# INFERENCE FOR PLANT-CAPTURE

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A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1998

University of St. Andrews, Scotland

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A thesis submitted to the University of St. Andrews for the degree of Doctor of Philosophy

by

Jonathan Ashbridge

September 1997

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I would like to thank my research supervisor Dr. I. B. J. Goudie for his advice, help and encouragement during the work which led to the production of this thesis.

I would like to acknowledge the support of the Engineering and Physical Sciences Research Council.

Abstract

When investigating the dynamics of an animal population, a primary objective is to obtain reasonable estimates of abundance or population size. This thesis concentrates on the problem of obtaining point estimates of abundance from capture-recapture data and on how such estimation can be improved by using the method of plant-capture.

Plant-capture constitutes a natural generalisation of capture-recapture. In a plant-capture study a pre-marked population of known size is added to the target population of unknown size. The capture-recapture experiment is then carried out on the augmented population.

Chapter 1 considers the addition of planted individuals to target populations which behave according to the standard capture-recapture model $M_0$. Chapter 2 investigates an analogous model based on sampling in continuous time. In each of these chapters, distributional results are derived under the assumption that the behaviour of the plants is indistinguishable from that of members of the target population. Maximum likelihood estimators and other new estimators are proposed for each model. The results suggest that the use of plants is beneficial, and furthermore that the new estimators perform more satisfactorily than the maximum likelihood estimators.

Chapter 3 introduces, initially in the absence of plants, a new class of estimators, described as coverage adjusted estimators, for the standard capture-recapture model $M_h$. These new estimators are shown, through simulation and real life data, to compare favourably with estimators that have previously been proposed. Plant-capture versions of these new estimators are then derived and the usefulness of the plants is demonstrated through simulation.

Chapter 4 describes how the approach taken in chapter 3 can be modified to produce a new estimator for the analogous continuous time model. This estimator is then shown through simulation to be preferable to estimators that have previously been proposed.

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## Introduction

Capture-recapture methods can be used to estimate population size and other fundamental demographic variables. The most popular class of models that describe the behaviour of closed populations were first introduced as a set by Pollock(1974, 1976) and later more fully described in a wildlife monograph by Otis et al. (1978). Another important reference for this class of models is White et al. (1982). Each model within the class requires a sequence of t samples to be taken from the population. After each sample is taken animals within the sample not previously caught each receive a unique tag so that they can be recognised if recaptured in a later sample. After each sample is taken all animals are released. In each of their models Otis et al. (1978) allow the capture probabilities to vary due to time(t), due to heterogeneity (h) between the capture probabilities of the different animals, and due to a behavioural (b) response to the traps used. A total of eight possible models within this class results from the fact that each of these three factors can be present or absent. The sampling scheme considered within the Otis et al. (1978) monograph is referred to as discrete time sampling.

There is a continuous time sampling analogue of each of the models described by Otis et al. (1978), in which the population is under continuous observation for some period of time, with the animals being seen according to independent Poisson processes. For this continuous time sampling procedure one animal is seen at a time and animals seen for the first time receive a unique tag so that they may be subsequently recognised.

Plant-capture constitutes a natural generalisation of capture-recapture. In a plant-capture study a pre-marked population of known size is first added to the target population of unknown size. The capture-recapture experiment is then carried out on the augmented population. Under the assumption that members of the planted population behave in an identical manner to those of the target population, one obtains, through sightings of the plants, additional information which can improve estimation of target population size.

This thesis concentrates on closed populations which behave according to two of the eight closed capture-recapture models described by Otis et al. (1978). The most basic model $M_0$ is considered in chapter 1, and the important heterogeneity model $M_h$ is considered within chapter 3. While chapter 1 is largely concerned with the case where plants are present, the emphasis in chapter 3 is on proposing an improved class of estimators for the standard model $M_h$.

The work contained within chapters 2 and 4, which considers the continuous time analogues of the models considered in chapters 1 and 3 respectively, could also be applied in a software reliability context where the population in question is one of errors or 'bugs' in a computer program.

The assumption that members of the planted and target populations behave in an identical manner is central to most plant-capture methodology. Indeed one should only apply plant-capture methods when there are adequate grounds for believing that this assumption is a reasonable approximation to reality.

The use of plants to assist population size estimation has been considered in a number of quite different situations. Change-in-ratio methods have been widely used to estimate the abundance of animal populations, see Seber(1982, chapter 9). It was Kelker(1940) who first introduced the idea that the size of a wildlife population could be estimated from a knowledge of sex ratios before and after a differential kill of the sexes. Rupp(1966) recognised that the theory was still valid when the ratios are changed by the insertion of planted individuals. In a software reliability context the idea of introducing plants into the population prior to sampling has been considered by Mills(1972), and by Duran & Wiorkowski(1981) who speak of ' deliberately seeding errors into the software' prior to testing. Laska & Meisner(1993) have described how the U.S. Census Bureau used a plant-capture method in an attempt to estimate the size of a selected component population of homeless people. Laska & Meisner(1993) state that 'there are many potential applications' of their methodology. Martin et al. (1995) have also investigated using a plant-capture method for estimating the size of the street dwelling population. Goudie(1995) has considered the use of plants in order to improve stopping rules for determining , within a specified error probability, when all members of a target population have been seen. Yip(1996) describes a martingale based approach for estimating population size from plant-capture data. Norris and Pollock(1996b) have considered the use of plants in connection with the heterogeneity model $M_h$.

N.B. Although every effort has been made to ensure that the notation used throughout this thesis remains consistent, the notation used to denote estimators must be viewed as being specific to each chapter.

# Chapter 1 : Plant-Capture Applied to the Model $M_0$ : Discrete Time Sampling Procedure

## § 1.1 : Introduction

This chapter considers how the method of plant-capture may be used to aid the problem of estimating population size in a multiple capture-recapture experiment when the population in question behaves according to the standard capture-recapture model known as $M_0$. The model $M_0$ is one of the set of models described by Otis et al. (1978) for capture-recapture data in closed populations.

The discrete time sampling procedure considered within this chapter essentially constitutes what is known in the literature as a Schnabel Census with random sample sizes, see Schnabel(1938) or, for a more comprehensive review, Seber(1982). In the absence of plants the most commonly used estimator for the model $M_0$, under discrete time sampling, is the maximum likelihood estimator, which was first considered by Darroch(1958) and later by Otis et al. (1978).

## § 1.2 : Sampling Procedure and Assumptions

The sampling procedure may be described as follows. Prior to the commencement of the experiment it is assumed that the target population, whose size N we wish to estimate, is augmented by the insertion of a known number R of planted individuals. Each planted individual is assumed to have received a unique tag prior to its release. A sequence of t sampling experiments is then carried out on the augmented population which is assumed to be closed and of size N+R. Independently of other animals and independently of its previous capture history animal i (i=1,2,...,N+R) is captured in sample j (j=1,2,...,t) with probability p. After each sample is taken every animal within that sample not previously marked receives a unique tag before its release so that it may be recognised on subsequent trapping occasions. The experiment generates an N+R by t matrix A where

$$a_{ij} = \begin{cases} 1 & \text{if animal i is caught on sampling occasion j} \\ 0 & \text{if animal i is not caught on sampling occasion j} \end{cases}$$

$$i = 1, 2, \ldots, N+R.$$
$$j = 1, 2, \ldots, t.$$

The sample space is the set of such matrices.

In the absence of plants, the sampling procedure considered here is the one most commonly used in practice.

Please note that the above implies that the behaviour of the planted individuals is assumed to be indistinguishable from the behaviour of members of the original population.

## § 1.3 : The Sufficient Statistics

In order to obtain the sufficient statistics some notation is needed :

Let $X_w^{(1)}$ ≡ the number of animals from the target population with capture history w. For example, for $t = 3$, $X_{101}^{(1)}$ is the number of animals seen on the first and third but not the second capture periods.

$X_w^{(2)}$ ≡ the number of animals from the planted population with capture history w.

$X_1 = \sum_w X_w^{(1)}$ ≡ the number of distinct animals seen from the target population.

$X_2 = \sum_w X_w^{(2)}$ ≡ the number of distinct animals seen from the planted population.

N.B. $\sum_w$ is used to represent the summation over all w except $w = \underline{0}$.

$X = X_1 + X_2$ ≡ the number of distinct animals seen from the augmented population.

$i_w$ ≡ the number of ones in w.

$Z_1 = \sum_w i_w X_w^{(1)}$ ≡ the total number of captures from the target population.

$Z_2 = \sum_w i_w X_w^{(2)}$ ≡ the total number of captures from the planted population.

$Z = Z_1 + Z_2$ ≡ the total number of captures from the augmented population.

And let $\{X_w^{(i)}\}$ denote the vector of the $X_w^{(i)}$'s, except for the unobservable $X_{\underline{0}}^{(i)}$, i=1, 2. For the moment if we consider only the target population it is seen that the distribution of $\{X_w^{(1)}, X_{\underline{0}}^{(1)}\}$ is multinomial, with $2^t$ cells and N trials. Given $\{X_w^{(1)}\}$ and N one can easily deduce the unobservable value of $X_{\underline{0}}^{(1)}$ so we consider $\text{Prob}(\{X_w^{(1)}\})$ in place of $\text{Prob}(\{X_w^{(1)}, X_{\underline{0}}^{(1)}\})$.

$$\text{Prob}\left(\left\{X_w^{(1)}\right\}\right) = \frac{N!}{(N-x_1)!\left(\prod_w X_w^{(1)}!\right)} \left(P_{\underline{0}}\right)^{N-x_1} \prod_w \left(P_w\right)^{X_w^{(1)}},$$

where $P_w$ denotes the cell probability of capture history w, and $\prod_w$ is used to represent the product over all w except $w = \underline{0}$.

$$= \frac{N!}{(N-x_1)!\left(\prod_w X_w^{(1)}!\right)} \left[(1-p)^t\right]^{N-x_1} \prod_w \left[p^{i_w}(1-p)^{t-i_w}\right]^{X_w^{(1)}}$$

$$= \frac{N!}{(N-x_1)!\left(\prod_w X_w^{(1)}!\right)} \left[(1-p)^t\right]^{N-x_1} p^{\sum_w i_w X_w^{(1)}} (1-p)^{\sum_w (t-i_w) X_w^{(1)}}$$

$$= \frac{N!}{(N-x_1)!\left(\prod_w X_w^{(1)}!\right)} \left[(1-p)^t\right]^{N-x_1} p^{z_1}(1-p)^{tx_1 - z_1}$$

$$= \frac{N!}{(N-x_1)!\left(\prod_w X_w^{(1)}!\right)} p^{z_1}(1-p)^{tN-z_1}.$$

This result was first obtained by Darroch(1958).

In a similar way it may be shown that

$$\text{Prob}\left(\left\{X_w^{(2)}\right\}\right) = \frac{R!}{(R-x_2)!\left(\prod_w X_w^{(2)}!\right)} p^{z_2}(1-p)^{tR-z_2}.$$

In view of the independence between target and planted populations we may write

$$\text{Prob}\left(\left\{X_w^{(1)}, X_w^{(2)}\right\}\right) = \text{Prob}\left(\left\{X_w^{(1)}\right\}\right).\text{Prob}\left(\left\{X_w^{(2)}\right\}\right)$$

$$= \frac{N!}{(N-x_1)!\left(\prod_w X_w^{(1)}!\right)} p^{z_1}(1-p)^{tN-z_1} \frac{R!}{(R-x_2)!\left(\prod_w X_w^{(2)}!\right)} p^{z_2}(1-p)^{tR-z_2}$$

$$= \frac{N!}{(N-x_1)!\left(\prod_w X_w^{(1)}!\right)} \frac{R!}{(R-x_2)!\left(\prod_w X_w^{(2)}!\right)} p^{z}(1-p)^{t(N+R)-z}$$

which implies that

$$L\left(N, p \middle| X_w^{(1)}, X_w^{(2)}\right) \propto \frac{N!}{(N-x_1)!} p^{z}(1-p)^{t(N+R)-z}$$

where $L\left(N, p \middle| X_w^{(1)}, X_w^{(2)}\right)$ represents the likelihood function for N and p. Hence by the Neyman-Pearson factorisation theorem it is seen that the sufficient statistics for N and p are in fact $x_1$ and $z = z_1 + z_2$.

§ 1.4 : The Distribution Function of the Sufficient Statistics

The joint distribution function of $X_1$ and Z may be obtained as follows :

$$\begin{aligned}\mathrm{Prob}(X_1 = x_1, Z = z) &= \mathrm{Prob}(Z = z)\mathrm{Prob}(X_1 = x_1|Z = z)\\ &= \mathrm{Prob}(Z = z)\sum_{z_1}\mathrm{Prob}(X_1 = x_1|Z_1 = z_1, Z = z)\mathrm{Prob}(Z_1 = z_1|Z = z)\\ &= \mathrm{Prob}(Z = z)\sum_{z_1}\mathrm{Prob}(X_1 = x_1|Z_1 = z_1)\mathrm{Prob}(Z_1 = z_1|Z = z)\end{aligned}$$

$$= \mathrm{Prob}(Z = z)\sum_{z_1}\frac{\mathrm{Prob}(Z_1 = z_1|X_1 = x_1)\mathrm{Prob}(X_1 = x_1)}{\mathrm{Prob}(Z_1 = z_1)}\mathrm{Prob}(Z_1 = z_1|Z = z). \qquad (1.1)$$

From the above assumptions it is known that

$X_1 \sim \mathrm{Bin}\left(N, 1-(1-p)^t\right)$,

$Z_1 \sim \mathrm{Bin}(Nt, p)$

and that $Z \sim \mathrm{Bin}((N+R)t, p)$.

The distribution of $Z_1|Z$ is hypergeometric. And we may observe that $Z_1|X_1 = x_1$ is the sum of $x_1$ zero truncated Binomial random variables, the probability function of which is derived in appendix 3.

Now from (1.1) it follows that

$\mathrm{Prob}(X_1 = x_1, Z = z) =$

$$\binom{(N+R)t}{z}p^z(1-p)^{(N+R)t-z}\sum_{z_1}\frac{\dfrac{p^{z_1}(1-p)^{tx_1-z_1}}{\left[1-(1-p)^t\right]^{x_1}}\sum_{j=0}^{x_1}\binom{x_1}{j}\binom{tj}{z_1}(-1)^{x_1-j}\binom{N}{x_1}\left[1-(1-p)^t\right]^{x_1}\left[(1-p)^t\right]^{N-x_1}}{\binom{Nt}{z_1}p^{z_1}(1-p)^{Nt-z_1}}\frac{\binom{Nt}{z_1}\binom{Rt}{z-z_1}}{\binom{(N+R)t}{z}}$$

$$\begin{aligned}&= \binom{N}{x_1}p^z(1-p)^{(N+R)t-z}\sum_{j=0}^{x_1}\binom{x_1}{j}\left[\sum_{z_1}\binom{tj}{z_1}\binom{Rt}{z-z_1}\right](-1)^{x_1-j}\\ &= \binom{N}{x_1}p^z(1-p)^{(N+R)t-z}\sum_{j=0}^{x_1}\binom{x_1}{j}\binom{tj+Rt}{z}(-1)^{x_1-j}, \qquad (1.2)\end{aligned}$$

$x_1 = 0, 1, 2, \ldots, N.$

$z = x_1,\ x_1+1,\ x_1+2, \ldots,\ tx_1 + tR.$

The joint probability function, given by (1.2), of the sufficient statistics $X_1$ and Z has not previously appeared in the literature for values of R greater than or equal to zero. However, the conditional distribution of $X_1$ given Z has previously been considered in an urn model context. For R = 0 the conditional distribution of $X_1$ given Z is identical to the distribution of the number of occupied urns where there is a limit r on the capacity of each urn, Romanovsky(1934). Charalambides(1981) generalised this result of Romanovsky(1934) by introducing a control urn of capacity s, where s is not necessarily equal to r. When $\frac{s}{r}$ is integer valued, the situation considered by Charalambides(1981), with $R=\frac{s}{r}$, is probabilistically equivalent to the one considered here, and so led Charalambides(1981) to derive the conditional distribution of $X_1$ given $Z$ : for values of R greater than or equal to zero.

N.B. The summation which appears in (1.2) is of importance within this chapter and is considered in more detail in the following section.

## § 1.5 : The $\delta$ - Numbers

The $\delta$-numbers are defined as follows

$$\delta(x_1,z:t,R)=\sum_{j=0}^{x_1}\binom{x_1}{j}\binom{tj+tR}{z}(-1)^{x_1-j},$$
$$x_1=0,1,2,\dots\dots$$
$$z=x_1,x_1+1,x_1+2,\dots\dots,tx_1+tR.$$

Within this chapter these $\delta$-numbers are of importance since they appear in the joint probability function of $X_1$ and Z, as given by equation (1.2).

The $\delta$-numbers are multiples of a subset of the Gould-Hopper numbers, see Gould and Hopper(1962) , Charalambides(1979) and Charalambides and Singh(1988). Explicitly the Gould-Hopper number is defined as $G(z,x_1,t,s)=\frac{1}{x_1!}\left[\Delta^{x_1}(ty+s)_z\right]_{y=0}$. When $\frac{s}{t}$ is integer valued, the Gould-Hopper number $G(z,x_1,t,s)=\frac{z!}{x_1!}\delta\left(x_1,z:t,\frac{s}{t}\right)$.

The Gould-Hopper numbers are a generalisation of the C-numbers. The C-numbers have been extensively studied, see Charalambides and Singh(1988), and are defined as $C(z,x_1,t)=\frac{1}{x_1!}\left[\Delta^{x_1}(ty)_z\right]_{y=0}$ . The relationship between the $\delta$-numbers and the C-numbers is given by

$$C(z,x_1,t)=\frac{z!}{x_1!}\delta(x_1,z:t,R=0).$$

In order to investigate the distributional properties of the estimators which are considered further on in this chapter it is necessary to evaluate the $\delta$-numbers over some particular range of parameter values. This can lead to computational problems since the form of the $\delta$-numbers is not desirable from a computational point of view. That is the alternating sign within the summation means that, for large values of N, R and t, very large numbers are repeatedly being added to and in particular subtracted from one another, and this is a major source of rounding error. To help avoid this, and other significant computational problems, one may consider the following 'triangular' recurrence relation of the $\delta$-numbers.

$$z\delta(x_1,z:t,R)=(tx_1+tR-z+1)\delta(x_1,z-1:t,R)+tx_1\delta(x_1-1,z-1:t,R). \tag{1.3}$$

A direct proof of this is as follows

$$\begin{aligned}
&(tx_1+tR-z+1)\delta(x_1,z-1:t,R)+tx_1\delta(x_1-1,z-1:t,R)\\
&\quad=(tx_1+tR-z+1)\sum_{j=0}^{x_1}\binom{x_1}{j}\binom{tj+tR}{z-1}(-1)^{x_1-j}+tx_1\sum_{j=0}^{x_1-1}\binom{x_1-1}{j}\binom{tj+tR}{z-1}(-1)^{x_1-1-j}\\
&\quad=(tx_1+tR-z+1)\binom{tx_1+tR}{z-1}+(tx_1+tR-z+1)\sum_{j=0}^{x_1-1}\binom{x_1}{j}\binom{tj+tR}{z-1}(-1)^{x_1-j}\\
&\qquad\qquad+tx_1\sum_{j=0}^{x_1-1}\binom{x_1-1}{j}\binom{tj+tR}{z-1}(-1)^{x_1-1-j}\\
&\quad=(tx_1+tR-z+1)\frac{z}{(tx_1+tR-z+1)}\binom{tx_1+tR}{z}\\
&\qquad+\sum_{j=0}^{x_1-1}(-1)^{x_1-j}\binom{tj+tR}{z-1}\left[(tx_1+tR-z+1)\binom{x_1}{j}-tx_1\binom{x_1-1}{j}\right]\\
&\quad=z\binom{tx_1+tR}{z}\\
&\qquad+\sum_{j=0}^{x_1-1}(-1)^{x_1-j}\binom{x_1}{j}\binom{tj+tR}{z}\frac{z}{(tj+tR-z+1)}\left[(tx_1+tR-z+1)-tx_1\frac{(x_1-j)}{x_1}\right]\\
&\quad=z\binom{tx_1+tR}{z}+\sum_{j=0}^{x_1-1}(-1)^{x_1-j}\binom{x_1}{j}\binom{tj+tR}{z}\frac{z}{(tj+tR-z+1)}[tj+tR-z+1]\\
&\quad=z\binom{tx_1+tR}{z}+z\sum_{j=0}^{x_1-1}(-1)^{x_1-j}\binom{x_1}{j}\binom{tj+tR}{z}\\
&\quad=z\sum_{j=0}^{x_1}\binom{x_1}{j}\binom{tj+tR}{z}(-1)^{x_1-j}\\
&\quad=z\delta(x_1,z:t,R).
\end{aligned}$$

Equation (1.3), when R=0, essentially reduces to the recurrence relation of the C-numbers as given by equation (3.25) in Charalambides and Singh(1988).

The triangular recurrence relation (1.3) along with the initial conditions

$$\delta(0,z:t,R)=\binom{tR}{z},\ \delta(x_1,x_1:t,R)=t^{x_1}\ \text{ and }\ \delta(x_1,tx_1+tR:t,R)=1 \tag{1.3a}$$

enables one to evaluate the required $\delta$-numbers without having to perform any subtraction operations whatsoever, and hence one can more easily avoid computational rounding error.

N.B. The first and third initial conditions are easy to show directly. The second can be shown to hold as follows. Firstly substituting $z = x_1$ into (1.3) implies that $\delta(x_1,x_1:t,R)=t\delta(x_1-1,x_1-1:t,R)$, then after observing that $\delta(0,0:t,R)=1$ it is easy to see that $\delta(x_1,x_1:t,R)=t^{x_1}$ for all $x_1 \geq 0$.

<u>Comments</u>

Using (1.3), one can show that a similar 'triangular' recurrence relation exists between the probabilities of the joint distribution of $X_1$ and Z, given by equation (1.2). It can be shown that

$$P_{x_1,z}=\frac{p}{z(1-p)}\left[(tx_1+tR-z+1)P_{x_1,z-1}+t(N-x_1+1)P_{x_1-1,z-1}\right], \tag{1.4}$$

where $P_{x_1,z}=\text{Prob}(X_1=x_1,Z=z)$.

It is also straightforward to show that (1.4) is subject to the initial conditions

$$P_{0,z}=p^z(1-p)^{tN+tR-z}\binom{tR}{z}, \qquad z=0,1,2,\ldots\ldots,tR. \tag{1.4a}$$

$$P_{x_1,x_1}=\binom{N}{x_1}p^{x_1}(1-p)^{tN+tR-x_1}t^{x_1} \qquad x_1=0,1,2,\ldots\ldots,N. \tag{1.4b}$$

and

$$P_{x_1,tx_1+tR}=\binom{N}{x_1}p^{tx_1+tR}(1-p)^{tN-tx_1}. \qquad x_1=0,1,2,\ldots\ldots,N. \tag{1.4c}$$

Again in an attempt to avoid numerical computational problems, one can determine the initial conditions (1.4a) and (1.4b) using the following recurrence relations :

(i) $$P_{0,z}=\frac{p}{(1-p)}\left[\frac{tR-z+1}{z}\right]P_{0,z-1}, \qquad z = 1, 2, \ldots, tR.$$

(ii) $$P_{x_1,x_1}=\frac{tp}{(1-p)}\left[\frac{N-x_1+1}{x_1}\right]P_{x_1-1,x_1-1}, \qquad x_1 = 1, 2, \ldots, N.$$

Where the appropriate initial condition for both (i) and (ii) is $P_{0,0}=(1-p)^{t(N+R)}$.

( Technically (1.4c) is not an 'initial condition', since the $P_{x_1,tx_1+tR}$ can be generated using (1.4) along with (1.4a) and (1.4b). (1.4c) is included for completeness. )

## § 1.6 : The Maximum Likelihood Estimator

From equation 1.2 it follows that the joint likelihood for N and p is given by

$$L(N,p) \propto \frac{N!}{(N-x_1)!} p^z (1-p)^{(N+R)t-z}. \qquad (1.5)$$

This is maximised over p as follows :

$$\frac{\partial L}{\partial p} \propto -p^z(t(N+R)-z)(1-p)^{t(N+R)-z-1} + zp^{z-1}(1-p)^{t(N+R)-z}$$

equate to zero to obtain $\hat{p}$ :

$$p^z(t(N+R)-z)(1-p)^{t(N+R)-z-1} = zp^{z-1}(1-p)^{t(N+R)-z}$$

$$p(t(N+R)-z) = z(1-p)$$

$$\Rightarrow \qquad \hat{p} = \frac{z}{t(N+R)}.$$

$\hat{p}$ is now substituted into (1.5) to obtain the profile likelihood for N :

$$L(N) \propto \frac{N!}{(N-x_1)!}\left[\frac{z}{t(N+R)}\right]^z \left[1-\frac{z}{t(N+R)}\right]^{(N+R)t-z}.$$

It is more convenient to consider the log-profile-likelihood from this point. It is easily shown that the log-profile-likelihood may be written as

$$l(N) \propto \ln\left[\frac{N!}{(N-x_1)!}\right] - t(N+R)\ln[t(N+R)] + [t(N+R)-z]\ln[t(N+R)-z]. \qquad (1.6)$$

Due to numerical complications, which can occur for larger values of $\hat{N}$, it was found that the most satisfactory way of calculating the value of the maximum likelihood estimator is as follows :

After observing that the likelihood function is uni-modal it is seen that $\hat{N} = k$, where k is the smallest integer in the set $\{x_1, x_1+1, x_1+2, .......\}$ to satisfy the condition

$$L(k) > L(k+1)$$

$$\Leftrightarrow \qquad l(k) > l(k+1)$$

$$\Leftrightarrow \qquad \ln\left[\frac{k!}{(k-x_1)!}\right] - t(k+R)\ln[t(k+R)] + [t(k+R)-z]\ln[t(k+R)-z]$$

$$> \ln\left[\frac{(k+1)!}{(k+1-x_1)!}\right] - t(k+1+R)\ln[t(k+1+R)] + [t(k+1+R)-z]\ln[t(k+1+R)-z]$$

from (1.6)

$$\Leftrightarrow \qquad \ln\left[\frac{k!}{(k-x_1)!}\right] - \ln\left[\frac{(k+1)!}{(k+1-x_1)!}\right]$$

$$> t(k+R)\ln[t(k+R)] - [t(k+R)-z]\ln[t(k+R)-z]$$

$$- \; t(k+1+R)\ln[t(k+1+R)] + [t(k+1+R)-z]\ln[t(k+1+R)-z]$$

$$\Leftrightarrow \quad \ln\left[\frac{(k+1-x_1)}{k+1}\right]+[t(k+R)-z]\ln[t(k+R)-z]-t(k+R)\ln[t(k+R)]$$
$$> [t(k+1+R)-z]\ln[t(k+1+R)-z]-t(k+1+R)\ln[t(k+1+R)].$$

N.B. Once $\hat{N}$ has been determined, this value may then be used in the calculation of the maximum likelihood estimate of p : $\hat{p}=\frac{z}{t(\hat{N}+R)}$.

## § 1.7 : A Peterson-Type Estimator

This section introduces an estimator of population size which is only dependent upon the observed numbers of distinct animals seen from the target and planted populations. The estimator is derived from the conditional distribution of $X_1$ given X. From the assumptions stated above one may deduce that

$$X_1 \sim \text{Bin}\left(N, 1-(1-p)^t\right),$$
$$X_2 \sim \text{Bin}\left(R, 1-(1-p)^t\right)$$

and that $\quad X \sim \text{Bin}\left(N+R, 1-(1-p)^t\right).$

It is then easy to show that the distribution of $X_1|X$ is in fact hypergeometric with probability function

$$\text{Prob}(X_1=x_1|X=x)=\binom{N}{x_1}\binom{R}{x-x_1}\bigg/\binom{N+R}{x}, \qquad \max(0,x-R)\le x_1\le \min(N,x).$$

The likelihood function for N based on this probability function is maximised by the Peterson-type estimator $\tilde{N}_P = RX_1/X_2$. To avoid introducing an estimator which becomes infinite when $X_2$=0, the estimator $\tilde{N}_P$ is now slightly modified. That is from this point consideration is given to the estimator $\hat{N}_P=\left[0.5+\frac{(R+1)X_1}{(X_2+1)}\right]$, where [.] denotes the integer part of.

## § 1.8 : A Conditionally Unbiased Estimator

This section introduces the Conditionally Unbiased Estimator $\tilde{N}_u$, which is an estimator of population size N defined by

$$\tilde{N}_u = \left(\frac{z+1}{t}\right)\frac{\delta(x_1, z+1{:}t, R)}{\delta(x_1, z{:}t, R)} - \left(\frac{Rt-z}{t}\right), \qquad (1.7)$$

$$\text{where} \quad \delta(x_1, z{:}t, R) = \sum_{j=0}^{x_1} \binom{x_1}{j}\binom{tj+tR}{z}(-1)^{x_1-j},$$

as defined in section 1.5.

This Conditionally Unbiased Estimator (CUE) was derived from the conditional distribution of $X_1$ given Z. As previously mentioned in section 1.4, the conditional distribution of $X_1$ given Z has appeared in the urn model literature. Charalambides(1981) considered a situation which in some respects may be described as a generalisation of the one discussed here - this being the reason why the conditional distribution of $X_1$ given Z can be obtained from his work. In addition, Charalambides(1981) introduced an estimator which is essentially equivalent to $\tilde{N}_u$ : for values of R geater than or equal to zero. He shows it is a minimum variance unbiased estimator with respect to the conditional distribution of $X_1$ given Z, provided that $Z \geq N$.

In the absence of plants an estimator very similar to $\tilde{N}_u$ has previously been considered in a capture-recapture context : Pathak(1964) derived an estimator in terms of $X_1$ and $\underline{n} = \{n_1, n_2, ...., n_t\}$, where the $n_i$ are the number of animals seen on the ith sampling occasion. Pathak(1964) assumed the $n_i$ to be known constants. A special case of Pathak's estimator is obtained when all the $n_i$ are equal to one : in this situation Berg(1974) showed that Pathak's estimator reduces to a ratio of Stirling numbers of the second kind. This latter result is consistent with the work of Harris(1968). Berg continued his work on Pathak's estimator ; a problem associated with the estimator of Pathak(1964) is that it can be very difficult to compute : being a ratio of two rapidly growing sumations. To overcome this problem, in the situation where all the $n_i$ are equal to one, Berg(1975) derived a recurrence relation which enables one to more easily evaluate the estimate produced by Pathak's estimator. In Berg(1976) the result for this latter special case was extended to include the general multiple-capture census. These recurrence relations for Pathak's estimator were given as functions of $X_1$ and $\underline{n} = \{n_1, n_2, ...., n_t\}$. The work of Berg(1974, 1975, 1976) provided the motivation for much of the work presented within section 1.8a of this chapter and of section 2.10a in chapter 2.

In order to prove that, provided that the condition $Z \geq N$ holds, $\tilde{N}_u$ is in fact unbiased over the conditional distribution of $X_1$ given Z, as a first step and to make the following proof more straightforward, the probability function of $X_1$ given Z is obtained explicitly :

$$\mathrm{Prob}(X_1 = x_1|Z = z) = \frac{\mathrm{Prob}(X_1 = x_1, Z = z)}{\mathrm{Prob}(Z = z)}$$

$$= \frac{\binom{N}{x_1} p^z (1-p)^{Nt+Rt-z} \delta(x_1, z{:}t, R)}{\binom{Nt+Rt}{z} p^z (1-p)^{Nt+Rt-z}}.$$

This follows from equation (1.2) and the fact that $Z \sim \mathrm{Bin}(Nt+Rt, p)$. Hence the probability distribution function of $X_1$ given Z can be written as

$$\mathrm{Prob}(X_1 = x_1|Z = z) = \frac{\binom{N}{x_1} \delta(x_1, z{:}t, R)}{\binom{Nt+Rt}{z}}. \tag{1.8}$$

This is essentially identical to the probability function (2.8), on page 604 of Charalambides(1981).

Now the expectation of $\tilde{N}_u$ taken over the conditional distribution of $X_1$ given Z is given by

$$\mathrm{E}(\tilde{N}_u) = \sum_{x_1} \tilde{N}_u \mathrm{Prob}(X_1 = x_1|Z = z)$$

$$= \sum_{x_1} \left( \left(\frac{z+1}{t}\right) \frac{\delta(x_1, z+1{:}t, R)}{\delta(x_1, z{:}t, R)} - \left(\frac{Rt-z}{t}\right) \right) \mathrm{Prob}(X_1 = x_1|Z = z)$$

$$= \left( \sum_{x_1} \left(\frac{z+1}{t}\right) \frac{\delta(x_1, z+1{:}t, R)}{\delta(x_1, z{:}t, R)} \mathrm{Prob}(X_1|Z) \right) - \left(\frac{Rt-z}{t}\right)$$

$$= \left( \sum_{x_1} \left(\frac{z+1}{t}\right) \frac{\delta(x_1, z+1{:}t, R)}{\delta(x_1, z{:}t, R)} \frac{\binom{N}{x_1} \delta(x_1, z{:}t, R)}{\binom{Nt+Rt}{z}} \right) - \left(\frac{Rt-z}{t}\right) \qquad \text{using (1.8)}$$

$$= \left( \left(\frac{z+1}{t}\right) \sum_{x_1} \frac{\binom{N}{x_1} \delta(x_1, z+1{:}t, R)}{\binom{Nt+Rt}{z}} \right) - \left(\frac{Rt-z}{t}\right)$$

$$= \left( \left(\frac{z+1}{t}\right) \frac{\binom{Nt+Rt}{z+1}}{\binom{Nt+Rt}{z}} \sum_{x_1} \frac{\binom{N}{x_1} \delta(x_1, z+1{:}t, R)}{\binom{Nt+Rt}{z+1}} \right) - \left(\frac{Rt-z}{t}\right)$$

$$= \left( \left(\frac{z+1}{t}\right) \left(\frac{Nt+Rt-z}{z+1}\right) .1 \right) - \left(\frac{Rt-z}{t}\right) \qquad \text{if } Z \geq N$$

$$= N.$$

This shows that $\tilde{N}_u$ is unbiased over the conditional distribution of $X_1$ given Z, provided that the condition $Z \geq N$ holds. Furthermore, again provided that $Z \geq N$, since $X_1$ and Z are sufficient, it follows that $\tilde{N}_u$ is the minimum variance unbiased estimator, Rao(1952).

In view of the fact that population size N is integer valued, in later sections consideration is given to the following slightly modified version of $\tilde{N}_u$ :

$$\hat{N}_u = \left[0.5 + \left(\frac{z+1}{t}\right)\frac{\delta(x_1, z+1{:}t, R)}{\delta(x_1, z{:}t, R)} - \left(\frac{Rt - z}{t}\right)\right],$$

where the square brackets have been used to denote the integer part.

§ 1.8a : A Note on the Evaluation of the CUE

Direct use of equation (1.7) to evaluate the estimates produced by the estimator $\tilde{N}_u$ can often be difficult, and involve very cumbersome computation. This is due to the fact that the $\delta$-numbers, present within (1.7), grow rapidly with increasing arguments. To overcome this computational problem, a recurrence relation linking the $\tilde{N}_u$ is stated and proved. To make the following proof more easily read some shorthand notation is necessary.

Let $$N_{x_1,z} = \tilde{N}_u = \left(\frac{z+1}{t}\right)\frac{\delta(x_1, z+1{:}t, R)}{\delta(x_1, z{:}t, R)} - \left(\frac{Rt - z}{t}\right)$$

and let $$\delta_{x_1,z} = \delta(x_1, z{:}t, R).$$

The $N_{x_1,z}$ are then subject to the following recurrence relation

$$N_{x_1,z} = x_1 + \left(\frac{tN_{x_1-1,z-1} + Rt - z + 1}{tN_{x_1,z-1} + Rt - z + 1}\right)\left(N_{x_1,z-1} - x_1\right), \tag{1.9}$$

with initial conditions $$N_{0,z} = 0 \qquad \text{for} \quad z = 0, 1, 2, \ldots, tR, \tag{1.10}$$

and $$N_{x_1,x_1} = \frac{x_1}{2t}\left[2Rt + tx_1 - x_1 + t + 1\right] \qquad \text{for} \quad x_1 \geq 0. \tag{1.11}$$

Proof of (1.9) :

$$x_1+\left(\frac{tN_{x_1-1,z-1}+Rt-z+1}{tN_{x_1,z-1}+Rt-z+1}\right)\left(N_{x_1,z-1}-x_1\right)$$

$$=x_1+\left(\frac{z\dfrac{\delta_{x_1-1,z}}{\delta_{x_1-1,z-1}}-Rt+z-1+Rt-z+1}{z\dfrac{\delta_{x_1,z}}{\delta_{x_1,z-1}}-Rt+z-1+Rt-z+1}\right)\left(\frac{z}{t}\frac{\delta_{x_1,z}}{\delta_{x_1,z-1}}-\frac{(Rt-z+1)}{t}-x_1\right)$$

$$=x_1+\frac{\delta_{x_1-1,z}}{\delta_{x_1-1,z-1}}\frac{\delta_{x_1,z-1}}{\delta_{x_1,z}}\frac{\left(z\delta_{x_1,z}-(tx_1+Rt-z+1)\delta_{x_1,z-1}\right)}{t\delta_{x_1,z-1}}$$

$$=x_1+\frac{\delta_{x_1-1,z}}{t\delta_{x_1,z}}\frac{\left(z\delta_{x_1,z}-(tx_1+Rt-z+1)\delta_{x_1,z-1}\right)}{\delta_{x_1-1,z-1}}$$

$$=x_1+\frac{\delta_{x_1-1,z}}{t\delta_{x_1,z}}\frac{\left(tx_1\delta_{x_1-1,z-1}\right)}{\delta_{x_1-1,z-1}} \qquad \text{using (1.3)}$$

$$=x_1+\frac{tx_1\delta_{x_1-1,z}}{t\delta_{x_1,z}}$$

$$=x_1+\frac{\left((z+1)\delta_{x_1,z+1}-(tx_1+Rt-z)\delta_{x_1,z}\right)}{t\delta_{x_1,z}} \qquad \text{using (1.3) with z replaced by z+1}$$

$$=x_1+\frac{(z+1)\delta_{x_1,z+1}}{t\delta_{x_1,z}}-\frac{(tx_1+Rt-z)}{t}$$

$$=x_1+\frac{(z+1)\delta_{x_1,z+1}}{t\delta_{x_1,z}}-\frac{(Rt-z)}{t}-x_1$$

$$=\left(\frac{z+1}{t}\right)\frac{\delta_{x_1,z+1}}{\delta_{x_1,z}}-\left(\frac{Rt-z}{t}\right)$$

$$=N_{x_1,z}.$$

Proof of (1.10) :

$$N_{0,z}=\left(\frac{z+1}{t}\right)\frac{\delta_{0,z+1}}{\delta_{0,z}}-\left(\frac{Rt-z}{t}\right)$$

$$=\left(\frac{z+1}{t}\right)\frac{\binom{Rt}{z+1}}{\binom{Rt}{z}}-\left(\frac{Rt-z}{t}\right)$$

$$=\left(\frac{z+1}{t}\right)\frac{\left(\dfrac{Rt-z}{z+1}\right)\binom{Rt}{z}}{\binom{Rt}{z}}-\left(\frac{Rt-z}{t}\right)$$

$$=0.$$

Proof of (1.11) :

As a first step in this proof it is necessary to prove the identity

$$\delta_{x_1,x_1+1} = \frac{t^{x_1}}{2}\left[2Rt + tx_1 - x_1\right]. \tag{1.12}$$

The identity (1.12) may be proved by induction :

Anchor : (1.12) is clearly true for $x_1 = 0$, since $\delta_{0,1} = Rt$.

Assume true for $x_1 = k$, i.e. assume $\delta_{k,k+1} = \frac{t^k}{2}[2Rt + tk - k]$.

Then

$$\begin{aligned}
\delta_{k+1,k+2} &= \frac{1}{k+2}\left[\left(t(k+1) + Rt - (k+2) + 1\right)\delta_{k+1,k+1} + t(k+1)\delta_{k,k+1}\right] && \text{using (1.3)} \\
&= \frac{1}{k+2}\left[(tk + t + Rt - k - 1)t^{k+1} + t(k+1)\frac{t^k}{2}[2Rt + tk - k]\right] && \text{using assumption} \\
&= \frac{t^{k+1}}{2(k+2)}\left[2(tk + t + Rt - k - 1) + (k+1)(2Rt + tk - k)\right] \\
&= \frac{t^{k+1}}{2(k+2)}\left[3tk + 2t - 3k - 2 + 4Rt + 2kRt + tk^2 - k^2\right] \\
&= \frac{t^{k+1}}{2(k+2)}\left[(k+2)\left(2Rt + t(k+1) - (k+1)\right)\right] \\
&= \frac{t^{k+1}}{2}\left[2Rt + t(k+1) - (k+1)\right].
\end{aligned}$$

This shows that, if (1.12) is true for $x_1 = k$, then it must also be true for $x_1 = k+1$. Since it has been shown that (1.12) is true for $x_1 = 0$, it follows by induction that (1.12) holds for all $x_1 \geq 0$.

The proof of (1.11) may now be completed :

$$\begin{aligned}
N_{x_1,x_1} &= \left(\frac{x_1+1}{t}\right)\frac{\delta_{x_1,x_1+1}}{\delta_{x_1,x_1}} - \left(\frac{Rt - x_1}{t}\right) \\
&= \left(\frac{x_1+1}{t}\right)\frac{\frac{t^{x_1}}{2}\left[2Rt + tx_1 - x_1\right]}{t^{x_1}} - \left(\frac{Rt - x_1}{t}\right) \\
&= \frac{1}{2t}\left[(x_1+1)(2Rt + tx_1 - x_1) - 2(Rt - x_1)\right] \\
&= \frac{1}{2t}\left[2x_1Rt + tx_1^2 - x_1^2 + tx_1 + x_1\right] \\
&= \frac{x_1}{2t}\left[2Rt + tx_1 - x_1 + t + 1\right].
\end{aligned}$$

## § 1.9 : A Comparison of All Three Estimators

In order to compare the performance of the three estimators which have so far been discussed we consider their mean, standard deviation and root mean square error conditional on the event $C = \{Z > X_1\}$. This conditioning is necessary since the maximum likelihood estimator $\hat{N}$ yields infinite estimates when $Z = X_1$. It is important to note however that both the Peterson-type estimator $\hat{N}_P$ and the CUE $\hat{N}_u$ produce finite estimates with probability one. The unconditional performance of $\hat{N}_P$ and $\hat{N}_u$ is considered later on in section 1.10.

Conditional on the event $C = \{Z > X_1\}$, the mean, standard deviation and root mean square error of each estimator are presented in tables 1.1a,b,c, 1.2a,b,c, 1.3a,b,c and 1.4a,b,c. These tables summarise the performance of the estimators for each combination from the following factorial design :

$$N = \begin{matrix} 10 \\ 25 \\ 50 \\ 100 \end{matrix} \times \ t = \begin{matrix} 5 \\ 10 \\ 15 \\ 20 \end{matrix} \times \ p = \begin{matrix} 0.05 \\ 0.10 \\ 0.20 \end{matrix} \times \ R = \begin{matrix} 0 \\ 5 \\ 10 \\ 25 \\ 50 \\ 100 \end{matrix}.$$

Note however that, for each value of population size N, only values of R up to and including N are considered; this is done for obvious practical reasons.

The notation used within each table is as follows :

Statistics

| | | |
|---|---|---|
| exp. | $\equiv$ | mean or expectation. |
| s.d. | $\equiv$ | standard deviation. |
| rmse | $\equiv$ | root mean square error. |
| P(inf mle) | $\equiv$ | $1 - \text{Prob}(C) = \text{Prob}(\overline{C}) = \text{Prob}(Z = X_1)$, which is the probability of the maximum likelihood estimator producing an infinite estimate. |

Estimators

| | | | |
|---|---|---|---|
| X1 | $\equiv$ | $X_1$, | the number of distinct individuals seen from the target population. |
| P | $\equiv$ | $\hat{N}_P$, | the Peterson-type estimator of section 1.7. |
| CUE | $\equiv$ | $\hat{N}_u$, | the conditionally unbiased estimator of section 1.8. |
| MLE | $\equiv$ | $\hat{N}$, | the maximum likelihood estimator of section 1.6. |

It is straightforward to obtain the distributions of both $\hat{N}$ and $\hat{N}_u$ given C. In order to obtain the conditional distribution of the Peterson-type estimator $\hat{N}_P$ given C we need to derive the conditional distribution of $X_1$ and $X_2$ given C. This may be done as follows :

C is defined as being the event $\{Z > X_1\}$.
Let $\overline{C}$ be the complementary event $\{Z = X_1\}$.
$\overline{C}$ occurs $\Leftrightarrow$ $X_2 = 0$ and each animal in target population is seen at most once.

Now $\text{Prob}(X_2 = 0) = \left[(1-p)^t\right]^R$. (1.13)

( This follows from the fact that $X_2 \sim \text{Bin}\left(R,\ 1-(1-p)^t\right)$. )

Let $Y_i$ = the number of sightings of animal i, it follows that $Y_i \sim \text{Bin}(t,p)$.

It may then be observed that

Prob( each animal in target population is seen at most once )

$$= \prod_{i=1}^{N} \text{Prob}(Y_i \le 1)$$

$$= \left[(1-p+tp)(1-p)^{t-1}\right]^N. \tag{1.14}$$

Use of (1.13) and (1.14) implies that

$$\text{Prob}(C) = 1 - \text{Prob}(\overline{C})$$
$$= 1 - \left[(1-p)^t\right]^R \left[(1-p+tp)(1-p)^{t-1}\right]^N.$$

Now

$$\text{Prob}(X_1 = x_1, X_2 = x_2 | Z > X_1) = \frac{\text{Prob}(X_1 = x_1, X_2 = x_2, Z > X_1)}{\text{Prob}(Z > X_1)}$$
$$= \frac{\text{Prob}(X_1 = x_1, X_2 = x_2)\text{Prob}(Z > X_1 | X_1 = x_1, X_2 = x_2)}{\text{Prob}(Z > X_1)}$$
$$= \frac{\text{Prob}(X_1 = x_1)\text{Prob}(X_2 = x_2)\text{Prob}(Z > X_1 | X_1 = x_1, X_2 = x_2)}{\text{Prob}(Z > X_1)}.$$

It is clear that $\text{Prob}(Z > X_1 | X_1 = x_1, X_2 = x_2) = 1$ if $X_2 > 0$.

When $X_2 = 0$ it may be observed that $Z|X_1, X_2 \equiv Z_1|X_1$ . It is known that the distribution of $Z_1|X_1$ may be characterised as being the sum of $X_1$ zero truncated Binomial random variables, the distribution of which is derived in appendix 3.

Explicitly the probability function of $Z_1|X_1$ is given by

$$\text{Prob}(Z_1 = z_1 | X_1 = x_1) = \frac{p^{z_1}(1-p)^{tx_1 - z_1}}{\left[1-(1-p)^t\right]^{x_1}} \delta(x_1, z_1; t, 0).$$

It follows that

$$\begin{aligned}\text{Prob}\left(Z > x_1 \middle| X_1 = x_1, X_2 = x_2\right) &= \text{Prob}\left(Z_1 > x_1 \middle| X_1 = x_1\right)\\ &= 1 - \text{Prob}\left(Z_1 = x_1 \middle| X_1 = x_1\right)\\ &= 1 - \frac{p^{x_1}(1-p)^{tx_1 - x_1}}{\left[1-(1-p)^t\right]^{x_1}}\delta(x_1, x_1; t, 0)\\ &= 1 - \frac{t^{x_1}p^{x_1}(1-p)^{tx_1-x_1}}{\left[1-(1-p)^t\right]^{x_1}}, \qquad \text{using 1.3a.}\end{aligned}$$

Using the notation $\tilde{P}(C) = \text{Prob}\left(Z > X_1 \middle| X_1 = x_1, X_2 = x_2\right)$ then allows one to write :

$$\text{Prob}\left(X_1{=}x_1, X_2{=}x_2 \middle| Z{>}X_1\right) = \frac{\binom{N}{x_1}\left[1-(1-p)^t\right]^{x_1}\left[(1-p)^t\right]^{N-x_1}\binom{R}{x_2}\left[1-(1-p)^t\right]^{x_2}\left[(1-p)^t\right]^{R-x_2}\tilde{P}(C)}{P(C)},$$

$$x_1 = \begin{cases} 0,1,2,\ldots,N & \text{for} \quad R>0 \\ 1,2,\ldots,N & \text{for} \quad R=0 \end{cases},$$

$$x_2 = \begin{cases} 1,2,\ldots,R & \text{for} \quad R>0,\ x_1=0 \\ 0,1,2,\ldots,R & \text{for} \quad R>0,\ x_1>0, \\ 0 & \text{for} \quad R=0 \end{cases}$$

$$\text{where} \quad \tilde{P}(C) = \begin{cases} 1 & \text{for} \quad X_2>0 \\ 1 - \dfrac{t^{x_1}p^{x_1}(1-p)^{tx_1-x_1}}{\left[1-(1-p)^t\right]^{x_1}} & \text{for} \quad X_2=0 \end{cases}$$

$$\text{and} \qquad P(C) = \text{Prob}(Z>X_1) = 1 - \left[(1-p)^t\right]^R\left[(1-p+tp)(1-p)^{t-1}\right]^N.$$

## § 1.9a : Discussion

Let us firstly compare the performance of the Peterson-type estimator $\hat{N}_p$ to that of the CUE $\hat{N}_u$ ; the comparison between these two estimators is straightforward in situations with or without plants. In the absence of plants , i.e. when R = 0, since $\hat{N}_p$ reduces to $X_1$, the number of distinct individuals seen from the target population, one would expect $\hat{N}_u$ to clearly outperform $\hat{N}_p$. This is broadly true, in that for the great majority of situations considered, when R = 0, the CUE generally possesses both a better mean and root mean square error. In the remaining three situations where the root mean square error of $\hat{N}_u$ is marginally greater then that of $\hat{N}_p$, the CUE is less biased. When sampling with plants, i.e. when R > 0, the CUE is again seen to be clearly a better alternative to $\hat{N}_p$. When R>0 both estimators have very small bias. However the estimator $\hat{N}_u$ is in almost all situations less biased than $\hat{N}_p$, and where its bias is worse the difference is minimal. The standard deviation of $\hat{N}_u$ is always less than that of $\hat{N}_p$, the difference between these two statistics being appreciable when the number of plants is small relative to population size. In terms of root mean square error, when $R > 0$, $\hat{N}_u$ is uniformly better than $\hat{N}_p$ ; the root mean square error of $\hat{N}_u$ is more appreciably better than that of $\hat{N}_p$ when R is small relative to N.

In situations where only a very small amount of information is available the maximum likelihood estimator $\hat{N}$ has a tendency to be positively biased, sometimes extremely so. This is most noticeable when considering the larger population sizes. In contrast to this the CUE tends to be negatively biased when only a very small amount of information is present. As more information becomes available both $\hat{N}$ and $\hat{N}_u$ each perform extremely well in terms of mean , with $\hat{N}_u$ on all but a few occasions being the less biased of the two. With regard to bias, it appears that, of the two estimators $\hat{N}$ and $\hat{N}_u$, the CUE behaves in a far more desirable way. To illustrate this consider table 1.4a, in which $N = 100$ and $p = 0.05$. Consider the situation where R is equal to zero : for $t =$ 5, 10, 15 and 20 the mean values taken by $\hat{N}$ are respectively 115, 108, 102 and 101 ; whereas the corresponding mean values taken by $\hat{N}_u$ are respectively 80.6, 99.9, 100 and 100. This shows how the bias of the estimators can alter as more information is gained through additional sampling occasions. To show that the estimators respond in a similar way as information is gained through the planted individuals consider the column giving results for $t = 5$ : for $R = 0, 5, 10, 25$ and 50 the mean values taken by $\hat{N}$ are respectively 115, 123, 122, 111 and 105 ; whereas the corresponding mean values taken by $\hat{N}_u$ are respectively 80.6, 92.0, 97.1, 99.9 and 100. These examples highlight in particular the general feature that the mean of the CUE improves uniformly with more information whereas that of the MLE is less predictable.

In all but a few situations, the CUE exhibits a smaller standard deviation than the MLE. In particular, when the number of sampling occasions is small the standard deviation of $\hat{N}_u$ tends to be significantly smaller than that of $\hat{N}$. The MLE only has a smaller standard deviation than that of $\hat{N}_u$ in a few situations, wherein $p = 0.20$, and notably in these situations one would expect to see on average at least 96% of the target population.

Since, on the whole, the CUE tends to posses both a smaller absolute bias and standard deviation, it necessarily follows that $\hat{N}_u$ usually also has the smaller root mean square error. When only a small amount of information is available $\hat{N}_u$, in terms of root mean square error, is seen to significantly outperform $\hat{N}$. Whereas, as more information is gained the two estimators are seen to behave more closely in terms of root mean square error, although again with $\hat{N}_u$ tending to be ahead.

In the above discussion a deliberate attempt has been made not to place too much emphasis on mean square error. This being due to the fact that mean square error is known to reward negative bias. So that when only a small proportion of the population is seen during sampling, that is when $\hat{N}_u$ is negatively biased with 'small' variance and $\hat{N}$ is positively biased with 'large' variance, one would expect mean square error to perhaps unfairly favour the estimator $\hat{N}_u$. It is true that this can occasionally happen : for an example consider table 1.2a. When $N = 25$, $p = 0.05$, $R = 5$ and $t = 5$ the mean, standard deviation and root mean square error of $\hat{N}_u$ are respectively 18.1, 9.47 and 11.7; whereas the corresponding values for $\hat{N}$ are respectively 26.2, 17.4 and 17.4. In this situation, on the basis of root mean square error alone, one would choose the CUE, however it could be argued that an alternative loss criterion which places more weight on the mean of an estimator might more sensibly favour the MLE. Examples of this type however are few and far between. Generally the mean of $\hat{N}_u$ is as good as or better than that of $\hat{N}$, and as a result of $\hat{N}_u$ also tending to have a smaller standard deviation, it may be concluded that one should always use the CUE in preference to the MLE.

Table 1.1a

| N = 10 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.05 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 2.96 | 1.24 | 7.15 | 4.40 | 1.46 | 5.79 | 5.54 | 1.51 | 4.71 | 6.48 | 1.49 | 3.82 |
| | P | 2.96 | 1.24 | 7.15 | 4.40 | 1.46 | 5.79 | 5.54 | 1.51 | 4.71 | 6.48 | 1.49 | 3.82 |
| | CUE | 3.93 | 2.32 | 6.50 | 6.63 | 3.31 | 4.72 | 8.46 | 3.51 | 3.83 | 9.47 | 3.32 | 3.36 |
| | MLE | 4.30 | 3.02 | 6.45 | 7.92 | 5.03 | 5.44 | 9.79 | 5.44 | 5.44 | 10.4 | 4.95 | 4.97 |
| | P(inf mle) | 0.7957 | | | 0.4063 | | | 0.1534 | | | 0.0465 | | |
| | | | | | | | | | | | | | |
| 5 | X1 | 2.31 | 1.33 | 7.80 | 4.03 | 1.55 | 6.17 | 5.37 | 1.58 | 4.89 | 6.42 | 1.52 | 3.89 |
| | P | 6.56 | 5.11 | 6.16 | 9.24 | 5.90 | 5.95 | 9.93 | 5.36 | 5.36 | 10.0 | 4.50 | 4.50 |
| | CUE | 6.60 | 4.54 | 5.67 | 9.27 | 5.00 | 5.05 | 9.87 | 4.18 | 4.18 | 10.0 | 3.25 | 3.25 |
| | MLE | 9.30 | 7.64 | 7.67 | 11.3 | 8.12 | 8.22 | 10.6 | 6.08 | 6.11 | 10.0 | 4.10 | 4.10 |
| | P(inf mle) | 0.2207 | | | 0.0313 | | | 0.0033 | | | 0.0003 | | |
| | | | | | | | | | | | | | |
| 10 | X1 | 2.27 | 1.32 | 7.84 | 4.01 | 1.55 | 6.18 | 5.37 | 1.58 | 4.89 | 6.42 | 1.52 | 3.89 |
| | P | 8.63 | 6.74 | 6.88 | 9.95 | 5.96 | 5.96 | 10.0 | 4.53 | 4.53 | 10.0 | 3.56 | 3.56 |
| | CUE | 8.52 | 6.14 | 6.31 | 9.87 | 5.26 | 5.26 | 9.96 | 3.84 | 3.84 | 9.99 | 2.93 | 2.93 |
| | MLE | 11.6 | 10.5 | 10.7 | 11.0 | 7.57 | 7.64 | 10.1 | 4.56 | 4.56 | 9.77 | 3.16 | 3.17 |
| | P(inf mle) | 0.0612 | | | 0.0024 | | | 0.0001 | | | 0.0000 | | |

Table 1.1b

| N = 10 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.10 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 4.49 | 1.47 | 5.71 | 6.57 | 1.48 | 3.73 | 7.95 | 1.28 | 2.42 | 8.78 | 1.03 | 1.60 |
| | P | 4.49 | 1.47 | 5.71 | 6.57 | 1.48 | 3.73 | 7.95 | 1.28 | 2.42 | 8.78 | 1.03 | 1.60 |
| | CUE | 6.62 | 3.10 | 4.59 | 9.38 | 3.21 | 3.27 | 9.97 | 2.31 | 2.31 | 10.0 | 1.57 | 1.57 |
| | MLE | 7.60 | 4.43 | 5.04 | 10.2 | 4.93 | 4.93 | 9.94 | 3.09 | 3.09 | 9.61 | 1.72 | 1.76 |
| | P(inf mle) | 0.4275 | | | 0.0467 | | | 0.0025 | | | 0.0001 | | |
| | | | | | | | | | | | | | |
| 5 | X1 | 4.11 | 1.55 | 6.09 | 6.51 | 1.51 | 3.80 | 7.94 | 1.28 | 2.42 | 8.78 | 1.03 | 1.60 |
| | P | 9.27 | 5.82 | 5.87 | 10.0 | 4.40 | 4.40 | 10.1 | 3.02 | 3.02 | 10.1 | 2.16 | 2.16 |
| | CUE | 9.29 | 4.82 | 4.87 | 9.99 | 3.20 | 3.20 | 9.99 | 2.02 | 2.02 | 10.0 | 1.34 | 1.34 |
| | MLE | 11.3 | 7.99 | 8.09 | 10.1 | 3.98 | 3.98 | 9.64 | 2.12 | 2.15 | 9.55 | 1.43 | 1.50 |
| | P(inf mle) | 0.0307 | | | 0.0002 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 10 | X1 | 4.10 | 1.55 | 6.11 | 6.51 | 1.51 | 3.80 | 7.94 | 1.28 | 2.42 | 8.78 | 1.03 | 1.60 |
| | P | 9.96 | 5.85 | 5.85 | 10.1 | 3.47 | 3.47 | 10.1 | 2.30 | 2.30 | 10.1 | 1.63 | 1.63 |
| | CUE | 9.90 | 5.13 | 5.13 | 10.0 | 2.88 | 2.88 | 10.0 | 1.88 | 1.88 | 10.0 | 1.29 | 1.29 |
| | MLE | 10.9 | 7.43 | 7.47 | 9.81 | 3.10 | 3.11 | 9.59 | 1.92 | 1.96 | 9.51 | 1.36 | 1.44 |
| | P(inf mle) | 0.0022 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |

Table 1.1c

| N = 10 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.20 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 6.78 | 1.46 | 3.53 | 8.93 | 0.98 | 1.45 | 9.65 | 0.58 | 0.68 | 9.88 | 0.34 | 0.36 |
| | P | 6.78 | 1.46 | 3.53 | 8.93 | 0.98 | 1.45 | 9.65 | 0.58 | 0.68 | 9.88 | 0.34 | 0.36 |
| | CUE | 9.58 | 3.08 | 3.11 | 9.98 | 1.44 | 1.44 | 9.86 | 0.75 | 0.76 | 9.90 | 0.35 | 0.37 |
| | MLE | 10.2 | 4.50 | 4.51 | 9.59 | 1.56 | 1.61 | 9.69 | 0.63 | 0.70 | 9.89 | 0.34 | 0.36 |
| | P(inf mle) | 0.0475 | | | 0.0001 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 5 | X1 | 6.72 | 1.48 | 3.60 | 8.93 | 0.98 | 1.45 | 9.65 | 0.58 | 0.68 | 9.88 | 0.34 | 0.36 |
| | P | 10.1 | 4.20 | 4.20 | 10.1 | 2.00 | 2.00 | 10.0 | 1.07 | 1.07 | 10.0 | 0.60 | 0.60 |
| | CUE | 10.1 | 3.06 | 3.06 | 10.0 | 1.29 | 1.29 | 9.85 | 0.73 | 0.75 | 9.89 | 0.34 | 0.36 |
| | MLE | 10.0 | 3.76 | 3.76 | 9.52 | 1.33 | 1.42 | 9.66 | 0.60 | 0.69 | 9.88 | 0.34 | 0.36 |
| | P(inf mle) | 0.0002 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 10 | X1 | 6.72 | 1.48 | 3.60 | 8.93 | 0.98 | 1.45 | 9.65 | 0.58 | 0.68 | 9.88 | 0.34 | 0.36 |
| | P | 10.1 | 3.30 | 3.30 | 10.0 | 1.52 | 1.52 | 10.0 | 0.84 | 0.84 | 10.0 | 0.48 | 0.48 |
| | CUE | 10.0 | 2.74 | 2.74 | 10.0 | 1.21 | 1.21 | 9.80 | 0.70 | 0.73 | 9.89 | 0.34 | 0.36 |
| | MLE | 9.75 | 2.95 | 2.96 | 9.50 | 1.27 | 1.37 | 9.66 | 0.59 | 0.68 | 9.88 | 0.34 | 0.36 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |

Table 1.2a

| N = 25 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.05 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| 0 | X1 | 6.24 | 2.03 | 18.9 | 10.2 | 2.41 | 15.0 | 13.4 | 2.48 | 11.8 | 16.0 | 2.40 | 9.28 |
| | P | 6.24 | 2.03 | 18.9 | 10.2 | 2.41 | 15.0 | 13.4 | 2.48 | 11.8 | 16.0 | 2.40 | 9.28 |
| | CUE | 11.5 | 5.91 | 14.7 | 21.2 | 8.89 | 9.65 | 24.5 | 8.35 | 8.36 | 25.0 | 6.39 | 6.39 |
| | MLE | 15.1 | 9.53 | 13.8 | 27.1 | 15.5 | 15.6 | 27.5 | 13.4 | 13.6 | 25.9 | 8.44 | 8.49 |
| | P(inf mle) | 0.5648 | | | 0.1052 | | | 0.0092 | | | 0.0005 | | |
| 5 | X1 | 5.74 | 2.09 | 19.4 | 10.0 | 2.45 | 15.2 | 13.4 | 2.49 | 11.8 | 16.0 | 2.40 | 9.28 |
| | P | 17.5 | 11.0 | 13.3 | 23.7 | 13.5 | 13.6 | 24.8 | 12.2 | 12.2 | 25.0 | 10.1 | 10.1 |
| | CUE | 18.1 | 9.47 | 11.7 | 24.4 | 10.4 | 10.4 | 25.0 | 7.68 | 7.68 | 25.0 | 5.52 | 5.52 |
| | MLE | 26.2 | 17.4 | 17.4 | 28.7 | 17.2 | 17.6 | 26.2 | 9.99 | 10.1 | 25.2 | 6.14 | 6.15 |
| | P(inf mle) | 0.1567 | | | 0.0081 | | | 0.0002 | | | 0.0000 | | |
| 10 | X1 | 5.68 | 2.09 | 19.4 | 10.0 | 2.45 | 15.2 | 13.4 | 2.49 | 11.8 | 16.0 | 2.40 | 9.28 |
| | P | 22.1 | 14.3 | 14.6 | 24.9 | 12.9 | 12.9 | 25.0 | 9.62 | 9.62 | 25.0 | 7.44 | 7.44 |
| | CUE | 22.1 | 12.1 | 12.5 | 24.9 | 9.96 | 9.96 | 25.0 | 6.85 | 6.85 | 25.0 | 5.09 | 5.09 |
| | MLE | 30.1 | 22.3 | 22.9 | 27.4 | 14.1 | 14.3 | 25.6 | 7.80 | 7.82 | 25.0 | 5.36 | 5.36 |
| | P(inf mle) | 0.0435 | | | 0.0006 | | | 0.0000 | | | 0.0000 | | |
| 25 | X1 | 5.66 | 2.09 | 19.5 | 10.0 | 2.45 | 15.2 | 13.4 | 2.49 | 11.8 | 16.0 | 2.40 | 9.28 |
| | P | 24.9 | 14.5 | 14.5 | 25.0 | 9.13 | 9.13 | 25.0 | 6.77 | 6.77 | 25.0 | 5.39 | 5.39 |
| | CUE | 24.9 | 13.4 | 13.4 | 25.0 | 8.19 | 8.19 | 25.0 | 5.85 | 5.85 | 25.0 | 4.51 | 4.51 |
| | MLE | 28.5 | 19.6 | 19.9 | 25.6 | 8.98 | 9.00 | 25.0 | 6.08 | 6.08 | 24.7 | 4.63 | 4.64 |
| | P(inf mle) | 0.0009 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |

Table 1.2b

| N = 25 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.10 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 10.4 | 2.41 | 14.8 | 16.3 | 2.38 | 9.04 | 19.9 | 2.02 | 5.53 | 22.0 | 1.63 | 3.45 |
| | P | 10.4 | 2.41 | 14.8 | 16.3 | 2.38 | 9.04 | 19.9 | 2.02 | 5.53 | 22.0 | 1.63 | 3.45 |
| | CUE | 20.9 | 8.44 | 9.39 | 25.0 | 6.25 | 6.25 | 25.0 | 3.56 | 3.56 | 25.0 | 2.29 | 2.29 |
| | MLE | 26.5 | 14.7 | 14.8 | 25.9 | 8.29 | 8.34 | 24.8 | 3.78 | 3.79 | 24.6 | 2.35 | 2.38 |
| | P(inf mle) | 0.1195 | | | 0.0005 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 5 | X1 | 10.2 | 2.46 | 15.0 | 16.3 | 2.38 | 9.04 | 19.9 | 2.02 | 5.53 | 22.0 | 1.63 | 3.45 |
| | P | 23.7 | 13.4 | 13.5 | 25.0 | 9.92 | 9.92 | 25.0 | 6.70 | 6.70 | 25.0 | 4.78 | 4.78 |
| | CUE | 24.3 | 10.3 | 10.3 | 25.0 | 5.41 | 5.41 | 25.0 | 3.26 | 3.26 | 25.0 | 2.21 | 2.21 |
| | MLE | 28.6 | 17.2 | 17.6 | 25.2 | 6.05 | 6.05 | 24.7 | 3.36 | 3.37 | 24.6 | 2.23 | 2.27 |
| | P(inf mle) | 0.0086 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 10 | X1 | 10.2 | 2.46 | 15.0 | 16.3 | 2.38 | 9.04 | 19.9 | 2.02 | 5.53 | 22.0 | 1.63 | 3.45 |
| | P | 24.9 | 12.7 | 12.7 | 25.0 | 7.26 | 7.26 | 25.0 | 4.90 | 4.90 | 24.9 | 3.54 | 3.54 |
| | CUE | 24.9 | 9.94 | 9.94 | 25.0 | 4.96 | 4.96 | 25.0 | 3.14 | 3.14 | 25.0 | 2.14 | 2.14 |
| | MLE | 27.2 | 14.1 | 14.3 | 25.0 | 5.24 | 5.24 | 24.6 | 3.18 | 3.20 | 24.5 | 2.16 | 2.20 |
| | P(inf mle) | 0.0006 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 25 | X1 | 10.2 | 2.46 | 15.0 | 16.3 | 2.38 | 9.04 | 19.9 | 2.02 | 5.53 | 22.0 | 1.63 | 3.45 |
| | P | 25.0 | 8.95 | 8.95 | 25.0 | 5.27 | 5.27 | 25.0 | 3.63 | 3.63 | 25.1 | 2.57 | 2.57 |
| | CUE | 25.0 | 8.06 | 8.06 | 25.0 | 4.41 | 4.41 | 25.0 | 2.91 | 2.91 | 25.0 | 2.05 | 2.05 |
| | MLE | 25.6 | 8.83 | 8.85 | 24.8 | 4.53 | 4.53 | 24.6 | 2.93 | 2.96 | 24.5 | 2.06 | 2.11 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |

Table 1.2c

| N = 25 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.20 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 16.8 | 2.35 | 8.52 | 22.3 | 1.55 | 3.10 | 24.1 | 0.92 | 1.27 | 24.7 | 0.53 | 0.61 |
| | P | 16.8 | 2.35 | 8.52 | 22.3 | 1.55 | 3.10 | 24.1 | 0.92 | 1.27 | 24.7 | 0.53 | 0.61 |
| | CUE | 24.9 | 6.12 | 6.12 | 25.0 | 2.14 | 2.14 | 25.1 | 1.07 | 1.07 | 24.8 | 0.61 | 0.64 |
| | MLE | 25.7 | 8.15 | 8.18 | 24.6 | 2.18 | 2.22 | 24.4 | 1.08 | 1.21 | 24.7 | 0.54 | 0.61 |
| | P(inf mle) | 0.0005 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 5 | X1 | 16.8 | 2.35 | 8.52 | 22.3 | 1.55 | 3.10 | 24.1 | 0.92 | 1.27 | 24.7 | 0.53 | 0.61 |
| | P | 25.1 | 9.45 | 9.45 | 25.0 | 4.44 | 4.44 | 25.0 | 2.40 | 2.40 | 25.0 | 1.34 | 1.34 |
| | CUE | 25.1 | 5.26 | 5.26 | 25.0 | 2.05 | 2.05 | 25.1 | 1.03 | 1.04 | 24.8 | 0.60 | 0.64 |
| | MLE | 25.2 | 5.80 | 5.81 | 24.5 | 2.07 | 2.12 | 24.4 | 1.08 | 1.21 | 24.7 | 0.53 | 0.61 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 10 | X1 | 16.8 | 2.35 | 8.52 | 22.3 | 1.55 | 3.10 | 24.1 | 0.92 | 1.27 | 24.7 | 0.53 | 0.61 |
| | P | 25.0 | 6.88 | 6.88 | 24.9 | 3.29 | 3.29 | 25.0 | 1.86 | 1.86 | 25.0 | 1.11 | 1.11 |
| | CUE | 25.0 | 4.81 | 4.81 | 25.0 | 1.99 | 1.99 | 25.1 | 1.02 | 1.02 | 24.8 | 0.58 | 0.63 |
| | MLE | 24.9 | 5.04 | 5.04 | 24.5 | 2.01 | 2.07 | 24.4 | 1.07 | 1.22 | 24.7 | 0.53 | 0.61 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 25 | X1 | 16.8 | 2.35 | 8.52 | 22.3 | 1.55 | 3.10 | 24.1 | 0.92 | 1.27 | 24.7 | 0.53 | 0.61 |
| | P | 25.0 | 5.02 | 5.02 | 25.1 | 2.38 | 2.38 | 25.0 | 1.32 | 1.32 | 25.0 | 0.76 | 0.76 |
| | CUE | 25.0 | 4.26 | 4.26 | 25.0 | 1.91 | 1.91 | 25.1 | 0.98 | 0.98 | 24.7 | 0.56 | 0.62 |
| | MLE | 24.7 | 4.33 | 4.34 | 24.5 | 1.92 | 1.98 | 24.4 | 1.06 | 1.22 | 24.7 | 0.53 | 0.61 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |

Table 1.3a

| N = 50 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.05 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 11.7 | 2.90 | 38.4 | 20.1 | 3.45 | 30.1 | 26.8 | 3.53 | 23.4 | 32.1 | 3.39 | 18.2 |
| | P | 11.7 | 2.90 | 38.4 | 20.1 | 3.45 | 30.1 | 26.8 | 3.53 | 23.4 | 32.1 | 3.39 | 18.2 |
| | CUE | 30.1 | 13.4 | 24.0 | 48.6 | 18.3 | 18.3 | 49.9 | 12.7 | 12.7 | 50.0 | 8.64 | 8.64 |
| | MLE | 43.3 | 24.4 | 25.4 | 57.7 | 31.7 | 32.7 | 52.3 | 16.2 | 16.3 | 50.6 | 9.37 | 9.39 |
| | P(inf mle) | | 0.3190 | | | 0.0111 | | | 0.0001 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 5 | X1 | 11.4 | 2.95 | 38.7 | 20.1 | 3.47 | 30.1 | 26.8 | 3.53 | 23.4 | 32.1 | 3.39 | 18.2 |
| | P | 37.0 | 20.8 | 24.5 | 47.7 | 25.9 | 26.0 | 49.6 | 23.4 | 23.4 | 50.0 | 19.5 | 19.5 |
| | CUE | 40.8 | 18.7 | 20.9 | 49.8 | 17.7 | 17.7 | 50.0 | 11.3 | 11.3 | 50.0 | 8.01 | 8.01 |
| | MLE | 57.9 | 36.3 | 37.1 | 55.2 | 26.3 | 26.8 | 51.4 | 12.8 | 12.9 | 50.3 | 8.43 | 8.44 |
| | P(inf mle) | | 0.0885 | | | 0.0009 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 10 | X1 | 11.3 | 2.96 | 38.8 | 20.1 | 3.47 | 30.1 | 26.8 | 3.53 | 23.4 | 32.1 | 3.39 | 18.2 |
| | P | 45.4 | 27.5 | 27.9 | 49.9 | 24.3 | 24.3 | 50.0 | 17.9 | 17.9 | 50.0 | 13.9 | 13.9 |
| | CUE | 46.3 | 22.3 | 22.6 | 50.0 | 16.0 | 16.0 | 50.0 | 10.4 | 10.4 | 50.0 | 7.58 | 7.58 |
| | MLE | 61.5 | 41.8 | 43.4 | 53.4 | 20.6 | 20.9 | 50.9 | 11.3 | 11.3 | 50.2 | 7.88 | 7.88 |
| | P(inf mle) | | 0.0245 | | | 0.0001 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 25 | X1 | 11.3 | 2.96 | 38.8 | 20.1 | 3.47 | 30.1 | 26.8 | 3.53 | 23.4 | 32.1 | 3.39 | 18.2 |
| | P | 49.9 | 25.7 | 25.7 | 50.0 | 15.9 | 15.9 | 50.0 | 11.8 | 11.8 | 50.0 | 9.36 | 9.36 |
| | CUE | 49.8 | 22.5 | 22.5 | 50.0 | 13.0 | 13.0 | 50.0 | 9.07 | 9.07 | 50.0 | 6.87 | 6.87 |
| | MLE | 56.8 | 33.1 | 33.7 | 51.4 | 14.2 | 14.2 | 50.3 | 9.41 | 9.41 | 49.9 | 7.00 | 7.00 |
| | P(inf mle) | | 0.0005 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 50 | X1 | 11.3 | 2.96 | 38.8 | 20.1 | 3.47 | 30.1 | 26.8 | 3.53 | 23.4 | 32.1 | 3.39 | 18.2 |
| | P | 50.0 | 19.7 | 19.7 | 50.0 | 12.5 | 12.5 | 50.0 | 9.43 | 9.43 | 50.0 | 7.55 | 7.55 |
| | CUE | 50.0 | 18.6 | 18.6 | 50.0 | 11.3 | 11.3 | 50.0 | 8.18 | 8.18 | 50.0 | 6.33 | 6.33 |
| | MLE | 52.7 | 21.2 | 21.3 | 50.5 | 11.7 | 11.7 | 50.0 | 8.32 | 8.32 | 49.8 | 6.40 | 6.40 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |

Table 1.3b

| N = 50 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.10 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 20.5 | 3.46 | 29.7 | 32.6 | 3.37 | 17.8 | 39.7 | 2.86 | 10.7 | 43.9 | 2.31 | 6.50 |
| | P | 20.5 | 3.46 | 29.7 | 32.6 | 3.37 | 17.8 | 39.7 | 2.86 | 10.7 | 43.9 | 2.31 | 6.50 |
| | CUE | 48.2 | 18.1 | 18.2 | 50.0 | 8.54 | 8.54 | 50.0 | 4.86 | 4.86 | 50.0 | 3.21 | 3.21 |
| | MLE | 57.7 | 31.8 | 32.7 | 50.6 | 9.29 | 9.31 | 49.8 | 4.97 | 4.97 | 49.6 | 3.24 | 3.27 |
| | P(inf mle) | 0.0143 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 5 | X1 | 20.5 | 3.48 | 29.7 | 32.6 | 3.37 | 17.8 | 39.7 | 2.86 | 10.7 | 43.9 | 2.31 | 6.50 |
| | P | 47.9 | 25.8 | 25.9 | 50.0 | 19.1 | 19.1 | 50.0 | 12.8 | 12.8 | 50.0 | 9.09 | 9.09 |
| | CUE | 49.8 | 17.7 | 17.7 | 50.0 | 7.89 | 7.89 | 50.0 | 4.71 | 4.71 | 50.0 | 3.14 | 3.14 |
| | MLE | 55.4 | 26.8 | 27.3 | 50.2 | 8.30 | 8.31 | 49.7 | 4.76 | 4.77 | 49.6 | 3.15 | 3.18 |
| | P(inf mle) | 0.0010 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 10 | X1 | 20.5 | 3.48 | 29.7 | 32.6 | 3.37 | 17.8 | 39.7 | 2.86 | 10.7 | 43.9 | 2.31 | 6.50 |
| | P | 49.9 | 23.9 | 23.9 | 50.0 | 13.5 | 13.5 | 50.0 | 9.07 | 9.07 | 50.0 | 6.58 | 6.58 |
| | CUE | 50.0 | 16.0 | 16.0 | 50.0 | 7.48 | 7.48 | 50.0 | 4.55 | 4.55 | 50.0 | 3.08 | 3.08 |
| | MLE | 53.4 | 20.9 | 21.1 | 50.1 | 7.78 | 7.78 | 49.7 | 4.61 | 4.62 | 49.6 | 3.10 | 3.14 |
| | P(inf mle) | 0.0001 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 25 | X1 | 20.5 | 3.48 | 29.7 | 32.6 | 3.37 | 17.8 | 39.7 | 2.86 | 10.7 | 43.9 | 2.31 | 6.50 |
| | P | 50.0 | 15.6 | 15.6 | 50.0 | 9.15 | 9.15 | 50.0 | 6.30 | 6.30 | 50.1 | 4.54 | 4.54 |
| | CUE | 50.0 | 12.9 | 12.9 | 50.0 | 6.75 | 6.75 | 50.0 | 4.31 | 4.31 | 50.0 | 2.97 | 2.97 |
| | MLE | 51.3 | 14.1 | 14.2 | 49.9 | 6.86 | 6.86 | 49.6 | 4.33 | 4.35 | 49.5 | 2.97 | 3.01 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 50 | X1 | 20.5 | 3.48 | 29.7 | 32.6 | 3.37 | 17.8 | 39.7 | 2.86 | 10.7 | 43.9 | 2.31 | 6.50 |
| | P | 50.0 | 12.3 | 12.3 | 50.0 | 7.39 | 7.39 | 50.0 | 5.13 | 5.13 | 50.1 | 3.68 | 3.68 |
| | CUE | 50.0 | 11.2 | 11.2 | 50.0 | 6.20 | 6.20 | 50.0 | 4.09 | 4.09 | 50.0 | 2.88 | 2.88 |
| | MLE | 50.5 | 11.6 | 11.6 | 49.7 | 6.25 | 6.26 | 49.6 | 4.10 | 4.12 | 49.5 | 2.89 | 2.92 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |

Table 1.3c

| N = 50 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.20 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 33.6 | 3.32 | 16.7 | 44.6 | 2.19 | 5.80 | 48.2 | 1.30 | 2.19 | 49.4 | 0.75 | 0.95 |
| | P | 33.6 | 3.32 | 16.7 | 44.6 | 2.19 | 5.80 | 48.2 | 1.30 | 2.19 | 49.4 | 0.75 | 0.95 |
| | CUE | 49.9 | 8.30 | 8.30 | 50.0 | 2.98 | 2.98 | 50.0 | 1.50 | 1.50 | 50.0 | 0.92 | 0.92 |
| | MLE | 50.5 | 9.05 | 9.07 | 49.6 | 3.00 | 3.03 | 49.5 | 1.48 | 1.55 | 49.5 | 0.78 | 0.95 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 5 | X1 | 33.6 | 3.32 | 16.7 | 44.6 | 2.19 | 5.80 | 48.2 | 1.30 | 2.19 | 49.4 | 0.75 | 0.95 |
| | P | 50.0 | 18.2 | 18.2 | 50.0 | 8.43 | 8.43 | 50.0 | 4.56 | 4.56 | 50.0 | 2.56 | 2.56 |
| | CUE | 50.0 | 7.66 | 7.66 | 50.0 | 2.91 | 2.91 | 50.0 | 1.49 | 1.49 | 50.0 | 0.92 | 0.92 |
| | MLE | 50.2 | 8.07 | 8.07 | 49.6 | 2.92 | 2.95 | 49.5 | 1.46 | 1.54 | 49.5 | 0.78 | 0.95 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 10 | X1 | 33.6 | 3.32 | 16.7 | 44.6 | 2.19 | 5.80 | 48.2 | 1.30 | 2.19 | 49.4 | 0.75 | 0.95 |
| | P | 50.0 | 12.8 | 12.8 | 50.0 | 6.12 | 6.12 | 50.0 | 3.35 | 3.35 | 50.0 | 1.88 | 1.88 |
| | CUE | 49.9 | 7.20 | 7.20 | 50.0 | 2.86 | 2.86 | 50.0 | 1.49 | 1.49 | 50.1 | 0.91 | 0.91 |
| | MLE | 50.0 | 7.50 | 7.50 | 49.5 | 2.88 | 2.91 | 49.5 | 1.46 | 1.54 | 49.4 | 0.77 | 0.95 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 25 | X1 | 33.6 | 3.32 | 16.7 | 44.6 | 2.19 | 5.80 | 48.2 | 1.30 | 2.19 | 49.4 | 0.75 | 0.95 |
| | P | 50.0 | 8.71 | 8.71 | 50.1 | 4.21 | 4.21 | 50.0 | 2.31 | 2.31 | 50.0 | 1.32 | 1.32 |
| | CUE | 50.0 | 6.46 | 6.46 | 50.0 | 2.76 | 2.76 | 50.0 | 1.47 | 1.47 | 50.1 | 0.91 | 0.92 |
| | MLE | 49.8 | 6.57 | 6.57 | 49.5 | 2.78 | 2.82 | 49.5 | 1.43 | 1.52 | 49.4 | 0.77 | 0.95 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |
| | | | | | | | | | | | | | |
| 50 | X1 | 33.6 | 3.32 | 16.7 | 44.6 | 2.19 | 5.80 | 48.2 | 1.30 | 2.19 | 49.4 | 0.75 | 0.95 |
| | P | 50.0 | 7.05 | 7.05 | 50.1 | 3.40 | 3.40 | 50.0 | 1.86 | 1.86 | 50.0 | 1.07 | 1.07 |
| | CUE | 50.0 | 5.96 | 5.96 | 50.0 | 2.68 | 2.68 | 50.0 | 1.44 | 1.44 | 50.1 | 0.91 | 0.92 |
| | MLE | 49.7 | 6.02 | 6.03 | 49.5 | 2.68 | 2.72 | 49.5 | 1.42 | 1.51 | 49.4 | 0.76 | 0.95 |
| | P(inf mle) | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | | |

Table 1.4a

| N = 100 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.05 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 22.8 | 4.15 | 77.3 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | P | 22.8 | 4.15 | 77.3 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | CUE | 80.6 | 33.2 | 38.5 | 99.9 | 29.1 | 29.1 | 100 | 17.2 | 17.2 | 100 | 11.9 | 11.9 |
| | MLE | 115 | 66.4 | 68.0 | 108 | 39.2 | 39.9 | 102 | 18.5 | 18.6 | 101 | 12.3 | 12.3 |
| | P(inf mle) | | 0.1018 | | | 0.0001 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 5 | X1 | 22.7 | 4.18 | 77.4 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | P | 77.2 | 40.1 | 46.1 | 95.4 | 50.2 | 50.5 | 99.1 | 45.7 | 45.7 | 99.9 | 38.1 | 38.1 |
| | CUE | 92.0 | 39.8 | 40.6 | 100 | 26.3 | 26.3 | 100 | 16.2 | 16.2 | 100 | 11.5 | 11.5 |
| | MLE | 123 | 77.9 | 81.2 | 105 | 31.9 | 32.4 | 101 | 17.2 | 17.2 | 100 | 11.8 | 11.8 |
| | P(inf mle) | | 0.0282 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 10 | X1 | 22.6 | 4.18 | 77.5 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | P | 93.1 | 54.5 | 54.9 | 99.7 | 46.8 | 46.8 | 100 | 34.5 | 34.5 | 100 | 26.6 | 26.6 |
| | CUE | 97.1 | 42.6 | 42.7 | 100 | 24.3 | 24.3 | 100 | 15.5 | 15.5 | 100 | 11.1 | 11.1 |
| | MLE | 122 | 77.6 | 80.6 | 104 | 27.7 | 28.0 | 101 | 16.2 | 16.2 | 100 | 11.4 | 11.4 |
| | P(inf mle) | | 0.0078 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 25 | X1 | 22.6 | 4.18 | 77.5 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | P | 99.8 | 48.0 | 48.0 | 100 | 29.2 | 29.2 | 100 | 21.5 | 21.5 | 100 | 17.1 | 17.1 |
| | CUE | 99.9 | 38.5 | 38.5 | 100 | 20.7 | 20.7 | 100 | 14.0 | 14.0 | 100 | 10.4 | 10.4 |
| | MLE | 111 | 54.2 | 55.4 | 102 | 22.2 | 22.3 | 101 | 14.4 | 14.4 | 100 | 10.5 | 10.5 |
| | P(inf mle) | | 0.0002 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 50 | X1 | 22.6 | 4.18 | 77.5 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | P | 100 | 34.4 | 34.4 | 100 | 21.7 | 21.7 | 100 | 16.3 | 16.3 | 100 | 13.1 | 13.1 |
| | CUE | 100 | 30.9 | 30.9 | 100 | 18.0 | 18.0 | 100 | 12.7 | 12.7 | 100 | 9.65 | 9.65 |
| | MLE | 105 | 35.1 | 35.5 | 101 | 18.7 | 18.7 | 100 | 12.9 | 12.9 | 99.9 | 9.73 | 9.73 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 100 | X1 | 22.6 | 4.18 | 77.5 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | P | 100 | 26.9 | 26.9 | 100 | 17.5 | 17.5 | 100 | 13.2 | 13.2 | 100 | 10.6 | 10.6 |
| | CUE | | | | | | | | | | | | |
| | MLE | 102 | 26.9 | 27.0 | 100 | 16.1 | 16.1 | 100 | 11.6 | 11.6 | 99.7 | 8.98 | 8.99 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |

Table 1.4b

| N = 100 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.10 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| 0 | X1 | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | P | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | CUE | 99.9 | 29.8 | 29.8 | 100 | 11.7 | 11.7 | 100 | 6.83 | 6.83 | 100 | 4.50 | 4.50 |
| | MLE | 108 | 41.4 | 42.2 | 100 | 12.2 | 12.2 | 99.8 | 6.89 | 6.89 | 99.6 | 4.51 | 4.53 |
| | P(inf mle) | | 0.0002 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 5 | X1 | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | P | 95.8 | 50.2 | 50.4 | 99.9 | 37.3 | 37.3 | 100 | 25.1 | 25.1 | 100 | 17.7 | 17.7 |
| | CUE | 100 | 26.7 | 26.7 | 100 | 11.3 | 11.3 | 100 | 6.69 | 6.69 | 100 | 4.46 | 4.46 |
| | MLE | 106 | 33.0 | 33.5 | 100 | 11.6 | 11.6 | 99.7 | 6.75 | 6.75 | 99.6 | 4.47 | 4.49 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 10 | X1 | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | P | 99.8 | 46.0 | 46.0 | 100 | 26.0 | 26.0 | 100 | 17.4 | 17.4 | 100 | 12.5 | 12.5 |
| | CUE | 100 | 24.5 | 24.5 | 100 | 11.0 | 11.0 | 100 | 6.58 | 6.58 | 100 | 4.41 | 4.41 |
| | MLE | 104 | 28.3 | 28.6 | 100 | 11.2 | 11.2 | 99.7 | 6.63 | 6.64 | 99.6 | 4.43 | 4.45 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 25 | X1 | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | P | 100 | 28.6 | 28.6 | 100 | 16.7 | 16.7 | 100 | 11.5 | 11.5 | 100 | 8.39 | 8.39 |
| | CUE | 100 | 20.7 | 20.7 | 100 | 10.2 | 10.2 | 100 | 6.33 | 6.33 | 100 | 4.31 | 4.31 |
| | MLE | 102 | 22.2 | 22.3 | 100 | 10.4 | 10.4 | 99.7 | 6.36 | 6.37 | 99.6 | 4.32 | 4.34 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 50 | X1 | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | P | 100 | 21.3 | 21.3 | 100 | 12.8 | 12.8 | 100 | 8.86 | 8.86 | 100 | 6.47 | 6.47 |
| | CUE | 100 | 17.9 | 17.9 | 100 | 9.48 | 9.48 | 100 | 6.07 | 6.07 | 100 | 4.19 | 4.19 |
| | MLE | 101 | 18.6 | 18.6 | 99.9 | 9.56 | 9.56 | 99.6 | 6.08 | 6.09 | 99.5 | 4.19 | 4.22 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| | | | | | | | | | | | | | |
| 100 | X1 | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | P | 100 | 17.2 | 17.2 | 100 | 10.4 | 10.4 | 100 | 7.22 | 7.22 | 100 | 5.28 | 5.28 |
| | CUE | | | | | | | | | | | | |
| | MLE | 100 | 15.9 | 15.9 | 99.7 | 8.81 | 8.81 | 99.6 | 5.78 | 5.79 | 99.5 | 4.06 | 4.09 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |

Table 1.4c

| N = 100 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.20 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| 0 | X1 | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | P | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | CUE | 100 | 11.4 | 11.4 | 100 | 4.17 | 4.17 | 100 | 2.08 | 2.08 | 100 | 1.12 | 1.12 |
| | MLE | 100 | 11.8 | 11.8 | 99.6 | 4.19 | 4.21 | 99.5 | 2.08 | 2.14 | 99.5 | 1.19 | 1.27 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| 5 | X1 | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | P | 100 | 35.6 | 35.6 | 100 | 16.5 | 16.5 | 100 | 8.85 | 8.85 | 100 | 4.99 | 4.99 |
| | CUE | 100 | 11.0 | 11.0 | 100 | 4.14 | 4.14 | 100 | 2.07 | 2.07 | 99.9 | 1.12 | 1.12 |
| | MLE | 100 | 11.3 | 11.3 | 99.6 | 4.14 | 4.16 | 99.5 | 2.07 | 2.13 | 99.5 | 1.18 | 1.27 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| 10 | X1 | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | P | 100 | 24.6 | 24.6 | 100 | 11.7 | 11.7 | 100 | 6.43 | 6.43 | 100 | 3.61 | 3.61 |
| | CUE | 100 | 10.6 | 10.6 | 100 | 4.09 | 4.09 | 100 | 2.06 | 2.06 | 99.9 | 1.11 | 1.12 |
| | MLE | 100 | 10.9 | 10.9 | 99.6 | 4.10 | 4.12 | 99.5 | 2.06 | 2.12 | 99.6 | 1.18 | 1.26 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| 25 | X1 | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | P | 100 | 15.9 | 15.9 | 100 | 7.80 | 7.80 | 100 | 4.27 | 4.27 | 100 | 2.40 | 2.40 |
| | CUE | 99.9 | 9.84 | 9.84 | 100 | 4.00 | 4.00 | 100 | 2.05 | 2.05 | 99.9 | 1.11 | 1.11 |
| | MLE | 99.9 | 9.98 | 9.98 | 99.5 | 4.02 | 4.04 | 99.5 | 2.05 | 2.11 | 99.6 | 1.18 | 1.25 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| 50 | X1 | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | P | 100 | 12.2 | 12.2 | 100 | 6.01 | 6.01 | 100 | 3.25 | 3.25 | 100 | 1.85 | 1.85 |
| | CUE | 100 | 9.11 | 9.11 | 100 | 3.89 | 3.89 | 100 | 2.02 | 2.02 | 99.9 | 1.10 | 1.11 |
| | MLE | 99.8 | 9.18 | 9.19 | 99.5 | 3.91 | 3.93 | 99.5 | 2.03 | 2.09 | 99.6 | 1.17 | 1.24 |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |
| 100 | X1 | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | P | 100 | 9.92 | 9.92 | 100 | 4.90 | 4.90 | 100 | 2.62 | 2.62 | 100 | 1.51 | 1.51 |
| | CUE | | | | | | | | | | | | |
| | MLE | 99.7 | 8.44 | 8.44 | 99.5 | 3.78 | 3.81 | 99.5 | 2.00 | 2.06 | | | |
| | P(inf mle) | | 0.0000 | | | 0.0000 | | | 0.0000 | | | 0.0000 | |

## § 1.10 : The Unconditional Performance of the CUE and the Peterson-Type Estimators

In the previous section, in order to compare the relative performance of all three estimators considered within this chapter, it was necessary to consider the distribution of each conditional on the event $C=\{Z>X_1\}$. This was necessary because the MLE is known to yield infinite estimates when $Z=X_1$, and this would result in the MLE having an infinite mean, standard deviation and mean square error over the entire joint distribution of $X_1$ and Z. The discussion of the previous section recommended that, conditional on the event $C=\{Z>X_1\}$, one should favour the CUE $\hat{N}_u$. In view of this, and due to the fact that the CUE and Peterson-type estimators are both finite with probability one, the distributional properties of $\hat{N}_u$ and $\hat{N}_p$ are now presented unconditionally over the entire joint distribution of $X_1$ and Z. The results are contained in tables 1.5a,b,c, 1.6a,b,c, 1.7a,b,c and 1.8a,b,c. The values of N, t, p and R which are considered are identical to those of section 1.9. Notation is the same as in previous section.

## § 1.10a : Discussion

The performance, and relative performance, of the estimators over the complete sample space is seen to be very similar to their performance in the previous section. As one would expect, the difference is most noticeable when the probability of $X_1$ being equal to Z is large.

Tables 1.5 to 1.8 clearly indicate that the overall performance of the CUE is superior to that of $\hat{N}_p$. In the absence of plants $\hat{N}_p$ reduces to $X_1$, so it is not surprising that $\hat{N}_u$ is seen to clearly outperform $\hat{N}_p$ in this situation. When R>0, both perform very well in terms of bias, with $\hat{N}_u$ almost always being the less biased of the two. The standard deviation of $\hat{N}_u$ is, in all but four of the situations considered, less than that of $\hat{N}_p$: the difference, when not in favour of $\hat{N}_u$, is small. The standard deviation of $\hat{N}_p$ tends to be large when R, greater than zero, is small relative to N, and it is in these situations that the standard deviation of $\hat{N}_u$ is appreciably smaller than that of $\hat{N}_p$. In each of the four situations where, for R>0, the standard deviation of $\hat{N}_u$ is greater than that of $\hat{N}_p$, the CUE exhibits a smaller bias. In conclusion, one should always use the CUE in preference to the Peterson-type estimator $\hat{N}_p$.

Table 1.5a

| N = 10 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.05 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 2.26 | 1.32 | 7.85 | 4.01 | 1.55 | 6.18 | 5.37 | 1.58 | 4.89 | 6.42 | 1.52 | 3.89 |
| | | | | | | | | | | | | | |
| 0 | P | 2.26 | 1.32 | 7.85 | 4.01 | 1.55 | 6.18 | 5.37 | 1.58 | 4.89 | 6.42 | 1.52 | 3.89 |
| | CUE | 3.70 | 3.13 | 7.04 | 7.31 | 4.60 | 5.33 | 9.10 | 4.62 | 4.71 | 9.80 | 4.02 | 4.03 |
| | | | | | | | | | | | | | |
| 5 | P | 7.87 | 6.29 | 6.64 | 9.60 | 6.35 | 6.36 | 9.98 | 5.46 | 5.46 | 10.0 | 4.51 | 4.51 |
| | CUE | 8.24 | 6.79 | 7.02 | 9.78 | 6.14 | 6.15 | 9.95 | 4.49 | 4.49 | 10.0 | 3.31 | 3.31 |
| | | | | | | | | | | | | | |
| 10 | P | 9.50 | 8.16 | 8.18 | 10.0 | 6.16 | 6.16 | 10.0 | 4.54 | 4.54 | 10.0 | 3.56 | 3.56 |
| | CUE | 9.50 | 8.11 | 8.12 | 9.95 | 5.59 | 5.59 | 9.97 | 3.87 | 3.87 | 9.99 | 2.93 | 2.93 |

Table 1.5b

| N = 10 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.10 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 4.10 | 1.56 | 6.11 | 6.51 | 1.51 | 3.80 | 7.94 | 1.28 | 2.42 | 8.78 | 1.03 | 1.60 |
| | | | | | | | | | | | | | |
| 0 | P | 4.10 | 1.56 | 6.11 | 6.51 | 1.51 | 3.80 | 7.94 | 1.28 | 2.42 | 8.78 | 1.03 | 1.60 |
| | CUE | 7.27 | 4.41 | 5.18 | 9.71 | 3.94 | 3.95 | 10.0 | 2.44 | 2.44 | 10.0 | 1.58 | 1.58 |
| | | | | | | | | | | | | | |
| 5 | P | 9.64 | 6.31 | 6.32 | 10.1 | 4.42 | 4.42 | 10.1 | 3.02 | 3.02 | 10.1 | 2.16 | 2.16 |
| | CUE | 9.81 | 6.00 | 6.01 | 10.0 | 3.25 | 3.25 | 9.99 | 2.02 | 2.02 | 10.0 | 1.34 | 1.34 |
| | | | | | | | | | | | | | |
| 10 | P | 10.0 | 6.05 | 6.05 | 10.1 | 3.47 | 3.48 | 10.1 | 2.30 | 2.30 | 10.1 | 1.63 | 1.63 |
| | CUE | 9.97 | 5.45 | 5.45 | 10.0 | 2.88 | 2.88 | 10.0 | 1.88 | 1.88 | 10.0 | 1.29 | 1.29 |

Table 1.5c

| N = 10 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.20 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 6.72 | 1.48 | 3.60 | 8.93 | 0.98 | 1.45 | 9.65 | 0.58 | 0.68 | 9.88 | 0.34 | 0.36 |
| | | | | | | | | | | | | | |
| 0 | P | 6.72 | 1.48 | 3.60 | 8.93 | 0.98 | 1.45 | 9.65 | 0.58 | 0.68 | 9.88 | 0.34 | 0.36 |
| | CUE | 9.91 | 3.79 | 3.79 | 9.99 | 1.45 | 1.45 | 9.86 | 0.75 | 0.76 | 9.90 | 0.35 | 0.37 |
| | | | | | | | | | | | | | |
| 5 | P | 10.1 | 4.21 | 4.22 | 10.1 | 2.00 | 2.00 | 10.0 | 1.07 | 1.07 | 10.0 | 0.60 | 0.60 |
| | CUE | 10.1 | 3.10 | 3.10 | 10.0 | 1.29 | 1.29 | 9.85 | 0.73 | 0.75 | 9.89 | 0.34 | 0.36 |
| | | | | | | | | | | | | | |
| 10 | P | 10.1 | 3.30 | 3.30 | 10.0 | 1.52 | 1.52 | 10.0 | 0.84 | 0.84 | 10.0 | 0.48 | 0.48 |
| | CUE | 10.0 | 2.74 | 2.74 | 10.0 | 1.21 | 1.21 | 9.80 | 0.70 | 0.73 | 9.89 | 0.34 | 0.36 |

Table 1.6a

| N = 25 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.05 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 5.66 | 2.09 | 19.5 | 10.0 | 2.45 | 15.2 | 13.4 | 2.49 | 11.8 | 16.0 | 2.40 | 9.28 |
| | | | | | | | | | | | | | |
| 0 | P | 5.66 | 2.09 | 19.5 | 10.0 | 2.45 | 15.2 | 13.4 | 2.49 | 11.8 | 16.0 | 2.40 | 9.28 |
| | CUE | 13.8 | 8.98 | 14.3 | 23.3 | 12.3 | 12.4 | 24.9 | 9.56 | 9.56 | 25.0 | 6.56 | 6.56 |
| | | | | | | | | | | | | | |
| 5 | P | 19.7 | 12.3 | 13.4 | 23.9 | 13.8 | 13.8 | 24.8 | 12.2 | 12.2 | 25.0 | 10.1 | 10.1 |
| | CUE | 21.8 | 14.6 | 15.0 | 24.9 | 12.0 | 12.0 | 25.0 | 7.82 | 7.82 | 25.0 | 5.53 | 5.53 |
| | | | | | | | | | | | | | |
| 10 | P | 23.6 | 16.4 | 16.4 | 25.0 | 13.1 | 13.1 | 25.0 | 9.62 | 9.62 | 25.0 | 7.44 | 7.44 |
| | CUE | 24.1 | 16.4 | 16.4 | 25.0 | 10.3 | 10.3 | 25.0 | 6.86 | 6.86 | 25.0 | 5.09 | 5.09 |
| | | | | | | | | | | | | | |
| 25 | P | 25.0 | 14.9 | 14.9 | 25.0 | 9.13 | 9.13 | 25.0 | 6.77 | 6.77 | 25.0 | 5.39 | 5.39 |
| | CUE | 25.0 | 14.1 | 14.1 | 25.0 | 8.19 | 8.19 | 25.0 | 5.85 | 5.85 | 25.0 | 4.51 | 4.51 |

Table 1.6b

| N = 25 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.10 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 10.2 | 2.46 | 15.0 | 16.3 | 2.38 | 9.04 | 19.9 | 2.02 | 5.53 | 22.0 | 1.63 | 3.45 |
| | | | | | | | | | | | | | |
| 0 | P | 10.2 | 2.46 | 15.0 | 16.3 | 2.38 | 9.04 | 19.9 | 2.02 | 5.53 | 22.0 | 1.63 | 3.45 |
| | CUE | 23.1 | 12.0 | 12.1 | 25.1 | 6.43 | 6.43 | 25.0 | 3.56 | 3.56 | 25.0 | 2.29 | 2.29 |
| | | | | | | | | | | | | | |
| 5 | P | 24.0 | 13.7 | 13.8 | 25.0 | 9.92 | 9.92 | 25.0 | 6.70 | 6.70 | 25.0 | 4.78 | 4.78 |
| | CUE | 24.9 | 12.0 | 12.0 | 25.0 | 5.41 | 5.41 | 25.0 | 3.26 | 3.26 | 25.0 | 2.21 | 2.21 |
| | | | | | | | | | | | | | |
| 10 | P | 25.0 | 12.8 | 12.8 | 25.0 | 7.26 | 7.26 | 25.0 | 4.90 | 4.90 | 24.9 | 3.54 | 3.54 |
| | CUE | 25.0 | 10.3 | 10.3 | 25.0 | 4.96 | 4.96 | 25.0 | 3.14 | 3.14 | 25.0 | 2.14 | 2.14 |
| | | | | | | | | | | | | | |
| 25 | P | 25.0 | 8.95 | 8.95 | 25.0 | 5.27 | 5.27 | 25.0 | 3.63 | 3.63 | 25.1 | 2.57 | 2.57 |
| | CUE | 25.0 | 8.06 | 8.06 | 25.0 | 4.41 | 4.41 | 25.0 | 2.91 | 2.91 | 25.0 | 2.05 | 2.05 |

Table 1.6c

| N = 25 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.20 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 16.8 | 2.35 | 8.52 | 22.3 | 1.55 | 3.10 | 24.1 | 0.92 | 1.27 | 24.7 | 0.53 | 0.61 |
| | | | | | | | | | | | | | |
| 0 | P | 16.8 | 2.35 | 8.52 | 22.3 | 1.55 | 3.10 | 24.1 | 0.92 | 1.27 | 24.7 | 0.53 | 0.61 |
| | CUE | 25.0 | 6.30 | 6.30 | 25.0 | 2.14 | 2.14 | 25.1 | 1.07 | 1.07 | 24.8 | 0.61 | 0.64 |
| | | | | | | | | | | | | | |
| 5 | P | 25.1 | 9.45 | 9.45 | 25.0 | 4.44 | 4.44 | 25.0 | 2.40 | 2.40 | 25.0 | 1.34 | 1.34 |
| | CUE | 25.1 | 5.26 | 5.26 | 25.0 | 2.05 | 2.05 | 25.1 | 1.03 | 1.04 | 24.8 | 0.60 | 0.64 |
| | | | | | | | | | | | | | |
| 10 | P | 25.0 | 6.88 | 6.88 | 24.9 | 3.29 | 3.29 | 25.0 | 1.86 | 1.86 | 25.0 | 1.11 | 1.11 |
| | CUE | 25.0 | 4.81 | 4.81 | 25.0 | 1.99 | 1.99 | 25.1 | 1.02 | 1.02 | 24.8 | 0.58 | 0.63 |
| | | | | | | | | | | | | | |
| 25 | P | 25.0 | 5.02 | 5.02 | 25.1 | 2.38 | 2.38 | 25.0 | 1.32 | 1.32 | 25.0 | 0.76 | 0.76 |
| | CUE | 25.0 | 4.26 | 4.26 | 25.0 | 1.91 | 1.91 | 25.1 | 0.98 | 0.98 | 24.7 | 0.56 | 0.62 |

Table 1.7a

| N = 50 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.05 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 11.3 | 2.96 | 38.8 | 20.1 | 3.47 | 30.1 | 26.8 | 3.53 | 23.4 | 32.1 | 3.39 | 18.2 |
| | | | | | | | | | | | | | |
| 0 | P | 11.3 | 2.96 | 38.8 | 20.1 | 3.47 | 30.1 | 26.8 | 3.53 | 23.4 | 32.1 | 3.39 | 18.2 |
| | CUE | 37.4 | 21.5 | 24.9 | 50.0 | 21.8 | 21.8 | 49.9 | 12.8 | 12.8 | 50.0 | 8.65 | 8.65 |
| | | | | | | | | | | | | | |
| 5 | P | 39.3 | 21.8 | 24.3 | 47.8 | 25.9 | 26.0 | 49.6 | 23.4 | 23.4 | 50.0 | 19.5 | 19.5 |
| | CUE | 46.5 | 28.3 | 28.5 | 50.0 | 18.6 | 18.6 | 50.0 | 11.3 | 11.3 | 50.0 | 8.01 | 8.01 |
| | | | | | | | | | | | | | |
| 10 | P | 47.1 | 29.6 | 29.7 | 49.9 | 24.3 | 24.3 | 50.0 | 17.9 | 17.9 | 50.0 | 13.9 | 13.9 |
| | CUE | 49.0 | 29.2 | 29.2 | 50.0 | 16.1 | 16.1 | 50.0 | 10.4 | 10.4 | 50.0 | 7.58 | 7.58 |
| | | | | | | | | | | | | | |
| 25 | P | 50.0 | 26.2 | 26.2 | 50.0 | 15.9 | 15.9 | 50.0 | 11.8 | 11.8 | 50.0 | 9.36 | 9.36 |
| | CUE | 49.9 | 23.4 | 23.4 | 50.0 | 13.0 | 13.0 | 50.0 | 9.07 | 9.07 | 50.0 | 6.87 | 6.87 |
| | | | | | | | | | | | | | |
| 50 | P | 50.0 | 19.7 | 19.7 | 50.0 | 12.5 | 12.5 | 50.0 | 9.43 | 9.43 | 50.0 | 7.55 | 7.55 |
| | CUE | 50.0 | 18.6 | 18.6 | 50.0 | 11.3 | 11.3 | 50.0 | 8.18 | 8.18 | 50.0 | 6.33 | 6.33 |

Table 1.7b

| N = 50 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.10 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 20.5 | 3.48 | 29.7 | 32.6 | 3.37 | 17.8 | 39.7 | 2.86 | 10.7 | 43.9 | 2.31 | 6.50 |
| | | | | | | | | | | | | | |
| 0 | P | 20.5 | 3.48 | 29.7 | 32.6 | 3.37 | 17.8 | 39.7 | 2.86 | 10.7 | 43.9 | 2.31 | 6.50 |
| | CUE | 49.6 | 22.0 | 22.0 | 50.0 | 8.54 | 8.54 | 50.0 | 4.86 | 4.86 | 50.0 | 3.21 | 3.21 |
| | | | | | | | | | | | | | |
| 5 | P | 47.9 | 25.9 | 26.0 | 50.0 | 19.1 | 19.1 | 50.0 | 12.8 | 12.8 | 50.0 | 9.09 | 9.09 |
| | CUE | 50.0 | 18.8 | 18.8 | 50.0 | 7.89 | 7.89 | 50.0 | 4.71 | 4.71 | 50.0 | 3.14 | 3.14 |
| | | | | | | | | | | | | | |
| 10 | P | 49.9 | 23.9 | 23.9 | 50.0 | 13.5 | 13.5 | 50.0 | 9.07 | 9.07 | 50.0 | 6.58 | 6.58 |
| | CUE | 50.1 | 16.2 | 16.2 | 50.0 | 7.48 | 7.48 | 50.0 | 4.55 | 4.55 | 50.0 | 3.08 | 3.08 |
| | | | | | | | | | | | | | |
| 25 | P | 50.0 | 15.6 | 15.6 | 50.0 | 9.15 | 9.15 | 50.0 | 6.30 | 6.30 | 50.1 | 4.54 | 4.54 |
| | CUE | 50.0 | 12.9 | 12.9 | 50.0 | 6.75 | 6.75 | 50.0 | 4.31 | 4.31 | 50.0 | 2.97 | 2.97 |
| | | | | | | | | | | | | | |
| 50 | P | 50.0 | 12.3 | 12.3 | 50.0 | 7.39 | 7.39 | 50.0 | 5.13 | 5.13 | 50.1 | 3.68 | 3.68 |
| | CUE | 50.0 | 11.2 | 11.2 | 50.0 | 6.20 | 6.20 | 50.0 | 4.09 | 4.09 | 50.0 | 2.88 | 2.88 |

Table 1.7c

| N = 50 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.20 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 33.6 | 3.32 | 16.7 | 44.6 | 2.19 | 5.80 | 48.2 | 1.30 | 2.19 | 49.4 | 0.75 | 0.95 |
| | | | | | | | | | | | | | |
| 0 | P | 33.6 | 3.32 | 16.7 | 44.6 | 2.19 | 5.80 | 48.2 | 1.30 | 2.19 | 49.4 | 0.75 | 0.95 |
| | CUE | 49.9 | 8.30 | 8.30 | 50.0 | 2.98 | 2.98 | 50.0 | 1.50 | 1.50 | 50.0 | 0.92 | 0.92 |
| | | | | | | | | | | | | | |
| 5 | P | 50.0 | 18.2 | 18.2 | 50.0 | 8.43 | 8.43 | 50.0 | 4.56 | 4.56 | 50.0 | 2.56 | 2.56 |
| | CUE | 50.0 | 7.66 | 7.66 | 50.0 | 2.91 | 2.91 | 50.0 | 1.49 | 1.49 | 50.0 | 0.92 | 0.92 |
| | | | | | | | | | | | | | |
| 10 | P | 50.0 | 12.8 | 12.8 | 50.0 | 6.12 | 6.12 | 50.0 | 3.35 | 3.35 | 50.0 | 1.88 | 1.88 |
| | CUE | 49.9 | 7.20 | 7.20 | 50.0 | 2.86 | 2.86 | 50.0 | 1.49 | 1.49 | 50.1 | 0.91 | 0.91 |
| | | | | | | | | | | | | | |
| 25 | P | 50.0 | 8.71 | 8.71 | 50.1 | 4.21 | 4.21 | 50.0 | 2.31 | 2.31 | 50.0 | 1.32 | 1.32 |
| | CUE | 50.0 | 6.46 | 6.46 | 50.0 | 2.76 | 2.76 | 50.0 | 1.47 | 1.47 | 50.1 | 0.91 | 0.92 |
| | | | | | | | | | | | | | |
| 50 | P | 50.0 | 7.05 | 7.05 | 50.1 | 3.40 | 3.40 | 50.0 | 1.86 | 1.86 | 50.0 | 1.07 | 1.07 |
| | CUE | 50.0 | 5.96 | 5.96 | 50.0 | 2.68 | 2.68 | 50.0 | 1.44 | 1.44 | 50.1 | 0.91 | 0.92 |

Table 1.8a

| N = 100 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.05 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 22.6 | 4.18 | 77.5 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | | | | | | | | | | | | | |
| 0 | P | 22.6 | 4.18 | 77.5 | 40.1 | 4.90 | 60.1 | 53.7 | 4.99 | 46.6 | 64.2 | 4.80 | 36.2 |
| | CUE | 92.0 | 51.6 | 52.2 | 100 | 29.6 | 29.6 | 100 | 17.2 | 17.2 | 100 | 11.9 | 11.9 |
| | | | | | | | | | | | | | |
| 5 | P | 78.6 | 40.5 | 45.8 | 95.5 | 50.2 | 50.5 | 99.1 | 45.7 | 45.7 | 99.9 | 38.1 | 38.1 |
| | CUE | 97.8 | 54.1 | 54.2 | 100 | 26.4 | 26.4 | 100 | 16.2 | 16.2 | 100 | 11.5 | 11.5 |
| | | | | | | | | | | | | | |
| 10 | P | 94.1 | 55.7 | 56.0 | 99.7 | 46.8 | 46.8 | 100 | 34.5 | 34.5 | 100 | 26.6 | 26.6 |
| | CUE | 99.4 | 51.1 | 51.1 | 100 | 24.3 | 24.3 | 100 | 15.5 | 15.5 | 100 | 11.1 | 11.1 |
| | | | | | | | | | | | | | |
| 25 | P | 99.9 | 48.4 | 48.4 | 100 | 29.2 | 29.2 | 100 | 21.5 | 21.5 | 100 | 17.1 | 17.1 |
| | CUE | 100 | 39.4 | 39.4 | 100 | 20.7 | 20.7 | 100 | 14.0 | 14.0 | 100 | 10.4 | 10.4 |
| | | | | | | | | | | | | | |
| 50 | P | 100 | 34.4 | 34.4 | 100 | 21.7 | 21.7 | 100 | 16.3 | 16.3 | 100 | 13.1 | 13.1 |
| | CUE | 100 | 30.9 | 30.9 | 100 | 18.0 | 18.0 | 100 | 12.7 | 12.7 | 100 | 9.65 | 9.65 |
| | | | | | | | | | | | | | |
| 100 | P | 100 | 26.9 | 26.9 | 100 | 17.5 | 17.5 | 100 | 13.2 | 13.2 | 100 | 10.6 | 10.6 |
| | CUE | | | | | | | | | | | | |

Table 1.8b

| N = 100 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.10 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | | | | | | | | | | | | | |
| 0 | P | 41.0 | 4.92 | 59.3 | 65.1 | 4.77 | 35.2 | 79.4 | 4.04 | 21.0 | 87.8 | 3.27 | 12.6 |
| | CUE | 100 | 30.5 | 30.5 | 100 | 11.7 | 11.7 | 100 | 6.83 | 6.83 | 100 | 4.50 | 4.50 |
| | | | | | | | | | | | | | |
| 5 | P | 95.8 | 50.2 | 50.4 | 99.9 | 37.3 | 37.3 | 100 | 25.1 | 25.1 | 100 | 17.7 | 17.7 |
| | CUE | 100 | 26.9 | 26.9 | 100 | 11.3 | 11.3 | 100 | 6.69 | 6.69 | 100 | 4.46 | 4.46 |
| | | | | | | | | | | | | | |
| 10 | P | 99.8 | 46.0 | 46.0 | 100 | 26.0 | 26.0 | 100 | 17.4 | 17.4 | 100 | 12.5 | 12.5 |
| | CUE | 100 | 24.5 | 24.5 | 100 | 11.0 | 11.0 | 100 | 6.58 | 6.58 | 100 | 4.41 | 4.41 |
| | | | | | | | | | | | | | |
| 25 | P | 100 | 28.6 | 28.6 | 100 | 16.7 | 16.7 | 100 | 11.5 | 11.5 | 100 | 8.39 | 8.39 |
| | CUE | 100 | 20.7 | 20.7 | 100 | 10.2 | 10.2 | 100 | 6.33 | 6.33 | 100 | 4.31 | 4.31 |
| | | | | | | | | | | | | | |
| 50 | P | 100 | 21.3 | 21.3 | 100 | 12.8 | 12.8 | 100 | 8.86 | 8.86 | 100 | 6.47 | 6.47 |
| | CUE | 100 | 17.9 | 17.9 | 100 | 9.48 | 9.48 | 100 | 6.07 | 6.07 | 100 | 4.19 | 4.19 |
| | | | | | | | | | | | | | |
| 100 | P | 100 | 17.2 | 17.2 | 100 | 10.4 | 10.4 | 100 | 7.22 | 7.22 | 100 | 5.28 | 5.28 |
| | CUE | | | | | | | | | | | | |

Table 1.8c

| N = 100 | | t | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| p = 0.20 | | 5 | | | 10 | | | 15 | | | 20 | | |
| R | Estimator | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse | exp. | s.d. | rmse |
| | | | | | | | | | | | | | |
| >=0 | X1 | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | | | | | | | | | | | | | |
| 0 | P | 67.2 | 4.69 | 33.1 | 89.3 | 3.10 | 11.2 | 96.5 | 1.84 | 3.97 | 98.8 | 1.07 | 1.57 |
| | CUE | 100 | 11.4 | 11.4 | 100 | 4.17 | 4.17 | 100 | 2.08 | 2.08 | 100 | 1.12 | 1.12 |
| | | | | | | | | | | | | | |
| 5 | P | 100 | 35.6 | 35.6 | 100 | 16.5 | 16.5 | 100 | 8.85 | 8.85 | 100 | 4.99 | 4.99 |
| | CUE | 100 | 11.0 | 11.0 | 100 | 4.14 | 4.14 | 100 | 2.07 | 2.07 | 99.9 | 1.12 | 1.12 |
| | | | | | | | | | | | | | |
| 10 | P | 100 | 24.6 | 24.6 | 100 | 11.7 | 11.7 | 100 | 6.43 | 6.43 | 100 | 3.61 | 3.61 |
| | CUE | 100 | 10.6 | 10.6 | 100 | 4.09 | 4.09 | 100 | 2.06 | 2.06 | 99.9 | 1.11 | 1.12 |
| | | | | | | | | | | | | | |
| 25 | P | 100 | 15.9 | 15.9 | 100 | 7.80 | 7.80 | 100 | 4.27 | 4.27 | 100 | 2.40 | 2.40 |
| | CUE | 99.9 | 9.84 | 9.84 | 100 | 4.00 | 4.00 | 100 | 2.05 | 2.05 | 99.9 | 1.11 | 1.11 |
| | | | | | | | | | | | | | |
| 50 | P | 100 | 12.2 | 12.2 | 100 | 6.01 | 6.01 | 100 | 3.25 | 3.25 | 100 | 1.85 | 1.85 |
| | CUE | 100 | 9.11 | 9.11 | 100 | 3.89 | 3.89 | 100 | 2.02 | 2.02 | 99.9 | 1.10 | 1.11 |
| | | | | | | | | | | | | | |
| 100 | P | 100 | 9.92 | 9.92 | 100 | 4.90 | 4.90 | 100 | 2.62 | 2.62 | 100 | 1.51 | 1.51 |
| | CUE | | | | | | | | | | | | |

## § 1.11 : The Performance of Plant-Capture When Applied to the Model $M_0$ : Under Discrete Time Sampling

In previous sections it has been argued that the overall performance of the CUE $\hat{N}_u$ should always be considered superior to that of the MLE and Peterson-type estimators. Rather than only discussing the way in which the information gained through plants may improve the performance of the CUE, this section considers how the method of plant-capture can affect the performance of all the estimators described within this chapter. This approach is taken since, in spite of the evidence of the previous sections, it is believed that more traditionally minded practitioners may still prefer to use the MLE. The following discussion is based on an inspection of all the 24 tables of this chapter.

It has previously been mentioned that mean square error is known to reward negative bias, and that this characteristic can lead to incorrect conclusions being drawn, that is if one places too much emphasis on mean square error alone. When comparing the performance of estimators, one should always, where possible, consider firstly their mean and standard deviation, and only then should one consider mean square error, or alternative loss functions such as mean absolute deviation. This approach is taken in the following discussion ; consideration of mean square error alone can lead to counter intuitive conclusions. For example, consider the performance of the CUE in table 1.6a, where $N = 25$, $p = 0.05$ and $t = 5$. As R is increased from 0 to 10 the mean square error of $\hat{N}_u$ increases from 14.3 to 16.4 ! However, only when one considers the way in which the bias of $\hat{N}_u$ is being significantly reduced can one see that the extra information gained from the plants is in fact improving the performance of the CUE.

This last example is quite typical of the way in which the information gained from plants enhances the performance of the estimators in situations where very little information is gained from the target population, however in many of these situations the improvement in bias is accompanied by a reduction in mean square error.

Except for situations where only a small amount of information is available, the CUE $\hat{N}_u$ is usually unbiased, and where not its bias is negligible. In those situations where only a small amount of information is available, $\hat{N}_u$ tends to be negatively biased, with this bias reducing significantly and uniformly as the number of plants is increased. This behaviour is intuitively very reasonable, since the CUE is unbiased conditional on the event $Z = Z_1 + Z_2 \geq N$. That is because $Z_2 \sim \text{Bin}(Rt, p)$, the event $Z = Z_1 + Z_2 \geq N$ is more and more likely to occur as R is increased. The standard deviation of $\hat{N}_u$ is generally seen to reduce uniformly as more and more plants are used. Where the standard deviation of the CUE is not reduced by an increase in R, this is always due to its bias being significantly improved.

In the absence of plants, the Peterson-type estimator reduces to $X_1$, commonly referred to as the 'enumeration estimator'. For this reason, $\hat{N}_p$ is only considered here when plants are used. In terms of bias, the Peterson-type estimator behaves in a very similar way to the CUE, although $\hat{N}_u$ is on almost all occasions less biased. When the number of plants is small relative to the size of the target population, $\hat{N}_p$ tends to have a relatively large variance, however this is on almost all occasions seen to reduce uniformly as R is increased.

The CUE is seen to utilise the information gained through the plants in a very 'smooth' way. That is, as R is increased, usually either the mean of $\hat{N}_u$ is significantly improved at the expense of a slight increase in standard deviation or both the bias and the standard deviation are reduced. The behaviour of the MLE when in situations where little information is available is less predictable as more plants are introduced. Consider for example table 1.3a, in which $N = 50$ and $p = 0.05$. When the number of sampling occasions is equal to 5, for $R = 0, 5, 10, 25$ and 50 the mean and standard deviation of $\hat{N}$ are respectively 43.3, 24.4; 57.9, 36.3; 61.5, 41.8; 56.8, 33.1 and 52.7, 21.2 : the corresponding values taken by $\hat{N}_u$ are respectively 30.1, 13.4; 40.8, 18.7; 46.3, 22.3; 49.8, 22.5 and 50.0, 18.6. The CUE in this example behaves in the manner described above, i.e. as R is increased its performance improves 'smoothly'. However, as the number of plants is increased from 0 to 5 to 10, both the mean and standard deviation of the MLE are seen to become worse ! This result appears counter intuitive, that is until one considers the way in which the value of R affects the probability of obtaining a finite MLE. In the above situation, where $t = 5$, for $R = 0, 5, 10, 25$ and 50, the probabilities of obtaining an infinite MLE are respectively 0.3190, 0.0885, 0.0245, 0.0005 and 0.0000. In other words the introduction of plants is seen to dramatically improve the probability of obtaining a useful MLE. When this advantage is considered along with the performance of the MLE, it can be argued that even in situations where, as in the above example, very little information is obtained from the target population, the presence of plants is beneficial to the overall performance of the MLE. Other than those extreme situations in which very little information is present, an increase in the number of plants is generally seen to improve the performance of the MLE via a reduction in both bias and standard deviation. And where both statistics are not improved, one of the two is.

In conclusion, the introduction of plants can be seen to enhance the performance of all three of the estimators which have been considered within this chapter, this being under the assumption that the planted individuals do indeed behave in an identical manner to members of the target population. In particular the plants are seen to be of most use when only little information has been gained from the target population. Furthermore, on the basis of the above discussions, whether sampling with or without plants, it is recommended that the CUE be considered superior to both the MLE and Peterson-type estimator.

## Chapter 2 : A Plant-Capture Approach for Sequential Tagging

### § 2.1 : Introduction

This chapter considers how the method of plant-capture may be used to aid the problem of estimating population size when the population in question behaves according to a continuous time analogue of the standard capture-recapture model known as $M_0$. The model $M_0$ is one of the sequence of models described by Otis et al. (1978) for capture-recapture data in closed populations. The sampling procedure considered within this chapter assumes that the population in question is under constant observation for some period of time, and that individuals are seen one at a time. Existing methods for estimating the value of N are based on either truncated sampling, in which sampling continues for a fixed predetermined amount of time , or censored sampling, in which sampling continues until a predetermined number of tagged individuals have been seen. Within this chapter consideration is given to the problem of estimation under the more commonly used method of truncated sampling. In the absence of plants this version of the problem has previously been studied by Nayak(1988), who derived a maximum likelihood estimator.

### § 2.2 : Sampling Procedure and Assumptions

Prior to the commencement of the experiment it is assumed that the target population, whose size N we wish to estimate, is augmented by the insertion of a known number R of planted individuals. Each planted individual is assumed to have received a unique tag prior to its release. Sightings of any particular member of the target population form a homogeneous Poisson process of rate $\lambda$. The augmented population, of size N+R, is randomly mixed. It is assumed that the planted individuals behave exactly as members of the target population, so that the augmented population constitutes N+R independent homogeneous Poisson processes each of rate $\lambda$. One member of the augmented population is randomly selected at a time : individuals that are seen for the first time receive a unique tag, so that they may be recognised on subsequent occasions. Individuals having been seen are then immediately released into the population. The augmented population is assumed to be closed and under continuous observation during the predetermined time period $[0, \tau]$.

## § 2.3 : A Note on Software Reliability

In the above, N has been referred to as being the size of ' a population '. More specifically, N may represent the size of a wildlife population, in which case the above sampling procedure constitutes a sequential Schnabel census with samples of size one, see Schnabel(1938) or, for a more comprehensive review, Seber(1982). The theory discussed here however is equally applicable to the problem of estimating the original number of faults , N, in a reliability system. Only the interpretation of the theory in both cases is a little different : probabilistically both situations are identical. In the situation where N represents the unknown number of errors in a piece of computer software the above model was originally proposed by Jelinski & Moranda (1972). Other models which aim to describe the stochastic failures of a piece of software have been proposed however the Jelinski & Moranda model is commonly regarded as being central to the topic of software reliability, see Langberg & Singpurwalla (1985). Originally attempts to estimate the value of N based upon the Jelinski & Moranda model assumed that only the times at which errors were first detected would be recorded. Nayak(1988) introduced a design called recapture debugging in which he developed a sampling procedure, which is analogous to the sequential Schnabel census, in an attempt to get extra information from the population prior to estimating the value of N. In Nayak's model the software is assumed to originally contain N errors. Whenever an error is detected it is corrected, without further errors being inserted, but a counter is added to record how often that area of the software is accessed during the remainder of sampling time. This ensures that recapture debugging uses the available sampling time more efficiently.

## § 2.4 : The Sufficient Statistics

Nayak(1988) determined the sufficient statistics for the situation in which no plants are present. This section utilizes the theory of Nayak(1988) in order to determine the sufficient statistics for situations in which the number of plants is greater than or equal to zero. The following notation is used :

$X_1$ = the number of distinct unplanted individuals seen in time $[0, \tau]$.

$X_2$ = the number of distinct planted individuals seen in time $[0, \tau]$.

$Z_1$ = the total number of sightings made from the target population in time $[0,\tau]$.

$Z_2$ = the total number of sightings made from the planted population in time $[0,\tau]$.

$Z$ = the total number of sightings made from the augmented population in time $[0,\tau]$.

Within this section it is more convenient to let

$N^{(1)} = N$ = the size of the target population

and $N^{(2)} = R$ = the size of the planted population.

Suppose that the individuals within each population are labelled as 1, 2, 3, .... according to the order in which they were seen.

Consider firstly the target population.

Let

$T^{(1)}_{(i)}$ = the time at which individual i is first seen, for $i = 1, 2, ..., X_1$. It follows that $0 \le T^{(1)}_{(1)} \le T^{(1)}_{(2)} \le .... \le T^{(1)}_{(X_1)} \le \tau$.

and let $M^{(1)}_i$ = the number of times individual i is recaptured after its initial capture, $i = 1, 2, ..., X_1$. In other words $M^{(1)}_i = Y^{(1)}_i - 1$, where $Y^{(1)}_i$ is the total number of times that individual i is seen during the time interval $[0,\tau]$.

Similarly for the planted population let

$T^{(2)}_{(i)}$ = the time at which individual i is first seen, for $i = 1, 2, ..., X_2$. It follows that $0 \le T^{(2)}_{(1)} \le T^{(2)}_{(2)} \le .... \le T^{(2)}_{(X_2)} \le \tau$.

and let $M^{(2)}_i$ = the number of times individual i is recaptured after its initial capture, $i = 1, 2, ..., X_2$. In other words $M^{(2)}_i = Y^{(2)}_i - 1$, where $Y^{(2)}_i$ is the total number of times that individual i is seen during the time interal $[0,\tau]$.

Now define the vectors

$$U_{(1)} = \left(X_1, T^{(1)}_{(1)}, T^{(1)}_{(2)}, ...., T^{(1)}_{(X_1)}\right)$$
$$V_{(1)} = \left(M^{(1)}_1, M^{(1)}_2, ...., M^{(1)}_{X_1}\right)$$

and

$$U_{(2)} = \left(X_2, T^{(2)}_{(1)}, T^{(2)}_{(2)}, ...., T^{(2)}_{(X_2)}\right)$$
$$V_{(2)} = \left(M^{(2)}_1, M^{(2)}_2, ...., M^{(2)}_{X_2}\right).$$

By the independence of the target and planted populations, it follows that

$$\begin{aligned}\operatorname{Prob}\left(U_{(1)}, V_{(1)}, U_{(2)}, V_{(2)}\right) &= \prod_{i=1}^{2} \operatorname{Prob}\left(U_{(i)} = u_{(i)}, V_{(i)} = v_{(i)}\right) \\ &= \prod_{i=1}^{2} \operatorname{Prob}\left(V_{(i)} = v_{(i)} \middle| U_{(i)} = u_{(i)}\right) \operatorname{Prob}\left(U_{(i)} = u_{(i)}\right). \qquad (2.1)\end{aligned}$$

Under the model, the following distributional results hold :

(i) Given $U_{(i)}$, $M^{(i)}_j \sim P\left(\lambda\left(\tau - t^{(i)}_{(j)}\right)\right)$, for $j = 1, 2, ..., X_i$; $i = 1, 2$, where $P(\lambda)$ denotes a Poisson distribution with mean $\lambda$.

This implies that

$$\mathrm{Prob}\left(V_{(i)}\middle|U_{(i)}\right)=\prod_{j=1}^{X_i}\mathrm{Prob}\left(M_j^{(i)}=m_j^{(i)}\right) \qquad \text{by independence}$$

$$=\prod_{j=1}^{X_i}\frac{\left[\lambda\left(\tau-t_{(j)}^{(i)}\right)\right]^{m_j^{(i)}}\exp\left[-\lambda\left(\tau-t_{(j)}^{(i)}\right)\right]}{m_j^{(i)}!}$$

$$=\frac{\lambda^{m^{(i)}}\exp\left[-\lambda X_i\tau+\lambda\sum_{j=1}^{X_i}t_{(j)}^{(i)}\right]\prod_{j=1}^{X_i}\left(\tau-t_{(j)}^{(i)}\right)^{m_j^{(i)}}}{\prod_{j=1}^{X_i}m_j^{(i)}!},$$

$$\text{where} \quad m^{(i)}=\sum_{j=1}^{X_i}m_j^{(i)}.$$

(ii) $\mathrm{Prob}\left(U_{(i)}\right)=\mathrm{Prob}\left(X_{(i)}=x_{(i)}\right)\mathrm{Prob}\left(T_{(1)}^{(i)},T_{(2)}^{(i)},\ldots,T_{(X_i)}^{(i)}\middle|X_{(i)}=x_{(i)}\right)$,

where $\quad X_{(i)}\sim \mathrm{Bin}\left(N^{(i)},1-\exp[-\lambda\tau]\right)$.

Given that an individual is seen by time $\tau$, its conditional time to detection has probability density function $\dfrac{\lambda\exp[-\lambda t]}{1-\exp[-\lambda\tau]}, \qquad 0\le t\le\tau.$

It then follows that the joint conditional probability distribution function of the order statistics of the $X_i$ seen by time $\tau$ is

$$X_i!\prod_{j=1}^{X_i}\frac{\lambda\exp\left[-\lambda t_{(j)}^{(i)}\right]}{1-\exp[-\lambda\tau]}=\frac{X_i!\lambda^{X_i}\exp\left[-\lambda\sum_{j=1}^{X_i}t_{(j)}^{(i)}\right]}{\left(1-\exp[-\lambda\tau]\right)^{X_i}}, \qquad \text{for } i=1,2.$$

Hence

$$\mathrm{Prob}\left(U_{(i)}=u_{(i)}\right)=\binom{N^{(i)}}{X_i}\left(1-\exp[-\lambda\tau]\right)^{X_i}\left(\exp[-\lambda\tau]\right)^{N^{(i)}-X_i}\frac{X_i!\lambda^{X_i}\exp\left[-\lambda\sum_{j=1}^{X_i}t_{(j)}^{(i)}\right]}{\left(1-\exp[-\lambda\tau]\right)^{X_i}}$$

$$=\binom{N^{(i)}}{X_i}\left(\exp[-\lambda\tau]\right)^{N^{(i)}-X_i}X_i!\lambda^{X_i}\exp\left[-\lambda\sum_{j=1}^{X_i}t_{(j)}^{(i)}\right].$$

Substituting these results into equation (2.1) yields the following :

$$\mathrm{Prob}\left(U_{(1)},V_{(1)},U_{(2)},V_{(2)}\right)=\prod_{i=1}^{2}\mathrm{Prob}\left(V_{(i)}=v_{(i)}\middle|U_{(i)}=u_{(i)}\right)\mathrm{Prob}\left(U_{(i)}=u_{(i)}\right)$$

$$=\prod_{j=1}^{2}\frac{\lambda^{m^{(i)}}\exp\left[-\lambda X_i\tau+\lambda\sum_{j=1}^{X_i}t_{(j)}^{(i)}\right]\prod_{j=1}^{X_i}\left(\tau-t_{(j)}^{(i)}\right)^{m_j^{(i)}}}{\prod_{j=1}^{X_i}m_j^{(i)}!}\binom{N^{(i)}}{X_i}\left(\exp[-\lambda\tau]\right)^{N^{(i)}-X_i}X_i!\lambda^{X_i}\exp\left[-\lambda\sum_{j=1}^{X_i}t_{(j)}^{(i)}\right]$$

$$= \prod_{i=1}^{2} \frac{N^{(i)}!\lambda^{X_i+m^{(i)}} \exp\left[-N^{(i)}\lambda\tau\right] \prod_{j=1}^{x_i} \left(\tau - t_{(j)}^{(i)}\right)^{m_j^{(i)}}}{\left(N^{(i)} - X_i\right)! \prod_{j=1}^{x_i} m_j^{(i)}!}$$

$$= \prod_{i=1}^{2} \frac{N^{(i)}!\lambda^{Z_i} \exp\left[-N^{(i)}\lambda\tau\right] \prod_{j=1}^{x_i} \left(\tau - t_{(j)}^{(i)}\right)^{m_j^{(i)}}}{\left(N^{(i)} - X_i\right)! \prod_{j=1}^{x_i} m_j^{(i)}!}.$$

It follows that the likelihood function for $\lambda$ and $N = N^{(1)}$ may be written as

$$L\left(\lambda, N^{(1)}\right) \propto \binom{N^{(1)}}{X_1} \lambda^{Z_1+Z_2} \exp\left[-\left(N^{(1)} + N^{(2)}\right)\lambda\tau\right].$$

Hence, by the Neyman-Pearson factorisation theorem, the sufficient statistics for $\lambda$ and $N = N^{(1)}$ are $X_1$ and $Z = Z_1 + Z_2$.

## § 2.5 : The Distribution Function of the Sufficient Statistics

The most direct way of deriving the joint probability function of $X_1$ and Z is to consider the decomposition

$$\text{Prob}\left(X_1 = x_1, Z = z\right) = \text{Prob}\left(X_1 = x_1 \middle| Z = z\right)\text{Prob}(Z = z). \tag{2.2}$$

Firstly, from the above assumptions it follows that Z has a Poisson distribution with parameter $(N + R)\lambda\tau$.

Explicitly $$\text{Prob}(Z = z) = \frac{\left[(N + R)\lambda\tau\right]^z \exp\left[-(N + R)\lambda\tau\right]}{z!}, \quad z = 0, 1, 2, \ldots \tag{2.3}$$

The conditional distribution of $X_1$ given Z has previously appeared in an urn model context, see Johnson and Kotz(1977) p.122. Suppose one thinks of the N+R members of the augmented population as N+R urns and that each time an individual is seen a ball is placed into the urn representing it. Initially let N of these urns be empty and R contain one ball. So that at any subsequent time the number of tagged individuals in the population is represented by the number of urns containing at least one ball. Now the probability of $X_1$ given Z is the probability that $X_1$ of the initially empty N urns contain at least one ball given that Z balls have been randomly allocated to the N+R urns, the balls being allocated to the urns in such a way that the probability of a ball being allocated to any one urn is $\frac{1}{N + R}$. The distribution of $X_1$ given Z is then exactly the variation of the classical occupancy situation discussed in Johnson and Kotz (1977) p.122, where a derivation of the probability function of $X_1$ given Z may be found.

Alternatively one may obtain the probability function of $X_1$ given Z more directly as follows.

$$\operatorname{Prob}(X_1 = x_1|Z = z) = \sum_{z_1=0}^{z} \operatorname{Prob}(X_1 = x_1|Z_1 = z_1, Z = z)\operatorname{Prob}(Z_1 = z_1|Z = z),$$

using the theorem of total conditional probability.

$$= \sum_{z_1=0}^{z} \operatorname{Prob}(X_1 = x_1|Z_1 = z_1)\operatorname{Prob}(Z_1 = z_1|Z = z). \qquad (2.3a)$$

From assumptions, it is known that $Z_1$ has a Poisson distribution with mean $N\lambda\tau$. Since Z has a Poisson distribution with mean $(N+R)\lambda\tau$, it is easy to show that the distribution of $Z_1|Z$ is Binomial : explicitly $Z_1|Z \sim \operatorname{Bin}\left(Z, \frac{N}{N+R}\right)$.

The conditional distribution of $X_1$ given $Z_1$ constitutes what is known in the literature as a Classical Occupancy distribution. For completeness, the Classical Occupancy distribution is described in Appendix 1, wherein its probability function is derived.

Explicitly

$$\operatorname{Prob}(X_1 = x_1|Z_1 = z_1) = N^{-z_1}\binom{N}{x_1}x_1!S(x_1, z_1),$$

$$x_1 = 0, 1, 2, ...., \min(N, z_1),$$

where $S(x_1, z_1) = \frac{1}{x_1!}\sum_{k=0}^{x_1}\binom{x_1}{k}(-1)^k(x_1-k)^{z_1}$ is a Stirling Number of the second kind.

The conditional distribution of $X_1$ given Z may now be obtained as follows :

$$\begin{aligned}
\operatorname{Prob}(X_1 = x_1|Z = z) &= \sum_{z_1=0}^{z} \operatorname{Prob}(X_1 = x_1|Z_1 = z_1)\operatorname{Prob}(Z_1 = z_1|Z = z) \quad \text{from (2.3a)} \\
&= \sum_{z_1=0}^{z} N^{-z_1}\binom{N}{x_1}x_1!S(x_1, z_1)\binom{z}{z_1}\left(\frac{N}{N+R}\right)^{z_1}\left(\frac{R}{N+R}\right)^{z-z_1} \\
&= \binom{N}{x_1}\frac{x_1!}{(N+R)^z}\sum_{z_1=0}^{z} S(x_1, z_1)\binom{z}{z_1}R^{z-z_1} \\
&= \binom{N}{x_1}\frac{x_1!}{(N+R)^z}\sum_{z_1=0}^{z}\left(\frac{1}{x_1!}\sum_{k=0}^{x_1}\binom{x_1}{k}(-1)^k(x_1-k)^{z_1}\right)\binom{z}{z_1}R^{z-z_1} \\
&= \binom{N}{x_1}\frac{1}{(N+R)^z}\sum_{k=0}^{x_1}\binom{x_1}{k}(-1)^k\sum_{z_1=0}^{z}\binom{z}{z_1}(x_1-k)^{z_1}R^{z-z_1} \\
&= \binom{N}{x_1}\frac{1}{(N+R)^z}\sum_{k=0}^{x_1}\binom{x_1}{k}(-1)^k(R+x_1-k)^z, \qquad (2.4)
\end{aligned}$$

$$x_1 = 0, 1, 2, ...., \min(N, z).$$

Substitution of (2.3) and (2.4) into equation (2.2) then yields the joint probability function for $X_1$ and Z :

$$\begin{aligned}
\mathrm{Prob}(X_1 = x_1, Z = z) &= \mathrm{Prob}(X_1 = x_1 | Z = z)\mathrm{Prob}(Z = z) \\
&= \binom{N}{x_1}\frac{1}{(N+R)^z}\sum_{k=0}^{x_1}\binom{x_1}{k}(-1)^k(R+x_1-k)^z\frac{[(N+R)\lambda\tau]^z\exp[-(N+R)\lambda\tau]}{z!} \\
&= \frac{(\lambda\tau)^z\exp[-(N+R)\lambda\tau]}{z!}\binom{N}{x_1}\sum_{k=0}^{x_1}\binom{x_1}{k}(-1)^k(R+x_1-k)^z \qquad (2.5)
\end{aligned}$$

$$x_1 = 0, 1, 2, ...., N,$$
$$z = x_1, x_1 + 1, x_1 + 2, ......$$

§ 2.6 : The Q - Numbers

The Q - numbers are defined as follows :

$$Q(x_1, z; R) = \sum_{k=0}^{x_1}\binom{x_1}{k}(-1)^k(R + x_1 - k)^z,$$

$$x_1 = 0, 1, 2, ....$$
$$z = x_1, x_1+1, x_1+2, ....$$

These Q - numbers are of importance since they appear within the joint probability distribution function of the sufficient statistics, as given in the previous section.

The Q - numbers are a generalisation of the Stirling numbers of the second kind. Explicitly the relationship is given by the equation

$$Q(x_1, z; R = 0) = x_1!S(x_1, z). \qquad (2.6)$$

In order to investigate the distributional properties of the estimators which are considered further on in this chapter it is necessary to evaluate the Q - numbers over some particular range of parameter values. This can lead to computational problems since the form of the Q - numbers is clearly not desirable from a computational point of view. That is the alternating sign within the summation means that very large numbers are repeatedly being added to and in particular subtracted from one another, and this is a major source of rounding error. To help avoid this, and other significant computational problems, one may consider the following 'triangular' recurrence relation of the Q - numbers :

$$Q(x_1, z; R) = x_1 Q(x_1 - 1, z - 1; R) + (R + x_1)Q(x_1, z - 1; R). \qquad (2.7)$$

A direct proof of (2.7) is as follows :

$$
\begin{aligned}
& x_1 Q(x_1 - 1, z - 1; R) + (R + x_1) Q(x_1, z - 1; R) \\
&= x_1 \sum_{k=0}^{x_1 - 1} \binom{x_1 - 1}{k} (-1)^k (R + x_1 - 1 - k)^{z-1} + (R + x_1) \sum_{k=0}^{x_1} \binom{x_1}{k} (-1)^k (R + x_1 - k)^{z-1} \\
&= x_1 \sum_{j=1}^{x_1} \binom{x_1 - 1}{j - 1} (-1)^{j-1} (R + x_1 - j)^{z-1} + (R + x_1) \sum_{k=1}^{x_1} \binom{x_1}{k} (-1)^k (R + x_1 - k)^{z-1} \\
&\qquad + (R + x_1)^z \\
&= (R + x_1)^z + \sum_{k=1}^{x_1} \left\{ -x_1 \binom{x_1 - 1}{k - 1} + (R + x_1) \binom{x_1}{k} \right\} (-1)^k (R + x_1 - k)^{z-1} \\
&= (R + x_1)^z + \sum_{k=1}^{x_1} \left\{ -x_1 \frac{(x_1 - 1)!}{(k - 1)!(x_1 - k)!} + (R + x_1) \binom{x_1}{k} \right\} (-1)^k (R + x_1 - k)^{z-1} \\
&= (R + x_1)^z + \sum_{k=1}^{x_1} \left\{ -k \frac{x_1!}{k!(x_1 - k)!} + (R + x_1) \binom{x_1}{k} \right\} (-1)^k (R + x_1 - k)^{z-1} \\
&= (R + x_1)^z + \sum_{k=1}^{x_1} \left\{ (R + x_1 - k) \binom{x_1}{k} \right\} (-1)^k (R + x_1 - k)^{z-1} \\
&= (R + x_1)^z + \sum_{k=1}^{x_1} \binom{x_1}{k} (-1)^k (R + x_1 - k)^z \\
&= \sum_{k=0}^{x_1} \binom{x_1}{k} (-1)^k (R + x_1 - k)^z \\
&= Q(x_1, z; R).
\end{aligned}
$$

Upon using the identity (2.6), the recurrence relation (2.7), when $R = 0$, can be shown to reduce to the well known relationship between Stirling numbers of the second kind, namely

$$S(x_1, z) = S(x_1 - 1, z - 1) + x_1 S(x_1, z - 1).$$

The triangular recurrence relation (2.7) along with the initial conditions

$$Q(0, z; R) = R^z \text{ and } Q(x_1, x_1; R) = x_1! \tag{2.7a}$$

enables one to evaluate the required Q - numbers without having to perform any subtraction operations whatsoever, and hence one can more easily avoid computational rounding error.

N.B. The first initial condition is easy to show directly. The second can be shown to hold as follows. Firstly substituting $z = x_1$ into (2.7) implies that $Q(x_1, x_1; R) = x_1 Q(x_1 - 1, x_1 - 1; R)$, then after observing that $Q(0,0;R) = 1$ it is easy to see that $Q(x_1, x_1; R) = x_1!$ for all $x_1 \geq 0$.

Comments

Using (2.7), one can show that a similar 'triangular' recurrence relation exists between the probabilities of the joint distribution of $X_1$ and Z, given by equation (2.5). It can be shown that

$$P_{X_1,Z} = \frac{\lambda\tau}{z}\left[(R + x_1)P_{X_1,Z-1} + (N - x_1 + 1)P_{X_1-1,Z-1}\right] \tag{2.8}$$

where $P_{X_1,Z} = \text{Prob}(X_1 = x_1, Z = z)$.

It is also straightforward to show that (2.8) is subject to the initial conditions

$$P_{0,Z} = \frac{(\lambda\tau)^z \exp[-(N+R)\lambda\tau]}{z!}R^z \qquad z = 0, 1, 2, \ldots.. \tag{2.8a}$$

and $$P_{X_1,X_1} = (\lambda\tau)^{x_1}\exp[-(N+R)\lambda\tau]\binom{N}{x_1} \qquad x_1 = 0, 1, 2, \ldots., N. \tag{2.8b}$$

Again in an attempt to avoid numerical computational problems, one may use the following recurrence relations to determine the intitial conditions (2.8a) and (2.8b) :

(i) $$P_{0,Z} = \left[\frac{(\lambda\tau)R}{z}\right]P_{0,Z-1} \qquad z = 1, 2, 3, \ldots.$$

(ii) $$P_{X_1,X_1} = \left[\frac{(\lambda\tau)(N - x_1 + 1)}{x_1}\right]P_{X_1-1,X_1-1} \qquad x_1 = 1, 2, 3, \ldots., N.$$

Where the appropriate initial condition for both (i) and (ii) is $P_{0,0} = \exp[-(N+R)\lambda\tau]$.

§ 2.7 : The Maximum Likelihood Estimator

It follows from equation (2.5) that the joint likelihood function for N and $\lambda$ may be written as

$$L(N,\lambda) \propto \lambda^z \exp[-(N+R)\lambda\tau]\binom{N}{x_1}. \tag{2.9}$$

This is now maximised over $\lambda$ :

$$\frac{\partial L}{\partial\lambda} \propto \lambda^z(-(N+R)\tau)\exp[-(N+R)\lambda\tau] + z\lambda^{z-1}\exp[-(N+R)\lambda\tau],$$

equate to zero to obtain $\hat{\lambda}$ :

$$\hat{\lambda}^z(N+R)\tau\exp\left[-(N+R)\hat{\lambda}\tau\right] = z\hat{\lambda}^{z-1}\exp\left[-(N+R)\hat{\lambda}\tau\right]$$

$$\Rightarrow \qquad \hat{\lambda} = \frac{z}{(N+R)\tau}.$$

$\hat{\lambda}$ may now be substituted into (2.9) to obtain the profile likelihood for N :

$$L(N) \propto \binom{N}{x_1}(N+R)^{-z}, \qquad N = x_1,\ x_1+1,\ x_1+2, \ldots..$$

There is no closed form expression for the value of N which maximises this profile likelihood function. However, for given values of $x_1$ and z, one may determine the estimate produced by the maximum likelihood estimator $\hat{N}$, using the following method.

Firstly, if $z = x_1$ the profile likelihood for N is clearly increasing and hence $\hat{N} = \infty$.

Now if $z > x_1$ one may observe that the profile likelihood function is uni-modal. Hence $\hat{N} = k$, where k is the smallest integer in the set $\{x_1, x_1+1, x_1+2, \ldots\ldots..\}$ able to satisfy the condition $L(k) > L(k+1) \quad \Leftrightarrow \quad \frac{x_1}{k+1} + \left(\frac{k+R}{k+R+1}\right)^z < 1.$

Once $\hat{N}$ has been determined, this value may then be used in the calculation of the maximum likelihood estimate of $\lambda$ : $\quad \hat{\lambda} = \frac{z}{\left(\hat{N}+R\right)\tau}.$

§ 2.8 : The Harmonic Mean Estimator

The Harmonic mean estimator was first considered by Joe and Reid (1985). Explicitly a point estimate of N is given by

$$\hat{N}_h = \left[ 0.5 + \left\{ \frac{2}{\frac{1}{n_1} + \frac{1}{n_2}} \right\} \right] \qquad \text{where [.] denotes the integer part.}$$

The values of $n_1$ and $n_2$ are defined as follows :

$$n_1 = \inf\{\ N \geq x : L(N|x) \geq cL(\hat{N}|x)\ \}$$

and $\quad n_2 = \sup\{\ N \geq x : L(N|x) \geq cL(\hat{N}|x)\ \},$

where $c \in (0,1]$.

## § 2.9 : A Peterson-Type Estimator

This section introduces an estimator of population size which is only dependent upon the observed numbers of distinct animals seen from the target and planted populations. The estimator is derived from the conditional distribution of $X_1$ given X. Within section 2.4 the following distributional results were observed to be true

$$X_1 \sim \mathrm{Bin}(N, 1-\exp[-\lambda\tau]),$$
$$X_2 \sim \mathrm{Bin}(R, 1-\exp[-\lambda\tau])$$

and
$$X \sim \mathrm{Bin}(N+R, 1-\exp[-\lambda\tau]).$$

It is then easy to show that the distribution of $X_1|X$ is in fact hypergeometric with probability function

$$\mathrm{Prob}(X_1 = x_1|X = x) = \binom{N}{x_1}\binom{R}{x-x_1}\Big/\binom{N+R}{x}, \qquad \max(0, x-R) \le x_1 \le \min(N, x).$$

The likelihood function for N based on this probability function is maximised by the Peterson-type estimator $\tilde{N}_P = RX_1/X_2$. To avoid an estimator which becomes infinite when $X_2$=0, $\tilde{N}_P$ is now slightly modified : from this point consideration is given to the estimator $\hat{N}_P = \left[0.5 + \frac{(R+1)X_1}{(X_2+1)}\right]$, where [.] denotes the integer part of.

## §2.9a : A New Estimator for Homogeneous Populations

A simple closed form estimator may be found by considering the expected value of $f_1$ ~ the number of animals seen exactly once during the experiment.
From the assumptions made above it follows that

$$f_1 = \sum_{i=1}^{N+R} I(X_i = 1) \quad \text{where} \quad I(X_i = 1) = \begin{cases} 1 & \text{w.p. } \lambda\tau.\exp(-\lambda\tau) \\ 0 & \text{w.p. } 1-\lambda\tau.\exp(-\lambda\tau) \end{cases}.$$

The expected value of $f_1$ is then given by

$$\begin{aligned} E[f_1] &= \sum_{i=1}^{N+R} E[I(X_i = 1)] \\ &= \sum_{i=1}^{N+R} \lambda\tau.\exp(-\lambda\tau) \\ &= (N+R)\lambda\tau.\exp(-\lambda\tau). \end{aligned}$$

By equating $f_1$ to its expected value one can obtain the equation

$$f_1 = (N+R)\lambda\tau.\exp\left(\frac{-(N+R)\lambda\tau}{(N+R)}\right) \qquad (2.9a)$$

The total number of sightings Z has a Poisson distribution with mean $(N+R)\lambda\tau$. Hence z may be used as an estimate of $(N+R)\lambda\tau$, and substituting this into equation (2.9a) yields

$$f_1 = z.\exp\left(\frac{-z}{N+R}\right).$$

Solving this for N gives $N = \dfrac{z}{\ln\left\{\dfrac{z}{f_1}\right\}} - R$, and so a point estimate of population size N is given by $\hat{N}_{fz} = \dfrac{z}{\ln\left\{\dfrac{z}{f_1}\right\}} - R$.

Care must be taken with this estimator since the estimate it produces is only dependent upon the values of $f_1$ and z. That is one may obtain a distorted view of population size if one observes the population only through the values of $f_1$ and z. This problem only occurs when sampling for a very long period of time and so in most practical situations it is expected that the above estimator may be considered for use in connection with the majority of experiments. Consider, for example, the data set contained within Table 4.2 of Seber(1982) p.137 which gives the capture-recapture data from a population of butterflies : from Craig(1953). For this data set R=0, $f_1$=258, $f_2$=72, $f_3$=11 and z=435. The maximum likelihood estimate and Darroch & Ratcliff(1980) estimate of population size are 853 and 838 respectively. The above estimator's estimate of population size is $\hat{N}_{fz} = \dfrac{435}{\ln\left\{\dfrac{435}{258}\right\}} = 833$.

Simulation results have shown that, provided no more than about 80% of the population is seen during the experiment, the estimator $\hat{N}_{fz}$ competes very well with the maximum likelihood estimator and the estimator of Darroch & Ratcliff(1980). The behaviour of $\hat{N}_{fz}$, in situations where less than about 80% of the population is seen, is in fact very similar to the behaviour of the maximum likelihood estimator - although the performance of the maximum likelihood estimator is on the whole marginally superior. The performance of $\hat{N}_{fz}$ diminishes after sampling has continued for a very long time, and is unsatisfactory when more than about 80% of the population is seen during the experiment. For this reason and the fact that $\hat{N}_{fz}$ is not a function of the sufficient statistics alone, the estimator $\hat{N}_{fz}$ is not considered any further at this stage.

## § 2.10 : A Conditionally Unbiased Estimator

This section introduces the Conditionally Unbiased Estimator (CUE) $\tilde{N}_u$, which is an estimator of population size N defined by

$$\tilde{N}_u = \frac{Q(x_1, z+1; R)}{Q(x_1, z; R)} - R. \tag{2.10}$$

where $Q(x_1, z; R) = \sum_{k=0}^{x_1} \binom{x_1}{k} (-1)^k (R + x_1 - k)^z$,

as defined in section 2.6.

The CUE $\tilde{N}_u$ was derived from the conditional distribution of $X_1$ given Z. When R=0, $\tilde{N}_u$ reduces to an estimator which has previously been considered in an urn model context, Harris(1968). In the absence of plants Harris(1968) showed that, provided $Z \geq N$, $\tilde{N}_u$ is a minimum variance unbiased estimator of N with respect to the conditional distribution of $X_1$ given Z. As previously mentioned in section 1.8 of chapter 1, when R=0, $\tilde{N}_u$ is equivalent to a special case of the estimator proposed by Pathak(1964). Berg(1975), using the notation of Pathak(1964), derived a recurrence relation for $\tilde{N}_u$ in the R = 0 situation. In section 2.10a a recurrence relation for $\tilde{N}_u$ is derived which allows for values of R greater than or equal to zero.

For values of R greater than or equal to zero, the CUE $\tilde{N}_u$ can be shown to be unbiased with respect to the conditional distribution of $X_1$ given Z, provided that $Z \geq N$, as follows.

$$E(\tilde{N}_u) = \sum_{x_1} \tilde{N}_u \text{Prob}(X_1 = x_1 | Z = z)$$

$$= \sum_{x_1} \left[ \frac{Q(x_1, z+1; R)}{Q(x_1, z; R)} - R \right] \binom{N}{x_1} \frac{1}{(N+R)^z} Q(x_1, z; R),$$

this follows from equation (2.4).

$$= \sum_{x_1} \frac{Q(x_1, z+1; R)}{Q(x_1, z; R)} \binom{N}{x_1} \frac{1}{(N+R)^z} Q(x_1, z; R) - R$$

$$= (N+R) \sum_{x_1} \binom{N}{x_1} \frac{1}{(N+R)^{z+1}} Q(x_1, z+1; R) - R$$

$$= (N+R).1 - R \qquad \text{if } Z \geq N.$$

$$= N.$$

Hence $\tilde{N}_u$ is unbiased over the conditional distribution of $X_1$ given Z, provided that the condition $Z \geq N$ holds. Furthermore, again provided that $Z \geq N$, since $X_1$ is sufficient for N with respect to the conditional distribution of $X_1$ given Z, it follows that $\tilde{N}_u$ is the minimum variance unbiased estimator of N with respect to the conditional distribution of $X_1$ given Z , Lehmann and Scheffe(1950).

In view of the fact that population size N is integer valued, in later sections consideration is given to the following slightly modified version of $\tilde{N}_u$ :

$$\hat{N}_u = \left[ 0.5 + \frac{Q(x_1, z+1; R)}{Q(x_1, z; R)} - R \right],$$

where the square brackets have been used to denote the integer part.

§ 2.10a : A Note on the Evaluation of the CUE

Direct use of equation (2.10) to evaluate the estimates produced by the estimator $\tilde{N}_u$ can often be difficult, and involve very cumbersome computation. This is due to the fact that the Q - numbers, present within (2.10), grow rapidly with increasing arguments. To overcome this computational problem, a recurrence relation linking the $\tilde{N}_u$ is now stated and proved.

To make the following proof more easily read some shorthand notation is necessary.

Let $\qquad Q_{x_1,z} = Q(x_1, z; R)$

and let $\qquad N_{x_1,z} = \tilde{N}_u = \dfrac{Q(x_1, z+1; R)}{Q(x_1, z; R)} - R = \dfrac{Q_{x_1,z+1}}{Q_{x_1,z}} - R.$

The $N_{x_1,z}$ are then subject to the following recurrence relation

$$N_{x_1,z} = x_1 + \left( \frac{N_{x_1-1,z-1} + R}{N_{x_1,z-1} + R} \right) \left( N_{x_1,z-1} - x_1 \right), \qquad (2.11)$$

with initial conditions $\qquad N_{0,z} = 0 \qquad$ for $z = 0, 1, 2, \ldots\ldots \qquad (2.12)$

and $\qquad N_{x_1,x_1} = \dfrac{x_1}{2}\left[ x_1 + 2R + 1 \right] \qquad$ for $x_1 \geq 0. \qquad (2.13)$

N.B. Substituting R = 0 into (2.11), (2.12) and (2.13) yields the 'Property 5' of Berg(1975) p.92.

Proof of (2.11) :

$$
\begin{aligned}
& x_1 + \left(\frac{N_{x_1-1,z-1}+R}{N_{x_1,z-1}+R}\right)\left(N_{x_1,z-1}-x_1\right) \\
&= x_1 + \left(\frac{\dfrac{Q_{x_1-1,z}}{Q_{x_1-1,z-1}}-R+R}{\dfrac{Q_{x_1,z}}{Q_{x_1,z-1}}-R+R}\right)\left(\frac{Q_{x_1,z}}{Q_{x_1,z-1}}-R-x_1\right) \\
&= x_1 + \frac{Q_{x_1-1,z}}{Q_{x_1-1,z-1}}\frac{Q_{x_1,z-1}}{Q_{x_1,z}}\frac{\left(Q_{x_1,z}-\left(R+x_1\right)Q_{x_1,z-1}\right)}{Q_{x_1,z-1}} \\
&= x_1 + \frac{Q_{x_1-1,z}}{Q_{x_1-1,z-1}}\frac{Q_{x_1,z-1}}{Q_{x_1,z}}\frac{\left(x_1 Q_{x_1-1,z-1}\right)}{Q_{x_1,z-1}} && \text{using (2.7)} \\
&= x_1 + \frac{x_1 Q_{x_1-1,z}}{Q_{x_1,z}} \\
&= x_1 + \frac{\left(Q_{x_1,z+1}-\left(R+x_1\right)Q_{x_1,z}\right)}{Q_{x_1,z}} && \text{using (2.7) with z replaced by z+1} \\
&= x_1 + \frac{Q_{x_1,z+1}}{Q_{x_1,z}}-\left(R+x_1\right) \\
&= \frac{Q_{x_1,z+1}}{Q_{x_1,z}}-R \\
&= N_{x_1,z}.
\end{aligned}
$$

Proof of (2.12) :

$$
\begin{aligned}
N_{0,z} &= \frac{Q_{0,z+1}}{Q_{0,z}}-R \\
&= \frac{R^{z+1}}{R^z}-R && \text{using (2.7a)} \\
&= 0.
\end{aligned}
$$

Proof of (2.13) :

As a first step in this proof it is necessary to prove the following identity

$$Q_{x_1,x_1+1} = \frac{x_1!}{2}(2R + x_1)(x_1 + 1), \qquad x_1 \geq 0. \tag{2.14}$$

The proof of (2.14) is by induction :

Anchor : (2.14) is clearly true for $x_1 = 0$, since $Q_{0,1} = R$.

Assume true for $x_1 = k$, i.e. assume $Q_{k,k+1} = \frac{k!}{2}(2R + k)(k + 1)$.

Then

$$\begin{aligned}
Q_{k+1,k+2} &= (k+1)Q_{k,k+1} + (R + k + 1)Q_{k+1,k+1} && \text{using (2.7)} \\
&= (k+1)\frac{k!}{2}(2R + k)(k + 1) + (R + k + 1)(k + 1)! && \text{using assumption} \\
& && \text{and (2.7a)} \\
&= \frac{(k+1)!}{2}\left[(2R + k)(k + 1) + 2(R + k + 1)\right] \\
&= \frac{(k+1)!}{2}\left[2Rk + 4R + k^2 + 3k + 2\right] \\
&= \frac{(k+1)!}{2}\left[(2R + k)(k + 2) + k + 2\right] \\
&= \frac{(k+1)!}{2}\left[(2R + (k + 1))((k + 1) + 1)\right].
\end{aligned}$$

This shows that, if (2.14) is true for $x_1 = k$, then it must also be true for $x_1 = k + 1$. Since it has been shown that (2.14) is true for $x_1 = 0$, it follows by induction that (2.14) holds for all $x_1 \geq 0$.

The proof of (2.13) may now be completed :

$$\begin{aligned}
N_{x_1,x_1} &= \frac{Q_{x_1,x_1+1}}{Q_{x_1,x_1}} - R \\
&= \frac{\frac{x_1!}{2}(2R + x_1)(x_1 + 1)}{x_1!} - R && \text{using (2.14) and (2.7a)} \\
&= \frac{1}{2}(2R + x_1)(x_1 + 1) - R \\
&= \frac{1}{2}\left[2Rx_1 + 2R + x_1^2 + x_1 - 2R\right] \\
&= \frac{x_1}{2}\left[x_1 + 2R + 1\right].
\end{aligned}$$

## § 2.11 : A Comparison of All Four Estimators

In order to compare the performance of the four estimators which have so far been discussed, consideration is given to their mean, standard deviation and root mean square error conditional on the event $C = \{Z > X_1\}$. This conditioning is necessary since the maximum likelihood estimator $\hat{N}$ yields infinite estimates when $Z = X_1$. It is important to note however that the Peterson-type estimator $\hat{N}_P$, harmonic mean estimator $\hat{N}_h$ and conditionally unbiased estimator $\hat{N}_u$ produce finite estimates with probability one. The unconditional performance of $\hat{N}_P$, $\hat{N}_h$ and $\hat{N}_u$ is considered later on.

Conditional on the event $C=\{Z>X_1\}$, the mean, standard deviation and root mean square error of each estimator are presented in tables 2.1a,b,c, 2.2a,b,c and 2.3a,b,c. These tables summarise the performance of the estimators for each combination from the following factorial design :

$$N = \begin{matrix} 10 \\ 25 \\ 50 \end{matrix} \times \lambda = 1 \times \tau = \log\left(\frac{1}{1-\rho}\right) : \rho = \begin{matrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \\ 0.5 \\ 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \end{matrix} \times R = \begin{matrix} 0 \\ 5 \\ 10 \\ 25 \\ 50 \end{matrix} \qquad \text{where } \rho = E\left(\frac{X_1}{N}\right).$$

Please note however that, for each value of population size N, only values of R up to and including N are considered; this is done for obvious practical reasons.
The notation used within each table is as follows :

Statistics

| | | |
|---|---|---|
| exp. | $\equiv$ | mean or expectation. |
| s.d. | $\equiv$ | standard deviation. |
| rmse | $\equiv$ | root mean square error. |
| P(inf mle) | $\equiv$ | $1 - \text{Prob}(C) = \text{Prob}(\overline{C}) = \text{Prob}(Z = X_1)$, which is the probability of the maximum likelihood estimator producing an infinite estimate. |

Estimators

| | | | |
|---|---|---|---|
| X1 | $\equiv$ | $X_1$, | the number of distinct individuals seen from the target population. |
| P | $\equiv$ | $\hat{N}_p$, | the Peterson-type estimator of section 2.9. |
| CUE | $\equiv$ | $\hat{N}_u$, | the conditionally unbiased estimator of section 2.10. |
| MLE | $\equiv$ | $\hat{N}$, | the maximum likelihood estimator of section 2.7. |
| HME | $\equiv$ | $\hat{N}_h$, | the harmonic mean estimator, Joe & Reed(1985) |

It is straightforward to obtain the distributions of both $\hat{N}$ and $\hat{N}_u$ given the event C. In order to obtain the conditional distribution of the Peterson-type estimator $\hat{N}_P$ given the event C, one must derive the conditional distribution of $X_1$ and $X_2$ given C. The following proof was provided by I. B. J. Goudie ( pers. com. ).

Firstly recall that C is defined as being the event $\{Z>X_1\}$ and that $\overline{C}$ is used to denote the complementary event $\{Z=X_1\}$.

$\overline{C}$ occurs $\Leftrightarrow$ $X_2=0$ and each individual in target population is seen at most once.

Now $\quad \text{Prob}(X_2=0)=\left[\exp(-\lambda\tau)\right]^R$ (2.15)

( This is because we know that $X_2 \sim \text{Bin}(R, 1-\exp(-\lambda\tau))$ )

Let $Y_i$ = the number of sightings of individual i. It follows that $Y_i \sim P(\lambda\tau)$.

It may then be observed that

Prob( each individual in target population is seen at most once )

$$= \prod_{i=1}^{N} \text{Prob}(Y_i \le 1)$$

$$= \left[(\lambda\tau+1)\exp(-\lambda\tau)\right]^N. \qquad (2.16)$$

Use of (2.15) and (2.16) implies that

$$\begin{aligned}\text{Prob}(C) &= 1-\text{Prob}(\overline{C})\\ &= 1-\left[\exp(-\lambda\tau)\right]^R\left[(\lambda\tau+1)\exp(-\lambda\tau)\right]^N.\end{aligned}$$

Now

$$\begin{aligned}\text{Prob}(X_1=x_1, X_2=x_2 | Z>X_1) &= \frac{\text{Prob}(X_1=x_1, X_2=x_2, Z>X_1)}{\text{Prob}(Z>X_1)}\\ &= \frac{\text{Prob}(X_1=x_1, X_2=x_2)\text{Prob}(Z>X_1 | X_1=x_1, X_2=x_2)}{\text{Prob}(Z>X_1)}\\ &= \frac{\text{Prob}(X_1=x_1)\text{Prob}(X_2=x_2)\text{Prob}(Z>X_1 | X_1=x_1, X_2=x_2)}{\text{Prob}(Z>X_1)} \qquad (2.17)\end{aligned}$$

It is clear that $\text{Prob}(Z>X_1|X_1=x_1,X_2=x_2)=1$ if $X_2>0$.

When $X_2=0$ it may be observed that $Z|X_1,X_2 \equiv Z_1|X_1$. The distribution of $Z_1|X_1$ may be characterised as being the sum of $X_1$ zero truncated Poisson random variables: this distribution is known in the literature as a Stirling distribution of the second kind; the probability function of this distribution is derived in appendix 2.
Explicitly the probability function of $Z_1|X_1$ is given by

$$\text{Prob}(Z_1 = z_1|X_1 = x_1) = \frac{x_1!}{z_1!}\frac{(\lambda\tau)^{z_1} S(x_1,z_1)}{(\exp(\lambda\tau)-1)^{x_1}},$$

$$z_1 = x_1,\ x_1+1,\ x_1+2,\ .....$$

It follows that

$$\begin{aligned}\text{Prob}(Z>x_1|X_1=x_1,X_2=x_2) &= \text{Prob}(Z_1>x_1|X_1=x_1)\\ &= 1-\text{Prob}(Z_1=x_1|X_1=x_1)\\ &= 1-\frac{(\lambda\tau)^{x_1}}{(\exp(\lambda\tau)-1)^{x_1}}.\end{aligned}$$

Following on from equation (2.17), and using the notation $\tilde{P}(C)=\text{Prob}(Z>X_1|X_1=x_1,X_2=x_2)$, allows one to write :

$$\text{Prob}(X_1 = x_1, X_2 = x_2|Z > X_1) = \frac{\text{Prob}(X_1 = x_1)\text{Prob}(X_2 = x_2)\text{Prob}(Z > X_1|X_1 = x_1, X_2 = x_2)}{\text{Prob}(Z > X_1)}$$

$$= \frac{\binom{N}{x_1}[1-\exp(-\lambda\tau)]^{x_1}[\exp(-\lambda\tau)]^{N-x_1}\binom{R}{x_2}[1-\exp(-\lambda\tau)]^{x_2}[\exp(-\lambda\tau)]^{R-x_2}\tilde{P}(C)}{P(C)},$$

$$x_1=\begin{cases}0,1,2,...,N & \text{for} \quad R>0\\ 1,2,...,N & \text{for} \quad R=0\end{cases}'$$

$$x_2=\begin{cases}1,2,...,R & \text{for} \quad R>0,\ x_1=0\\ 0,1,2,...,R & \text{for} \quad R>0,\ x_1>0\\ 0 & \text{for} \quad R=0\end{cases},$$

$$\text{where} \quad \tilde{P}(C)=\begin{cases}1 & \text{for} \quad X_2>0\\ 1-\dfrac{(\lambda\tau)^{x_1}}{(\exp(\lambda\tau)-1)^{x_1}} & \text{for} \quad X_2=0\end{cases}$$

$$\text{and} \quad P(C) = \text{Prob}(Z > X_1) = 1-[\exp(-\lambda\tau)]^R[(\lambda\tau+1)\exp(-\lambda\tau)]^N.$$

§ 2.11a : The c Constant

The Harmonic Mean Estimator, as defined by Joe & Reid(1985), depends upon a constant $c \in (0,1]$. There is no natural or obvious value that this constant should take. The problem of deciding which single value of c should be used in connection with the HME is therefore highly subjective. It is for this reason that the following tables illustrate the performance of the HME when used in connection with a range of possible c values . In practice however it would be necessary to have chosen a priori an appropriate value of c. Hence we now aim to recommend a single value of c that may be considered suitable for general use. This is not straightforward: in some situations the performance of the HME depends heavily upon the value taken by the c constant, whereas in other situations varying the constant has little effect. Joe & Reid(1985), although still concerned with population size estimation, considered a different problem to the one described here. It is worthwhile to note however that one conclusion they reached, bearing in mind that they only considered populations of size $N = 10$ when $R = 0$, was that changing c did not substantially affect the performance of the HME. The same conclusion may be drawn from tables 2.1a,b,c when R is equal to zero. So this, at least, is broadly consistent with Joe & Reid(1985). However when one considers situations in which $N = 10$, $R > 0$ and in particular those in which $N > 10$, $R >= 0$ one can see that varying the c constant can in fact produce significant change in the performance of the HME, and that the effect of the constant is most significant for the smaller values of $\tau$. For these small values of $\tau$, which represent situations in which little information is available, the likelihood functions can be very spread out, almost flat and although remaining unimodal, are certainly not peaked, their shape representing the paucity of information. It is in each of these situations that the shape of the likelihood function can allow the harmonic mean estimate to differ greatly from the maximum likelihood estimate. The reason for this being that, due to the shape of the likelihood function, it is in each of these situations that the c constant has most effect upon the resulting harmonic mean estimate. Consequently, for small values of $\tau$, the performance of the HME can be greatly affected by changing the c constant. For the larger values of $\tau$, which represent situations in which a large amount of information has been obtained, the likelihood functions become peaked. In each of these situations the estimate produced by the HME is therefore not likely to differ greatly from the maximum likelihood estimate. This is the reason why, for the larger values of $\tau$, the HME and MLE perform in a very similar way. And furthermore, due to the peakedness of the likelihood functons, the c constant has little affect upon each resulting harmonic mean estimate. Hence for the larger values of $\tau$ the c constant does not significantly affect the performance of the HME.

Regarding the way in which the c constant affects the HME when its performance is in turn being compared with that of the MLE, the general trend is that for small values of c the HME, when compared with the MLE, tends to have both a smaller mean and variance, and that as c is increased the mean and variance of the HME increase to those of the MLE. For the larger values of c, as one would expect, the performance of the HME is very similar to that of the MLE : in particular when $c = 1$ the HME and MLE are equivalent.

In view of this behaviour, when seeking to choose an appropriate value for the c constant, one must be careful not to place too much emphasis on mean square error. It would be inappropriate to base a choice of c solely upon the effect that this constant may have upon the mean square error of the HME, since this loss function is known to favour estimators possessing negative bias and small variance. Hence consideration of mean square error alone would lead one to choose an unduly small value for the c constant.

Based on an inspection of tables 2.1a,b,c, 2.2a,b,c and 2.3a,b,c it is believed that for general use the most appropriate value of the c constant should be $c = 0.5$. In reaching this decision consideration was given to mean, standard deviation and mean square error. Joe & Reed(1985) recommended use of the HME with a c value of 0.5. Hence from this point when reference is made to the HME a c value of 0.5 is to be assumed.

## § 2.11b : Discussion of Relative Performance of Estimators

The reasons underlying the differing performance of the MLE and HME have been discussed above. For the largest values of $\tau$ considered in the tables the MLE and HME perform in a very similar way. However for the smaller values of $\tau$ the HME can in some situations be seen to perform significantly better than the MLE : particularly in terms of mean square error, although it is important to note that this is often at the expense of negative bias. On the whole, within the range of population sizes covered in the tables, it can be said that the performance of the HME is either better than or as good as that of the MLE. Whilst it is important to bear in mind the fact that the HME tends to possess a comparatively small mean square error as a result of its negative bias, it is recommended that the HME be preferred to the MLE.

At this stage it is important to observe that, given the proportion of the population seen during the experiment, the performance and relative performance of the MLE, CUE and Peterson-type estimators can be seen to be essentially identical under both the discrete time sampling procedure of the previous chapter and the continuous time sampling procedure of this chapter. Hence the conclusions reached in chapter 1 regarding the relative performance of the MLE, CUE and Peterson-type estimators are valid here. So that firstly, on the basis of tables 2.1,a,b,c, 2.2a,b,c and 2.3a,b,c, it can be argued that the CUE is clearly a better alternative to the Peterson-type estimator. And that secondly, for the reasons stated fully in chapter 1, one should always use the CUE in preference to the MLE.

The comparison between the HME and CUE is quite straightforward, this being a result of the fact that both estimators behave in a very similar way. When little information is available both the HME and CUE tend to be negatively biased with small variance, this bias then reduces smoothly as more information is obtained. In view of the fact that both the HME and MLE behave in a very similar way with regard to bias, it is worthwhile to note that in all but one of the one-hundred and eight situations considered in the tables, that the standard deviation of the CUE is smaller than that of the HME. For values of $\tau$ greater than 0.36, the absolute bias of the CUE is only greater than that of the HME in six out of the seventy-two situations considered, and in terms of mean square error the CUE is uniformly better. For values of $\tau$ less than or equal to 0.36, both the CUE and HME can be negatively biased, and in particular for these small values of $\tau$ the CUE can be more negatively biased than the HME. The standard deviation of the CUE however remains smaller than that of the HME, and consequently, for values of $\tau$ less than or equal to 0.36, the HME possesses a smaller mean square error than the CUE in only nine out of the thirty-six situations considered, and in none of these nine cases is the reduction in mean square error particularly large. In view of the above discussion, it is believed that one should always use the CUE in preference to the HME. This conclusion has been based entirely on consideration of performance : it is worthwhile to note however that for any given data set the estimate produced by the CUE is usually much easier to calculate than the harmonic mean estimate.

Table 2.1a

| N = 10 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | x10^4 | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.11 | 0 | 9452 | exp. | 1.91 | 1.91 | 2.25 | 2.48 | 3.78 | 3.77 | 3.21 | 2.80 | 2.89 | 2.85 | 2.86 | 2.67 | 2.48 |
| | | | s.d. | 0.91 | 0.91 | 1.60 | 2.07 | 1.79 | 1.81 | 1.54 | 1.86 | 2.11 | 2.00 | 2.04 | 2.36 | 2.06 |
| | | | rmse | 8.14 | 8.14 | 7.91 | 7.80 | 6.47 | 6.49 | 6.97 | 7.44 | 7.42 | 7.43 | 7.43 | 7.70 | 7.80 |
| | 5 | 5453 | exp. | 1.10 | 3.50 | 3.39 | 5.43 | 2.36 | 2.39 | 2.82 | 2.89 | 3.25 | 3.88 | 4.27 | 4.36 | 4.98 |
| | | | s.d. | 0.99 | 3.73 | 3.42 | 5.76 | 2.40 | 2.53 | 3.10 | 3.34 | 3.90 | 3.99 | 4.53 | 4.96 | 5.12 |
| | | | rmse | 8.95 | 7.49 | 7.44 | 7.35 | 8.01 | 8.02 | 7.83 | 7.86 | 7.80 | 7.31 | 7.30 | 7.51 | 7.17 |
| | 10 | 3146 | exp. | 1.06 | 5.40 | 5.27 | 8.34 | 2.60 | 3.17 | 3.35 | 4.12 | 4.42 | 5.61 | 6.03 | 6.81 | 7.52 |
| | | | s.d. | 0.97 | 5.61 | 5.25 | 9.24 | 2.92 | 3.89 | 4.31 | 4.81 | 5.50 | 6.24 | 7.01 | 7.60 | 8.33 |
| | | | rmse | 8.99 | 7.26 | 7.06 | 9.39 | 7.95 | 7.86 | 7.93 | 7.60 | 7.83 | 7.63 | 8.06 | 8.24 | 8.69 |
| 0.22 | 0 | 8094 | exp. | 2.70 | 2.70 | 3.71 | 4.31 | 5.25 | 5.25 | 4.61 | 4.39 | 4.69 | 4.58 | 4.61 | 4.63 | 4.29 |
| | | | s.d. | 1.18 | 1.18 | 2.49 | 3.40 | 2.31 | 2.40 | 2.49 | 2.68 | 3.14 | 3.04 | 3.16 | 3.49 | 3.37 |
| | | | rmse | 7.39 | 7.39 | 6.76 | 6.63 | 5.28 | 5.32 | 5.93 | 6.22 | 6.17 | 6.21 | 6.25 | 6.40 | 6.63 |
| | 5 | 2694 | exp. | 2.04 | 6.06 | 6.03 | 8.92 | 4.54 | 4.69 | 5.33 | 5.58 | 6.29 | 6.77 | 7.36 | 7.84 | 8.27 |
| | | | s.d. | 1.27 | 5.02 | 4.50 | 7.67 | 3.41 | 3.77 | 4.34 | 4.83 | 5.48 | 5.62 | 6.20 | 6.70 | 7.08 |
| | | | rmse | 8.06 | 6.38 | 6.00 | 7.75 | 6.44 | 6.52 | 6.37 | 6.55 | 6.62 | 6.49 | 6.74 | 7.04 | 7.29 |
| | 10 | 897 | exp. | 1.99 | 8.17 | 8.07 | 11.46 | 4.81 | 5.72 | 6.28 | 6.94 | 7.47 | 8.57 | 9.20 | 9.94 | 10.72 |
| | | | s.d. | 1.26 | 6.78 | 6.18 | 10.72 | 3.91 | 4.94 | 5.69 | 6.25 | 7.06 | 7.69 | 8.37 | 9.00 | 9.83 |
| | | | rmse | 8.11 | 7.02 | 6.47 | 10.82 | 6.50 | 6.53 | 6.80 | 6.96 | 7.50 | 7.83 | 8.41 | 9.00 | 9.86 |
| 0.36 | 0 | 5915 | exp. | 3.57 | 3.57 | 5.45 | 6.50 | 6.79 | 6.83 | 6.36 | 6.21 | 6.70 | 6.59 | 6.63 | 6.76 | 6.45 |
| | | | s.d. | 1.36 | 1.36 | 3.16 | 4.57 | 2.69 | 2.91 | 3.34 | 3.49 | 3.92 | 4.00 | 4.21 | 4.35 | 4.48 |
| | | | rmse | 6.57 | 6.57 | 5.54 | 5.75 | 4.19 | 4.30 | 4.94 | 5.15 | 5.13 | 5.26 | 5.39 | 5.43 | 5.72 |
| | 5 | 978 | exp. | 3.06 | 8.12 | 8.17 | 10.96 | 6.64 | 6.90 | 7.55 | 7.95 | 8.70 | 9.07 | 9.59 | 10.16 | 10.41 |
| | | | s.d. | 1.45 | 5.80 | 4.98 | 8.59 | 3.88 | 4.41 | 5.01 | 5.58 | 6.15 | 6.48 | 7.02 | 7.49 | 8.12 |
| | | | rmse | 7.09 | 6.10 | 5.30 | 8.64 | 5.13 | 5.39 | 5.58 | 5.95 | 6.28 | 6.55 | 7.03 | 7.49 | 8.13 |
| | 10 | 162 | exp. | 3.03 | 9.58 | 9.50 | 11.70 | 6.65 | 7.46 | 8.12 | 8.69 | 9.15 | 9.77 | 10.34 | 10.78 | 11.35 |
| | | | s.d. | 1.45 | 6.79 | 6.02 | 9.76 | 4.16 | 5.01 | 5.68 | 6.27 | 6.89 | 7.34 | 7.88 | 8.43 | 9.13 |
| | | | rmse | 7.12 | 6.81 | 6.04 | 9.91 | 5.34 | 5.61 | 5.99 | 6.41 | 6.94 | 7.34 | 7.88 | 8.46 | 9.23 |

Table 2.1b

| N = 10 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | x10^4 | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.51 | 0 | 3757 | exp. | 4.37 | 4.37 | 6.99 | 8.35 | 8.11 | 8.20 | 7.95 | 7.82 | 8.32 | 8.31 | 8.30 | 8.46 | 8.26 |
| | | | s.d. | 1.46 | 1.46 | 3.52 | 5.27 | 2.95 | 3.26 | 3.83 | 4.01 | 4.34 | 4.58 | 4.84 | 4.93 | 5.16 |
| | | | rmse | 5.82 | 5.82 | 4.63 | 5.52 | 3.51 | 3.73 | 4.35 | 4.57 | 4.65 | 4.88 | 5.13 | 5.16 | 5.45 |
| | 5 | 293 | exp. | 4.01 | 9.27 | 9.29 | 11.31 | 8.07 | 8.35 | 8.87 | 9.27 | 9.83 | 10.11 | 10.42 | 10.90 | 10.95 |
| | | | s.d. | 1.55 | 5.97 | 4.91 | 8.19 | 3.87 | 4.42 | 4.99 | 5.51 | 5.97 | 6.36 | 6.81 | 7.22 | 7.85 |
| | | | rmse | 6.18 | 6.01 | 4.96 | 8.30 | 4.32 | 4.72 | 5.12 | 5.55 | 5.97 | 6.36 | 6.82 | 7.27 | 7.90 |
| | 10 | 23 | exp. | 4.00 | 9.96 | 9.92 | 10.90 | 7.90 | 8.43 | 8.91 | 9.35 | 9.66 | 9.93 | 10.30 | 10.56 | 10.80 |
| | | | s.d. | 1.55 | 6.00 | 5.20 | 7.49 | 3.92 | 4.52 | 4.96 | 5.37 | 5.78 | 6.01 | 6.37 | 6.71 | 7.17 |
| | | | rmse | 6.20 | 6.00 | 5.20 | 7.55 | 4.45 | 4.78 | 5.07 | 5.41 | 5.79 | 6.01 | 6.37 | 6.74 | 7.22 |
| 0.69 | 0 | 1915 | exp. | 5.20 | 5.20 | 8.34 | 9.73 | 9.31 | 9.41 | 9.30 | 9.20 | 9.58 | 9.65 | 9.57 | 9.74 | 9.62 |
| | | | s.d. | 1.51 | 1.51 | 3.64 | 5.56 | 3.10 | 3.44 | 3.98 | 4.24 | 4.49 | 4.79 | 5.10 | 5.21 | 5.45 |
| | | | rmse | 5.03 | 5.03 | 4.00 | 5.57 | 3.17 | 3.49 | 4.05 | 4.31 | 4.51 | 4.81 | 5.12 | 5.22 | 5.47 |
| | 5 | 61 | exp. | 4.99 | 9.83 | 9.80 | 10.85 | 9.11 | 9.26 | 9.58 | 9.85 | 10.15 | 10.32 | 10.45 | 10.74 | 10.63 |
| | | | s.d. | 1.58 | 5.62 | 4.41 | 6.71 | 3.56 | 4.00 | 4.46 | 4.83 | 5.15 | 5.49 | 5.78 | 6.08 | 6.52 |
| | | | rmse | 5.25 | 5.62 | 4.42 | 6.76 | 3.67 | 4.07 | 4.47 | 4.83 | 5.15 | 5.50 | 5.80 | 6.12 | 6.55 |
| | 10 | 2 | exp. | 4.98 | 10.01 | 10.03 | 10.24 | 8.86 | 9.11 | 9.37 | 9.61 | 9.77 | 9.88 | 10.08 | 10.20 | 10.19 |
| | | | s.d. | 1.58 | 4.93 | 4.20 | 5.21 | 3.44 | 3.79 | 4.03 | 4.24 | 4.46 | 4.55 | 4.72 | 4.89 | 5.14 |
| | | | rmse | 5.26 | 4.93 | 4.20 | 5.22 | 3.62 | 3.90 | 4.07 | 4.26 | 4.46 | 4.55 | 4.72 | 4.90 | 5.15 |
| 0.92 | 0 | 688 | exp. | 6.11 | 6.11 | 9.33 | 10.37 | 10.32 | 10.32 | 10.21 | 10.13 | 10.31 | 10.39 | 10.24 | 10.37 | 10.27 |
| | | | s.d. | 1.51 | 1.51 | 3.45 | 5.30 | 3.05 | 3.35 | 3.78 | 4.09 | 4.29 | 4.57 | 4.88 | 5.02 | 5.22 |
| | | | rmse | 4.18 | 4.18 | 3.51 | 5.31 | 3.07 | 3.37 | 3.78 | 4.09 | 4.30 | 4.59 | 4.89 | 5.03 | 5.22 |
| | 5 | 7 | exp. | 6.02 | 10.02 | 9.98 | 10.24 | 9.87 | 9.80 | 9.90 | 10.01 | 10.10 | 10.15 | 10.19 | 10.23 | 10.09 |
| | | | s.d. | 1.55 | 4.86 | 3.58 | 4.69 | 3.02 | 3.31 | 3.58 | 3.80 | 3.94 | 4.19 | 4.27 | 4.47 | 4.68 |
| | | | rmse | 4.27 | 4.86 | 3.58 | 4.70 | 3.03 | 3.31 | 3.59 | 3.80 | 3.95 | 4.19 | 4.28 | 4.47 | 4.68 |
| | 10 | 0 | exp. | 6.01 | 10.03 | 10.03 | 9.87 | 9.57 | 9.58 | 9.65 | 9.75 | 9.81 | 9.81 | 9.91 | 9.90 | 9.75 |
| | | | s.d. | 1.55 | 3.91 | 3.25 | 3.59 | 2.82 | 3.03 | 3.16 | 3.27 | 3.33 | 3.39 | 3.42 | 3.53 | 3.65 |
| | | | rmse | 4.28 | 3.91 | 3.25 | 3.59 | 2.86 | 3.06 | 3.18 | 3.28 | 3.34 | 3.39 | 3.42 | 3.53 | 3.66 |

Table 2.1c

| N = 10 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) x10^4 | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 1.20 | 0 | 163 | exp. | 7.01 | 7.01 | 9.82 | 10.27 | 10.92 | 10.76 | 10.54 | 10.45 | 10.42 | 10.45 | 10.28 | 10.30 | 10.15 |
| | | | s.d. | 1.43 | 1.43 | 2.96 | 4.39 | 2.76 | 2.98 | 3.24 | 3.53 | 3.67 | 3.88 | 4.11 | 4.23 | 4.35 |
| | | | rmse | 3.31 | 3.31 | 2.97 | 4.39 | 2.91 | 3.07 | 3.29 | 3.56 | 3.69 | 3.90 | 4.12 | 4.24 | 4.36 |
| | 5 | 0 | exp. | 6.99 | 10.06 | 10.02 | 9.87 | 10.33 | 10.10 | 10.02 | 10.03 | 10.02 | 9.96 | 9.96 | 9.83 | 9.66 |
| | | | s.d. | 1.45 | 3.96 | 2.72 | 3.11 | 2.43 | 2.56 | 2.72 | 2.83 | 2.89 | 3.02 | 2.99 | 3.13 | 3.17 |
| | | | rmse | 3.34 | 3.96 | 2.72 | 3.12 | 2.45 | 2.56 | 2.72 | 2.83 | 2.89 | 3.02 | 2.99 | 3.14 | 3.19 |
| | 10 | 0 | exp. | 6.99 | 10.08 | 9.99 | 9.68 | 10.06 | 9.86 | 9.85 | 9.83 | 9.86 | 9.75 | 9.80 | 9.69 | 9.47 |
| | | | s.d. | 1.45 | 3.08 | 2.50 | 2.61 | 2.24 | 2.36 | 2.47 | 2.53 | 2.54 | 2.60 | 2.59 | 2.68 | 2.68 |
| | | | rmse | 3.34 | 3.08 | 2.50 | 2.63 | 2.24 | 2.37 | 2.47 | 2.54 | 2.55 | 2.61 | 2.59 | 2.70 | 2.73 |
| 1.61 | 0 | 15 | exp. | 8.00 | 8.00 | 9.98 | 9.82 | 11.11 | 10.79 | 10.49 | 10.32 | 10.15 | 10.12 | 9.95 | 9.85 | 9.68 |
| | | | s.d. | 1.26 | 1.26 | 2.17 | 2.84 | 2.19 | 2.27 | 2.37 | 2.55 | 2.58 | 2.71 | 2.81 | 2.87 | 2.87 |
| | | | rmse | 2.36 | 2.36 | 2.17 | 2.85 | 2.46 | 2.41 | 2.42 | 2.57 | 2.58 | 2.71 | 2.81 | 2.87 | 2.88 |
| | 5 | 0 | exp. | 8.00 | 10.08 | 10.01 | 9.65 | 10.62 | 10.30 | 10.11 | 9.98 | 9.91 | 9.75 | 9.71 | 9.51 | 9.40 |
| | | | s.d. | 1.26 | 2.96 | 1.91 | 2.04 | 1.82 | 1.84 | 1.92 | 1.97 | 2.03 | 2.05 | 2.05 | 2.13 | 2.04 |
| | | | rmse | 2.37 | 2.96 | 1.91 | 2.07 | 1.93 | 1.86 | 1.92 | 1.97 | 2.04 | 2.07 | 2.07 | 2.19 | 2.13 |
| | 10 | 0 | exp. | 8.00 | 10.09 | 9.98 | 9.57 | 10.46 | 10.15 | 9.99 | 9.86 | 9.80 | 9.63 | 9.64 | 9.46 | 9.33 |
| | | | s.d. | 1.26 | 2.25 | 1.82 | 1.84 | 1.70 | 1.72 | 1.79 | 1.82 | 1.89 | 1.89 | 1.92 | 1.99 | 1.84 |
| | | | rmse | 2.37 | 2.26 | 1.82 | 1.89 | 1.76 | 1.73 | 1.79 | 1.83 | 1.90 | 1.93 | 1.95 | 2.06 | 1.96 |
| 2.30 | 0 | 0 | exp. | 9.00 | 9.00 | 10.00 | 9.56 | 10.89 | 10.51 | 10.17 | 9.98 | 9.80 | 9.69 | 9.59 | 9.47 | 9.44 |
| | | | s.d. | 0.95 | 0.95 | 1.31 | 1.39 | 1.47 | 1.46 | 1.44 | 1.46 | 1.42 | 1.44 | 1.44 | 1.41 | 1.38 |
| | | | rmse | 1.38 | 1.38 | 1.31 | 1.46 | 1.72 | 1.55 | 1.45 | 1.46 | 1.44 | 1.48 | 1.50 | 1.51 | 1.49 |
| | 5 | 0 | exp. | 9.00 | 10.06 | 10.03 | 9.51 | 10.64 | 10.30 | 10.06 | 9.82 | 9.70 | 9.54 | 9.42 | 9.29 | 9.35 |
| | | | s.d. | 0.95 | 1.92 | 1.21 | 1.25 | 1.25 | 1.27 | 1.27 | 1.26 | 1.29 | 1.24 | 1.25 | 1.19 | 1.18 |
| | | | rmse | 1.38 | 1.92 | 1.21 | 1.34 | 1.41 | 1.31 | 1.27 | 1.27 | 1.32 | 1.33 | 1.38 | 1.39 | 1.35 |
| | 10 | 0 | exp. | 9.00 | 10.04 | 10.04 | 9.47 | 10.57 | 10.22 | 9.97 | 9.78 | 9.61 | 9.50 | 9.36 | 9.24 | 9.31 |
| | | | s.d. | 0.95 | 1.46 | 1.15 | 1.21 | 1.21 | 1.20 | 1.20 | 1.22 | 1.23 | 1.22 | 1.20 | 1.15 | 1.15 |
| | | | rmse | 1.38 | 1.46 | 1.15 | 1.32 | 1.34 | 1.22 | 1.20 | 1.24 | 1.29 | 1.32 | 1.36 | 1.38 | 1.34 |

Table 2.2a

| N = 25 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | x10^4 | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.11 | 0 | 8685 | exp. | 3.44 | 3.44 | 5.53 | 6.98 | 6.85 | 6.97 | 6.51 | 6.43 | 7.03 | 6.92 | 7.09 | 7.24 | 6.92 |
| | | | s.d. | 1.48 | 1.48 | 3.78 | 5.71 | 3.16 | 3.45 | 4.05 | 4.26 | 4.73 | 4.87 | 5.19 | 5.35 | 5.57 |
| | | | rmse | 21.61 | 21.61 | 19.83 | 18.90 | 18.43 | 18.36 | 18.92 | 19.06 | 18.58 | 18.72 | 18.65 | 18.55 | 18.91 |
| | 5 | 5011 | exp. | 2.73 | 9.29 | 9.44 | 15.22 | 6.97 | 7.48 | 8.68 | 9.36 | 10.55 | 11.30 | 12.40 | 13.21 | 14.14 |
| | | | s.d. | 1.55 | 7.20 | 6.56 | 11.62 | 5.13 | 5.95 | 6.70 | 7.51 | 8.33 | 8.77 | 9.52 | 10.21 | 11.04 |
| | | | rmse | 22.32 | 17.29 | 16.89 | 15.18 | 18.74 | 18.51 | 17.64 | 17.35 | 16.67 | 16.27 | 15.80 | 15.60 | 15.49 |
| | 10 | 2891 | exp. | 2.66 | 13.75 | 13.73 | 22.13 | 8.13 | 10.12 | 11.33 | 12.76 | 14.13 | 16.03 | 17.31 | 18.81 | 20.45 |
| | | | s.d. | 1.54 | 10.57 | 9.47 | 17.48 | 6.54 | 8.12 | 9.38 | 10.45 | 11.87 | 12.74 | 13.74 | 14.93 | 16.21 |
| | | | rmse | 22.40 | 15.44 | 14.72 | 17.72 | 18.09 | 16.95 | 16.58 | 16.09 | 16.10 | 15.58 | 15.74 | 16.16 | 16.84 |
| | 25 | 555 | exp. | 2.61 | 21.22 | 21.10 | 31.09 | 11.06 | 13.60 | 16.10 | 18.21 | 20.47 | 22.36 | 24.65 | 26.70 | 28.69 |
| | | | s.d. | 1.53 | 16.67 | 15.64 | 28.59 | 10.02 | 12.40 | 14.68 | 16.41 | 18.32 | 20.23 | 22.08 | 24.10 | 26.20 |
| | | | rmse | 22.44 | 17.09 | 16.12 | 29.23 | 17.16 | 16.85 | 17.17 | 17.76 | 18.87 | 20.40 | 22.09 | 24.16 | 26.46 |
| 0.22 | 0 | 5894 | exp. | 5.55 | 5.55 | 11.13 | 15.16 | 11.54 | 12.10 | 12.52 | 12.78 | 13.66 | 13.99 | 14.42 | 14.78 | 14.88 |
| | | | s.d. | 1.92 | 1.92 | 6.22 | 10.30 | 5.22 | 5.84 | 6.79 | 7.42 | 7.79 | 8.39 | 8.95 | 9.37 | 9.91 |
| | | | rmse | 19.54 | 19.54 | 15.20 | 14.25 | 14.44 | 14.16 | 14.21 | 14.29 | 13.76 | 13.84 | 13.86 | 13.86 | 14.16 |
| | 5 | 1962 | exp. | 5.04 | 16.21 | 17.02 | 25.39 | 13.61 | 14.98 | 16.68 | 18.00 | 19.54 | 20.61 | 21.91 | 23.08 | 24.33 |
| | | | s.d. | 1.99 | 10.56 | 9.50 | 17.35 | 7.55 | 8.84 | 10.02 | 11.01 | 12.03 | 13.03 | 14.15 | 15.00 | 16.44 |
| | | | rmse | 20.06 | 13.74 | 12.41 | 17.35 | 13.67 | 13.36 | 13.03 | 13.05 | 13.21 | 13.75 | 14.49 | 15.12 | 16.45 |
| | 10 | 653 | exp. | 4.97 | 21.00 | 21.14 | 29.93 | 15.10 | 17.61 | 19.46 | 21.18 | 22.76 | 24.20 | 25.67 | 27.03 | 28.68 |
| | | | s.d. | 1.99 | 14.18 | 12.11 | 22.40 | 9.19 | 10.93 | 12.28 | 13.80 | 15.27 | 16.54 | 17.82 | 19.34 | 20.86 |
| | | | rmse | 20.13 | 14.73 | 12.71 | 22.93 | 13.50 | 13.20 | 13.48 | 14.32 | 15.43 | 16.56 | 17.83 | 19.45 | 21.18 |
| | 25 | 24 | exp. | 4.94 | 24.75 | 24.71 | 29.41 | 17.02 | 19.25 | 21.24 | 22.73 | 24.07 | 25.33 | 26.44 | 27.56 | 28.55 |
| | | | s.d. | 1.99 | 15.61 | 14.38 | 22.21 | 10.65 | 12.41 | 13.85 | 15.11 | 16.30 | 17.38 | 18.55 | 19.64 | 20.89 |
| | | | rmse | 20.16 | 15.61 | 14.38 | 22.64 | 13.31 | 13.68 | 14.35 | 15.28 | 16.33 | 17.39 | 18.61 | 19.81 | 21.19 |
| 0.36 | 0 | 2690 | exp. | 7.90 | 7.90 | 17.36 | 23.62 | 17.24 | 18.19 | 19.10 | 19.86 | 20.67 | 21.47 | 22.03 | 22.70 | 23.11 |
| | | | s.d. | 2.23 | 2.23 | 8.21 | 14.35 | 7.10 | 8.04 | 8.94 | 9.93 | 10.59 | 11.41 | 12.12 | 12.90 | 13.62 |
| | | | rmse | 17.24 | 17.24 | 11.21 | 14.42 | 10.52 | 10.54 | 10.71 | 11.18 | 11.44 | 11.94 | 12.48 | 13.11 | 13.75 |
| | 5 | 445 | exp. | 7.60 | 21.33 | 22.49 | 29.56 | 19.39 | 21.02 | 22.56 | 23.78 | 24.93 | 26.07 | 27.03 | 27.93 | 28.86 |
| | | | s.d. | 2.29 | 12.92 | 11.05 | 19.81 | 8.91 | 10.33 | 11.58 | 12.72 | 13.88 | 14.99 | 16.22 | 17.28 | 18.65 |
| | | | rmse | 17.55 | 13.43 | 11.33 | 20.32 | 10.52 | 11.08 | 11.83 | 12.78 | 13.88 | 15.03 | 16.35 | 17.53 | 19.04 |
| | 10 | 74 | exp. | 7.57 | 24.24 | 24.39 | 29.54 | 19.99 | 22.12 | 23.34 | 24.57 | 25.52 | 26.40 | 27.31 | 28.04 | 28.95 |
| | | | s.d. | 2.30 | 14.80 | 11.96 | 19.99 | 9.51 | 11.04 | 12.20 | 13.38 | 14.41 | 15.54 | 16.60 | 17.67 | 18.85 |
| | | | rmse | 17.59 | 14.82 | 11.98 | 20.50 | 10.75 | 11.41 | 12.31 | 13.39 | 14.42 | 15.61 | 16.76 | 17.93 | 19.26 |
| | 25 | 0 | exp. | 7.56 | 25.01 | 24.95 | 26.66 | 20.25 | 21.79 | 22.88 | 23.68 | 24.39 | 25.04 | 25.54 | 25.98 | 26.36 |
| | | | s.d. | 2.30 | 11.75 | 10.64 | 12.94 | 8.85 | 9.66 | 10.24 | 10.76 | 11.20 | 11.57 | 11.95 | 12.26 | 12.62 |
| | | | rmse | 17.59 | 11.75 | 10.64 | 13.05 | 10.05 | 10.18 | 10.45 | 10.84 | 11.22 | 11.57 | 11.96 | 12.30 | 12.70 |

Table 2.2b

| N = 25 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | x10^4 | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.51 | 0 | 865 | exp. | 10.13 | 10.13 | 21.78 | 27.85 | 21.70 | 22.73 | 23.55 | 24.35 | 25.03 | 25.81 | 26.30 | 26.96 | 27.30 |
| | | | s.d. | 2.41 | 2.41 | 9.18 | 16.16 | 7.90 | 9.05 | 10.02 | 11.03 | 11.94 | 12.81 | 13.56 | 14.46 | 15.30 |
| | | | rmse | 15.06 | 15.06 | 9.73 | 16.41 | 8.56 | 9.33 | 10.13 | 11.05 | 11.94 | 12.84 | 13.62 | 14.59 | 15.47 |
| | 5 | 68 | exp. | 10.00 | 23.70 | 24.49 | 28.57 | 22.53 | 23.78 | 24.72 | 25.47 | 26.09 | 26.86 | 27.38 | 27.80 | 28.28 |
| | | | s.d. | 2.45 | 13.58 | 10.43 | 16.99 | 8.61 | 9.86 | 10.83 | 11.82 | 12.70 | 13.47 | 14.40 | 15.24 | 16.15 |
| | | | rmse | 15.20 | 13.64 | 10.44 | 17.36 | 8.96 | 9.94 | 10.84 | 11.83 | 12.74 | 13.60 | 14.60 | 15.50 | 16.48 |
| | 10 | 5 | exp. | 9.99 | 24.93 | 24.98 | 27.31 | 22.39 | 23.81 | 24.36 | 25.10 | 25.52 | 25.96 | 26.40 | 26.71 | 27.07 |
| | | | s.d. | 2.45 | 13.00 | 9.83 | 13.79 | 8.30 | 9.29 | 9.96 | 10.61 | 11.16 | 11.69 | 12.27 | 12.73 | 13.27 |
| | | | rmse | 15.21 | 13.00 | 9.83 | 13.98 | 8.69 | 9.37 | 9.98 | 10.61 | 11.17 | 11.73 | 12.34 | 12.85 | 13.43 |
| | 25 | 0 | exp. | 9.99 | 25.04 | 25.00 | 25.62 | 22.08 | 23.06 | 23.72 | 24.10 | 24.54 | 24.82 | 25.13 | 25.36 | 25.53 |
| | | | s.d. | 2.45 | 9.17 | 8.16 | 8.90 | 7.27 | 7.65 | 7.92 | 8.12 | 8.28 | 8.45 | 8.58 | 8.69 | 8.84 |
| | | | rmse | 15.21 | 9.17 | 8.16 | 8.92 | 7.84 | 7.90 | 8.03 | 8.17 | 8.30 | 8.45 | 8.58 | 8.70 | 8.86 |
| 0.69 | 0 | 161 | exp. | 12.50 | 12.50 | 24.20 | 28.04 | 24.50 | 25.16 | 25.66 | 26.12 | 26.61 | 26.94 | 27.24 | 27.65 | 27.71 |
| | | | s.d. | 2.48 | 2.48 | 8.84 | 14.65 | 7.79 | 8.79 | 9.64 | 10.44 | 11.23 | 11.92 | 12.55 | 13.26 | 13.97 |
| | | | rmse | 12.75 | 12.75 | 8.88 | 14.96 | 7.80 | 8.79 | 9.66 | 10.50 | 11.35 | 12.08 | 12.75 | 13.52 | 14.23 |
| | 5 | 5 | exp. | 12.46 | 24.66 | 24.94 | 26.67 | 24.18 | 24.75 | 25.19 | 25.51 | 25.80 | 26.17 | 26.38 | 26.47 | 26.60 |
| | | | s.d. | 2.50 | 12.80 | 8.41 | 11.55 | 7.40 | 8.09 | 8.65 | 9.18 | 9.59 | 9.94 | 10.44 | 10.81 | 11.20 |
| | | | rmse | 12.78 | 12.80 | 8.41 | 11.67 | 7.44 | 8.10 | 8.65 | 9.19 | 9.63 | 10.01 | 10.53 | 10.91 | 11.31 |
| | 10 | 0 | exp. | 12.46 | 25.03 | 25.00 | 25.86 | 23.82 | 24.50 | 24.71 | 25.07 | 25.28 | 25.46 | 25.62 | 25.73 | 25.81 |
| | | | s.d. | 2.50 | 10.52 | 7.50 | 8.86 | 6.76 | 7.29 | 7.54 | 7.80 | 8.03 | 8.17 | 8.41 | 8.56 | 8.74 |
| | | | rmse | 12.79 | 10.52 | 7.50 | 8.91 | 6.86 | 7.31 | 7.54 | 7.80 | 8.03 | 8.18 | 8.43 | 8.59 | 8.78 |
| | 25 | 0 | exp. | 12.46 | 25.01 | 24.99 | 25.11 | 23.37 | 23.87 | 24.22 | 24.45 | 24.66 | 24.76 | 24.91 | 25.04 | 25.09 |
| | | | s.d. | 2.50 | 7.34 | 6.39 | 6.68 | 5.95 | 6.15 | 6.29 | 6.37 | 6.43 | 6.50 | 6.54 | 6.57 | 6.69 |
| | | | rmse | 12.79 | 7.34 | 6.39 | 6.68 | 6.17 | 6.26 | 6.33 | 6.39 | 6.44 | 6.50 | 6.54 | 6.57 | 6.69 |
| 0.92 | 0 | 12 | exp. | 15.04 | 15.04 | 24.92 | 26.41 | 25.72 | 25.83 | 25.98 | 26.06 | 26.27 | 26.22 | 26.33 | 26.43 | 26.33 |
| | | | s.d. | 2.45 | 2.45 | 7.09 | 10.01 | 6.63 | 7.18 | 7.62 | 8.04 | 8.40 | 8.73 | 9.07 | 9.37 | 9.72 |
| | | | rmse | 10.26 | 10.26 | 7.09 | 10.11 | 6.67 | 7.22 | 7.69 | 8.11 | 8.50 | 8.82 | 9.17 | 9.48 | 9.81 |
| | 5 | 0 | exp. | 15.04 | 24.99 | 25.00 | 25.50 | 25.06 | 25.14 | 25.20 | 25.30 | 25.39 | 25.53 | 25.56 | 25.55 | 25.49 |
| | | | s.d. | 2.45 | 10.98 | 6.18 | 7.15 | 5.82 | 6.07 | 6.27 | 6.46 | 6.58 | 6.69 | 6.89 | 6.97 | 7.10 |
| | | | rmse | 10.26 | 10.98 | 6.18 | 7.16 | 5.82 | 6.08 | 6.27 | 6.47 | 6.59 | 6.71 | 6.91 | 6.99 | 7.12 |
| | 10 | 0 | exp. | 15.04 | 25.04 | 25.01 | 25.16 | 24.68 | 24.85 | 24.96 | 25.03 | 25.06 | 25.12 | 25.19 | 25.18 | 25.12 |
| | | | s.d. | 2.45 | 8.22 | 5.63 | 6.06 | 5.30 | 5.54 | 5.63 | 5.74 | 5.77 | 5.84 | 5.92 | 5.98 | 6.09 |
| | | | rmse | 10.26 | 8.22 | 5.63 | 6.06 | 5.31 | 5.54 | 5.63 | 5.74 | 5.77 | 5.84 | 5.92 | 5.98 | 6.09 |
| | 25 | 0 | exp. | 15.04 | 25.01 | 24.99 | 24.84 | 24.25 | 24.42 | 24.56 | 24.65 | 24.71 | 24.77 | 24.79 | 24.85 | 24.75 |
| | | | s.d. | 2.45 | 5.89 | 4.97 | 5.10 | 4.77 | 4.86 | 4.93 | 4.96 | 4.99 | 5.01 | 5.01 | 5.07 | 5.11 |
| | | | rmse | 10.26 | 5.89 | 4.97 | 5.10 | 4.83 | 4.89 | 4.95 | 4.97 | 5.00 | 5.01 | 5.01 | 5.08 | 5.12 |

Table 2.2c

| N = 25 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | x10^4 | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 1.20 | 0 | 0 | exp. | 17.47 | 17.47 | 24.99 | 25.29 | 25.99 | 25.82 | 25.71 | 25.60 | 25.61 | 25.47 | 25.45 | 25.41 | 25.27 |
| | | | s.d. | 2.29 | 2.29 | 5.09 | 5.94 | 5.01 | 5.22 | 5.33 | 5.46 | 5.51 | 5.61 | 5.74 | 5.81 | 5.94 |
| | | | rmse | 7.87 | 7.87 | 5.09 | 5.95 | 5.11 | 5.28 | 5.38 | 5.50 | 5.55 | 5.63 | 5.76 | 5.82 | 5.95 |
| | 5 | 0 | exp. | 17.47 | 25.06 | 24.99 | 24.96 | 25.42 | 25.30 | 25.20 | 25.16 | 25.15 | 25.16 | 25.10 | 25.03 | 24.91 |
| | | | s.d. | 2.29 | 8.86 | 4.52 | 4.81 | 4.42 | 4.50 | 4.56 | 4.62 | 4.64 | 4.68 | 4.76 | 4.83 | 4.82 |
| | | | rmse | 7.87 | 8.86 | 4.52 | 4.81 | 4.44 | 4.51 | 4.56 | 4.63 | 4.65 | 4.68 | 4.76 | 4.83 | 4.82 |
| | 10 | 0 | exp. | 17.47 | 25.04 | 25.01 | 24.81 | 25.11 | 25.08 | 25.06 | 25.02 | 24.98 | 24.96 | 24.91 | 24.86 | 24.73 |
| | | | s.d. | 2.29 | 6.43 | 4.25 | 4.39 | 4.12 | 4.21 | 4.22 | 4.28 | 4.27 | 4.31 | 4.36 | 4.43 | 4.42 |
| | | | rmse | 7.87 | 6.43 | 4.25 | 4.39 | 4.12 | 4.21 | 4.22 | 4.28 | 4.27 | 4.31 | 4.36 | 4.43 | 4.43 |
| | 25 | 0 | exp. | 17.47 | 25.02 | 25.01 | 24.67 | 24.79 | 24.74 | 24.73 | 24.74 | 24.75 | 24.75 | 24.76 | 24.66 | 24.54 |
| | | | s.d. | 2.29 | 4.71 | 3.86 | 3.91 | 3.79 | 3.81 | 3.84 | 3.85 | 3.85 | 3.84 | 3.90 | 3.97 | 3.89 |
| | | | rmse | 7.87 | 4.71 | 3.86 | 3.92 | 3.80 | 3.82 | 3.85 | 3.86 | 3.86 | 3.84 | 3.91 | 3.98 | 3.92 |
| 1.61 | 0 | 0 | exp. | 20.00 | 20.00 | 24.99 | 24.75 | 25.90 | 25.59 | 25.38 | 25.20 | 25.15 | 25.06 | 24.92 | 24.83 | 24.66 |
| | | | s.d. | 2.00 | 2.00 | 3.37 | 3.53 | 3.39 | 3.46 | 3.46 | 3.47 | 3.48 | 3.53 | 3.58 | 3.57 | 3.59 |
| | | | rmse | 5.38 | 5.38 | 3.37 | 3.54 | 3.51 | 3.51 | 3.48 | 3.48 | 3.48 | 3.53 | 3.58 | 3.58 | 3.61 |
| | 5 | 0 | exp. | 20.00 | 25.04 | 24.97 | 24.71 | 25.56 | 25.30 | 25.21 | 25.05 | 24.96 | 24.90 | 24.79 | 24.68 | 24.55 |
| | | | s.d. | 2.00 | 6.56 | 3.14 | 3.22 | 3.10 | 3.14 | 3.16 | 3.19 | 3.19 | 3.20 | 3.28 | 3.26 | 3.28 |
| | | | rmse | 5.38 | 6.56 | 3.14 | 3.24 | 3.15 | 3.15 | 3.17 | 3.19 | 3.19 | 3.20 | 3.29 | 3.27 | 3.31 |
| | 10 | 0 | exp. | 20.00 | 25.01 | 25.03 | 24.63 | 25.36 | 25.12 | 25.08 | 24.99 | 24.86 | 24.81 | 24.69 | 24.58 | 24.48 |
| | | | s.d. | 2.00 | 4.80 | 3.02 | 3.06 | 2.96 | 3.02 | 2.98 | 2.99 | 3.03 | 3.04 | 3.12 | 3.08 | 3.10 |
| | | | rmse | 5.38 | 4.80 | 3.02 | 3.08 | 2.98 | 3.03 | 2.98 | 2.99 | 3.03 | 3.05 | 3.14 | 3.11 | 3.15 |
| | 25 | 0 | exp. | 20.00 | 25.03 | 25.01 | 24.58 | 25.07 | 24.92 | 24.81 | 24.87 | 24.79 | 24.69 | 24.60 | 24.47 | 24.41 |
| | | | s.d. | 2.00 | 3.56 | 2.82 | 2.85 | 2.80 | 2.79 | 2.77 | 2.76 | 2.86 | 2.88 | 2.89 | 2.85 | 2.92 |
| | | | rmse | 5.38 | 3.56 | 2.82 | 2.88 | 2.80 | 2.79 | 2.77 | 2.76 | 2.87 | 2.89 | 2.92 | 2.90 | 2.98 |
| 2.30 | 0 | 0 | exp. | 22.49 | 22.49 | 25.00 | 24.56 | 25.73 | 25.33 | 25.10 | 24.91 | 24.77 | 24.71 | 24.57 | 24.47 | 24.32 |
| | | | s.d. | 1.50 | 1.50 | 2.00 | 2.02 | 1.94 | 2.00 | 2.06 | 2.07 | 2.08 | 2.08 | 2.05 | 2.07 | 2.03 |
| | | | rmse | 2.92 | 2.92 | 2.00 | 2.07 | 2.08 | 2.02 | 2.07 | 2.07 | 2.09 | 2.10 | 2.10 | 2.14 | 2.14 |
| | 5 | 0 | exp. | 22.49 | 25.03 | 24.99 | 24.55 | 25.61 | 25.19 | 25.01 | 24.87 | 24.66 | 24.67 | 24.54 | 24.44 | 24.28 |
| | | | s.d. | 1.50 | 4.27 | 1.93 | 1.94 | 1.85 | 1.90 | 1.98 | 2.00 | 1.99 | 1.96 | 1.96 | 1.99 | 1.93 |
| | | | rmse | 2.92 | 4.27 | 1.93 | 1.99 | 1.95 | 1.91 | 1.98 | 2.00 | 2.02 | 1.99 | 2.01 | 2.07 | 2.06 |
| | 10 | 0 | exp. | 22.49 | 24.93 | 25.00 | 24.54 | 25.51 | 25.12 | 24.94 | 24.86 | 24.58 | 24.65 | 24.53 | 24.41 | 24.24 |
| | | | s.d. | 1.50 | 3.16 | 1.88 | 1.89 | 1.80 | 1.85 | 1.92 | 1.97 | 1.95 | 1.89 | 1.89 | 1.95 | 1.87 |
| | | | rmse | 2.92 | 3.16 | 1.88 | 1.94 | 1.87 | 1.85 | 1.92 | 1.97 | 2.00 | 1.92 | 1.95 | 2.03 | 2.02 |
| | 25 | 0 | exp. | 22.49 | 25.05 | 25.00 | 24.53 | 25.40 | 25.05 | 24.81 | 24.79 | 24.54 | 24.59 | 24.51 | 24.37 | 24.21 |
| | | | s.d. | 1.50 | 2.29 | 1.81 | 1.83 | 1.72 | 1.79 | 1.84 | 1.90 | 1.90 | 1.78 | 1.81 | 1.88 | 1.78 |
| | | | rmse | 2.92 | 2.29 | 1.81 | 1.89 | 1.76 | 1.79 | 1.85 | 1.91 | 1.95 | 1.82 | 1.87 | 1.98 | 1.95 |

Table 2.3a

| N = 50 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | x10^4 | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.11 | 0 | 7543 | exp. | 5.99 | 5.99 | 13.22 | 19.04 | 13.23 | 14.07 | 14.85 | 15.38 | 16.50 | 17.04 | 17.75 | 18.28 | 18.59 |
| | | | s.d. | 2.12 | 2.12 | 7.72 | 13.20 | 6.50 | 7.34 | 8.41 | 9.32 | 9.80 | 10.60 | 11.24 | 11.91 | 12.60 |
| | | | rmse | 44.06 | 44.06 | 37.58 | 33.65 | 37.34 | 36.67 | 36.15 | 35.85 | 34.91 | 34.62 | 34.15 | 33.88 | 33.84 |
| | 5 | 4352 | exp. | 5.40 | 19.87 | 21.55 | 35.05 | 16.80 | 18.90 | 21.40 | 23.33 | 25.62 | 27.30 | 29.40 | 31.14 | 33.33 |
| | | | s.d. | 2.18 | 12.58 | 12.00 | 22.13 | 9.62 | 11.28 | 12.85 | 14.07 | 15.39 | 16.72 | 18.11 | 19.29 | 20.93 |
| | | | rmse | 44.65 | 32.65 | 30.88 | 26.71 | 34.56 | 33.09 | 31.35 | 30.15 | 28.83 | 28.19 | 27.42 | 26.98 | 26.76 |
| | 10 | 2511 | exp. | 5.29 | 28.72 | 29.32 | 47.41 | 20.39 | 24.48 | 27.54 | 30.60 | 33.54 | 36.30 | 38.90 | 41.67 | 44.70 |
| | | | s.d. | 2.17 | 18.74 | 16.11 | 31.32 | 12.50 | 14.89 | 16.93 | 19.09 | 21.12 | 22.92 | 24.87 | 27.07 | 29.15 |
| | | | rmse | 44.76 | 28.36 | 26.21 | 31.42 | 32.14 | 29.55 | 28.13 | 27.22 | 26.78 | 26.71 | 27.24 | 28.32 | 29.63 |
| | 25 | 482 | exp. | 5.22 | 43.06 | 43.06 | 62.98 | 28.07 | 33.47 | 38.23 | 42.10 | 45.86 | 49.21 | 52.77 | 56.12 | 59.50 |
| | | | s.d. | 2.16 | 28.66 | 25.75 | 49.31 | 18.60 | 22.71 | 26.06 | 29.22 | 32.37 | 35.49 | 38.71 | 42.05 | 45.50 |
| | | | rmse | 44.83 | 29.48 | 26.66 | 50.99 | 28.75 | 28.09 | 28.60 | 30.27 | 32.63 | 35.50 | 38.81 | 42.49 | 46.48 |
| | 50 | 31 | exp. | 5.21 | 49.27 | 49.19 | 60.97 | 31.54 | 37.16 | 41.22 | 44.75 | 48.06 | 50.99 | 53.62 | 56.13 | 58.63 |
| | | | s.d. | 2.16 | 31.66 | 30.17 | 48.60 | 21.73 | 25.82 | 29.08 | 31.88 | 34.70 | 37.34 | 40.04 | 42.80 | 45.60 |
| | | | rmse | 44.84 | 31.67 | 30.19 | 49.82 | 28.52 | 28.84 | 30.38 | 32.31 | 34.75 | 37.35 | 40.21 | 43.24 | 46.41 |
| 0.22 | 0 | 3473 | exp. | 10.33 | 10.33 | 28.86 | 42.89 | 27.12 | 29.51 | 31.61 | 33.58 | 35.34 | 37.12 | 38.51 | 40.16 | 41.46 |
| | | | s.d. | 2.76 | 2.76 | 13.75 | 25.40 | 11.57 | 13.52 | 15.17 | 16.75 | 18.26 | 19.62 | 20.96 | 22.51 | 23.98 |
| | | | rmse | 39.77 | 39.77 | 25.22 | 26.38 | 25.64 | 24.55 | 23.84 | 23.45 | 23.41 | 23.47 | 23.90 | 24.57 | 25.45 |
| | 5 | 1156 | exp. | 9.99 | 34.38 | 38.98 | 56.90 | 32.84 | 36.65 | 39.92 | 42.77 | 45.37 | 47.87 | 50.27 | 52.49 | 54.87 |
| | | | s.d. | 2.81 | 19.49 | 18.64 | 36.17 | 14.93 | 17.66 | 20.06 | 22.30 | 24.57 | 26.77 | 28.99 | 31.33 | 33.73 |
| | | | rmse | 40.11 | 24.98 | 21.65 | 36.82 | 22.75 | 22.13 | 22.45 | 23.44 | 25.00 | 26.85 | 28.99 | 31.43 | 34.08 |
| | 10 | 385 | exp. | 9.91 | 43.51 | 44.77 | 61.82 | 36.32 | 41.06 | 44.39 | 47.46 | 50.02 | 52.57 | 55.07 | 57.27 | 59.72 |
| | | | s.d. | 2.81 | 26.96 | 22.20 | 42.47 | 17.48 | 20.74 | 23.64 | 26.22 | 28.77 | 31.40 | 34.08 | 36.82 | 39.56 |
| | | | rmse | 40.19 | 27.73 | 22.80 | 44.08 | 22.20 | 22.58 | 24.29 | 26.34 | 28.77 | 31.50 | 34.45 | 37.53 | 40.74 |
| | 25 | 14 | exp. | 9.88 | 49.62 | 49.60 | 58.41 | 39.76 | 43.80 | 46.65 | 49.02 | 51.06 | 52.79 | 54.39 | 55.78 | 57.22 |
| | | | s.d. | 2.81 | 27.85 | 24.19 | 37.70 | 19.14 | 22.09 | 24.34 | 26.40 | 28.33 | 30.16 | 32.04 | 33.86 | 35.79 |
| | | | rmse | 40.22 | 27.85 | 24.20 | 38.63 | 21.71 | 22.95 | 24.57 | 26.42 | 28.35 | 30.29 | 32.34 | 34.35 | 36.51 |
| | 50 | 0 | exp. | 9.87 | 50.02 | 49.98 | 53.56 | 40.21 | 43.56 | 45.70 | 47.43 | 48.90 | 50.15 | 51.07 | 52.08 | 52.87 |
| | | | s.d. | 2.81 | 21.76 | 20.40 | 24.10 | 17.25 | 18.75 | 19.84 | 20.59 | 21.32 | 21.99 | 22.60 | 23.12 | 23.63 |
| | | | rmse | 40.22 | 21.76 | 20.40 | 24.37 | 19.83 | 19.82 | 20.30 | 20.75 | 21.35 | 21.99 | 22.63 | 23.21 | 23.80 |
| 0.36 | 0 | 724 | exp. | 15.26 | 15.26 | 43.30 | 58.29 | 41.09 | 44.11 | 46.53 | 48.55 | 50.52 | 52.17 | 53.78 | 55.46 | 56.80 |
| | | | s.d. | 3.21 | 3.21 | 18.44 | 34.74 | 15.36 | 18.07 | 20.53 | 22.54 | 24.61 | 26.55 | 28.59 | 30.64 | 32.77 |
| | | | rmse | 34.89 | 34.89 | 19.62 | 35.71 | 17.76 | 19.00 | 20.82 | 22.58 | 24.62 | 26.64 | 28.84 | 31.12 | 33.47 |
| | 5 | 120 | exp. | 15.14 | 43.84 | 48.16 | 59.82 | 43.88 | 47.09 | 49.37 | 51.37 | 53.03 | 54.61 | 56.15 | 57.36 | 58.67 |
| | | | s.d. | 3.24 | 24.55 | 20.90 | 37.23 | 17.10 | 20.02 | 22.44 | 24.62 | 26.75 | 28.77 | 30.79 | 32.91 | 35.08 |
| | | | rmse | 35.01 | 25.31 | 20.98 | 38.50 | 18.16 | 20.23 | 22.45 | 24.66 | 26.92 | 29.14 | 31.39 | 33.72 | 36.14 |
| | 10 | 20 | exp. | 15.12 | 48.93 | 49.57 | 57.50 | 44.62 | 47.71 | 49.53 | 51.24 | 52.53 | 53.61 | 54.80 | 55.80 | 56.71 |
| | | | s.d. | 3.25 | 28.16 | 20.78 | 33.01 | 17.18 | 19.61 | 21.67 | 23.42 | 25.06 | 26.69 | 28.28 | 29.83 | 31.44 |
| | | | rmse | 35.03 | 28.18 | 20.79 | 33.85 | 18.01 | 19.74 | 21.67 | 23.45 | 25.18 | 26.93 | 28.68 | 30.39 | 32.14 |
| | 25 | 0 | exp. | 15.12 | 50.02 | 49.98 | 53.12 | 44.76 | 47.02 | 48.38 | 49.43 | 50.32 | 51.08 | 51.74 | 52.24 | 52.80 |
| | | | s.d. | 3.25 | 20.61 | 17.20 | 20.59 | 15.10 | 16.27 | 17.08 | 17.82 | 18.36 | 18.86 | 19.38 | 19.76 | 20.19 |
| | | | rmse | 35.03 | 20.61 | 17.20 | 20.83 | 15.98 | 16.55 | 17.15 | 17.83 | 18.36 | 18.89 | 19.46 | 19.89 | 20.38 |
| | 50 | 0 | exp. | 15.12 | 50.03 | 50.03 | 51.36 | 44.61 | 46.48 | 47.59 | 48.47 | 49.19 | 49.79 | 50.29 | 50.71 | 51.06 |
| | | | s.d. | 3.25 | 15.81 | 14.50 | 15.49 | 13.28 | 13.86 | 14.21 | 14.46 | 14.72 | 14.91 | 15.06 | 15.23 | 15.34 |
| | | | rmse | 35.03 | 15.81 | 14.50 | 15.55 | 14.33 | 14.30 | 14.41 | 14.54 | 14.74 | 14.91 | 15.06 | 15.24 | 15.38 |

Table 2.3b

| N = 50 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | x10^4 | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.51 | 0 | 75 | exp. | 20.00 | 20.00 | 48.94 | 57.49 | 47.97 | 49.94 | 51.45 | 52.56 | 53.59 | 54.53 | 55.32 | 56.11 | 56.84 |
| | | | s.d. | 3.45 | 3.45 | 18.32 | 30.95 | 15.70 | 18.07 | 20.04 | 21.70 | 23.29 | 24.86 | 26.32 | 27.88 | 29.40 |
| | | | rmse | 30.20 | 30.20 | 18.35 | 31.85 | 15.83 | 18.07 | 20.10 | 21.85 | 23.57 | 25.27 | 26.86 | 28.54 | 30.18 |
| | 5 | 6 | exp. | 19.98 | 47.68 | 49.87 | 54.91 | 48.02 | 49.68 | 50.73 | 51.63 | 52.34 | 53.05 | 53.67 | 54.09 | 54.53 |
| | | | s.d. | 3.46 | 25.92 | 17.26 | 24.98 | 15.04 | 16.83 | 18.18 | 19.34 | 20.35 | 21.28 | 22.18 | 23.14 | 24.09 |
| | | | rmse | 30.22 | 26.02 | 17.27 | 25.46 | 15.17 | 16.84 | 18.20 | 19.41 | 20.49 | 21.50 | 22.48 | 23.50 | 24.51 |
| | 10 | 0 | exp. | 19.98 | 49.87 | 50.02 | 53.17 | 47.68 | 49.21 | 50.07 | 50.80 | 51.31 | 51.82 | 52.28 | 52.66 | 52.99 |
| | | | s.d. | 3.46 | 24.39 | 15.65 | 19.74 | 13.97 | 15.09 | 16.00 | 16.69 | 17.29 | 17.85 | 18.37 | 18.84 | 19.29 |
| | | | rmse | 30.22 | 24.39 | 15.65 | 19.99 | 14.16 | 15.11 | 16.00 | 16.70 | 17.34 | 17.95 | 18.51 | 19.03 | 19.52 |
| | 25 | 0 | exp. | 19.98 | 50.03 | 50.03 | 51.38 | 47.19 | 48.39 | 49.08 | 49.63 | 50.06 | 50.43 | 50.71 | 50.95 | 51.28 |
| | | | s.d. | 3.46 | 15.97 | 12.89 | 13.98 | 11.97 | 12.46 | 12.79 | 13.05 | 13.28 | 13.46 | 13.62 | 13.76 | 13.88 |
| | | | rmse | 30.22 | 15.97 | 12.89 | 14.05 | 12.30 | 12.57 | 12.83 | 13.06 | 13.28 | 13.46 | 13.64 | 13.79 | 13.94 |
| | 50 | 0 | exp. | 19.98 | 50.01 | 50.00 | 50.51 | 46.82 | 47.92 | 48.55 | 49.04 | 49.41 | 49.72 | 49.99 | 50.23 | 50.44 |
| | | | s.d. | 3.46 | 12.57 | 11.25 | 11.66 | 10.67 | 10.94 | 11.13 | 11.23 | 11.33 | 11.40 | 11.49 | 11.56 | 11.60 |
| | | | rmse | 30.22 | 12.57 | 11.25 | 11.67 | 11.13 | 11.14 | 11.22 | 11.27 | 11.35 | 11.41 | 11.49 | 11.56 | 11.61 |
| 0.69 | 0 | 3 | exp. | 24.92 | 24.92 | 49.94 | 53.20 | 50.20 | 50.96 | 51.50 | 51.87 | 52.15 | 52.51 | 52.65 | 52.88 | 53.10 |
| | | | s.d. | 3.53 | 3.53 | 14.12 | 19.02 | 12.96 | 14.12 | 14.97 | 15.69 | 16.31 | 16.94 | 17.43 | 17.96 | 18.50 |
| | | | rmse | 25.33 | 25.33 | 14.12 | 19.29 | 12.96 | 14.16 | 15.04 | 15.80 | 16.45 | 17.12 | 17.63 | 18.19 | 18.76 |
| | 5 | 0 | exp. | 24.92 | 49.28 | 50.01 | 51.90 | 49.60 | 50.31 | 50.62 | 50.91 | 51.14 | 51.38 | 51.57 | 51.74 | 51.84 |
| | | | s.d. | 3.54 | 24.49 | 12.51 | 14.67 | 11.68 | 12.39 | 12.88 | 13.27 | 13.56 | 13.82 | 14.07 | 14.32 | 14.50 |
| | | | rmse | 25.33 | 24.50 | 12.51 | 14.80 | 11.69 | 12.39 | 12.90 | 13.30 | 13.61 | 13.89 | 14.16 | 14.42 | 14.61 |
| | 10 | 0 | exp. | 24.92 | 50.02 | 49.99 | 51.25 | 49.19 | 49.85 | 50.11 | 50.45 | 50.68 | 50.88 | 51.02 | 51.13 | 51.25 |
| | | | s.d. | 3.54 | 19.63 | 11.47 | 12.67 | 10.82 | 11.29 | 11.62 | 11.85 | 12.01 | 12.20 | 12.34 | 12.46 | 12.56 |
| | | | rmse | 25.33 | 19.63 | 11.47 | 12.73 | 10.85 | 11.29 | 11.62 | 11.85 | 12.03 | 12.24 | 12.38 | 12.51 | 12.62 |
| | 25 | 0 | exp. | 24.92 | 50.02 | 50.02 | 50.52 | 48.58 | 49.16 | 49.53 | 49.75 | 49.94 | 50.12 | 50.24 | 50.36 | 50.48 |
| | | | s.d. | 3.54 | 12.77 | 9.90 | 10.35 | 9.51 | 9.71 | 9.90 | 9.99 | 10.07 | 10.14 | 10.21 | 10.27 | 10.32 |
| | | | rmse | 25.33 | 12.77 | 9.90 | 10.36 | 9.61 | 9.75 | 9.91 | 9.99 | 10.07 | 10.14 | 10.22 | 10.28 | 10.33 |
| | 50 | 0 | exp. | 24.92 | 50.00 | 50.00 | 50.06 | 48.20 | 48.78 | 49.13 | 49.35 | 49.57 | 49.72 | 49.85 | 49.94 | 50.05 |
| | | | s.d. | 3.54 | 10.20 | 8.90 | 9.07 | 8.59 | 8.73 | 8.82 | 8.86 | 8.91 | 8.95 | 8.99 | 9.04 | 9.10 |
| | | | rmse | 25.33 | 10.20 | 8.90 | 9.07 | 8.78 | 8.82 | 8.86 | 8.88 | 8.93 | 8.95 | 9.00 | 9.04 | 9.10 |
| 0.92 | 0 | 0 | exp. | 30.07 | 30.07 | 50.00 | 51.02 | 50.80 | 50.89 | 50.85 | 50.96 | 50.98 | 51.06 | 51.03 | 51.07 | 51.03 |
| | | | s.d. | 3.46 | 3.46 | 9.78 | 10.89 | 9.51 | 9.89 | 10.10 | 10.30 | 10.43 | 10.55 | 10.64 | 10.73 | 10.85 |
| | | | rmse | 20.22 | 20.22 | 9.78 | 10.94 | 9.54 | 9.93 | 10.14 | 10.35 | 10.47 | 10.60 | 10.69 | 10.78 | 10.90 |
| | 5 | 0 | exp. | 30.07 | 49.89 | 50.00 | 50.59 | 50.27 | 50.40 | 50.48 | 50.46 | 50.51 | 50.59 | 50.60 | 50.67 | 50.60 |
| | | | s.d. | 3.46 | 21.09 | 8.95 | 9.56 | 8.70 | 8.93 | 9.10 | 9.20 | 9.30 | 9.38 | 9.45 | 9.48 | 9.57 |
| | | | rmse | 20.22 | 21.09 | 8.95 | 9.58 | 8.70 | 8.94 | 9.12 | 9.22 | 9.32 | 9.40 | 9.47 | 9.51 | 9.59 |
| | 10 | 0 | exp. | 30.07 | 50.04 | 50.00 | 50.35 | 49.96 | 50.14 | 50.13 | 50.28 | 50.31 | 50.32 | 50.34 | 50.40 | 50.33 |
| | | | s.d. | 3.46 | 15.31 | 8.44 | 8.85 | 8.18 | 8.45 | 8.49 | 8.60 | 8.63 | 8.70 | 8.75 | 8.79 | 8.85 |
| | | | rmse | 20.22 | 15.31 | 8.44 | 8.86 | 8.18 | 8.45 | 8.50 | 8.60 | 8.64 | 8.70 | 8.76 | 8.80 | 8.86 |
| | 25 | 0 | exp. | 30.07 | 50.00 | 50.00 | 50.02 | 49.46 | 49.63 | 49.75 | 49.84 | 49.89 | 49.93 | 49.98 | 50.07 | 49.97 |
| | | | s.d. | 3.46 | 10.22 | 7.58 | 7.75 | 7.39 | 7.50 | 7.57 | 7.62 | 7.65 | 7.68 | 7.73 | 7.73 | 7.80 |
| | | | rmse | 20.22 | 10.22 | 7.58 | 7.75 | 7.41 | 7.51 | 7.57 | 7.62 | 7.65 | 7.68 | 7.73 | 7.73 | 7.80 |
| | 50 | 0 | exp. | 30.07 | 50.03 | 49.99 | 49.82 | 49.12 | 49.34 | 49.49 | 49.60 | 49.69 | 49.70 | 49.74 | 49.84 | 49.71 |
| | | | s.d. | 3.46 | 8.23 | 6.97 | 7.05 | 6.82 | 6.90 | 6.92 | 6.95 | 6.98 | 7.01 | 7.00 | 7.05 | 7.07 |
| | | | rmse | 20.22 | 8.23 | 6.97 | 7.05 | 6.88 | 6.93 | 6.94 | 6.96 | 6.99 | 7.02 | 7.01 | 7.05 | 7.07 |

Table 2.3c

| N = 50 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | 1-P(C) | | X1 | P | CUE | MLE | HME | | | | | | | | |
| | | x10^4 | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| | | | | | | | | | | | | | | | | |
| 1.20 | 0 | 0 | exp. | 34.94 | 34.94 | 50.00 | 50.15 | 50.88 | 50.71 | 50.53 | 50.48 | 50.45 | 50.40 | 50.36 | 50.29 | 50.12 |
| | | | s.d. | 3.24 | 3.24 | 6.91 | 7.23 | 6.88 | 7.01 | 7.03 | 7.10 | 7.16 | 7.17 | 7.18 | 7.23 | 7.27 |
| | | | rmse | 15.41 | 15.41 | 6.91 | 7.23 | 6.94 | 7.05 | 7.05 | 7.12 | 7.17 | 7.18 | 7.19 | 7.23 | 7.27 |
| | 5 | 0 | exp. | 34.94 | 50.04 | 50.01 | 50.00 | 50.57 | 50.41 | 50.35 | 50.25 | 50.21 | 50.18 | 50.18 | 50.11 | 49.95 |
| | | | s.d. | 3.24 | 17.02 | 6.52 | 6.74 | 6.48 | 6.52 | 6.61 | 6.63 | 6.67 | 6.71 | 6.71 | 6.75 | 6.78 |
| | | | rmse | 15.41 | 17.02 | 6.52 | 6.74 | 6.51 | 6.54 | 6.62 | 6.63 | 6.68 | 6.71 | 6.71 | 6.75 | 6.78 |
| | 10 | 0 | exp. | 34.94 | 50.04 | 50.01 | 49.93 | 50.34 | 50.22 | 50.15 | 50.12 | 50.05 | 50.06 | 50.06 | 50.02 | 49.84 |
| | | | s.d. | 3.24 | 11.96 | 6.27 | 6.42 | 6.21 | 6.27 | 6.29 | 6.33 | 6.36 | 6.39 | 6.39 | 6.43 | 6.46 |
| | | | rmse | 15.41 | 11.96 | 6.27 | 6.42 | 6.22 | 6.27 | 6.29 | 6.33 | 6.36 | 6.39 | 6.39 | 6.43 | 6.47 |
| | 25 | 0 | exp. | 34.94 | 50.03 | 50.01 | 49.77 | 49.95 | 49.91 | 49.88 | 49.87 | 49.85 | 49.86 | 49.90 | 49.81 | 49.66 |
| | | | s.d. | 3.24 | 8.17 | 5.79 | 5.87 | 5.70 | 5.77 | 5.80 | 5.81 | 5.85 | 5.85 | 5.86 | 5.91 | 5.89 |
| | | | rmse | 15.41 | 8.17 | 5.79 | 5.87 | 5.70 | 5.77 | 5.80 | 5.82 | 5.86 | 5.85 | 5.86 | 5.91 | 5.90 |
| | 50 | 0 | exp. | 34.94 | 50.04 | 50.00 | 49.67 | 49.67 | 49.67 | 49.70 | 49.69 | 49.68 | 49.75 | 49.78 | 49.65 | 49.55 |
| | | | s.d. | 3.24 | 6.62 | 5.42 | 5.45 | 5.36 | 5.40 | 5.39 | 5.41 | 5.47 | 5.44 | 5.47 | 5.49 | 5.50 |
| | | | rmse | 15.41 | 6.62 | 5.42 | 5.46 | 5.37 | 5.41 | 5.40 | 5.42 | 5.48 | 5.45 | 5.48 | 5.50 | 5.52 |
| 1.61 | 0 | 0 | exp. | 40.01 | 40.01 | 50.01 | 49.74 | 50.79 | 50.54 | 50.30 | 50.19 | 50.14 | 50.06 | 49.93 | 49.78 | 49.63 |
| | | | s.d. | 2.83 | 2.83 | 4.66 | 4.75 | 4.68 | 4.71 | 4.74 | 4.76 | 4.75 | 4.77 | 4.78 | 4.78 | 4.77 |
| | | | rmse | 10.39 | 10.39 | 4.66 | 4.76 | 4.75 | 4.74 | 4.75 | 4.76 | 4.75 | 4.77 | 4.78 | 4.79 | 4.78 |
| | 5 | 0 | exp. | 40.01 | 50.05 | 49.99 | 49.70 | 50.61 | 50.32 | 50.18 | 50.10 | 50.06 | 49.96 | 49.85 | 49.71 | 49.55 |
| | | | s.d. | 2.83 | 12.57 | 4.51 | 4.55 | 4.52 | 4.52 | 4.58 | 4.56 | 4.58 | 4.59 | 4.60 | 4.60 | 4.59 |
| | | | rmse | 10.39 | 12.57 | 4.51 | 4.56 | 4.56 | 4.53 | 4.58 | 4.56 | 4.58 | 4.59 | 4.60 | 4.61 | 4.61 |
| | 10 | 0 | exp. | 40.01 | 50.02 | 49.99 | 49.68 | 50.47 | 50.23 | 50.11 | 50.03 | 50.00 | 49.92 | 49.79 | 49.65 | 49.51 |
| | | | s.d. | 2.83 | 8.90 | 4.39 | 4.42 | 4.38 | 4.42 | 4.42 | 4.44 | 4.45 | 4.47 | 4.48 | 4.46 | 4.44 |
| | | | rmse | 10.39 | 8.90 | 4.39 | 4.43 | 4.40 | 4.43 | 4.42 | 4.44 | 4.45 | 4.47 | 4.48 | 4.47 | 4.46 |
| | 25 | 0 | exp. | 40.01 | 50.00 | 50.00 | 49.61 | 50.27 | 50.06 | 49.94 | 49.91 | 49.90 | 49.82 | 49.68 | 49.57 | 49.43 |
| | | | s.d. | 2.83 | 6.19 | 4.16 | 4.19 | 4.13 | 4.15 | 4.21 | 4.21 | 4.20 | 4.22 | 4.22 | 4.18 | 4.18 |
| | | | rmse | 10.39 | 6.19 | 4.16 | 4.21 | 4.14 | 4.15 | 4.21 | 4.21 | 4.20 | 4.23 | 4.24 | 4.21 | 4.22 |
| | 50 | 0 | exp. | 40.01 | 50.01 | 50.00 | 49.57 | 50.03 | 49.90 | 49.81 | 49.84 | 49.83 | 49.70 | 49.59 | 49.52 | 49.36 |
| | | | s.d. | 2.83 | 5.04 | 3.97 | 3.99 | 3.97 | 3.96 | 4.02 | 4.01 | 4.01 | 4.01 | 4.03 | 3.97 | 3.94 |
| | | | rmse | 10.39 | 5.04 | 3.97 | 4.02 | 3.97 | 3.96 | 4.02 | 4.01 | 4.02 | 4.02 | 4.05 | 4.00 | 3.99 |
| 2.30 | 0 | 0 | exp. | 44.99 | 44.99 | 50.01 | 49.57 | 50.65 | 50.36 | 50.13 | 49.98 | 49.81 | 49.67 | 49.56 | 49.47 | 49.39 |
| | | | s.d. | 2.12 | 2.12 | 2.77 | 2.79 | 2.85 | 2.86 | 2.84 | 2.81 | 2.83 | 2.81 | 2.83 | 2.79 | 2.87 |
| | | | rmse | 5.44 | 5.44 | 2.77 | 2.83 | 2.93 | 2.89 | 2.85 | 2.81 | 2.84 | 2.83 | 2.87 | 2.84 | 2.93 |
| | 5 | 0 | exp. | 44.99 | 50.01 | 50.00 | 49.56 | 50.60 | 50.30 | 50.04 | 49.92 | 49.78 | 49.64 | 49.53 | 49.45 | 49.37 |
| | | | s.d. | 2.12 | 8.09 | 2.73 | 2.74 | 2.82 | 2.79 | 2.78 | 2.76 | 2.77 | 2.76 | 2.78 | 2.72 | 2.82 |
| | | | rmse | 5.44 | 8.09 | 2.73 | 2.77 | 2.88 | 2.80 | 2.78 | 2.76 | 2.78 | 2.79 | 2.82 | 2.78 | 2.89 |
| | 10 | 0 | exp. | 44.99 | 50.05 | 50.00 | 49.55 | 50.54 | 50.24 | 50.00 | 49.86 | 49.76 | 49.63 | 49.51 | 49.43 | 49.35 |
| | | | s.d. | 2.12 | 5.88 | 2.69 | 2.70 | 2.79 | 2.74 | 2.74 | 2.72 | 2.72 | 2.72 | 2.75 | 2.67 | 2.77 |
| | | | rmse | 5.44 | 5.88 | 2.69 | 2.74 | 2.84 | 2.75 | 2.74 | 2.73 | 2.73 | 2.74 | 2.80 | 2.73 | 2.85 |
| | 25 | 0 | exp. | 44.99 | 50.09 | 50.00 | 49.53 | 50.42 | 50.17 | 49.93 | 49.79 | 49.69 | 49.60 | 49.44 | 49.41 | 49.32 |
| | | | s.d. | 2.12 | 4.04 | 2.61 | 2.62 | 2.68 | 2.64 | 2.67 | 2.65 | 2.62 | 2.62 | 2.67 | 2.58 | 2.70 |
| | | | rmse | 5.44 | 4.05 | 2.61 | 2.66 | 2.71 | 2.64 | 2.67 | 2.66 | 2.64 | 2.65 | 2.73 | 2.64 | 2.79 |
| | 50 | 0 | exp. | 44.99 | 50.06 | 50.00 | 49.52 | 50.32 | 50.13 | 49.92 | 49.74 | 49.62 | 49.60 | 49.40 | 49.41 | 49.27 |
| | | | s.d. | 2.12 | 3.26 | 2.54 | 2.55 | 2.59 | 2.57 | 2.59 | 2.60 | 2.54 | 2.54 | 2.62 | 2.49 | 2.63 |
| | | | rmse | 5.44 | 3.26 | 2.54 | 2.60 | 2.61 | 2.58 | 2.60 | 2.61 | 2.57 | 2.57 | 2.68 | 2.56 | 2.73 |

Table 2.4a

| N = 10 | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | P(Z<=1) | | X1 | P | CUE | HME | | | | | | | | |
| | | x10^4 | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| | | | | | | | | | | | | | | | |
| 0.11 | 0 | 6990 | exp. | 2.24 | 2.24 | 3.69 | 4.57 | 4.65 | 4.99 | 5.36 | 5.91 | 7.72 | 10.47 | 14.27 | 28.11 |
| | | | s.d. | 0.67 | 0.67 | 2.04 | 1.65 | 1.98 | 2.70 | 3.61 | 4.73 | 5.71 | 8.41 | 13.46 | 27.75 |
| | | | rmse | 7.78 | 7.78 | 6.63 | 5.68 | 5.71 | 5.69 | 5.87 | 6.25 | 6.15 | 8.42 | 14.12 | 33.14 |
| | 5 | 5089 | exp. | 1.69 | 7.26 | 7.65 | 5.08 | 6.03 | 8.37 | 9.96 | 13.14 | 17.80 | 24.50 | 37.68 | 76.69 |
| | | | s.d. | 0.91 | 5.56 | 6.34 | 4.25 | 5.94 | 8.76 | 11.41 | 15.98 | 22.27 | 32.73 | 54.40 | 118.6 |
| | | | rmse | 8.36 | 6.20 | 6.77 | 6.50 | 7.14 | 8.91 | 11.41 | 16.28 | 23.60 | 35.80 | 61.04 | 136.0 |
| | 10 | 3546 | exp. | 1.43 | 8.97 | 9.05 | 4.71 | 6.49 | 7.93 | 10.27 | 12.76 | 17.31 | 23.15 | 34.51 | 66.84 |
| | | | s.d. | 0.95 | 8.47 | 8.86 | 5.36 | 8.41 | 11.76 | 15.60 | 21.47 | 29.89 | 44.41 | 72.94 | 159.2 |
| | | | rmse | 8.63 | 8.53 | 8.91 | 7.53 | 9.11 | 11.94 | 15.60 | 21.65 | 30.77 | 46.32 | 76.95 | 169.1 |
| 0.22 | 0 | 3546 | exp. | 2.68 | 2.68 | 4.82 | 5.61 | 5.85 | 6.39 | 7.10 | 8.20 | 10.16 | 13.72 | 19.36 | 37.62 |
| | | | s.d. | 0.96 | 0.96 | 3.11 | 2.58 | 3.22 | 4.18 | 5.43 | 7.17 | 9.16 | 13.26 | 21.29 | 44.60 |
| | | | rmse | 7.38 | 7.38 | 6.05 | 5.09 | 5.25 | 5.52 | 6.16 | 7.39 | 9.16 | 13.77 | 23.26 | 52.46 |
| | 5 | 1586 | exp. | 2.25 | 8.15 | 8.59 | 6.18 | 7.10 | 9.04 | 10.48 | 13.21 | 16.65 | 21.93 | 32.05 | 61.21 |
| | | | s.d. | 1.16 | 6.20 | 6.93 | 4.93 | 6.81 | 9.51 | 12.43 | 17.04 | 23.71 | 34.65 | 56.96 | 124.3 |
| | | | rmse | 7.84 | 6.47 | 7.07 | 6.23 | 7.41 | 9.56 | 12.44 | 17.34 | 24.62 | 36.65 | 61.08 | 134.4 |
| | 10 | 663 | exp. | 2.09 | 9.55 | 9.59 | 5.72 | 7.19 | 8.39 | 9.81 | 11.42 | 14.14 | 17.44 | 23.47 | 40.06 |
| | | | s.d. | 1.22 | 8.37 | 8.45 | 5.42 | 7.86 | 10.63 | 13.85 | 18.61 | 25.46 | 37.16 | 60.53 | 131.1 |
| | | | rmse | 8.01 | 8.38 | 8.46 | 6.91 | 8.35 | 10.75 | 13.86 | 18.67 | 25.79 | 37.89 | 62.01 | 134.5 |
| 0.36 | 0 | 1257 | exp. | 3.35 | 3.35 | 6.36 | 7.06 | 7.51 | 8.17 | 9.11 | 10.70 | 12.81 | 16.80 | 23.78 | 44.97 |
| | | | s.d. | 1.25 | 1.25 | 4.12 | 3.37 | 4.32 | 5.57 | 7.23 | 9.49 | 12.60 | 18.08 | 29.20 | 62.03 |
| | | | rmse | 6.77 | 6.77 | 5.50 | 4.47 | 4.99 | 5.86 | 7.28 | 9.51 | 12.90 | 19.32 | 32.29 | 71.21 |
| | 5 | 289 | exp. | 3.10 | 9.04 | 9.41 | 7.43 | 8.15 | 9.47 | 10.55 | 12.39 | 14.37 | 17.47 | 23.26 | 39.19 |
| | | | s.d. | 1.41 | 6.53 | 6.81 | 5.02 | 6.75 | 9.02 | 11.66 | 15.59 | 21.43 | 31.02 | 50.43 | 109.5 |
| | | | rmse | 7.05 | 6.60 | 6.83 | 5.64 | 7.00 | 9.03 | 11.68 | 15.77 | 21.87 | 31.90 | 52.14 | 113.3 |
| | 10 | 61 | exp. | 3.04 | 9.92 | 9.90 | 6.90 | 7.87 | 8.71 | 9.50 | 10.27 | 11.36 | 12.69 | 14.65 | 19.76 |
| | | | s.d. | 1.44 | 7.44 | 7.04 | 4.82 | 6.34 | 8.02 | 9.99 | 12.77 | 16.80 | 23.74 | 37.82 | 80.56 |
| | | | rmse | 7.11 | 7.44 | 7.04 | 5.73 | 6.69 | 8.12 | 10.00 | 12.78 | 16.86 | 23.89 | 38.11 | 81.15 |

Table 2.4b

| N = 10 | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | P(Z<=1) x10^4 | | X1 | P | CUE | HME | | | | | | | | |
| | | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.51 | 0 | 372 | exp. | 4.12 | 4.12 | 7.82 | 8.46 | 9.02 | 9.67 | 10.59 | 12.28 | 14.27 | 17.87 | 24.63 | 44.53 |
| | | | s.d. | 1.44 | 1.44 | 4.68 | 3.77 | 4.92 | 6.36 | 8.30 | 10.81 | 14.63 | 21.02 | 34.09 | 73.01 |
| | | | rmse | 6.06 | 6.06 | 5.16 | 4.08 | 5.01 | 6.37 | 8.32 | 11.05 | 15.24 | 22.45 | 37.10 | 80.76 |
| | 5 | 41 | exp. | 4.01 | 9.62 | 9.79 | 8.39 | 8.87 | 9.66 | 10.35 | 11.35 | 12.30 | 13.66 | 16.26 | 22.69 |
| | | | s.d. | 1.54 | 6.35 | 6.01 | 4.55 | 5.85 | 7.46 | 9.38 | 12.13 | 16.25 | 23.10 | 36.95 | 79.34 |
| | | | rmse | 6.18 | 6.36 | 6.02 | 4.83 | 5.96 | 7.47 | 9.39 | 12.20 | 16.41 | 23.38 | 37.48 | 80.35 |
| | 10 | 4 | exp. | 4.00 | 10.02 | 10.00 | 7.94 | 8.51 | 9.03 | 9.51 | 9.88 | 10.24 | 10.76 | 11.30 | 12.42 |
| | | | s.d. | 1.55 | 6.18 | 5.52 | 4.11 | 4.93 | 5.70 | 6.61 | 7.84 | 9.54 | 12.64 | 19.12 | 39.35 |
| | | | rmse | 6.20 | 6.18 | 5.52 | 4.60 | 5.14 | 5.78 | 6.63 | 7.85 | 9.55 | 12.67 | 19.17 | 39.43 |
| 0.69 | 0 | 80 | exp. | 5.02 | 5.02 | 9.02 | 9.68 | 10.18 | 10.67 | 11.34 | 12.65 | 14.18 | 16.63 | 21.64 | 36.03 |
| | | | s.d. | 1.54 | 1.54 | 4.75 | 3.83 | 4.99 | 6.43 | 8.39 | 10.85 | 14.74 | 21.19 | 34.33 | 73.64 |
| | | | rmse | 5.22 | 5.22 | 4.85 | 3.84 | 5.00 | 6.47 | 8.50 | 11.17 | 15.32 | 22.20 | 36.25 | 78.10 |
| | 5 | 4 | exp. | 4.99 | 9.92 | 9.94 | 9.20 | 9.40 | 9.80 | 10.15 | 10.57 | 10.91 | 11.32 | 12.18 | 13.76 |
| | | | s.d. | 1.58 | 5.76 | 4.87 | 3.84 | 4.61 | 5.51 | 6.57 | 8.01 | 10.24 | 13.95 | 21.63 | 45.43 |
| | | | rmse | 5.26 | 5.76 | 4.87 | 3.92 | 4.65 | 5.52 | 6.57 | 8.03 | 10.28 | 14.02 | 21.74 | 45.59 |
| | 10 | 0 | exp. | 4.98 | 10.02 | 10.04 | 8.86 | 9.12 | 9.38 | 9.62 | 9.80 | 9.91 | 10.13 | 10.28 | 10.36 |
| | | | s.d. | 1.58 | 4.96 | 4.25 | 3.47 | 3.86 | 4.15 | 4.47 | 4.85 | 5.27 | 6.12 | 8.03 | 14.64 |
| | | | rmse | 5.26 | 4.96 | 4.25 | 3.65 | 3.96 | 4.20 | 4.48 | 4.85 | 5.27 | 6.12 | 8.03 | 14.64 |
| 0.92 | 0 | 10 | exp. | 6.02 | 6.02 | 9.74 | 10.57 | 10.80 | 10.97 | 11.28 | 11.94 | 12.74 | 13.82 | 16.34 | 23.36 |
| | | | s.d. | 1.54 | 1.54 | 4.27 | 3.54 | 4.48 | 5.62 | 7.21 | 9.15 | 12.29 | 17.49 | 28.06 | 59.85 |
| | | | rmse | 4.27 | 4.27 | 4.27 | 3.59 | 4.55 | 5.70 | 7.33 | 9.35 | 12.59 | 17.90 | 28.77 | 61.32 |
| | 5 | 0 | exp. | 6.01 | 10.03 | 10.00 | 9.88 | 9.82 | 9.93 | 10.06 | 10.16 | 10.23 | 10.32 | 10.44 | 10.53 |
| | | | s.d. | 1.55 | 4.89 | 3.68 | 3.08 | 3.45 | 3.84 | 4.23 | 4.70 | 5.50 | 6.76 | 9.62 | 18.94 |
| | | | rmse | 4.28 | 4.89 | 3.68 | 3.09 | 3.45 | 3.84 | 4.23 | 4.70 | 5.51 | 6.76 | 9.63 | 18.95 |
| | 10 | 0 | exp. | 6.01 | 10.03 | 10.03 | 9.57 | 9.58 | 9.65 | 9.75 | 9.81 | 9.82 | 9.92 | 9.90 | 9.76 |
| | | | s.d. | 1.55 | 3.91 | 3.26 | 2.83 | 3.03 | 3.17 | 3.29 | 3.36 | 3.44 | 3.54 | 3.81 | 4.78 |
| | | | rmse | 4.28 | 3.91 | 3.26 | 2.86 | 3.06 | 3.19 | 3.30 | 3.36 | 3.44 | 3.54 | 3.81 | 4.79 |

Table 2.4c

| N = 10 | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | P(Z<=1) x10^4 | | X1 | P | CUE | HME | | | | | | | | |
| | | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| | | | | | | | | | | | | | | | |
| 1.20 | 0 | 1 | exp. | 6.99 | 6.99 | 9.97 | 11.02 | 10.93 | 10.80 | 10.83 | 10.95 | 11.21 | 11.41 | 12.17 | 14.21 |
| | | | s.d. | 1.45 | 1.45 | 3.38 | 2.99 | 3.54 | 4.20 | 5.17 | 6.28 | 8.16 | 11.29 | 17.69 | 37.20 |
| | | | rmse | 3.34 | 3.34 | 3.38 | 3.15 | 3.66 | 4.28 | 5.24 | 6.36 | 8.25 | 11.37 | 17.83 | 37.44 |
| | 5 | 0 | exp. | 6.99 | 10.06 | 10.02 | 10.33 | 10.10 | 10.02 | 10.04 | 10.02 | 9.97 | 9.97 | 9.85 | 9.69 |
| | | | s.d. | 1.45 | 3.96 | 2.73 | 2.44 | 2.57 | 2.75 | 2.88 | 2.98 | 3.19 | 3.34 | 3.96 | 6.09 |
| | | | rmse | 3.34 | 3.96 | 2.73 | 2.46 | 2.58 | 2.75 | 2.88 | 2.98 | 3.19 | 3.34 | 3.96 | 6.10 |
| | 10 | 0 | exp. | 6.99 | 10.08 | 9.99 | 10.06 | 9.86 | 9.85 | 9.83 | 9.86 | 9.75 | 9.80 | 9.69 | 9.47 |
| | | | s.d. | 1.45 | 3.08 | 2.50 | 2.24 | 2.36 | 2.47 | 2.53 | 2.55 | 2.60 | 2.59 | 2.69 | 2.71 |
| | | | rmse | 3.34 | 3.08 | 2.50 | 2.24 | 2.37 | 2.47 | 2.54 | 2.55 | 2.61 | 2.60 | 2.70 | 2.77 |
| 1.61 | 0 | 0 | exp. | 8.00 | 8.00 | 10.00 | 11.12 | 10.82 | 10.52 | 10.37 | 10.22 | 10.21 | 10.09 | 10.07 | 10.16 |
| | | | s.d. | 1.26 | 1.26 | 2.26 | 2.24 | 2.40 | 2.59 | 2.94 | 3.25 | 3.86 | 4.89 | 7.12 | 14.23 |
| | | | rmse | 2.37 | 2.37 | 2.26 | 2.50 | 2.53 | 2.64 | 2.97 | 3.25 | 3.87 | 4.89 | 7.12 | 14.23 |
| | 5 | 0 | exp. | 8.00 | 10.08 | 10.01 | 10.62 | 10.30 | 10.11 | 9.98 | 9.91 | 9.75 | 9.71 | 9.51 | 9.40 |
| | | | s.d. | 1.26 | 2.96 | 1.91 | 1.82 | 1.84 | 1.92 | 1.97 | 2.04 | 2.06 | 2.06 | 2.15 | 2.15 |
| | | | rmse | 2.37 | 2.96 | 1.91 | 1.93 | 1.86 | 1.92 | 1.97 | 2.04 | 2.07 | 2.08 | 2.21 | 2.23 |
| | 10 | 0 | exp. | 8.00 | 10.09 | 9.98 | 10.46 | 10.15 | 9.99 | 9.86 | 9.80 | 9.63 | 9.64 | 9.46 | 9.33 |
| | | | s.d. | 1.26 | 2.25 | 1.82 | 1.70 | 1.72 | 1.79 | 1.82 | 1.89 | 1.89 | 1.92 | 1.99 | 1.84 |
| | | | rmse | 2.37 | 2.26 | 1.82 | 1.76 | 1.73 | 1.79 | 1.83 | 1.90 | 1.93 | 1.95 | 2.06 | 1.96 |
| 2.30 | 0 | 0 | exp. | 9.00 | 9.00 | 10.00 | 10.89 | 10.51 | 10.17 | 9.99 | 9.80 | 9.69 | 9.59 | 9.47 | 9.45 |
| | | | s.d. | 0.95 | 0.95 | 1.31 | 1.47 | 1.46 | 1.44 | 1.47 | 1.45 | 1.49 | 1.53 | 1.64 | 2.25 |
| | | | rmse | 1.38 | 1.38 | 1.31 | 1.72 | 1.55 | 1.45 | 1.47 | 1.46 | 1.52 | 1.59 | 1.72 | 2.31 |
| | 5 | 0 | exp. | 9.00 | 10.06 | 10.03 | 10.64 | 10.30 | 10.06 | 9.82 | 9.70 | 9.54 | 9.42 | 9.29 | 9.35 |
| | | | s.d. | 0.95 | 1.92 | 1.21 | 1.25 | 1.27 | 1.27 | 1.26 | 1.29 | 1.24 | 1.25 | 1.19 | 1.18 |
| | | | rmse | 1.38 | 1.92 | 1.21 | 1.41 | 1.31 | 1.27 | 1.27 | 1.32 | 1.33 | 1.38 | 1.39 | 1.35 |
| | 10 | 0 | exp. | 9.00 | 10.04 | 10.04 | 10.57 | 10.22 | 9.97 | 9.78 | 9.61 | 9.50 | 9.36 | 9.24 | 9.31 |
| | | | s.d. | 0.95 | 1.46 | 1.15 | 1.21 | 1.20 | 1.20 | 1.22 | 1.23 | 1.22 | 1.20 | 1.15 | 1.15 |
| | | | rmse | 1.38 | 1.46 | 1.15 | 1.34 | 1.22 | 1.20 | 1.24 | 1.29 | 1.32 | 1.36 | 1.38 | 1.34 |

Table 2.5a

| N = 25 | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | P(Z<=1) | | X1 | P | CUE | HME | | | | | | | | |
| | | x10^4 | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.11 | 0 | 2397 | exp. | 3.19 | 3.19 | 6.92 | 7.26 | 7.98 | 9.27 | 10.86 | 13.10 | 16.53 | 22.78 | 33.72 | 67.32 |
| | | | s.d. | 1.25 | 1.25 | 5.01 | 4.08 | 5.46 | 7.00 | 9.16 | 12.02 | 15.89 | 22.42 | 36.08 | 75.74 |
| | | | rmse | 21.84 | 21.84 | 18.76 | 18.21 | 17.87 | 17.22 | 16.85 | 16.91 | 18.00 | 22.53 | 37.12 | 86.76 |
| | 5 | 1586 | exp. | 2.97 | 13.63 | 15.48 | 11.00 | 13.89 | 18.51 | 22.77 | 29.71 | 39.08 | 53.96 | 83.09 | 169.1 |
| | | | s.d. | 1.37 | 8.43 | 10.86 | 8.21 | 11.98 | 16.24 | 21.70 | 29.46 | 41.18 | 60.30 | 99.02 | 216.3 |
| | | | rmse | 22.07 | 14.15 | 14.44 | 16.23 | 16.34 | 17.49 | 21.81 | 29.84 | 43.52 | 66.89 | 114.8 | 259.9 |
| | 10 | 1032 | exp. | 2.84 | 18.97 | 19.90 | 12.18 | 16.78 | 21.19 | 26.46 | 33.34 | 43.35 | 58.45 | 87.43 | 171.6 |
| | | | s.d. | 1.43 | 14.04 | 15.50 | 11.15 | 16.44 | 22.93 | 30.78 | 41.93 | 58.16 | 85.93 | 141.7 | 309.2 |
| | | | rmse | 22.21 | 15.29 | 16.32 | 16.99 | 18.38 | 23.25 | 30.81 | 42.75 | 60.99 | 92.21 | 154.8 | 342.2 |
| | 25 | 266 | exp. | 2.66 | 23.97 | 24.02 | 12.90 | 16.60 | 20.42 | 24.23 | 28.82 | 34.17 | 42.13 | 55.55 | 91.66 |
| | | | s.d. | 1.50 | 21.15 | 21.13 | 14.06 | 20.16 | 27.00 | 35.51 | 47.29 | 64.83 | 93.73 | 152.3 | 329.4 |
| | | | rmse | 22.39 | 21.17 | 21.15 | 18.55 | 21.84 | 27.39 | 35.52 | 47.44 | 65.48 | 95.28 | 155.4 | 336.1 |
| 0.22 | 0 | 266 | exp. | 5.05 | 5.05 | 13.31 | 12.70 | 14.87 | 17.36 | 20.63 | 25.05 | 31.27 | 41.59 | 61.63 | 120.7 |
| | | | s.d. | 1.90 | 1.90 | 9.06 | 7.25 | 10.18 | 13.11 | 17.46 | 22.67 | 31.11 | 44.58 | 72.30 | 154.9 |
| | | | rmse | 20.04 | 20.04 | 14.79 | 14.27 | 14.37 | 15.17 | 18.00 | 22.67 | 31.73 | 47.57 | 81.05 | 182.1 |
| | 5 | 103 | exp. | 4.98 | 18.48 | 21.08 | 16.43 | 19.80 | 23.86 | 28.11 | 33.89 | 41.38 | 53.15 | 75.41 | 140.2 |
| | | | s.d. | 1.95 | 11.56 | 14.57 | 11.24 | 16.20 | 21.62 | 28.75 | 38.66 | 53.71 | 78.69 | 128.8 | 280.3 |
| | | | rmse | 20.11 | 13.27 | 15.08 | 14.14 | 17.01 | 21.65 | 28.92 | 39.67 | 56.15 | 83.58 | 138.3 | 303.1 |
| | 10 | 39 | exp. | 4.95 | 22.93 | 23.70 | 16.91 | 20.57 | 23.80 | 27.26 | 31.23 | 36.25 | 43.67 | 56.95 | 94.25 |
| | | | s.d. | 1.98 | 16.34 | 16.83 | 12.73 | 17.66 | 23.34 | 30.66 | 40.73 | 55.67 | 80.72 | 131.3 | 283.7 |
| | | | rmse | 20.14 | 16.47 | 16.88 | 15.08 | 18.20 | 23.37 | 30.75 | 41.21 | 56.80 | 82.86 | 135.2 | 292.0 |
| | 25 | 2 | exp. | 4.94 | 24.97 | 24.96 | 17.19 | 19.52 | 21.63 | 23.27 | 24.82 | 26.37 | 27.97 | 30.07 | 34.00 |
| | | | s.d. | 1.99 | 16.42 | 15.48 | 11.43 | 14.01 | 16.60 | 19.62 | 23.62 | 29.50 | 39.55 | 60.34 | 125.0 |
| | | | rmse | 20.16 | 16.42 | 15.48 | 13.84 | 15.04 | 16.93 | 19.69 | 23.62 | 29.53 | 39.67 | 60.55 | 125.3 |
| 0.36 | 0 | 12 | exp. | 7.57 | 7.57 | 20.15 | 19.03 | 21.86 | 24.73 | 28.39 | 32.84 | 39.35 | 49.30 | 68.88 | 125.9 |
| | | | s.d. | 2.29 | 2.29 | 11.98 | 9.72 | 13.60 | 17.76 | 23.68 | 31.25 | 43.18 | 62.75 | 102.4 | 221.6 |
| | | | rmse | 17.58 | 17.58 | 12.93 | 11.40 | 13.96 | 17.76 | 23.92 | 32.22 | 45.50 | 67.29 | 111.4 | 243.5 |
| | 5 | 2 | exp. | 7.56 | 22.15 | 24.19 | 20.62 | 23.08 | 25.54 | 27.95 | 30.74 | 34.32 | 39.31 | 48.23 | 73.22 |
| | | | s.d. | 2.29 | 13.48 | 14.57 | 11.47 | 15.53 | 20.06 | 25.86 | 33.96 | 46.03 | 66.34 | 107.0 | 230.0 |
| | | | rmse | 17.59 | 13.78 | 14.59 | 12.27 | 15.65 | 20.07 | 26.03 | 34.44 | 46.96 | 67.87 | 109.5 | 235.0 |
| | 10 | 0 | exp. | 7.56 | 24.60 | 24.90 | 20.36 | 22.71 | 24.19 | 25.74 | 27.15 | 28.67 | 30.66 | 33.53 | 40.86 |
| | | | s.d. | 2.30 | 15.46 | 13.71 | 10.79 | 13.58 | 16.55 | 20.30 | 25.35 | 32.97 | 45.78 | 71.90 | 151.7 |
| | | | rmse | 17.59 | 15.46 | 13.71 | 11.74 | 13.77 | 16.57 | 20.31 | 25.44 | 33.18 | 46.13 | 72.40 | 152.5 |
| | 25 | 0 | exp. | 7.56 | 25.02 | 24.95 | 20.25 | 21.79 | 22.89 | 23.69 | 24.40 | 25.06 | 25.57 | 26.03 | 26.48 |
| | | | s.d. | 2.30 | 11.79 | 10.69 | 8.88 | 9.73 | 10.37 | 10.99 | 11.62 | 12.33 | 13.45 | 15.84 | 24.93 |
| | | | rmse | 17.59 | 11.79 | 10.69 | 10.07 | 10.25 | 10.59 | 11.07 | 11.63 | 12.33 | 13.46 | 15.87 | 24.98 |

Table 2.5b

| N = 25 | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | P(Z<=1) | | X1 | P | CUE | HME | | | | | | | | | |
| | | x10^4 | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| | | | | | | | | | | | | | | | | |
| 0.51 | 0 | 0 | exp. | 9.99 | 9.99 | 23.59 | 22.96 | 25.04 | 26.97 | 29.30 | 31.95 | 35.75 | 41.19 | 51.76 | 81.72 |
| | | | s.d. | 2.45 | 2.45 | 12.37 | 10.14 | 13.74 | 17.78 | 23.17 | 30.43 | 41.49 | 59.76 | 96.85 | 208.6 |
| | | | rmse | 15.21 | 15.21 | 12.45 | 10.35 | 13.74 | 17.89 | 23.56 | 31.21 | 42.86 | 61.92 | 100.5 | 216.1 |
| | 5 | 0 | exp. | 9.99 | 23.89 | 24.90 | 22.83 | 24.27 | 25.42 | 26.43 | 27.41 | 28.71 | 30.09 | 32.24 | 37.89 |
| | | | s.d. | 2.45 | 13.76 | 11.84 | 9.62 | 11.95 | 14.40 | 17.51 | 21.72 | 27.99 | 38.75 | 60.56 | 127.4 |
| | | | rmse | 15.21 | 13.80 | 11.84 | 9.87 | 11.97 | 14.41 | 17.56 | 21.86 | 28.23 | 39.08 | 61.00 | 128.1 |
| | 10 | 0 | exp. | 9.99 | 24.97 | 25.03 | 22.43 | 23.87 | 24.45 | 25.22 | 25.68 | 26.19 | 26.73 | 27.25 | 28.24 |
| | | | s.d. | 2.45 | 13.10 | 10.16 | 8.52 | 9.77 | 10.81 | 12.04 | 13.57 | 15.84 | 19.83 | 28.32 | 55.93 |
| | | | rmse | 15.21 | 13.10 | 10.16 | 8.90 | 9.83 | 10.83 | 12.04 | 13.59 | 15.88 | 19.91 | 28.41 | 56.03 |
| | 25 | 0 | exp. | 9.99 | 25.04 | 25.00 | 22.08 | 23.06 | 23.72 | 24.10 | 24.54 | 24.82 | 25.13 | 25.36 | 25.53 |
| | | | s.d. | 2.45 | 9.17 | 8.16 | 7.27 | 7.66 | 7.93 | 8.12 | 8.29 | 8.46 | 8.61 | 8.77 | 9.17 |
| | | | rmse | 15.21 | 9.17 | 8.16 | 7.84 | 7.90 | 8.03 | 8.17 | 8.30 | 8.46 | 8.61 | 8.77 | 9.18 |
| 0.69 | 0 | 0 | exp. | 12.46 | 12.46 | 24.78 | 24.93 | 25.89 | 26.71 | 27.58 | 28.62 | 29.78 | 31.43 | 34.53 | 42.63 |
| | | | s.d. | 2.50 | 2.50 | 10.44 | 8.90 | 11.15 | 13.68 | 16.90 | 21.33 | 28.01 | 39.18 | 62.06 | 131.6 |
| | | | rmse | 12.79 | 12.79 | 10.44 | 8.90 | 11.18 | 13.78 | 17.09 | 21.64 | 28.42 | 39.70 | 62.78 | 132.8 |
| | 5 | 0 | exp. | 12.46 | 24.68 | 24.99 | 24.21 | 24.80 | 25.26 | 25.61 | 25.94 | 26.36 | 26.65 | 26.92 | 27.57 |
| | | | s.d. | 2.50 | 12.82 | 8.67 | 7.58 | 8.48 | 9.35 | 10.36 | 11.59 | 13.40 | 16.75 | 23.82 | 46.88 |
| | | | rmse | 12.79 | 12.83 | 8.67 | 7.62 | 8.49 | 9.35 | 10.37 | 11.63 | 13.47 | 16.83 | 23.89 | 46.95 |
| | 10 | 0 | exp. | 12.46 | 25.03 | 25.00 | 23.83 | 24.50 | 24.72 | 25.08 | 25.29 | 25.47 | 25.64 | 25.75 | 25.86 |
| | | | s.d. | 2.50 | 10.53 | 7.52 | 6.77 | 7.32 | 7.60 | 7.91 | 8.22 | 8.52 | 9.10 | 10.26 | 14.91 |
| | | | rmse | 12.79 | 10.53 | 7.52 | 6.87 | 7.34 | 7.61 | 7.91 | 8.22 | 8.53 | 9.13 | 10.28 | 14.94 |
| | 25 | 0 | exp. | 12.46 | 25.01 | 24.99 | 23.37 | 23.87 | 24.22 | 24.45 | 24.66 | 24.76 | 24.91 | 25.04 | 25.09 |
| | | | s.d. | 2.50 | 7.34 | 6.39 | 5.95 | 6.15 | 6.29 | 6.37 | 6.43 | 6.50 | 6.54 | 6.57 | 6.69 |
| | | | rmse | 12.79 | 7.34 | 6.39 | 6.17 | 6.26 | 6.33 | 6.39 | 6.44 | 6.50 | 6.54 | 6.57 | 6.69 |
| 0.92 | 0 | 0 | exp. | 15.04 | 15.04 | 24.99 | 25.77 | 25.91 | 26.10 | 26.22 | 26.50 | 26.53 | 26.79 | 27.17 | 27.93 |
| | | | s.d. | 2.45 | 2.45 | 7.43 | 6.86 | 7.69 | 8.54 | 9.58 | 10.97 | 13.10 | 16.85 | 24.87 | 50.37 |
| | | | rmse | 10.26 | 10.26 | 7.43 | 6.91 | 7.74 | 8.61 | 9.66 | 11.07 | 13.19 | 16.94 | 24.96 | 50.45 |
| | 5 | 0 | exp. | 15.04 | 24.99 | 25.00 | 25.06 | 25.14 | 25.20 | 25.30 | 25.40 | 25.54 | 25.57 | 25.56 | 25.52 |
| | | | s.d. | 2.45 | 10.98 | 6.20 | 5.83 | 6.10 | 6.31 | 6.53 | 6.71 | 6.93 | 7.37 | 8.14 | 11.46 |
| | | | rmse | 10.26 | 10.98 | 6.20 | 5.83 | 6.10 | 6.31 | 6.54 | 6.72 | 6.95 | 7.39 | 8.16 | 11.47 |
| | 10 | 0 | exp. | 15.04 | 25.04 | 25.01 | 24.68 | 24.85 | 24.96 | 25.03 | 25.06 | 25.12 | 25.19 | 25.18 | 25.12 |
| | | | s.d. | 2.45 | 8.22 | 5.63 | 5.30 | 5.54 | 5.63 | 5.74 | 5.78 | 5.84 | 5.93 | 6.01 | 6.23 |
| | | | rmse | 10.26 | 8.22 | 5.63 | 5.31 | 5.54 | 5.63 | 5.74 | 5.78 | 5.85 | 5.94 | 6.02 | 6.23 |
| | 25 | 0 | exp. | 15.04 | 25.01 | 24.99 | 24.25 | 24.42 | 24.56 | 24.65 | 24.71 | 24.77 | 24.79 | 24.85 | 24.75 |
| | | | s.d. | 2.45 | 5.89 | 4.97 | 4.77 | 4.86 | 4.93 | 4.96 | 4.99 | 5.01 | 5.01 | 5.07 | 5.11 |
| | | | rmse | 10.26 | 5.89 | 4.97 | 4.83 | 4.89 | 4.95 | 4.97 | 5.00 | 5.01 | 5.01 | 5.08 | 5.12 |

Table 2.5c

| N = 25 | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | P(Z<=1) | | X1 | P | CUE | HME | | | | | | | | |
| | | x10^4 | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| | | | | | | | | | | | | | | | |
| 1.20 | 0 | 0 | exp. | 17.47 | 17.47 | 24.99 | 25.99 | 25.82 | 25.71 | 25.61 | 25.62 | 25.49 | 25.47 | 25.44 | 25.32 |
| | | | s.d. | 2.29 | 2.29 | 5.11 | 5.03 | 5.25 | 5.40 | 5.58 | 5.72 | 5.99 | 6.49 | 7.59 | 12.02 |
| | | | rmse | 7.87 | 7.87 | 5.11 | 5.12 | 5.32 | 5.44 | 5.61 | 5.75 | 6.01 | 6.51 | 7.61 | 12.03 |
| | 5 | 0 | exp. | 17.47 | 25.06 | 24.99 | 25.42 | 25.30 | 25.20 | 25.16 | 25.15 | 25.16 | 25.10 | 25.03 | 24.91 |
| | | | s.d. | 2.29 | 8.86 | 4.52 | 4.42 | 4.50 | 4.56 | 4.62 | 4.65 | 4.68 | 4.77 | 4.84 | 4.91 |
| | | | rmse | 7.87 | 8.86 | 4.52 | 4.44 | 4.51 | 4.56 | 4.63 | 4.65 | 4.69 | 4.77 | 4.84 | 4.91 |
| | 10 | 0 | exp. | 17.47 | 25.04 | 25.01 | 25.11 | 25.08 | 25.06 | 25.02 | 24.98 | 24.96 | 24.91 | 24.86 | 24.73 |
| | | | s.d. | 2.29 | 6.43 | 4.25 | 4.12 | 4.21 | 4.22 | 4.28 | 4.27 | 4.31 | 4.36 | 4.43 | 4.42 |
| | | | rmse | 7.87 | 6.43 | 4.25 | 4.12 | 4.21 | 4.22 | 4.28 | 4.27 | 4.31 | 4.36 | 4.43 | 4.43 |
| | 25 | 0 | exp. | 17.47 | 25.02 | 25.01 | 24.79 | 24.74 | 24.73 | 24.74 | 24.75 | 24.75 | 24.76 | 24.66 | 24.54 |
| | | | s.d. | 2.29 | 4.71 | 3.86 | 3.79 | 3.81 | 3.84 | 3.85 | 3.85 | 3.84 | 3.90 | 3.97 | 3.89 |
| | | | rmse | 7.87 | 4.71 | 3.86 | 3.80 | 3.82 | 3.85 | 3.86 | 3.86 | 3.84 | 3.91 | 3.98 | 3.92 |
| 1.61 | 0 | 0 | exp. | 20.00 | 20.00 | 24.99 | 25.90 | 25.59 | 25.38 | 25.20 | 25.15 | 25.06 | 24.92 | 24.83 | 24.66 |
| | | | s.d. | 2.00 | 2.00 | 3.37 | 3.39 | 3.46 | 3.46 | 3.47 | 3.48 | 3.53 | 3.58 | 3.59 | 3.65 |
| | | | rmse | 5.38 | 5.38 | 3.37 | 3.51 | 3.51 | 3.48 | 3.48 | 3.49 | 3.53 | 3.59 | 3.59 | 3.67 |
| | 5 | 0 | exp. | 20.00 | 25.04 | 24.97 | 25.56 | 25.30 | 25.21 | 25.05 | 24.96 | 24.90 | 24.79 | 24.68 | 24.55 |
| | | | s.d. | 2.00 | 6.56 | 3.14 | 3.10 | 3.14 | 3.16 | 3.19 | 3.19 | 3.20 | 3.28 | 3.26 | 3.28 |
| | | | rmse | 5.38 | 6.56 | 3.14 | 3.15 | 3.15 | 3.17 | 3.19 | 3.19 | 3.20 | 3.29 | 3.27 | 3.31 |
| | 10 | 0 | exp. | 20.00 | 25.01 | 25.03 | 25.36 | 25.12 | 25.08 | 24.99 | 24.86 | 24.81 | 24.69 | 24.58 | 24.48 |
| | | | s.d. | 2.00 | 4.80 | 3.02 | 2.96 | 3.02 | 2.98 | 2.99 | 3.03 | 3.04 | 3.12 | 3.08 | 3.10 |
| | | | rmse | 5.38 | 4.80 | 3.02 | 2.98 | 3.03 | 2.98 | 2.99 | 3.03 | 3.05 | 3.14 | 3.11 | 3.15 |
| | 25 | 0 | exp. | 20.00 | 25.03 | 25.01 | 25.07 | 24.92 | 24.81 | 24.87 | 24.79 | 24.69 | 24.60 | 24.47 | 24.41 |
| | | | s.d. | 2.00 | 3.56 | 2.82 | 2.80 | 2.79 | 2.77 | 2.76 | 2.86 | 2.88 | 2.89 | 2.85 | 2.92 |
| | | | rmse | 5.38 | 3.56 | 2.82 | 2.80 | 2.79 | 2.77 | 2.76 | 2.87 | 2.89 | 2.92 | 2.90 | 2.98 |
| 2.30 | 0 | 0 | exp. | 22.49 | 22.49 | 25.00 | 25.73 | 25.33 | 25.10 | 24.91 | 24.77 | 24.71 | 24.57 | 24.47 | 24.32 |
| | | | s.d. | 1.50 | 1.50 | 2.00 | 1.94 | 2.00 | 2.06 | 2.07 | 2.08 | 2.08 | 2.05 | 2.07 | 2.03 |
| | | | rmse | 2.92 | 2.92 | 2.00 | 2.08 | 2.02 | 2.07 | 2.07 | 2.09 | 2.10 | 2.10 | 2.14 | 2.14 |
| | 5 | 0 | exp. | 22.49 | 25.03 | 24.99 | 25.61 | 25.19 | 25.01 | 24.87 | 24.66 | 24.67 | 24.54 | 24.44 | 24.28 |
| | | | s.d. | 1.50 | 4.27 | 1.93 | 1.85 | 1.90 | 1.98 | 2.00 | 1.99 | 1.96 | 1.96 | 1.99 | 1.93 |
| | | | rmse | 2.92 | 4.27 | 1.93 | 1.95 | 1.91 | 1.98 | 2.00 | 2.02 | 1.99 | 2.01 | 2.07 | 2.06 |
| | 10 | 0 | exp. | 22.49 | 24.93 | 25.00 | 25.51 | 25.12 | 24.94 | 24.86 | 24.58 | 24.65 | 24.53 | 24.41 | 24.24 |
| | | | s.d. | 1.50 | 3.16 | 1.88 | 1.80 | 1.85 | 1.92 | 1.97 | 1.95 | 1.89 | 1.89 | 1.95 | 1.87 |
| | | | rmse | 2.92 | 3.16 | 1.88 | 1.87 | 1.85 | 1.92 | 1.97 | 2.00 | 1.92 | 1.95 | 2.03 | 2.02 |
| | 25 | 0 | exp. | 22.49 | 25.05 | 25.00 | 25.40 | 25.05 | 24.81 | 24.79 | 24.54 | 24.59 | 24.51 | 24.37 | 24.21 |
| | | | s.d. | 1.50 | 2.29 | 1.81 | 1.72 | 1.79 | 1.84 | 1.90 | 1.90 | 1.78 | 1.81 | 1.88 | 1.78 |
| | | | rmse | 2.92 | 2.29 | 1.81 | 1.76 | 1.79 | 1.85 | 1.91 | 1.95 | 1.82 | 1.87 | 1.98 | 1.95 |

Table 2.6a

| N = 50 | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | P(Z<=1) | | X1 | P | CUE | HME | | | | | | | | |
| | | x10^4 | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.11 | 0 | 266 | exp. | 5.33 | 5.33 | 16.48 | 15.08 | 18.50 | 22.34 | 27.51 | 34.16 | 43.95 | 60.03 | 91.58 | 184.9 |
| | | | s.d. | 2.06 | 2.06 | 11.72 | 9.43 | 13.48 | 17.36 | 23.03 | 29.93 | 40.94 | 58.44 | 94.25 | 201.1 |
| | | | rmse | 44.72 | 44.72 | 35.52 | 36.18 | 34.26 | 32.65 | 32.19 | 33.86 | 41.38 | 59.29 | 103.0 | 242.1 |
| | 5 | 166 | exp. | 5.28 | 24.49 | 30.78 | 23.47 | 30.54 | 38.75 | 48.03 | 60.86 | 78.65 | 107.2 | 162.5 | 326.4 |
| | | | s.d. | 2.10 | 13.41 | 20.26 | 16.07 | 23.74 | 31.92 | 42.85 | 57.85 | 80.64 | 118.6 | 194.8 | 424.6 |
| | | | rmse | 44.77 | 28.82 | 27.93 | 31.02 | 30.70 | 33.84 | 42.89 | 58.86 | 85.58 | 131.7 | 224.9 | 506.6 |
| | 10 | 103 | exp. | 5.26 | 35.48 | 38.99 | 27.57 | 36.36 | 45.20 | 55.53 | 68.65 | 86.69 | 115.0 | 169.3 | 327.3 |
| | | | s.d. | 2.12 | 22.83 | 27.14 | 21.36 | 31.07 | 42.61 | 57.38 | 77.80 | 108.3 | 159.6 | 262.8 | 573.4 |
| | | | rmse | 44.80 | 27.06 | 29.29 | 30.98 | 33.93 | 42.88 | 57.64 | 80.01 | 114.4 | 172.3 | 288.6 | 637.0 |
| | 25 | 24 | exp. | 5.22 | 47.27 | 47.86 | 31.55 | 39.12 | 46.36 | 53.46 | 61.64 | 71.56 | 85.98 | 111.1 | 179.8 |
| | | | s.d. | 2.15 | 35.57 | 35.71 | 26.52 | 37.45 | 49.67 | 65.23 | 86.52 | 118.3 | 171.1 | 277.5 | 598.8 |
| | | | rmse | 44.83 | 35.67 | 35.78 | 32.30 | 38.99 | 49.80 | 65.32 | 87.30 | 120.2 | 174.8 | 284.2 | 612.7 |
| | 50 | 2 | exp. | 5.21 | 49.90 | 49.86 | 32.01 | 37.91 | 42.29 | 46.23 | 50.08 | 53.82 | 57.79 | 62.96 | 73.44 |
| | | | s.d. | 2.16 | 34.09 | 33.07 | 23.92 | 30.14 | 36.40 | 43.72 | 53.56 | 68.04 | 92.50 | 142.8 | 297.6 |
| | | | rmse | 44.84 | 34.09 | 33.07 | 29.93 | 32.48 | 37.21 | 43.88 | 53.56 | 68.14 | 92.83 | 143.3 | 298.5 |
| 0.22 | 0 | 2 | exp. | 9.88 | 9.88 | 35.81 | 32.11 | 38.98 | 45.94 | 54.59 | 65.30 | 80.87 | 105.1 | 152.7 | 292.2 |
| | | | s.d. | 2.81 | 2.81 | 21.72 | 17.86 | 25.48 | 34.23 | 45.48 | 61.13 | 84.71 | 123.9 | 203.0 | 441.1 |
| | | | rmse | 40.22 | 40.22 | 25.94 | 25.27 | 27.76 | 34.47 | 45.71 | 63.02 | 90.16 | 135.6 | 227.5 | 503.2 |
| | 5 | 1 | exp. | 9.87 | 36.66 | 45.33 | 37.71 | 44.80 | 51.77 | 59.35 | 68.55 | 80.88 | 99.64 | 134.5 | 235.0 |
| | | | s.d. | 2.81 | 20.16 | 28.41 | 22.74 | 32.54 | 43.79 | 58.05 | 77.94 | 107.6 | 157.2 | 256.6 | 556.3 |
| | | | rmse | 40.22 | 24.17 | 28.79 | 25.84 | 32.95 | 43.82 | 58.80 | 80.12 | 111.9 | 164.8 | 270.1 | 586.2 |
| | 10 | 0 | exp. | 9.87 | 45.65 | 48.40 | 39.13 | 45.54 | 50.84 | 56.39 | 62.37 | 69.96 | 80.83 | 99.73 | 152.2 |
| | | | s.d. | 2.81 | 29.12 | 30.25 | 23.93 | 32.83 | 43.24 | 56.14 | 73.95 | 100.5 | 144.7 | 233.6 | 502.1 |
| | | | rmse | 40.22 | 29.44 | 30.30 | 26.29 | 33.14 | 43.24 | 56.50 | 74.97 | 102.4 | 148.0 | 238.9 | 512.4 |
| | 25 | 0 | exp. | 9.87 | 49.89 | 49.92 | 40.00 | 44.17 | 47.18 | 49.75 | 52.05 | 54.16 | 56.39 | 59.05 | 64.25 |
| | | | s.d. | 2.81 | 28.81 | 25.86 | 20.43 | 24.65 | 28.72 | 33.53 | 39.85 | 49.18 | 65.10 | 98.12 | 201.2 |
| | | | rmse | 40.22 | 28.81 | 25.86 | 22.75 | 25.33 | 28.86 | 33.53 | 39.90 | 49.35 | 65.41 | 98.54 | 201.7 |
| | 50 | 0 | exp. | 9.87 | 50.02 | 49.98 | 40.21 | 43.57 | 45.70 | 47.44 | 48.91 | 50.16 | 51.08 | 52.11 | 52.93 |
| | | | s.d. | 2.81 | 21.79 | 20.44 | 17.27 | 18.79 | 19.93 | 20.74 | 21.59 | 22.48 | 23.59 | 25.57 | 33.23 |
| | | | rmse | 40.22 | 21.79 | 20.44 | 19.85 | 19.87 | 20.39 | 20.90 | 21.61 | 22.48 | 23.62 | 25.65 | 33.36 |
| 0.36 | 0 | 0 | exp. | 15.12 | 15.12 | 47.34 | 44.22 | 49.36 | 54.19 | 59.29 | 65.48 | 73.42 | 85.40 | 107.8 | 171.3 |
| | | | s.d. | 3.25 | 3.25 | 25.79 | 21.17 | 29.32 | 38.76 | 50.56 | 66.91 | 91.39 | 132.2 | 214.2 | 461.8 |
| | | | rmse | 35.03 | 35.03 | 25.93 | 21.94 | 29.33 | 38.98 | 51.41 | 68.68 | 94.35 | 136.8 | 221.9 | 477.5 |
| | 5 | 0 | exp. | 15.12 | 44.27 | 49.56 | 44.98 | 48.83 | 51.85 | 54.76 | 57.68 | 61.12 | 65.72 | 73.01 | 92.53 |
| | | | s.d. | 3.25 | 24.79 | 25.22 | 20.50 | 26.62 | 33.39 | 41.73 | 53.12 | 70.10 | 98.56 | 156.2 | 331.2 |
| | | | rmse | 35.03 | 25.45 | 25.22 | 21.11 | 26.64 | 33.44 | 42.00 | 53.67 | 70.98 | 99.80 | 157.9 | 334.0 |
| | 10 | 0 | exp. | 15.12 | 49.12 | 49.92 | 44.90 | 48.15 | 50.14 | 52.08 | 53.67 | 55.19 | 57.11 | 59.56 | 64.78 |
| | | | s.d. | 3.25 | 28.50 | 22.53 | 18.54 | 22.30 | 26.25 | 30.84 | 36.97 | 46.15 | 61.73 | 93.94 | 193.9 |
| | | | rmse | 35.03 | 28.51 | 22.53 | 19.23 | 22.38 | 26.25 | 30.91 | 37.15 | 46.44 | 62.14 | 94.43 | 194.5 |
| | 25 | 0 | exp. | 15.12 | 50.02 | 49.98 | 44.76 | 47.02 | 48.39 | 49.44 | 50.33 | 51.10 | 51.76 | 52.28 | 52.87 |
| | | | s.d. | 3.25 | 20.63 | 17.24 | 15.13 | 16.34 | 17.19 | 18.01 | 18.70 | 19.48 | 20.63 | 22.82 | 31.66 |
| | | | rmse | 35.03 | 20.63 | 17.24 | 16.01 | 16.61 | 17.26 | 18.02 | 18.70 | 19.51 | 20.70 | 22.93 | 31.79 |
| | 50 | 0 | exp. | 15.12 | 50.03 | 50.03 | 44.61 | 46.48 | 47.59 | 48.47 | 49.19 | 49.79 | 50.29 | 50.71 | 51.06 |
| | | | s.d. | 3.25 | 15.81 | 14.50 | 13.28 | 13.86 | 14.21 | 14.46 | 14.72 | 14.91 | 15.06 | 15.23 | 15.35 |
| | | | rmse | 35.03 | 15.81 | 14.50 | 14.33 | 14.30 | 14.41 | 14.54 | 14.74 | 14.91 | 15.06 | 15.25 | 15.39 |

Table 2.6b

| N = 50 | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tau$ | R | P(Z<=1) | | X1 | P | CUE | HME | | | | | | | | |
| | | x10^4 | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 0.51 | 0 | 0 | exp. | 19.98 | 19.98 | 49.75 | 48.62 | 50.96 | 52.90 | 54.54 | 56.30 | 58.31 | 60.86 | 65.16 | 76.36 |
| | | | s.d. | 3.46 | 3.46 | 21.10 | 17.85 | 22.36 | 27.27 | 33.21 | 41.34 | 53.58 | 74.18 | 116.2 | 244.7 |
| | | | rmse | 30.22 | 30.22 | 21.11 | 17.90 | 22.38 | 27.42 | 33.52 | 41.82 | 54.22 | 74.97 | 117.2 | 246.1 |
| | 5 | 0 | exp. | 19.98 | 47.71 | 49.98 | 48.11 | 49.82 | 50.92 | 51.89 | 52.69 | 53.53 | 54.37 | 55.23 | 56.98 |
| | | | s.d. | 3.46 | 25.95 | 17.96 | 15.56 | 17.90 | 20.05 | 22.45 | 25.51 | 30.06 | 37.99 | 55.05 | 109.7 |
| | | | rmse | 30.22 | 26.05 | 17.96 | 15.67 | 17.90 | 20.07 | 22.53 | 25.65 | 30.26 | 38.24 | 55.30 | 109.9 |
| | 10 | 0 | exp. | 19.98 | 49.87 | 50.03 | 47.69 | 49.23 | 50.09 | 50.83 | 51.35 | 51.87 | 52.36 | 52.78 | 53.25 |
| | | | s.d. | 3.46 | 24.41 | 15.78 | 14.06 | 15.28 | 16.33 | 17.26 | 18.28 | 19.63 | 21.85 | 26.84 | 45.19 |
| | | | rmse | 30.22 | 24.41 | 15.78 | 14.25 | 15.30 | 16.33 | 17.28 | 18.33 | 19.72 | 21.98 | 26.99 | 45.30 |
| | 25 | 0 | exp. | 19.98 | 50.03 | 50.03 | 47.19 | 48.39 | 49.08 | 49.63 | 50.06 | 50.43 | 50.71 | 50.95 | 51.28 |
| | | | s.d. | 3.46 | 15.97 | 12.89 | 11.97 | 12.46 | 12.79 | 13.05 | 13.29 | 13.46 | 13.63 | 13.78 | 13.98 |
| | | | rmse | 30.22 | 15.97 | 12.89 | 12.30 | 12.57 | 12.83 | 13.06 | 13.29 | 13.47 | 13.65 | 13.81 | 14.03 |
| | 50 | 0 | exp. | 19.98 | 50.01 | 50.00 | 46.82 | 47.92 | 48.55 | 49.04 | 49.41 | 49.72 | 49.99 | 50.23 | 50.44 |
| | | | s.d. | 3.46 | 12.57 | 11.25 | 10.67 | 10.94 | 11.13 | 11.23 | 11.33 | 11.40 | 11.49 | 11.56 | 11.60 |
| | | | rmse | 30.22 | 12.57 | 11.25 | 11.13 | 11.14 | 11.22 | 11.27 | 11.35 | 11.41 | 11.49 | 11.56 | 11.61 |
| 0.69 | 0 | 0 | exp. | 24.92 | 24.92 | 49.98 | 50.24 | 51.01 | 51.58 | 51.98 | 52.29 | 52.71 | 52.94 | 53.34 | 54.09 |
| | | | s.d. | 3.54 | 3.54 | 14.44 | 13.19 | 14.62 | 15.85 | 17.18 | 18.84 | 21.35 | 25.68 | 35.38 | 67.78 |
| | | | rmse | 25.33 | 25.33 | 14.44 | 13.19 | 14.65 | 15.93 | 17.29 | 18.98 | 21.52 | 25.84 | 35.53 | 67.90 |
| | 5 | 0 | exp. | 24.92 | 49.28 | 50.01 | 49.60 | 50.31 | 50.62 | 50.92 | 51.15 | 51.39 | 51.59 | 51.77 | 51.89 |
| | | | s.d. | 3.54 | 24.49 | 12.53 | 11.70 | 12.43 | 12.96 | 13.40 | 13.79 | 14.24 | 14.93 | 16.42 | 22.47 |
| | | | rmse | 25.33 | 24.50 | 12.53 | 11.71 | 12.43 | 12.97 | 13.43 | 13.84 | 14.31 | 15.02 | 16.52 | 22.55 |
| | 10 | 0 | exp. | 24.92 | 50.02 | 49.99 | 49.19 | 49.85 | 50.11 | 50.45 | 50.68 | 50.88 | 51.02 | 51.13 | 51.26 |
| | | | s.d. | 3.54 | 19.63 | 11.47 | 10.82 | 11.30 | 11.63 | 11.85 | 12.03 | 12.23 | 12.39 | 12.61 | 13.20 |
| | | | rmse | 25.33 | 19.63 | 11.47 | 10.85 | 11.30 | 11.63 | 11.86 | 12.05 | 12.26 | 12.44 | 12.66 | 13.26 |
| | 25 | 0 | exp. | 24.92 | 50.02 | 50.02 | 48.58 | 49.16 | 49.53 | 49.75 | 49.94 | 50.12 | 50.24 | 50.36 | 50.48 |
| | | | s.d. | 3.54 | 12.77 | 9.90 | 9.51 | 9.71 | 9.90 | 9.99 | 10.07 | 10.14 | 10.21 | 10.27 | 10.32 |
| | | | rmse | 25.33 | 12.77 | 9.90 | 9.61 | 9.75 | 9.91 | 9.99 | 10.07 | 10.14 | 10.22 | 10.28 | 10.33 |
| | 50 | 0 | exp. | 24.92 | 50.00 | 50.00 | 48.20 | 48.78 | 49.13 | 49.35 | 49.57 | 49.72 | 49.85 | 49.94 | 50.05 |
| | | | s.d. | 3.54 | 10.20 | 8.90 | 8.59 | 8.73 | 8.82 | 8.86 | 8.91 | 8.95 | 8.99 | 9.04 | 9.10 |
| | | | rmse | 25.33 | 10.20 | 8.90 | 8.78 | 8.82 | 8.86 | 8.88 | 8.93 | 8.95 | 9.00 | 9.04 | 9.10 |
| 0.92 | 0 | 0 | exp. | 30.07 | 30.07 | 50.00 | 50.80 | 50.89 | 50.85 | 50.96 | 50.98 | 51.06 | 51.03 | 51.07 | 51.03 |
| | | | s.d. | 3.46 | 3.46 | 9.78 | 9.51 | 9.90 | 10.12 | 10.33 | 10.48 | 10.64 | 10.83 | 11.20 | 12.84 |
| | | | rmse | 20.22 | 20.22 | 9.78 | 9.54 | 9.94 | 10.15 | 10.38 | 10.52 | 10.69 | 10.88 | 11.25 | 12.89 |
| | 5 | 0 | exp. | 30.07 | 49.89 | 50.00 | 50.27 | 50.40 | 50.48 | 50.46 | 50.51 | 50.59 | 50.60 | 50.67 | 50.60 |
| | | | s.d. | 3.46 | 21.09 | 8.95 | 8.70 | 8.93 | 9.10 | 9.20 | 9.30 | 9.38 | 9.45 | 9.49 | 9.62 |
| | | | rmse | 20.22 | 21.09 | 8.95 | 8.70 | 8.94 | 9.12 | 9.22 | 9.32 | 9.40 | 9.47 | 9.52 | 9.64 |
| | 10 | 0 | exp. | 30.07 | 50.04 | 50.00 | 49.96 | 50.14 | 50.13 | 50.28 | 50.31 | 50.32 | 50.34 | 50.40 | 50.33 |
| | | | s.d. | 3.46 | 15.31 | 8.44 | 8.18 | 8.45 | 8.49 | 8.60 | 8.63 | 8.70 | 8.75 | 8.79 | 8.85 |
| | | | rmse | 20.22 | 15.31 | 8.44 | 8.18 | 8.45 | 8.50 | 8.60 | 8.64 | 8.70 | 8.76 | 8.80 | 8.86 |
| | 25 | 0 | exp. | 30.07 | 50.00 | 50.00 | 49.46 | 49.63 | 49.75 | 49.84 | 49.89 | 49.93 | 49.98 | 50.07 | 49.97 |
| | | | s.d. | 3.46 | 10.22 | 7.58 | 7.39 | 7.50 | 7.57 | 7.62 | 7.65 | 7.68 | 7.73 | 7.73 | 7.80 |
| | | | rmse | 20.22 | 10.22 | 7.58 | 7.41 | 7.51 | 7.57 | 7.62 | 7.65 | 7.68 | 7.73 | 7.73 | 7.80 |
| | 50 | 0 | exp. | 30.07 | 50.03 | 49.99 | 49.12 | 49.34 | 49.49 | 49.60 | 49.69 | 49.70 | 49.74 | 49.84 | 49.71 |
| | | | s.d. | 3.46 | 8.23 | 6.97 | 6.82 | 6.90 | 6.92 | 6.95 | 6.98 | 7.01 | 7.00 | 7.05 | 7.07 |
| | | | rmse | 20.22 | 8.23 | 6.97 | 6.88 | 6.93 | 6.94 | 6.96 | 6.99 | 7.02 | 7.01 | 7.05 | 7.07 |

Table 2.6c

N = 50

| $\tau$ | R | P(Z<=1) x10^4 | | X1 | P | CUE | HME | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| 1.20 | 0 | 0 | exp. | 34.94 | 34.94 | 50.00 | 50.88 | 50.71 | 50.53 | 50.48 | 50.45 | 50.40 | 50.36 | 50.29 | 50.12 |
| | | | s.d. | 3.24 | 3.24 | 6.91 | 6.88 | 7.01 | 7.03 | 7.10 | 7.16 | 7.17 | 7.18 | 7.23 | 7.28 |
| | | | rmse | 15.41 | 15.41 | 6.91 | 6.94 | 7.05 | 7.05 | 7.12 | 7.17 | 7.18 | 7.19 | 7.23 | 7.28 |
| | 5 | 0 | exp. | 34.94 | 50.04 | 50.01 | 50.57 | 50.41 | 50.35 | 50.25 | 50.21 | 50.18 | 50.18 | 50.11 | 49.95 |
| | | | s.d. | 3.24 | 17.02 | 6.52 | 6.48 | 6.52 | 6.61 | 6.63 | 6.67 | 6.71 | 6.71 | 6.75 | 6.78 |
| | | | rmse | 15.41 | 17.02 | 6.52 | 6.51 | 6.54 | 6.62 | 6.63 | 6.68 | 6.71 | 6.71 | 6.75 | 6.78 |
| | 10 | 0 | exp. | 34.94 | 50.04 | 50.01 | 50.34 | 50.22 | 50.15 | 50.12 | 50.05 | 50.06 | 50.06 | 50.02 | 49.84 |
| | | | s.d. | 3.24 | 11.96 | 6.27 | 6.21 | 6.27 | 6.29 | 6.33 | 6.36 | 6.39 | 6.39 | 6.43 | 6.46 |
| | | | rmse | 15.41 | 11.96 | 6.27 | 6.22 | 6.27 | 6.29 | 6.33 | 6.36 | 6.39 | 6.39 | 6.43 | 6.47 |
| | 25 | 0 | exp. | 34.94 | 50.03 | 50.01 | 49.95 | 49.91 | 49.88 | 49.87 | 49.85 | 49.86 | 49.90 | 49.81 | 49.66 |
| | | | s.d. | 3.24 | 8.17 | 5.79 | 5.70 | 5.77 | 5.80 | 5.81 | 5.85 | 5.85 | 5.86 | 5.91 | 5.89 |
| | | | rmse | 15.41 | 8.17 | 5.79 | 5.70 | 5.77 | 5.80 | 5.82 | 5.86 | 5.85 | 5.86 | 5.91 | 5.90 |
| | 50 | 0 | exp. | 34.94 | 50.04 | 50.00 | 49.67 | 49.67 | 49.70 | 49.69 | 49.68 | 49.75 | 49.78 | 49.65 | 49.55 |
| | | | s.d. | 3.24 | 6.62 | 5.42 | 5.36 | 5.40 | 5.39 | 5.41 | 5.47 | 5.44 | 5.47 | 5.49 | 5.50 |
| | | | rmse | 15.41 | 6.62 | 5.42 | 5.37 | 5.41 | 5.40 | 5.42 | 5.48 | 5.45 | 5.48 | 5.50 | 5.52 |
| 1.61 | 0 | 0 | exp. | 40.01 | 40.01 | 50.01 | 50.79 | 50.54 | 50.30 | 50.19 | 50.14 | 50.06 | 49.93 | 49.78 | 49.63 |
| | | | s.d. | 2.83 | 2.83 | 4.66 | 4.68 | 4.71 | 4.74 | 4.76 | 4.75 | 4.77 | 4.78 | 4.78 | 4.77 |
| | | | rmse | 10.39 | 10.39 | 4.66 | 4.75 | 4.74 | 4.75 | 4.76 | 4.75 | 4.77 | 4.78 | 4.79 | 4.78 |
| | 5 | 0 | exp. | 40.01 | 50.05 | 49.99 | 50.61 | 50.32 | 50.18 | 50.10 | 50.06 | 49.96 | 49.85 | 49.71 | 49.55 |
| | | | s.d. | 2.83 | 12.57 | 4.51 | 4.52 | 4.52 | 4.58 | 4.56 | 4.58 | 4.59 | 4.60 | 4.60 | 4.59 |
| | | | rmse | 10.39 | 12.57 | 4.51 | 4.56 | 4.53 | 4.58 | 4.56 | 4.58 | 4.59 | 4.60 | 4.61 | 4.61 |
| | 10 | 0 | exp. | 40.01 | 50.02 | 49.99 | 50.47 | 50.23 | 50.11 | 50.03 | 50.00 | 49.92 | 49.79 | 49.65 | 49.51 |
| | | | s.d. | 2.83 | 8.90 | 4.39 | 4.38 | 4.42 | 4.42 | 4.44 | 4.45 | 4.47 | 4.48 | 4.46 | 4.44 |
| | | | rmse | 10.39 | 8.90 | 4.39 | 4.40 | 4.43 | 4.42 | 4.44 | 4.45 | 4.47 | 4.48 | 4.47 | 4.46 |
| | 25 | 0 | exp. | 40.01 | 50.00 | 50.00 | 50.27 | 50.06 | 49.94 | 49.91 | 49.90 | 49.82 | 49.68 | 49.57 | 49.43 |
| | | | s.d. | 2.83 | 6.19 | 4.16 | 4.13 | 4.15 | 4.21 | 4.21 | 4.20 | 4.22 | 4.22 | 4.18 | 4.18 |
| | | | rmse | 10.39 | 6.19 | 4.16 | 4.14 | 4.15 | 4.21 | 4.21 | 4.20 | 4.23 | 4.24 | 4.21 | 4.22 |
| | 50 | 0 | exp. | 40.01 | 50.01 | 50.00 | 50.03 | 49.90 | 49.81 | 49.84 | 49.83 | 49.70 | 49.59 | 49.52 | 49.36 |
| | | | s.d. | 2.83 | 5.04 | 3.97 | 3.97 | 3.96 | 4.02 | 4.01 | 4.01 | 4.01 | 4.03 | 3.97 | 3.94 |
| | | | rmse | 10.39 | 5.04 | 3.97 | 3.97 | 3.96 | 4.02 | 4.01 | 4.02 | 4.02 | 4.05 | 4.00 | 3.99 |
| 2.30 | 0 | 0 | exp. | 44.99 | 44.99 | 50.01 | 50.65 | 50.36 | 50.13 | 49.98 | 49.81 | 49.67 | 49.56 | 49.47 | 49.39 |
| | | | s.d. | 2.12 | 2.12 | 2.77 | 2.85 | 2.86 | 2.84 | 2.81 | 2.83 | 2.81 | 2.83 | 2.79 | 2.87 |
| | | | rmse | 5.44 | 5.44 | 2.77 | 2.93 | 2.89 | 2.85 | 2.81 | 2.84 | 2.83 | 2.87 | 2.84 | 2.93 |
| | 5 | 0 | exp. | 44.99 | 50.01 | 50.00 | 50.60 | 50.30 | 50.04 | 49.92 | 49.78 | 49.64 | 49.53 | 49.45 | 49.37 |
| | | | s.d. | 2.12 | 8.09 | 2.73 | 2.82 | 2.79 | 2.78 | 2.76 | 2.77 | 2.76 | 2.78 | 2.72 | 2.82 |
| | | | rmse | 5.44 | 8.09 | 2.73 | 2.88 | 2.80 | 2.78 | 2.76 | 2.78 | 2.79 | 2.82 | 2.78 | 2.89 |
| | 10 | 0 | exp. | 44.99 | 50.05 | 50.00 | 50.54 | 50.24 | 50.00 | 49.86 | 49.76 | 49.63 | 49.51 | 49.43 | 49.35 |
| | | | s.d. | 2.12 | 5.88 | 2.69 | 2.79 | 2.74 | 2.74 | 2.72 | 2.72 | 2.72 | 2.75 | 2.67 | 2.77 |
| | | | rmse | 5.44 | 5.88 | 2.69 | 2.84 | 2.75 | 2.74 | 2.73 | 2.73 | 2.74 | 2.80 | 2.73 | 2.85 |
| | 25 | 0 | exp. | 44.99 | 50.09 | 50.00 | 50.42 | 50.17 | 49.93 | 49.79 | 49.69 | 49.60 | 49.44 | 49.41 | 49.32 |
| | | | s.d. | 2.12 | 4.04 | 2.61 | 2.68 | 2.64 | 2.67 | 2.65 | 2.62 | 2.62 | 2.67 | 2.58 | 2.70 |
| | | | rmse | 5.44 | 4.05 | 2.61 | 2.71 | 2.64 | 2.67 | 2.66 | 2.64 | 2.65 | 2.73 | 2.64 | 2.79 |
| | 50 | 0 | exp. | 44.99 | 50.06 | 50.00 | 50.32 | 50.13 | 49.92 | 49.74 | 49.62 | 49.60 | 49.40 | 49.41 | 49.27 |
| | | | s.d. | 2.12 | 3.26 | 2.54 | 2.59 | 2.57 | 2.59 | 2.60 | 2.54 | 2.54 | 2.62 | 2.49 | 2.63 |
| | | | rmse | 5.44 | 3.26 | 2.54 | 2.61 | 2.58 | 2.60 | 2.61 | 2.57 | 2.57 | 2.68 | 2.56 | 2.73 |

§ 2.13 : The Performance of Plant-Capture
When Applied to the Model $M_0$ : Under Continuous Time Sampling

It has been argued that, conditional on the event $C=\{Z>X_1\}$, the performance of the CUE should be considered superior to that of the HME, and that the performance of the HME should in turn be considered superior to that of the MLE and Peterson-type estimators. In addition to this the distributional properties of the CUE, HME and Peterson-type estimators were investigated conditional on the less restrictive event $A=\{Z>1\}$. The conclusion drawn from the latter investigation was that, conditional on the event $A=\{Z>1\}$, one should again consider the CUE to be the estimator whose performance is most desirable. Rather than only discussing the way in which the information gained through plants may improve the performance of the CUE, this section considers how the method of plant-capture can affect the performance of all the estimators described within this chapter. This approach is taken since, in spite of the evidence of the previous sections, it is believed that more traditionally minded practitioners may still prefer to use the MLE. The following discussion is based on an inspection of all the 18 tables of this chapter.

It has previously been mentioned that mean square error is known to reward negative bias and that this characteristic can lead to incorrect conclusions being drawn if one places too much emphasis on mean square error alone. When comparing the performance of estimators, one should always, where possible, consider firstly their mean and standard deviation, and only then should one consider mean square error, or alternative loss functions such as mean absolute deviation. This approach is taken in the following discussion because consideration of mean square error alone can lead to counter intuitive conclusions. For example, consider the performance of the CUE in table 2.2a, where $N = 25$ and $\tau=0.36$. As R is increased from 0 to 5 to 10 the root mean square error of $\hat{N}_u$ increases from 11.21 to 11.33 to 11.98! However, only when one considers the way in which the bias of $\hat{N}_u$ is being significantly reduced can one see that the extra information gained from the plants is in fact improving the overall performance of the CUE.

This last example is quite typical of the way in which the information gained from plants enhances the performance of the estimators in situations where very little information is gained from the target population. In many of these situations, however, the improvement in bias is accompanied by a reduction in mean square error.

Except for a very few situations where only a small amount of information is available, the CUE $\hat{N}_u$ is usually unbiased, and where not its bias is negligible. In those situations where only a small amount of information is available, $\hat{N}_u$ tends to be negatively biased, with this bias reducing appreciably and monotonically as the number of plants is increased. This behaviour is intuitively very reasonable, since the CUE is unbiased conditional on the event $Z = Z_1 + Z_2 \geq N$. Now as $Z_2 \sim P(R\lambda\tau)$, the event $Z = Z_1 + Z_2 \geq N$ is more and more likely to occur as R is increased. The standard deviation of $\hat{N}_u$ is generally seen to reduce monotonically as more and more plants are used. Where the standard deviation of the CUE is not reduced by an increase in R, this is always due to its bias being significantly improved.

In the absence of plants, the Peterson-type estimator reduces to $X_1$, commonly referred to as the 'enumeration estimator'. For this reason, $\hat{N}_p$ is only considered here when plants are used. In terms of bias, the Peterson-type estimator behaves in a very similar way to the CUE, although $\hat{N}_u$ is on almost all occasions less biased. When the number of plants is small relative to the size of the target population, $\hat{N}_p$ tends to have a relatively large variance. On almost all occasions, however, this is seen to reduce uniformly as R is increased.

The CUE is seen to utilize the information gained through the plants in a very 'smooth' way. That is, as R is increased, usually either the mean of $\hat{N}_u$ is significantly improved at the expense of a slight increase in standard deviation or both the bias and the standard deviation are reduced. The behaviour of the MLE in situations where little information is available is less predictable as more plants are introduced. For an example consider again table 2.2a, in which N = 25 and $\tau$=0.36. As the number of plants is increased from 0 to 25 the mean and standard deviation of $\hat{N}$ are respectively 23.62, 14.35; 29.56, 19.81; 29.54, 19.99 and 26.66, 12.94 : the corresponding values taken by $\hat{N}_u$ are respectively 17.36, 8.21; 22.49, 11.05; 24.39, 11.96 and 24.95, 10.64.. The CUE in this example behaves in the manner described above, i.e. as R is increased its' performance improves 'smoothly'. However, as the number of plants is increased from 0 to 5 to 10, the standard deviation of the MLE is seen to increase, whilst its mean either becomes worse or is not significantly improved ! This result appears counter intuitive, that is until one considers the way in which the value of R affects the probability of obtaining a finite MLE. In the above situation, wherein N = 25 and $\tau$=0.36, for R = 0, 5, 10 and 25, the probabilities of obtaining an infinite MLE are respectively 0.2690, 0.0445, 0.0074 and 0.0000. In other words the introduction of plants is seen to drastically improve the probability of obtaining a useful MLE. When this advantage is considered along with the performance of the MLE, it can be argued that even in situations where, as in the above example, very little information is obtained from the target population, the presence of plants is beneficial to the overall performance of the MLE. Other than those extreme situations in which very little information is present, an increase in the number of plants is generally seen to improve the performance of the MLE via a reduction in both bias and standard deviation. And where both statistics are not improved, one of the two is.

In conclusion, the introduction of plants can be seen to enhance the performance of all four of the estimators which have been considered within this chapter, this being under the assumption that the planted individuals do indeed behave in an identical manner to members of the target population. In particular the plants are seen to be of most use when only little information has been gained from the target population. Furthermore, whether sampling with or without plants, it is recommended that, conditional on the event $C = \{Z > X_1\}$, the performance of the CUE should be considered superior to that of the HME, MLE and Peterson-type estimators.

# Chapter 3 : Estimation Under the Capture-Recapture Model $M_h$ : Discrete Time Sampling Procedure.

## §3.1 : Introduction

This chapter introduces, initially in the absence of plants, a new class of estimators for the standard capture-recapture model $M_h$. The model $M_h$ is one of the set of models discussed in the wildlife monograph by Otis et al. (1978) for capture-recapture data in closed populations. In section 3.7 it is shown how these new estimators can be modified so as to utilize the information gained from planted individuals.

In the previous two chapters consideration was given to a capture-recapture model which assumed that each animal in the target population was equally likely to be caught. Indeed in the large statistical literature on capture-recapture methods the majority of work present is seen to adopt this central assumption, see Seber(1982). However, although the assumption that every animal in the population is equally likely to be caught is convenient from a mathematical point of view, in practice this will rarely be the case. In particular if the population under investigation possesses significant heterogeneity between capture probabilities then, in experiments where the true population size is known, the usual estimators, for example the maximum likelihood estimators as described in chapters 1 and 2, have been shown to become extremely negatively biased, see Edwards & Eberhardt(1973) or Cormack(1966).

The discrete time sampling procedure considered within this chapter is identical to the one considered in chapter 1 : it essentially constitutes what is known in the literature as a Schnabel Census with random sample sizes, see Schnabel(1938) or, for a more comprehensive review, Seber(1982). The sampling procedure considered here is the one most commonly used in practice, and is described in detail in the following section.

## §3.2 : Sampling Procedure and Assumptions

A sequence of t sampling experiments is carried out on the target population which is assumed to be closed and of size N. Independently of other animals and independently of its previous capture history animal i ( $i = 1, 2, ..., N$ ) is captured in sample j ( $j = 1, 2, ..., t$ )

with probability $p_i$. After each sample is taken every animal within that sample which has not previously been marked receives a unique tag before its immediate release so that it may be recognised on subsequent trapping occasions. The experiment generates an N by t matrix A where

$$a_{ij} = \begin{cases} 1 & \text{if animal i is caught on sampling occasion j} \\ 0 & \text{if animal i is not caught on sampling occasion j} \end{cases}$$

$$i = 1, 2, ...., N.$$
$$j = 1, 2, ...., t.$$

The sample space is the set of such matrices.

It is assumed that the $p_i$, for i = 1, 2, ..., N, are a random sample from some probability distribution f(p), $p \in [0,1]$, with c.d.f. F(p).

## §3.3 : A Class of Coverage Adjusted Estimators for the Model $M_h$

Probabilistic results for the model $M_h$ are covered in detail by Otis et al (1978), Overton(1969) and Pollock & Otto(1983). The following derivation of an approximate maximum likelihood estimator borrows heavily from this previous work : the approach taken here is almost identical to that of Overton(1969) and Pollock & Otto(1983). At this point it is necessary to introduce some notation :

$t \equiv$ number of sampling occasions.

$X \equiv$ number of distinct animals seen.

$Z \equiv$ total number of sightings made.

$p_i \equiv$ capture probability of animal i, i = 1, 2,..., N.

$X_i \equiv$ number of sightings of the $i^{th}$ animal, i = 1, 2, ..., N.

$f_k = \sum_{i=1}^{N} I(X_i = k) \equiv$ number of animals seen exactly k times, k= 0, 1, 2, ...., t.

$$C = \frac{\sum_{i=1}^{N} p_i I(X_i > 0)}{\sum_{i=1}^{N} p_i} \equiv \text{' Sample Coverage '.}$$

Let the set $S_x = \{s_k : k = 1,2,..,x\}$, where $s_k \in \{1,2,3,...,N\}$ for all k, denote the set of the indexes of the x distinct animals seen during the sampling period.

Under this model all of the available information is contained within the vector of capture frequencies $(f_1, f_2, \ldots, f_t)$, see Pollock et al (1990).

The joint probability distribution of the sufficient statistics, $\{f_i : i = 1,2,\ldots,t\}$, is multinomial:

$$\mathrm{Prob}(f_1, f_2, \ldots, f_t | F) = \binom{N}{N-x, f_1, f_2, \ldots, f_t} (\pi_0)^{N-x} \prod_{j=1}^{t} (\pi_j)^{f_j} \tag{3.1}$$

$$\text{where } \pi_j = \int_0^1 \binom{t}{j} p^j (1-p)^{t-j} dF(p)$$

$$j = 0,1,2,\ldots,t.$$

From equation (3.1), assuming that F(p) is known exactly, the profile likelihood for N can be written as

$$\begin{aligned} L(N) &= \frac{N!}{(N-x)!} \{\pi_0\}^{N-x} \\ &= \frac{N!}{(N-x)!} \left\{ E\left[(1-p)^t\right] \right\}^{N-x}, \end{aligned} \tag{3.2}$$

$$\text{since } \pi_0 = \int_0^1 (1-p)^t dF(p) = E\left[(1-p)^t\right].$$

If one again assumes that F(p) is known exactly, so that $E\left[(1-p)^t\right]$ may be viewed as a known constant, it follows from equation (3.2) that an approximate maximum likelihood estimate may be obtained by equating L(N) to L(N-1) :

$$\begin{aligned} L(N) = L(N\text{-}1) \quad &\Leftrightarrow \quad \frac{N!}{(N-x)!} \left\{ E\left[(1-p)^t\right] \right\}^{N-x} = \frac{(N-1)!}{(N-1-x)!} \left\{ E\left[(1-p)^t\right] \right\}^{N-1-x} \\ &\Leftrightarrow \quad N\left\{ E\left[(1-p)^t\right] \right\} = (N-x) \\ &\Leftrightarrow \quad N - N\left\{ E\left[(1-p)^t\right] \right\} = x \\ &\Leftrightarrow \quad N = \frac{x}{1 - \left\{ E\left[(1-p)^t\right] \right\}}. \end{aligned} \tag{3.3}$$

This expression is of no direct use in practice since $E\left[(1-p)^t\right]$ is not known. However an estimate of this quantity may be obtained : Overton(1969) used a method based on a theorem of Horvitz & Thompson(1952) ; Pollock & Otto(1983) obtained exactly the same result using the theory of weighted distributions as follows :

$$\begin{aligned} E\left[(1-p)^t\right] = \pi_0 &= \int_0^1 (1-p)^t dF(p) \\ &= \int_0^1 (1-p)^t f(p) dp. \end{aligned}$$

Let $f^w(p)$ be the probability density function of the capture probabilities of all x animals captured during the experiment. When $f^w(p)$ is derived from f(p) as a weighted distribution, see Patil & Rao(1978), with weight $w(p)=1-(1-p)^t$, the probability of capture at least once,

$$f^w(p)=\frac{w(p)f(p)}{\mu}$$

$$=\frac{\{1-(1-p)^t\}f(p)}{1-E\left[(1-p)^t\right]}.$$

Now one may observe that, using the properties of weighted distributions, an unbiased estimator of $\left\{1-E\left[(1-p)^t\right]\right\}^{-1}$ is given by $\frac{1}{x}\sum_{i\in S_x}\frac{1}{1-(1-p_i)^t}$.

It then follows from equation (3.3) that

$$\hat{N}=\sum_{i\in S_x}\frac{1}{1-(1-p_i)^t} \tag{3.4}$$

would be an unbiased estimator of population size N if the capture probabilities of the animals seen during the experiment were known exactly. At this point it is also worthwhile to note that, since the probability of animal i being seen at least once during the experiment is $1-(1-p_i)^t$ for i = 1, 2,..., N, taking an expectation of (3.4), when the $p_i$ for $i\in S_x$ are known, would show $\hat{N}$ to be an unbiased estimator of N. However since these capture probabilities are clearly not known exactly the approach taken here is to estimate the $p_i$ and in doing so obtain an estimator of N by substituting these estimates of capture probability into equation (3.4).

It is now necessary to estimate the capture probabilities of the animals which were seen during the experiment :

Overton(1969) used the fact that under the model $X_i \sim Bin(t,p_i)$. Based on this distribution the maximum likelihood estimate of the capture probability of animal i is given by $\hat{p}_i^{(1)}=\frac{x_i}{t}$. Overton(1969) then substituted the estimates $\hat{p}_i^{(1)}$ into equation (3.4) to produce the estimator $\hat{N}_O$, defined by

$$\hat{N}_O=\sum_{i\in S_x}\frac{1}{1-\left[1-\hat{p}_i^{(1)}\right]^t}$$

$$=\sum_{i\in S_x}\frac{1}{1-\left(1-\frac{x_i}{t}\right)^t}$$

$$=\sum_{i=1}^{t}\frac{f_i}{1-\left(1-\frac{i}{t}\right)^t}.$$

The estimates $\hat{p}_i^{(1)}$ are intuitively reasonable estimates of capture probability - essentially being 'the number of times the animal was seen divided by the number of times the animal could have been seen'. However this method of estimating capture probability does not make full use of all of the available information. In order to obtain better estimates of capture probability one may proceed as follows :

In addition to estimating the capture probability of each animal in turn, via the $\hat{p}_i^{(1)}$, we are also able to obtain independently an estimate of the sum of the capture probabilities of the animals seen during sampling. The approach taken here is to use this latter estimate to scale the $\hat{p}_i^{(1)}$ in an appropriate manner :

Let

$$\hat{p}_i^{(2)} \propto \frac{X_i}{t} \Rightarrow \hat{p}_i^{(2)} = k\frac{X_i}{t}, \qquad \text{where } k = \text{constant}. \tag{3.5}$$

It then follows from equation (3.5) that

$$\sum_{i \in S_x} \hat{p}_i^{(2)} = \sum_{i \in S_x} \frac{k.X_i}{t} = \frac{k}{t}\sum_{i \in S_x} X_i = k\frac{Z}{t}. \tag{3.6}$$

For the reasons stated above we now set

$$\sum_{i \in S_x} \hat{p}_i^{(2)} = \sum_{i \in S_x} p_i = \sum_{i=1}^{N} p_i I(X_i > 0) = C\sum_{i=1}^{N} p_i, \quad \text{where } C = \frac{\sum_{i=1}^{N} p_i I(X_i > 0)}{\sum_{i=1}^{N} p_i}. \tag{3.7}$$

From assumptions,

$$\begin{aligned} X_i \sim \mathrm{Bin}(t, p_i) &\Rightarrow E[X_i] = t.p_i \\ &\Rightarrow E\left[\sum_{i=1}^{N} X_i\right] = E[Z] = t\sum_{i=1}^{N} p_i \end{aligned} \tag{3.8}$$

$\Rightarrow$ an estimate of $\sum_{i=1}^{N} p_i$ is given by $\frac{Z}{t}$. The value of sample coverage C may be estimated by $\hat{C}_1$, $\hat{C}_2$ or $\hat{C}_3$ : please refer to appendix 4.

Now from equation (3.7) we require that $\sum_{i \in S_x} \hat{p}_i^{(2)} = \hat{C}_j \frac{Z}{t}$, $\qquad j = 1, 2 \text{ or } 3. \qquad (3.9)$

Combining equations (3.6) and (3.9) enables one to determine the value of the constant k as follows :

$$\sum_{i \in S_x} \hat{p}_i^{(2)} = k\frac{Z}{t} = \hat{C}_j \frac{Z}{t} \quad \Rightarrow \quad k = \hat{C}_j, \qquad j = 1, 2 \text{ or } 3,$$

$$\Rightarrow \quad \text{Estimate } p_i \text{ by} \quad \hat{p}_{i,j}^{(2)} = \hat{C}_j \frac{X_i}{t}, \qquad i \in S_x, \qquad j = 1, 2 \text{ or } 3.$$

Explicitly

$$\hat{p}_{i,1}^{(2)} = \hat{C}_1 \frac{x_i}{t} = \left(1 - \frac{f_1}{z}\right)\frac{x_i}{t},$$

$$\hat{p}_{i,2}^{(2)} = \hat{C}_2 \frac{x_i}{t} = \left(1 - \frac{f_1}{z} + \frac{2}{(t-1)}\frac{f_2}{z}\right)\frac{x_i}{t}$$

and
$$\hat{p}_{i,3}^{(2)} = \hat{C}_3 \frac{x_i}{t} = \left(1 - \frac{f_1}{z} + \frac{2}{(t-1)}\frac{f_2}{z} - \frac{6}{(t-1)(t-2)}\frac{f_3}{z}\right)\frac{x_i}{t}.$$

These three estimates of capture probability may each be substituted into equation (3.4) to produce a corresponding Coverage Adjusted Estimator ( CAE ):

$$\begin{aligned}\hat{N}_{ca1} &= \sum_{i \in S_x} \frac{1}{1 - \left[1 - \hat{p}_{i,1}^{(2)}\right]^t} \\ &= \sum_{i \in S_x} \frac{1}{1 - \left(1 - \left(1 - \frac{f_1}{z}\right)\frac{x_i}{t}\right)^t} \\ &= \sum_{i=1}^{t} \frac{f_i}{1 - \left(1 - \left(1 - \frac{f_1}{z}\right)\frac{i}{t}\right)^t}.\end{aligned}$$

Similarly

$$\hat{N}_{ca2} = \sum_{i=1}^{t} \frac{f_i}{1 - \left(1 - \left(1 - \frac{f_1}{z} + \frac{2}{(t-1)}\frac{f_2}{z}\right)\frac{i}{t}\right)^t}$$

and
$$\hat{N}_{ca3} = \sum_{i=1}^{t} \frac{f_i}{1 - \left(1 - \left(1 - \frac{f_1}{z} + \frac{2}{(t-1)}\frac{f_2}{z} - \frac{6}{(t-1)(t-2)}\frac{f_3}{z}\right)\frac{i}{t}\right)^t}.$$

## §3.4 : Other Estimators for the Model $M_h$

Within the introduction to this chapter it was stated that the majority of capture-recapture work has been based on the assumption that capture probabilities are equal for all animals in the population being trapped. This is true. However, over the years, a number of authors have considered heterogeneous populations. Overton(1969) introduced the above Horvitz-Thompson type estimator which allows capture probabilities to vary between animals. The performance of this estimator was, however, not considered to be reasonable, so that, up to this date and beyond, a viable estimation procedure for heterogeneous populations was still not available. Burnham and Overton(1978) sought to rectify this less than ideal situation by introducing a nonparametric jackknife estimator of population size, aimed at heterogeneous populations. The Burnham and Overton(1978) paper stimulated interest in the topic and since its appearance many other authors have proposed estimators for the model $M_h$. Pollock and Otto(1983) considered a first order jackknife of the estimator proposed by Overton(1969). Smith and van Belle(1984) considered bootstrapping based on the enumeration estimator. Chao(1989), Chao, Lee and Jeng(1992) and, more recently, Norris and Pollock(1996a) have also proposed estimators for the model $M_h$.

To date, the estimators most commonly favoured have been the jackknife estimators of Burnham and Overton(1978). Lee and Chao(1994), however, assert that the estimators of Chao, Lee and Jeng(1992) are to be preferred, except when the heterogeneity is very mild. In the latter case they recommend the maximum likelihood estimator for the model $M_0$, details of which may be found in chapter 1 of this thesis.

## §3.5 : Simulation Study

A simulation study was carried out in order to investigate the properties of each estimator. In each simulation the capture probabilities of the N animals were drawn as a random sample from some probability distribution with mean E(p), variance Var(p) and coefficient of variation sqrt[Var(p)]/E(p). Live trapping was then simulated on this population. Each table consists of six cells, with each cell depicting the results for one of t = 5, 10, 15, 20, 25 or 30 sampling occasions. For each value of t one thousand simulations were carried out : a different set of capture probabilities was used each time. The values shown in the tables are mostly averages. As many of the estimators are only finite if at least one recapture occurs, any data set not meeting this condition was discarded. The simulation procedure continued until one thousand data sets for which the condition did hold had been generated.

In tables 3.5.1a, 3.5.1b and 3.5.1c the capture probabilities of the animals were drawn from a uniform distribution on the interval (0, alpha), symbolised by p ~ U(0, alpha).

In tables 3.5.2a to 3.5.2i the distribution considered is Beta : symbolised by p ~ Beta (alpha, beta). The most comprehensive simulation study to appear in the literature to date was carried out by Burnham and Overton(1979), and they considered mainly Beta distributions. It appeared that they essentially varied the parameters of each Beta distribution in such a way so as to achieve particular values for the expectation of p. That is prime consideration was given to E(p). It has, however, been known for many years, Cormack(1966) and Edwards and Eberhardt(1973), that the performance of the estimators is dependent not only on the mean but also the coefficient of variation of the distribution of p. Hence within this simulation study prime consideration is given jointly to both the mean and the coefficient of variation of each Beta distribution; the parameters of the distribution leading to these values are considered to be of only secondary importance. It was therefore decided to choose the parameters of each Beta distribution systematically in such a way so as to investigate the dependence of the estimators performance on both mean and coefficient of variation. In practice it is believed that if the model $M_h$ is chosen as an appropriate model to fit the data, using for example the testing procedures described by Otis et al.(1978), then one may expect to see a true coefficient of variation approximately in the range of $0.55 \pm 0.25$. Since, if the true coefficient of variation was below say 0.3 one would expect a choice of model $M_0$. Whereas if there appeared to be a very large coefficient of variation, say greater than 0.8, it would be reasonable to assume that model $M_h$ would be rejected anyway - in favour of perhaps $M_{th}$ or $M_{bh}$. It is believed that in most practical situations one may expect the expectation of the distribution of p to vary between 0.04 and 0.20. For the above reasons tables 3.5.2a to 3.5.2i cover the following nine points in the ( E(p), sqrt[Var(p)]/E(p)) plane :

$$\begin{array}{rcl} & 0.04 & & & 0.3 \\ \mathrm{E(p)} = & 0.12 & \times & \mathrm{sqrt[Var(p)]/E(p)} = & 0.55. \\ & 0.20 & & & 0.8 \end{array}$$

At each point in the above grid it necessarily follows that the parameters of the Beta distribution satisfy the equations $\mathrm{alpha} = \frac{1-\mathrm{ep}}{(\mathrm{cv})^2} - \mathrm{ep}$ and $\mathrm{beta} = \frac{(1-\mathrm{ep})^2}{\mathrm{ep}(\mathrm{cv})^2} - (1-\mathrm{ep})$,

where $\mathrm{ep} = \mathrm{E(p)}$ and $\mathrm{cv} = \mathrm{sqrt[Var(p)]/E(p)}$.

The range of detection probabilities covered by the above grid is consistent with the simulation studies of Burnham and Overton(1979), Chao, Lee and Jeng(1992) and Lee and Chao(1994).

In order to further investigate the robustness of the estimators over the entire subset of the ( E(p), sqrt[Var(p)]/E(p)) plane, which is considered here to be appropriate for the model $M_h$, the simulations of tables 3.5.3a, 3.5.3b and 3.5.3c were carried out in a slightly different way. At the beginning of each simulation the value of E(p) was selected as a random observation from a uniform distribution on the interval (0.04, 0.20). The value of the coefficient of variation was selected as a random observation from a uniform distribution on the interval (0.3, 0.8). The distribution of p was chosen to be Beta(alpha, beta), and so to achieve the required values of ep and cv it was required that $\text{alpha} = \frac{1-\text{ep}}{(\text{cv})^2} - \text{ep}$ and $\text{beta} = \frac{(1-\text{ep})^2}{\text{ep}(\text{cv})^2} - (1-\text{ep})$. The capture probabilities of the N animals were drawn as a random sample from this beta distribution and live trapping was then simulated in the usual way.

## Notation and Estimators

N $\equiv$ population size.

t $\equiv$ number of sampling occasions.

The following estimators are considered within the simulation study. The notation used for each estimator is stated, and where possible a detailed expression for the estimator is given.

x $\equiv$ number of distinct individuals seen.

mle $\equiv \hat{N}$ the maximum likelihood estimator for the model $M_0$, for details please refer to chapter 1.

boot $\equiv \hat{N}_B = x + \sum_{i=1}^{t} f_i \left(1 - \frac{i}{t}\right)^t$ the bootstrap estimator of Smith and van Belle(1984).

dr1 $\equiv \hat{N}_{0,1} = \frac{x}{\hat{C}_1}$ represents the estimator proposed by Darroch & Ratcliff(1980) for the classical species problem.

$$\text{ac1} = \hat{N}_{ac1} = \frac{x}{\hat{C}_1} + \frac{f_1}{\hat{C}_1}\hat{\gamma}_1^2$$

$$\text{ac2} = \hat{N}_{ac2} = \frac{x}{\hat{C}_2} + \frac{f_1}{\hat{C}_2}\hat{\gamma}_2^2$$

$$\text{ac3} = \hat{N}_{ac3} = \frac{x}{\hat{C}_3} + \frac{f_1}{\hat{C}_3}\hat{\gamma}_3^2$$

the three estimators proposed by Chao, Lee and Jeng(1992), see also Lee and Chao(1994).

$$\hat{\gamma}_i^2 = \max\left\{ \frac{\hat{N}_{0,i} t \sum_{k=2}^{t} k(k-1) f_k}{(t-1)\left[\sum_{k=1}^{t} k f_k\right]^2} - 1, 0 \right\}, \quad i = 1, 2, 3.$$

$$\text{O} \equiv \hat{N}_O = \sum_{i=1}^{t} \frac{f_i}{1-\left(1-\frac{i}{t}\right)^t}$$ the estimator of Overton(1969).

$$\text{ca1} \equiv \hat{N}_{ca1} = \sum_{i=1}^{t} \frac{f_i}{1-\left(1-\left(1-\frac{f_1}{z}\right)\frac{i}{t}\right)^t}.$$

$$\text{ca2} \equiv \hat{N}_{ca2} = \sum_{i=1}^{t} \frac{f_i}{1-\left(1-\left(1-\frac{f_1}{z}+\frac{2}{(t-1)}\frac{f_2}{z}\right)\frac{i}{t}\right)^t}.$$ the CAEs of section 3.3.

$$\text{ca3} \equiv \hat{N}_{ca3} = \sum_{i=1}^{t} \frac{f_i}{1-\left(1-\left(1-\frac{f_1}{z}+\frac{2}{(t-1)}\frac{f_2}{z}-\frac{6}{(t-1)(t-2)}\frac{f_3}{z}\right)\frac{i}{t}\right)^t}.$$

$\text{pojac} \equiv \hat{N}_U$ the first order jackknife of the estimator proposed by Overton(1969), considered by Pollock and Otto(1983). Goudie(1996, pers.comm.) noted a typographical error in that paper . That is $\hat{N}_U$ is in fact explicitly given by

$$\hat{N}_U = f_1\left\{t.a_{1,t} - \frac{(t-1)^2}{t}a_{1,t-1}\right\} + \sum_{i=2}^{t} f_i\left[t.a_{i,t} - \frac{(t-1)}{t}\left\{i.a_{i-1,t-1} + (t-i)a_{i,t-1}\right\}\right],$$

$$\text{where} \quad a_{i,t} = \left\{1-\left(1-\frac{i}{t}\right)^t\right\}^{-1}.$$

$\text{jac1} \equiv \hat{N}_{J1} = x + \frac{(t-1)}{t}f_1$ the first order jackknife estimator of Burnham & Overton(1978).

$\text{jacseq} \equiv \hat{N}_{Jk}$ the jackknife of order k - where k is chosen from the set (1,2,3,4) according to the procedure of Burnham & Overton(1978).

$\text{jacint} \equiv \hat{N}_J$ the interpolated jackknife estimator of Burnham & Overton(1979).

N.B. The selection procedure proposed by Burnham and Overton(1978) is not entirely objective. That is if the fourth order jackknife is rejected their recommendation is to select the jackknife of whichever of the first three orders appears most appropriate. In order to avoid this subjectivity it was decided that, in the event of the selection procedure rejecting the fourth order jackknife, the estimator $\hat{N}_{Jk}$ should be made equivalent to $\hat{N}_{J1}$. That is the selection procedure has been slightly modified, but only to the extent of always selecting the first order jackknife when the fourth order is rejected. Alternative modifications of the selection procedure were investigated. However, the above version of the sequential jackknife is the one whose performance was best. Some of the alternative modifications of the sequential jackknife that were considered may merit future consideration.

When the selection procedure chooses the first order jackknife then the interpolated jackknife $\hat{N}_J$ is equal to $\hat{N}_{J1}$. When the selection procedure chooses the jackknife of order k, for k = 2, 3 or 4, then $\hat{N}_J$ is a weighted average of the jackknives of orders k and k-1. When the selection procedure rejects the fourth order jackknife, $\hat{N}_J$ is equal to $\hat{N}_{J1}$.

s.d. $\equiv$ standard deviation.

rmse $\equiv$ root mean square error.

Pr( inf mle ) $\equiv$ the probability that an estimator in the set {mle,dr1,ac1,ca1} is infinite : each estimator in this set is infinite if and only if $1 - \frac{f_1}{z} = 0$.

N.B. All results are given conditional on $1 - \frac{f_1}{z} > 0$.

$$c \equiv C = \frac{\sum_{i=1}^{N} p_i I(X_i > 0)}{\sum_{i=1}^{N} p_i} \equiv \text{' Sample Coverage '}.$$

ch1 $\equiv \hat{C}_1$

ch2 $\equiv \hat{C}_2$ — the three estimators of sample coverage proposed by Chao, Lee and Jeng(1992), see section 3.3.

ch3 $\equiv \hat{C}_3$

cv1 $\equiv \hat{\gamma}_1$

cv2 $\equiv \hat{\gamma}_2$ — the three estimators of the coefficient of variation proposed by Chao, Lee and Jeng(1992), where $\hat{\gamma}_i$ , for i = 1, 2, 3, are as above.

cv3 $\equiv \hat{\gamma}_3$

N.B. Two of the estimators which were discussed in section 3.4 have not been included in the simulation study. The estimator introduced by Chao(1989) produces infinite estimates when no animals are seen exactly twice during the sampling experiment, and so to avoid imposing additional constraints on each simulation this estimator was omitted from the study. The nonparametric maximum likelihood estimator of Norris and Pollock(1996a) was also ommitted : Norris and Pollock(1996a) noted that although possessing small bias, the variance of their estimator was usually large when compared to the estimators of Chao, Lee and Jeng(1992).

Discussion

The following discussion is based on an inspection of all fifteen tables, but with particular attention being paid to tables 3.5.3a, 3.5.3b and 3.5.3c, which are believed to give the best overall view of how the estimators perform in practice.

Firstly, as one would expect, the estimators designed for the model $M_0$, namely the maximum likelihood estimator $\hat{N}$ and the Darroch and Ratcliff estimator $\hat{N}_{0,1}$, perform well in situations when the heterogeneity is mild. However these estimators are seen to become negatively biased when the coefficient of variation becomes significantly large, and it is these situations in which their performance is unsatisfactory.

The bootstrap estimator of Smith and van Belle(1984) does not perform well. In almost all situations it is negatively biased, even when a large proportion of the population is seen during the experiment. And when a small proportion of the population is seen, its negative bias is extreme. The performance of the first order jackknife estimator $\hat{N}_{J1}$ generally dominates that of the bootstrap estimator $\hat{N}_B$. This is an intuitively reasonable outcome : both $\hat{N}_{J1}$ and $\hat{N}_B$ are based on the enumeration estimator x. But whereas x is an ideal estimator to jackknife, being biased with small variance, it is not ideal for bootstrapping. For point estimation, it would be best to bootstrap an estimator with small bias and large variance. Within this context, however, it is believed that bootstrapping would be of most use in obtaining confidence intervals.

In terms of bias, the Coverage Adjusted estimators clearly perform better than the Overton estimator $\hat{N}_O$ - particularly when sample coverage is small. The reason for this being that the estimators $\hat{p}_i^{(1)}$ tend to overestimate capture probability and consequently the estimator $\hat{N}_O$, which directly incorporates the $\hat{p}_i^{(1)}$, has a tendency to always underestimate population size. When sample coverage is small, the estimators $\hat{p}_i^{(1)}$ are particularly positively biased and so $\hat{N}_O$ is particularly negatively biased. The estimators $\hat{p}_{i,j}^{(2)}$ are reasonable estimators of capture probability for most values of sample coverage and as a result of this $\hat{N}_{ca1}$, $\hat{N}_{ca2}$ and $\hat{N}_{ca3}$ generally possess an acceptable mean for all values of sample coverage. As one would expect, the Coverage Adjusted estimators perform in a very similar way to the Overton estimator $\hat{N}_O$ when a large proportion of the population is seen during sampling. This is easily explained by the fact that, for each i and j, the value of $\hat{p}_{i,j}^{(2)}$ tends towards that of $\hat{p}_i^{(1)}$ as t is increased - since $\hat{C}_j \to 1$ as $t \to \infty$, for j=1,2,3. As a consequence of the significant improvement in mean which $\hat{N}_{ca1}$, $\hat{N}_{ca2}$ and $\hat{N}_{ca3}$ have over $\hat{N}_O$, the Coverage Adjusted estimators, although having a larger variance than $\hat{N}_O$, usually posses a much smaller mean square error.

Consider the relative performance of the Coverage Adjusted estimators and the estimators of Chao, Lee and Jeng(1992). It is, as a first step, worthwhile to note that, in most situations, the standard deviations of $\hat{N}_{ca1}$, $\hat{N}_{ca2}$ and $\hat{N}_{ca3}$ are less than those of $\hat{N}_{ac1}$, $\hat{N}_{ac2}$ and $\hat{N}_{ac3}$ respectively. With reference to tables 3.5.2a, 3.5.2d and 3.5.2g one may observe that in situations wherein the coefficient of variation is small, the coverage adjusted estimators can become positively biased, whereas in contrast $\hat{N}_{ac1}$, $\hat{N}_{ac2}$ and $\hat{N}_{ac3}$ each tend to possess a very good mean value. Despite this, in situations of this type wherein less than about 60% of the population is seen during the experiment, the coverage adjusted estimators, owing to their smaller variance, are seen to be performing best in terms of mean square error. With reference to tables 3.5.2b,c, 3.5.2e,f and 3.5.2h,i one may observe that in situations wherein a moderate to large coefficient of variation is present, the coverage adjusted estimators tend to perform well : they generally posses a good mean value and a relatively small variance. Consequently in these situations the coverage adjusted estimators tend to perform better than the estimators of Chao, Lee and Jeng(1992) in terms of both mean and variance. Hence for the majority of situations in which the model $M_h$ would seem to be the most appropriate choice it is seen that the coverage adjusted estimators tend to perform better than the estimators of Chao, Lee and Jeng(1992). Tables 3.5.3a,b and c support this conclusion.

It is clear from tables 3.5.1a, 3.5.1b and 3.5.1c that the estimators of Chao, Lee and Jeng(1992) do not respond well when the heterogeneity results from the capture probabilities having a uniform distribution. In situations of this type, even when a large proportion of the population is seen during the experiment, the estimators of Chao, Lee and Jeng(1992) can be very negatively biased. In contrast to this the coverage adjusted estimators perform particularly well in tables 3.5.1a, 3.5.1b and 3.5.1c.

When considering the problem of estimating sample coverage it is seen that the estimators $\hat{C}_2$ and $\hat{C}_3$ are to be preferred to $\hat{C}_1$. This is particularly true when the number of sampling occasions is small. For this reason, along with observing the performance of the estimators in the tables, it is recommended that $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ be preferred to $\hat{N}_{ca1}$. In a similar way it is believed that $\hat{N}_{ac2}$, $\hat{N}_{ac3}$ should be preferred to $\hat{N}_{ac1}$.

The comparison between $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ and the first order jackknife estimator $\hat{N}_{J1}$ is seen to depend mainly upon the value of sample coverage - or equivalently upon our estimate of sample coverage since this quantity may be estimated very well. If sample coverage is greater than say 0.7 then $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ and $\hat{N}_{J1}$ all perform very well in that they have small bias and relatively small variance. However if sample coverage is less than 0.7 the first order jackknife tends to be very negatively biased whereas $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ continue to achieve a good mean value. The first order jackknife estimator has a very small variance and other than x, the enumeration estimator, and $\hat{N}_B$, the bootstrap estimator, usually has the smallest variance of all of the estimators. For this reason, even though $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ have a better mean than $\hat{N}_{J1}$, when sample coverage is less than 0.7, the first order jackknife estimator can have a smaller mean square error. Even though this is true, it is believed that overall the Coverage Adjusted estimators $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ are to be preferred to the first order jackknife estimator.

The sequential jackknife estimator is generally seen to be superior to the first order jackknife in terms of mean - particularly when sample coverage is small. This is due to the fact that the sequential selection procedure developed by Burnham & Overton(1978) generally works well : when sample coverage is low high order jackknives are usually chosen whereas when sample coverage is high low order jackknifes are usually chosen. However even with a lot of data the sequential selection procedure used to determine the sequential jackknife can be unpredictable. Rosenberg, Overton and Anthony(1995) stated that, when capture probabilities are low and heterogeneous, the selection procedure should be 'treated with caution'. The performance of the selection procedure results in the sequential jackknife estimator having a high variance even with good data. A good indication as to which jackknife estimator would be most appropriate is sample coverage. That is if our estimate of sample coverage is high, say above 0.7, then the first order, or second order, jacknife estimator should be considered most appropriate. If however sample coverage is smaller then the sequential selection procedure of Burnham & Overton(1978) is to date the best way of deciding which order jackknife to choose.

Consider the interpolated jackknife estimator. Due to the imprecise and often unpredictable nature of the sequential selection procedure, it is believed that the interpolated jackknife does not differ greatly enough from the sequential jackknife to warrant consideration as an estimator in its own right. In other words the difference between the interpolated jackknife and the sequential jackknife, in any one given situation, is believed to be insignificant when compared with the variance of the sequential jackknife estimator.

Of the jackknife estimators then, the first order jackknife is to be preferred if sample coverage is high, say above 0.7, whereas if sample coverage is small then the sequential jackknife should be considered a more appropriate choice.

Our conclusion above was that $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ are to be preferred to the first order jackknife. The above discussion now also implies that $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ are to be preferred to the sequential jackknife estimator when sample coverage is above 0.7. It remains to consider how $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ compare to the sequential jackknife when sample coverage is small, or rather less than about 0.7. When sample coverage is very small, usually for $t = 5$ sampling occasions, the sequential jackknife estimator, although being negatively biased, can have a smaller mean square error than $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ - although in this situation $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ tend to have a much better mean. As sample coverage becomes larger, or as t is increased, the sequential jackknife estimator becomes less biased but, as mentioned above, does have a relatively large variance. As a result of this the Coverage Adjusted estimators $\hat{N}_{ca2}$, $\hat{N}_{ca3}$ generally perform far better than the sequential jackknife estimator in terms of mean square error - whilst at the same time performing in very much the same way in terms of bias.

Table 3.5.1a

N = 100 : p ~ U(0, 0.08) : E(p) = 0.04 : sqrt[Var(p)]/E(p) = 0.5774

Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 18.35 | -81.65 | 3.759 | 81.738 |
| mle | 82.56 | -17.44 | 46.494 | 49.656 |
| dr1 | 106.82 | 6.82 | 60.535 | 60.918 |
| boot | 23.77 | -76.23 | 4.951 | 76.388 |
| ac1 | 126.91 | 26.91 | 77.129 | 81.689 |
| ac2 | 92.28 | -7.72 | 56.302 | 56.829 |
| ac3 | 96.32 | -3.68 | 67.336 | 67.436 |
| o | 26.36 | -73.64 | 5.494 | 73.841 |
| ca1 | 109.29 | 9.29 | 60.930 | 61.634 |
| ca2 | 90.27 | -9.73 | 48.950 | 49.907 |
| ca3 | 91.27 | -8.73 | 49.453 | 50.218 |
| pojac | 45.67 | -54.33 | 10.066 | 55.251 |
| jac1 | 31.25 | -68.75 | 6.613 | 69.069 |
| jacseq | 48.03 | -51.97 | 11.405 | 53.209 |
| jacint | 45.22 | -54.78 | 10.816 | 55.843 |

Pr(inf mle) , mean jacknife order = 0.159, 3.839
c, ch1, ch2, ch3 = 0.241, 0.218 , 0.267 , 0.264
cvh1, cvh2, cvh3 = 0.392289, 0.089107, 0.102428

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 31.67 | -68.33 | 4.462 | 68.479 |
| mle | 85.35 | -14.65 | 42.942 | 45.373 |
| dr1 | 95.75 | -4.25 | 49.241 | 49.425 |
| boot | 40.95 | -59.05 | 5.822 | 59.338 |
| ac1 | 105.16 | 5.16 | 57.724 | 57.954 |
| ac2 | 93.65 | -6.35 | 50.023 | 50.424 |
| ac3 | 94.40 | -5.60 | 50.611 | 50.921 |
| o | 45.64 | -54.36 | 6.537 | 54.751 |
| ca1 | 101.53 | 1.53 | 49.580 | 49.603 |
| ca2 | 94.50 | -5.50 | 44.863 | 45.198 |
| ca3 | 94.85 | -5.15 | 44.923 | 45.218 |
| pojac | 76.41 | -23.59 | 12.339 | 26.624 |
| jac1 | 53.95 | -46.05 | 7.915 | 46.722 |
| jacseq | 70.08 | -29.92 | 16.732 | 34.280 |
| jacint | 65.58 | -34.42 | 15.758 | 37.856 |

Pr(inf mle) , mean jacknife order = 0.002, 2.276
c, ch1, ch2, ch3 = 0.411, 0.373 , 0.405 , 0.403
cvh1, cvh2, cvh3 = 0.237140, 0.155685, 0.162834

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 42.52 | -57.48 | 4.839 | 57.687 |
| mle | 78.74 | -21.26 | 16.358 | 26.826 |
| dr1 | 85.17 | -14.83 | 18.381 | 23.619 |
| boot | 54.08 | -45.92 | 6.254 | 46.344 |
| ac1 | 92.34 | -7.66 | 24.448 | 25.621 |
| ac2 | 86.51 | -13.49 | 22.495 | 26.232 |
| ac3 | 86.87 | -13.13 | 22.631 | 26.165 |
| o | 59.91 | -40.09 | 7.008 | 40.698 |
| ca1 | 93.34 | -6.66 | 19.133 | 20.260 |
| ca2 | 90.23 | -9.77 | 18.176 | 20.635 |
| ca3 | 90.43 | -9.57 | 18.223 | 20.582 |
| pojac | 94.18 | -5.82 | 13.327 | 14.543 |
| jac1 | 69.50 | -30.50 | 8.583 | 31.681 |
| jacseq | 90.59 | -9.41 | 20.410 | 22.475 |
| jacint | 82.17 | -17.83 | 19.160 | 26.174 |

Pr(inf mle) , mean jacknife order = 0.000, 2.277
c, ch1, ch2, ch3 = 0.542, 0.514 , 0.538 , 0.537
cvh1, cvh2, cvh3 = 0.247527, 0.188130, 0.192005

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 50.86 | -49.14 | 4.995 | 49.394 |
| mle | 78.89 | -21.11 | 11.824 | 24.200 |
| dr1 | 83.82 | -16.18 | 13.111 | 20.828 |
| boot | 63.66 | -36.34 | 6.352 | 36.888 |
| ac1 | 89.53 | -10.47 | 17.792 | 20.645 |
| ac2 | 85.93 | -14.07 | 16.828 | 21.936 |
| ac3 | 86.14 | -13.86 | 16.893 | 21.851 |
| o | 70.00 | -30.00 | 7.119 | 30.834 |
| ca1 | 93.57 | -6.43 | 14.069 | 15.468 |
| ca2 | 91.80 | -8.20 | 13.628 | 15.904 |
| ca3 | 91.91 | -8.09 | 13.659 | 15.877 |
| pojac | 103.49 | 3.49 | 13.670 | 14.109 |
| jac1 | 79.79 | -20.21 | 8.416 | 21.890 |
| jacseq | 96.64 | -3.36 | 20.593 | 20.865 |
| jacint | 88.07 | -11.93 | 18.357 | 21.891 |

Pr(inf mle) , mean jacknife order = 0.000, 1.962
c, ch1, ch2, ch3 = 0.640, 0.615 , 0.633 , 0.632
cvh1, cvh2, cvh3 = 0.246666, 0.204019, 0.206731

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 57.43 | -42.57 | 4.916 | 42.854 |
| mle | 79.25 | -20.75 | 9.281 | 22.733 |
| dr1 | 83.32 | -16.68 | 10.163 | 19.531 |
| boot | 70.68 | -29.32 | 6.125 | 29.952 |
| ac1 | 87.97 | -12.03 | 13.113 | 17.799 |
| ac2 | 85.52 | -14.48 | 12.553 | 19.167 |
| ac3 | 85.66 | -14.34 | 12.583 | 19.076 |
| o | 77.11 | -22.89 | 6.810 | 23.884 |
| ca1 | 93.99 | -6.01 | 11.164 | 12.678 |
| ca2 | 92.88 | -7.12 | 10.956 | 13.065 |
| ca3 | 92.94 | -7.06 | 10.951 | 13.030 |
| pojac | 107.30 | 7.30 | 13.360 | 15.222 |
| jac1 | 86.48 | -13.52 | 8.248 | 15.837 |
| jacseq | 97.18 | -2.82 | 17.041 | 17.272 |
| jacint | 90.38 | -9.62 | 13.392 | 16.491 |

Pr(inf mle) , mean jacknife order = 0.000, 1.640
c, ch1, ch2, ch3 = 0.711, 0.694 , 0.708 , 0.707
cvh1, cvh2, cvh3 = 0.247270, 0.216243, 0.217988

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 62.77 | -37.23 | 4.831 | 37.547 |
| mle | 79.65 | -20.35 | 7.746 | 21.775 |
| dr1 | 83.56 | -16.44 | 8.482 | 18.500 |
| boot | 76.06 | -23.94 | 5.938 | 24.665 |
| ac1 | 88.25 | -11.75 | 11.351 | 16.338 |
| ac2 | 86.54 | -13.46 | 11.007 | 17.388 |
| ac3 | 86.61 | -13.39 | 11.031 | 17.350 |
| o | 82.39 | -17.61 | 6.591 | 18.799 |
| ca1 | 94.79 | -5.21 | 9.581 | 10.908 |
| ca2 | 94.07 | -5.93 | 9.430 | 11.137 |
| ca3 | 94.11 | -5.89 | 9.429 | 11.115 |
| pojac | 109.03 | 9.03 | 13.096 | 15.905 |
| jac1 | 90.89 | -9.11 | 8.037 | 12.149 |
| jacseq | 98.57 | -1.43 | 17.026 | 17.086 |
| jacint | 93.72 | -6.28 | 12.975 | 14.415 |

Pr(inf mle) , mean jacknife order = 0.000, 1.456
c, ch1, ch2, ch3 = 0.768, 0.754 , 0.765 , 0.764
cvh1, cvh2, cvh3 = 0.279265, 0.255061, 0.256413

Table 3.5.1b

N = 100 : p ~ U(0, 0.24) : E(p) = 0.12 : sqrt[Var(p)]/E(p) = 0.5774

Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 44.23 | -55.77 | 4.999 | 55.993 |
| mle | 78.95 | -21.05 | 16.384 | 26.675 |
| dr1 | 94.79 | -5.21 | 21.210 | 21.839 |
| boot | 55.23 | -44.77 | 6.294 | 45.209 |
| ac1 | 109.06 | 9.06 | 30.741 | 32.047 |
| ac2 | 84.62 | -15.38 | 22.439 | 27.205 |
| ac3 | 89.38 | -10.62 | 25.269 | 27.408 |
| o | 60.23 | -39.77 | 6.886 | 40.358 |
| ca1 | 100.44 | 0.44 | 21.731 | 21.735 |
| ca2 | 88.44 | -11.56 | 18.137 | 21.508 |
| ca3 | 90.65 | -9.35 | 18.786 | 20.984 |
| pojac | 93.12 | -6.88 | 12.266 | 14.066 |
| jac1 | 68.98 | -31.02 | 8.095 | 32.061 |
| jacseq | 91.69 | -8.31 | 15.121 | 17.252 |
| jacint | 87.55 | -12.45 | 15.027 | 19.514 |

Pr(inf mle) , mean jacknife order = 0.000, 3.378
c, ch1, ch2, ch3 = 0.566, 0.483 , 0.571 , 0.552
cvh1, cvh2, cvh3 = 0.397431, 0.166725, 0.213716

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 63.76 | -36.24 | 4.710 | 36.548 |
| mle | 78.85 | -21.15 | 7.264 | 22.364 |
| dr1 | 85.11 | -14.89 | 8.387 | 17.085 |
| boot | 76.38 | -23.62 | 5.775 | 24.321 |
| ac1 | 90.53 | -9.47 | 11.169 | 14.640 |
| ac2 | 84.68 | -15.32 | 10.136 | 18.373 |
| ac3 | 85.71 | -14.29 | 10.344 | 17.639 |
| o | 82.12 | -17.88 | 6.334 | 18.973 |
| ca1 | 94.67 | -5.33 | 9.319 | 10.737 |
| ca2 | 92.33 | -7.67 | 8.882 | 11.732 |
| ca3 | 92.73 | -7.27 | 8.961 | 11.540 |
| pojac | 107.64 | 7.64 | 12.277 | 14.460 |
| jac1 | 90.22 | -9.78 | 7.562 | 12.364 |
| jacseq | 97.07 | -2.93 | 14.213 | 14.512 |
| jacint | 92.78 | -7.22 | 11.316 | 13.422 |

Pr(inf mle) , mean jacknife order = 0.000, 1.494
c, ch1, ch2, ch3 = 0.781, 0.752 , 0.788 , 0.782
cvh1, cvh2, cvh3 = 0.320567, 0.234481, 0.250624

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 74.46 | -25.54 | 4.249 | 25.889 |
| mle | 81.76 | -18.24 | 5.159 | 18.956 |
| dr1 | 86.33 | -13.67 | 5.844 | 14.869 |
| boot | 86.05 | -13.95 | 5.092 | 14.852 |
| ac1 | 90.54 | -9.46 | 7.664 | 12.174 |
| ac2 | 88.32 | -11.68 | 7.409 | 13.834 |
| ac3 | 88.69 | -11.31 | 7.457 | 13.550 |
| o | 91.12 | -8.88 | 5.629 | 10.512 |
| ca1 | 96.50 | -3.50 | 6.801 | 7.649 |
| ca2 | 95.77 | -4.23 | 6.687 | 7.911 |
| ca3 | 95.90 | -4.10 | 6.715 | 7.866 |
| pojac | 107.11 | 7.11 | 11.790 | 13.765 |
| jac1 | 97.09 | -2.91 | 6.747 | 7.346 |
| jacseq | 99.28 | -0.72 | 10.428 | 10.453 |
| jacint | 97.96 | -2.04 | 8.215 | 8.466 |

Pr(inf mle) , mean jacknife order = 0.000, 1.157
c, ch1, ch2, ch3 = 0.880, 0.864 , 0.880 , 0.877
cvh1, cvh2, cvh3 = 0.347381, 0.312934, 0.318786

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 80.16 | -19.84 | 3.938 | 20.228 |
| mle | 83.59 | -16.41 | 4.312 | 16.964 |
| dr1 | 87.37 | -12.63 | 4.745 | 13.492 |
| boot | 90.08 | -9.92 | 4.622 | 10.941 |
| ac1 | 91.08 | -8.92 | 6.026 | 10.767 |
| ac2 | 90.07 | -9.93 | 5.927 | 11.562 |
| ac3 | 90.19 | -9.81 | 5.925 | 11.458 |
| o | 94.29 | -5.71 | 5.076 | 7.639 |
| ca1 | 96.89 | -3.11 | 5.682 | 6.480 |
| ca2 | 96.59 | -3.41 | 5.645 | 6.593 |
| ca3 | 96.63 | -3.37 | 5.656 | 6.583 |
| pojac | 103.83 | 3.83 | 10.559 | 11.232 |
| jac1 | 98.59 | -1.41 | 6.226 | 6.384 |
| jacseq | 99.30 | -0.70 | 7.886 | 7.917 |
| jacint | 98.94 | -1.06 | 6.930 | 7.010 |

Pr(inf mle) , mean jacknife order = 0.000, 1.052
c, ch1, ch2, ch3 = 0.925, 0.918 , 0.926 , 0.925
cvh1, cvh2, cvh3 = 0.391368, 0.377440, 0.379452

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 84.04 | -15.96 | 3.624 | 16.369 |
| mle | 85.59 | -14.41 | 3.767 | 14.895 |
| dr1 | 88.86 | -11.14 | 4.107 | 11.878 |
| boot | 92.54 | -7.46 | 4.184 | 8.555 |
| ac1 | 92.12 | -7.88 | 5.039 | 9.350 |
| ac2 | 91.62 | -8.38 | 5.013 | 9.764 |
| ac3 | 91.69 | -8.31 | 5.017 | 9.707 |
| o | 96.06 | -3.94 | 4.610 | 6.064 |
| ca1 | 97.48 | -2.52 | 4.937 | 5.542 |
| ca2 | 97.38 | -2.62 | 4.925 | 5.581 |
| ca3 | 97.38 | -2.62 | 4.921 | 5.573 |
| pojac | 102.11 | 2.11 | 9.811 | 10.036 |
| jac1 | 99.19 | -0.81 | 5.504 | 5.563 |
| jacseq | 100.07 | 0.07 | 8.429 | 8.430 |
| jacint | 99.73 | -0.27 | 7.053 | 7.058 |

Pr(inf mle) , mean jacknife order = 0.000, 1.059
c, ch1, ch2, ch3 = 0.951, 0.946 , 0.950 , 0.950
cvh1, cvh2, cvh3 = 0.419827, 0.413151, 0.413980

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 86.39 | -13.61 | 3.374 | 14.021 |
| mle | 86.95 | -13.05 | 3.467 | 13.500 |
| dr1 | 89.77 | -10.23 | 3.659 | 10.865 |
| boot | 93.66 | -6.34 | 3.847 | 7.419 |
| ac1 | 92.61 | -7.39 | 4.345 | 8.574 |
| ac2 | 92.31 | -7.69 | 4.312 | 8.815 |
| ac3 | 92.34 | -7.66 | 4.315 | 8.791 |
| o | 96.65 | -3.35 | 4.174 | 5.350 |
| ca1 | 97.49 | -2.51 | 4.377 | 5.045 |
| ca2 | 97.44 | -2.56 | 4.359 | 5.057 |
| ca3 | 97.44 | -2.56 | 4.361 | 5.056 |
| pojac | 100.80 | 0.80 | 8.583 | 8.620 |
| jac1 | 99.48 | -0.52 | 4.825 | 4.853 |
| jacseq | 100.20 | 0.20 | 7.334 | 7.336 |
| jacint | 99.95 | -0.05 | 6.289 | 6.289 |

Pr(inf mle) , mean jacknife order = 0.000, 1.052
c, ch1, ch2, ch3 = 0.964, 0.962 , 0.965 , 0.965
cvh1, cvh2, cvh3 = 0.434079, 0.430362, 0.430754

Table 3.5.1c

N = 100 : p ~ U(0, 0.40) : E(p) = 0.20 : sqrt[Var(p)]/E(p) = 0.5774

Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 60.25 | -39.75 | 5.004 | 40.061 |
| mle | 77.62 | -22.38 | 8.244 | 23.854 |
| dr1 | 89.21 | -10.79 | 10.410 | 14.994 |
| boot | 72.21 | -27.79 | 6.059 | 28.446 |
| ac1 | 98.17 | -1.83 | 14.773 | 14.887 |
| ac2 | 81.09 | -18.91 | 11.519 | 22.144 |
| ac3 | 87.08 | -12.92 | 13.519 | 18.702 |
| o | 77.42 | -22.58 | 6.584 | 23.524 |
| ca1 | 96.00 | -4.00 | 11.048 | 11.752 |
| ca2 | 88.81 | -11.19 | 9.750 | 14.842 |
| ca3 | 91.37 | -8.63 | 10.252 | 13.400 |
| pojac | 106.00 | 6.00 | 11.642 | 13.098 |
| jac1 | 85.66 | -14.34 | 7.623 | 16.245 |
| jacseq | 97.95 | -2.05 | 14.057 | 14.206 |
| jacint | 93.02 | -6.98 | 13.689 | 15.366 |

Pr(inf mle) , mean jacknife order = 0.000, 2.242
c, ch1, ch2, ch3 = 0.749, 0.680 , 0.775 , 0.738
cvh1, cvh2, cvh3 = 0.398147, 0.175534, 0.259839

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 77.20 | -22.80 | 4.149 | 23.175 |
| mle | 81.89 | -18.11 | 4.661 | 18.698 |
| dr1 | 86.96 | -13.04 | 5.332 | 14.088 |
| boot | 87.50 | -12.50 | 4.867 | 13.410 |
| ac1 | 91.05 | -8.95 | 6.796 | 11.237 |
| ac2 | 88.16 | -11.84 | 6.492 | 13.507 |
| ac3 | 89.01 | -10.99 | 6.575 | 12.803 |
| o | 91.88 | -8.12 | 5.336 | 9.716 |
| ca1 | 95.82 | -4.18 | 6.199 | 7.475 |
| ca2 | 94.96 | -5.04 | 6.069 | 7.890 |
| ca3 | 95.20 | -4.80 | 6.093 | 7.754 |
| pojac | 104.73 | 4.73 | 10.596 | 11.604 |
| jac1 | 97.03 | -2.97 | 6.352 | 7.010 |
| jacseq | 98.36 | -1.64 | 9.042 | 9.190 |
| jacint | 97.62 | -2.38 | 7.623 | 7.986 |

Pr(inf mle) , mean jacknife order = 0.000, 1.110
c, ch1, ch2, ch3 = 0.905, 0.889 , 0.911 , 0.904
cvh1, cvh2, cvh3 = 0.381294, 0.339151, 0.352298

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 84.52 | -15.48 | 3.611 | 15.900 |
| mle | 85.70 | -14.30 | 3.689 | 14.771 |
| dr1 | 89.19 | -10.81 | 4.010 | 11.532 |
| boot | 92.51 | -7.49 | 4.108 | 8.540 |
| ac1 | 92.35 | -7.65 | 4.845 | 9.054 |
| ac2 | 91.48 | -8.52 | 4.758 | 9.757 |
| ac3 | 91.68 | -8.32 | 4.782 | 9.600 |
| o | 95.80 | -4.20 | 4.430 | 6.106 |
| ca1 | 97.11 | -2.89 | 4.723 | 5.536 |
| ca2 | 96.91 | -3.09 | 4.675 | 5.603 |
| ca3 | 96.96 | -3.04 | 4.690 | 5.588 |
| pojac | 101.85 | 1.85 | 9.160 | 9.345 |
| jac1 | 99.02 | -0.98 | 5.401 | 5.489 |
| jacseq | 99.63 | -0.37 | 7.360 | 7.369 |
| jacint | 99.41 | -0.59 | 6.521 | 6.547 |

Pr(inf mle) , mean jacknife order = 0.000, 1.048
c, ch1, ch2, ch3 = 0.955, 0.948 , 0.955 , 0.953
cvh1, cvh2, cvh3 = 0.420097, 0.408829, 0.411348

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 88.14 | -11.86 | 3.175 | 12.276 |
| mle | 88.18 | -11.82 | 3.182 | 12.244 |
| dr1 | 90.91 | -9.09 | 3.413 | 9.708 |
| boot | 94.52 | -5.48 | 3.628 | 6.571 |
| ac1 | 93.69 | -6.31 | 4.105 | 7.524 |
| ac2 | 93.32 | -6.68 | 4.067 | 7.818 |
| ac3 | 93.39 | -6.61 | 4.080 | 7.768 |
| o | 97.17 | -2.83 | 3.958 | 4.868 |
| ca1 | 97.75 | -2.25 | 4.116 | 4.689 |
| ca2 | 97.69 | -2.31 | 4.101 | 4.710 |
| ca3 | 97.70 | -2.30 | 4.108 | 4.708 |
| pojac | 101.01 | 1.01 | 8.395 | 8.456 |
| jac1 | 99.47 | -0.53 | 4.634 | 4.664 |
| jacseq | 99.97 | -0.03 | 6.082 | 6.082 |
| jacint | 99.79 | -0.21 | 5.485 | 5.489 |

Pr(inf mle) , mean jacknife order = 0.000, 1.043
c, ch1, ch2, ch3 = 0.973, 0.970 , 0.973 , 0.972
cvh1, cvh2, cvh3 = 0.456086, 0.451684, 0.452397

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 90.43 | -9.57 | 2.898 | 9.997 |
| mle | 90.43 | -9.57 | 2.898 | 9.997 |
| dr1 | 92.25 | -7.75 | 3.045 | 8.326 |
| boot | 95.68 | -4.32 | 3.270 | 5.418 |
| ac1 | 94.64 | -5.36 | 3.552 | 6.433 |
| ac2 | 94.44 | -5.56 | 3.547 | 6.597 |
| ac3 | 94.47 | -5.53 | 3.555 | 6.577 |
| o | 97.84 | -2.16 | 3.571 | 4.174 |
| ca1 | 98.17 | -1.83 | 3.689 | 4.119 |
| ca2 | 98.14 | -1.86 | 3.681 | 4.123 |
| ca3 | 98.15 | -1.85 | 3.684 | 4.124 |
| pojac | 100.71 | 0.71 | 7.480 | 7.513 |
| jac1 | 99.97 | -0.03 | 4.209 | 4.209 |
| jacseq | 100.48 | 0.48 | 5.974 | 5.993 |
| jacint | 100.27 | 0.27 | 5.091 | 5.099 |

Pr(inf mle) , mean jacknife order = 0.000, 1.043
c, ch1, ch2, ch3 = 0.982, 0.980 , 0.982 , 0.982
cvh1, cvh2, cvh3 = 0.474867, 0.472717, 0.472992

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 91.80 | -8.20 | 2.725 | 8.644 |
| mle | 91.80 | -8.20 | 2.725 | 8.644 |
| dr1 | 93.04 | -6.96 | 2.778 | 7.496 |
| boot | 96.20 | -3.80 | 2.992 | 4.838 |
| ac1 | 95.09 | -4.91 | 3.144 | 5.830 |
| ac2 | 94.98 | -5.02 | 3.126 | 5.912 |
| ac3 | 94.99 | -5.01 | 3.128 | 5.908 |
| o | 97.98 | -2.02 | 3.220 | 3.799 |
| ca1 | 98.14 | -1.86 | 3.245 | 3.740 |
| ca2 | 98.13 | -1.87 | 3.237 | 3.739 |
| ca3 | 98.13 | -1.87 | 3.239 | 3.739 |
| pojac | 99.95 | -0.05 | 6.717 | 6.718 |
| jac1 | 99.75 | -0.25 | 3.890 | 3.898 |
| jacseq | 100.17 | 0.17 | 5.324 | 5.326 |
| jacint | 100.00 | 0.00 | 4.658 | 4.658 |

Pr(inf mle) , mean jacknife order = 0.000, 1.041
c, ch1, ch2, ch3 = 0.987, 0.987 , 0.987 , 0.987
cvh1, cvh2, cvh3 = 0.486945, 0.485736, 0.485865

Table 3.5.2a

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.04 : sqrt[Var(p)]/E(p) = 0.30
alpha = 10.6267 : beta = 255.0400 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 18.54 | -81.46 | 3.889 | 81.554 |
| mle | 90.88 | -9.12 | 49.735 | 50.564 |
| dr1 | 117.40 | 17.40 | 64.761 | 67.058 |
| boot | 24.07 | -75.93 | 5.113 | 76.101 |
| ac1 | 138.62 | 38.62 | 82.613 | 91.194 |
| ac2 | 99.58 | -0.42 | 60.354 | 60.355 |
| ac3 | 103.03 | 3.03 | 72.228 | 72.291 |
| o | 26.69 | -73.31 | 5.678 | 73.533 |
| ca1 | 119.93 | 19.93 | 65.153 | 68.132 |
| ca2 | 98.53 | -1.47 | 52.395 | 52.416 |
| ca3 | 99.33 | -0.67 | 52.938 | 52.942 |
| pojac | 46.53 | -53.47 | 10.298 | 54.449 |
| jac1 | 31.75 | -68.25 | 6.805 | 68.584 |
| jacseq | 49.11 | -50.89 | 11.546 | 52.180 |
| jacint | 46.22 | -53.78 | 10.931 | 54.882 |

Pr(inf mle) , mean jacknife order = 0.199, 3.874
c, ch1, ch2, ch3 = 0.201, 0.200 , 0.247 , 0.245
cvh1, cvh2, cvh3 = 0.388403, 0.063613, 0.073429

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 33.11 | -66.89 | 4.712 | 67.059 |
| mle | 106.62 | 6.62 | 60.500 | 60.861 |
| dr1 | 119.83 | 19.83 | 69.456 | 72.232 |
| boot | 43.11 | -56.89 | 6.173 | 57.223 |
| ac1 | 131.80 | 31.80 | 81.177 | 87.185 |
| ac2 | 116.86 | 16.86 | 70.074 | 72.075 |
| ac3 | 117.74 | 17.74 | 70.972 | 73.156 |
| o | 48.20 | -51.80 | 6.952 | 52.260 |
| ca1 | 125.91 | 25.91 | 69.804 | 74.458 |
| ca2 | 116.55 | 16.55 | 63.108 | 65.242 |
| ca3 | 116.90 | 16.90 | 63.205 | 65.427 |
| pojac | 82.65 | -17.35 | 13.150 | 21.768 |
| jac1 | 57.43 | -42.57 | 8.434 | 43.396 |
| jacseq | 74.06 | -25.94 | 18.849 | 32.065 |
| jacint | 70.22 | -29.78 | 17.273 | 34.427 |

Pr(inf mle) , mean jacknife order = 0.001, 2.226
c, ch1, ch2, ch3 = 0.355, 0.320 , 0.349 , 0.348
cvh1, cvh2, cvh3 = 0.218941, 0.143147, 0.148654

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 45.01 | -54.99 | 5.061 | 55.225 |
| mle | 97.74 | -2.26 | 24.917 | 25.020 |
| dr1 | 105.47 | 5.47 | 27.831 | 28.364 |
| boot | 57.88 | -42.12 | 6.569 | 42.626 |
| ac1 | 113.80 | 13.80 | 34.950 | 37.577 |
| ac2 | 105.95 | 5.95 | 31.898 | 32.448 |
| ac3 | 106.34 | 6.34 | 32.067 | 32.688 |
| o | 64.46 | -35.54 | 7.361 | 36.292 |
| ca1 | 114.34 | 14.34 | 28.445 | 31.854 |
| ca2 | 109.90 | 9.90 | 26.867 | 28.633 |
| ca3 | 110.09 | 10.09 | 26.918 | 28.749 |
| pojac | 105.04 | 5.04 | 14.212 | 15.078 |
| jac1 | 75.68 | -24.32 | 9.006 | 25.930 |
| jacseq | 101.76 | 1.76 | 22.761 | 22.829 |
| jacint | 93.33 | -6.67 | 21.716 | 22.717 |

Pr(inf mle) , mean jacknife order = 0.000, 2.418
c, ch1, ch2, ch3 = 0.480, 0.447 , 0.470 , 0.469
cvh1, cvh2, cvh3 = 0.224203, 0.163480, 0.167183

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 54.18 | -45.82 | 5.121 | 46.103 |
| mle | 94.23 | -5.77 | 14.863 | 15.945 |
| dr1 | 99.54 | -0.46 | 16.519 | 16.525 |
| boot | 68.63 | -31.37 | 6.499 | 32.033 |
| ac1 | 105.61 | 5.61 | 21.633 | 22.350 |
| ac2 | 100.90 | 0.90 | 20.221 | 20.241 |
| ac3 | 101.11 | 1.11 | 20.297 | 20.327 |
| o | 75.91 | -24.09 | 7.275 | 25.163 |
| ca1 | 110.20 | 10.20 | 17.361 | 20.137 |
| ca2 | 107.69 | 7.69 | 16.741 | 18.423 |
| ca3 | 107.80 | 7.80 | 16.770 | 18.496 |
| pojac | 116.44 | 16.44 | 13.986 | 21.581 |
| jac1 | 87.41 | -12.59 | 8.745 | 15.327 |
| jacseq | 110.61 | 10.61 | 21.733 | 24.183 |
| jacint | 99.81 | -0.19 | 20.533 | 20.534 |

Pr(inf mle) , mean jacknife order = 0.000, 2.171
c, ch1, ch2, ch3 = 0.573, 0.554 , 0.573 , 0.572
cvh1, cvh2, cvh3 = 0.207953, 0.164737, 0.166782

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 62.54 | -37.46 | 5.034 | 37.801 |
| mle | 95.02 | -4.98 | 11.623 | 12.645 |
| dr1 | 99.34 | -0.66 | 12.628 | 12.646 |
| boot | 78.09 | -21.91 | 6.366 | 22.820 |
| ac1 | 104.41 | 4.41 | 16.128 | 16.721 |
| ac2 | 101.11 | 1.11 | 15.336 | 15.377 |
| ac3 | 101.26 | 1.26 | 15.386 | 15.438 |
| o | 85.75 | -14.25 | 7.111 | 15.923 |
| ca1 | 111.46 | 11.46 | 13.604 | 17.788 |
| ca2 | 109.81 | 9.81 | 13.272 | 16.507 |
| ca3 | 109.90 | 9.90 | 13.285 | 16.567 |
| pojac | 124.52 | 24.52 | 14.026 | 28.251 |
| jac1 | 97.31 | -2.69 | 8.426 | 8.845 |
| jacseq | 114.89 | 14.89 | 19.640 | 24.649 |
| jacint | 104.49 | 4.49 | 16.856 | 17.445 |

Pr(inf mle) , mean jacknife order = 0.000, 1.894
c, ch1, ch2, ch3 = 0.657, 0.635 , 0.650 , 0.649
cvh1, cvh2, cvh3 = 0.212644, 0.179258, 0.180817

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 68.86 | -31.14 | 4.672 | 31.487 |
| mle | 94.47 | -5.53 | 9.086 | 10.634 |
| dr1 | 97.94 | -2.06 | 9.910 | 10.120 |
| boot | 84.61 | -15.39 | 5.860 | 16.466 |
| ac1 | 101.92 | 1.92 | 12.537 | 12.684 |
| ac2 | 99.61 | -0.39 | 12.069 | 12.075 |
| ac3 | 99.71 | -0.29 | 12.070 | 12.073 |
| o | 92.19 | -7.81 | 6.552 | 10.197 |
| ca1 | 110.81 | 10.81 | 10.943 | 15.378 |
| ca2 | 109.72 | 9.72 | 10.736 | 14.483 |
| ca3 | 109.77 | 9.77 | 10.755 | 14.530 |
| pojac | 125.88 | 25.88 | 13.818 | 29.334 |
| jac1 | 102.86 | 2.86 | 8.151 | 8.640 |
| jacseq | 113.53 | 13.53 | 18.674 | 23.061 |
| jacint | 106.37 | 6.37 | 14.180 | 15.546 |

Pr(inf mle) , mean jacknife order = 0.000, 1.564
c, ch1, ch2, ch3 = 0.719, 0.707 , 0.719 , 0.719
cvh1, cvh2, cvh3 = 0.204100, 0.176759, 0.177875

Table 3.5.2b

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.04 : sqrt[Var(p)]/E(p) = 0.55
alpha = 3.1336 : beta = 75.2053 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 18.22 | -81.78 | 3.865 | 81.869 |
| mle | 84.19 | -15.81 | 47.739 | 50.289 |
| dr1 | 109.19 | 9.19 | 62.120 | 62.796 |
| boot | 23.64 | -76.36 | 5.081 | 76.526 |
| ac1 | 130.47 | 30.47 | 79.963 | 85.570 |
| ac2 | 94.81 | -5.19 | 59.216 | 59.443 |
| ac3 | 98.89 | -1.11 | 71.136 | 71.145 |
| o | 26.19 | -73.81 | 5.633 | 74.029 |
| ca1 | 111.65 | 11.65 | 62.526 | 63.601 |
| ca2 | 92.20 | -7.80 | 50.324 | 50.925 |
| ca3 | 93.20 | -6.80 | 50.960 | 51.412 |
| pojac | 45.52 | -54.48 | 10.278 | 55.436 |
| jac1 | 31.10 | -68.90 | 6.781 | 69.234 |
| jacseq | 47.96 | -52.04 | 11.612 | 53.318 |
| jacint | 45.14 | -54.86 | 10.999 | 55.952 |

Pr(inf mle) , mean jacknife order = 0.168, 3.860
c, ch1, ch2, ch3 = 0.231, 0.211 , 0.259 , 0.256
cvh1, cvh2, cvh3 = 0.400661, 0.090000, 0.103173

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 32.10 | -67.90 | 4.472 | 68.044 |
| mle | 87.35 | -12.65 | 36.223 | 38.369 |
| dr1 | 98.65 | -1.35 | 41.271 | 41.293 |
| boot | 41.60 | -58.40 | 5.839 | 58.690 |
| ac1 | 110.49 | 10.49 | 51.213 | 52.277 |
| ac2 | 98.37 | -1.63 | 45.155 | 45.184 |
| ac3 | 99.19 | -0.81 | 45.578 | 45.586 |
| o | 46.41 | -53.59 | 6.570 | 53.996 |
| ca1 | 104.55 | 4.55 | 41.639 | 41.886 |
| ca2 | 97.39 | -2.61 | 37.659 | 37.749 |
| ca3 | 97.76 | -2.24 | 37.702 | 37.768 |
| pojac | 78.36 | -21.64 | 12.261 | 24.874 |
| jac1 | 55.01 | -44.99 | 7.912 | 45.678 |
| jacseq | 70.80 | -29.20 | 16.902 | 33.736 |
| jacint | 66.65 | -33.35 | 15.480 | 36.766 |

Pr(inf mle) , mean jacknife order = 0.000, 2.228
c, ch1, ch2, ch3 = 0.396, 0.359 , 0.390 , 0.388
cvh1, cvh2, cvh3 = 0.271022, 0.191507, 0.198043

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 43.17 | -56.83 | 4.901 | 57.038 |
| mle | 82.04 | -17.96 | 18.550 | 25.821 |
| dr1 | 89.06 | -10.94 | 20.892 | 23.584 |
| boot | 55.06 | -44.94 | 6.332 | 45.388 |
| ac1 | 97.78 | -2.22 | 28.030 | 28.117 |
| ac2 | 91.42 | -8.58 | 25.737 | 27.128 |
| ac3 | 91.78 | -8.22 | 25.903 | 27.177 |
| o | 61.07 | -38.93 | 7.095 | 39.571 |
| ca1 | 97.45 | -2.55 | 21.618 | 21.767 |
| ca2 | 94.16 | -5.84 | 20.537 | 21.351 |
| ca3 | 94.34 | -5.66 | 20.574 | 21.338 |
| pojac | 96.84 | -3.16 | 13.602 | 13.965 |
| jac1 | 71.05 | -28.95 | 8.701 | 30.232 |
| jacseq | 93.33 | -6.67 | 20.541 | 21.596 |
| jacint | 84.95 | -15.05 | 19.713 | 24.801 |

Pr(inf mle) , mean jacknife order = 0.000, 2.314
c, ch1, ch2, ch3 = 0.524, 0.501 , 0.525 , 0.523
cvh1, cvh2, cvh3 = 0.273519, 0.214161, 0.217915

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 51.60 | -48.40 | 4.932 | 48.648 |
| mle | 82.31 | -17.69 | 13.525 | 22.269 |
| dr1 | 88.09 | -11.91 | 14.888 | 19.063 |
| boot | 64.87 | -35.13 | 6.357 | 35.699 |
| ac1 | 95.78 | -4.22 | 19.940 | 20.381 |
| ac2 | 91.72 | -8.28 | 18.774 | 20.520 |
| ac3 | 91.95 | -8.05 | 18.850 | 20.499 |
| o | 71.50 | -28.50 | 7.137 | 29.378 |
| ca1 | 98.13 | -1.87 | 15.800 | 15.909 |
| ca2 | 96.22 | -3.78 | 15.283 | 15.743 |
| ca3 | 96.31 | -3.69 | 15.295 | 15.734 |
| pojac | 107.66 | 7.66 | 14.091 | 16.038 |
| jac1 | 81.91 | -18.09 | 8.598 | 20.028 |
| jacseq | 101.75 | 1.75 | 20.643 | 20.717 |
| jacint | 92.08 | -7.92 | 18.771 | 20.372 |

Pr(inf mle) , mean jacknife order = 0.000, 2.085
c, ch1, ch2, ch3 = 0.613, 0.595 , 0.613 , 0.612
cvh1, cvh2, cvh3 = 0.290091, 0.246089, 0.248383

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 58.96 | -41.04 | 4.881 | 41.329 |
| mle | 83.18 | -16.82 | 9.752 | 19.444 |
| dr1 | 88.20 | -11.80 | 10.808 | 16.005 |
| boot | 73.00 | -27.00 | 6.080 | 27.681 |
| ac1 | 95.16 | -4.84 | 14.980 | 15.743 |
| ac2 | 92.39 | -7.61 | 14.363 | 16.255 |
| ac3 | 92.53 | -7.47 | 14.393 | 16.217 |
| o | 79.85 | -20.15 | 6.764 | 21.254 |
| ca1 | 99.43 | -0.57 | 11.784 | 11.797 |
| ca2 | 98.23 | -1.77 | 11.557 | 11.691 |
| ca3 | 98.28 | -1.72 | 11.572 | 11.699 |
| pojac | 113.70 | 13.70 | 13.381 | 19.151 |
| jac1 | 90.17 | -9.83 | 8.144 | 12.768 |
| jacseq | 104.69 | 4.69 | 19.744 | 20.293 |
| jacint | 96.24 | -3.76 | 16.388 | 16.813 |

Pr(inf mle) , mean jacknife order = 0.000, 1.803
c, ch1, ch2, ch3 = 0.689, 0.674 , 0.688 , 0.687
cvh1, cvh2, cvh3 = 0.307286, 0.274258, 0.275945

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 64.52 | -35.48 | 4.726 | 35.790 |
| mle | 83.87 | -16.13 | 8.123 | 18.057 |
| dr1 | 88.51 | -11.49 | 8.921 | 14.548 |
| boot | 78.76 | -21.24 | 5.870 | 22.038 |
| ac1 | 95.14 | -4.86 | 12.624 | 13.529 |
| ac2 | 93.14 | -6.86 | 12.235 | 14.030 |
| ac3 | 93.23 | -6.77 | 12.261 | 14.005 |
| o | 85.61 | -14.39 | 6.535 | 15.805 |
| ca1 | 100.42 | 0.42 | 9.998 | 10.007 |
| ca2 | 99.64 | -0.36 | 9.823 | 9.829 |
| ca3 | 99.69 | -0.31 | 9.829 | 9.834 |
| pojac | 116.30 | 16.30 | 13.270 | 21.022 |
| jac1 | 95.24 | -4.76 | 8.044 | 9.345 |
| jacseq | 105.53 | 5.53 | 17.556 | 18.407 |
| jacint | 99.03 | -0.97 | 13.355 | 13.390 |

Pr(inf mle) , mean jacknife order = 0.000, 1.583
c, ch1, ch2, ch3 = 0.743, 0.732 , 0.743 , 0.743
cvh1, cvh2, cvh3 = 0.326660, 0.301499, 0.302791

Table 3.5.2c

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.04 : sqrt[Var(p)]/E(p) = 0.80
alpha = 1.4600 : beta = 35.0400 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 17.87 | -82.13 | 3.825 | 82.220 |
| mle | 70.25 | -29.75 | 42.584 | 51.945 |
| dr1 | 90.87 | -9.13 | 55.401 | 56.149 |
| boot | 23.06 | -76.94 | 5.008 | 77.106 |
| ac1 | 108.53 | 8.53 | 72.010 | 72.513 |
| ac2 | 80.48 | -19.52 | 54.294 | 57.696 |
| ac3 | 84.47 | -15.53 | 64.876 | 66.709 |
| o | 25.54 | -74.47 | 5.539 | 74.671 |
| ca1 | 93.21 | -6.79 | 55.756 | 56.168 |
| ca2 | 77.55 | -22.45 | 44.917 | 50.214 |
| ca3 | 78.56 | -21.44 | 45.492 | 50.289 |
| pojac | 43.77 | -56.23 | 10.049 | 57.121 |
| jac1 | 30.20 | -69.80 | 6.657 | 70.114 |
| jacseq | 45.65 | -54.35 | 11.452 | 55.546 |
| jacint | 43.05 | -56.95 | 10.839 | 57.976 |

Pr(inf mle) , mean jacknife order = 0.101, 3.764
c, ch1, ch2, ch3 = 0.279, 0.247 , 0.302 , 0.298
cvh1, cvh2, cvh3 = 0.389301, 0.116463, 0.133231

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 30.63 | -69.37 | 4.531 | 69.518 |
| mle | 70.16 | -29.84 | 26.266 | 39.756 |
| dr1 | 79.98 | -20.02 | 30.484 | 36.471 |
| boot | 39.36 | -60.64 | 5.852 | 60.925 |
| ac1 | 92.11 | -7.89 | 41.191 | 41.939 |
| ac2 | 82.62 | -17.38 | 37.183 | 41.043 |
| ac3 | 83.44 | -16.56 | 37.827 | 41.293 |
| o | 43.76 | -56.24 | 6.560 | 56.621 |
| ca1 | 85.54 | -14.46 | 30.928 | 34.141 |
| ca2 | 80.41 | -19.59 | 28.276 | 34.398 |
| ca3 | 80.81 | -19.19 | 28.399 | 34.277 |
| pojac | 72.13 | -27.87 | 12.220 | 30.428 |
| jac1 | 51.47 | -48.53 | 7.911 | 49.172 |
| jacseq | 66.53 | -33.47 | 16.585 | 37.356 |
| jacint | 61.97 | -38.03 | 15.637 | 41.120 |

Pr(inf mle) , mean jacknife order = 0.000, 2.241
c, ch1, ch2, ch3 = 0.454, 0.416 , 0.448 , 0.446
cvh1, cvh2, cvh3 = 0.324446, 0.244188, 0.252078

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 40.38 | -59.62 | 4.854 | 59.817 |
| mle | 68.18 | -31.82 | 14.435 | 34.941 |
| dr1 | 74.94 | -25.06 | 16.400 | 29.949 |
| boot | 51.02 | -48.98 | 6.231 | 49.373 |
| ac1 | 84.67 | -15.33 | 23.747 | 28.263 |
| ac2 | 79.59 | -20.41 | 22.100 | 30.081 |
| ac3 | 79.95 | -20.05 | 22.261 | 29.957 |
| o | 56.37 | -43.63 | 6.987 | 44.190 |
| ca1 | 82.66 | -17.34 | 17.225 | 24.442 |
| ca2 | 80.28 | -19.72 | 16.424 | 25.667 |
| ca3 | 80.45 | -19.55 | 16.486 | 25.576 |
| pojac | 87.23 | -12.77 | 13.359 | 18.481 |
| jac1 | 65.00 | -35.00 | 8.485 | 36.009 |
| jacseq | 83.38 | -16.62 | 19.042 | 25.276 |
| jacint | 75.77 | -24.23 | 17.788 | 30.059 |

Pr(inf mle) , mean jacknife order = 0.000, 2.200
c, ch1, ch2, ch3 = 0.577, 0.554 , 0.577 , 0.576
cvh1, cvh2, cvh3 = 0.348919, 0.295317, 0.299306

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 48.08 | -51.92 | 5.010 | 52.163 |
| mle | 68.94 | -31.06 | 9.913 | 32.605 |
| dr1 | 74.95 | -25.05 | 11.203 | 27.442 |
| boot | 59.79 | -40.21 | 6.301 | 40.705 |
| ac1 | 84.79 | -15.21 | 17.384 | 23.101 |
| ac2 | 81.64 | -18.36 | 16.684 | 24.805 |
| ac3 | 81.85 | -18.15 | 16.734 | 24.684 |
| o | 65.58 | -34.42 | 7.005 | 35.127 |
| ca1 | 84.17 | -15.83 | 12.168 | 19.969 |
| ca2 | 82.89 | -17.11 | 11.883 | 20.833 |
| ca3 | 82.96 | -17.04 | 11.897 | 20.782 |
| pojac | 96.21 | -3.79 | 13.253 | 13.784 |
| jac1 | 74.67 | -25.33 | 8.312 | 26.663 |
| jacseq | 89.81 | -10.19 | 19.565 | 22.057 |
| jacint | 82.05 | -17.95 | 17.261 | 24.905 |

Pr(inf mle) , mean jacknife order = 0.000, 1.929
c, ch1, ch2, ch3 = 0.662, 0.649 , 0.665 , 0.664
cvh1, cvh2, cvh3 = 0.397515, 0.362089, 0.364475

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 54.44 | -45.56 | 5.106 | 45.845 |
| mle | 71.24 | -28.76 | 8.341 | 29.942 |
| dr1 | 76.93 | -23.07 | 9.398 | 24.907 |
| boot | 66.78 | -33.22 | 6.308 | 33.811 |
| ac1 | 86.66 | -13.34 | 14.950 | 20.038 |
| ac2 | 84.50 | -15.50 | 14.483 | 21.212 |
| ac3 | 84.62 | -15.38 | 14.517 | 21.146 |
| o | 72.80 | -27.20 | 6.964 | 28.079 |
| ca1 | 87.17 | -12.83 | 10.464 | 16.553 |
| ca2 | 86.35 | -13.65 | 10.311 | 17.109 |
| ca3 | 86.38 | -13.62 | 10.314 | 17.082 |
| pojac | 102.32 | 2.32 | 13.080 | 13.285 |
| jac1 | 81.82 | -18.18 | 8.383 | 20.020 |
| jacseq | 94.84 | -5.16 | 18.529 | 19.233 |
| jacint | 87.83 | -12.17 | 15.494 | 19.702 |

Pr(inf mle) , mean jacknife order = 0.000, 1.785
c, ch1, ch2, ch3 = 0.729, 0.712 , 0.724 , 0.723
cvh1, cvh2, cvh3 = 0.431791, 0.406248, 0.407800

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 59.33 | -40.67 | 4.739 | 40.949 |
| mle | 72.57 | -27.43 | 7.309 | 28.391 |
| dr1 | 77.94 | -22.06 | 8.305 | 23.571 |
| boot | 71.80 | -28.20 | 5.865 | 28.808 |
| ac1 | 87.45 | -12.55 | 12.928 | 18.020 |
| ac2 | 85.91 | -14.09 | 12.649 | 18.934 |
| ac3 | 86.00 | -14.00 | 12.656 | 18.873 |
| o | 77.79 | -22.21 | 6.535 | 23.151 |
| ca1 | 88.78 | -11.22 | 9.418 | 14.652 |
| ca2 | 88.22 | -11.78 | 9.324 | 15.020 |
| ca3 | 88.25 | -11.75 | 9.328 | 15.000 |
| pojac | 104.80 | 4.80 | 13.099 | 13.951 |
| jac1 | 86.14 | -13.86 | 8.055 | 16.028 |
| jacseq | 95.88 | -4.12 | 18.433 | 18.888 |
| jacint | 90.20 | -9.80 | 14.496 | 17.496 |

Pr(inf mle) , mean jacknife order = 0.000, 1.582
c, ch1, ch2, ch3 = 0.775, 0.765 , 0.774 , 0.773
cvh1, cvh2, cvh3 = 0.464550, 0.447307, 0.448278

Table 3.5.2d

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.12 : sqrt[Var(p)]/E(p) = 0.30
alpha = 9.6578 : beta = 70.8237 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 46.24 | -53.76 | 4.736 | 53.970 |
| mle | 95.65 | -4.35 | 22.571 | 22.986 |
| dr1 | 115.87 | 15.87 | 29.166 | 33.205 |
| boot | 58.36 | -41.64 | 6.024 | 42.070 |
| ac1 | 134.23 | 34.23 | 40.991 | 53.403 |
| ac2 | 101.96 | 1.96 | 29.620 | 29.684 |
| ac3 | 106.89 | 6.89 | 33.012 | 33.724 |
| o | 63.93 | -36.07 | 6.642 | 36.672 |
| ca1 | 121.90 | 21.90 | 29.583 | 36.810 |
| ca2 | 105.64 | 5.64 | 24.374 | 25.018 |
| ca3 | 107.95 | 7.95 | 25.065 | 26.295 |
| pojac | 102.05 | 2.05 | 12.207 | 12.378 |
| jac1 | 73.89 | -26.11 | 7.912 | 27.278 |
| jacseq | 102.80 | 2.80 | 14.878 | 15.139 |
| jacint | 98.35 | -1.65 | 14.351 | 14.445 |

Pr(inf mle) , mean jacknife order = 0.000, 3.716
c, ch1, ch2, ch3 = 0.494, 0.417 , 0.499 , 0.486
cvh1, cvh2, cvh3 = 0.399709, 0.151290, 0.191827

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 70.09 | -29.91 | 4.451 | 30.235 |
| mle | 93.91 | -6.09 | 8.598 | 10.536 |
| dr1 | 100.91 | 0.91 | 9.949 | 9.990 |
| boot | 85.27 | -14.73 | 5.559 | 15.749 |
| ac1 | 106.33 | 6.33 | 13.201 | 14.638 |
| ac2 | 98.32 | -1.68 | 11.668 | 11.789 |
| ac3 | 99.44 | -0.56 | 11.952 | 11.965 |
| o | 92.26 | -7.74 | 6.168 | 9.895 |
| ca1 | 111.95 | 11.95 | 10.778 | 16.093 |
| ca2 | 108.31 | 8.31 | 10.175 | 13.135 |
| ca3 | 108.81 | 8.81 | 10.247 | 13.515 |
| pojac | 125.75 | 25.75 | 12.721 | 28.717 |
| jac1 | 102.54 | 2.54 | 7.487 | 7.905 |
| jacseq | 113.91 | 13.91 | 16.183 | 21.337 |
| jacint | 106.66 | 6.66 | 13.236 | 14.815 |

Pr(inf mle) , mean jacknife order = 0.000, 1.708
c, ch1, ch2, ch3 = 0.735, 0.698 , 0.739 , 0.733
cvh1, cvh2, cvh3 = 0.259418, 0.163912, 0.177511

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 82.68 | -17.32 | 3.697 | 17.706 |
| mle | 95.11 | -4.89 | 5.163 | 7.112 |
| dr1 | 99.23 | -0.77 | 5.877 | 5.928 |
| boot | 96.83 | -3.17 | 4.530 | 5.529 |
| ac1 | 102.38 | 2.38 | 7.616 | 7.981 |
| ac2 | 99.11 | -0.89 | 7.180 | 7.235 |
| ac3 | 99.55 | -0.45 | 7.231 | 7.245 |
| o | 103.01 | 3.01 | 5.053 | 5.880 |
| ca1 | 111.21 | 11.21 | 6.787 | 13.103 |
| ca2 | 109.99 | 9.99 | 6.610 | 11.981 |
| ca3 | 110.16 | 10.16 | 6.634 | 12.138 |
| pojac | 121.99 | 21.99 | 11.734 | 24.922 |
| jac1 | 110.14 | 10.14 | 6.394 | 11.991 |
| jacseq | 111.76 | 11.76 | 9.266 | 14.973 |
| jacint | 110.55 | 10.55 | 7.148 | 12.741 |

Pr(inf mle) , mean jacknife order = 0.000, 1.110
c, ch1, ch2, ch3 = 0.854, 0.835 , 0.855 , 0.852
cvh1, cvh2, cvh3 = 0.245586, 0.195656, 0.202314

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 89.72 | -10.28 | 3.078 | 10.732 |
| mle | 96.06 | -3.94 | 3.666 | 5.379 |
| dr1 | 98.94 | -1.06 | 4.156 | 4.288 |
| boot | 101.59 | 1.59 | 3.732 | 4.057 |
| ac1 | 101.00 | 1.00 | 5.190 | 5.284 |
| ac2 | 99.51 | -0.49 | 5.068 | 5.092 |
| ac3 | 99.69 | -0.31 | 5.070 | 5.080 |
| o | 106.43 | 6.43 | 4.233 | 7.700 |
| ca1 | 109.96 | 9.96 | 5.076 | 11.177 |
| ca2 | 109.51 | 9.51 | 4.996 | 10.743 |
| ca3 | 109.57 | 9.57 | 5.016 | 10.802 |
| pojac | 113.13 | 13.13 | 10.752 | 16.973 |
| jac1 | 110.68 | 10.68 | 5.618 | 12.068 |
| jacseq | 110.79 | 10.79 | 6.572 | 12.634 |
| jacint | 110.72 | 10.72 | 6.050 | 12.310 |

Pr(inf mle) , mean jacknife order = 0.000, 1.019
c, ch1, ch2, ch3 = 0.917, 0.907 , 0.918 , 0.917
cvh1, cvh2, cvh3 = 0.244734, 0.218949, 0.222307

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 93.84 | -6.16 | 2.429 | 6.617 |
| mle | 97.01 | -2.99 | 2.729 | 4.048 |
| dr1 | 99.13 | -0.87 | 3.028 | 3.151 |
| boot | 103.31 | 3.31 | 2.971 | 4.449 |
| ac1 | 100.55 | 0.55 | 3.662 | 3.703 |
| ac2 | 99.80 | -0.20 | 3.609 | 3.614 |
| ac3 | 99.91 | -0.09 | 3.635 | 3.636 |
| o | 106.92 | 6.92 | 3.442 | 7.730 |
| ca1 | 108.50 | 8.50 | 3.873 | 9.341 |
| ca2 | 108.30 | 8.30 | 3.821 | 9.140 |
| ca3 | 108.33 | 8.33 | 3.831 | 9.164 |
| pojac | 105.83 | 5.83 | 9.427 | 11.084 |
| jac1 | 108.81 | 8.81 | 4.506 | 9.900 |
| jacseq | 108.80 | 8.80 | 5.814 | 10.546 |
| jacint | 108.88 | 8.88 | 5.234 | 10.311 |

Pr(inf mle) , mean jacknife order = 0.000, 1.023
c, ch1, ch2, ch3 = 0.953, 0.947 , 0.953 , 0.952
cvh1, cvh2, cvh3 = 0.252211, 0.237973, 0.239814

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 96.10 | -3.90 | 1.999 | 4.383 |
| mle | 97.58 | -2.42 | 2.127 | 3.222 |
| dr1 | 99.28 | -0.72 | 2.319 | 2.429 |
| boot | 103.57 | 3.57 | 2.409 | 4.307 |
| ac1 | 100.28 | 0.28 | 2.682 | 2.696 |
| ac2 | 99.86 | -0.14 | 2.660 | 2.664 |
| ac3 | 99.90 | -0.10 | 2.665 | 2.667 |
| o | 106.24 | 6.24 | 2.781 | 6.835 |
| ca1 | 106.97 | 6.97 | 2.996 | 7.589 |
| ca2 | 106.88 | 6.88 | 2.985 | 7.503 |
| ca3 | 106.89 | 6.89 | 2.992 | 7.513 |
| pojac | 100.92 | 0.92 | 7.620 | 7.675 |
| jac1 | 107.28 | 7.28 | 3.634 | 8.140 |
| jacseq | 107.21 | 7.21 | 3.716 | 8.114 |
| jacint | 107.25 | 7.25 | 3.647 | 8.119 |

Pr(inf mle) , mean jacknife order = 0.000, 1.006
c, ch1, ch2, ch3 = 0.972, 0.968 , 0.972 , 0.971
cvh1, cvh2, cvh3 = 0.263439, 0.255651, 0.256595

Table 3.5.2e

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.12 : sqrt[Var(p)]/E(p) = 0.55
alpha = 2.7891 : beta = 20.4533 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 44.38 | -55.62 | 4.957 | 55.838 |
| mle | 82.29 | -17.71 | 18.973 | 25.955 |
| dr1 | 99.86 | -0.14 | 24.817 | 24.817 |
| boot | 55.59 | -44.41 | 6.250 | 44.849 |
| ac1 | 117.32 | 17.32 | 36.357 | 40.273 |
| ac2 | 90.36 | -9.64 | 26.688 | 28.377 |
| ac3 | 95.64 | -4.36 | 29.838 | 30.155 |
| o | 60.74 | -39.26 | 6.881 | 39.858 |
| ca1 | 105.59 | 5.59 | 25.275 | 25.887 |
| ca2 | 92.85 | -7.15 | 21.018 | 22.202 |
| ca3 | 95.19 | -4.81 | 21.700 | 22.227 |
| pojac | 95.02 | -4.98 | 12.392 | 13.354 |
| jac1 | 69.78 | -30.22 | 8.117 | 31.286 |
| jacseq | 94.60 | -5.40 | 15.220 | 16.150 |
| jacint | 90.41 | -9.59 | 14.890 | 17.713 |

Pr(inf mle) , mean jacknife order = 0.000, 3.524
c, ch1, ch2, ch3 = 0.541, 0.463 , 0.547 , 0.529
cvh1, cvh2, cvh3 = 0.432891, 0.193849, 0.242735

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 66.19 | -33.81 | 4.667 | 34.132 |
| mle | 84.08 | -15.92 | 7.696 | 17.687 |
| dr1 | 91.69 | -8.31 | 9.041 | 12.281 |
| boot | 79.97 | -20.03 | 5.754 | 20.843 |
| ac1 | 99.80 | -0.20 | 13.156 | 13.158 |
| ac2 | 92.87 | -7.13 | 11.977 | 13.940 |
| ac3 | 94.02 | -5.98 | 12.200 | 13.589 |
| o | 86.33 | -13.67 | 6.364 | 15.074 |
| ca1 | 102.00 | 2.00 | 9.926 | 10.125 |
| ca2 | 99.24 | -0.76 | 9.457 | 9.487 |
| ca3 | 99.69 | -0.31 | 9.519 | 9.524 |
| pojac | 116.77 | 16.77 | 12.503 | 20.920 |
| jac1 | 95.67 | -4.33 | 7.612 | 8.759 |
| jacseq | 106.02 | 6.02 | 15.629 | 16.749 |
| jacint | 99.74 | -0.26 | 12.584 | 12.587 |

Pr(inf mle) , mean jacknife order = 0.000, 1.693
c, ch1, ch2, ch3 = 0.764, 0.725 , 0.762 , 0.755
cvh1, cvh2, cvh3 = 0.376647, 0.292608, 0.306845

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 77.55 | -22.45 | 4.182 | 22.832 |
| mle | 86.60 | -13.40 | 5.447 | 14.467 |
| dr1 | 91.99 | -8.01 | 6.236 | 10.153 |
| boot | 90.52 | -9.48 | 5.094 | 10.761 |
| ac1 | 98.45 | -1.55 | 9.055 | 9.187 |
| ac2 | 95.74 | -4.26 | 8.695 | 9.683 |
| ac3 | 96.11 | -3.89 | 8.748 | 9.572 |
| o | 96.30 | -3.70 | 5.655 | 6.756 |
| ca1 | 103.28 | 3.28 | 7.265 | 7.972 |
| ca2 | 102.37 | 2.37 | 7.116 | 7.501 |
| ca3 | 102.52 | 2.52 | 7.122 | 7.556 |
| pojac | 116.46 | 16.46 | 12.258 | 20.520 |
| jac1 | 103.39 | 3.39 | 7.000 | 7.777 |
| jacseq | 106.56 | 6.56 | 12.026 | 13.700 |
| jacint | 104.50 | 4.50 | 9.165 | 10.209 |

Pr(inf mle) , mean jacknife order = 0.000, 1.215
c, ch1, ch2, ch3 = 0.863, 0.844 , 0.862 , 0.859
cvh1, cvh2, cvh3 = 0.409232, 0.376264, 0.381227

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 84.17 | -15.83 | 3.608 | 16.231 |
| mle | 88.76 | -11.24 | 4.125 | 11.970 |
| dr1 | 93.09 | -6.91 | 4.623 | 8.314 |
| boot | 95.54 | -4.46 | 4.316 | 6.209 |
| ac1 | 98.39 | -1.61 | 6.372 | 6.572 |
| ac2 | 97.10 | -2.90 | 6.253 | 6.894 |
| ac3 | 97.26 | -2.74 | 6.283 | 6.856 |
| o | 100.42 | 0.42 | 4.804 | 4.822 |
| ca1 | 103.86 | 3.86 | 5.586 | 6.793 |
| ca2 | 103.51 | 3.51 | 5.529 | 6.549 |
| ca3 | 103.56 | 3.56 | 5.545 | 6.591 |
| pojac | 112.34 | 12.34 | 10.963 | 16.510 |
| jac1 | 105.64 | 5.64 | 6.082 | 8.292 |
| jacseq | 106.49 | 6.49 | 8.239 | 10.491 |
| jacint | 105.96 | 5.96 | 6.825 | 9.058 |

Pr(inf mle) , mean jacknife order = 0.000, 1.064
c, ch1, ch2, ch3 = 0.914, 0.905 , 0.914 , 0.913
cvh1, cvh2, cvh3 = 0.440186, 0.425102, 0.427043

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 88.46 | -11.54 | 3.183 | 11.972 |
| mle | 90.72 | -9.28 | 3.372 | 9.869 |
| dr1 | 94.28 | -5.72 | 3.726 | 6.825 |
| boot | 98.21 | -1.79 | 3.760 | 4.163 |
| ac1 | 98.78 | -1.22 | 5.082 | 5.226 |
| ac2 | 98.09 | -1.91 | 5.047 | 5.398 |
| ac3 | 98.17 | -1.83 | 5.052 | 5.373 |
| o | 102.28 | 2.28 | 4.182 | 4.763 |
| ca1 | 104.12 | 4.12 | 4.611 | 6.182 |
| ca2 | 103.94 | 3.94 | 4.601 | 6.057 |
| ca3 | 103.96 | 3.96 | 4.599 | 6.071 |
| pojac | 108.72 | 8.72 | 9.874 | 13.175 |
| jac1 | 105.72 | 5.72 | 5.261 | 7.769 |
| jacseq | 106.17 | 6.17 | 7.302 | 9.563 |
| jacint | 105.99 | 5.99 | 6.358 | 8.735 |

Pr(inf mle) , mean jacknife order = 0.000, 1.034
c, ch1, ch2, ch3 = 0.944, 0.938 , 0.944 , 0.943
cvh1, cvh2, cvh3 = 0.461711, 0.454075, 0.454955

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 91.22 | -8.78 | 2.889 | 9.241 |
| mle | 92.24 | -7.76 | 2.989 | 8.317 |
| dr1 | 95.19 | -4.81 | 3.219 | 5.788 |
| boot | 99.52 | -0.48 | 3.384 | 3.418 |
| ac1 | 98.96 | -1.04 | 4.208 | 4.335 |
| ac2 | 98.55 | -1.45 | 4.203 | 4.447 |
| ac3 | 98.59 | -1.41 | 4.198 | 4.429 |
| o | 102.85 | 2.85 | 3.750 | 4.708 |
| ca1 | 103.89 | 3.89 | 3.975 | 5.561 |
| ca2 | 103.81 | 3.81 | 3.971 | 5.500 |
| ca3 | 103.81 | 3.81 | 3.968 | 5.502 |
| pojac | 105.69 | 5.69 | 8.806 | 10.486 |
| jac1 | 105.54 | 5.54 | 4.469 | 7.120 |
| jacseq | 105.89 | 5.89 | 6.711 | 8.926 |
| jacint | 105.80 | 5.80 | 5.830 | 8.222 |

Pr(inf mle) , mean jacknife order = 0.000, 1.026
c, ch1, ch2, ch3 = 0.961, 0.959 , 0.962 , 0.961
cvh1, cvh2, cvh3 = 0.478040, 0.473689, 0.474141

Table 3.5.2f

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.12 : sqrt[Var(p)]/E(p) = 0.80
alpha = 1.2550 : beta = 9.2033 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 41.41 | -58.59 | 4.907 | 58.794 |
| mle | 65.67 | -34.33 | 12.967 | 36.694 |
| dr1 | 79.63 | -20.37 | 17.179 | 26.647 |
| boot | 51.26 | -48.74 | 6.155 | 49.132 |
| ac1 | 94.82 | -5.18 | 26.984 | 27.477 |
| ac2 | 75.08 | -24.92 | 20.942 | 32.555 |
| ac3 | 80.31 | -19.69 | 23.870 | 30.941 |
| o | 55.69 | -44.31 | 6.754 | 44.822 |
| ca1 | 84.86 | -15.14 | 17.754 | 23.333 |
| ca2 | 76.27 | -23.73 | 15.299 | 28.234 |
| ca3 | 78.47 | -21.53 | 16.045 | 26.853 |
| pojac | 84.49 | -15.51 | 12.072 | 19.653 |
| jac1 | 63.42 | -36.58 | 7.977 | 37.440 |
| jacseq | 82.48 | -17.52 | 15.060 | 23.106 |
| jacint | 78.58 | -21.42 | 15.046 | 26.176 |

Pr(inf mle) , mean jacknife order = 0.000, 3.191
c, ch1, ch2, ch3 = 0.602, 0.534 , 0.620 , 0.595
cvh1, cvh2, cvh3 = 0.477044, 0.263912, 0.324121

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 59.92 | -40.08 | 4.822 | 40.369 |
| mle | 71.02 | -28.98 | 6.699 | 29.742 |
| dr1 | 78.49 | -21.51 | 7.916 | 22.920 |
| boot | 71.57 | -28.43 | 5.834 | 29.026 |
| ac1 | 88.64 | -11.36 | 12.303 | 16.746 |
| ac2 | 83.57 | -16.43 | 11.461 | 20.030 |
| ac3 | 84.51 | -15.49 | 11.630 | 19.368 |
| o | 76.95 | -23.05 | 6.387 | 23.919 |
| ca1 | 87.65 | -12.35 | 8.820 | 15.178 |
| ca2 | 85.86 | -14.14 | 8.508 | 16.506 |
| ca3 | 86.18 | -13.82 | 8.567 | 16.261 |
| pojac | 102.35 | 2.35 | 11.965 | 12.193 |
| jac1 | 84.80 | -15.20 | 7.490 | 16.948 |
| jacseq | 93.77 | -6.23 | 15.055 | 16.293 |
| jacint | 88.75 | -11.25 | 12.396 | 16.737 |

Pr(inf mle) , mean jacknife order = 0.000, 1.664
c, ch1, ch2, ch3 = 0.794, 0.766 , 0.797 , 0.791
cvh1, cvh2, cvh3 = 0.494638, 0.437884, 0.449143

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 70.36 | -29.64 | 4.569 | 29.990 |
| mle | 75.77 | -24.23 | 5.344 | 24.814 |
| dr1 | 81.77 | -18.23 | 6.144 | 19.241 |
| boot | 81.59 | -18.41 | 5.463 | 19.199 |
| ac1 | 91.26 | -8.74 | 9.449 | 12.873 |
| ac2 | 89.21 | -10.79 | 9.128 | 14.136 |
| ac3 | 89.51 | -10.49 | 9.173 | 13.938 |
| o | 86.70 | -13.30 | 6.005 | 14.588 |
| ca1 | 91.97 | -8.03 | 7.198 | 10.784 |
| ca2 | 91.36 | -8.64 | 7.102 | 11.182 |
| ca3 | 91.46 | -8.54 | 7.116 | 11.116 |
| pojac | 106.26 | 6.26 | 11.855 | 13.406 |
| jac1 | 93.29 | -6.71 | 7.119 | 9.784 |
| jacseq | 97.81 | -2.19 | 13.200 | 13.380 |
| jacint | 95.25 | -4.75 | 10.512 | 11.534 |

Pr(inf mle) , mean jacknife order = 0.000, 1.321
c, ch1, ch2, ch3 = 0.877, 0.862 , 0.875 , 0.873
cvh1, cvh2, cvh3 = 0.552546, 0.532779, 0.535884

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 76.16 | -23.84 | 4.372 | 24.238 |
| mle | 78.81 | -21.19 | 4.698 | 21.702 |
| dr1 | 83.86 | -16.14 | 5.265 | 16.979 |
| boot | 86.43 | -13.57 | 5.137 | 14.507 |
| ac1 | 92.77 | -7.23 | 7.917 | 10.724 |
| ac2 | 91.73 | -8.27 | 7.798 | 11.369 |
| ac3 | 91.84 | -8.16 | 7.806 | 11.289 |
| o | 91.03 | -8.97 | 5.624 | 10.589 |
| ca1 | 93.99 | -6.01 | 6.271 | 8.686 |
| ca2 | 93.72 | -6.28 | 6.231 | 8.844 |
| ca3 | 93.75 | -6.25 | 6.229 | 8.824 |
| pojac | 105.64 | 5.64 | 11.327 | 12.653 |
| jac1 | 96.50 | -3.50 | 6.860 | 7.702 |
| jacseq | 99.14 | -0.86 | 11.773 | 11.805 |
| jacint | 97.68 | -2.32 | 9.217 | 9.504 |

Pr(inf mle) , mean jacknife order = 0.000, 1.191
c, ch1, ch2, ch3 = 0.917, 0.909 , 0.916 , 0.916
cvh1, cvh2, cvh3 = 0.596817, 0.587209, 0.588433

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 80.54 | -19.46 | 4.055 | 19.879 |
| mle | 81.67 | -18.33 | 4.202 | 18.802 |
| dr1 | 85.84 | -14.16 | 4.568 | 14.879 |
| boot | 89.69 | -10.31 | 4.686 | 11.325 |
| ac1 | 93.82 | -6.18 | 6.739 | 9.143 |
| ac2 | 93.27 | -6.73 | 6.661 | 9.471 |
| ac3 | 93.33 | -6.67 | 6.657 | 9.426 |
| o | 93.69 | -6.31 | 5.149 | 8.142 |
| ca1 | 95.43 | -4.57 | 5.510 | 7.159 |
| ca2 | 95.28 | -4.72 | 5.485 | 7.234 |
| ca3 | 95.30 | -4.70 | 5.496 | 7.230 |
| pojac | 104.24 | 4.24 | 10.192 | 11.037 |
| jac1 | 97.84 | -2.16 | 6.096 | 6.467 |
| jacseq | 99.20 | -0.80 | 9.596 | 9.629 |
| jacint | 98.53 | -1.47 | 8.083 | 8.216 |

Pr(inf mle) , mean jacknife order = 0.000, 1.097
c, ch1, ch2, ch3 = 0.942, 0.939 , 0.943 , 0.942
cvh1, cvh2, cvh3 = 0.622433, 0.617217, 0.617767

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 83.61 | -16.39 | 3.739 | 16.815 |
| mle | 83.94 | -16.06 | 3.817 | 16.512 |
| dr1 | 87.55 | -12.45 | 4.067 | 13.102 |
| boot | 91.86 | -8.14 | 4.289 | 9.198 |
| ac1 | 94.88 | -5.12 | 5.821 | 7.755 |
| ac2 | 94.54 | -5.46 | 5.767 | 7.940 |
| ac3 | 94.58 | -5.42 | 5.774 | 7.922 |
| o | 95.42 | -4.58 | 4.643 | 6.523 |
| ca1 | 96.53 | -3.47 | 4.900 | 6.004 |
| ca2 | 96.46 | -3.54 | 4.893 | 6.038 |
| ca3 | 96.47 | -3.53 | 4.892 | 6.035 |
| pojac | 103.30 | 3.30 | 9.243 | 9.812 |
| jac1 | 98.99 | -1.01 | 5.291 | 5.386 |
| jacseq | 99.98 | -0.02 | 8.044 | 8.044 |
| jacint | 99.49 | -0.51 | 6.495 | 6.515 |

Pr(inf mle) , mean jacknife order = 0.000, 1.072
c, ch1, ch2, ch3 = 0.957, 0.955 , 0.958 , 0.958
cvh1, cvh2, cvh3 = 0.646150, 0.642928, 0.643218

Table 3.5.2g

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.20 : sqrt[Var(p)]/E(p) = 0.30
alpha = 8.6889 : beta = 34.7556 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 65.25 | -34.75 | 4.852 | 35.091 |
| mle | 93.53 | -6.47 | 11.050 | 12.804 |
| dr1 | 108.38 | 8.38 | 14.514 | 16.762 |
| boot | 79.51 | -20.49 | 6.067 | 21.367 |
| ac1 | 119.75 | 19.75 | 21.104 | 28.905 |
| ac2 | 96.08 | -3.92 | 15.462 | 15.953 |
| ac3 | 102.12 | 2.12 | 17.790 | 17.916 |
| o | 85.83 | -14.17 | 6.651 | 15.649 |
| ca1 | 116.16 | 16.16 | 15.129 | 22.134 |
| ca2 | 105.25 | 5.25 | 12.979 | 13.999 |
| ca3 | 108.12 | 8.12 | 13.594 | 15.834 |
| pojac | 123.05 | 23.05 | 12.717 | 26.323 |
| jac1 | 96.21 | -3.79 | 7.919 | 8.780 |
| jacseq | 116.11 | 16.11 | 16.370 | 22.965 |
| jacint | 110.60 | 10.60 | 16.933 | 19.979 |

Pr(inf mle) , mean jacknife order = 0.000, 2.745
c, ch1, ch2, ch3 = 0.687, 0.609 , 0.707 , 0.678
cvh1, cvh2, cvh3 = 0.376778, 0.130889, 0.195731

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 86.65 | -13.35 | 3.423 | 13.786 |
| mle | 95.49 | -4.51 | 4.283 | 6.219 |
| dr1 | 100.25 | 0.25 | 4.945 | 4.951 |
| boot | 99.46 | -0.54 | 4.084 | 4.120 |
| ac1 | 103.08 | 3.08 | 6.282 | 6.997 |
| ac2 | 98.64 | -1.36 | 5.825 | 5.981 |
| ac3 | 99.72 | -0.28 | 5.923 | 5.930 |
| o | 104.81 | 4.81 | 4.547 | 6.621 |
| ca1 | 110.86 | 10.86 | 5.752 | 12.294 |
| ca2 | 109.31 | 9.31 | 5.541 | 10.838 |
| ca3 | 109.72 | 9.72 | 5.591 | 11.212 |
| pojac | 118.82 | 18.82 | 10.334 | 21.467 |
| jac1 | 110.75 | 10.75 | 5.601 | 12.121 |
| jacseq | 111.33 | 11.33 | 6.853 | 13.242 |
| jacint | 110.89 | 10.89 | 5.906 | 12.390 |

Pr(inf mle) , mean jacknife order = 0.000, 1.046
c, ch1, ch2, ch3 = 0.891, 0.865 , 0.895 , 0.888
cvh1, cvh2, cvh3 = 0.265636, 0.190820, 0.209235

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 94.31 | -5.69 | 2.306 | 6.141 |
| mle | 96.92 | -3.08 | 2.506 | 3.968 |
| dr1 | 99.48 | -0.52 | 2.803 | 2.852 |
| boot | 103.24 | 3.24 | 2.778 | 4.268 |
| ac1 | 100.91 | 0.91 | 3.371 | 3.492 |
| ac2 | 99.63 | -0.37 | 3.307 | 3.327 |
| ac3 | 99.92 | -0.08 | 3.330 | 3.331 |
| o | 106.61 | 6.61 | 3.169 | 7.326 |
| ca1 | 108.07 | 8.07 | 3.537 | 8.809 |
| ca2 | 107.78 | 7.78 | 3.518 | 8.536 |
| ca3 | 107.84 | 7.84 | 3.515 | 8.596 |
| pojac | 105.60 | 5.60 | 8.578 | 10.244 |
| jac1 | 108.64 | 8.64 | 4.335 | 9.666 |
| jacseq | 108.36 | 8.36 | 4.873 | 9.678 |
| jacint | 108.58 | 8.58 | 4.397 | 9.645 |

Pr(inf mle) , mean jacknife order = 0.000, 1.028
c, ch1, ch2, ch3 = 0.958, 0.948 , 0.959 , 0.956
cvh1, cvh2, cvh3 = 0.266271, 0.241715, 0.247371

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 97.34 | -2.66 | 1.581 | 3.093 |
| mle | 97.88 | -2.12 | 1.674 | 2.698 |
| dr1 | 99.45 | -0.55 | 1.809 | 1.889 |
| boot | 103.15 | 3.15 | 1.954 | 3.708 |
| ac1 | 100.21 | 0.21 | 2.085 | 2.095 |
| ac2 | 99.78 | -0.22 | 2.075 | 2.087 |
| ac3 | 99.87 | -0.13 | 2.075 | 2.079 |
| o | 105.12 | 5.12 | 2.274 | 5.599 |
| ca1 | 105.52 | 5.52 | 2.415 | 6.029 |
| ca2 | 105.45 | 5.45 | 2.403 | 5.953 |
| ca3 | 105.45 | 5.45 | 2.410 | 5.963 |
| pojac | 98.65 | -1.35 | 6.681 | 6.815 |
| jac1 | 105.54 | 5.54 | 2.970 | 6.287 |
| jacseq | 105.54 | 5.54 | 2.970 | 6.287 |
| jacint | 105.54 | 5.54 | 2.970 | 6.287 |

Pr(inf mle) , mean jacknife order = 0.000, 1.000
c, ch1, ch2, ch3 = 0.982, 0.979 , 0.983 , 0.982
cvh1, cvh2, cvh3 = 0.272016, 0.263308, 0.265115

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 98.76 | -1.24 | 1.120 | 1.670 |
| mle | 98.76 | -1.24 | 1.120 | 1.670 |
| dr1 | 99.73 | -0.27 | 1.222 | 1.251 |
| boot | 102.52 | 2.52 | 1.410 | 2.889 |
| ac1 | 100.14 | 0.14 | 1.392 | 1.399 |
| ac2 | 99.98 | -0.02 | 1.381 | 1.381 |
| ac3 | 100.01 | 0.01 | 1.392 | 1.392 |
| o | 103.67 | 3.67 | 1.672 | 4.037 |
| ca1 | 103.80 | 3.80 | 1.731 | 4.173 |
| ca2 | 103.77 | 3.77 | 1.732 | 4.153 |
| ca3 | 103.78 | 3.78 | 1.732 | 4.159 |
| pojac | 96.72 | -3.28 | 5.048 | 6.022 |
| jac1 | 103.48 | 3.48 | 2.354 | 4.203 |
| jacseq | 103.48 | 3.48 | 2.354 | 4.203 |
| jacint | 103.48 | 3.48 | 2.354 | 4.203 |

Pr(inf mle) , mean jacknife order = 0.000, 1.000
c, ch1, ch2, ch3 = 0.992, 0.990 , 0.992 , 0.992
cvh1, cvh2, cvh3 = 0.282084, 0.278758, 0.279393

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 99.35 | -0.65 | 0.846 | 1.070 |
| mle | 99.35 | -0.65 | 0.846 | 1.070 |
| dr1 | 99.77 | -0.23 | 0.998 | 1.025 |
| boot | 101.81 | 1.81 | 1.137 | 2.141 |
| ac1 | 100.07 | 0.07 | 1.065 | 1.068 |
| ac2 | 99.97 | -0.03 | 1.075 | 1.076 |
| ac3 | 99.98 | -0.02 | 1.078 | 1.078 |
| o | 102.52 | 2.52 | 1.341 | 2.850 |
| ca1 | 102.55 | 2.55 | 1.364 | 2.894 |
| ca2 | 102.55 | 2.55 | 1.363 | 2.889 |
| ca3 | 102.55 | 2.55 | 1.364 | 2.890 |
| pojac | 96.61 | -3.39 | 3.984 | 5.230 |
| jac1 | 102.13 | 2.13 | 1.845 | 2.821 |
| jacseq | 102.13 | 2.13 | 1.845 | 2.821 |
| jacint | 102.13 | 2.13 | 1.845 | 2.821 |

Pr(inf mle) , mean jacknife order = 0.000, 1.000
c, ch1, ch2, ch3 = 0.996, 0.995 , 0.996 , 0.996
cvh1, cvh2, cvh3 = 0.291759, 0.290382, 0.290617

Table 3.5.2h

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.20 : sqrt[Var(p)]/E(p) = 0.55
alpha = 2.4446 : beta = 9.7785 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 61.39 | -38.61 | 4.934 | 38.928 |
| mle | 81.45 | -18.55 | 8.772 | 20.518 |
| dr1 | 94.76 | -5.24 | 11.527 | 12.664 |
| boot | 74.08 | -25.92 | 5.962 | 26.595 |
| ac1 | 106.79 | 6.79 | 17.836 | 19.084 |
| ac2 | 87.47 | -12.53 | 13.919 | 18.727 |
| ac3 | 93.94 | -6.06 | 16.092 | 17.194 |
| o | 79.66 | -20.34 | 6.479 | 21.348 |
| ca1 | 101.91 | 1.91 | 12.167 | 12.316 |
| ca2 | 93.90 | -6.10 | 10.643 | 12.268 |
| ca3 | 96.61 | -3.39 | 11.195 | 11.698 |
| pojac | 111.72 | 11.72 | 11.786 | 16.621 |
| jac1 | 88.71 | -11.29 | 7.546 | 13.576 |
| jacseq | 105.02 | 5.02 | 14.976 | 15.796 |
| jacint | 99.90 | -0.10 | 15.139 | 15.139 |

Pr(inf mle) , mean jacknife order = 0.000, 2.581
c, ch1, ch2, ch3 = 0.724, 0.653 , 0.746 , 0.711
cvh1, cvh2, cvh3 = 0.438492, 0.218674, 0.296805

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 80.74 | -19.26 | 3.884 | 19.649 |
| mle | 86.73 | -13.27 | 4.586 | 14.040 |
| dr1 | 92.76 | -7.24 | 5.351 | 9.002 |
| boot | 92.47 | -7.53 | 4.638 | 8.839 |
| ac1 | 98.88 | -1.12 | 7.565 | 7.647 |
| ac2 | 95.26 | -4.74 | 7.191 | 8.615 |
| ac3 | 96.18 | -3.82 | 7.287 | 8.228 |
| o | 97.52 | -2.48 | 5.104 | 5.675 |
| ca1 | 102.75 | 2.75 | 6.250 | 6.826 |
| ca2 | 101.64 | 1.64 | 6.106 | 6.321 |
| ca3 | 101.94 | 1.94 | 6.124 | 6.424 |
| pojac | 114.11 | 14.11 | 11.217 | 18.027 |
| jac1 | 103.70 | 3.70 | 6.235 | 7.252 |
| jacseq | 105.69 | 5.69 | 9.650 | 11.200 |
| jacint | 104.41 | 4.41 | 7.631 | 8.812 |

Pr(inf mle) , mean jacknife order = 0.000, 1.160
c, ch1, ch2, ch3 = 0.894, 0.871 , 0.896 , 0.889
cvh1, cvh2, cvh3 = 0.434239, 0.391012, 0.403107

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 88.65 | -11.35 | 3.202 | 11.794 |
| mle | 90.36 | -9.64 | 3.399 | 10.218 |
| dr1 | 94.21 | -5.79 | 3.770 | 6.906 |
| boot | 97.76 | -2.24 | 3.759 | 4.377 |
| ac1 | 98.48 | -1.52 | 4.905 | 5.134 |
| ac2 | 97.33 | -2.67 | 4.813 | 5.506 |
| ac3 | 97.59 | -2.41 | 4.840 | 5.406 |
| o | 101.50 | 1.50 | 4.171 | 4.432 |
| ca1 | 103.17 | 3.17 | 4.561 | 5.555 |
| ca2 | 102.92 | 2.92 | 4.513 | 5.374 |
| ca3 | 102.96 | 2.96 | 4.525 | 5.409 |
| pojac | 107.61 | 7.61 | 9.529 | 12.191 |
| jac1 | 105.04 | 5.04 | 5.202 | 7.241 |
| jacseq | 105.55 | 5.55 | 6.930 | 8.879 |
| jacint | 105.33 | 5.33 | 6.100 | 8.101 |

Pr(inf mle) , mean jacknife order = 0.000, 1.043
c, ch1, ch2, ch3 = 0.949, 0.941 , 0.950 , 0.948
cvh1, cvh2, cvh3 = 0.460934, 0.448011, 0.450664

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 92.68 | -7.32 | 2.534 | 7.745 |
| mle | 92.88 | -7.12 | 2.611 | 7.580 |
| dr1 | 95.76 | -4.24 | 2.783 | 5.075 |
| boot | 99.75 | -0.25 | 2.934 | 2.944 |
| ac1 | 99.05 | -0.95 | 3.569 | 3.694 |
| ac2 | 98.62 | -1.38 | 3.512 | 3.774 |
| ac3 | 98.69 | -1.31 | 3.520 | 3.755 |
| o | 102.54 | 2.54 | 3.322 | 4.181 |
| ca1 | 103.24 | 3.24 | 3.500 | 4.769 |
| ca2 | 103.16 | 3.16 | 3.493 | 4.708 |
| ca3 | 103.17 | 3.17 | 3.497 | 4.720 |
| pojac | 104.34 | 4.34 | 8.150 | 9.233 |
| jac1 | 104.56 | 4.56 | 4.093 | 6.125 |
| jacseq | 104.72 | 4.72 | 4.762 | 6.704 |
| jacint | 104.64 | 4.64 | 4.350 | 6.361 |

Pr(inf mle) , mean jacknife order = 0.000, 1.014
c, ch1, ch2, ch3 = 0.972, 0.968 , 0.972 , 0.971
cvh1, cvh2, cvh3 = 0.488531, 0.483369, 0.484236

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 94.95 | -5.05 | 2.210 | 5.510 |
| mle | 94.95 | -5.05 | 2.208 | 5.508 |
| dr1 | 96.77 | -3.23 | 2.368 | 4.002 |
| boot | 100.42 | 0.42 | 2.575 | 2.609 |
| ac1 | 99.28 | -0.72 | 2.993 | 3.078 |
| ac2 | 99.05 | -0.95 | 2.975 | 3.123 |
| ac3 | 99.08 | -0.92 | 2.981 | 3.120 |
| o | 102.54 | 2.54 | 2.890 | 3.849 |
| ca1 | 102.86 | 2.86 | 2.993 | 4.138 |
| ca2 | 102.82 | 2.82 | 2.983 | 4.108 |
| ca3 | 102.83 | 2.83 | 2.982 | 4.111 |
| pojac | 102.03 | 2.03 | 7.387 | 7.660 |
| jac1 | 104.05 | 4.05 | 3.633 | 5.437 |
| jacseq | 104.35 | 4.35 | 5.123 | 6.722 |
| jacint | 104.24 | 4.24 | 4.455 | 6.149 |

Pr(inf mle) , mean jacknife order = 0.000, 1.023
c, ch1, ch2, ch3 = 0.983, 0.981 , 0.983 , 0.983
cvh1, cvh2, cvh3 = 0.500606, 0.498106, 0.498458

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 96.24 | -3.76 | 1.905 | 4.212 |
| mle | 96.24 | -3.76 | 1.905 | 4.212 |
| dr1 | 97.37 | -2.63 | 2.004 | 3.306 |
| boot | 100.51 | 0.51 | 2.244 | 2.301 |
| ac1 | 99.24 | -0.76 | 2.467 | 2.580 |
| ac2 | 99.14 | -0.86 | 2.457 | 2.601 |
| ac3 | 99.16 | -0.84 | 2.459 | 2.600 |
| o | 102.10 | 2.10 | 2.497 | 3.261 |
| ca1 | 102.25 | 2.25 | 2.585 | 3.430 |
| ca2 | 102.24 | 2.24 | 2.581 | 3.420 |
| ca3 | 102.24 | 2.24 | 2.581 | 3.420 |
| pojac | 100.39 | 0.39 | 6.248 | 6.261 |
| jac1 | 103.06 | 3.06 | 3.286 | 4.491 |
| jacseq | 103.31 | 3.31 | 4.678 | 5.733 |
| jacint | 103.24 | 3.24 | 4.191 | 5.298 |

Pr(inf mle) , mean jacknife order = 0.000, 1.021
c, ch1, ch2, ch3 = 0.989, 0.988 , 0.990 , 0.989
cvh1, cvh2, cvh3 = 0.509534, 0.508207, 0.508373

Table 3.5.2i

N = 100 : p ~ Beta(alpha,beta) : E(p) = 0.20 : sqrt[Var(p)]/E(p) = 0.80
alpha = 1.0500 : beta = 4.2000 : Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 55.83 | -44.17 | 5.006 | 44.453 |
| mle | 67.41 | -32.59 | 7.423 | 33.426 |
| dr1 | 78.41 | -21.59 | 9.559 | 23.616 |
| boot | 66.33 | -33.67 | 6.077 | 34.215 |
| ac1 | 89.48 | -10.52 | 14.635 | 18.022 |
| ac2 | 76.00 | -24.00 | 12.122 | 26.891 |
| ac3 | 82.13 | -17.87 | 13.736 | 22.540 |
| o | 70.88 | -29.12 | 6.599 | 29.856 |
| ca1 | 84.63 | -15.37 | 10.255 | 18.479 |
| ca2 | 79.64 | -20.36 | 9.222 | 22.356 |
| ca3 | 81.90 | -18.10 | 9.682 | 20.526 |
| pojac | 96.13 | -3.87 | 11.716 | 12.340 |
| jac1 | 78.16 | -21.84 | 7.655 | 23.147 |
| jacseq | 89.73 | -10.27 | 14.295 | 17.598 |
| jacint | 85.59 | -14.41 | 13.700 | 19.879 |

Pr(inf mle) , mean jacknife order = 0.000, 2.301
c, ch1, ch2, ch3 = 0.772, 0.717 , 0.798 , 0.759
cvh1, cvh2, cvh3 = 0.498770, 0.332861, 0.412624

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 72.25 | -27.75 | 4.456 | 28.108 |
| mle | 75.34 | -24.66 | 4.804 | 25.119 |
| dr1 | 81.48 | -18.52 | 5.508 | 19.320 |
| boot | 82.22 | -17.78 | 5.195 | 18.519 |
| ac1 | 89.88 | -10.12 | 7.987 | 12.892 |
| ac2 | 87.37 | -12.63 | 7.693 | 14.792 |
| ac3 | 88.03 | -11.97 | 7.764 | 14.265 |
| o | 86.61 | -13.39 | 5.632 | 14.529 |
| ca1 | 90.38 | -9.62 | 6.391 | 11.548 |
| ca2 | 89.69 | -10.31 | 6.290 | 12.078 |
| ca3 | 89.89 | -10.11 | 6.319 | 11.923 |
| pojac | 103.10 | 3.10 | 10.748 | 11.187 |
| jac1 | 92.35 | -7.65 | 6.613 | 10.109 |
| jacseq | 95.52 | -4.48 | 11.066 | 11.940 |
| jacint | 93.80 | -6.20 | 8.925 | 10.866 |

Pr(inf mle) , mean jacknife order = 0.000, 1.266
c, ch1, ch2, ch3 = 0.904, 0.887 , 0.906 , 0.901
cvh1, cvh2, cvh3 = 0.558968, 0.533515, 0.540548

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 79.64 | -20.36 | 4.133 | 20.778 |
| mle | 80.28 | -19.72 | 4.235 | 20.173 |
| dr1 | 84.65 | -15.35 | 4.593 | 16.020 |
| boot | 88.10 | -11.90 | 4.698 | 12.794 |
| ac1 | 91.98 | -8.02 | 6.373 | 10.240 |
| ac2 | 91.12 | -8.88 | 6.274 | 10.873 |
| ac3 | 91.28 | -8.72 | 6.308 | 10.762 |
| o | 91.76 | -8.24 | 5.085 | 9.682 |
| ca1 | 93.33 | -6.67 | 5.408 | 8.587 |
| ca2 | 93.11 | -6.89 | 5.390 | 8.745 |
| ca3 | 93.16 | -6.84 | 5.400 | 8.716 |
| pojac | 102.39 | 2.39 | 9.853 | 10.140 |
| jac1 | 96.04 | -3.96 | 6.039 | 7.219 |
| jacseq | 97.47 | -2.53 | 8.921 | 9.274 |
| jacint | 96.77 | -3.23 | 7.498 | 8.165 |

Pr(inf mle) , mean jacknife order = 0.000, 1.117
c, ch1, ch2, ch3 = 0.947, 0.941 , 0.948 , 0.947
cvh1, cvh2, cvh3 = 0.612518, 0.604065, 0.605672

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 84.00 | -16.00 | 3.653 | 16.410 |
| mle | 84.02 | -15.98 | 3.653 | 16.389 |
| dr1 | 87.22 | -12.78 | 3.890 | 13.354 |
| boot | 91.30 | -8.70 | 4.117 | 9.623 |
| ac1 | 93.77 | -6.23 | 5.302 | 8.180 |
| ac2 | 93.35 | -6.65 | 5.263 | 8.482 |
| ac3 | 93.40 | -6.60 | 5.259 | 8.441 |
| o | 94.42 | -5.58 | 4.458 | 7.138 |
| ca1 | 95.26 | -4.74 | 4.640 | 6.634 |
| ca2 | 95.18 | -4.82 | 4.624 | 6.678 |
| ca3 | 95.19 | -4.81 | 4.625 | 6.670 |
| pojac | 101.88 | 1.88 | 8.731 | 8.930 |
| jac1 | 97.56 | -2.44 | 5.213 | 5.756 |
| jacseq | 98.56 | -1.44 | 7.901 | 8.031 |
| jacint | 98.20 | -1.80 | 6.925 | 7.155 |

Pr(inf mle) , mean jacknife order = 0.000, 1.080
c, ch1, ch2, ch3 = 0.966, 0.963 , 0.967 , 0.966
cvh1, cvh2, cvh3 = 0.645492, 0.641543, 0.642095

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 86.83 | -13.17 | 3.325 | 13.588 |
| mle | 86.83 | -13.17 | 3.325 | 13.588 |
| dr1 | 89.04 | -10.96 | 3.487 | 11.505 |
| boot | 93.12 | -6.88 | 3.752 | 7.837 |
| ac1 | 94.89 | -5.11 | 4.750 | 6.976 |
| ac2 | 94.66 | -5.34 | 4.732 | 7.137 |
| ac3 | 94.69 | -5.31 | 4.729 | 7.112 |
| o | 95.83 | -4.17 | 4.080 | 5.836 |
| ca1 | 96.30 | -3.70 | 4.181 | 5.581 |
| ca2 | 96.27 | -3.73 | 4.179 | 5.602 |
| ca3 | 96.27 | -3.73 | 4.178 | 5.599 |
| pojac | 101.61 | 1.61 | 8.431 | 8.583 |
| jac1 | 98.59 | -1.41 | 4.676 | 4.885 |
| jacseq | 99.80 | -0.20 | 8.459 | 8.461 |
| jacint | 99.30 | -0.70 | 6.758 | 6.794 |

Pr(inf mle) , mean jacknife order = 0.000, 1.092
c, ch1, ch2, ch3 = 0.977, 0.975 , 0.977 , 0.977
cvh1, cvh2, cvh3 = 0.668869, 0.666814, 0.667053

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 88.92 | -11.08 | 3.281 | 11.554 |
| mle | 88.92 | -11.08 | 3.281 | 11.554 |
| dr1 | 90.54 | -9.46 | 3.435 | 10.061 |
| boot | 94.52 | -5.48 | 3.683 | 6.605 |
| ac1 | 95.84 | -4.16 | 4.538 | 6.157 |
| ac2 | 95.72 | -4.28 | 4.513 | 6.217 |
| ac3 | 95.73 | -4.27 | 4.516 | 6.212 |
| o | 96.87 | -3.13 | 4.008 | 5.083 |
| ca1 | 97.16 | -2.84 | 4.088 | 4.978 |
| ca2 | 97.14 | -2.86 | 4.085 | 4.987 |
| ca3 | 97.14 | -2.86 | 4.082 | 4.983 |
| pojac | 101.22 | 1.22 | 8.184 | 8.274 |
| jac1 | 99.42 | -0.58 | 4.741 | 4.776 |
| jacseq | 100.31 | 0.31 | 7.469 | 7.476 |
| jacint | 99.94 | -0.06 | 6.192 | 6.193 |

Pr(inf mle) , mean jacknife order = 0.000, 1.073
c, ch1, ch2, ch3 = 0.983, 0.982 , 0.983 , 0.983
cvh1, cvh2, cvh3 = 0.688996, 0.687744, 0.687866

Table 3.5.3a

N = 50 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.20 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 21.55 | -28.45 | 7.080 | 29.314 |
| mle | 42.18 | -7.82 | 18.684 | 20.254 |
| dr1 | 52.62 | 2.62 | 24.766 | 24.904 |
| boot | 26.80 | -23.20 | 8.453 | 24.694 |
| ac1 | 61.34 | 11.34 | 32.831 | 34.735 |
| ac2 | 47.80 | -2.20 | 24.259 | 24.359 |
| ac3 | 50.46 | 0.46 | 27.527 | 27.531 |
| o | 29.21 | -20.79 | 9.026 | 22.664 |
| ca1 | 55.34 | 5.34 | 24.970 | 25.534 |
| ca2 | 48.25 | -1.75 | 20.175 | 20.251 |
| ca3 | 49.25 | -0.75 | 20.491 | 20.505 |
| pojac | 44.91 | -5.09 | 12.793 | 13.767 |
| jac1 | 33.39 | -16.61 | 10.018 | 19.399 |
| jacseq | 43.19 | -6.81 | 13.212 | 14.863 |
| jacint | 40.83 | -9.17 | 12.680 | 15.648 |

Pr(inf mle) , mean jacknife order = 0.049, 2.866
c, ch1, ch2, ch3 = 0.527, 0.457 , 0.536 , 0.519
cvh1, cvh2, cvh3 = 0.387640, 0.184828, 0.224229

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 31.20 | -18.80 | 7.846 | 20.372 |
| mle | 41.98 | -8.02 | 10.290 | 13.047 |
| dr1 | 46.34 | -3.66 | 11.678 | 12.239 |
| boot | 37.48 | -12.52 | 8.538 | 15.150 |
| ac1 | 50.72 | 0.72 | 14.664 | 14.681 |
| ac2 | 47.21 | -2.79 | 13.169 | 13.462 |
| ac3 | 47.72 | -2.28 | 13.330 | 13.524 |
| o | 40.39 | -9.61 | 8.825 | 13.050 |
| ca1 | 51.05 | 1.05 | 12.146 | 12.192 |
| ca2 | 49.40 | -0.60 | 11.172 | 11.188 |
| ca3 | 49.62 | -0.38 | 11.186 | 11.192 |
| pojac | 54.40 | 4.40 | 10.647 | 11.520 |
| jac1 | 44.72 | -5.28 | 9.045 | 10.475 |
| jacseq | 49.00 | -1.00 | 10.997 | 11.043 |
| jacint | 46.98 | -3.02 | 10.076 | 10.518 |

Pr(inf mle) , mean jacknife order = 0.000, 1.474
c, ch1, ch2, ch3 = 0.721, 0.692 , 0.725 , 0.720
cvh1, cvh2, cvh3 = 0.348524, 0.284018, 0.295033

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 36.47 | -13.53 | 7.308 | 15.380 |
| mle | 42.84 | -7.16 | 6.583 | 9.727 |
| dr1 | 45.90 | -4.10 | 7.190 | 8.276 |
| boot | 42.51 | -7.49 | 7.259 | 10.428 |
| ac1 | 49.33 | -0.67 | 9.472 | 9.495 |
| ac2 | 47.78 | -2.22 | 8.883 | 9.155 |
| ac3 | 47.95 | -2.05 | 8.941 | 9.174 |
| o | 45.26 | -4.74 | 7.188 | 8.610 |
| ca1 | 51.06 | 1.06 | 7.746 | 7.818 |
| ca2 | 50.42 | 0.42 | 7.523 | 7.534 |
| ca3 | 50.48 | 0.48 | 7.531 | 7.546 |
| pojac | 55.68 | 5.68 | 8.947 | 10.600 |
| jac1 | 48.73 | -1.27 | 7.043 | 7.156 |
| jacseq | 51.61 | 1.61 | 9.700 | 9.833 |
| jacint | 50.23 | 0.23 | 8.650 | 8.653 |

Pr(inf mle) , mean jacknife order = 0.000, 1.276
c, ch1, ch2, ch3 = 0.819, 0.801 , 0.818 , 0.816
cvh1, cvh2, cvh3 = 0.369175, 0.336881, 0.341120

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 40.12 | -9.88 | 6.321 | 11.729 |
| mle | 43.88 | -6.12 | 4.750 | 7.747 |
| dr1 | 46.28 | -3.72 | 4.891 | 6.145 |
| boot | 45.62 | -4.38 | 5.788 | 7.259 |
| ac1 | 49.10 | -0.90 | 6.315 | 6.379 |
| ac2 | 48.30 | -1.70 | 6.087 | 6.321 |
| ac3 | 48.36 | -1.64 | 6.107 | 6.323 |
| o | 48.08 | -1.92 | 5.568 | 5.891 |
| ca1 | 51.37 | 1.37 | 5.435 | 5.605 |
| ca2 | 51.09 | 1.09 | 5.331 | 5.440 |
| ca3 | 51.11 | 1.11 | 5.351 | 5.466 |
| pojac | 55.25 | 5.25 | 8.130 | 9.679 |
| jac1 | 50.85 | 0.85 | 5.411 | 5.477 |
| jacseq | 52.34 | 2.34 | 7.350 | 7.713 |
| jacint | 51.55 | 1.55 | 6.363 | 6.549 |

Pr(inf mle) , mean jacknife order = 0.000, 1.139
c, ch1, ch2, ch3 = 0.879, 0.868 , 0.878 , 0.877
cvh1, cvh2, cvh3 = 0.398419, 0.379708, 0.381657

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 42.28 | -7.72 | 5.667 | 9.578 |
| mle | 44.73 | -5.27 | 4.403 | 6.865 |
| dr1 | 46.67 | -3.33 | 4.350 | 5.476 |
| boot | 47.18 | -2.82 | 4.899 | 5.651 |
| ac1 | 49.08 | -0.92 | 5.239 | 5.319 |
| ac2 | 48.59 | -1.41 | 5.127 | 5.319 |
| ac3 | 48.63 | -1.37 | 5.127 | 5.308 |
| o | 49.27 | -0.73 | 4.656 | 4.713 |
| ca1 | 51.41 | 1.41 | 4.728 | 4.933 |
| ca2 | 51.26 | 1.26 | 4.665 | 4.832 |
| ca3 | 51.27 | 1.27 | 4.668 | 4.838 |
| pojac | 54.06 | 4.06 | 7.636 | 8.646 |
| jac1 | 51.54 | 1.54 | 4.597 | 4.848 |
| jacseq | 52.62 | 2.62 | 6.753 | 7.244 |
| jacint | 52.08 | 2.08 | 5.777 | 6.141 |

Pr(inf mle) , mean jacknife order = 0.000, 1.095
c, ch1, ch2, ch3 = 0.913, 0.906 , 0.912 , 0.912
cvh1, cvh2, cvh3 = 0.414950, 0.404248, 0.405307

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 43.84 | -6.16 | 4.813 | 7.814 |
| mle | 45.34 | -4.66 | 3.746 | 5.976 |
| dr1 | 46.98 | -3.02 | 3.540 | 4.653 |
| boot | 48.16 | -1.84 | 3.966 | 4.374 |
| ac1 | 49.16 | -0.84 | 4.173 | 4.256 |
| ac2 | 48.87 | -1.13 | 4.111 | 4.263 |
| ac3 | 48.89 | -1.11 | 4.111 | 4.257 |
| o | 49.95 | -0.05 | 3.781 | 3.781 |
| ca1 | 51.30 | 1.30 | 3.846 | 4.061 |
| ca2 | 51.21 | 1.21 | 3.825 | 4.011 |
| ca3 | 51.22 | 1.22 | 3.827 | 4.016 |
| pojac | 53.11 | 3.11 | 7.296 | 7.931 |
| jac1 | 51.83 | 1.83 | 3.932 | 4.338 |
| jacseq | 52.41 | 2.41 | 5.454 | 5.961 |
| jacint | 52.10 | 2.10 | 4.556 | 5.016 |

Pr(inf mle) , mean jacknife order = 0.000, 1.053
c, ch1, ch2, ch3 = 0.938, 0.933 , 0.937 , 0.937
cvh1, cvh2, cvh3 = 0.441960, 0.434874, 0.435427

Table 3.5.3b

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.20 ) : cv ~ U( 0.30, 0.80 )

Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 42.64 | -57.36 | 13.551 | 58.937 |
| mle | 84.19 | -15.81 | 31.991 | 35.687 |
| dr1 | 102.71 | 2.71 | 41.912 | 41.999 |
| boot | 53.11 | -46.89 | 16.039 | 49.555 |
| ac1 | 120.56 | 20.56 | 57.043 | 60.636 |
| ac2 | 93.03 | -6.97 | 41.537 | 42.117 |
| ac3 | 98.48 | -1.52 | 47.533 | 47.557 |
| o | 57.87 | -42.13 | 17.125 | 45.477 |
| ca1 | 108.09 | 8.09 | 42.122 | 42.892 |
| ca2 | 94.58 | -5.42 | 34.184 | 34.611 |
| ca3 | 96.83 | -3.17 | 34.948 | 35.092 |
| pojac | 89.31 | -10.69 | 23.287 | 25.623 |
| jac1 | 66.22 | -33.78 | 18.789 | 38.650 |
| jacseq | 88.35 | -11.65 | 22.917 | 25.709 |
| jacint | 84.28 | -15.72 | 22.223 | 27.221 |

Pr(inf mle) , mean jacknife order = 0.010, 3.405
c, ch1, ch2, ch3 = 0.520, 0.450 , 0.529 , 0.511
cvh1, cvh2, cvh3 = 0.422743, 0.185882, 0.231433

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 62.74 | -37.26 | 14.389 | 39.944 |
| mle | 84.12 | -15.88 | 14.334 | 21.393 |
| dr1 | 91.98 | -8.02 | 15.993 | 17.891 |
| boot | 75.47 | -24.53 | 15.397 | 28.962 |
| ac1 | 100.46 | 0.46 | 20.090 | 20.095 |
| ac2 | 93.33 | -6.67 | 17.749 | 18.963 |
| ac3 | 94.38 | -5.62 | 18.046 | 18.900 |
| o | 81.40 | -18.60 | 15.712 | 24.351 |
| ca1 | 101.51 | 1.51 | 16.659 | 16.728 |
| ca2 | 98.37 | -1.63 | 15.481 | 15.567 |
| ca3 | 98.78 | -1.22 | 15.536 | 15.584 |
| pojac | 110.17 | 10.17 | 16.278 | 19.195 |
| jac1 | 90.17 | -9.83 | 15.662 | 18.489 |
| jacseq | 100.97 | 0.97 | 17.986 | 18.012 |
| jacint | 95.86 | -4.14 | 16.729 | 17.234 |

Pr(inf mle) , mean jacknife order = 0.001, 1.725
c, ch1, ch2, ch3 = 0.726, 0.692 , 0.726 , 0.720
cvh1, cvh2, cvh3 = 0.373758, 0.297725, 0.309859

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 73.87 | -26.13 | 13.730 | 29.517 |
| mle | 86.04 | -13.96 | 9.899 | 17.110 |
| dr1 | 91.21 | -8.79 | 10.298 | 13.542 |
| boot | 85.89 | -14.11 | 13.252 | 19.357 |
| ac1 | 97.32 | -2.68 | 12.416 | 12.701 |
| ac2 | 94.29 | -5.71 | 11.756 | 13.067 |
| ac3 | 94.66 | -5.34 | 11.830 | 12.981 |
| o | 91.30 | -8.70 | 12.887 | 15.551 |
| ca1 | 101.51 | 1.51 | 11.098 | 11.200 |
| ca2 | 100.35 | 0.35 | 10.827 | 10.833 |
| ca3 | 100.47 | 0.47 | 10.855 | 10.865 |
| pojac | 111.10 | 11.10 | 12.965 | 17.068 |
| jac1 | 98.16 | -1.84 | 11.886 | 12.028 |
| jacseq | 104.14 | 4.14 | 14.143 | 14.738 |
| jacint | 101.00 | 1.00 | 12.961 | 13.000 |

Pr(inf mle) , mean jacknife order = 0.000, 1.373
c, ch1, ch2, ch3 = 0.829, 0.812 , 0.829 , 0.827
cvh1, cvh2, cvh3 = 0.391570, 0.355868, 0.360293

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 80.08 | -19.92 | 11.881 | 23.193 |
| mle | 87.73 | -12.27 | 8.498 | 14.923 |
| dr1 | 91.89 | -8.11 | 8.403 | 11.682 |
| boot | 91.06 | -8.94 | 10.555 | 13.835 |
| ac1 | 97.34 | -2.66 | 9.659 | 10.018 |
| ac2 | 95.73 | -4.27 | 9.362 | 10.289 |
| ac3 | 95.90 | -4.10 | 9.377 | 10.234 |
| o | 95.87 | -4.13 | 9.952 | 10.775 |
| ca1 | 102.02 | 2.02 | 9.060 | 9.283 |
| ca2 | 101.46 | 1.46 | 8.952 | 9.070 |
| ca3 | 101.51 | 1.51 | 8.960 | 9.087 |
| pojac | 110.04 | 10.04 | 13.029 | 16.451 |
| jac1 | 101.39 | 1.39 | 9.131 | 9.236 |
| jacseq | 105.22 | 5.22 | 12.673 | 13.705 |
| jacint | 102.97 | 2.97 | 10.942 | 11.338 |

Pr(inf mle) , mean jacknife order = 0.000, 1.234
c, ch1, ch2, ch3 = 0.880, 0.871 , 0.881 , 0.880
cvh1, cvh2, cvh3 = 0.417228, 0.397975, 0.400036

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 84.86 | -15.14 | 10.141 | 18.218 |
| mle | 89.86 | -10.14 | 7.009 | 12.326 |
| dr1 | 93.35 | -6.65 | 6.656 | 9.409 |
| boot | 94.76 | -5.24 | 8.358 | 9.866 |
| ac1 | 98.03 | -1.97 | 7.071 | 7.341 |
| ac2 | 97.10 | -2.90 | 7.008 | 7.585 |
| ac3 | 97.19 | -2.81 | 7.024 | 7.566 |
| o | 99.02 | -0.98 | 7.694 | 7.756 |
| ca1 | 102.94 | 2.94 | 7.119 | 7.701 |
| ca2 | 102.63 | 2.63 | 7.048 | 7.522 |
| ca3 | 102.66 | 2.66 | 7.060 | 7.545 |
| pojac | 109.08 | 9.08 | 12.037 | 15.077 |
| jac1 | 103.43 | 3.43 | 7.195 | 7.971 |
| jacseq | 105.55 | 5.55 | 10.255 | 11.658 |
| jacint | 104.23 | 4.23 | 8.820 | 9.781 |

Pr(inf mle) , mean jacknife order = 0.000, 1.137
c, ch1, ch2, ch3 = 0.916, 0.908 , 0.915 , 0.914
cvh1, cvh2, cvh3 = 0.440923, 0.429554, 0.430571

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 87.30 | -12.70 | 9.544 | 15.887 |
| mle | 90.65 | -9.35 | 7.110 | 11.746 |
| dr1 | 93.49 | -6.51 | 6.531 | 9.222 |
| boot | 95.99 | -4.01 | 7.585 | 8.580 |
| ac1 | 97.61 | -2.39 | 6.668 | 7.083 |
| ac2 | 97.00 | -3.00 | 6.668 | 7.311 |
| ac3 | 97.06 | -2.94 | 6.662 | 7.284 |
| o | 99.62 | -0.38 | 6.951 | 6.961 |
| ca1 | 102.20 | 2.20 | 6.631 | 6.986 |
| ca2 | 102.02 | 2.02 | 6.596 | 6.899 |
| ca3 | 102.03 | 2.03 | 6.597 | 6.904 |
| pojac | 105.54 | 5.54 | 11.401 | 12.674 |
| jac1 | 102.98 | 2.98 | 6.626 | 7.265 |
| jacseq | 104.43 | 4.43 | 10.207 | 11.128 |
| jacint | 103.71 | 3.71 | 8.516 | 9.289 |

Pr(inf mle) , mean jacknife order = 0.000, 1.088
c, ch1, ch2, ch3 = 0.935, 0.933 , 0.937 , 0.937
cvh1, cvh2, cvh3 = 0.449766, 0.442131, 0.442714

Table 3.5.3c

N = 200 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.20 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 86.53 | -113.47 | 26.388 | 116.503 |
| mle | 165.60 | -34.40 | 57.866 | 67.319 |
| dr1 | 199.03 | -0.97 | 74.760 | 74.767 |
| boot | 107.57 | -92.43 | 31.023 | 97.496 |
| ac1 | 233.07 | 33.07 | 102.399 | 107.606 |
| ac2 | 178.36 | -21.64 | 73.940 | 77.041 |
| ac3 | 188.63 | -11.37 | 85.094 | 85.850 |
| o | 117.13 | -82.87 | 33.012 | 89.200 |
| ca1 | 209.97 | 9.97 | 74.935 | 75.595 |
| ca2 | 184.46 | -15.54 | 60.754 | 62.710 |
| ca3 | 188.87 | -11.13 | 62.014 | 63.005 |
| pojac | 179.88 | -20.12 | 43.406 | 47.844 |
| jac1 | 133.84 | -66.16 | 35.953 | 75.300 |
| jacseq | 179.58 | -20.42 | 41.279 | 46.054 |
| jacint | 172.64 | -27.36 | 40.590 | 48.951 |

Pr(inf mle) , mean jacknife order = 0.001, 3.747
c, ch1, ch2, ch3 = 0.528, 0.461 , 0.540 , 0.521
cvh1, cvh2, cvh3 = 0.451279, 0.186030, 0.240712

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 125.67 | -74.33 | 28.278 | 79.527 |
| mle | 167.32 | -32.68 | 21.086 | 38.895 |
| dr1 | 181.67 | -18.33 | 22.622 | 29.117 |
| boot | 150.99 | -49.01 | 29.879 | 57.401 |
| ac1 | 197.51 | -2.49 | 26.871 | 26.986 |
| ac2 | 183.18 | -16.82 | 23.309 | 28.744 |
| ac3 | 185.23 | -14.77 | 23.698 | 27.925 |
| o | 162.69 | -37.31 | 30.222 | 48.012 |
| ca1 | 200.63 | 0.63 | 24.017 | 24.025 |
| ca2 | 194.66 | -5.34 | 22.742 | 23.360 |
| ca3 | 195.46 | -4.54 | 22.864 | 23.311 |
| pojac | 219.50 | 19.50 | 27.714 | 33.888 |
| jac1 | 180.01 | -19.99 | 29.476 | 35.613 |
| jacseq | 204.85 | 4.85 | 31.358 | 31.731 |
| jacint | 194.51 | -5.49 | 29.055 | 29.570 |

Pr(inf mle) , mean jacknife order = 0.000, 1.991
c, ch1, ch2, ch3 = 0.729, 0.696 , 0.730 , 0.724
cvh1, cvh2, cvh3 = 0.393350, 0.309276, 0.322405

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 145.87 | -54.13 | 26.502 | 60.270 |
| mle | 170.95 | -29.05 | 16.640 | 33.475 |
| dr1 | 180.50 | -19.50 | 16.307 | 25.416 |
| boot | 170.07 | -29.93 | 25.349 | 39.224 |
| ac1 | 192.27 | -7.73 | 17.246 | 18.900 |
| ac2 | 186.18 | -13.82 | 16.805 | 21.758 |
| ac3 | 186.91 | -13.09 | 16.890 | 21.371 |
| o | 181.02 | -18.98 | 24.434 | 30.941 |
| ca1 | 201.17 | 1.17 | 17.689 | 17.727 |
| ca2 | 198.84 | -1.16 | 17.553 | 17.591 |
| ca3 | 199.08 | -0.92 | 17.567 | 17.591 |
| pojac | 222.16 | 22.16 | 20.654 | 30.289 |
| jac1 | 195.03 | -4.97 | 21.917 | 22.474 |
| jacseq | 210.24 | 10.24 | 23.356 | 25.500 |
| jacint | 202.58 | 2.58 | 22.618 | 22.765 |

Pr(inf mle) , mean jacknife order = 0.000, 1.565
c, ch1, ch2, ch3 = 0.820, 0.808 , 0.825 , 0.823
cvh1, cvh2, cvh3 = 0.398615, 0.360282, 0.364733

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 161.08 | -38.92 | 23.350 | 45.384 |
| mle | 176.75 | -23.25 | 14.150 | 27.216 |
| dr1 | 184.22 | -15.78 | 13.351 | 20.668 |
| boot | 182.94 | -17.06 | 20.101 | 26.364 |
| ac1 | 194.39 | -5.61 | 13.666 | 14.775 |
| ac2 | 191.24 | -8.76 | 13.613 | 16.188 |
| ac3 | 191.56 | -8.44 | 13.630 | 16.030 |
| o | 192.54 | -7.46 | 18.424 | 19.878 |
| ca1 | 204.42 | 4.42 | 14.012 | 14.694 |
| ca2 | 203.32 | 3.32 | 13.913 | 14.304 |
| ca3 | 203.43 | 3.43 | 13.909 | 14.325 |
| pojac | 220.20 | 20.20 | 19.749 | 28.247 |
| jac1 | 203.38 | 3.38 | 15.476 | 15.841 |
| jacseq | 211.36 | 11.36 | 18.658 | 21.846 |
| jacint | 206.72 | 6.72 | 16.873 | 18.162 |

Pr(inf mle) , mean jacknife order = 0.000, 1.309
c, ch1, ch2, ch3 = 0.882, 0.873 , 0.883 , 0.882
cvh1, cvh2, cvh3 = 0.423520, 0.403792, 0.405877

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 168.78 | -31.22 | 20.761 | 37.490 |
| mle | 179.55 | -20.45 | 13.210 | 24.344 |
| dr1 | 185.93 | -14.07 | 12.010 | 18.499 |
| boot | 188.42 | -11.58 | 16.524 | 20.177 |
| ac1 | 195.27 | -4.73 | 11.486 | 12.423 |
| ac2 | 193.38 | -6.62 | 11.443 | 13.221 |
| ac3 | 193.54 | -6.46 | 11.444 | 13.143 |
| o | 196.88 | -3.12 | 14.660 | 14.989 |
| ca1 | 204.90 | 4.90 | 12.276 | 13.219 |
| ca2 | 204.30 | 4.30 | 12.154 | 12.892 |
| ca3 | 204.35 | 4.35 | 12.160 | 12.915 |
| pojac | 216.53 | 16.53 | 20.624 | 26.431 |
| jac1 | 205.58 | 5.58 | 12.450 | 13.642 |
| jacseq | 210.69 | 10.69 | 17.248 | 20.293 |
| jacint | 207.54 | 7.54 | 14.430 | 16.281 |

Pr(inf mle) , mean jacknife order = 0.000, 1.222
c, ch1, ch2, ch3 = 0.913, 0.907 , 0.913 , 0.912
cvh1, cvh2, cvh3 = 0.448544, 0.437062, 0.438125

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 175.87 | -24.13 | 18.085 | 30.158 |
| mle | 182.99 | -17.01 | 12.003 | 20.821 |
| dr1 | 188.17 | -11.83 | 10.431 | 15.771 |
| boot | 193.05 | -6.95 | 13.236 | 14.949 |
| ac1 | 196.20 | -3.80 | 8.998 | 9.768 |
| ac2 | 195.05 | -4.95 | 9.056 | 10.319 |
| ac3 | 195.13 | -4.87 | 9.050 | 10.275 |
| o | 200.27 | 0.27 | 11.391 | 11.394 |
| ca1 | 205.43 | 5.43 | 10.044 | 11.419 |
| ca2 | 205.09 | 5.09 | 9.952 | 11.179 |
| ca3 | 205.12 | 5.12 | 9.967 | 11.206 |
| pojac | 212.96 | 12.96 | 18.640 | 22.701 |
| jac1 | 207.01 | 7.01 | 9.875 | 12.109 |
| jacseq | 209.77 | 9.77 | 12.719 | 16.038 |
| jacint | 207.76 | 7.76 | 10.756 | 13.266 |

Pr(inf mle) , mean jacknife order = 0.000, 1.129
c, ch1, ch2, ch3 = 0.938, 0.933 , 0.937 , 0.937
cvh1, cvh2, cvh3 = 0.460833, 0.453616, 0.454193

Table 3.5.3c

N = 200 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.20 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

Number of sampling occasions, t = .... 5

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 86.53 | -113.47 | 26.388 | 116.503 |
| mle | 165.60 | -34.40 | 57.866 | 67.319 |
| dr1 | 199.03 | -0.97 | 74.760 | 74.767 |
| boot | 107.57 | -92.43 | 31.023 | 97.496 |
| ac1 | 233.07 | 33.07 | 102.399 | 107.606 |
| ac2 | 178.36 | -21.64 | 73.940 | 77.041 |
| ac3 | 188.63 | -11.37 | 85.094 | 85.850 |
| o | 117.13 | -82.87 | 33.012 | 89.200 |
| ca1 | 209.97 | 9.97 | 74.935 | 75.595 |
| ca2 | 184.46 | -15.54 | 60.754 | 62.710 |
| ca3 | 188.87 | -11.13 | 62.014 | 63.005 |
| pojac | 179.88 | -20.12 | 43.406 | 47.844 |
| jac1 | 133.84 | -66.16 | 35.953 | 75.300 |
| jacseq | 179.58 | -20.42 | 41.279 | 46.054 |
| jacint | 172.64 | -27.36 | 40.590 | 48.951 |

Pr(inf mle) , mean jacknife order = 0.001, 3.747
c, ch1, ch2, ch3 = 0.528, 0.461 , 0.540 , 0.521
cvh1, cvh2, cvh3 = 0.451279, 0.186030, 0.240712

Number of sampling occasions, t = .... 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 125.67 | -74.33 | 28.278 | 79.527 |
| mle | 167.32 | -32.68 | 21.086 | 38.895 |
| dr1 | 181.67 | -18.33 | 22.622 | 29.117 |
| boot | 150.99 | -49.01 | 29.879 | 57.401 |
| ac1 | 197.51 | -2.49 | 26.871 | 26.986 |
| ac2 | 183.18 | -16.82 | 23.309 | 28.744 |
| ac3 | 185.23 | -14.77 | 23.698 | 27.925 |
| o | 162.69 | -37.31 | 30.222 | 48.012 |
| ca1 | 200.63 | 0.63 | 24.017 | 24.025 |
| ca2 | 194.66 | -5.34 | 22.742 | 23.360 |
| ca3 | 195.46 | -4.54 | 22.864 | 23.311 |
| pojac | 219.50 | 19.50 | 27.714 | 33.888 |
| jac1 | 180.01 | -19.99 | 29.476 | 35.613 |
| jacseq | 204.85 | 4.85 | 31.358 | 31.731 |
| jacint | 194.51 | -5.49 | 29.055 | 29.570 |

Pr(inf mle) , mean jacknife order = 0.000, 1.991
c, ch1, ch2, ch3 = 0.729, 0.696 , 0.730 , 0.724
cvh1, cvh2, cvh3 = 0.393350, 0.309276, 0.322405

Number of sampling occasions, t = .... 15

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 145.87 | -54.13 | 26.502 | 60.270 |
| mle | 170.95 | -29.05 | 16.640 | 33.475 |
| dr1 | 180.50 | -19.50 | 16.307 | 25.416 |
| boot | 170.07 | -29.93 | 25.349 | 39.224 |
| ac1 | 192.27 | -7.73 | 17.246 | 18.900 |
| ac2 | 186.18 | -13.82 | 16.805 | 21.758 |
| ac3 | 186.91 | -13.09 | 16.890 | 21.371 |
| o | 181.02 | -18.98 | 24.434 | 30.941 |
| ca1 | 201.17 | 1.17 | 17.689 | 17.727 |
| ca2 | 198.84 | -1.16 | 17.553 | 17.591 |
| ca3 | 199.08 | -0.92 | 17.567 | 17.591 |
| pojac | 222.16 | 22.16 | 20.654 | 30.289 |
| jac1 | 195.03 | -4.97 | 21.917 | 22.474 |
| jacseq | 210.24 | 10.24 | 23.356 | 25.500 |
| jacint | 202.58 | 2.58 | 22.618 | 22.765 |

Pr(inf mle) , mean jacknife order = 0.000, 1.565
c, ch1, ch2, ch3 = 0.820, 0.808 , 0.825 , 0.823
cvh1, cvh2, cvh3 = 0.398615, 0.360282, 0.364733

Number of sampling occasions, t = .... 20

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 161.08 | -38.92 | 23.350 | 45.384 |
| mle | 176.75 | -23.25 | 14.150 | 27.216 |
| dr1 | 184.22 | -15.78 | 13.351 | 20.668 |
| boot | 182.94 | -17.06 | 20.101 | 26.364 |
| ac1 | 194.39 | -5.61 | 13.666 | 14.775 |
| ac2 | 191.24 | -8.76 | 13.613 | 16.188 |
| ac3 | 191.56 | -8.44 | 13.630 | 16.030 |
| o | 192.54 | -7.46 | 18.424 | 19.878 |
| ca1 | 204.42 | 4.42 | 14.012 | 14.694 |
| ca2 | 203.32 | 3.32 | 13.913 | 14.304 |
| ca3 | 203.43 | 3.43 | 13.909 | 14.325 |
| pojac | 220.20 | 20.20 | 19.749 | 28.247 |
| jac1 | 203.38 | 3.38 | 15.476 | 15.841 |
| jacseq | 211.36 | 11.36 | 18.658 | 21.846 |
| jacint | 206.72 | 6.72 | 16.873 | 18.162 |

Pr(inf mle) , mean jacknife order = 0.000, 1.309
c, ch1, ch2, ch3 = 0.882, 0.873 , 0.883 , 0.882
cvh1, cvh2, cvh3 = 0.423520, 0.403792, 0.405877

Number of sampling occasions, t = .... 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 168.78 | -31.22 | 20.761 | 37.490 |
| mle | 179.55 | -20.45 | 13.210 | 24.344 |
| dr1 | 185.93 | -14.07 | 12.010 | 18.499 |
| boot | 188.42 | -11.58 | 16.524 | 20.177 |
| ac1 | 195.27 | -4.73 | 11.486 | 12.423 |
| ac2 | 193.38 | -6.62 | 11.443 | 13.221 |
| ac3 | 193.54 | -6.46 | 11.444 | 13.143 |
| o | 196.88 | -3.12 | 14.660 | 14.989 |
| ca1 | 204.90 | 4.90 | 12.276 | 13.219 |
| ca2 | 204.30 | 4.30 | 12.154 | 12.892 |
| ca3 | 204.35 | 4.35 | 12.160 | 12.915 |
| pojac | 216.53 | 16.53 | 20.624 | 26.431 |
| jac1 | 205.58 | 5.58 | 12.450 | 13.642 |
| jacseq | 210.69 | 10.69 | 17.248 | 20.293 |
| jacint | 207.54 | 7.54 | 14.430 | 16.281 |

Pr(inf mle) , mean jacknife order = 0.000, 1.222
c, ch1, ch2, ch3 = 0.913, 0.907 , 0.913 , 0.912
cvh1, cvh2, cvh3 = 0.448544, 0.437062, 0.438125

Number of sampling occasions, t = .... 30

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x | 175.87 | -24.13 | 18.085 | 30.158 |
| mle | 182.99 | -17.01 | 12.003 | 20.821 |
| dr1 | 188.17 | -11.83 | 10.431 | 15.771 |
| boot | 193.05 | -6.95 | 13.236 | 14.949 |
| ac1 | 196.20 | -3.80 | 8.998 | 9.768 |
| ac2 | 195.05 | -4.95 | 9.056 | 10.319 |
| ac3 | 195.13 | -4.87 | 9.050 | 10.275 |
| o | 200.27 | 0.27 | 11.391 | 11.394 |
| ca1 | 205.43 | 5.43 | 10.044 | 11.419 |
| ca2 | 205.09 | 5.09 | 9.952 | 11.179 |
| ca3 | 205.12 | 5.12 | 9.967 | 11.206 |
| pojac | 212.96 | 12.96 | 18.640 | 22.701 |
| jac1 | 207.01 | 7.01 | 9.875 | 12.109 |
| jacseq | 209.77 | 9.77 | 12.719 | 16.038 |
| jacint | 207.76 | 7.76 | 10.756 | 13.266 |

Pr(inf mle) , mean jacknife order = 0.000, 1.129
c, ch1, ch2, ch3 = 0.938, 0.933 , 0.937 , 0.937
cvh1, cvh2, cvh3 = 0.460833, 0.453616, 0.454193

## §3.6 : Some Standard Data Sets

The aim of this section is to assess how the estimators perform on various standard data sets which have previously been studied in the literature. Of particular interest in this section is the fact that some of the following data sets were obtained from experiments carried out on populations of known size. It is also of interest to see how the results of this section, being obtained from real life populations, compare with the results of the simulation study of section 3.5. It is noted however that it would be unwise to draw firm conclusions from a small number of data sets.

The notation used in the following tables is identical to that of the previous section, with the following additions :

N $\equiv$ population size ( if known ).

f1 $\equiv$ the number of animals seen exactly once.

f2 $\equiv$ the number of animals seen exactly twice.

.

.

ft $\equiv$ the number of animals seen exactly t times.

x $\equiv$ number of distinct individuals seen.

z $\equiv$ total number of sightings.

Carothers (1973) carried out a capture-recapture experiment on a population of 420 taxicabs working in Edinburgh. The whole data set, as illustrated in table 3.6.1, was presented along with various subsets. Tables 3.6.2a and 3.6.2b show how the estimators perform on all 42 subsets. A number of the subsets intersect and so not all of the estimates are independent. For details of how the different subsets were obtained please refer to Carothers(1973). The whole taxicab data set, along with the snowshoe hare data of Burnham and Cushwa, appeared in the Otis et al. (1978) wildlife monograph, wherein for both sets of data the model selection procedure of Otis et al. (1978) was shown to choose the model $M_h$. The model selection procedure of Otis et al. (1978) also selects the model $M_h$ as being most appropriate for the meadow vole trapping data of Pollock et al. (1990). The model $M_h$ has been judged appropriate by Norris and Pollock(1996) for the eastern chipmunk data of Mares et al. (1981). It is impossible to apply the model selection procedure of Otis et al. (1978) to the mud turtle data of Chao(1989) : this data set is included so as to illustrate through a real data example how the estimators perform in a situation wherein sample coverage is very small.

The performance and relative performance of the estimators in connection with the standard data sets is almost entirely consistent with the simulation study of section 3.5. One may firstly observe that the standard data sets illustrate the tendency of the maximum likelihood estimator $\hat{N}$ and Darroch and Ratcliff estimator dr1 to underestimate in the presence of heterogeneity. The bootstrap estimator of Smith and van Belle(1984) is negatively biased, and particularly so when sample coverage is small. Also immediately apparent is the way in which the coverage adjusted estimators clearly perform much better than the Overton estimator. The Overton estimator $\hat{N}_0$ can be severely negatively biased and consequently its performance tends to be unacceptable, unless that is, as with the data of Mares et al. (1981) and Pollock et al. (1990), an extremely large proportion of the population is seen during the experiment. $\hat{N}_0$ is particularly negatively biased when sample coverage is small whereas in this situation, for the reasons discussed in the previous section, the coverage adjusted estimators $\hat{N}_{ca1}$, $\hat{N}_{ca2}$ and $\hat{N}_{ca3}$ are still able to provide reasonable estimates of population size.

The estimators ac1 and ca1 both make use of ch1 as an estimator of sample coverage. In the same way ac2, ca2 and ac3, ca3 incorporate ch2 and ch3 respectively. In tables 3.6.2a and 3.6.2b it is therefore reasonable to compare ca1 to ac1, ca2 to ac2 and ca3 to ac3. In order to highlight this comparison, the cell containing the better estimate is shaded in each case. Looking at the shaded cells in tables 3.6.2a and 3.6.2b shows quite clearly that, for this particular population, the coverage adjusted estimators for each given estimate of sample coverage, tend to perform better than the corresponding ac1, ac2 or ac3. Explicitly, on 25 of the 42 subsets, the estimate provided by ca1 is closer to the true value of 420 than that given by ac1. Similarly on 35 of these 42 subsets, the estimators ca2 and ca3 improve on ac2 and ac3 respectively. This is consistent with the simulation study. However, for the complete taxicab data, as shown in table 3.6.1, ac1 is actually more accurate than ca1. For this data set, ca2 and ca3 are respectively more accurate than ac2 and ac3, so it is perhaps worthwhile to recall at this point that the simulation study recommended use of either ca2 or ca3 in favour of ca1.

The complete taxicab data of table 3.6.1 also illustrates how, even with good data, the sequential selection procedure of Burnham and Overton(1978) can mislead. That is, for the taxicab data, the sequential selection procedure chooses the third order jackknife as being the most appropriate - whereas this estimator clearly overestimates by a relatively large amount. The meadow vole data appears to provide a similar example. The discussion of the previous section recommended that, if estimates of sample coverage were above 0.7, then one should consider simply ignoring the sequential selection procedure of Burnham and Overton(1978) and always choose the first order jackknife. The standard data sets do lend support to this suggestion.

The mud turtle data set illustrates how, when sample coverage is very small, the jackknife estimators can possess extreme negative bias - for example, for the mud turtle data, even the fifth order jackknife estimator provides an estimate of 491 ( Chao(1989) concluded that, if equal catchability were a reasonable assumption, then there were about 800 turtles in the habitat). This data set, therefore, also highlights the extreme negative bias of the bootstrap and Overton estimators when sample coverage is very small.

It has been noted that it would be unwise to draw firm conclusions from a small number of data sets. With this in mind, it is still, however, pleasing to observe that the performance of the estimators in connection with the real life data is consistent with their performance within the simulation study. And that furthermore, the recommendations of the previous section are supported by the performance of the estimators on the standard data sets.

Table 3.6.1 : Standard Data Sets

| Source | Mares et al.(1981) | Carothers (1973) | Burnham and Cushwa. (see text) | Pollock et al. (1990) | A.Chao (1989) |
|---|---|---|---|---|---|
| Description | Eastern Chipmunks | Taxicab Data | Snowshoe Hare | Meadow Vole | Mud Turtle |
| N | 82 | 420 | - | - | - |
| t | 13 | 10 | 6 | 5 | 40 |
| f1 | 14 | 142 | 25 | 29 | 94 |
| f2 | 13 | 81 | 22 | 15 | 5 |
| f3 | 18 | 49 | 13 | 15 | 0 |
| f4 | 12 | 7 | 5 | 16 | 0 |
| f5 | 7 | 3 | 1 | 27 | 0 |
| f6 | 5 | 1 | 2 | - | 0 |
| f7 | 1 | 0 | - | - | 0 |
| f8 | 1 | 0 | - | - | 0 |
| f9 to f(t) | 0 | 0 | - | - | 0 |
| | | | | | |
| x | 71 | 283 | 68 | 102 | 99 |
| z | 222 | 500 | 145 | 303 | 104 |
| | | | | | |
| mle | 73 | 368 | 75 | 103 | 1011 |
| dr1 | 76 | 395 | 82 | 113 | 1030 |
| | | | | | |
| boot | 78 | 343 | 79 | 113 | 134 |
| | | | | | |
| ac1 | 77 | 416 | 89 | 123 | 1030 |
| ac2 | 76 | 386 | 81 | 118 | 1004 |
| ac3 | 77 | 393 | 84 | 123 | 1004 |
| | | | | | |
| O | 81 | 370 | 83 | 118 | 153 |
| ca1 | 82 | 439 | 90 | 121 | 1053 |
| ca2 | 82 | 427 | 87 | 120 | 1028 |
| ca3 | 82 | 429 | 88 | 121 | 1028 |
| | | | | | |
| pojac | 86 | 504 | 100 | 144 | 294 |
| | | | | | |
| jac1 | 84 | 411 | 89 | 125 | 191 |
| jacseq | 84 | 495 | 89 | 142 | 191 |
| jacint | 84 | 469 | 89 | 138 | 191 |
| order jac | 1 | 3 | 1 | 3 | 1 |
| | | | | | |
| ch1 | 0.937 | 0.716 | 0.828 | 0.904 | 0.096 |
| ch2 | 0.947 | 0.752 | 0.888 | 0.929 | 0.099 |
| ch3 | 0.943 | 0.744 | 0.861 | 0.904 | 0.099 |
| | | | | | |
| cvh1 | 0.331 | 0.327 | 0.478 | 0.555 | 0.000 |
| cvh2 | 0.313 | 0.232 | 0.380 | 0.523 | 0.000 |
| cvh3 | 0.320 | 0.256 | 0.425 | 0.555 | 0.000 |

# Table 3.6.2a : Taxicab Data : N = 420

| | Sampling Scheme A | | | | | | | Sampling Scheme B | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Data Subset alpha | | | | | | | Data Subset alpha | | | | | | |
| Subset | a | b | c | d | e | f | g | a | b | c | d | e | f | g |
| t | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| f1 | 65 | 73 | 75 | 109 | 112 | 117 | 135 | 78 | 67 | 71 | 112 | 106 | 102 | 116 |
| f2 | 12 | 8 | 7 | 24 | 28 | 24 | 42 | 5 | 9 | 7 | 22 | 28 | 26 | 48 |
| f3 | 0 | 0 | 0 | 3 | 2 | 4 | 9 | 0 | 0 | 1 | 0 | 3 | 3 | 6 |
| f4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 2 |
| f5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| x | 77 | 81 | 82 | 136 | 142 | 145 | 187 | 83 | 76 | 79 | 136 | 137 | 131 | 172 |
| z | 89 | 89 | 89 | 166 | 174 | 177 | 250 | 88 | 85 | 88 | 164 | 171 | 163 | 238 |
| mle | 242 | 373 | 428 | 330 | 339 | 352 | 341 | 593 | 299 | 321 | 347 | 306 | 295 | 290 |
| dr1 | 286 | 451 | 521 | 396 | 399 | 428 | 407 | 730 | 359 | 409 | 429 | 360 | 350 | 336 |
| boot | 99 | 106 | 107 | 174 | 181 | 185 | 235 | 109 | 99 | 103 | 174 | 174 | 166 | 214 |
| ac1 | 305 | 506 | 594 | 455 | 436 | 507 | 471 | 853 | 396 | 527 | 555 | 399 | 392 | 371 |
| ac2 | 228 | 360 | 417 | 327 | 325 | 363 | 352 | 584 | 287 | 368 | 389 | 297 | 289 | 280 |
| ac3 | 228 | 360 | 417 | 335 | 329 | 382 | 373 | 584 | 287 | 386 | 389 | 302 | 295 | 286 |
| O | 110 | 117 | 119 | 191 | 199 | 204 | 256 | 121 | 109 | 114 | 192 | 191 | 183 | 233 |
| ca1 | 296 | 462 | 533 | 414 | 417 | 448 | 431 | 742 | 370 | 420 | 448 | 379 | 367 | 357 |
| ca2 | 244 | 377 | 433 | 354 | 353 | 384 | 379 | 601 | 302 | 355 | 381 | 323 | 314 | 314 |
| ca3 | 244 | 377 | 433 | 360 | 357 | 393 | 389 | 601 | 302 | 363 | 381 | 328 | 319 | 318 |
| pojac | 185 | 205 | 210 | 316 | 325 | 338 | 403 | 218 | 189 | 200 | 322 | 309 | 297 | 352 |
| jac1 | 129 | 139 | 142 | 223 | 232 | 239 | 295 | 145 | 130 | 136 | 226 | 222 | 213 | 265 |
| jacseq | 192 | 217 | 223 | 325 | 332 | 350 | 407 | 233 | 199 | 213 | 332 | 315 | 303 | 345 |
| jacint | 183 | 205 | 211 | 311 | 318 | 334 | 392 | 219 | 189 | 201 | 317 | 302 | 291 | 336 |
| order jac | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| ch1 | 0.270 | 0.180 | 0.157 | 0.343 | 0.356 | 0.339 | 0.460 | 0.114 | 0.212 | 0.193 | 0.317 | 0.380 | 0.374 | 0.513 |
| ch2 | 0.337 | 0.225 | 0.197 | 0.416 | 0.437 | 0.407 | 0.544 | 0.142 | 0.265 | 0.233 | 0.384 | 0.462 | 0.454 | 0.613 |
| ch3 | 0.337 | 0.225 | 0.197 | 0.407 | 0.431 | 0.395 | 0.526 | 0.142 | 0.265 | 0.227 | 0.384 | 0.453 | 0.445 | 0.601 |
| cov1 | 0.285 | 0.371 | 0.389 | 0.431 | 0.345 | 0.478 | 0.469 | 0.423 | 0.343 | 0.566 | 0.596 | 0.374 | 0.391 | 0.394 |
| cov2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.155 | 0.177 | 0.000 | 0.000 | 0.308 | 0.345 | 0.000 | 0.000 | 0.000 |
| cov3 | 0.000 | 0.000 | 0.000 | 0.037 | 0.000 | 0.231 | 0.258 | 0.000 | 0.000 | 0.350 | 0.345 | 0.000 | 0.000 | 0.000 |
| | Data Subset beta | | | | | | | Data Subset beta | | | | | | |
| Subset | a | b | c | d | e | f | g | a | b | c | d | e | f | g |
| t | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| f1 | 75 | 75 | 71 | 126 | 114 | 115 | 145 | 66 | 68 | 70 | 105 | 106 | 100 | 115 |
| f2 | 6 | 7 | 8 | 19 | 25 | 26 | 42 | 9 | 10 | 10 | 28 | 25 | 27 | 50 |
| f3 | 1 | 0 | 0 | 2 | 3 | 2 | 7 | 2 | 0 | 0 | 2 | 3 | 3 | 6 |
| f4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| f5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| x | 82 | 82 | 79 | 147 | 142 | 143 | 194 | 77 | 78 | 80 | 135 | 135 | 131 | 172 |
| z | 90 | 89 | 87 | 170 | 173 | 173 | 250 | 90 | 88 | 90 | 167 | 169 | 167 | 237 |
| mle | 381 | 428 | 355 | 464 | 347 | 360 | 391 | 227 | 287 | 301 | 311 | 298 | 272 | 293 |
| dr1 | 492 | 521 | 430 | 568 | 416 | 427 | 462 | 289 | 343 | 360 | 364 | 362 | 327 | 334 |
| boot | 107 | 107 | 103 | 190 | 181 | 183 | 245 | 99 | 101 | 104 | 172 | 172 | 166 | 214 |
| ac1 | 657 | 594 | 482 | 679 | 477 | 475 | 519 | 372 | 376 | 395 | 394 | 438 | 384 | 357 |
| ac2 | 462 | 417 | 344 | 471 | 344 | 348 | 385 | 269 | 275 | 288 | 297 | 316 | 277 | 277 |
| ac3 | 488 | 417 | 344 | 488 | 351 | 353 | 396 | 288 | 275 | 288 | 301 | 328 | 286 | 283 |
| O | 119 | 119 | 114 | 210 | 200 | 201 | 268 | 110 | 112 | 115 | 189 | 189 | 182 | 232 |
| ca1 | 504 | 533 | 441 | 589 | 436 | 446 | 487 | 300 | 354 | 371 | 382 | 381 | 344 | 356 |
| ca2 | 426 | 433 | 359 | 496 | 371 | 377 | 422 | 258 | 290 | 304 | 323 | 328 | 297 | 311 |
| ca3 | 437 | 433 | 359 | 504 | 378 | 381 | 432 | 266 | 290 | 304 | 327 | 334 | 302 | 315 |
| pojac | 211 | 210 | 200 | 359 | 330 | 332 | 428 | 188 | 193 | 198 | 306 | 309 | 293 | 349 |
| jac1 | 142 | 142 | 136 | 248 | 233 | 235 | 310 | 130 | 132 | 136 | 219 | 220 | 211 | 264 |
| jacseq | 225 | 223 | 211 | 376 | 340 | 341 | 433 | 198 | 202 | 208 | 311 | 316 | 298 | 341 |
| jacint | 212 | 211 | 200 | 357 | 325 | 326 | 417 | 188 | 191 | 197 | 298 | 303 | 286 | 333 |
| order jac | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| ch1 | 0.167 | 0.157 | 0.184 | 0.259 | 0.341 | 0.335 | 0.420 | 0.267 | 0.227 | 0.222 | 0.371 | 0.373 | 0.401 | 0.515 |
| ch2 | 0.200 | 0.197 | 0.230 | 0.315 | 0.413 | 0.410 | 0.504 | 0.317 | 0.284 | 0.278 | 0.455 | 0.447 | 0.482 | 0.620 |
| ch3 | 0.194 | 0.197 | 0.230 | 0.309 | 0.405 | 0.405 | 0.490 | 0.306 | 0.284 | 0.278 | 0.449 | 0.438 | 0.473 | 0.608 |
| cov1 | 0.606 | 0.389 | 0.368 | 0.478 | 0.427 | 0.374 | 0.405 | 0.580 | 0.329 | 0.333 | 0.329 | 0.518 | 0.479 | 0.317 |
| cov2 | 0.373 | 0.000 | 0.000 | 0.101 | 0.000 | 0.000 | 0.000 | 0.355 | 0.000 | 0.000 | 0.000 | 0.241 | 0.152 | 0.000 |
| cov3 | 0.414 | 0.000 | 0.000 | 0.171 | 0.000 | 0.000 | 0.000 | 0.408 | 0.000 | 0.000 | 0.000 | 0.282 | 0.206 | 0.000 |

**Table 3.6.2b : Taxicab Data : N = 420**

| | Sampling Scheme A | | | | | | | Sampling Scheme B | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Data Subset gamma | | | | | | | Data Subset gamma | | | | | | |
| Subset | a | b | c | d | e | f | g | a | b | c | d | e | f | g |
| | | | | | | | | | | | | | | |
| t | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| f1 | 101 | 115 | 110 | 143 | 127 | 145 | 142 | 100 | 103 | 98 | 110 | 112 | 113 | 104 |
| f2 | 33 | 27 | 27 | 62 | 73 | 67 | 81 | 30 | 26 | 29 | 67 | 62 | 61 | 67 |
| f3 | 4 | 3 | 4 | 19 | 19 | 19 | 49 | 6 | 6 | 6 | 20 | 24 | 19 | 51 |
| f4 | 0 | 0 | 0 | 3 | 3 | 1 | 7 | 0 | 0 | 1 | 3 | 8 | 7 | 12 |
| f5 | 0 | 0 | 0 | 0 | 1 | 2 | 3 | 0 | 0 | 0 | 3 | 0 | 2 | 6 |
| f6 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| f7 to f10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | | | | | | | | | | | | | | |
| x | 138 | 145 | 141 | 227 | 223 | 234 | 283 | 136 | 135 | 134 | 203 | 206 | 202 | 241 |
| z | 179 | 178 | 176 | 336 | 347 | 350 | 500 | 178 | 173 | 178 | 331 | 340 | 330 | 475 |
| | | | | | | | | | | | | | | |
| mle | 299 | 380 | 347 | 368 | 335 | 373 | 368 | 287 | 304 | 272 | 287 | 287 | 285 | 288 |
| dr1 | 317 | 410 | 376 | 395 | 352 | 400 | 395 | 310 | 334 | 298 | 304 | 307 | 307 | 309 |
| | | | | | | | | | | | | | | |
| boot | 177 | 188 | 182 | 284 | 276 | 292 | 343 | 174 | 174 | 171 | 249 | 252 | 249 | 286 |
| | | | | | | | | | | | | | | |
| ac1 | 317 | 421 | 391 | 412 | 357 | 421 | 416 | 321 | 357 | 322 | 317 | 320 | 329 | 324 |
| ac2 | 289 | 374 | 345 | 369 | 328 | 376 | 386 | 286 | 310 | 281 | 287 | 292 | 299 | 306 |
| ac3 | 291 | 375 | 346 | 372 | 330 | 380 | 393 | 288 | 313 | 284 | 290 | 296 | 302 | 311 |
| | | | | | | | | | | | | | | |
| O | 196 | 210 | 203 | 312 | 300 | 320 | 370 | 193 | 193 | 190 | 271 | 274 | 270 | 306 |
| ca1 | 341 | 436 | 402 | 434 | 388 | 439 | 439 | 335 | 358 | 322 | 337 | 340 | 340 | 343 |
| ca2 | 319 | 405 | 375 | 414 | 371 | 419 | 427 | 314 | 337 | 304 | 324 | 328 | 328 | 336 |
| ca3 | 320 | 406 | 376 | 416 | 373 | 421 | 429 | 316 | 338 | 305 | 325 | 330 | 329 | 338 |
| | | | | | | | | | | | | | | |
| pojac | 316 | 354 | 340 | 467 | 425 | 475 | 504 | 313 | 320 | 308 | 375 | 384 | 383 | 395 |
| | | | | | | | | | | | | | | |
| jac1 | 229 | 249 | 240 | 356 | 337 | 365 | 411 | 226 | 228 | 222 | 302 | 307 | 304 | 335 |
| jacseq | 320 | 249 | 240 | 463 | 387 | 465 | 495 | 347 | 228 | 342 | 342 | 352 | 351 | 370 |
| jacint | 312 | 249 | 240 | 435 | 345 | 436 | 469 | 331 | 228 | 326 | 306 | 323 | 319 | 349 |
| order jac | 3 | 1 | 1 | 3 | 2 | 3 | 3 | 4 | 1 | 4 | 2 | 2 | 2 | 2 |
| | | | | | | | | | | | | | | |
| ch1 | 0.436 | 0.354 | 0.375 | 0.574 | 0.634 | 0.586 | 0.716 | 0.438 | 0.405 | 0.449 | 0.668 | 0.671 | 0.658 | 0.781 |
| ch2 | 0.477 | 0.388 | 0.409 | 0.615 | 0.681 | 0.628 | 0.752 | 0.476 | 0.438 | 0.486 | 0.713 | 0.711 | 0.699 | 0.812 |
| ch3 | 0.475 | 0.386 | 0.407 | 0.611 | 0.676 | 0.624 | 0.744 | 0.473 | 0.435 | 0.483 | 0.708 | 0.705 | 0.694 | 0.803 |
| | | | | | | | | | | | | | | |
| cov1 | 0.000 | 0.186 | 0.228 | 0.256 | 0.160 | 0.295 | 0.327 | 0.212 | 0.300 | 0.329 | 0.281 | 0.273 | 0.358 | 0.340 |
| cov2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.116 | 0.232 | 0.000 | 0.083 | 0.160 | 0.105 | 0.116 | 0.249 | 0.269 |
| cov3 | 0.000 | 0.000 | 0.000 | 0.049 | 0.000 | 0.144 | 0.256 | 0.000 | 0.117 | 0.178 | 0.135 | 0.148 | 0.263 | 0.290 |

## § 3.7 : Plant-Capture Applied to the Model $M_h$

### § 3.7.1 : Introduction

Within the previous sections of this chapter consideration was given to the standard problem of estimating population size from capture-recapture data, in the absence of plants. The aim of this section is to show how an initial insertion of planted individuals into the population prior to the beginning of the capture-recapture experiment can enhance point estimation of population size. This is done by deriving a Peterson-type estimator in section 3.7.4, and in section 3.7.5 it is shown how the coverage adjusted estimators of section 3.3 can be modified so as to enable them to utilize the information gained from planted individuals.

### § 3.7.2 : Sampling Procedure, Assumptions and Some Additional Notation

The sampling procedure considered within this section is almost identical to the one described in section 3.2. The only difference being that prior to the beginning of the experiment it is assumed that a known number of R pre-marked individuals have been mixed with the target population, of size N. It is assumed that the planted individuals behave in an identical manner to those of the target population.

Once the planted animals have mixed with the target population a sequence of t sampling experiments are carried out on the augmented population which is assumed to be closed and of size N+R. Independently of other animals and independently of its previous capture history animal i (i=1,2,...,N+R) is captured in sample j (j=1,2,...,t) with probability $p_i$. After each sample is taken every animal within that sample which has not previously been marked receives a unique tag before its immediate release so that it may be recognised on subsequent trapping occasions. The experiment generates an N+R by t matrix A where

$$a_{ij} = \begin{cases} 1 & \text{if animal i is caught on sampling occasion j} \\ 0 & \text{if animal i is not caught on sampling occasion j} \end{cases}$$

$$i = 1, 2, ...., N+R.$$

$$j = 1, 2, ...., t.$$

The sample space is the set of such matrices.

It is assumed that the $p_i$, for i= 1, 2, ..., N+R, are a random sample from some probability distribution f(p), $p \in [0,1]$, with c.d.f. F(p).

At this point it is necessary to introduce some additional notation .

As in chapters 1 and 2, let

$X_1 \equiv$ number of distinct animals seen from target population,

$X_2 \equiv$ number of distinct animals seen from planted population

and let $X = X_1 + X_2 \equiv$ number of distinct animals seen from augmented population.

Now the frequencies for the target, planted and augmented populations are written as

$f_k \equiv$ number of animals from target population seen exactly k times,

$f_k^* \equiv$ number of animals from planted population seen exactly k times

and $f_k^a = f_k + f_k^* \equiv$ number of animals from augmented population seen exactly k times.

Finally let

$$Z_a = \sum_{k=1}^{t} k f_k^a \equiv \text{total number of sightings made.}$$

§ 3.7.3 : Some Distribution Theory

By the independence of the target and planted populations, it follows from equation (3.1) that the joint probability distribution function of $\{X_1, X_2, f_1, f_2, \ldots, f_t, f_1^*, f_2^*, \ldots, f_t^*\}$ may be written as

$$\text{Prob}\{X_1, X_2, f_1, f_2, \ldots, f_t, f_1^*, f_2^*, \ldots, f_t^*\} =$$

$$\binom{N}{N - x_1, f_1, f_2, \ldots, f_t} (\pi_0)^{N-x_1} \prod_{j=1}^{t} (\pi_j)^{f_j} \cdot \binom{R}{R - x_2, f_1^*, f_2^*, \ldots, f_t^*} (\pi_0)^{R-x_2} \prod_{j=1}^{t} (\pi_j)^{f_j^*}, \quad (3.10)$$

$$\text{where } \pi_j = \int_0^1 \binom{t}{j} p^j (1-p)^{t-j} dF(p),$$

$$j = 0, 1, 2, \ldots, t.$$

It can be shown ( K. Pollock pers.com. ) that this probability function may be decomposed as follows :

$$\text{Prob}\{X_1, X_2, f_1, f_2, \ldots, f_t, f_1^*, f_2^*, \ldots, f_t^*\} = P_1 . P_2 . P_3 . P_4, \quad (3.11)$$

$$\text{where } P_1 = \frac{\binom{N}{\sum f_i} \binom{R}{\sum f_i^*}}{\binom{N+R}{\sum f_i + \sum f_i^*}},$$

$$P_2 = \binom{N+R}{\sum f_i + \sum f_i^*} (1 - \pi_0)^{\Sigma f_i + \Sigma f_i^*} (\pi_0)^{N+R-\Sigma f_i - \Sigma f_i^*},$$

$$P_3 = \binom{\sum f_i + \sum f_i^*}{f_1 + f_1^*, \ldots, f_t + f_t^*} \prod_{i=1}^{t} \left( \frac{\pi_i}{1 - \pi_0} \right)^{f_i + f_i^*}$$

and $$P_4 = \frac{\binom{\sum f_i}{f_1, f_2, \ldots, f_t} \binom{\sum f_i^*}{f_1^*, f_2^*, \ldots, f_t^*}}{\binom{\sum f_i + \sum f_i^*}{f_1 + f_1^*, \ldots, f_t + f_t^*}}.$$

N.B. Each summation is to be evaluated over the range i = 1, 2, ..., t.

## § 3.7.4 : A Peterson-Type Estimator

The Peterson-type estimator proposed here is derived from the $P_1$ component of the probability function (3.11) : this $P_1$ component is in fact a hypergeometric density function. Let $L(N)$ denote the likelihood function for N based on this hypergeometric density. By equating $L(N)$ to $L(N-1)$ one can show that the likelihood function for N, based on the hypergeometric distribution $P_1$, is in fact maximised by the Peterson-type estimator $\tilde{N}_P = \frac{RX_1}{X_2}$. In order to avoid introducing an estimator which becomes infinite when $X_2 = 0$, and in view of the fact that population size N is integer valued, $\tilde{N}_P$ is slightly modified : from this point consideration is given to the estimator $\hat{N}_P = \left[ 0.5 + \frac{(R+1)X_1}{X_2 + 1} \right]$, where [.] denotes the integer part.

## § 3.7.5 : Plant-Capture Versions of the Overton and Coverage Adjusted Estimators

Of the estimators considered within this chapter that were initially designed for the standard problem of estimating population size in the absence of plants, other than the nonparametric maximum likelihood estimator of Norris and Pollock(1996), only the Overton estimator and coverage adjusted estimators of section 3.3 can be extended naturally in such a way that allows them to be able to utilize the extra information gained from the planted individuals. The plant-capture versions of the Overton and coverage adjusted estimators are obtained via an approach almost identical to the one used to derive them in the absence of plants : hence the following derivation is explained only briefly.

From equation 3.10 it follows that the likelihood function for N may be written as

$$L(N) \propto \frac{N!}{(N-x_1)!}(\pi_0)^{N-x_1}$$
$$= \frac{N!}{(N-x_1)!}\left\{E\left[(1-p)^t\right]\right\}^{N-x_1}.$$

If one assumes for the moment that F(p) is known exactly, so that $E\left[(1-p)^t\right]$ can be viewed as a known constant, then, by equating L(N) to L(N-1), one can show that an approximate maximum likelihood estimate is given by

$$\hat{N} = \frac{x_1}{1-E\left[(1-p)^t\right]}. \tag{3.12}$$

Using the theory of weighted distributions, and assuming that the capture probabilities of the animals seen during the experiment are known exactly, one can show that an unbiased estimator of $\left\{1-E\left[(1-p)^t\right]\right\}^{-1}$ is given by $\frac{1}{x}\sum_{i\in S_x^a}\frac{1}{1-(1-p_i)^t}$ : the set $S_x^a = \{s_k : k = 1,2,..,x\}$, where $s_k \in \{1,2,3,\ldots,N+R\}$ for all k, represents the set of the indices of the x distinct animals seen during the sampling period.

Now assuming that the $p_i$ for all $i \in S_x^a$ are known, use of equation (3.12) suggests the estimator

$$\hat{N} = \frac{x_1}{x}\sum_{i\in S_x^a}\frac{1}{1-(1-p_i)^t}. \tag{3.13}$$

A plant-capture version of the Overton estimator can now be obtained by substituting into equation (3.13) the maximum likelihood estimates of the $p_i$ :

$$\hat{N}_0^a = \frac{x_1}{x}\sum_{i=1}^{t}\frac{f_i^a}{1-\left(1-\frac{i}{t}\right)^t}.$$

Similarly substituting coverage-adjusted estimates of the capture-probabilities into equation (3.13) yields the three following plant-capture versions of the coverage adjusted estimators of section 3.3 :

$$\hat{N}_{ca1}^a = \frac{x_1}{x}\sum_{i=1}^{t}\frac{f_i^a}{1-\left(1-\left(1-\frac{f_1^a}{z_a}\right)\frac{i}{t}\right)^t},$$

$$\hat{N}_{ca2}^a = \frac{x_1}{x}\sum_{i=1}^{t}\frac{f_i^a}{1-\left(1-\left(1-\frac{f_1^a}{z_a}+\frac{2}{(t-1)}\frac{f_2^a}{z_a}\right)\frac{i}{t}\right)^t}$$

and $$\hat{N}_{ca3}^a = \frac{x_1}{x}\sum_{i=1}^{t}\frac{f_i^a}{1-\left(1-\left(1-\frac{f_1^a}{z_a}+\frac{2}{(t-1)}\frac{f_2^a}{z_a}-\frac{6}{(t-1)(t-2)}\frac{f_3^a}{z_a}\right)\frac{i}{t}\right)^t}.$$

N.B. In the absence of plants, i.e. when R = 0, the estimators $\hat{N}_0^a$, $\hat{N}_{ca1}^a$, $\hat{N}_{ca2}^a$ and $\hat{N}_{ca3}^a$ do in fact reduce to $\hat{N}_0$, $\hat{N}_{ca1}$, $\hat{N}_{ca2}$ and $\hat{N}_{ca3}$ respectively.

§ 3.7.6 : Plant-Capture Simulation Study

A simulation study was carried out in order to investigate the performance of the plant-capture estimators. The simulations were carried out in an almost identical manner to those of tables 3.5.3a, b and c. In each simulation the capture probabilities of the N+R animals were drawn as a random sample from some probability distribution p with mean E(p), variance Var(p) and coefficient of variation sqrt[Var(p)]/E(p). At the beginning of each simulation the value of E(p) was selected as a random observation from a uniform distribution on some interval ( . , . ). Similarly the value of the coefficient of variation was selected as a random observation from a uniform distribution on some interval ( . , . ). The distribution of p was chosen to be Beta(alpha, beta), where, as in section 3.3, $\text{alpha} = \frac{1-\text{ep}}{(\text{cv})^2} - \text{ep}$ and $\text{beta} = \frac{(1-\text{ep})^2}{\text{ep}(\text{cv})^2} - (1-\text{ep})$. The capture probabilities of the N+R animals were drawn as a random sample from this beta distribution and live trapping was then simulated in the usual way. Each table is split into two columns : each column depicting the results for a certain number of sampling occasions. Each column is split into four cells : the first cell shows how the estimators perform in the absence of planted individuals ; the second, third and fourth cells illustrate respectively the performance of the estimators in the presence of 10, 25 and 50 plants. For each value of t and R one thousand simulations were carried out : a different set of capture probabilities was used each time. The values shown in the tables are mostly averages. As many of the estimators are only finite if at least one recapture occurs, any data set not meeting this condition was discarded. The simulation procedure continued until one thousand data sets for which the condition did hold had been generated. This simulation study considers various target populations of size N = 100. The results of the simulations are presented in tables 3.7.1a,b, 3.7.2a,b, 3.7.3a,b, 3.7.4a,b, 3.7.5a,b and 3.7.6a,b. The notation used within these tables is identical to the notation of section 3.5, with the following additions :

x1 $\equiv X_1$.

x2 $\equiv X_2$.

pet $\equiv \hat{N}_P$.

mlea $\equiv$ the maximum likelihood estimator for the model $M_0$, for details please refer to chapter 1.

dr1a $\equiv \hat{N}^a_{0,1} = \dfrac{x_1}{1 - \dfrac{f^a_1}{z_a}}$.

$$
\begin{array}{ll}
\text{Oa} & \equiv \hat{N}_0^a. \\
\text{ca1a} & \equiv \hat{N}_{ca1}^a. \\
\text{ca2a} & \equiv \hat{N}_{ca2}^a. \\
\text{ca3a} & \equiv \hat{N}_{ca3}^a.
\end{array}
$$

$$
\text{ca} \quad \equiv C^a = \frac{\sum_{i \in S_x^a} p_i}{\sum_{i=1}^{N+R} p_i}.
$$

$$
\text{ch1a} \quad \equiv 1 - \frac{f_1^a}{z_a}.
$$

$$
\text{ch2a} \quad \equiv 1 - \frac{f_1^a}{z_a} + \frac{2}{(t-1)}\frac{f_2^a}{z_a}.
$$

$$
\text{ch3a} \quad \equiv 1 - \frac{f_1^a}{z_a} + \frac{2}{(t-1)}\frac{f_2^a}{z_a} - \frac{6}{(t-1)(t-2)}\frac{f_3^a}{z_a}.
$$

## Discussion

An important characteristic of the Peterson-type estimator $\hat{N}_P$ is that it remains virtually unbiased in the presence of heterogeneity : consistent unbiasedness in the presence of heterogeneity is something no other existing estimator is able to achieve. However a major disadvantage of the Peterson-type estimator is that it can have a very large variance when the number of plants is small relative to the size of the target population.

The plant-capture versions of the Overton and coverage adjusted estimators can be seen to improve as the number of plants is increased from $R = 0$ : the beneficial effect of the plants is most noticeable when the number of sampling occasions is small. However it is disappointing to observe that this improvement due to the plants is on the whole only marginal. There is scope for more work here since it is believed that the plants may be used in a more efficient way to improve point estimation in this situation, perhaps through considering an estimator in the form of a weighted average of the Peterson-type and coverage adjusted estimators.

**Table 3.7.1a**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.20 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

**t = 5**

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 43.73 | -56.27 | 13.481 | 57.863 |
| mle | 84.22 | -15.78 | 31.533 | 35.263 |
| dr1 | 102.45 | 2.45 | 41.631 | 41.703 |
| boot | 54.37 | -45.63 | 15.953 | 48.342 |
| ac1 | 120.29 | 20.29 | 61.026 | 64.311 |
| ac2 | 93.14 | -6.86 | 49.668 | 50.139 |
| ac3 | 98.88 | -1.12 | 64.136 | 64.146 |
| o | 59.19 | -40.81 | 17.038 | 44.220 |
| ca1 | 107.96 | 7.96 | 41.846 | 42.597 |
| ca2 | 94.71 | -5.29 | 34.429 | 34.833 |
| ca3 | 96.94 | -3.06 | 35.909 | 36.039 |
| pojac | 90.80 | -9.20 | 23.253 | 25.007 |
| jac1 | 67.59 | -32.41 | 18.682 | 37.408 |
| jacseq | 89.50 | -10.50 | 22.841 | 25.139 |
| jacint | 85.36 | -14.64 | 22.199 | 26.592 |

Pr(inf mle) , mean jacknife order = 0.012, 3.369
c, ch1, ch2, ch3 = 0.533, 0.463 , 0.543 , 0.525

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 4.27 | | | |
| pet | 101.46 | 1.46 | 45.752 | 45.775 |
| mlea | 88.18 | -11.82 | 29.154 | 31.461 |
| dr1a | 101.80 | 1.80 | 41.163 | 41.202 |
| Oa | 59.18 | -40.82 | 17.017 | 44.227 |
| ca1a | 107.31 | 7.31 | 41.370 | 42.011 |
| ca2a | 94.09 | -5.91 | 33.761 | 34.274 |
| ca3a | 96.26 | -3.74 | 34.821 | 35.021 |

ca,ch1a,ch2a,ch3a = 0.533, 0.464 , 0.545 , 0.526

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 10.76 | | | |
| pet | 99.72 | -0.28 | 30.925 | 30.926 |
| mlea | 88.56 | -11.44 | 25.440 | 27.895 |
| dr1a | 99.40 | -0.60 | 32.645 | 32.650 |
| Oa | 58.20 | -41.80 | 16.886 | 45.085 |
| ca1a | 104.81 | 4.81 | 32.909 | 33.258 |
| ca2a | 91.92 | -8.08 | 26.879 | 28.069 |
| ca3a | 94.01 | -5.99 | 27.482 | 28.127 |

ca,ch1a,ch2a,ch3a = 0.525, 0.456 , 0.536 , 0.518

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 21.33 | | | |
| pet | 99.71 | -0.29 | 22.434 | 22.436 |
| mlea | 89.34 | -10.66 | 17.691 | 20.656 |
| dr1a | 97.37 | -2.63 | 29.613 | 29.730 |
| Oa | 57.62 | -42.38 | 16.648 | 45.530 |
| ca1a | 102.76 | 2.76 | 29.890 | 30.017 |
| ca2a | 90.11 | -9.89 | 24.400 | 26.329 |
| ca3a | 92.16 | -7.84 | 24.878 | 26.083 |

ca,ch1a,ch2a,ch3a = 0.522, 0.458 , 0.538 , 0.520

**t = 10**

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 62.49 | -37.51 | 14.459 | 40.200 |
| mle | 83.92 | -16.08 | 14.706 | 21.791 |
| dr1 | 91.64 | -8.36 | 16.622 | 18.607 |
| boot | 75.11 | -24.89 | 15.455 | 29.302 |
| ac1 | 99.91 | -0.09 | 20.910 | 20.910 |
| ac2 | 92.74 | -7.26 | 18.404 | 19.786 |
| ac3 | 93.73 | -6.27 | 18.682 | 19.705 |
| o | 80.94 | -19.06 | 15.787 | 24.747 |
| ca1 | 101.08 | 1.08 | 17.353 | 17.386 |
| ca2 | 97.94 | -2.06 | 16.086 | 16.218 |
| ca3 | 98.33 | -1.67 | 16.155 | 16.241 |
| pojac | 109.21 | 9.21 | 17.046 | 19.377 |
| jac1 | 89.59 | -10.41 | 15.797 | 18.918 |
| jacseq | 100.98 | 0.98 | 18.939 | 18.965 |
| jacint | 95.81 | -4.19 | 17.394 | 17.891 |

Pr(inf mle) , mean jacknife order = 0.000, 1.755
c, ch1, ch2, ch3 = 0.728, 0.693 , 0.727 , 0.721

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 6.28 | | | |
| pet | 99.58 | -0.42 | 32.501 | 32.503 |
| mlea | 84.83 | -15.17 | 12.202 | 19.470 |
| dr1a | 91.30 | -8.70 | 15.292 | 17.593 |
| Oa | 80.93 | -19.07 | 15.738 | 24.725 |
| ca1a | 100.75 | 0.75 | 16.068 | 16.086 |
| ca2a | 97.64 | -2.36 | 15.032 | 15.217 |
| ca3a | 98.06 | -1.94 | 15.106 | 15.230 |

ca,ch1a,ch2a,ch3a = 0.728, 0.694 , 0.728 , 0.722

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 15.79 | | | |
| pet | 99.83 | -0.17 | 19.123 | 19.124 |
| mlea | 86.35 | -13.65 | 11.566 | 17.892 |
| dr1a | 91.19 | -8.81 | 13.905 | 16.462 |
| Oa | 81.61 | -18.39 | 15.679 | 24.167 |
| ca1a | 100.70 | 0.70 | 14.810 | 14.826 |
| ca2a | 97.65 | -2.35 | 13.973 | 14.168 |
| ca3a | 98.07 | -1.93 | 14.072 | 14.204 |

ca,ch1a,ch2a,ch3a = 0.728, 0.697 , 0.731 , 0.725

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 31.46 | | | |
| pet | 100.18 | 0.18 | 15.010 | 15.011 |
| mlea | 87.13 | -12.87 | 10.974 | 16.916 |
| dr1a | 90.46 | -9.54 | 13.787 | 16.764 |
| Oa | 81.41 | -18.59 | 15.811 | 24.407 |
| ca1a | 99.92 | -0.08 | 14.617 | 14.617 |
| ca2a | 96.98 | -3.02 | 13.841 | 14.166 |
| ca3a | 97.39 | -2.61 | 13.912 | 14.155 |

ca,ch1a,ch2a,ch3a = 0.732, 0.702 , 0.735 , 0.729

**Table 3.7.1b**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.20 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

**t = 15**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 73.23 | -26.77 | 13.483 | 29.973 |
| mle | 85.59 | -14.41 | 10.032 | 17.558 |
| dr1 | 90.97 | -9.03 | 10.457 | 13.818 |
| boot | 85.39 | -14.61 | 13.151 | 19.655 |
| ac1 | 97.61 | -2.39 | 13.327 | 13.539 |
| ac2 | 94.62 | -5.38 | 12.678 | 13.773 |
| ac3 | 94.95 | -5.05 | 12.759 | 13.721 |
| o | 90.90 | -9.10 | 12.901 | 15.788 |
| ca1 | 101.35 | 1.35 | 11.460 | 11.539 |
| ca2 | 100.19 | 0.19 | 11.216 | 11.218 |
| ca3 | 100.32 | 0.32 | 11.247 | 11.251 |
| pojac | 112.04 | 12.04 | 14.037 | 18.495 |
| jac1 | 98.05 | -1.95 | 12.191 | 12.345 |
| jacseq | 104.72 | 4.72 | 15.443 | 16.150 |
| jacint | 101.11 | 1.11 | 13.996 | 14.040 |

Pr(inf mle) , mean jacknife order = 0.000, 1.409
c, ch1, ch2, ch3 = 0.824, 0.807 , 0.824 , 0.821

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 7.32 | | | |
| pet | 100.51 | 0.51 | 29.352 | 29.356 |
| mlea | 86.33 | -13.67 | 9.614 | 16.715 |
| dr1a | 90.95 | -9.05 | 10.244 | 13.669 |
| Oa | 90.95 | -9.05 | 12.886 | 15.749 |
| ca1a | 101.36 | 1.36 | 11.233 | 11.314 |
| ca2a | 100.20 | 0.20 | 11.045 | 11.047 |
| ca3a | 100.33 | 0.33 | 11.058 | 11.063 |

ca,ch1a,ch2a,ch3a = 0.824, 0.806 , 0.823 , 0.821

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 18.37 | | | |
| pet | 99.39 | -0.61 | 14.424 | 14.437 |
| mlea | 86.95 | -13.05 | 8.894 | 15.791 |
| dr1a | 90.68 | -9.32 | 9.523 | 13.326 |
| Oa | 90.81 | -9.19 | 12.609 | 15.601 |
| ca1a | 101.08 | 1.08 | 10.371 | 10.427 |
| ca2a | 99.91 | -0.09 | 10.182 | 10.183 |
| ca3a | 100.05 | 0.05 | 10.207 | 10.207 |

ca,ch1a,ch2a,ch3a = 0.822, 0.807 , 0.825 , 0.823

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 36.72 | | | |
| pet | 99.96 | -0.04 | 10.907 | 10.907 |
| mlea | 88.03 | -11.97 | 8.640 | 14.764 |
| dr1a | 90.80 | -9.20 | 9.778 | 13.427 |
| Oa | 90.98 | -9.02 | 13.033 | 15.849 |
| ca1a | 101.13 | 1.13 | 10.652 | 10.712 |
| ca2a | 99.97 | -0.03 | 10.503 | 10.503 |
| ca3a | 100.11 | 0.11 | 10.515 | 10.515 |

ca,ch1a,ch2a,ch3a = 0.825, 0.809 , 0.826 , 0.824

**t = 20**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 80.84 | -19.16 | 11.925 | 22.571 |
| mle | 88.45 | -11.55 | 8.310 | 14.228 |
| dr1 | 92.59 | -7.41 | 8.086 | 10.970 |
| boot | 91.76 | -8.24 | 10.481 | 13.332 |
| ac1 | 98.02 | -1.98 | 9.004 | 9.219 |
| ac2 | 96.43 | -3.57 | 8.703 | 9.406 |
| ac3 | 96.60 | -3.40 | 8.732 | 9.370 |
| o | 96.55 | -3.45 | 9.784 | 10.373 |
| ca1 | 102.69 | 2.69 | 8.743 | 9.147 |
| ca2 | 102.14 | 2.14 | 8.598 | 8.861 |
| ca3 | 102.19 | 2.19 | 8.599 | 8.874 |
| pojac | 110.24 | 10.24 | 12.736 | 16.345 |
| jac1 | 101.93 | 1.93 | 8.751 | 8.961 |
| jacseq | 105.41 | 5.41 | 11.587 | 12.786 |
| jacint | 103.33 | 3.33 | 9.921 | 10.464 |

Pr(inf mle) , mean jacknife order = 0.000, 1.205
c, ch1, ch2, ch3 = 0.885, 0.873 , 0.883 , 0.882

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 8.06 | | | |
| pet | 100.20 | 0.20 | 17.981 | 17.982 |
| mlea | 88.81 | -11.19 | 7.850 | 13.668 |
| dr1a | 92.56 | -7.44 | 7.864 | 10.826 |
| Oa | 96.52 | -3.48 | 9.751 | 10.354 |
| ca1a | 102.64 | 2.64 | 8.483 | 8.886 |
| ca2a | 102.11 | 2.11 | 8.364 | 8.627 |
| ca3a | 102.16 | 2.16 | 8.372 | 8.645 |

ca,ch1a,ch2a,ch3a = 0.885, 0.873 , 0.883 , 0.882

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 20.12 | | | |
| pet | 99.75 | -0.25 | 12.362 | 12.364 |
| mlea | 88.81 | -11.19 | 8.163 | 13.849 |
| dr1a | 92.00 | -8.00 | 8.169 | 11.436 |
| Oa | 96.16 | -3.84 | 10.065 | 10.774 |
| ca1a | 102.20 | 2.20 | 8.827 | 9.096 |
| ca2a | 101.62 | 1.62 | 8.719 | 8.867 |
| ca3a | 101.68 | 1.68 | 8.733 | 8.893 |

ca,ch1a,ch2a,ch3a = 0.881, 0.871 , 0.881 , 0.880

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 40.02 | | | |
| pet | 99.86 | -0.14 | 9.089 | 9.090 |
| mlea | 89.19 | -10.81 | 7.426 | 13.113 |
| dr1a | 91.94 | -8.06 | 7.943 | 11.318 |
| Oa | 95.78 | -4.22 | 10.246 | 11.083 |
| ca1a | 102.07 | 2.07 | 8.513 | 8.760 |
| ca2a | 101.51 | 1.51 | 8.425 | 8.559 |
| ca3a | 101.55 | 1.55 | 8.432 | 8.574 |

ca,ch1a,ch2a,ch3a = 0.878, 0.868 , 0.878 , 0.877

**Table 3.7.2a**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.08 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

**t = 5** | **t = 10**

**R = 0** | **R = 0**

| Estimator | mean | bias | s.d. | rmse | Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|---|---|---|---|---|
| x1 | 25.42 | -74.58 | 6.215 | 74.835 | x1 | 43.35 | -56.65 | 7.569 | 57.154 |
| mle | 93.50 | -6.50 | 60.073 | 60.424 | mle | 83.51 | -16.49 | 22.556 | 27.940 |
| dr1 | 119.20 | 19.20 | 77.840 | 80.172 | dr1 | 93.10 | -6.90 | 26.190 | 27.083 |
| boot | 32.75 | -67.25 | 7.975 | 67.724 | boot | 54.98 | -45.02 | 9.313 | 45.968 |
| ac1 | 144.70 | 44.70 | 103.042 | 112.320 | ac1 | 103.51 | 3.51 | 34.139 | 34.319 |
| ac2 | 107.13 | 7.13 | 76.544 | 76.876 | ac2 | 92.97 | -7.03 | 29.771 | 30.590 |
| ac3 | 112.57 | 12.57 | 87.982 | 88.875 | ac3 | 93.89 | -6.11 | 30.242 | 30.853 |
| o | 36.17 | -63.83 | 8.772 | 64.428 | o | 60.73 | -39.27 | 10.165 | 40.565 |
| ca1 | 122.66 | 22.66 | 78.218 | 81.433 | ca1 | 100.79 | 0.79 | 26.841 | 26.853 |
| ca2 | 102.71 | 2.71 | 63.041 | 63.099 | ca2 | 95.32 | -4.68 | 24.674 | 25.114 |
| ca3 | 104.32 | 4.32 | 63.757 | 63.903 | ca3 | 95.75 | -4.25 | 24.806 | 25.167 |
| pojac | 61.40 | -38.60 | 15.079 | 41.439 | pojac | 95.83 | -4.17 | 16.357 | 16.881 |
| jac1 | 42.61 | -57.39 | 10.347 | 58.312 | jac1 | 70.45 | -29.55 | 11.636 | 31.760 |
| jacseq | 63.94 | -36.06 | 16.395 | 39.612 | jacseq | 90.39 | -9.61 | 21.049 | 23.138 |
| jacint | 60.54 | -39.46 | 15.604 | 42.434 | jacint | 83.80 | -16.20 | 20.126 | 25.836 |

t = 5: Pr(inf mle) , mean jacknife order = 0.035, 3.843
c, ch1, ch2, ch3 = 0.322, 0.267 , 0.324 , 0.318

t = 10: Pr(inf mle) , mean jacknife order = 0.000, 2.333
c, ch1, ch2, ch3 = 0.531, 0.488 , 0.524 , 0.521

**R = 10** | **R = 10**

| Estimator | mean | bias | s.d. | rmse | Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|---|---|---|---|---|
| x2 | 2.56 | | | | x2 | 4.35 | | | |
| pet | 96.04 | -3.96 | 58.198 | 58.332 | pet | 99.33 | -0.67 | 43.319 | 43.325 |
| mlea | 96.44 | -3.56 | 49.175 | 49.304 | mlea | 85.79 | -14.21 | 20.274 | 24.759 |
| dr1a | 118.58 | 18.58 | 78.542 | 80.710 | dr1a | 92.36 | -7.64 | 24.624 | 25.780 |
| Oa | 36.18 | -63.82 | 8.757 | 64.416 | Oa | 60.72 | -39.28 | 10.178 | 40.579 |
| ca1a | 122.10 | 22.10 | 78.866 | 81.905 | ca1a | 100.08 | 0.08 | 25.283 | 25.283 |
| ca2a | 102.22 | 2.22 | 63.627 | 63.665 | ca2a | 94.70 | -5.30 | 23.232 | 23.829 |
| ca3a | 103.83 | 3.83 | 64.320 | 64.433 | ca3a | 95.14 | -4.86 | 23.338 | 23.840 |

t = 5: ca,ch1a,ch2a,ch3a = 0.323, 0.266 , 0.323 , 0.317

t = 10: ca,ch1a,ch2a,ch3a = 0.530, 0.489 , 0.526 , 0.522

**R = 25** | **R = 25**

| Estimator | mean | bias | s.d. | rmse | Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|---|---|---|---|---|
| x2 | 6.24 | | | | x2 | 10.69 | | | |
| pet | 102.02 | 2.02 | 50.825 | 50.865 | pet | 99.92 | -0.08 | 28.566 | 28.566 |
| mlea | 95.24 | -4.76 | 33.507 | 33.844 | mlea | 87.72 | -12.28 | 18.091 | 21.865 |
| dr1a | 111.19 | 11.19 | 62.047 | 63.049 | dr1a | 92.10 | -7.90 | 24.230 | 25.485 |
| Oa | 36.02 | -63.98 | 8.500 | 64.547 | Oa | 60.05 | -39.95 | 10.573 | 41.321 |
| ca1a | 114.66 | 14.66 | 62.257 | 63.961 | ca1a | 99.72 | -0.28 | 24.875 | 24.876 |
| ca2a | 96.12 | -3.88 | 50.120 | 50.270 | ca2a | 94.33 | -5.67 | 22.785 | 23.479 |
| ca3a | 97.64 | -2.36 | 50.741 | 50.796 | ca3a | 94.76 | -5.24 | 22.917 | 23.509 |

t = 5: ca,ch1a,ch2a,ch3a = 0.321, 0.271 , 0.329 , 0.322

t = 10: ca,ch1a,ch2a,ch3a = 0.523, 0.485 , 0.521 , 0.517

**R = 50** | **R = 50**

| Estimator | mean | bias | s.d. | rmse | Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|---|---|---|---|---|
| x2 | 12.81 | | | | x2 | 21.27 | | | |
| pet | 100.42 | 0.42 | 29.729 | 29.732 | pet | 100.06 | 0.06 | 21.945 | 21.945 |
| mlea | 95.80 | -4.20 | 27.879 | 28.193 | mlea | 89.57 | -10.43 | 16.066 | 19.153 |
| dr1a | 112.04 | 12.04 | 64.877 | 65.985 | dr1a | 91.19 | -8.81 | 22.032 | 23.726 |
| Oa | 36.88 | -63.12 | 8.480 | 63.689 | Oa | 59.57 | -40.43 | 10.378 | 41.743 |
| ca1a | 115.60 | 15.60 | 65.124 | 66.967 | ca1a | 98.76 | -1.24 | 22.765 | 22.799 |
| ca2a | 96.91 | -3.09 | 52.538 | 52.628 | ca2a | 93.42 | -6.58 | 20.880 | 21.893 |
| ca3a | 98.36 | -1.64 | 53.195 | 53.220 | ca3a | 93.82 | -6.18 | 20.964 | 21.855 |

t = 5: ca,ch1a,ch2a,ch3a = 0.327, 0.273 , 0.332 , 0.325

t = 10: ca,ch1a,ch2a,ch3a = 0.518, 0.480 , 0.516 , 0.513

**Table 3.7.2b**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.08 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

**t = 15**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 55.39 | -44.61 | 8.641 | 45.441 |
| mle | 83.06 | -16.94 | 13.585 | 21.712 |
| dr1 | 89.38 | -10.62 | 14.853 | 18.259 |
| boot | 68.76 | -31.24 | 10.195 | 32.860 |
| ac1 | 97.07 | -2.93 | 19.723 | 19.939 |
| ac2 | 91.86 | -8.14 | 18.282 | 20.011 |
| ac3 | 92.25 | -7.75 | 18.385 | 19.952 |
| o | 75.30 | -24.70 | 10.922 | 27.011 |
| ca1 | 99.45 | -0.55 | 15.977 | 15.986 |
| ca2 | 97.00 | -3.00 | 15.318 | 15.610 |
| ca3 | 97.18 | -2.82 | 15.343 | 15.601 |
| pojac | 109.60 | 9.60 | 16.280 | 18.902 |
| jac1 | 85.33 | -14.67 | 12.126 | 19.032 |
| jacseq | 102.19 | 2.19 | 20.580 | 20.695 |
| jacint | 93.61 | -6.39 | 18.131 | 19.225 |

Pr(inf mle) , mean jacknife order = 0.000, 1.978
c, ch1, ch2, ch3 = 0.653, 0.628 , 0.652 , 0.650

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 5.47 | | | |
| pet | 101.33 | 1.33 | 38.281 | 38.305 |
| mlea | 84.78 | -15.22 | 12.840 | 19.912 |
| dr1a | 89.21 | -10.79 | 14.318 | 17.928 |
| Oa | 75.28 | -24.72 | 10.908 | 27.023 |
| ca1a | 99.23 | -0.77 | 15.428 | 15.447 |
| ca2a | 96.83 | -3.17 | 14.812 | 15.148 |
| ca3a | 96.99 | -3.01 | 14.834 | 15.136 |

ca,ch1a,ch2a,ch3a = 0.652, 0.629 , 0.653 , 0.651

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 13.63 | | | |
| pet | 100.42 | 0.42 | 21.638 | 21.642 |
| mlea | 85.86 | -14.14 | 12.765 | 19.047 |
| dr1a | 88.66 | -11.34 | 14.830 | 18.668 |
| Oa | 74.39 | -25.61 | 10.769 | 27.782 |
| ca1a | 98.62 | -1.38 | 15.861 | 15.921 |
| ca2a | 96.20 | -3.80 | 15.236 | 15.703 |
| ca3a | 96.39 | -3.61 | 15.268 | 15.690 |

ca,ch1a,ch2a,ch3a = 0.650, 0.626 , 0.649 , 0.647

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 27.53 | | | |
| pet | 100.07 | 0.07 | 15.404 | 15.404 |
| mlea | 87.84 | -12.16 | 11.279 | 16.583 |
| dr1a | 88.89 | -11.11 | 13.607 | 17.565 |
| Oa | 75.05 | -24.95 | 10.429 | 27.043 |
| ca1a | 98.95 | -1.05 | 14.659 | 14.696 |
| ca2a | 96.50 | -3.50 | 14.046 | 14.475 |
| ca3a | 96.70 | -3.30 | 14.087 | 14.469 |

ca,ch1a,ch2a,ch3a = 0.651, 0.627 , 0.651 , 0.649

**t = 20**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 64.02 | -35.98 | 8.237 | 36.910 |
| mle | 83.67 | -16.33 | 10.933 | 19.656 |
| dr1 | 89.08 | -10.92 | 11.539 | 15.889 |
| boot | 77.89 | -22.11 | 9.313 | 23.988 |
| ac1 | 96.35 | -3.65 | 14.424 | 14.878 |
| ac2 | 93.22 | -6.78 | 13.714 | 15.300 |
| ac3 | 93.45 | -6.55 | 13.763 | 15.244 |
| o | 84.50 | -15.50 | 9.826 | 18.349 |
| ca1 | 100.39 | 0.39 | 12.672 | 12.677 |
| ca2 | 99.09 | -0.91 | 12.337 | 12.370 |
| ca3 | 99.19 | -0.81 | 12.366 | 12.392 |
| pojac | 114.82 | 14.82 | 14.612 | 20.813 |
| jac1 | 93.88 | -6.12 | 10.541 | 12.188 |
| jacseq | 104.72 | 4.72 | 18.001 | 18.611 |
| jacint | 98.29 | -1.71 | 14.820 | 14.919 |

Pr(inf mle) , mean jacknife order = 0.000, 1.620
c, ch1, ch2, ch3 = 0.739, 0.724 , 0.740 , 0.739

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 6.47 | | | |
| pet | 99.07 | -0.93 | 26.791 | 26.808 |
| mlea | 84.75 | -15.25 | 10.550 | 18.544 |
| dr1a | 89.02 | -10.98 | 11.395 | 15.823 |
| Oa | 84.50 | -15.50 | 9.834 | 18.357 |
| ca1a | 100.35 | 0.35 | 12.528 | 12.533 |
| ca2a | 99.06 | -0.94 | 12.193 | 12.229 |
| ca3a | 99.17 | -0.83 | 12.219 | 12.247 |

ca,ch1a,ch2a,ch3a = 0.740, 0.724 , 0.740 , 0.739

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 16.12 | | | |
| pet | 99.97 | -0.03 | 17.949 | 17.949 |
| mlea | 85.96 | -14.04 | 10.152 | 17.326 |
| dr1a | 88.85 | -11.15 | 10.795 | 15.520 |
| Oa | 84.87 | -15.13 | 9.714 | 17.981 |
| ca1a | 100.20 | 0.20 | 11.920 | 11.922 |
| ca2a | 98.93 | -1.07 | 11.647 | 11.696 |
| ca3a | 99.02 | -0.98 | 11.671 | 11.713 |

ca,ch1a,ch2a,ch3a = 0.745, 0.728 , 0.745 , 0.743

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 31.99 | | | |
| pet | 100.38 | 0.38 | 13.407 | 13.413 |
| mlea | 87.25 | -12.75 | 9.576 | 15.942 |
| dr1a | 88.98 | -11.02 | 10.654 | 15.329 |
| Oa | 84.78 | -15.22 | 9.929 | 18.173 |
| ca1a | 100.32 | 0.32 | 11.783 | 11.787 |
| ca2a | 99.07 | -0.93 | 11.519 | 11.556 |
| ca3a | 99.18 | -0.82 | 11.531 | 11.560 |

ca,ch1a,ch2a,ch3a = 0.744, 0.725 , 0.741 , 0.740

**Table 3.7.3a**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.08, 0.12 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

**t = 5**

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 38.41 | -61.59 | 6.104 | 61.894 |
| mle | 82.88 | -17.12 | 30.497 | 34.972 |
| dr1 | 101.63 | 1.63 | 39.343 | 39.377 |
| boot | 48.51 | -51.49 | 7.653 | 52.054 |
| ac1 | 119.88 | 19.88 | 53.137 | 56.735 |
| ac2 | 91.44 | -8.56 | 38.424 | 39.366 |
| ac3 | 96.33 | -3.67 | 42.291 | 42.450 |
| o | 53.18 | -46.82 | 8.398 | 47.568 |
| ca1 | 106.69 | 6.69 | 39.699 | 40.259 |
| ca2 | 92.36 | -7.64 | 32.280 | 33.171 |
| ca3 | 94.44 | -5.56 | 32.872 | 33.339 |
| pojac | 85.23 | -14.77 | 14.264 | 20.536 |
| jac1 | 61.52 | -38.48 | 9.804 | 39.705 |
| jacseq | 85.94 | -14.06 | 16.408 | 21.611 |
| jacint | 81.93 | -18.07 | 16.011 | 24.141 |

Pr(inf mle) , mean jacknife order = 0.000, 3.629
c, ch1, ch2, ch3 = 0.482, 0.409 , 0.487 , 0.473

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 3.77 | | | |
| pet | 100.24 | 0.24 | 45.962 | 45.963 |
| mlea | 86.05 | -13.95 | 23.444 | 27.283 |
| dr1a | 100.34 | 0.34 | 36.635 | 36.637 |
| Oa | 53.16 | -46.84 | 8.418 | 47.593 |
| ca1a | 105.43 | 5.43 | 37.028 | 37.424 |
| ca2a | 91.40 | -8.60 | 30.379 | 31.573 |
| ca3a | 93.48 | -6.52 | 31.110 | 31.787 |

ca,ch1a,ch2a,ch3a = 0.481, 0.410 , 0.488 , 0.474

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 9.74 | | | |
| pet | 99.14 | -0.86 | 42.684 | 42.693 |
| mlea | 87.90 | -12.10 | 19.936 | 23.319 |
| dr1a | 100.89 | 0.89 | 36.230 | 36.241 |
| Oa | 53.21 | -46.79 | 8.292 | 47.523 |
| ca1a | 105.97 | 5.97 | 36.585 | 37.069 |
| ca2a | 91.74 | -8.26 | 30.145 | 31.256 |
| ca3a | 93.77 | -6.23 | 31.035 | 31.654 |

ca,ch1a,ch2a,ch3a = 0.477, 0.407 , 0.486 , 0.472

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 19.41 | | | |
| pet | 99.92 | -0.08 | 23.052 | 23.052 |
| mlea | 90.44 | -9.56 | 17.863 | 20.261 |
| dr1a | 100.06 | 0.06 | 30.842 | 30.842 |
| Oa | 53.71 | -46.29 | 8.245 | 47.018 |
| ca1a | 105.21 | 5.21 | 31.292 | 31.722 |
| ca2a | 91.32 | -8.68 | 25.766 | 27.189 |
| ca3a | 93.39 | -6.61 | 26.364 | 27.180 |

ca,ch1a,ch2a,ch3a = 0.482, 0.409 , 0.487 , 0.473

**t = 10**

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 59.61 | -40.39 | 6.524 | 40.911 |
| mle | 82.83 | -17.17 | 12.361 | 21.154 |
| dr1 | 90.82 | -9.18 | 13.688 | 16.482 |
| boot | 73.09 | -26.91 | 7.904 | 28.051 |
| ac1 | 99.55 | -0.45 | 17.772 | 17.778 |
| ac2 | 91.67 | -8.33 | 15.766 | 17.829 |
| ac3 | 92.72 | -7.28 | 16.040 | 17.613 |
| o | 79.47 | -20.53 | 8.597 | 22.255 |
| ca1 | 100.54 | 0.54 | 14.680 | 14.690 |
| ca2 | 97.12 | -2.88 | 13.684 | 13.985 |
| ca3 | 97.56 | -2.44 | 13.788 | 14.003 |
| pojac | 112.61 | 12.61 | 14.717 | 19.377 |
| jac1 | 89.25 | -10.75 | 9.894 | 14.613 |
| jacseq | 103.19 | 3.19 | 18.129 | 18.407 |
| jacint | 95.76 | -4.24 | 16.080 | 16.631 |

Pr(inf mle) , mean jacknife order = 0.000, 1.898
c, ch1, ch2, ch3 = 0.702, 0.664 , 0.702 , 0.697

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 6.00 | | | |
| pet | 99.63 | -0.37 | 34.845 | 34.847 |
| mlea | 84.18 | -15.82 | 11.654 | 19.646 |
| dr1a | 90.70 | -9.30 | 13.203 | 16.146 |
| Oa | 79.47 | -20.53 | 8.581 | 22.251 |
| ca1a | 100.45 | 0.45 | 14.181 | 14.188 |
| ca2a | 97.01 | -2.99 | 13.212 | 13.547 |
| ca3a | 97.47 | -2.53 | 13.326 | 13.564 |

ca,ch1a,ch2a,ch3a = 0.703, 0.664 , 0.702 , 0.697

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 14.95 | | | |
| pet | 98.86 | -1.14 | 18.287 | 18.323 |
| mlea | 85.40 | -14.60 | 11.226 | 18.417 |
| dr1a | 90.34 | -9.66 | 13.127 | 16.299 |
| Oa | 78.98 | -21.02 | 8.811 | 22.792 |
| ca1a | 100.00 | 0.00 | 14.101 | 14.101 |
| ca2a | 96.54 | -3.46 | 13.222 | 13.666 |
| ca3a | 97.03 | -2.97 | 13.298 | 13.627 |

ca,ch1a,ch2a,ch3a = 0.698, 0.662 , 0.700 , 0.694

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 29.80 | | | |
| pet | 99.60 | -0.40 | 14.494 | 14.499 |
| mlea | 86.28 | -13.72 | 10.190 | 17.092 |
| dr1a | 89.33 | -10.67 | 11.809 | 15.918 |
| Oa | 78.94 | -21.06 | 8.577 | 22.736 |
| ca1a | 98.97 | -1.03 | 12.711 | 12.753 |
| ca2a | 95.69 | -4.31 | 11.945 | 12.698 |
| ca3a | 96.14 | -3.86 | 12.015 | 12.620 |

ca,ch1a,ch2a,ch3a = 0.702, 0.669 , 0.707 , 0.701

**Table 3.7.3b**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.08, 0.12 ) : cv ~ U( 0.30, 0.80 )
Number of simulations = 1000

## t = 15

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 71.70 | -28.30 | 6.528 | 29.045 |
| mle | 84.94 | -15.06 | 8.768 | 17.431 |
| dr1 | 90.67 | -9.33 | 9.164 | 13.078 |
| boot | 85.25 | -14.75 | 7.544 | 16.568 |
| ac1 | 97.64 | -2.36 | 11.379 | 11.621 |
| ac2 | 94.25 | -5.75 | 10.762 | 12.204 |
| ac3 | 94.67 | -5.33 | 10.825 | 12.067 |
| o | 91.47 | -8.53 | 8.088 | 11.756 |
| ca1 | 101.95 | 1.95 | 10.280 | 10.464 |
| ca2 | 100.69 | 0.69 | 10.008 | 10.031 |
| ca3 | 100.85 | 0.85 | 10.031 | 10.067 |
| pojac | 116.79 | 16.79 | 13.402 | 21.481 |
| jac1 | 99.74 | -0.26 | 9.109 | 9.112 |
| jacseq | 105.99 | 5.99 | 15.362 | 16.489 |
| jacint | 101.92 | 1.92 | 12.158 | 12.309 |

Pr(inf mle) , mean jacknife order = 0.000, 1.401
c, ch1, ch2, ch3 = 0.813, 0.793 , 0.813 , 0.811

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 7.22 | | | |
| pet | 99.21 | -0.79 | 20.766 | 20.782 |
| mlea | 85.54 | -14.46 | 8.519 | 16.779 |
| dr1a | 90.56 | -9.44 | 8.968 | 13.020 |
| Oa | 91.46 | -8.54 | 8.018 | 11.717 |
| ca1a | 101.86 | 1.86 | 10.058 | 10.228 |
| ca2a | 100.60 | 0.60 | 9.772 | 9.790 |
| ca3a | 100.75 | 0.75 | 9.808 | 9.836 |

ca,ch1a,ch2a,ch3a = 0.814, 0.794 , 0.814 , 0.811

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 17.90 | | | |
| pet | 99.99 | -0.01 | 14.464 | 14.464 |
| mlea | 86.27 | -13.73 | 8.640 | 16.219 |
| dr1a | 90.32 | -9.68 | 9.155 | 13.325 |
| Oa | 91.23 | -8.77 | 8.042 | 11.898 |
| ca1a | 101.60 | 1.60 | 10.279 | 10.402 |
| ca2a | 100.31 | 0.31 | 9.945 | 9.950 |
| ca3a | 100.47 | 0.47 | 9.981 | 9.992 |

ca,ch1a,ch2a,ch3a = 0.814, 0.794 , 0.815 , 0.812

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 35.73 | | | |
| pet | 99.29 | -0.71 | 11.155 | 11.178 |
| mlea | 86.14 | -13.86 | 8.132 | 16.074 |
| dr1a | 89.29 | -10.71 | 8.831 | 13.879 |
| Oa | 90.38 | -9.62 | 8.049 | 12.541 |
| ca1a | 100.43 | 0.43 | 9.886 | 9.896 |
| ca2a | 99.21 | -0.79 | 9.626 | 9.658 |
| ca3a | 99.34 | -0.66 | 9.642 | 9.664 |

ca,ch1a,ch2a,ch3a = 0.813, 0.797 , 0.817 , 0.814

## t = 20

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 79.35 | -20.65 | 6.281 | 21.588 |
| mle | 87.11 | -12.89 | 7.556 | 14.938 |
| dr1 | 91.64 | -8.36 | 7.466 | 11.206 |
| boot | 91.83 | -8.17 | 6.966 | 10.734 |
| ac1 | 97.50 | -2.50 | 8.449 | 8.811 |
| ac2 | 95.76 | -4.24 | 8.147 | 9.185 |
| ac3 | 95.95 | -4.05 | 8.168 | 9.115 |
| o | 97.35 | -2.65 | 7.357 | 7.820 |
| ca1 | 103.02 | 3.02 | 8.500 | 9.020 |
| ca2 | 102.46 | 2.46 | 8.355 | 8.710 |
| ca3 | 102.52 | 2.52 | 8.360 | 8.731 |
| pojac | 114.19 | 14.19 | 12.240 | 18.737 |
| jac1 | 103.83 | 3.83 | 8.057 | 8.920 |
| jacseq | 105.96 | 5.96 | 11.286 | 12.762 |
| jacint | 104.58 | 4.58 | 9.517 | 10.562 |

Pr(inf mle) , mean jacknife order = 0.000, 1.141
c, ch1, ch2, ch3 = 0.879, 0.867 , 0.878 , 0.877

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 7.91 | | | |
| pet | 100.14 | 0.14 | 16.522 | 16.523 |
| mlea | 87.45 | -12.55 | 7.409 | 14.570 |
| dr1a | 91.62 | -8.38 | 7.395 | 11.178 |
| Oa | 97.34 | -2.66 | 7.365 | 7.829 |
| ca1a | 102.97 | 2.97 | 8.414 | 8.923 |
| ca2a | 102.40 | 2.40 | 8.278 | 8.618 |
| ca3a | 102.45 | 2.45 | 8.300 | 8.655 |

ca,ch1a,ch2a,ch3a = 0.879, 0.867 , 0.879 , 0.878

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 19.68 | | | |
| pet | 100.18 | 0.18 | 11.612 | 11.613 |
| mlea | 87.50 | -12.50 | 7.350 | 14.499 |
| dr1a | 91.14 | -8.86 | 7.397 | 11.540 |
| Oa | 96.88 | -3.12 | 7.374 | 8.007 |
| ca1a | 102.50 | 2.50 | 8.375 | 8.739 |
| ca2a | 101.92 | 1.92 | 8.228 | 8.449 |
| ca3a | 101.99 | 1.99 | 8.247 | 8.483 |

ca,ch1a,ch2a,ch3a = 0.876, 0.867 , 0.878 , 0.877

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 39.71 | | | |
| pet | 100.26 | 0.26 | 9.184 | 9.188 |
| mlea | 88.35 | -11.65 | 6.725 | 13.454 |
| dr1a | 91.56 | -8.44 | 6.842 | 10.865 |
| Oa | 97.49 | -2.51 | 6.923 | 7.364 |
| ca1a | 102.91 | 2.91 | 7.729 | 8.260 |
| ca2a | 102.36 | 2.36 | 7.598 | 7.956 |
| ca3a | 102.41 | 2.41 | 7.598 | 7.971 |

ca,ch1a,ch2a,ch3a = 0.880, 0.870 , 0.881 , 0.880

**Table 3.7.4a**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.08 ) : cv ~ U( 0.55, 0.80 )
Number of simulations = 1000

**t = 5**

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 25.13 | -74.87 | 5.646 | 75.081 |
| mle | 83.32 | -16.68 | 44.193 | 47.234 |
| dr1 | 106.12 | 6.12 | 57.681 | 58.005 |
| boot | 32.26 | -67.74 | 7.137 | 68.111 |
| ac1 | 128.60 | 28.60 | 78.620 | 83.662 |
| ac2 | 96.33 | -3.67 | 61.388 | 61.498 |
| ac3 | 101.87 | 1.87 | 75.572 | 75.595 |
| o | 35.62 | -64.38 | 7.869 | 64.863 |
| ca1 | 109.48 | 9.48 | 57.940 | 58.710 |
| ca2 | 92.24 | -7.76 | 46.918 | 47.556 |
| ca3 | 93.90 | -6.10 | 48.028 | 48.414 |
| pojac | 60.00 | -40.00 | 13.165 | 42.111 |
| jac1 | 41.88 | -58.12 | 9.174 | 58.845 |
| jacseq | 62.17 | -37.83 | 14.297 | 40.440 |
| jacint | 58.88 | -41.12 | 13.681 | 43.338 |

Pr(inf mle) , mean jacknife order = 0.031, 3.808
c, ch1, ch2, ch3 = 0.347, 0.286 , 0.345 , 0.338

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 2.54 | | | |
| pet | 94.87 | -5.13 | 54.642 | 54.883 |
| mlea | 89.98 | -10.02 | 46.376 | 47.446 |
| dr1a | 104.75 | 4.75 | 57.335 | 57.532 |
| Oa | 35.62 | -64.38 | 7.871 | 64.856 |
| ca1a | 108.20 | 8.20 | 57.569 | 58.150 |
| ca2a | 91.15 | -8.85 | 46.319 | 47.157 |
| ca3a | 92.76 | -7.24 | 46.960 | 47.514 |

ca,ch1a,ch2a,ch3a = 0.347, 0.286 , 0.346 , 0.339

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 6.26 | | | |
| pet | 100.41 | 0.41 | 47.373 | 47.375 |
| mlea | 90.31 | -9.69 | 31.996 | 33.431 |
| dr1a | 100.96 | 0.96 | 54.172 | 54.181 |
| Oa | 35.57 | -64.43 | 7.736 | 64.895 |
| ca1a | 104.40 | 4.40 | 54.346 | 54.524 |
| ca2a | 87.95 | -12.05 | 43.902 | 45.525 |
| ca3a | 89.43 | -10.57 | 44.728 | 45.961 |

ca,ch1a,ch2a,ch3a = 0.349, 0.293 , 0.355 , 0.348

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 12.63 | | | |
| pet | 99.24 | -0.76 | 32.789 | 32.798 |
| mlea | 91.20 | -8.80 | 28.550 | 29.877 |
| dr1a | 97.46 | -2.54 | 52.389 | 52.451 |
| Oa | 35.48 | -64.53 | 8.121 | 65.034 |
| ca1a | 100.89 | 0.89 | 52.600 | 52.607 |
| ca2a | 85.32 | -14.68 | 42.695 | 45.148 |
| ca3a | 86.96 | -13.04 | 43.620 | 45.528 |

ca,ch1a,ch2a,ch3a = 0.348, 0.298 , 0.359 , 0.352

**t = 10**

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 41.52 | -58.48 | 7.378 | 58.942 |
| mle | 76.73 | -23.27 | 22.101 | 32.090 |
| dr1 | 86.02 | -13.98 | 25.424 | 29.016 |
| boot | 52.48 | -47.52 | 9.017 | 48.366 |
| ac1 | 96.75 | -3.25 | 32.547 | 32.708 |
| ac2 | 87.31 | -12.69 | 28.624 | 31.311 |
| ac3 | 88.26 | -11.74 | 29.039 | 31.322 |
| o | 57.88 | -42.12 | 9.786 | 43.243 |
| ca1 | 93.33 | -6.67 | 25.933 | 26.778 |
| ca2 | 88.60 | -11.40 | 23.676 | 26.276 |
| ca3 | 89.03 | -10.97 | 23.782 | 26.190 |
| pojac | 90.61 | -9.39 | 15.454 | 18.084 |
| jac1 | 66.98 | -33.02 | 11.147 | 34.848 |
| jacseq | 84.95 | -15.05 | 20.771 | 25.652 |
| jacint | 78.85 | -21.15 | 19.455 | 28.735 |

Pr(inf mle) , mean jacknife order = 0.000, 2.256
c, ch1, ch2, ch3 = 0.546, 0.507 , 0.542 , 0.539

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 4.11 | | | |
| pet | 100.88 | 0.88 | 48.008 | 48.016 |
| mlea | 80.12 | -19.88 | 19.218 | 27.649 |
| dr1a | 85.19 | -14.81 | 23.460 | 27.742 |
| Oa | 57.88 | -42.12 | 9.785 | 43.240 |
| ca1a | 92.52 | -7.48 | 24.000 | 25.138 |
| ca2a | 87.88 | -12.12 | 21.985 | 25.103 |
| ca3a | 88.29 | -11.71 | 22.087 | 24.998 |

ca,ch1a,ch2a,ch3a = 0.546, 0.509 , 0.544 , 0.540

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 10.45 | | | |
| pet | 99.42 | -0.58 | 28.889 | 28.895 |
| mlea | 82.43 | -17.57 | 16.469 | 24.080 |
| dr1a | 83.85 | -16.15 | 18.917 | 24.874 |
| Oa | 57.87 | -42.13 | 10.015 | 43.300 |
| ca1a | 91.19 | -8.81 | 19.623 | 21.511 |
| ca2a | 86.60 | -13.40 | 18.155 | 22.563 |
| ca3a | 87.01 | -12.99 | 18.225 | 22.382 |

ca,ch1a,ch2a,ch3a = 0.545, 0.509 , 0.545 , 0.541

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 21.00 | | | |
| pet | 99.10 | -0.90 | 19.977 | 19.997 |
| mlea | 84.37 | -15.63 | 14.276 | 21.165 |
| dr1a | 82.96 | -17.04 | 17.657 | 24.541 |
| Oa | 58.06 | -41.94 | 9.709 | 43.049 |
| ca1a | 90.32 | -9.68 | 18.386 | 20.778 |
| ca2a | 85.91 | -14.09 | 17.058 | 22.122 |
| ca3a | 86.31 | -13.69 | 17.138 | 21.936 |

ca,ch1a,ch2a,ch3a = 0.548, 0.515 , 0.551 , 0.547

**Table 3.7.4b**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.04, 0.08 ) : cv ~ U( 0.55, 0.80 )
Number of simulations = 1000

**R = 0**

| Estimator | t = 15 mean | t = 15 bias | t = 15 s.d. | t = 15 rmse | t = 20 mean | t = 20 bias | t = 20 s.d. | t = 20 rmse |
|---|---|---|---|---|---|---|---|---|
| x1 | 53.06 | -46.94 | 7.651 | 47.555 | 61.93 | -38.07 | 7.829 | 38.869 |
| mle | 76.08 | -23.92 | 11.391 | 26.496 | 78.22 | -21.78 | 9.050 | 23.586 |
| drl | 82.63 | -17.37 | 12.800 | 21.579 | 83.92 | -16.08 | 10.079 | 18.978 |
| boot | 65.60 | -34.40 | 8.879 | 35.529 | 74.98 | -25.02 | 8.763 | 26.511 |
| ac1 | 91.40 | -8.60 | 17.943 | 19.897 | 92.30 | -7.70 | 14.540 | 16.451 |
| ac2 | 86.73 | -13.27 | 16.738 | 21.360 | 89.56 | -10.44 | 13.988 | 17.456 |
| ac3 | 87.12 | -12.88 | 16.855 | 21.214 | 89.78 | -10.22 | 14.058 | 17.383 |
| o | 71.68 | -28.32 | 9.476 | 29.864 | 81.19 | -18.81 | 9.251 | 20.963 |
| ca1 | 92.20 | -7.80 | 13.735 | 15.793 | 94.75 | -5.25 | 11.230 | 12.395 |
| ca2 | 90.19 | -9.81 | 13.219 | 16.465 | 93.72 | -6.28 | 11.032 | 12.697 |
| ca3 | 90.36 | -9.64 | 13.293 | 16.421 | 93.80 | -6.20 | 11.043 | 12.667 |
| pojac | 103.49 | 3.49 | 14.214 | 14.635 | 109.63 | 9.63 | 14.275 | 17.217 |
| jacl | 81.01 | -18.99 | 10.484 | 21.695 | 90.07 | -9.93 | 9.996 | 14.087 |
| jacseq | 96.13 | -3.87 | 18.914 | 19.305 | 100.50 | 0.50 | 18.338 | 18.344 |
| jacint | 88.51 | -11.49 | 16.990 | 20.512 | 94.56 | -5.44 | 15.393 | 16.328 |

t = 15: Pr(inf mle) , mean jacknife order = 0.000, 1.929
c, ch1, ch2, ch3 = 0.671, 0.650 , 0.672 , 0.670

t = 20: Pr(inf mle) , mean jacknife order = 0.000, 1.620
c, ch1, ch2, ch3 = 0.758, 0.742 , 0.757 , 0.755

**R = 10**

| Estimator | t = 15 mean | t = 15 bias | t = 15 s.d. | t = 15 rmse | t = 20 mean | t = 20 bias | t = 20 s.d. | t = 20 rmse |
|---|---|---|---|---|---|---|---|---|
| x2 | 5.35 | | | | 6.22 | | | |
| pet | 99.64 | -0.36 | 36.714 | 36.716 | 99.51 | -0.49 | 26.940 | 26.945 |
| mlea | 78.30 | -21.70 | 11.045 | 24.347 | 79.65 | -20.35 | 8.838 | 22.183 |
| drla | 82.42 | -17.58 | 12.375 | 21.502 | 83.86 | -16.14 | 9.682 | 18.826 |
| Oa | 71.67 | -28.33 | 9.456 | 29.870 | 81.22 | -18.78 | 9.206 | 20.916 |
| cala | 91.98 | -8.02 | 13.328 | 15.554 | 94.69 | -5.31 | 10.803 | 12.040 |
| ca2a | 89.95 | -10.05 | 12.864 | 16.326 | 93.62 | -6.38 | 10.629 | 12.396 |
| ca3a | 90.12 | -9.88 | 12.892 | 16.241 | 93.69 | -6.31 | 10.638 | 12.369 |

t = 15: ca,ch1a,ch2a,ch3a = 0.671, 0.651 , 0.673 , 0.671

t = 20: ca,ch1a,ch2a,ch3a = 0.758, 0.742 , 0.757 , 0.756

**R = 25**

| Estimator | t = 15 mean | t = 15 bias | t = 15 s.d. | t = 15 rmse | t = 20 mean | t = 20 bias | t = 20 s.d. | t = 20 rmse |
|---|---|---|---|---|---|---|---|---|
| x2 | 13.32 | | | | 15.43 | | | |
| pet | 99.94 | -0.06 | 22.562 | 22.562 | 99.98 | -0.02 | 18.181 | 18.181 |
| mlea | 80.92 | -19.08 | 10.596 | 21.829 | 81.03 | -18.97 | 8.450 | 20.769 |
| drla | 83.02 | -16.98 | 11.600 | 20.561 | 83.68 | -16.32 | 9.198 | 18.734 |
| Oa | 71.94 | -28.06 | 9.693 | 29.692 | 81.07 | -18.93 | 9.037 | 20.980 |
| cala | 92.65 | -7.35 | 12.590 | 14.579 | 94.56 | -5.44 | 10.291 | 11.639 |
| ca2a | 90.61 | -9.39 | 12.164 | 15.368 | 93.48 | -6.52 | 10.112 | 12.034 |
| ca3a | 90.78 | -9.22 | 12.206 | 15.298 | 93.56 | -6.44 | 10.120 | 11.994 |

t = 15: ca,ch1a,ch2a,ch3a = 0.671, 0.646 , 0.668 , 0.666

t = 20: ca,ch1a,ch2a,ch3a = 0.756, 0.740 , 0.755 , 0.754

**R = 50**

| Estimator | t = 15 mean | t = 15 bias | t = 15 s.d. | t = 15 rmse | t = 20 mean | t = 20 bias | t = 20 s.d. | t = 20 rmse |
|---|---|---|---|---|---|---|---|---|
| x2 | 26.74 | | | | 30.65 | | | |
| pet | 99.48 | -0.52 | 16.772 | 16.780 | 99.98 | -0.02 | 13.508 | 13.508 |
| mlea | 82.32 | -17.68 | 10.042 | 20.329 | 82.31 | -17.69 | 8.528 | 19.635 |
| drla | 82.15 | -17.85 | 11.490 | 21.231 | 83.12 | -16.88 | 9.332 | 19.287 |
| Oa | 71.83 | -28.17 | 9.601 | 29.759 | 80.52 | -19.48 | 9.272 | 21.572 |
| cala | 91.76 | -8.24 | 12.455 | 14.934 | 93.91 | -6.09 | 10.458 | 12.102 |
| ca2a | 89.78 | -10.22 | 12.066 | 15.814 | 92.84 | -7.16 | 10.301 | 12.542 |
| ca3a | 89.95 | -10.05 | 12.104 | 15.731 | 92.93 | -7.07 | 10.319 | 12.508 |

t = 15: ca,ch1a,ch2a,ch3a = 0.674, 0.652 , 0.675 , 0.673

t = 20: ca,ch1a,ch2a,ch3a = 0.754, 0.740 , 0.755 , 0.754

**Table 3.7.5a**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.08, 0.12 ) : cv ~ U( 0.30, 0.55 )
Number of simulations = 1000

**t = 5**

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 39.50 | -60.50 | 5.928 | 60.793 |
| mle | 91.54 | -8.46 | 33.475 | 34.528 |
| dr1 | 112.01 | 12.01 | 43.487 | 45.114 |
| boot | 50.05 | -49.95 | 7.381 | 50.488 |
| ac1 | 130.78 | 30.78 | 59.479 | 66.970 |
| ac2 | 98.74 | -1.26 | 43.382 | 43.400 |
| ac3 | 103.28 | 3.28 | 48.258 | 48.370 |
| O | 54.94 | -45.06 | 8.079 | 45.777 |
| ca1 | 117.25 | 17.25 | 43.769 | 47.046 |
| ca2 | 100.63 | 0.63 | 35.581 | 35.586 |
| ca3 | 102.60 | 2.60 | 36.305 | 36.398 |
| pojac | 88.86 | -11.14 | 13.502 | 17.507 |
| jac1 | 63.77 | -36.23 | 9.354 | 37.420 |
| jacseq | 90.09 | -9.91 | 15.305 | 18.234 |
| jacint | 85.96 | -14.04 | 14.820 | 20.415 |

Pr(inf mle) , mean jacknife order = 0.000, 3.736
c, ch1, ch2, ch3 = 0.451, 0.384 , 0.461 , 0.449

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 3.94 | | | |
| pet | 98.41 | -1.59 | 40.982 | 41.013 |
| mlea | 92.29 | -7.71 | 24.523 | 25.706 |
| dr1a | 110.54 | 10.54 | 37.685 | 39.132 |
| Oa | 54.95 | -45.05 | 8.050 | 45.762 |
| ca1a | 115.76 | 15.76 | 38.019 | 41.155 |
| ca2a | 99.44 | -0.56 | 30.738 | 30.743 |
| ca3a | 101.38 | 1.38 | 31.222 | 31.252 |

ca,ch1a,ch2a,ch3a = 0.451, 0.384 , 0.461 , 0.450

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 9.89 | | | |
| pet | 99.69 | -0.31 | 37.212 | 37.213 |
| mlea | 94.41 | -5.59 | 21.601 | 22.313 |
| dr1a | 112.11 | 12.11 | 38.864 | 40.706 |
| Oa | 54.76 | -45.24 | 8.097 | 45.964 |
| ca1a | 117.35 | 17.35 | 39.245 | 42.909 |
| ca2a | 100.75 | 0.75 | 32.068 | 32.077 |
| ca3a | 102.80 | 2.80 | 32.761 | 32.880 |

ca,ch1a,ch2a,ch3a = 0.449, 0.375 , 0.450 , 0.439

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 19.66 | | | |
| pet | 99.25 | -0.75 | 21.666 | 21.679 |
| mlea | 94.42 | -5.58 | 17.742 | 18.599 |
| dr1a | 108.84 | 8.84 | 30.455 | 31.712 |
| Oa | 54.50 | -45.50 | 8.044 | 46.202 |
| ca1a | 114.07 | 14.07 | 30.822 | 33.882 |
| ca2a | 98.04 | -1.96 | 25.133 | 25.210 |
| ca3a | 99.97 | -0.03 | 25.582 | 25.582 |

ca,ch1a,ch2a,ch3a = 0.446, 0.377 , 0.452 , 0.441

**t = 10**

R = 0

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 61.49 | -38.51 | 6.317 | 39.025 |
| mle | 88.85 | -11.15 | 11.354 | 15.912 |
| dr1 | 97.02 | -2.98 | 13.174 | 13.506 |
| boot | 75.80 | -24.20 | 7.515 | 25.337 |
| ac1 | 104.96 | 4.96 | 18.056 | 18.725 |
| ac2 | 96.15 | -3.85 | 15.919 | 16.377 |
| ac3 | 97.24 | -2.76 | 16.232 | 16.465 |
| O | 82.57 | -17.43 | 8.112 | 19.225 |
| ca1 | 107.17 | 7.17 | 14.022 | 15.750 |
| ca2 | 103.15 | 3.15 | 13.117 | 13.490 |
| ca3 | 103.63 | 3.63 | 13.234 | 13.724 |
| pojac | 118.47 | 18.47 | 13.855 | 23.091 |
| jac1 | 93.10 | -6.90 | 9.277 | 11.562 |
| jacseq | 109.47 | 9.47 | 18.321 | 20.625 |
| jacint | 101.08 | 1.08 | 16.728 | 16.763 |

Pr(inf mle) , mean jacknife order = 0.000, 2.001
c, ch1, ch2, ch3 = 0.680, 0.640 , 0.680 , 0.674

R = 10

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 6.19 | | | |
| pet | 99.43 | -0.57 | 28.421 | 28.427 |
| mlea | 89.71 | -10.29 | 10.538 | 14.727 |
| dr1a | 96.78 | -3.22 | 12.609 | 13.013 |
| Oa | 82.57 | -17.43 | 8.126 | 19.228 |
| ca1a | 106.92 | 6.92 | 13.507 | 15.176 |
| ca2a | 102.93 | 2.93 | 12.685 | 13.019 |
| ca3a | 103.43 | 3.43 | 12.762 | 13.215 |

ca,ch1a,ch2a,ch3a = 0.681, 0.641 , 0.681 , 0.675

R = 25

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 15.47 | | | |
| pet | 99.28 | -0.72 | 16.695 | 16.711 |
| mlea | 90.48 | -9.52 | 9.721 | 13.606 |
| dr1a | 96.20 | -3.80 | 11.777 | 12.375 |
| Oa | 82.52 | -17.48 | 7.932 | 19.200 |
| ca1a | 106.30 | 6.30 | 12.621 | 14.108 |
| ca2a | 102.35 | 2.35 | 11.829 | 12.060 |
| ca3a | 102.84 | 2.84 | 11.935 | 12.268 |

ca,ch1a,ch2a,ch3a = 0.681, 0.645 , 0.685 , 0.679

R = 50

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 30.84 | | | |
| pet | 99.92 | -0.08 | 13.132 | 13.132 |
| mlea | 91.79 | -8.21 | 9.219 | 12.343 |
| dr1a | 96.46 | -3.54 | 11.542 | 12.072 |
| Oa | 82.75 | -17.25 | 8.042 | 19.029 |
| ca1a | 106.62 | 6.62 | 12.391 | 14.047 |
| ca2a | 102.64 | 2.64 | 11.642 | 11.939 |
| ca3a | 103.12 | 3.12 | 11.726 | 12.134 |

ca,ch1a,ch2a,ch3a = 0.681, 0.644 , 0.683 , 0.678

**Table 3.7.5b**

N = 100 : p ~ Beta(alpha,beta) : ep ~ U( 0.08, 0.12 ) : cv ~ U( 0.30, 0.55 )
Number of simulations = 1000

**t = 15**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 74.72 | -25.28 | 5.570 | 25.889 |
| mle | 90.47 | -9.53 | 6.899 | 11.761 |
| dr1 | 95.68 | -4.32 | 7.707 | 8.835 |
| boot | 89.09 | -10.91 | 6.190 | 12.540 |
| ac1 | 100.92 | 0.92 | 10.348 | 10.388 |
| ac2 | 97.15 | -2.85 | 9.738 | 10.146 |
| ac3 | 97.58 | -2.42 | 9.814 | 10.107 |
| o | 95.68 | -4.32 | 6.587 | 7.876 |
| ca1 | 107.47 | 7.47 | 8.673 | 11.449 |
| ca2 | 105.99 | 5.99 | 8.428 | 10.337 |
| ca3 | 106.15 | 6.15 | 8.445 | 10.448 |
| pojac | 121.64 | 21.64 | 12.742 | 25.110 |
| jac1 | 104.28 | 4.28 | 7.594 | 8.719 |
| jacseq | 110.08 | 10.08 | 13.984 | 17.236 |
| jacint | 106.30 | 6.30 | 10.537 | 12.277 |

Pr(inf mle) , mean jacknife order = 0.000, 1.355
c, ch1, ch2, ch3 = 0.806, 0.783 , 0.805 , 0.802

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 7.48 | | | |
| pet | 99.88 | -0.12 | 19.983 | 19.983 |
| mlea | 90.89 | -9.11 | 6.628 | 11.268 |
| dr1a | 95.59 | -4.41 | 7.444 | 8.654 |
| Oa | 95.66 | -4.34 | 6.512 | 7.823 |
| ca1a | 107.41 | 7.41 | 8.376 | 11.180 |
| ca2a | 105.89 | 5.89 | 8.144 | 10.051 |
| ca3a | 106.07 | 6.07 | 8.152 | 10.165 |

ca,ch1a,ch2a,ch3a = 0.806, 0.784 , 0.805 , 0.803

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 18.48 | | | |
| pet | 99.94 | -0.06 | 13.677 | 13.677 |
| mlea | 90.74 | -9.26 | 7.049 | 11.639 |
| dr1a | 94.64 | -5.36 | 7.798 | 9.461 |
| Oa | 94.60 | -5.40 | 6.863 | 8.731 |
| ca1a | 106.32 | 6.32 | 8.750 | 10.795 |
| ca2a | 104.81 | 4.81 | 8.509 | 9.775 |
| ca3a | 104.99 | 4.99 | 8.515 | 9.869 |

ca,ch1a,ch2a,ch3a = 0.800, 0.782 , 0.804 , 0.802

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 37.37 | | | |
| pet | 99.77 | -0.23 | 9.723 | 9.726 |
| mlea | 91.78 | -8.22 | 6.279 | 10.346 |
| dr1a | 95.33 | -4.67 | 6.983 | 8.400 |
| Oa | 95.50 | -4.50 | 6.599 | 7.990 |
| ca1a | 107.11 | 7.11 | 7.860 | 10.600 |
| ca2a | 105.64 | 5.64 | 7.660 | 9.512 |
| ca3a | 105.81 | 5.81 | 7.679 | 9.631 |

ca,ch1a,ch2a,ch3a = 0.805, 0.784 , 0.806 , 0.803

**t = 20**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 82.41 | -17.59 | 5.161 | 18.331 |
| mle | 91.71 | -8.29 | 5.494 | 9.942 |
| dr1 | 95.66 | -4.34 | 5.876 | 7.305 |
| boot | 95.51 | -4.49 | 5.476 | 7.084 |
| ac1 | 99.68 | -0.32 | 7.404 | 7.411 |
| ac2 | 97.74 | -2.26 | 7.158 | 7.506 |
| ac3 | 97.95 | -2.05 | 7.214 | 7.499 |
| o | 101.20 | 1.20 | 5.783 | 5.907 |
| ca1 | 107.40 | 7.40 | 6.822 | 10.068 |
| ca2 | 106.71 | 6.71 | 6.739 | 9.507 |
| ca3 | 106.77 | 6.77 | 6.744 | 9.558 |
| pojac | 117.08 | 17.08 | 11.998 | 20.876 |
| jac1 | 107.61 | 7.61 | 6.666 | 10.116 |
| jacseq | 109.05 | 9.05 | 9.280 | 12.960 |
| jacint | 108.02 | 8.02 | 7.389 | 10.908 |

Pr(inf mle) , mean jacknife order = 0.000, 1.097
c, ch1, ch2, ch3 = 0.875, 0.863 , 0.875 , 0.874

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 8.27 | | | |
| pet | 99.73 | -0.27 | 15.544 | 15.546 |
| mlea | 91.99 | -8.01 | 5.399 | 9.659 |
| dr1a | 95.69 | -4.31 | 5.762 | 7.198 |
| Oa | 101.26 | 1.26 | 5.737 | 5.874 |
| ca1a | 107.44 | 7.44 | 6.674 | 9.996 |
| ca2a | 106.78 | 6.78 | 6.583 | 9.451 |
| ca3a | 106.84 | 6.84 | 6.592 | 9.499 |

ca,ch1a,ch2a,ch3a = 0.875, 0.862 , 0.875 , 0.873

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 20.50 | | | |
| pet | 100.81 | 0.81 | 10.543 | 10.575 |
| mlea | 92.31 | -7.69 | 5.269 | 9.319 |
| dr1a | 95.60 | -4.40 | 5.572 | 7.102 |
| Oa | 101.39 | 1.39 | 5.659 | 5.826 |
| ca1a | 107.32 | 7.32 | 6.433 | 9.745 |
| ca2a | 106.69 | 6.69 | 6.344 | 9.217 |
| ca3a | 106.76 | 6.76 | 6.358 | 9.279 |

ca,ch1a,ch2a,ch3a = 0.877, 0.866 , 0.878 , 0.877

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 41.14 | | | |
| pet | 100.14 | 0.14 | 8.147 | 8.148 |
| mlea | 92.44 | -7.56 | 5.150 | 9.148 |
| dr1a | 95.57 | -4.43 | 5.410 | 6.990 |
| Oa | 101.24 | 1.24 | 5.561 | 5.696 |
| ca1a | 107.34 | 7.34 | 6.241 | 9.634 |
| ca2a | 106.69 | 6.69 | 6.152 | 9.091 |
| ca3a | 106.75 | 6.75 | 6.167 | 9.142 |

ca,ch1a,ch2a,ch3a = 0.875, 0.863 , 0.875 , 0.874

**Table 3.7.6a**

N = 100 : p ~ U( 0, 0.16) : E(p) = 0.08 : sqrt[Var(p)]/E(p) = 0.5774

Number of simulations = 1000

**t = 5**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 32.44 | -67.56 | 4.510 | 67.706 |
| mle | 83.09 | -16.91 | 35.492 | 39.317 |
| dr1 | 102.80 | 2.80 | 46.371 | 46.455 |
| boot | 41.33 | -58.67 | 5.789 | 58.952 |
| ac1 | 120.86 | 20.86 | 63.997 | 67.312 |
| ac2 | 91.23 | -8.77 | 48.615 | 49.400 |
| ac3 | 95.56 | -4.44 | 58.350 | 58.519 |
| o | 45.49 | -54.51 | 6.427 | 54.888 |
| ca1 | 107.11 | 7.11 | 46.654 | 47.193 |
| ca2 | 91.29 | -8.71 | 38.065 | 39.049 |
| ca3 | 92.98 | -7.02 | 39.085 | 39.711 |
| pojac | 74.64 | -25.36 | 11.693 | 27.925 |
| jac1 | 52.99 | -47.01 | 7.659 | 47.628 |
| jacseq | 76.15 | -23.85 | 13.895 | 27.606 |
| jacint | 72.40 | -27.60 | 13.428 | 30.696 |

Pr(inf mle) , mean jacknife order = 0.001, 3.723
c, ch1, ch2, ch3 = 0.421, 0.352 , 0.425 , 0.416

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 3.14 | | | |
| pet | 103.40 | 3.40 | 58.831 | 58.929 |
| mlea | 87.47 | -12.53 | 28.244 | 30.897 |
| dr1a | 101.37 | 1.37 | 43.967 | 43.988 |
| Oa | 45.48 | -54.52 | 6.412 | 54.898 |
| ca1a | 105.75 | 5.75 | 44.266 | 44.638 |
| ca2a | 90.22 | -9.78 | 36.486 | 37.775 |
| ca3a | 91.96 | -8.04 | 37.809 | 38.655 |

ca,ch1a,ch2a,ch3a = 0.420, 0.354 , 0.427 , 0.417

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 8.10 | | | |
| pet | 99.88 | -0.12 | 33.742 | 33.743 |
| mlea | 89.82 | -10.18 | 23.650 | 25.749 |
| dr1a | 101.24 | 1.24 | 42.967 | 42.985 |
| Oa | 45.45 | -54.55 | 6.520 | 54.941 |
| ca1a | 105.62 | 5.62 | 43.250 | 43.613 |
| ca2a | 90.09 | -9.91 | 35.090 | 36.464 |
| ca3a | 91.71 | -8.29 | 35.623 | 36.575 |

ca,ch1a,ch2a,ch3a = 0.420, 0.350 , 0.422 , 0.412

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 16.19 | | | |
| pet | 99.31 | -0.69 | 24.791 | 24.801 |
| mlea | 90.48 | -9.52 | 19.390 | 21.600 |
| dr1a | 97.25 | -2.75 | 33.498 | 33.611 |
| Oa | 45.20 | -54.80 | 6.424 | 55.178 |
| ca1a | 101.58 | 1.58 | 33.847 | 33.884 |
| ca2a | 86.85 | -13.15 | 27.855 | 30.803 |
| ca3a | 88.52 | -11.48 | 28.619 | 30.834 |

ca,ch1a,ch2a,ch3a = 0.418, 0.354 , 0.427 , 0.417

**t = 10**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 51.70 | -48.30 | 5.086 | 48.572 |
| mle | 78.57 | -21.43 | 11.588 | 24.363 |
| dr1 | 86.01 | -13.99 | 13.362 | 19.343 |
| boot | 64.19 | -35.81 | 6.389 | 36.374 |
| ac1 | 92.88 | -7.12 | 17.699 | 19.079 |
| ac2 | 84.67 | -15.33 | 15.402 | 21.731 |
| ac3 | 85.65 | -14.35 | 15.681 | 21.259 |
| o | 70.19 | -29.81 | 7.100 | 30.649 |
| ca1 | 94.69 | -5.31 | 14.201 | 15.161 |
| ca2 | 90.80 | -9.20 | 13.240 | 16.121 |
| ca3 | 91.26 | -8.74 | 13.323 | 15.936 |
| pojac | 103.01 | 3.01 | 13.154 | 13.494 |
| jac1 | 79.66 | -20.34 | 8.460 | 22.028 |
| jacseq | 95.57 | -4.43 | 17.518 | 18.070 |
| jacint | 87.79 | -12.21 | 16.026 | 20.147 |

Pr(inf mle) , mean jacknife order = 0.000, 2.056
c, ch1, ch2, ch3 = 0.648, 0.609 , 0.649 , 0.644

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 5.05 | | | |
| pet | 101.92 | 1.92 | 33.834 | 33.888 |
| mlea | 81.32 | -18.68 | 11.408 | 21.885 |
| dr1a | 86.00 | -14.00 | 12.963 | 19.079 |
| Oa | 70.21 | -29.79 | 7.103 | 30.628 |
| ca1a | 94.69 | -5.31 | 13.819 | 14.806 |
| ca2a | 90.81 | -9.19 | 12.915 | 15.851 |
| ca3a | 91.25 | -8.75 | 12.994 | 15.663 |

ca,ch1a,ch2a,ch3a = 0.647, 0.609 , 0.648 , 0.644

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 12.91 | | | |
| pet | 99.50 | -0.50 | 23.299 | 23.305 |
| mlea | 82.33 | -17.67 | 10.864 | 20.741 |
| dr1a | 84.53 | -15.47 | 11.978 | 19.565 |
| Oa | 69.55 | -30.45 | 6.835 | 31.212 |
| ca1a | 93.10 | -6.90 | 12.756 | 14.504 |
| ca2a | 89.29 | -10.71 | 11.935 | 16.039 |
| ca3a | 89.74 | -10.26 | 11.994 | 15.782 |

ca,ch1a,ch2a,ch3a = 0.647, 0.613 , 0.653 , 0.648

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 25.76 | | | |
| pet | 99.60 | -0.40 | 16.670 | 16.675 |
| mlea | 84.83 | -15.17 | 10.135 | 18.240 |
| dr1a | 84.81 | -15.19 | 10.997 | 18.750 |
| Oa | 69.70 | -30.30 | 6.424 | 30.974 |
| ca1a | 93.43 | -6.57 | 11.751 | 13.462 |
| ca2a | 89.61 | -10.39 | 11.034 | 15.157 |
| ca3a | 90.06 | -9.94 | 11.087 | 14.892 |

ca,ch1a,ch2a,ch3a = 0.646, 0.610 , 0.650 , 0.646

**Table 3.7.6b**

N = 100 : p ~ U( 0, 0.16) : E(p) = 0.08 : sqrt[Var(p)]/E(p) = 0.5774
Number of simulations = 1000

**t = 15**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 62.94 | -37.06 | 5.041 | 37.405 |
| mle | 78.77 | -21.23 | 7.575 | 22.540 |
| dr1 | 83.72 | -16.28 | 8.525 | 18.374 |
| boot | 75.82 | -24.18 | 6.172 | 24.958 |
| ac1 | 88.59 | -11.41 | 11.476 | 16.182 |
| ac2 | 84.97 | -15.03 | 10.803 | 18.508 |
| ac3 | 85.38 | -14.62 | 10.882 | 18.226 |
| o | 81.78 | -18.22 | 6.803 | 19.449 |
| ca1 | 94.05 | -5.95 | 9.554 | 11.255 |
| ca2 | 92.56 | -7.44 | 9.305 | 11.915 |
| ca3 | 92.74 | -7.26 | 9.336 | 11.829 |
| pojac | 107.36 | 7.36 | 12.925 | 14.874 |
| jac1 | 89.92 | -10.08 | 8.139 | 12.952 |
| jacseq | 96.28 | -3.72 | 14.260 | 14.737 |
| jacint | 91.92 | -8.08 | 10.884 | 13.557 |

Pr(inf mle) , mean jacknife order = 0.000, 1.429
c, ch1, ch2, ch3 = 0.772, 0.755 , 0.777 , 0.775

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 6.35 | | | |
| pet | 99.40 | -0.60 | 27.989 | 27.996 |
| mlea | 80.07 | -19.93 | 7.725 | 21.374 |
| dr1a | 83.66 | -16.34 | 8.465 | 18.402 |
| Oa | 81.78 | -18.22 | 6.815 | 19.449 |
| ca1a | 94.00 | -6.00 | 9.493 | 11.232 |
| ca2a | 92.50 | -7.50 | 9.231 | 11.893 |
| ca3a | 92.68 | -7.32 | 9.242 | 11.790 |

ca,ch1a,ch2a,ch3a = 0.772, 0.755 , 0.778 , 0.775

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 15.82 | | | |
| pet | 100.55 | 0.55 | 17.111 | 17.120 |
| mlea | 82.61 | -17.39 | 7.202 | 18.821 |
| dr1a | 84.74 | -15.26 | 7.818 | 17.144 |
| Oa | 82.76 | -17.24 | 6.232 | 18.334 |
| ca1a | 95.19 | -4.81 | 8.786 | 10.017 |
| ca2a | 93.66 | -6.34 | 8.522 | 10.622 |
| ca3a | 93.85 | -6.15 | 8.548 | 10.532 |

ca,ch1a,ch2a,ch3a = 0.776, 0.753 , 0.776 , 0.773

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 31.74 | | | |
| pet | 99.89 | -0.11 | 13.293 | 13.293 |
| mlea | 83.67 | -16.33 | 7.210 | 17.854 |
| dr1a | 84.35 | -15.65 | 7.592 | 17.397 |
| Oa | 82.43 | -17.57 | 6.186 | 18.625 |
| ca1a | 94.79 | -5.21 | 8.514 | 9.983 |
| ca2a | 93.28 | -6.72 | 8.290 | 10.674 |
| ca3a | 93.45 | -6.55 | 8.316 | 10.586 |

ca,ch1a,ch2a,ch3a = 0.776, 0.753 , 0.776 , 0.773

**t = 20**

**R = 0**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x1 | 71.08 | -28.92 | 4.479 | 29.268 |
| mle | 80.82 | -19.18 | 5.664 | 19.997 |
| dr1 | 85.05 | -14.95 | 6.230 | 16.199 |
| boot | 83.42 | -16.58 | 5.330 | 17.419 |
| ac1 | 89.41 | -10.59 | 8.233 | 13.415 |
| ac2 | 87.52 | -12.48 | 8.021 | 14.837 |
| ac3 | 87.72 | -12.28 | 8.050 | 14.682 |
| o | 88.93 | -11.07 | 5.844 | 12.514 |
| ca1 | 95.86 | -4.14 | 7.232 | 8.333 |
| ca2 | 95.19 | -4.81 | 7.140 | 8.607 |
| ca3 | 95.26 | -4.74 | 7.146 | 8.576 |
| pojac | 107.75 | 7.75 | 11.872 | 14.178 |
| jac1 | 95.78 | -4.22 | 6.958 | 8.138 |
| jacseq | 98.52 | -1.48 | 11.372 | 11.469 |
| jacint | 96.78 | -3.22 | 8.764 | 9.336 |

Pr(inf mle) , mean jacknife order = 0.000, 1.181
c, ch1, ch2, ch3 = 0.850, 0.837 , 0.850 , 0.849

**R = 10**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 7.09 | | | |
| pet | 100.02 | 0.02 | 21.859 | 21.859 |
| mlea | 81.56 | -18.44 | 5.611 | 19.271 |
| dr1a | 85.04 | -14.96 | 6.111 | 16.158 |
| Oa | 88.95 | -11.05 | 5.777 | 12.466 |
| ca1a | 95.85 | -4.15 | 7.067 | 8.197 |
| ca2a | 95.18 | -4.82 | 6.963 | 8.469 |
| ca3a | 95.23 | -4.77 | 6.983 | 8.455 |

ca,ch1a,ch2a,ch3a = 0.850, 0.837 , 0.850 , 0.849

**R = 25**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 17.83 | | | |
| pet | 99.77 | -0.23 | 14.789 | 14.790 |
| mlea | 82.55 | -17.45 | 6.138 | 18.494 |
| dr1a | 85.06 | -14.94 | 6.701 | 16.377 |
| Oa | 88.98 | -11.02 | 6.375 | 12.727 |
| ca1a | 95.88 | -4.12 | 7.713 | 8.743 |
| ca2a | 95.22 | -4.78 | 7.625 | 9.001 |
| ca3a | 95.29 | -4.71 | 7.633 | 8.968 |

ca,ch1a,ch2a,ch3a = 0.849, 0.837 , 0.850 , 0.849

**R = 50**

| Estimator | mean | bias | s.d. | rmse |
|---|---|---|---|---|
| x2 | 35.76 | | | |
| pet | 99.23 | -0.77 | 10.608 | 10.636 |
| mlea | 83.33 | -16.67 | 5.672 | 17.606 |
| dr1a | 85.03 | -14.97 | 6.039 | 16.144 |
| Oa | 88.91 | -11.09 | 5.719 | 12.480 |
| ca1a | 95.87 | -4.13 | 6.944 | 8.081 |
| ca2a | 95.17 | -4.83 | 6.883 | 8.406 |
| ca3a | 95.25 | -4.75 | 6.886 | 8.367 |

ca,ch1a,ch2a,ch3a = 0.850, 0.836 , 0.849 , 0.847

## Chapter 4 : Estimation Under The Capture-Recapture Model $M_h$ : Continuous Time Sampling Procedure

### § 4.1 : Introduction

Within this chapter a new estimator of population size is proposed for a continuous time sampling procedure. The population in question is assumed to behave according to a continuous time analogue of the standard capture-recapture model $M_h$. This chapter concentrates on the standard estimation problem when no plants are used.

The sampling procedure considered here is identical to the one of chapter 2. But whereas within chapter 2 it was assumed that each animal in the population behaved in exactly the same way, this strong condition is no longer imposed. As was the case in chapter 3 however, a link between the behaviour of the animals is still needed and this is described below.

The estimation problem considered here has previously been studied in a software reliability context by Chao, Ma and Yang(1993) and more recently, in a capture-recapture context, by Yip and Chao(1996).

### § 4.2 : Sampling Procedure, Assumptions, Some Notation and the Sufficient Statistics

It is assumed that one animal is seen at a time and that animals seen for the first time receive a unique tag so that they may be recognised if subsequently recaptured. Sampling stops after a fixed predetermined amount of time $\tau$. It is assumed that there are N animals in the population and that each animal is seen according to a Poisson process with rate $\lambda_i$, $i = 1, 2, ..., N$. It is further assumed that $\lambda_i = k.u_i$, for $i = 1, 2, ..., N$, where k is a constant and the $u_i$ form a random sample from some probability distribution with c.d.f. F(u), $u \in [0,1]$. Detection times for different animals are assumed to be independent.

Let

$X \equiv$ number of distinct animals seen in time $(0, \tau)$.

$Z \equiv$ total number of sightings made in time $(0, \tau)$.

$X_i \equiv$ number of sightings of the ith animal, $i = 1, 2, ..., N$.

$f_k = \sum_{i=1}^{N} I(X_i = k) \equiv$ number of animals seen exactly k times, $k = 0, 1, 2, ....$

$m \equiv$ the most number of times any one particular animal was seen.

$$\bar{\lambda} = \frac{1}{N}\sum_{i=1}^{N}\lambda_i$$

$$\gamma = \frac{1}{\bar{\lambda}}\left\{\frac{1}{N}\sum_{i=1}^{N}\left(\lambda_i - \bar{\lambda}\right)^2\right\}^{1/2} \quad \equiv \quad \text{the coefficient of variation .}$$

$$C = \frac{1}{N\bar{\lambda}}\sum_{i=1}^{N}\lambda_i I(X_i > 0) \quad \equiv \quad \text{sample coverage.}$$

$p_i = \dfrac{\lambda_i}{\sum_{j=1}^{N}\lambda_j} \equiv$ the probability of capture of the $i^{th}$ animal on any trapping

occasion, $i = 1, 2, ..., N$.

The set $S_x = \{s_k : k = 1,2,..,x\}$, where $s_k \in \{1,2,3,...,N\}$ for all k, is used to denote the set of the indices of the x distinct animals seen during the sampling period.

Nayak(1991) showed, in his proposition 2.2, that $(x\ ; X_1, X_2, ....., X_x)$ or equivalently $(m;\ f_1, f_2, ....., f_m)$ are complete and sufficient for the parameters, namely $\lambda_1, \lambda_2, ...., \lambda_N$ and N.

## § 4.3 : A New Coverage Adjusted Estimator for the Model $M_h$

Under this sampling procedure, no usable information is gained from observing the value of z alone, i.e. even if one were able to know the value of $\tau\sum_{i=1}^{N}\lambda_i$ exactly this would not aid estimation of population size N. (This is also true when looking at the model $M_0$ : under the model $M_0$ one obtains the same likelihood function for N from both the probability distribution of x given z and from the joint probability distribution of x and z, see chapter 2 ). For this reason the approach taken here is to derive an estimate of population size from the conditional distribution of the frequencies given Z. This is essentially equivalent to treating the continuous time data set as if it were obtained from a discrete time sampling experiment with z sampling occasions, where z is viewed as a known constant.

Consider the estimator

$$\hat{N} = \sum_{i \in S_x}\frac{1}{1-(1-p_i)^z}. \tag{4.1}$$

Under the assumptions described above, it is straightforward to show that $\hat{N}$ would be an unbiased estimator of population size N if the capture probabilities of the animals seen during the experiment were known exactly. However since these capture probabilities are clearly not known exactly the approach taken here is to estimate the $p_i$

and by doing so obtain an estimator of N by substituting these estimates of capture probability into equation (4.1).

It is now required to estimate the capture probabilities of the animals which were seen during the experiment :

One could use the same approach that Overton(1969) did in the discrete time version of the problem, and use the fact that, under the model, $X_i|Z \sim \mathrm{Bin}(z, p_i)$. Based on this distribution the maximum likelihood estimate of the capture probability of animal i is given by $\hat{p}_i^{(1)} = \frac{x_i}{z}$. On substituting the estimates $\hat{p}_i^{(1)}$ into equation (4.1) one may obtain the estimator $\hat{N}_O$, defined by

$$\begin{aligned}\hat{N}_O &= \sum_{i \in S_x} \frac{1}{1-\left[1-\hat{p}_i^{(1)}\right]^z} \\ &= \sum_{i \in S_x} \frac{1}{1-\left(1-\frac{x_i}{z}\right)^z} \\ &= \sum_{i=1}^{z} \frac{f_i}{1-\left(1-\frac{i}{z}\right)^z}.\end{aligned}$$

The $\hat{p}_i^{(1)}$ are intuitively reasonable estimates of capture probability. This method of estimating capture probability, however, as was the case in chapter 3, does not make full use of all of the available information. The main problem with using the $\hat{p}_i^{(1)}$ is that the sum of these estimates of capture probability is always equal to 1. Whereas they should sum to 1 only if the entire population was seen during the experiment, that is if $x = N$ . If $x < N$ then the sum of the capture probabilities of the animals seen during the sampling period must clearly be less than 1. Hence the estimates $\hat{p}_i^{(1)}$ always tend to overestimate capture probability. In order to obtain better estimates of capture probability one may proceed as follows :

The above discussion implies that, firstly, one can argue on intuitive grounds that the estimates of capture probability should be proportional to $\frac{x_i}{z}$, but that the sum of these estimates of capture probability should be strictly less than 1, unless $x = N$. The sum of the capture probabilities of the x animals seen during the experiment is in fact the quantity which has previously been referred to in the literature as 'sample coverage' : defined by $C = \frac{\sum_{i=1}^{N} \lambda_i I(X_i > 0)}{\sum_{i=1}^{N} \lambda_i}$. This quantity is well known in the capture-recapture literature, see Seber(1982). The fact that the sum of the capture probabilities of the animals seen during the experiment is given by C, sample coverage, may be seen directly .

A good estimator of sample coverage was presented in a software reliability context by Chao, Ma and Jeng(1993) : a derivation of the estimator is outlined in appendix 5. Explicitly a point estimate of sample coverage is given by $\hat{C} = 1 - \frac{f_1}{z}$.

Now returning to the question of how to estimate the $p_i$ for $i \in S_x$, one may proceed as follows :

It is required that $\hat{p}_i \propto \frac{x_i}{z}$

$$\Rightarrow \quad \hat{p}_i = k\frac{x_i}{z}, \qquad \text{where k is a constant.} \tag{4.2}$$

For the reasons stated above we now set

$$\sum_{i \in S_x} \hat{p}_i = \hat{C} = 1 - \frac{f_1}{z}$$

$$\Rightarrow \quad k\sum_{i \in S_x} \frac{x_i}{z} = 1 - \frac{f_1}{z} \qquad \text{using (4.2)}$$

$$\Rightarrow \quad k = 1 - \frac{f_1}{z}$$

$$\Rightarrow \quad \text{estimate the } p_i \text{ by} \quad \hat{p}_i^{(2)} = \left(1 - \frac{f_1}{z}\right)\frac{x_i}{z}, \qquad \text{for } i \in S_x.$$

These estimates of capture probability are now substituted into equation (4.1) to produce the coverage adjusted estimator $\hat{N}_{ca}$ : defined by

$$\begin{aligned}
\hat{N}_{ca} &= \sum_{i \in S_x} \frac{1}{1 - \left[1 - \hat{p}_i^{(2)}\right]^z} \\
&= \sum_{i \in S_x} \frac{1}{1 - \left[1 - \left(1 - \frac{f_1}{z}\right)\frac{x_i}{z}\right]^z} \\
&= \sum_{i=1}^{z} \frac{f_i}{1 - \left[1 - \left(1 - \frac{f_1}{z}\right)\frac{i}{z}\right]^z}.
\end{aligned}$$

## § 4.4 : Simulation Study

The results presented in this simulation study concentrate on situations in which the capture probabilities of the animals are obtained via a scaled random sample from a Beta distribution. For each value of N, results are also given for the situation in which each animal in the population is equally likely to be caught. Explicitly results are presented for situations in which the capture probabilities of the animals are obtained as follows :

Case 1 : $\lambda_i = c$ $i = 1, 2, 3, \ldots, N.$

Case 2 : $\lambda_i = c.u_i$ where $u_1, u_2, u_3, \ldots, u_N$ are a random sample from B(alpha, beta) : alpha > 0, beta > 0.

The constant c in each case is a normalising constant used to ensure that $\sum_{i=1}^{N} \lambda_i = 1$.

So that for case 1 $\sum_{i=1}^{N} \lambda_i = 1 \Rightarrow Nc = 1 \Rightarrow c = \frac{1}{N} \Rightarrow p_i = \lambda_i = \frac{1}{N}.$

Similarly for case 2 $p_i = \frac{\lambda_i}{\sum_{j=1}^{N} \lambda_j} = \lambda_i = c.u_i \Rightarrow c = \frac{1}{\sum_{j=1}^{N} u_j}.$

For each case 1000 simulations were generated where each simulation ended when the fixed predetermined stopping time $\tau$ was reached. On each resulting data set the estimates produced by each of the estimators were calculated and at the end of the 1000 simulations the mean and root mean square error of each estimator was determined. For each version of case 2, a different random sample was used to obtain a different set of capture probabilities in each of the 1000 simulations. Results are given for populations of size 100 and 400, for various stopping times. For each value of N four stopping times were considered. For N=100 results are given for $\tau$ = 41, 69, 139 and 230 : these are the times for which in the homogeneous case, case 1, one would expect to see the proportions 0.33, 0.5, 0.75 and 0.9 of the population. For N=400 results are given for $\tau$ = 89, 143, 277 and 644 : these are the times for which in the homogeneous case, case 1, one would expect to see the proportions 0.2, 0.3, 0.5 and 0.8 of the population. Results are conditional upon seeing at least one animal more than once during sampling, i.e. results are conditional upon the maximum likelihood estimator $\hat{N}$, of chapter 2, producing a finite estimate. In each table 'cv' represents the coefficient of variation of the population ; 'average D' gives the average number of distinct individuals seen. The stopping time $\tau$ is denoted by t.

Results are given for the following estimators :

mle $\equiv \hat{N}$ the maximum likelihood estimator described in chapter 2.

chao $\equiv \hat{N}_1 = \frac{x}{\hat{C}} + \frac{f_1}{\hat{C}}\hat{\gamma}^2$ is the estimator introduced by Chao, Ma and Yang(1993),

where $\hat{\gamma}^2 = \max\left\{\frac{\hat{N}_D}{z^2}\sum_i i(i-1)f_i - 1, 0\right\}.$

$\hat{N}_D = \frac{x}{\hat{C}}$ is the estimator proposed by Darroch and Ratcliff(1980).

nj1 $\equiv \hat{N}_{J1} = x + \left(\frac{z-1}{z}\right) f_1$ a first order jackknife estimator - when z is assumed to be a constant, this estimator is equivalent to the first order jackknife estimator of Burnham and Overton(1978, 1979).

bov $\equiv \hat{N}_j$ again under the assumption that z is a pre-chosen constant, this estimator is equivalent to the interpolated jackknife estimator of Burnham and Overton(1979). This interpolated jackknife is obtained as follows. When the selection procedure of Burnham and Overton(1978) chooses the first order jackknife then $\hat{N}_j$ is equal to $\hat{N}_{J1}$. When the selection procedure chooses the jackknife of order k, for k = 2, 3 or 4, then $\hat{N}_j$ is a weighted average of the jackknives of orders k and k-1. When the selection procedure rejects the fourth order jackknife, $\hat{N}_j$ is equal to $\hat{N}_{J1}$.

$$\text{O} \equiv \hat{N}_O = \sum_{i=1}^{z} \frac{f_i}{1-\left(1-\frac{i}{z}\right)^z}$$

and ca $\equiv$ $$\hat{N}_{ca} = \sum_{i=1}^{z} \frac{f_i}{1-\left[1-\left(1-\frac{f_1}{z}\right)\frac{i}{z}\right]^z}$$ represents the coverage adjusted estimator.

Discussion

In case 1 the maximum likelihood estimator $\hat{N}$ for homogeneous populations, from chapter 2, is seen to perform best when both mean and mean square error are considered together. However $\hat{N}$ becomes unacceptably negatively biased in most heterogeneous situations.

The estimators $\hat{p}_i^{(1)}$, as mentioned above, tend to produce estimates which overestimate the capture probabilities of the animals seen during the sampling period. For this reason the estimator $\hat{N}_O$, which directly incorporates the $\hat{p}_i^{(1)}$, has a tendency to underestimate population size. When sampling for a small amount of time the $\hat{p}_i^{(1)}$ are particularly positively biased. The reason for this is that the sum of the $\hat{p}_i^{(1)}$ is always equal to 1, whereas, as proved above, the sum of the capture probabilities of the animals seen during the experiment should sum to the random quantity C, sample coverage, which is equal to 1 only if x = N. For small sampling times one would expect the sum of the capture probabilities of the sighted animals, or equivalently the value of C, to be small - 'a lot less than 1'. Conversely for longer sampling times one would expect the value of C to be large - 'a lot closer to 1'. This is why the $\hat{p}_i^{(1)}$ overestimate more for small sampling times and less for longer sampling times. A direct consequence of the behaviour of the $\hat{p}_i^{(1)}$ is that the estimator $\hat{N}_O$ tends to be extremely negatively biased for small sampling times and less negatively biased for longer sampling times. Since $\hat{N}_{ca}$ incorporates the estimates $\hat{p}_i^{(2)}$, which are much more reasonable estimates of capture probability over all sampling times, this estimator is able to perform well, notably in terms of mean, for sampling times both long and short.

Given the above discussion, it is not surprising to observe from the following tables that the bias of $\hat{N}_{ca}$ is usually much smaller than that of $\hat{N}_O$ : particularly for the smaller sampling times considered. As one might also have expected $\hat{N}_O$ and $\hat{N}_{ca}$ are seen to perform in a very similar way for the longer sampling times. This is explained by the fact that, for each i, the value of $\hat{p}_i^{(2)}$ tends towards that of $\hat{p}_i^{(1)}$ as sampling time is increased ~ since $1-\frac{f_1}{z}\rightarrow 1$ as $\tau, z \rightarrow \infty$. In terms of root mean square error, due to the fact that $\hat{N}_{ca}$ generally possesses a much more realistic mean, the coverage adjusted estimator $\hat{N}_{ca}$ is seen on the whole to clearly out perform the Overton-type estimator $\hat{N}_O$.

The comparison between the first order jackknife estimator $\hat{N}_{J1}$ and the interpolated jackknife estimator $\hat{N}_j$ is also quite straightforward. As one would expect $\hat{N}_j$ generally has a better mean than $\hat{N}_{J1}$ whilst possessing a larger variance. This results in $\hat{N}_j$ being a far better alternative to $\hat{N}_{J1}$ for the smaller stopping times - but that for the longer stopping times $\hat{N}_{J1}$, due to its smaller variance, can occasionally improve upon $\hat{N}_j$ in terms of mean square error. In terms of overall performance though the interpolated jackknife estimator $\hat{N}_j$ is seen to be preferable to the first order jackknife estimator $\hat{N}_{J1}$.

For the smaller sampling times $\hat{N}_{ca}$ generally possesses a very good mean value whereas a feature of $\hat{N}_j$ is that in this situation it can be extremely negatively biased. For longer sampling times $\hat{N}_j$ and $\hat{N}_{ca}$, in situations where the level of heterogeneity is not extreme, have similar bias. However in situations where the coefficient of variation is very large, the negative bias of $\hat{N}_j$ tends to be slightly less than that of $\hat{N}_{ca}$. The comparison between $\hat{N}_j$ and $\hat{N}_{ca}$ in terms of mean square error is confused somewhat by the way in which mean square error works as a loss function. It has previously been discussed, see chapter 3, that since mean square error rewards negative bias to quite a large extent it is not an ideal loss function - and should not be used on its own. That is in deciding between which of the two estimators $\hat{N}_j$ and $\hat{N}_{ca}$ is performing best overall one must consider both mean and mean square error - or equivalently both mean and variance. For this reason the choice between the estimators $\hat{N}_j$ and $\hat{N}_{ca}$ is not straightforward and is particularly difficult due to the fact that these two estimators each behave in a very different way. For the smaller stopping times considered in the tables it is seen that $\hat{N}_{ca}$ usually has a very good mean but possesses a relatively large variance whereas $\hat{N}_j$ tends to be extremely negatively biased whilst possessing a small variance. For these small stopping times, the estimator $\hat{N}_{ca}$, in terms of mean square error, is generally seen to be a better alternative to $\hat{N}_j$. In situations of this type where $\hat{N}_{ca}$ has a larger mean square error than $\hat{N}_j$ it almost always exhibits a much more realistic mean value. For the longer sampling times, when the coefficient of variation is very large, $\hat{N}_j$ generally possesses a slightly better mean than $\hat{N}_{ca}$ and consequently can also have a slightly smaller mean square error.

For long stopping times, when the heterogeneity is mild, an important feature of $\hat{N}_1$ is that it tends to possess a good mean value. This is in contrast to the behaviour of the other estimators designed for heterogeneous populations. That is the estimators $\hat{N}_{J1}$, $\hat{N}_j$ and $\hat{N}_{ca}$ have a tendency to overestimate when the sampling time is long and the coefficient of variation is less than about 0.4. However, for reasonably large values of $\tau$, when the coefficient of variation is above about 0.4, the negative bias of $\hat{N}_1$ tends to be greater than that of $\hat{N}_{J1}$, $\hat{N}_j$ and $\hat{N}_{ca}$.

The comparison between the coverage adjusted estimator $\hat{N}_{ca}$ and the estimator of Chao and Yang(1993) is quite straightforward. Only when the coefficient of variation is very small and the sampling time is long does $\hat{N}_1$ perform better than $\hat{N}_{ca}$ : in this situation $\hat{N}_{ca}$ is positively biased and its variance is larger than that of the almost unbiased $\hat{N}_1$. For small to moderate sampling times, the variance of $\hat{N}_{ca}$ is smaller than that of $\hat{N}_1$ and consequently, even though when the heterogeneity is mild $\hat{N}_1$ can be less biased, the coverage adjusted $\hat{N}_{ca}$ tends to posses a mean square error smaller than that of $\hat{N}_1$. Finally, for large values of $\tau$, when the coefficient of variation is above about 0.4, the coverage adjusted estimator $\hat{N}_{ca}$ is generally less biased and also tends to have a mean square error smaller than that of $\hat{N}_1$.

Given the above evidence it is clear that the coverage adjusted estimator $\hat{N}_{ca}$ may at least be considered as a viable alternative to the estimator proposed by Chao, Ma and Yang(1993) and to the jackknife estimators. Furthermore it is believed that, particularly due to the performance of the estimators for the smaller stopping times, one should always use the coverage adjusted estimator $\hat{N}_{ca}$ in preference to either the jackknife estimators or the estimator of Chao, Ma and Yang(1993).

Table 4.4.1a

| N=100 | t = 41 | [ E(p) = 0.33 ] | | t = 69 | [ E(p) = 0.5 ] | | t = 139 | [ E(p) = 0.75 ] | | t = 230 | [ E(p) = 0.9 ] | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | |
| | average D = 34 | | | average D = 50 | | | average D = 75 | | | average D = 90 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case1 | mle | 115 | 59.74 | mle | 103 | 20.90 | mle | 100 | 8.25 | mle | 100 | 3.87 |
| const | chao | 125 | 70.68 | chao | 108 | 26.78 | chao | 103 | 10.49 | chao | 101 | 4.79 |
| cv=0.000 | nj1 | 60 | 40.87 | nj1 | 84 | 18.19 | nj1 | 110 | 12.56 | nj1 | 113 | 14.29 |
| | bov | 80 | 30.01 | bov | 100 | 22.80 | bov | 111 | 15.98 | bov | 113 | 14.32 |
| | O | 50 | 50.31 | O | 72 | 29.11 | O | 100 | 6.31 | O | 109 | 9.78 |
| | ca | 125 | 64.89 | ca | 115 | 26.78 | ca | 115 | 18.00 | ca | 113 | 14.05 |
| | | | | | | | | | | | | |
| | average D = 32 | | | average D = 46 | | | average D = 66 | | | average D = 78 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 82 | 35.85 | mle | 79 | 25.38 | mle | 80 | 20.66 | mle | 83 | 17.15 |
| a=1.0 | chao | 92 | 42.76 | chao | 86 | 23.52 | chao | 87 | 15.92 | chao | 90 | 11.98 |
| b=1.0 | nj1 | 55 | 45.63 | nj1 | 75 | 26.63 | nj1 | 94 | 9.57 | nj1 | 99 | 6.49 |
| cv=0.58 | bov | 70 | 35.82 | bov | 85 | 24.94 | bov | 96 | 12.19 | bov | 100 | 9.40 |
| | O | 47 | 53.91 | O | 65 | 36.14 | O | 86 | 15.41 | O | 94 | 7.79 |
| | ca | 92 | 34.31 | ca | 91 | 18.38 | ca | 95 | 9.45 | ca | 97 | 6.53 |
| | | | | | | | | | | | | |
| | average D = 33 | | | average D = 48 | | | average D = 72 | | | average D = 85 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 101 | 48.70 | mle | 92 | 18.96 | mle | 92 | 11.47 | mle | 92 | 8.69 |
| a=1.0 | chao | 110 | 58.04 | chao | 98 | 21.99 | chao | 95 | 10.87 | chao | 95 | 7.13 |
| b=0.25 | nj1 | 58 | 42.64 | nj1 | 80 | 21.34 | nj1 | 103 | 8.33 | nj1 | 106 | 8.77 |
| cv=0.33 | bov | 76 | 31.94 | bov | 93 | 22.04 | bov | 104 | 11.35 | bov | 107 | 9.28 |
| | O | 49 | 51.63 | O | 69 | 31.79 | O | 94 | 8.82 | O | 102 | 5.18 |
| | ca | 111 | 51.88 | ca | 105 | 19.66 | ca | 106 | 11.25 | ca | 106 | 7.93 |
| | | | | | | | | | | | | |
| | average D = 33 | | | average D = 47 | | | average D = 69 | | | average D = 82 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 93 | 42.90 | mle | 87 | 20.58 | mle | 86 | 15.36 | mle | 88 | 12.36 |
| a=1.0 | chao | 103 | 51.34 | chao | 93 | 21.68 | chao | 91 | 12.79 | chao | 93 | 9.25 |
| b=0.5 | nj1 | 57 | 43.84 | nj1 | 78 | 23.27 | nj1 | 99 | 7.67 | nj1 | 103 | 6.74 |
| cv=0.45 | bov | 74 | 33.61 | bov | 90 | 22.77 | bov | 101 | 11.33 | bov | 103 | 8.73 |
| | O | 48 | 52.56 | O | 67 | 33.42 | O | 90 | 11.57 | O | 99 | 5.02 |
| | ca | 103 | 44.40 | ca | 99 | 17.63 | ca | 101 | 8.74 | ca | 102 | 5.85 |
| | | | | | | | | | | | | |
| | average D = 32 | | | average D = 47 | | | average D = 67 | | | average D = 80 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 86 | 37.00 | mle | 83 | 22.97 | mle | 83 | 18.59 | mle | 85 | 15.26 |
| a=1.0 | chao | 95 | 40.25 | chao | 89 | 22.22 | chao | 88 | 14.92 | chao | 91 | 11.02 |
| b=0.75 | nj1 | 56 | 44.96 | nj1 | 76 | 25.05 | nj1 | 96 | 8.80 | nj1 | 100 | 6.34 |
| cv=0.52 | bov | 71 | 35.13 | bov | 86 | 24.02 | bov | 97 | 11.00 | bov | 101 | 8.19 |
| | O | 47 | 53.40 | O | 66 | 34.87 | O | 87 | 13.98 | O | 96 | 6.55 |
| | ca | 95 | 35.86 | ca | 95 | 17.55 | ca | 97 | 9.01 | ca | 99 | 5.95 |
| | | | | | | | | | | | | |
| | average D = 31 | | | average D = 43 | | | average D = 62 | | | average D = 74 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 70 | 39.45 | mle | 69 | 33.43 | mle | 72 | 28.34 | mle | 77 | 23.06 |
| a=1.0 | chao | 80 | 41.08 | chao | 78 | 27.64 | chao | 83 | 19.72 | chao | 88 | 13.83 |
| b=3.0 | nj1 | 52 | 48.35 | nj1 | 70 | 31.42 | nj1 | 87 | 14.78 | nj1 | 95 | 8.55 |
| cv=0.77 | bov | 65 | 39.60 | bov | 77 | 28.36 | bov | 90 | 16.34 | bov | 96 | 10.04 |
| | O | 44 | 56.06 | O | 60 | 40.13 | O | 80 | 21.14 | O | 89 | 12.15 |
| | ca | 80 | 34.23 | ca | 81 | 23.30 | ca | 87 | 15.12 | ca | 92 | 10.03 |

Table 4.4.1b

| N=100 | t = 41 [ E(p) = 0.33 ] | | | t = 69 [ E(p) = 0.5 ] | | | t = 139 [ E(p) = 0.75 ] | | | t = 230 [ E(p) = 0.9 ] | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | average D = 29 | | | average D = 41 | | | average D = 57 | | | average D = 66 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 61 | 43.84 | mle | 61 | 40.46 | mle | 64 | 36.61 | mle | 68 | 31.90 |
| a=0.5 | chao | 69 | 41.35 | chao | 68 | 34.96 | chao | 72 | 28.94 | chao | 77 | 23.76 |
| b=1.0 | nj1 | 49 | 51.01 | nj1 | 64 | 36.38 | nj1 | 78 | 23.46 | nj1 | 83 | 17.91 |
| cv=0.89 | bov | 59 | 44.45 | bov | 69 | 33.79 | bov | 80 | 23.30 | bov | 85 | 18.20 |
| | O | 42 | 58.13 | O | 56 | 43.99 | O | 72 | 28.94 | O | 79 | 21.79 |
| | ca | 70 | 36.86 | ca | 72 | 30.43 | ca | 77 | 24.41 | ca | 81 | 20.10 |
| | | | | | | | | | | | | |
| | average D = 31 | | | average D = 44 | | | average D = 63 | | | average D = 74 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 75 | 39.84 | mle | 72 | 30.63 | mle | 74 | 27.04 | mle | 78 | 22.67 |
| a=0.75 | chao | 84 | 40.32 | chao | 79 | 26.97 | chao | 81 | 20.99 | chao | 85 | 16.02 |
| b=1.0 | nj1 | 53 | 47.42 | nj1 | 71 | 30.05 | nj1 | 88 | 14.64 | nj1 | 93 | 9.30 |
| cv=0.70 | bov | 67 | 38.06 | bov | 78 | 27.53 | bov | 89 | 15.74 | bov | 95 | 11.51 |
| | O | 45 | 55.32 | O | 62 | 38.90 | O | 80 | 20.61 | O | 89 | 12.58 |
| | ca | 84 | 35.81 | ca | 84 | 21.39 | ca | 88 | 14.74 | ca | 91 | 10.64 |
| | | | | | | | | | | | | |
| | average D = 33 | | | average D = 48 | | | average D = 70 | | | average D = 83 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 94 | 40.71 | mle | 88 | 21.42 | mle | 88 | 14.25 | mle | 90 | 11.04 |
| a=1.5 | chao | 104 | 52.74 | chao | 95 | 24.59 | chao | 93 | 11.95 | chao | 95 | 7.91 |
| b=1.0 | nj1 | 57 | 43.69 | nj1 | 79 | 22.86 | nj1 | 100 | 7.70 | nj1 | 105 | 8.07 |
| cv=0.44 | bov | 75 | 32.84 | bov | 90 | 22.19 | bov | 102 | 11.96 | bov | 106 | 10.49 |
| | O | 48 | 52.45 | O | 68 | 33.10 | O | 91 | 10.74 | O | 100 | 4.89 |
| | ca | 104 | 42.85 | ca | 101 | 19.92 | ca | 103 | 9.25 | ca | 104 | 6.95 |
| | | | | | | | | | | | | |
| | average D = 33 | | | average D = 49 | | | average D = 73 | | | average D = 88 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 106 | 59.40 | mle | 97 | 18.67 | mle | 95 | 9.28 | mle | 96 | 5.68 |
| a=3.0 | chao | 116 | 69.68 | chao | 103 | 23.72 | chao | 99 | 10.19 | chao | 99 | 5.42 |
| b=1.0 | nj1 | 59 | 42.04 | nj1 | 82 | 19.93 | nj1 | 107 | 10.37 | nj1 | 111 | 12.08 |
| cv=0.26 | bov | 78 | 31.00 | bov | 97 | 22.88 | bov | 108 | 13.34 | bov | 111 | 12.75 |
| | O | 49 | 51.20 | O | 70 | 30.58 | O | 97 | 7.36 | O | 106 | 7.42 |
| | ca | 117 | 63.46 | ca | 109 | 22.09 | ca | 111 | 14.44 | ca | 110 | 11.32 |
| | | | | | | | | | | | | |
| | average D = 33 | | | average D = 47 | | | average D = 70 | | | average D = 84 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 93 | 44.56 | mle | 87 | 20.79 | mle | 88 | 14.13 | mle | 90 | 10.46 |
| a=2.0 | chao | 103 | 49.59 | chao | 94 | 20.77 | chao | 94 | 11.96 | chao | 96 | 7.13 |
| b=2.0 | nj1 | 57 | 43.73 | nj1 | 78 | 23.43 | nj1 | 101 | 7.88 | nj1 | 106 | 8.60 |
| cv=0.44 | bov | 74 | 33.44 | bov | 90 | 21.95 | bov | 102 | 11.50 | bov | 107 | 9.72 |
| | O | 48 | 52.47 | O | 67 | 33.59 | O | 92 | 10.55 | O | 101 | 4.83 |
| | ca | 103 | 45.56 | ca | 99 | 17.49 | ca | 103 | 9.64 | ca | 105 | 7.37 |
| | | | | | | | | | | | | |
| | average D = 33 | | | average D = 49 | | | average D = 73 | | | average D = 87 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 103 | 41.82 | mle | 94 | 18.84 | mle | 95 | 9.43 | mle | 95 | 6.11 |
| a=5.0 | chao | 112 | 49.36 | chao | 101 | 22.83 | chao | 99 | 9.88 | chao | 99 | 5.62 |
| b=5.0 | nj1 | 59 | 41.70 | nj1 | 81 | 20.54 | nj1 | 106 | 9.68 | nj1 | 111 | 12.15 |
| cv=0.30 | bov | 79 | 30.87 | bov | 95 | 22.37 | bov | 108 | 14.33 | bov | 111 | 13.01 |
| | O | 50 | 50.90 | O | 70 | 31.15 | O | 96 | 7.16 | O | 106 | 7.26 |
| | ca | 113 | 45.06 | ca | 107 | 20.61 | ca | 110 | 13.64 | ca | 110 | 11.23 |

Table 4.4.2a

| N=400 | t = 89 | [ E(p) = 0.2 ] | | t = 143 | [ E(p) = 0.3 ] | | t = 277 | [ E(p) = 0.5 ] | | t = 644 | [ E(p) = 0.8 ] | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | |
| | average D = 80 | | | average D = 120 | | | average D = 200 | | | average D = 320 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case1 | mle | 447 | 192.1 | mle | 413 | 87.28 | mle | 402 | 37.49 | mle | 400 | 12.86 |
| const | chao | 468 | 208.3 | chao | 430 | 102.3 | chao | 412 | 44.87 | chao | 403 | 15.49 |
| cv=0.000 | nj1 | 150 | 250.4 | nj1 | 219 | 181.6 | nj1 | 338 | 64.32 | nj1 | 449 | 51.01 |
| | bov | 168 | 236.1 | bov | 284 | 138.4 | bov | 439 | 73.25 | bov | 450 | 52.75 |
| | O | 122 | 278.1 | O | 181 | 219.5 | O | 288 | 112.5 | O | 415 | 18.39 |
| | ca | 470 | 200.4 | ca | 444 | 99.36 | ca | 449 | 63.07 | ca | 458 | 60.53 |
| | | | | | | | | | | | | |
| | average D = 77 | | | average D = 114 | | | average D = 184 | | | average D = 281 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 327 | 132.6 | mle | 314 | 101.7 | mle | 312 | 92.01 | mle | 325 | 75.56 |
| a=1.0 | chao | 350 | 137.8 | chao | 333 | 95.03 | chao | 331 | 77.60 | chao | 348 | 54.02 |
| b=1.0 | nj1 | 143 | 257.4 | nj1 | 203 | 197.4 | nj1 | 300 | 101.7 | nj1 | 384 | 21.70 |
| cv=0.58 | bov | 179 | 226.2 | bov | 284 | 131.4 | bov | 361 | 69.29 | bov | 387 | 23.05 |
| | O | 117 | 283.0 | O | 170 | 230.8 | O | 259 | 141.5 | O | 356 | 45.56 |
| | ca | 351 | 124.3 | ca | 345 | 79.66 | ca | 358 | 51.55 | ca | 382 | 23.05 |
| | | | | | | | | | | | | |
| | average D = 79 | | | average D = 118 | | | average D = 194 | | | average D = 305 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 399 | 156.4 | mle | 376 | 74.90 | mle | 365 | 47.01 | mle | 368 | 34.37 |
| a=1.0 | chao | 423 | 173.5 | chao | 394 | 86.28 | chao | 377 | 44.57 | chao | 377 | 27.97 |
| b=0.25 | nj1 | 148 | 252.4 | nj1 | 214 | 186.8 | nj1 | 324 | 78.02 | nj1 | 421 | 25.61 |
| cv=0.33 | bov | 169 | 234.7 | bov | 292 | 129.9 | bov | 408 | 61.68 | bov | 422 | 28.37 |
| | O | 121 | 279.5 | O | 177 | 223.2 | O | 278 | 123.1 | O | 390 | 15.40 |
| | ca | 423 | 160.5 | ca | 407 | 74.63 | ca | 411 | 36.08 | ca | 425 | 29.94 |
| | | | | | | | | | | | | |
| | average D = 78 | | | average D = 116 | | | average D = 189 | | | average D = 294 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 360 | 134.6 | mle | 345 | 83.76 | mle | 340 | 66.71 | mle | 349 | 52.77 |
| a=1.0 | chao | 380 | 144.3 | chao | 362 | 82.84 | chao | 354 | 59.34 | chao | 363 | 40.39 |
| b=0.5 | nj1 | 146 | 254.9 | nj1 | 209 | 191.7 | nj1 | 313 | 88.75 | nj1 | 404 | 15.08 |
| cv=0.45 | bov | 173 | 231.5 | bov | 287 | 131.1 | bov | 385 | 61.82 | bov | 405 | 18.08 |
| | O | 119 | 281.2 | O | 174 | 226.7 | O | 269 | 131.3 | O | 375 | 27.88 |
| | ca | 382 | 132.2 | ca | 376 | 70.20 | ca | 386 | 36.21 | ca | 405 | 16.45 |
| | | | | | | | | | | | | |
| | average D = 78 | | | average D = 115 | | | average D = 186 | | | average D = 286 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 346 | 148.0 | mle | 325 | 94.74 | mle | 324 | 81.01 | mle | 335 | 66.36 |
| a=1.0 | chao | 368 | 150.7 | chao | 342 | 90.73 | chao | 340 | 69.47 | chao | 354 | 48.75 |
| b=0.75 | nj1 | 144 | 256.1 | nj1 | 205 | 195.3 | nj1 | 305 | 96.10 | nj1 | 392 | 17.22 |
| cv=0.52 | bov | 176 | 229.3 | bov | 283 | 132.9 | bov | 369 | 64.80 | bov | 395 | 22.35 |
| | O | 118 | 282.0 | O | 171 | 229.3 | O | 264 | 137.1 | O | 364 | 38.34 |
| | ca | 369 | 143.0 | ca | 356 | 75.59 | ca | 369 | 42.96 | ca | 392 | 17.83 |
| | | | | | | | | | | | | |
| | average D = 75 | | | average D = 110 | | | average D = 174 | | | average D = 263 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 273 | 151.1 | mle | 267 | 139.8 | mle | 272 | 130.0 | mle | 296 | 104.7 |
| a=1.0 | chao | 295 | 141.2 | chao | 290 | 122.8 | chao | 304 | 101.5 | chao | 337 | 66.05 |
| b=3.0 | nj1 | 138 | 262.7 | nj1 | 193 | 207.5 | nj1 | 279 | 121.8 | nj1 | 360 | 42.89 |
| cv=0.77 | bov | 182 | 223.7 | bov | 268 | 143.0 | bov | 332 | 86.26 | bov | 368 | 41.73 |
| | O | 114 | 286.7 | O | 162 | 238.2 | O | 243 | 157.8 | O | 332 | 68.76 |
| | ca | 296 | 133.5 | ca | 299 | 110.9 | ca | 320 | 84.49 | ca | 355 | 47.50 |

Table 4.4.2b

| N=400 | t = 89 | | [ E(p) = 0.2 ] | t = 143 | | [ E(p) = 0.3 ] | t = 277 | | [ E(p) = 0.5 ] | t = 644 | | [ E(p) = 0.8 ] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | |
| | average D = 74 | | | average D = 107 | | | average D = 164 | | | average D = 239 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 238 | 175.3 | mle | 235 | 168.5 | mle | 240 | 160.7 | mle | 261 | 139.5 |
| a=0.5 | chao | 257 | 161.8 | chao | 254 | 152.6 | chao | 264 | 138.5 | chao | 293 | 108.1 |
| b=1.0 | nj1 | 133 | 267.0 | nj1 | 184 | 216.5 | nj1 | 257 | 143.5 | nj1 | 318 | 83.28 |
| cv=0.89 | bov | 181 | 223.9 | bov | 249 | 158.2 | bov | 289 | 119.1 | bov | 325 | 79.19 |
| | O | 111 | 289.7 | O | 156 | 244.8 | O | 226 | 174.2 | O | 296 | 104.3 |
| | ca | 260 | 155.8 | ca | 265 | 140.2 | ca | 283 | 119.2 | ca | 311 | 90.05 |
| | | | | | | | | | | | | |
| | average D = 76 | | | average D = 112 | | | average D = 177 | | | average D = 266 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 295 | 152.0 | mle | 282 | 126.6 | mle | 284 | 118.9 | mle | 301 | 99.80 |
| a=0.75 | chao | 318 | 149.4 | chao | 300 | 115.2 | chao | 305 | 100.1 | chao | 328 | 73.92 |
| b=1.0 | nj1 | 140 | 260.5 | nj1 | 196 | 204.1 | nj1 | 284 | 116.7 | nj1 | 359 | 40.02 |
| cv=0.70 | bov | 181 | 224.4 | bov | 274 | 138.1 | bov | 333 | 84.40 | bov | 364 | 42.07 |
| | O | 115 | 285.1 | O | 165 | 235.6 | O | 247 | 153.2 | O | 334 | 66.91 |
| | ca | 318 | 139.1 | ca | 313 | 100.2 | ca | 329 | 76.24 | ca | 356 | 46.68 |
| | | | | | | | | | | | | |
| | average D = 79 | | | average D = 117 | | | average D = 190 | | | average D = 297 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 370 | 135.4 | mle | 348 | 83.46 | mle | 345 | 62.85 | mle | 354 | 47.59 |
| a=1.5 | chao | 392 | 146.2 | chao | 364 | 82.85 | chao | 361 | 54.67 | chao | 371 | 33.70 |
| b=1.0 | nj1 | 146 | 254.3 | nj1 | 209 | 191.3 | nj1 | 316 | 86.21 | nj1 | 410 | 17.95 |
| cv=0.44 | bov | 174 | 231.3 | bov | 289 | 128.9 | bov | 394 | 59.45 | bov | 413 | 24.22 |
| | O | 120 | 280.7 | O | 174 | 226.4 | O | 271 | 129.6 | O | 380 | 23.25 |
| | ca | 393 | 134.6 | ca | 379 | 70.86 | ca | 392 | 34.32 | ca | 413 | 20.13 |
| | | | | | | | | | | | | |
| | average D = 79 | | | average D = 119 | | | average D = 197 | | | average D = 312 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 413 | 158.4 | mle | 390 | 74.41 | mle | 380 | 39.40 | mle | 383 | 21.39 |
| a=3.0 | chao | 436 | 174.0 | chao | 407 | 84.39 | chao | 393 | 42.21 | chao | 392 | 18.39 |
| b=1.0 | nj1 | 149 | 251.5 | nj1 | 216 | 184.7 | nj1 | 330 | 72.00 | nj1 | 435 | 38.27 |
| cv=0.26 | bov | 167 | 237.0 | bov | 289 | 133.2 | bov | 425 | 68.52 | bov | 437 | 41.51 |
| | O | 121 | 278.9 | O | 179 | 221.7 | O | 282 | 118.5 | O | 402 | 11.59 |
| | ca | 437 | 164.1 | ca | 421 | 79.32 | ca | 427 | 45.90 | ca | 442 | 44.96 |
| | | | | | | | | | | | | |
| | average D = 79 | | | average D = 117 | | | average D = 191 | | | average D = 298 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 369 | 153.7 | mle | 345 | 85.44 | mle | 347 | 61.22 | mle | 355 | 46.55 |
| a=2.0 | chao | 393 | 161.6 | chao | 363 | 83.02 | chao | 365 | 52.16 | chao | 375 | 30.02 |
| b=2.0 | nj1 | 146 | 254.3 | nj1 | 210 | 191.2 | nj1 | 317 | 85.35 | nj1 | 413 | 19.58 |
| cv=0.45 | bov | 175 | 230.1 | bov | 287 | 131.2 | bov | 397 | 58.57 | bov | 416 | 25.05 |
| | O | 120 | 280.7 | O | 174 | 226.2 | O | 272 | 129.0 | O | 382 | 21.89 |
| | ca | 393 | 152.7 | ca | 376 | 71.18 | ca | 395 | 34.72 | ca | 415 | 21.90 |
| | | | | | | | | | | | | |
| | average D = 79 | | | average D = 118 | | | average D = 196 | | | average D = 310 | | |
| | method | mean | rmse | method | mean | rmse | method | mean | rmse | method | mean | rmse |
| Case2 | mle | 415 | 168.8 | mle | 377 | 76.07 | mle | 377 | 41.42 | mle | 380 | 24.23 |
| a=5.0 | chao | 443 | 199.0 | chao | 395 | 84.30 | chao | 391 | 43.29 | chao | 392 | 18.91 |
| b=5.0 | nj1 | 148 | 252.7 | nj1 | 214 | 186.6 | nj1 | 329 | 73.47 | nj1 | 434 | 36.81 |
| cv=0.30 | bov | 166 | 237.2 | bov | 291 | 130.5 | bov | 420 | 66.15 | bov | 436 | 40.10 |
| | O | 120 | 279.8 | O | 177 | 223.1 | O | 281 | 119.6 | O | 400 | 11.92 |
| | ca | 440 | 177.6 | ca | 408 | 75.55 | ca | 424 | 44.46 | ca | 440 | 42.90 |

## Chapter 5 : Conclusions

### § 5.1 : Initial Objectives and Results Obtained

The initial aim of this thesis was to begin a systematic investigation into the method of plant-capture when applied to populations behaving according to the closed capture-recapture models of Otis et al. (1978) and their continuous time analogues.

Chapter 1 considers the addition of plants to target populations which behave according to the most basic Otis et al. (1978) model $M_0$. A plant-capture generalisation of the standard maximum likelihood estimator, that was discussed by Otis et al. (1978), is derived. Other new estimators are also introduced. A near-unbiased estimator, described as a conditionally unbiased estimator (CUE), which was originally considered in a rather simpler urn theory context, is shown to be more satisfactory than the maximum likelihood estimator and the Peterson-type estimator. This latter conclusion holds either in the presence or absence of planted individuals, and hence for the standard capture-recapture model it is recommended that the CUE be preferred to the usual maximum likelihood estimator of Otis et al. (1978).

In chapter 2 consideration is given to the use of plants in connection with a continuous time analogue of the Otis et al. (1978) model $M_0$. In the absence of plants this model is equivalent to the recapture debugging model of Nayak(1988). The maximum likelihood estimator of Nayak(1988), which was designed for situations in which no plants are used, is generalised and other new estimators are introduced. As well as estimators corresponding to those of chapter 1, harmonic mean estimators are considered. Again, however, in situations where plants are deployed and in those which no plants are used, it is the near-unbiased estimator, described as a conditionally unbiased estimator, which is seen to give the best results.

Difficulties arose when investigating the usefulness of plants when used in connection with populations behaving according to the important heterogeneity model $M_h$. The main problem was that the most commonly preferred estimators, which for the model $M_h$ have been the jackknife estimators of Burnham and Overton(1978, 1979) and the estimators of Chao, Lee and Jeng(1992), cannot easily be generalised in a way which allows them to use the extra information gained from the plants. The approach taken to overcome this problem was to seek new estimators for the standard capture-recapture problem with a view to finding an estimator that could be generalised for use in a plant-capture scenario. This approach proved fruitful and led to a number of new estimators and estimation procedures for the standard capture-recapture model $M_h$. Of the new estimators that were obtained, the most satisfactory were found to be the coverage adjusted estimators, which are presented in chapter 3. Also within chapter 3 it is shown how these coverage adjusted estimators can be modified in a way which allows them to utilize the information gained from the planted individuals. In the absence of plants, the coverage adjusted estimators are shown, through simulation and real data, to compare favourably with other estimators.

It was recognised that the approach taken in chapter 3 could be modified to produce a new estimator for the continuous time analogue of the Otis et al. (1978) model $M_h$. Within chapter 4, this new coverage adjusted estimator was shown using a simulation study to perform more satisfactorily than other estimators that have been proposed for this model.

The emphasis throughout this thesis has been on point estimation. Most practitioners, however, would also require reasonable variance estimates. An appropriate extension of the work contained in this thesis would, therefore, be to develop estimators of variance. This could be done analytically or using computer-intensive methods.

Throughout this thesis it has been concluded that the use of plants is beneficial. It is important to keep in mind that this conclusion has been reached under the assumption that the planted individuals behave in an identical manner to those of the target population. Further work should, therefore, investigate the robustness of the conclusions reached in the case where the behaviour of the planted individuals is different to those of the target population. The development of procedures to test whether this central assumption does indeed hold would also be appropriate. Some work has already been done in this area : in a personal communication K. Pollock showed that in the component $P_4$ of the probability function 3.11 of section 3.7.3 chapter 3, we have a multiple hypergeometric distribution which gives rise to a contingency table test of the assumption that the planted individuals behave in an identical manner to those of the target population. This test could be used in connection with target populations behaving according to the Otis et al. (1978) models $M_0$ and $M_h$. Similar contingency table tests would be appropriate for the models $M_b$ and $M_{bh}$, K. Pollock( pers. com. ).

Within each of chapters 1, 2 and 3 a Peterson-type estimator was derived. In each chapter it has exactly the same functional form, and is based only on the number of distinct individuals seen from the planted and target populations. An important feature of this estimator is that, when the plants do behave as members of the target population, it is on the whole unbiased. Within chapters 1 and 2, however, the Peterson-type estimator did not perform well in comparison to the other estimators : although its bias tended to be low, its variance, particularly when the number of plants used was small relative to the size of the target population, was relatively large. In chapter 3 the performance of the Peterson-type estimator improved in relation to that of the other estimators; the consistent near unbiasedness of the Peterson-type estimator becoming more useful in the presence of heterogeneity. Within chapters 1 and 2 each of the models considered had considerable structure which benefited the other estimators and enabled them to perform better than the Peterson-type estimator. This Peterson-type estimator is in fact suitable for use in plant-capture scenarios when the target population behaves according to any of the eight closed capture-recapture models of Otis et al. (1978) or their continuous time analogues. In each of these situations, so long as the planted individuals do behave in an identical manner to those of the target population, the Peterson-type estimator would remain nearly unbiased. Hence for the more complicated, less structured, models the Peterson-type estimator should prove to be a much stronger candidate.

## Appendix 1 : The Classical Occupancy Distribution

Suppose Z balls are dropped independently into N urns in such a way that the probability of any one ball being allocated to any one urn is 1/N. Let X denote the number of occupied urns after the Z balls have been dropped, i.e. X denotes the number of urns containing at least one ball after the Z balls have been dropped. The distribution of X given the value of Z is known as the Classical Occupancy distribution, see Johnson and Kotz(1977) p.110. The conditional probability function of X given Z can be written as $P(X = x|Z = z) = N^{-z}\binom{N}{x}x!S(x,z)$, $x=1,2,3.....,\min(N,z)$,

where S(x,z) is a Stirling number of the second kind defined by

$$S(x,z) = \frac{1}{x!}\sum_{k=0}^{x}\binom{x}{k}(-1)^k(x-k)^z.$$

Proof :

As a first step in this proof consider the following formula :

Boole's Formula : The Inclusion Exclusion Principle

$$P\left[\bigcup_{i=1}^{x}A_i\right] = \sum_{k=1}^{x}(-1)^{k-1}S_k,$$

$$\text{where } S_k = \sum_{\substack{\text{all subsets}\\ \text{of size } k}} P(A_{i1},A_{i2},A_{i3},....,A_{ik}).$$

Now define the events B and $A_i$ in the following way

B ≡ a particular N-x urns remain empty
( i.e. z balls are restricted to the other x urns )

$A_i$ ≡ a particular N-x urns remain empty but urn i of the remaining x urns also remains empty.

$$\begin{aligned} P[\text{ B, all x urns are occupied }] &= P[B] - P[\text{ B, at least one of the x urns is empty }] \\ &= P[B] - P\left[\bigcup_{i=1}^{x}A_i\right] \\ &= P[B] - \sum_{k=1}^{x}(-1)^{k-1}S_k, \qquad \text{using Boole's formula.} \\ &= \left(\frac{x}{N}\right)^z - \sum_{k=1}^{x}(-1)^{k-1}\left(\frac{x-k}{N}\right)^z\binom{x}{k}, \qquad \text{(A1.1)} \end{aligned}$$

$$\text{since } S_k = \sum_{\substack{\text{all subsets}\\ \text{of size } k}} P(A_{i1},A_{i2},A_{i3},....,A_{ik}) = \left(\frac{x-k}{N}\right)^z\binom{x}{k}.$$

From (A1.1) it follows that

$$\begin{aligned} \text{P[ B, all x urns are occupied ]} &= \left(\frac{1}{N}\right)^z \left[ x^z + \sum_{k=1}^{x} \binom{x}{k} (-1)^k (x-k)^z \right] \\ &= N^{-z} \sum_{k=0}^{x} \binom{x}{k} (-1)^k (x-k)^z \\ &= N^{-z} x! S(x,z). \end{aligned}$$

But a particular subset of size x can be chosen in $\binom{N}{x}$ ways. Hence

$$P(X = x | Z = z) = N^{-z} \binom{N}{x} x! S(x,z), \qquad x = 1, 2, ...., \min(N, z).$$

Appendix 2 : The Stirling distribution of the second kind

Suppose that $X_i$ has a Poisson distribution with mean $\lambda$. Then the random variable $Y=\sum_{i=1}^{n} X_i | X_i > 0$ is said to have the Stirling distribution of the second kind, see Patil et al(1984). That is a random variable is said to have the Stirling distribution of the second kind if it can be represented as the sum of a number of zero truncated Poisson random variables.

The probability function of $X_i$ is given by $P(X_i = x_i) = \dfrac{\lambda^{x_i}\exp(-\lambda)}{x_i!}$, $\quad x_i$=0,1,2,....

It follows that $\quad P(X_i = x_i | x_i > 0) = \dfrac{\lambda^{x_i}\exp(-\lambda)}{x_i!(1-\exp(-\lambda))}$, $\quad x_i$=1,2,3,....

The probability generating function of $X_i | X_i > 0$ is given by

$$\begin{aligned}
G_{X_i|X_i>0}(t) &= E\left(t^{X_i|X_i>0}\right) \\
&= \sum_{x_i=1}^{\infty} t^{x_i}.\frac{\lambda^{x_i}\exp(-\lambda)}{x_i!(1-\exp(-\lambda))} \\
&= \frac{\exp(-\lambda)}{1-\exp(-\lambda)}.\left[\sum_{x_i=0}^{\infty}\frac{(t\lambda)^{x_i}}{x_i!}-1\right] \\
&= \frac{\exp(-\lambda)}{1-\exp(-\lambda)}.\left[\exp(\lambda t)-1\right] \\
&= \frac{\exp(\lambda t)-1}{\exp(\lambda)-1}.
\end{aligned}$$

The probability generating function of Y is then

$$\begin{aligned}
G_Y(t) = G_{\sum_{i=1}^{n} X_i|X_i>0}(t) &= E\left(t^{\sum_{i=1}^{n} X_i|X_i>0}\right) = \prod_{i=1}^{n} G_{X_i|X_i>0}(t) \\
&= \left[\frac{\exp(\lambda t)-1}{\exp(\lambda)-1}\right]^n \\
&= \frac{1}{(\exp(\lambda)-1)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\exp(k\lambda t) \\
&= \frac{1}{(\exp(\lambda)-1)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\sum_{r=0}^{\infty}\frac{(k\lambda)^r t^r}{r!}.
\end{aligned}$$

The probability of Y taking the value r is equal to the coefficient of $t^r$ in the expansion of $G_Y(t)$. Hence

$$P(Y=r)=\frac{\lambda^r}{r!(\exp(\lambda)-1)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^r$$
$$=\frac{n!}{r!}.\frac{\lambda^r S(n,r)}{(\exp(\lambda)-1)^n}, \qquad r=n, n+1, n+2, ....$$

where S(n,r) is a Stirling number of the second kind.

## Appendix 3 : The Distribution Function of A Sum of Zero Truncated Binomial Random Variables

Suppose that $Y_j$, for j = 1, 2, ..., x, are independently and identically distributed Binomial random variables, each having parameters t and p. Here consideration is given to the distribution of $Q = \sum_{j=1}^{x} Y_j \big| Y_j > 0$. This distribution was first derived by Ahuja(1970) and, more recently, was inspected via a power series approach by Charalambides and Singh(1988). The approach taken here is to obtain the distribution of Q using probability generating functions :

Firstly, given that $Y_j \sim \mathrm{Bin}(t,p)$,

$$\mathrm{Prob}\left(Y_j = y_j\right) = \binom{t}{y_j} p^{y_j}(1-p)^{t-y_j}, \qquad y_j = 0,1,2,....,t.$$

$$\mathrm{Prob}\left(Y_j = y_j \big| Y_j > 0\right) = \frac{\binom{t}{y_j} p^{y_j}(1-p)^{t-y_j}}{1-(1-p)^t}, \qquad y_j = 1,2,....,t.$$

The probability generating function of $Y_j \big| Y_j > 0$ is then given by

$$\begin{aligned}
G_{Y_j|Y_j>0}(m) &= E\left(m^{Y_j|Y_j>0}\right) \\
&= \sum_{y_j=1}^{t} m^{y_j} . \frac{\binom{t}{y_j} p^{y_j}(1-p)^{t-y_j}}{1-(1-p)^t} \\
&= \frac{1}{1-(1-p)^t} \sum_{y_j=1}^{t} \binom{t}{y_j} (mp)^{y_j}(1-p)^{t-y_j} \\
&= \frac{1}{1-(1-p)^t} \left[ \sum_{y_j=0}^{t} \binom{t}{y_j} (mp)^{y_j}(1-p)^{t-y_j} - (1-p)^t \right] \\
&= \frac{1}{1-(1-p)^t} \left[ (1-p+mp)^t - (1-p)^t \right]. \qquad (A3.1)
\end{aligned}$$

The probability generating function of $Q = \sum_{j=1}^{x} Y_j \big| Y_j > 0$ may now be obtained as follows :

$$\begin{aligned}
G_Q(m) = E\left(m^Q\right) &= E\left( m^{\sum_{j=1}^{x} Y_j|Y_j>0} \right) = E\left( \prod_{j=1}^{x} m^{Y_j|Y_j>0} \right) \\
&= \prod_{j=1}^{x} E\left(m^{Y_j|Y_j>0}\right) \qquad \text{by independence} \\
&= \prod_{j=1}^{x} G_{Y_j|Y_j>0}(m)
\end{aligned}$$

$$= \prod_{j=1}^{x} \frac{1}{1-(1-p)^t}\left[(1-p+mp)^t - (1-p)^t\right], \qquad \text{from (A3.1)}$$

$$= \frac{1}{\left[1-(1-p)^t\right]^x}\left[(1-p+mp)^t - (1-p)^t\right]^x$$

$$= \frac{1}{\left[1-(1-p)^t\right]^x}\sum_{r=0}^{x}\binom{x}{r}(1-p+mp)^{tr}(-1)^{x-r}(1-p)^{tx-tr}$$

$$= \frac{1}{\left[1-(1-p)^t\right]^x}\sum_{r=0}^{x}\binom{x}{r}(-1)^{x-r}(1-p)^{tx-tr}\sum_{q=0}^{tr}\binom{tr}{q}(1-p)^{tr-q}p^q m^q.$$

It then follows that

$$\text{Prob}(Q=q) = \frac{1}{\left[1-(1-p)^t\right]^x}\sum_{r=0}^{x}\binom{x}{r}(-1)^{x-r}(1-p)^{tx-tr}\binom{tr}{q}(1-p)^{tr-q}p^q$$

$$= \frac{p^q(1-p)^{tx-q}}{\left[1-(1-p)^t\right]^x}\sum_{r=0}^{x}\binom{x}{r}\binom{tr}{q}(-1)^{x-r}, \qquad q = x, x+1, x+2, \ldots\ldots, tx.$$

## Appendix 4 : Estimation of Sample Coverage : Model $M_h$ : Discrete Time Sampling Procedure

This appendix describes how one may estimate the quantity referred to as 'sample coverage' within chapter 3. For notation and relevant background please refer to chapter 3. The three estimators of sample coverage that are described below were considered by Chao, Lee and Jeng(1992).

Sample Coverage (C), is defined as follows :

$$C = \frac{\sum_{i=1}^{N} p_i I(X_i > 0)}{\sum_{i=1}^{N} p_i}, \qquad \text{where} \qquad I(X_i > 0) = \begin{cases} 1 & \text{w.p.} \quad 1-(1-p_i)^t \\ 0 & \text{w.p.} \quad (1-p_i)^t \end{cases}.$$

As a first step to estimating this quantity, consider its expectation :

$$\begin{aligned} E(C) &= E\left[\frac{\sum_{i=1}^{N} p_i I(X_i > 0)}{\sum_{i=1}^{N} p_i}\right] \\ &= 1 - \frac{\sum_{i=1}^{N} p_i (1-p_i)^t}{\sum_{i=1}^{N} p_i}. \end{aligned} \tag{A4.1}$$

An estimate of C may be obtained from equation (A4.1) by substituting in estimates of $\sum_{i=1}^{N} p_i (1-p_i)^t$ and $\sum_{i=1}^{N} p_i$.

As in section 3.3 , using equation (3.8), an estimate of $\sum_{i=1}^{N} p_i$ is given by $\frac{z}{t}$.

To obtain an estimate of $\sum_{i=1}^{N} p_i (1-p_i)^t$ consider the expansion :

$$\begin{aligned} \sum_{i=1}^{N} p_i (1-p_i)^t &= \sum_{i=1}^{N} p_i (1-p_i)^{t-1} - \sum_{i=1}^{N} p_i^2 (1-p_i)^{t-1} \\ &= \frac{1}{\binom{t}{1}} E(f_1) - \sum_{i=1}^{N} p_i^2 (1-p_i)^{t-2} + \sum_{i=1}^{N} p_i^3 (1-p_i)^{t-2} \\ &= \frac{1}{\binom{t}{1}} E(f_1) - \frac{1}{\binom{t}{2}} E(f_2) + \sum_{i=1}^{N} p_i^3 (1-p_i)^{t-3} - \sum_{i=1}^{N} p_i^4 (1-p_i)^{t-3} \\ &= \frac{1}{\binom{t}{1}} E(f_1) - \frac{1}{\binom{t}{2}} E(f_2) + \frac{1}{\binom{t}{3}} E(f_3) + \ldots\ldots + \frac{(-1)^{t+1}}{\binom{t}{t}} E(f_t) + (-1)^{t+2} \sum_{i=1}^{N} p_i^{t+1}. \end{aligned}$$

That is

$$\sum_{i=1}^{N} p_i(1-p_i)^t = \sum_{j=1}^{t} \frac{(-1)^{j+1}}{\binom{t}{j}} E(f_j) + (-1)^{t+2} \sum_{i=1}^{N} p_i^{t+1}$$

$$\Rightarrow \quad \sum_{i=1}^{N} p_i(1-p_i)^t \approx \sum_{j=1}^{t} \frac{(-1)^{j+1}}{\binom{t}{j}} E(f_j)$$

$\Rightarrow$ an estimate of $\sum_{i=1}^{N} p_i(1-p_i)^t$ is given by $\sum_{j=1}^{t} \frac{(-1)^{j+1}}{\binom{t}{j}} f_j$.

Now returning to equation (A4.1) it is seen that an estimate of sample coverage C is given by

$$\hat{C} = 1 - \frac{\sum_{j=1}^{t} \frac{(-1)^{j+1}}{\binom{t}{j}} f_j}{\frac{z}{t}}$$

$$= 1 - \frac{t}{z} \sum_{j=1}^{t} \frac{(-1)^{j+1}}{\binom{t}{j}} f_j. \qquad \text{(A4.2)}$$

Computation has shown that, in almost all situations, only the first few terms in the above summation are significant. For this reason we take the approach of Chao, Lee and Jeng(1992) and truncate the summation in (A4.2) in order to obtain the following three estimators of sample coverage :

$$\hat{C}_1 = 1 - \frac{f_1}{z},$$

$$\hat{C}_2 = 1 - \frac{f_1}{z} + \frac{2}{(t-1)} \frac{f_2}{z}$$

and $$\hat{C}_3 = 1 - \frac{f_1}{z} + \frac{2}{(t-1)} \frac{f_2}{z} - \frac{6}{(t-1)(t-2)} \frac{f_3}{z}.$$

Using the whole summation of equation (A4.2) is not recommended. This is mainly due to the fact that the higher frequencies tend to posses large variability : elements in the summation which incorporate these quantities can occasionally distort the resulting estimate of sample coverage.

## Appendix 5 : Estimation of Sample Coverage : Model $M_h$ : Continuous Time Sampling Procedure

This appendix describes how one may estimate the quantity referred to as 'sample coverage' within chapter 4. For notation and relevent background please refer to chapter 4. The following derivation appeared in Chao, Ma and Yang(1993).

Firstly note that $$I(X_i > 0) = \begin{cases} 1 & \text{w.p.} \quad 1-\exp(-\lambda_i t) \\ 0 & \text{w.p.} \quad \exp(-\lambda_i t) \end{cases}.$$

$$\Rightarrow \quad E[I(X_i > 0)] = 1 - \exp(-\lambda_i t). \qquad \text{(A5.1)}$$

The expectation of sample coverage C may now be written as

$$E(C) = E\left[\frac{\sum_{i=1}^{N} \lambda_i I(X_i > 0)}{\sum_{j=1}^{N} \lambda_j}\right]$$

$$= \frac{\sum_{i=1}^{N} \lambda_i E[I(X_i > 0)]}{\sum_{j=1}^{N} \lambda_j}$$

$$= \frac{\sum_{i=1}^{N} \lambda_i [1 - \exp(-\lambda_i t)]}{\sum_{j=1}^{N} \lambda_j}, \qquad \text{from A5.1.}$$

$$= 1 - \frac{\sum_{i=1}^{N} \lambda_i \exp(-\lambda_i t)}{\sum_{j=1}^{N} \lambda_j}$$

$$= 1 - \frac{\sum_{i=1}^{N} \lambda_i t.\exp(-\lambda_i t)}{t\sum_{j=1}^{N} \lambda_j}. \qquad \text{(A5.2)}$$

Now $X_i \sim P(\lambda_i t)$

$$\Rightarrow z = \sum_{i=1}^{N} X_i \sim P\left(t\sum_{i=1}^{N} \lambda_i\right) \Rightarrow \text{z is a good estimate of } t\sum_{i=1}^{N} \lambda_i.$$

Consider the quantity $f_1 = \sum_{i=1}^{N} I(X_i = 1)$,

$$\text{where} \quad I(X_i = 1) = \begin{cases} 1 & \text{w.p.} \quad \lambda_i t.\exp(-\lambda_i t) \\ 0 & \text{w.p.} \quad 1 - \lambda_i t.\exp(-\lambda_i t) \end{cases}.$$

It follows that

$$E[f_1] = \sum_{i=1}^{N} E[I(X_i = 1)]$$
$$= \sum_{i=1}^{N} \lambda_i t.\exp(-\lambda_i t).$$

$\Rightarrow$ $f_1$ may be used as an estimator of $\sum_{i=1}^{N} \lambda_i t.\exp(-\lambda_i t)$.

Now using equation A5.2 it follows that the value of C may be estimated by $\hat{C} = 1 - \frac{f_1}{z}$.

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