{i;jkl}\right]Z^iZ^jZ^kZ^l\right\}\\
&\quad +O(n^{-\frac{3}{2}}).
\end{aligned}$$

Finally, we find that the observed Bartlett-corrected version of the modified geometric Wald statistic for testing simple null hypotheses is

$$\overset{\alpha}{\widetilde{\mathcal{W}}}{}' = \left\{1-\frac{1}{n}(\tilde{c}+\tilde{b}\,\overset{\alpha}{\widetilde{\mathcal{W}}}+\tilde{a}\overset{\alpha}{\widetilde{\mathcal{W}}}{}^2)\right\}\overset{\alpha}{\widetilde{\mathcal{W}}}, \tag{5.2.16}$$

where

$$\begin{aligned}
\tilde{a} &= \frac{1}{12p(p+2)(p+4)}\left(\left(\frac{3}{4}+\frac{9}{2}\alpha+\frac{27}{4}\alpha^2\right)\tau_{ijkl}+\left(\frac{1}{2}+3\alpha+\frac{9}{2}\alpha^2\right)\tau_{ikjl}\right)j^{i,j}j^{k,l},\\
\tilde{b} &= \frac{1}{12p(p+2)}\left(3\not{t}_{ijkl;}-6\alpha\not{t}_{i;jkl}+(3+3\alpha-6\alpha^2)\tau_{ijkl}+(5+6\alpha-3\alpha^2)\tau_{ikjl}\right)j^{i,j}j^{k,l},
\end{aligned}$$

$$\tilde{c} = \frac{1}{12p}\left(3\not{l}_{ijkl;} + 12\not{l}_{ik;jl} + 3\tau_{ijkl} + 2\tau_{ikjl}\right) j^{i,j} j^{k,l}.$$

Example 5.1:

We return to the Example 2.2 in Section 2.3.3 of the exponential distribution. Recall that the expected and the observed geometry coincide for full exponential families. In Section 2.3 we found that the geometric Wald statistic is

$$\overset{\alpha}{\mathcal{W}} = n\frac{1}{4}\left((1+\alpha)\left(1-\frac{\theta}{\hat{\theta}}\right)+(1-\alpha)\left(\frac{\hat{\theta}}{\theta}-1\right)\right)^2.$$

We find a Bartlett adjusted version of $\overset{\alpha}{\mathcal{W}}$ by calculating a, b and c in (5.2.15). The log-likelihood function of the exponential distribution in the canonical parameterisation is

$$l(\theta;\hat{\theta}) = \log\theta - \frac{\theta}{\hat{\theta}}.$$

Then,

$$\begin{aligned} l_{i;j}(\theta;\hat{\theta}) &= \tfrac{1}{\hat{\theta}^2} & \qquad l_{i;jkl}(\theta;\hat{\theta}) &= \tfrac{6}{\hat{\theta}^4} \\ l_{i;jk}(\theta;\hat{\theta}) &= \tfrac{-2}{\hat{\theta}^3} & \qquad l_{ij;kl}(\theta;\hat{\theta}) &= 0 \\ l_{ij;k}(\theta;\hat{\theta}) &= 0 & \qquad l_{ijk;l}(\theta;\hat{\theta}) &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} j^{i;j} &= \hat{\theta}^2 & \qquad \not{l}_{i;jkl} &= \tfrac{6}{\hat{\theta}^4} \\ \not{l}_{ijk} &= \tfrac{-2}{\hat{\theta}^3} & \qquad \not{l}_{ij;kl} &= 0 \\ \not{l}_{ijkl;} &= \tfrac{-6}{\hat{\theta}^4} & \qquad \tau_{ijkl} &= \tfrac{4}{\hat{\theta}^4}. \end{aligned}$$

We find that

$$\begin{aligned} a &= \frac{1}{12\times 3\times 5}\left(\left(\frac{3}{4}-\frac{9}{2}\alpha+\frac{27}{4}\alpha^2\right)4\frac{1}{\hat{\theta}^4}+\left(\frac{1}{2}-3\alpha+\frac{9}{2}\alpha^2\right)4\frac{1}{\hat{\theta}^4}\right)\left(\hat{\theta}^2\right)^2 \\ &= \frac{1}{36}(1-3\alpha)^2 \end{aligned}$$

$$\begin{aligned} b &= \frac{1}{12\times 3}\left(3\left(-6\frac{1}{\hat{\theta}^4}\right)+12\alpha\left(6\frac{1}{\hat{\theta}^4}\right)\right. \\ &\quad \left.+\left(3-6\alpha-6\alpha^2\right)4\frac{1}{\hat{\theta}^4}+\left(5-12\alpha-3\alpha^2\right)4\frac{1}{\hat{\theta}^4}\right)\left(\hat{\theta}^2\right)^2 \\ &= \frac{1}{18}\left(7-18\alpha^2\right) \end{aligned}$$

$$
\begin{aligned}
c &= \frac{1}{12}\left(3\left(-6\frac{1}{\hat{\theta}^4}\right)+3\left(4\frac{1}{\hat{\theta}^4}\right)+2\left(4\frac{1}{\hat{\theta}^4}\right)\right)\left(\hat{\theta}^2\right)^2\\
&= \frac{1}{6}.
\end{aligned}
$$

By (5.2.15) we find a Bartlett-adjusted geometric Wald statistic as

$$
\overset{\alpha}{\mathcal{W}}{}' = \left(1-n^{-1}\left\{\frac{1}{6}+\frac{1}{18}\left(7-18\alpha^2\right)\overset{\alpha}{\mathcal{W}}+\frac{1}{36}(1-3\alpha)^2\,\overset{\alpha}{\mathcal{W}}{}^2\right\}\right)\overset{\alpha}{\mathcal{W}},
$$

where

$$
\overset{\alpha}{\mathcal{W}}= n\frac{1}{4}\left((1+\alpha)\left(1-\frac{\tilde{\theta}}{\bar{\theta}}\right)+(1-\alpha)\left(\frac{\hat{\theta}}{\bar{\theta}}-1\right)\right)^2.
$$

To investigate the effect of the Bartlett-type adjustment, 1000 simulations were carried out, to estimate the cumulative probabilities of the adjusted geometric Wald statistics $\overset{\alpha}{\mathcal{W}}$ for the exponential distribution when testing the null hypothesis

$$
H_0 : \theta = 1,
$$

where θ is the canonical parameter. For sample sizes $n = 12$ and $n = 5$. The simulated cumulative probabilities corresponding to nominal cumulative probabilities of 0.900, 0.950, 0.975 and 0.990 are given in Table 2.

	n=12				n=5			
	nominal cumulative probability				nominal cumulative probability			
α	0.900	0.950	0.975	0.990	0.900	0.950	0.975	0.990
0	0.916	0.962	1.000	1.000	1.000	1.000	1.000	1.000
0.33	0.904	0.943	0.958	0.982	0.913	0.969	1.000	1.000
-0.33	0.940	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.5	0.898	0.936	0.953	0.975	0.894	0.938	0.972	1.000
-0.5	0.956	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1	0.897	0.936	0.963	1.000	0.900	0.939	0.977	1.000
-1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10	0.788	0.812	0.839	0.856	0.783	0.793	0.802	0.816
-10	0.787	0.805	0.819	0.852	0.758	0.775	0.786	0.802

Table 2: $P\left(\overset{\alpha}{\mathcal{W}}{}' \leq \chi^2_{1;c}\right)$ for $c = 0.90, 0.95, 0.975, 0.99$ based on 1000 simulations.

Comparing the entries in Table 2 with those in Table 1 in Section 2.3, indicates that there are slight improvements in the chi-squared approximation when α is positive. It is obvious, however, that for α negative the upper tail of the distribution of the adjusted Wald statistics is shorter than that of the the limiting χ^2 distribution.

Chapter 6

Discussion

In this thesis, a family of new and parameterisation-invariant versions of the Wald statistics has been defined. Ideas from differential geometry have been used as a guideline to defining test statistics with a geometrically intrinsic meaning. The distributional properties of the new statistics have been investigated, and Bartlett-type corrections have been applied in order to approximate the asymptotic behaviour of the statistics better by the limiting distribution.

The geometric Wald statistics form a wide family of test statistics. Here we have mainly looked at the expected geometric Wald statistics and the observed geometric Wald statistics, but the definition (2.3.5) allows any yoke to be used. Also, when a yoke has been decided on, there is a whole family of tests to choose from, depending on the value of α.

A few ideas for further investigation of the geometric Wald statistics introduced in this thesis include:

(i) which one of all these geometric Wald tests?

(ii) what happens if we use a yoke that is not one of the two likelihood yokes?

(iii) how do the geometric Wald statistics behave for different values of α?

(vi) is it only the $\overset{1}{\Gamma}_*$- and $\overset{-1}{\Gamma}_*$-parameterisations that have an intuitive meaning?

(v) what are the power properties of the geometric Wald statistics?

(vi) when do the geometric Wald statistics simplify?

Looking at special cases where the geometric Wald statistics simplify could give us some knowledge about the geometrical 'shape' of the test statistics – and about the special cases.

As mentioned in Section 2.3.3, the values $\alpha = 1$ and $\alpha = -1$ correspond to the canonical and the expectation parameterisations, respectively, in full exponential models. It would be interesting to investigate how this concept generalises to other statistical models. More research is also needed on the score tests (2.2.4) that we have come across a few times as being 'almost' one of the geometric Wald tests. The traditional definition of the score statistic is not consistent in its use of geometries. Comparing the traditional score statistic to the two geometrically 'correct' statistics, (i) the *observed score statistic*

$$\overset{-1}{\tilde{\mathcal{W}}} = l_\theta(\tilde{\theta}; \hat{\theta}, a) j^{-1}(\tilde{\theta}) l_{\theta^T}(\tilde{\theta}; \hat{\theta}, a),$$

or (ii) *the expected score statistic*

$$\overset{-1}{\tilde{\mathcal{W}}} = \mathbb{E}_{\hat{\theta}}[l_\theta(\tilde{\theta})] i^{-1}(\tilde{\theta}) \mathbb{E}_{\hat{\theta}}[l_{\theta^T}(\tilde{\theta})]$$

could give us a better understanding of one of the key instruments in parametrical statistics, - and (possibly) an improvement of the score test. Naturally, when the two geometries coincide (e.g. for full exponential models) the traditional score test has a 'correct' geometrical form.

In general, the importance of differential geometry in statistics is first and foremost that it encourages statisticians to concentrate on parameterisation-invariant procedures. Further, differential geometry can provide a geometrical insight and perhaps a better understanding of parametric statistical inference as we know it. A complete understanding of the correspondence between differential geometry and statistics has not yet been established. There is still a long way to go, but maybe this thesis will be just a small step on the way to complete this understanding.

Index of Notation

θ	general parameter	
ψ	interest parameter	
χ	nuisance parameter	
$\hat{\theta}$	unrestricted m.l.e. of θ	
$\tilde{\theta}$	m.l.e. of θ under H_0	
κ	(scaled) moments	(1.4.20)
λ	(scaled) cumulants	(1.4.22)
a	auxiliary statistic	
l	likelihood function	
i	(scaled) expected information	(1.3.5)
j	(scaled) observed information	(1.3.6)
$[\nu^{\alpha,\beta}]$	nuisance part of i (expected) or j (observed)	
f	expected likelihood yoke	(1.3.16)
g	typical yoke	(1.3.7)
	observed likelihood yoke	(1.3.17)
$\not{g}_{I;J}$	mixed derivatives of a yoke g at $(\theta;\theta)$	
$\overset{\alpha}{\Gamma}_i$	(normal) coordinate system	(2.3.7)
$\overset{\alpha}{\not\Gamma}_{ij}$, $\overset{\alpha}{\not\Gamma}_{i;j}$, *etc.*	derivatives of $\overset{\alpha}{\Gamma}_i$ at $(\theta;\theta)$	
$\not{t}$	tensors (expected)	(3.2.19)–(3.2.22)
	tensors (observed)	(4.1.5)–(4.1.8)
$\overset{1}{H}, \overset{-1}{H}$	embedding curvatures (expected)	(3.2.23)–(3.2.24)
	embedding curvatures (observed)	(4.1.9)–(4.1.10)
τ	tensors (expected)	(3.2.27)–(3.2.32)
	tensors (observed)	(4.1.13)–(4.1.18)
Z_i, Z_{ij}, Z_{ijk}	(normalised) derivatives of the score (expected)	(3.2.11)–(3.2.13)
Y_i, Y_{ij}, Y_{ijk}	tensorial versions of Z_i, Z_{ij}, Z_{ijk} (expected)	(3.2.16)–(3.2.18)
Z_i	(scaled) score (observed)	(4.1.2)
Z^i	Z_i lifted by i (expected), or by j (observed)	

W	traditional Wald statistic	(2.1.1)
$\overset{\alpha}{\mathcal{W}}$	geometric Wald statistic (general yoke)	(2.3.6)
$\overset{\alpha}{\widetilde{\mathcal{W}}}$	modified geometric Wald statistic (general yoke)	(2.3.9)
$M_{\overset{\alpha}{\mathcal{W}}}(t)$	moment generating function of $\overset{\alpha}{\mathcal{W}}$ (expected)	(3.5.90)
	moment generating function of $\overset{\alpha}{\mathcal{W}}$ (observed)	(4.4.62)
$M_{\overset{\alpha}{\widetilde{\mathcal{W}}}}(t)$	moment generating function of $\overset{\alpha}{\widetilde{\mathcal{W}}}$ (expected)	(3.5.99)
	moment generating function of $\overset{\alpha}{\widetilde{\mathcal{W}}}$ (observed)	(4.4.71)
$\overset{\alpha}{\mathcal{W}}{}'$	Bartlett adjusted $\overset{\alpha}{\mathcal{W}}$ (expected)	(3.5.91)
	Bartlett adjusted $\overset{\alpha}{\mathcal{W}}$ (observed)	(4.4.63)
$\overset{\alpha}{\widetilde{\mathcal{W}}}{}'$	Bartlett adjusted $\overset{\alpha}{\widetilde{\mathcal{W}}}$ (expected)	(3.5.100)
	Bartlett adjusted $\overset{\alpha}{\widetilde{\mathcal{W}}}$ (observed)	(4.4.72).

Bibliography

[1] Barndorff-Nielsen, O.E. (1986) Likelihood and observed geometries. *Ann. Statist.*, **14**, 856-873.

[2] Barndorff-Nielsen, O.E., Blæsild, P., Pace, L. and Salvan, A. (1991) Formulas for Geometric Expansions. *Research Report 207. Dept. Theor. Statist., Aarhus University.*

[3] Barndorff-Nielsen, O.E. and Blæsild, P. (1993) Orthogeodesic models. *Ann. Statist.*, **21**, 1018-1039.

[4] Barndorff-Nielsen, O.E. and Cox, D.R. (1984) Bartlett adjustments to the likelihood ratio statistic and the distribution of the maximum likelihood estimator. *J.R. Statist. Soc.*, **B**, **46**, 483-495.

[5] Barndorff-Nielsen, O.E. and Cox, D.R. (1989) Asymptotic Techniques for Use in Statistics. *Monographs on Statistics and Applied Probability 31, Chapman & Hall.*

[6] Barndorff-Nielsen, O.E. and Cox, D.R. (1994) Inference and Asymptotics. *Monographs on Statistics and Applied Probability 52, Chapman & Hall.*

[7] Barndorff-Nielsen, O.E. and Hall, P. (1988) On the level-error after Bartlett adjustment of the likelihood ratio statistic. *Biometrika*, **75**, 374-388.

[8] Barndorff-Nielsen, O.E. and Jupp, P.E. (1988) Differential geometry, profile likelihood, L-sufficiency and composite transformation models. *Ann. Statist.*, **16**, 1009-1043.

[9] Barndorff-Nielsen, O.E., Jupp, P.E. and Kendall, W.S. (1994) Stochastic calculus, statistical asymptotics, Taylor strings and phyla. *Annales de la Faculté des Sciences de Toulouse*, **III**, **1**.

[10] Blæsild, P. (1990) Yokes: orthogonal and extended normal coordinates. *Research Report 205. Dept. Theor. Statist., Aarhus University.*

[11] Blæsild, P. (1991) Yokes and tensors derived from yokes. *Ann. Inst. Statist. Math.*, **43**, 95-113.

[12] Breusch, R.S. and Schmidt, P. (1988) Alternative forms of the Wald test: How long is a piece of string? *Commun. Statist. - Theory Meth.*, **17**, 2789-2795.

[13] Buse, A. (1982) The likelihood ratio, Wald, and Lagrange multiplier test: an expository note. *The American Statistician*, **36**, 156-161.

[14] Cordeiro, G.M. and Ferrari, S.L.P. (1991) A modified score test statistic having chi-squared distribution to order n^{-1}. *Biometrika*, **78**, 573-582.

[15] Cox, D.R. and Hinkley, D.V. (1974) Theoretical Statistics. *Chapman & Hall.*

[16] Cribari-Neto, F. and Cordeiro, G.M. (1995) On Bartlett and Bartlett-type corrections. *Discussion Paper Series* **95-17**, *Department of Economics, Southern Illinois University.*

[17] Critchley, F., Marriott, P. and Salmon, M. (1994) On the differential geometry of the Wald test with nonlinear restrictions. *Working Paper* **94/45**, *Economics Department, European University Institute.*

[18] Critchley, F., Marriott, P. and Salmon, M. (1996) On the differential geometry of the Wald test with nonlinear restrictions. *Econometrica*, **64**, 1213-1222.

[19] Dagenais, M. and Dufour, J. (1991) Invariance, nonlinear models, and asymptotic tests. *Econometrica*, **59**, pp. 1601-1615.

[20] Efron, B. (1978) The geometry of exponential families. *Ann. Statist.*, **6**, 362-376.

[21] Ferrari, S.L.P. and Cribari-Neto, F. (1993) On the corrections to the Wald test of non-linear restrictions. *Economic Letters*, **42**, 321-326.

[22] Gregory, A.W. and Veall, M.R. (1985) Formulating Wald tests of nonlinear restrictions. *Econometrica*, **53**, 1465-1468.

[23] Hayakawa, T. and Puri, M.L. (1985) Asymptotic expansions of the distribution of some test statistics. *Ann. Inst. Statist. Math.*, **37**, 95-108.

[24] Lauritzen, S.L. (1987) Statistical Manifolds. *Differential Geometry in Statistical Inference, IMS Lecture Notes-Monograph series, Hayward, California.*

[25] Lawley, D.N. (1956) A general method for approximating to the distribution of the likelihood ratio criteria. *Biometrika*, **43**, 295-303.

[26] Le Cam, L. (1990) On the standard asymptotic confidence ellipsoids of Wald. *Int. Stat. Rev.*, **58**, 129-152.

[27] McCullagh, P. (1987) Tensor Methods in Statistics. *Chapman & Hall.*

[28] Mora, M. (1992) Geometric expansions for the distributions of the score vector and the maximum likelihood estimator. *Ann. Inst. Statist. Math.*, **44**, 63-83.

[29] Murray, M.K. and Rice, J.W. (1993) Differential Geometry and Statistics. *Monographs on Statistics and Applied Probability 48, Chapman & Hall.*

[30] Neyman, J. and Pearson, E.S. (1928) On the use and interpretation of certain test criteria for purposes of statistical inference. *Biometrika*, **20A**, 175-240 and 263-294.

[31] Phillips, P.C.B. and Park, J.Y. (1988) On the formulation of Wald tests of nonlinear restrictions. *Econometrica*, **56**, 1065-1083.

[32] Rao, C.R. (1948) Large sample tests of statistical hypotheses concerning several parameters with applications to problems of estimation. *Proc. Camb. Phil. Soc.*, **44**, 50-57.

[33] Wald, A. (1943) Tests of statistical hypotheses concerning several parameters when the number of observations is large. *Trans. Amer. Math. Soc.*, **54**, 426-482.