CONTRIBUTIONS TO THE THEORY OF OCKHAM ALGEBRAS

Jie Fang

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1992

CONTRIBUTIONS

TO THE THEORY OF

OCKHAM ALGEBRAS

BY

JIE FANG

A thesis submitted for the degree of Doctor of Philosophy

of the University of St Andrews

Department of Mathematical and

Computational Sciences

1991

Th A 1465

DECLARATIONS

I, Jie Fang, hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for a higher degree.

Signed Date 19 July 1991

I was admitted to the Faculty of Science of the University of St Andrews under Ordinance General No. 12 on 1/10/88 and as a candidate for the degree of Ph.D. on 1/10/89.

Signed Date 19 July 1991

In submitting this thesis to the University of St Andrews I understand that I am giving permission for it to be made available for use in accordance with the regulations of the University Library for the time being in force, subject to any copyright vested in the work not being affected thereby. I also understand that the title and abstract may be published, and that a copy of the work may be made and supplied to any *bona fide* library or research worker.

CERTIFICATION

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the Degree of Ph.D.

Signature of Supervisor .

Date 19 July 91

PREFACE

I am most deeply indebted to my supervisor Professor T. S. Blyth, under whose supervision this work has been carried out; for his constant encouragement and invaluable guidance, for introducing me to research in lattice theory, and for his great help in reading and correcting the manuscript.

I wish to express my appreciation to Professor J. C. Varlet for the opportunity of reading many of his manuscripts.

I also wish to express my thanks for the assistance I received from many of the staff in the Mathematical Institute, in particular Dr. C. M. Campbell, Professor J. M. Howie, Dr. G. M. Phillips, Dr. E. F. Robertson and Miss K. P. Dunne.

Many thanks are also due to the University of St Andrews, and to the Committee of Vice-Chancellors and Principals in U.K., for the research scholarship and financial support which enabled me to undertake research.

This thesis is dedicated to my parents and my wife whose encouragement and love have inspired me in pursuing my studies, and to the memory of my kind grandmother.

Finally, I would like also to dedicate this thesis to many friends of mine who have given me a lot of love and friendship.

ABSTRACT

In the first part of this thesis we consider particular ordered sets (connected and of small height) and determine the cardinality of the corresponding dual **MS** - algebra and of its set of fixed points.

The remainder of the thesis is devoted to a study of congruences of Ockham algebras and a generalised variety $\mathbf{K}_\omega$ of Ockham algebras that contains all of the Berman varieties $\mathbf{K}_{p,q}$. In particular we consider the congruences Φ_i (i = 1, 2, ...) defined on an Ockham algebra (L; f) by

$$(x, y) \in \Phi_i \iff f^i(x) = f^i(y)$$

and show that $(L; f) \in \mathbf{K}_\omega$ is subdirectly irreducible if and only if the lattice of congruences of L reduces to the chain

$$\omega = \Phi_0 \preceq \Phi_1 \preceq \Phi_2 \preceq \dots \preceq \Phi_\omega \prec \iota$$

where $\Phi_\omega = \bigvee_{i \geq 0} \Phi_i$. Finally we obtain a characterisation of the finite simple Ockham algebras.

CONTENTS

Declarations		i
Certification		ii
Preface		iii
Abstract		iv
1	INTRODUCTION	1
2	SOME FINITE MS - ALGEBRAS	11
2.1	Double fences	12
2.2	Extended double fences	19
2.3	Half tiaras	30
2.4	Tiaras	33
2.5	Double crowns	36
2.6	Tall double fences	41
3	CONGRUENCE LATTICES	47
4	SUBDIRECTLY IRREDUCIBLE OCKHAM ALGEBRAS	63
4.1	Weakly subdirectly irreducible Ockham algebras	64
4.2	The generalised variety $\mathbf{K}_\omega$	69
4.3	Totally ordered subdirectly irreducible Ockham algebras	83
4.4	Ockham algebras generated by finite subdirectly irreducible algebras	87
5	FINITE SIMPLE OCKHAM ALGEBRAS	91
	REFRENCES	107

CHAPTER 1

INTRODUCTION

An *Ockham algebra* is an algebra $< L; \wedge, \vee, f, 0, 1 >$ of type $< 2, 2, 1, 0, 0 >$ such that $< L; \wedge, \vee, 0, 1 >$ is a bounded distributive lattice and f is a unary operation defined on L such that, for all $x, y \in L$,

$$f(x \wedge y) = f(x) \vee f(y), \quad f(x \vee y) = f(x) \wedge f(y)$$

$$f(1) = 0, \quad f(0) = 1.$$

Thus f is a dual endomorphism on L. The class of Ockham algebras is a variety which will be denoted by **O**.

The study of these algebras has been initiated by J. Berman [2] who gave particular attention to certain subvarieties $\mathbf{K}_{p,q}$. The main results in [2] are that the class of Ockham algebras satisfies the congruence extension property and that $\mathbf{K}_{p,q}$ has only finitely many subdirectly irreducible algebras all of which are themselves finite. Afterwards, A. Urquhart [27] obtained a description of the dual spaces, based on H. A. Priestley's order-topological duality for bounded distributive lattices [23, 24]. This work was further developed by M. S. Goldberg [20], and by T.S. Blyth and J. C. Varlet [7] who introduced the notion of an **MS**-*algebra* as a common abstraction of a de

Morgan algebra and a Stone algebra. Blyth and Varlet proved that there are, up to isomorphism, nine subdirectly irreducible algebras in the subclass of **MS**-algebras and exhibited their Hasse diagrams. The methods employed in [7] were generalised by Beazer [5]. Beazer and, independently Sankappanavar [26] showed that there are, up to isomorphism, twenty subdirectly irreducible algebras in the class $\mathbf{K}_{1,1}$. Beazer also showed that $(L; f) \in \mathbf{K}_{1,1}$ is subdirectly irreducible if and only if its congruence lattice Con L reduces to the chain

$$\omega \preceq \Phi_1 \prec \iota,$$

where $(x, y) \in \Phi_1 \Leftrightarrow f(x) = f(y)$. As shown by Ramalho and Sequeira [25], this is also true for subdirectly irreducible algebras in $\mathbf{K}_{p,1}$.

Blyth and Varlet [10] and Beazer [4] showed the role that duality theory can play in the study of **MS**-algebras. Recently, Blyth and Varlet [11] described the **MS**-algebras dual to some Ockham spaces and basic connections between **MS**-algebras and **MS**-spaces.

In this thesis, we investigate further aspects. In chapter 2 we establish some results that are obtained by specifying the ordered set X and determining both the size of the **MS**-algebra $(L_X, ^o)$, in which ^{o} is given by $I \to I^o = X \setminus g^{-1}(I)$ where $g : X \to X$ is an antitone map such that $g^2 \leq id_X$, and the number of its fixed points. In chapter 3 we obtain properties of the congruence lattices of Ockham algebras, based on the congruence relations Φ_i $(i = 1, 2, ...)$ which are defined on (L; f) by

$$(x, y) \in \Phi_i \quad \Leftrightarrow \quad f^i(x) = f^i(y)$$

where $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $n \geq 0$.

In particular, we prove that if (L; f) is finite and if $\mathbf{K}_{p,q}$ is the smallest Berman class that contains L then each interval $[\Phi_i, \Phi_{i+1}]$ is boolean and Con L contains the vertical sum

$$[\omega, \Phi_1] \mathbin{\bar{\oplus}} [\Phi_1, \Phi_2] \mathbin{\bar{\oplus}} \cdots \mathbin{\bar{\oplus}} [\Phi_{q-1}, \Phi_q] \mathbin{\bar{\oplus}} [\Phi_q, \iota].$$

In chapter 4, a generalised variety $\mathbf{K}_\omega$ of Ockham algebras that contains all the Berman varieties $\mathbf{K}_{p,q}$ is introduced. We prove that in $\mathbf{K}_\omega$ an algebra is weakly subdirectly irreducible if and only if it is subdirectly irreducible; and that $L \in \mathbf{K}_\omega$ is subdirectly irreducible if and only if its lattice of congruences reduces to the chain

$$\omega = \Phi_0 \preceq \Phi_1 \preceq \Phi_2 \preceq \ldots \preceq \Phi_\omega \prec \iota$$

where, $\Phi_\omega = \bigvee_{i \geq 0} \Phi_i$ and, the symbol $\preceq$ means 'is covered by or is equal to' and the symbol $\prec$ means ' is covered by'. In chapter 5 we give a description of the structure of finite simple Ockham algebras.

We now recall here the main results and notions that we shall need.

The *Berman classes* $\mathbf{K}_{p,q}$ are defined for $p \geq 1$, $q \geq 0$ by the condition $f^q = f^{2p+q}$. These classes are related by the property

$$\mathbf{K}_{p,q} \subseteq \mathbf{K}_{p',q'} \quad \Leftrightarrow \quad p \mid p',\ q \leq q'.$$

A class $\mathcal{V}$ of algebras is said to be *locally finite* if every finitely generated member of $\mathcal{V}$ is finite. The following result was established by Berman [2].

Theorem 1.1 [2, Theorem 3] $\mathbf{K}_{p,q}$ is locally finite. ◊

It is easy to see that the class $\mathbf{K}_{1,0}$ is the class of de Morgan algebras [22], which are bounded distributive lattices M together with a unary operation $x \to f(x)$ such that

(M_1) $f(1) = 0$;

(M_2) $(\forall x \in M)$ $f^2(x) = x$;

(M_3) $(\forall x, y \in M)$ $f(x \wedge y) = f(x) \vee f(y)$.

The subclass of $\mathbf{K}_{1,0}$ given by the condition $x \wedge f(x) \leq y \vee f(y)$ is the class of Kleene algebras [16], which are de Morgan algebras K together with the condition

$$(\forall x, y \in K) \quad x \wedge f(x) \leq y \vee f(y).$$

The subclass of $\mathbf{K}_{1,0}$ given by the equation $x \wedge f(x) = 0$ is the class of Stone algebras [19], which are bounded distributive lattices S together with a unary operation $x \to f(x)$ such that

(S_1) $f(0) = 1$;

(S_2) $(\forall x \in S)$ $x \wedge f(x) = 0$;

(S_3) $(\forall x, y \in S)$ $f(x \wedge y) = f(x) \vee f(y)$.

An **MS**-algebra [7] is an Ockham algebra $< L; \wedge, \vee, f, 0, 1 >$ in which $x \leq f^2(x)$ for every $x \in L$; equivalently, in which f^2 is a closure. It has become the practice to denote the unary operation f by $x \to x^o$ when dealing with an **MS**-algebra, and to denote it by $x \to x^\sim$ when dealing with a general Ockham algebra. The following results are due to Blyth and Varlet [8].

Theorem 1.2 ([8, Theorem 2.1]) For an **MS**-algebra L, we have:

(1) $x = y \Leftrightarrow L \in \mathbf{T}$, the trivial class;

(2) $x \vee x^o = 1 \Leftrightarrow L \in \mathbf{B}$, the class of boolean algebras;

(2_d) $x \wedge x^o = 0 \Leftrightarrow L \in \mathbf{S}$, the class of Stone algebras;

(3) $x = x^{oo} \Leftrightarrow L \in \mathbf{M}$, the class of de Morgan algebras;

(4) $x \wedge x^o = x^{oo} \wedge x^o \Leftrightarrow (8_d)$ $x \wedge y^o \wedge y^{oo} = x^{oo} \wedge y^o \wedge y^{oo}$;

(4_d) $x \vee x^o = x^{oo} \vee x^o$;

(5) $(x \wedge x^o) \vee y \vee y^o = y \vee y^o \Leftrightarrow x \wedge x^o \leq y \vee y^o$;

(6) $(x \wedge x^o) \vee y^{oo} \vee y^o = y^{oo} \vee y^o \Leftrightarrow L^{oo} \in \mathbf{K}$, the class of Kleene algebras

$\Leftrightarrow x \wedge x^o \leq y^{oo} \vee y^o$;

(7) $(x \wedge x^o) \vee y \vee y^o = (x^{oo} \wedge x^o) \vee y \vee y^o$

$\Leftrightarrow (7_d)$ $(x \vee x^o) \wedge y \wedge y^o = (x^{oo} \vee x^o) \wedge y \wedge y^o$

$\Leftrightarrow (9_d)$ $(x \vee x^o) \wedge y^{oo} \wedge y^o = (x^{oo} \vee x^o) \wedge y^{oo} \wedge y^o$;

(8) $x \vee y^o \vee y^{oo} = x^{oo} \vee y^o \vee y^{oo} \Leftrightarrow x^o \vee x \vee y^o \vee y^{oo} = x^o \vee x^{oo} \vee y^o \vee y^{oo}$;

(9) $(x \wedge x^o) \vee y^{oo} \vee y^o = (x^{oo} \wedge x^o) \vee y^{oo} \vee y^o$. ◊

Theorem 1.3 ([8, Theorem 2.3]) The class of **MS**-algebras has only twenty subvarieties, and these are characterised by the identities indicated in Theorem 1.2, as follows:

T : (1); **B** : (2); **S**: (2_d); **K** : (3), (5); **S** ∨ **K** : (4), (5), (8); **M** : (3); **S** ∨ **M** : (4), (8); $\mathbf{K}_2$: (4), (5); $\mathbf{K}_2 \vee \mathbf{M}$: (4); $\mathbf{K}_1$: (4_d), (5); $\mathbf{M} \vee \mathbf{K}_1$: (4_d); $\mathbf{S} \vee \mathbf{K}_1$: (5), (8); $\mathbf{S} \vee \mathbf{M} \vee \mathbf{K}_1$: (7), (8); $\mathbf{K}_1 \vee \mathbf{K}_2$: (5); $\mathbf{M} \vee \mathbf{K}_1 \vee \mathbf{K}_2$: (7); $\mathbf{K}_3$: (6), (8); $\mathbf{M} \vee \mathbf{K}_3$: (8); $\mathbf{K}_2 \vee \mathbf{K}_3$: (6); $\mathbf{M} \vee \mathbf{K}_2 \vee \mathbf{K}_3$: (9); $\mathbf{M}_1$: none. ◊

If $(L; \sim)$ is an Ockham algebra then the set

$$S(L) = \{ x^{\sim} \mid x \in L \}$$

is a subalgebra of L. This subalgebra is a de Morgan subalgebra, precisely when $x^{\sim\sim\sim} = x^{\sim}$ for all $x \in L$, i.e., precisely when L belongs to the Berman class $\mathbf{K}_{1,1}$. When this is the case, we say that L has a de Morgan skeleton. Note that then we also have

$$S(L) = \{ x^{\sim\sim} \mid x \in L \}.$$

Every Ockham algebra $(L; \sim)$ contains a subalgebra with a de Morgan skeleton. The greatest such subalgebra is

$$M(L) = \{ x \in L \mid x^{\sim\sim\sim} = x^{\sim} \}.$$

It is clear that an **MS**-algebra L is a de Morgan algebra if and only if $L = L^{oo} = \{ x^{oo} \mid x \in L \}$, and is a Kleene algebra if, moreover, $x \wedge x^o \leq y \vee y^o$ for all $x, y \in L$.

We recall (see [13, 18, 20, 24, 27]) that if $(X, \leq)$ is an ordered set then an (order) ideal of X is a subset I of X such that if $x \in I$ and $y \leq x$ then $y \in I$. The principal ideal generated by $a \in X$, namely $\{x \in X \mid x \leq a\}$, will be denoted by $a^{\downarrow}$. $(X, \tau, \leq)$ is an *ordered space* if (X, τ) is a topological space, and $\leq$ an order on X. An ordered topological space X is *totally order disconnected* if for all $x, y \in X$ with $x \not\geq y$ there exists a clopen order ideal I of X with $x \in I$ and $y \notin I$. A map $f\colon X \to Y$, where X, Y are ordered sets, is isotone if $x_1 \leq x_2$ in X implies $f(x_1) \leq f(x_2)$ in Y. $(X; g)$ is an *Ockham space* if it is a compact totally order disconnected space endowed with a continuous antitone map $g\colon X \to X$. If $(X; g)$ is an Ockham space we can define a unary operation f on $\mathbf{O}(X)$, the bounded distributive lattice of clopen order ideals on X under set-theoretic intersection and union, by setting

$$f(I) = X \setminus g^{-1}(I)$$

for each $I \in \mathbf{O}(X)$. We thereby obtain an Ockham algebra. Conversely, if $(L; f)$ is an Ockham algebra then we can obtain an Ockham space by defining a map on the ordered set $\mathbf{P}(L)$ of prime ideals of L by setting

$$g(x) = \{ a \in L \mid f(a) \notin x \}$$

for each $x \in \mathbf{P}(L)$.

In particular, every finite ordered set $(X, \leq)$ as a topological space is discrete and every (order) ideal of X is clopen, so the antitone map $g\colon X \to X$ is continuous. For a finite ordered set $(X, \leq)$ with an antitone map $g\colon X \to X$ such that $g^2 \leq id_X$, $\mathbf{O}(X)$ is an **MS**-algebra whose ordered set of $\wedge$-irreducible elements (other than 1) is isomorphic to X. Moreover, every finite **MS**-algebra arises in this way. We replace $\mathbf{O}(X)$ by L_X when dealing with such a **MS**-space $(X; g)$. Actually, it can often be difficult to determine all the antitone maps g on an ordered set X such that $g^2 \leq id_X$, and a useful alternative is the following. As shown in [13, Theorem 1.1], the existence of such an antitone map g is equivalent to the existence of a dual closure map $f\colon X \to X$ such that Imf is self-dual. In fact, given such a map g, the map $f = g^2$ is such a dual closure; and given such a dual closure f, for every dual isomorphism $x \to \bar{x}$ on Imf the map $g\colon X \to X$ defined by $g(x) = \overline{f(x)}$ is antitone and such that $g^2 \leq id_X$.

The following results will be used in chapter 2.

Theorem 1.4 ([13, Corollary 1.2 of Theorem 1.1]) Let $(L; {}^{o})$ be the **MS**-algebra corresponding to $(X; g)$. Then $g^2 = id_X$ if and only if L is a de Morgan algebra. ◊

Theorem 1.5 ([13]) Let $(L; {}^{o})$ be the **MS**-algebra corresponding to $(X; g)$. Then the fixed points of L_X are the ideals I of X that satisfy

$$g(I) \subseteq X \setminus I \quad \text{and} \quad g(X \setminus I) \subseteq I. \quad \lozenge$$

We shall refer to such ideals as *distinguished ideals*. Thus $|L_X|$ is the number of ideals of X, and $|\mathrm{Fix}L_X|$ is the number of distinguished ideals of X. In the particular case where $g^2 = id_X$, a distinguished ideal I of X satisfies $g(I) = X \setminus I$ and $g(X \setminus I) = I$.

Theorem 1.6 ([13, Theorem 1.9]) $(L; {}^{o})$ is fixed point free if and only if $(X; g)$ has a fixed point. ◊

Theorem 1.7 ([11, Theorem 1]) All subvarieties of the class of **MS**-algebras are characterised by the following formulas of the corresponding dual space:

$\mathbf{B}: g^o = g$; $\mathbf{S}: g = g^2$; $\mathbf{K}: g^o = g^2 \gtrless g$; $\mathbf{S} \vee \mathbf{K}: g^o = g^2 \gtrless g$ or $g = g^2$; $\mathbf{M}: g^o = g^2$; $\mathbf{S} \vee \mathbf{M}: g^o = g^2$ or $g = g^2$; $\mathbf{K}_2: g^o = g^2 \leq g$ or $g^o > g$; $\mathbf{K}_2 \vee \mathbf{M}: g^o = g^2$ or $g^o > g$; $\mathbf{K}_1: g^o = g^2 \geq g$ or $g^o < g$; $\mathbf{M} \vee \mathbf{K}_1: g^o = g^2$ or $g^o < g$; $\mathbf{S} \vee \mathbf{K}_1: g^o = g^2 > g$ or $g = g^2$ or $g^o < g$; $\mathbf{S} \vee \mathbf{M} \vee \mathbf{K}_1: g^o = g^2$ or $g = g^2$ or $g^o < g$; $\mathbf{K}_1 \vee \mathbf{K}_2: g^o \gtrless g$; $\mathbf{M} \vee \mathbf{K}_1 \vee \mathbf{K}_2: g^o = g$ or $g^o > g$; $\mathbf{K}_3: g^o = g^2 > g$ or $g^2 \leq g$; $\mathbf{M} \vee \mathbf{K}_3: g^o = g^2$ or $g^2 \leq g$; $\mathbf{K}_2 \vee \mathbf{K}_3: g \gtrless g^2$; $\mathbf{M} \vee \mathbf{K}_2 \vee \mathbf{K}_3: g^o = g^2$ or $g \gtrless g^2$; $\mathbf{M}_1$: none. ◊

In dealing with the congruence lattice of an Ockham algebra L it is essential to distinguish between the elements of the lattice $\mathrm{Con}_{lat}\, L$ of lattice congruences and those of the lattice Con L of congruences of L. For this reason we denote elements of the former by the subscript 'lat'. The letters ω and ι stand for the equality relation and the universal relation respectively. If L is a lattice and if $a, b \in L$ then the *principal congruence* relation generated by a, b is defined to be

$$\theta_{lat}(a, b) = \bigwedge \{ \varphi \in \mathrm{Con}_{lat}\, L \mid (a, b) \in \varphi \}$$

i.e., the intersection in $\mathrm{Con}_{lat}\, L$ of all the lattice congruences that identify a and b.

For $a, b \in L$ we have

$$\theta_{lat}(a \wedge b, b) = \theta_{lat}(a, a \vee b).$$

In a distributive lattice, it is well known [19, Theorem 3] that, for $a, b \in L$,

$$(x, y) \in \theta_{lat}(a, b) \iff x \wedge a = y \wedge a \text{ and } x \vee b = y \vee b.$$

Moreover, the intersection of two principal lattice congruences is again a principal lattice congruence. Precisely, for $a \leq b$ and $c \leq d$, we have

$$\theta_{lat}(a, b) \wedge \theta_{lat}(c, d) = \theta_{lat}((a \vee c) \wedge b \wedge d, b \wedge d).$$

If L is an Ockham algebra and if $a, b \in L$ with $a \leq b$ then the principal congruence generated by a, b is defined to be

$$\theta(a, b) = \bigwedge \{ \varphi \in \mathrm{Con}\, L \mid (a, b) \in \varphi \}$$

Every Ockham congruence is in particular a lattice congruence. The following two important results were given by Berman [2].

Theorem 1.8 [2, Theorem 2] The class **O** of Ockham algebras enjoys the congruence extension property. ◊

Theorem 1.9 [2, Theorem 1] Let (L; f) be an Ockham algebra and let $a, b \in L$ with $a \leq b$. Then

$$\theta(a, b) = \bigvee_{n<\omega} \theta_{lat}\, (f^n(a), f^n(b)). \quad \lozenge$$

For a Berman class $\mathbf{K}_{p,q}$, Carvalho [17] has proved the following result:

Theorem 1.10 ([17, Proposition 1.1]). If $L \in \mathbf{K}_{p,0}$ is a finite Ockham algebra of height m, then

(1) Con L is a boolean lattice with at most m atoms.

(2) Con L has exactly m atoms if and only if L is a boolean algebra. ◊

An algebra L is said to be *subdirectly irreducible* if it has a smallest non-trivial congruence, i.e., Con L has an atom α such that if $\varphi \in$ Con L with $\varphi \neq \omega$ then $\varphi \geq \alpha$. Such an atom is called the *monolith* of L. In particular, if L is subdirectly irreducible then ω is $\wedge$-irreducible in the sense that for $\theta_1, \theta_2 \in$ Con L if $\theta_1 \wedge \theta_2 = \omega$ then either $\theta_1 = \omega$ or $\theta_2 = \omega$.

The following results were shown by Berman [2].

Theorem 1.11 [2, Lemma 1] Let (L; f) be a subdirectly irreducible Ockham algebra. If a, b $\in$ L are such that f(a) = b and f(b) = a, then either a = b or {a, b} = {0, 1}. Moreover, f has at most two fixed points. ◊

Theorem 1.12 [2, Theorem 7] If $L \in \mathbf{K}_{p,q}$ is subdirectly irreducible then L is finite. ◊

A subset Y of an Ockham space (X; g) is said to be a g-*subset* [27] if $x \in Y$ implies $g(x) \in Y$. Let $g^{\omega}(Y)$ denotes the least g-subset that contains Y, i.e., $g^{\omega}(Y) = \{g^n(x) \mid n \geq 0, x \in Y\}$.

Urquhart has proved [27, Theorem 6] that if (L; f) is an Ockham algebra with dual space $\mathbf{P}(L) = \langle X; \tau, \leq, g \rangle$ then L is subdirectly irreducible if and only if there exists some clopen subset U of X such that $\overline{g^{\omega}(\{x\})} = X$ for all $x \in U$. Moreover, (L; f) is simple if and only if $\overline{g^{\omega}(\{x\})} = X$ for all $x \in X$.

Finally, if for an Ockham algebra (L; f) we define

$$T_i(L) = \{ x \in L \mid f^i(x) = x \},$$

then the following results were determined by Ramalho and Sequeira [25].

Theorem 1.13 [25, Lemma 1] If (L; f) $\in \mathbf{K}_{p,0}$ is such that $T_2(L) = T_1(L) \cup \{0,1\}$ then L is simple. ◊

Theorem 1.14 [25, Lemma 2] If L $\in \mathbf{K}_{p,q}$ is subdirectly irreducible then $f^q(L) \in \mathbf{K}_{p,0}$ is simple. ◊

Theorem 1.15 [25, Lemma 3] Let L $\in \mathbf{K}_{p,1}$ be non-trivial. Then L is subdirectly irreducible if and only if Con L is a chain with at most 3 elements. ◊

CHAPTER 2

SOME FINITE MS-ALGEBRAS

In this Chapter we shall discuss some results that are obtained by specifying the ordered set X and determining both the size of the **MS**-algebra L_X and the number of its fixed points. This was begun by Blyth, Goossens and Varlet [13] where X was a fence or a crown. Here we consider similar, though more complicated, ordered sets. It turns out that in many cases the **MS**-algebras which arise from these ordered sets are de Morgan and Kleene algebras. All of the results that follow involve intricate combinatorial arguments, on the one hand in counting ideals and on the other in determining which of these are distinguished.

2.1 Double fences.

Here we shall be concerned with a particular sequence that is defined recursively by a second order difference equation, namely the sequence $(J_n)_{n\geq 0}$ given by

$$J_0 = J_1 = 1, \quad (\forall\ n \geq 2)\ J_n = 2J_{n-1} + J_{n-2}.$$

As we shall see, this sequence appears in many of the results. For this purpose, we first observe the following property of this sequence.

Theorem 2.1.1 $J_0 + J_1 + J_2 + \ldots + J_n = \frac{1}{2}(J_n + J_{n+1})$.

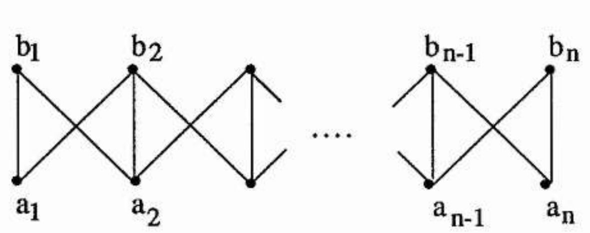
Proof Let $x_n = J_0 + J_1 + \ldots + J_n$ and observe that

$$\begin{aligned} 3x_n &= 2x_n + x_n \\ &= 2(J_0+J_1+ \ldots +J_n) + (J_0+J_1+ \ldots +J_n) \\ &= 2J_0 + (2J_1+J_0) + \ldots + (2J_n+J_{n-1}) + J_n \\ &= 2J_0 + J_2 + \ldots + J_{n+1} + J_n \\ &= x_n + J_{n+1} + J_n \quad (\text{since } J_0 = J_1 = 1). \end{aligned}$$

It follows that $x_n = \frac{1}{2}(J_{n+1} + J_n)$. ◊

Definition By a *double fence* we shall mean an ordered set of the form

DF_{2n} :



On DF_{2n} there is clearly only one dual closure f with a self-dual image, namely f = id. There are two dual isomorphisms on Imf = DF_{2n}, namely a reflection g_1 in the horizontal, and a rotation g_2 through 180^0; specifically, for each i,

$$g_1(a_i) = b_i, \qquad g_1(b_i) = a_i ;$$

$$g_2(a_i) = b_{n-i+1}, \qquad g_2(b_i) = a_{n-i+1}.$$

Since $g_1^2 = g_2^2 = \text{id}$, the corresponding **MS**-algebras $(L_{DF_{2n}}; g_1)$ and $(L_{DF_{2n}}; g_2)$ are de Morgan algebras. In fact, we can say more: since $g_1(x)$ and x are comparable for every x, it follows by Theorem 1.7 of Chapter 1 that $(L_{DF_{2n}}; g_1)$ is a Kleene algebra.

In what follows, for an ordered set X we shall denote by #(X) the number of ideals of X; by #(X; a) the number of ideals of X that contain the element a of X; by #(X; $\bar{a}$) the number of ideals that do not contain the element a; and by #(X; a, $\bar{b}$) the number of ideals that contain the element a but not the element b.

Theorem 2.1.2 $|L_{DF_{2n}}| = J_{n+1}$.

Proof Consider the element b_n of DF_{2n}. We have

(1) $\#(DF_{2n}; b_n) = \#(DF_{2n-2}; b_{n-1}) + |L_{DF_{2n-4}}|$;

(2) $\#(DF_{2n}; \overline{b}_n) = \#(DF_{2n}; a_n, \overline{b}_n) + \#(DF_{2n}; \overline{a}_n, \overline{b}_n)$

$= |L_{DF_{2n-2}}| + \#(DF_{2n-2}; \overline{b}_{n-1})$

$= |L_{DF_{2n-2}}| + [\,|L_{DF_{2n-2}}| - \#(DF_{2n-2}; b_{n-1})]$

$= 2\,|L_{DF_{2n-2}}| - \#(DF_{2n-2}; b_{n-1})$.

It follows from (1) and (2) that

$$|L_{DF_{2n}}| = \#(DF_{2n}; b_n) + \#(DF_{2n}; \overline{b}_n)$$

$$= 2\,|L_{DF_{2n-2}}| + |L_{DF_{2n-4}}|.$$

If we let $\alpha_n = |L_{DF_{2n}}|$ then we obtain the recurrence relation

$$\alpha_n = 2\alpha_{n-1} + \alpha_{n-2}.$$

Now we can see immediately that the ideals of DF_2 are

$$\varnothing,\ \{a_1\},\ \{a_1, b_1\};$$

and that the ideals of DF_4 are

$$\varnothing,\ \{a_1\},\ \{a_2\},\ \{a_1, a_2\},\ \{a_1, a_2, b_1\},\ \{a_1, a_2, b_2\},\ \{a_1, a_2, b_1, b_2\}.$$

So $\alpha_1 = |L_{DF_2}| = 3 = J_2$ and $\alpha_2 = |L_{DF_4}| = 7 = J_3$, and it follows that

$$|L_{DF_{2n}}| = \alpha_n = J_{n+1}. \quad \Diamond$$

Corollary 1 $\#(DF_{2n}; b_n) = \frac{1}{2}(J_{n-1} + J_n)$.

Proof By (1) above we have, for each n,

$$\alpha_{n-2} = |L_{DF_{2n-4}}| = \#(DF_{2n}; b_n) - \#(DF_{2n-2}; b_{n-1});$$

$$\alpha_{n-3} = |L_{DF_{2n-6}}| = \#(DF_{2n-2}; b_{n-1}) - \#(DF_{2n-4}; b_{n-2});$$

$$\cdots\cdots$$

$$\alpha_2 = |L_{DF_4}| = \#(DF_8; b_4) - \#(DF_6; b_3);$$

$$\alpha_1 = |L_{DF_2}| = \#(DF_6; b_3) - \#(DF_4; b_2).$$

Consequently,

$$\sum_{i=1}^{n-2} \alpha_i = \#(DF_{2n}; b_n) - \#(DF_4; b_2).$$

It is easily seen that the ideals of DF_4 that contain b_2 are

$$\{a_1, a_2, b_2\} \text{ and } \{a_1, a_2, b_1, b_2\},$$

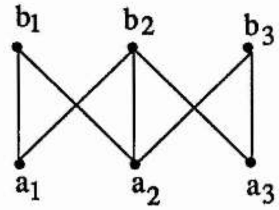
and so $\#(DF_4; b_2) = 2 = J_0 + J_1$. Thus we see that

$$\begin{aligned}\#(DF_{2n}; b_n) &= \#(DF_4; b_2) + \sum_{i=1}^{n-2} \alpha_i \\ &= J_0 + J_1 + \sum_{i=1}^{n-2} J_{i+1} \\ &= \frac{1}{2}(J_{n-1} + J_n). \quad \text{by Theorem 2.1.1.} \ \Diamond\end{aligned}$$

Corollary 2 $\#(DF_{2n}; a_n, \overline{b}_n) = |L_{DF_{2n-2}}| = \alpha_{n-1} = J_n$. $\Diamond$

By way of illustration, we consider the double fence

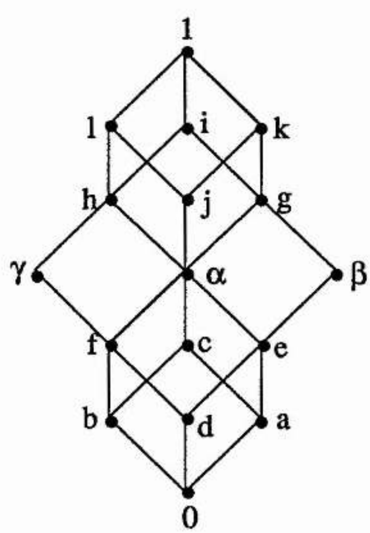
DF_6 :



By Theorem 2.1.2 we have

$$|L_{DF_6}| = J_4 = 17.$$

The Hasse diagram of L_{DF_6} is



The mappings g_1 and g_2 are given by

$$\begin{array}{rllllll} x: & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ g_1(x): & b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\ g_2(x): & b_3 & b_2 & b_1 & a_3 & a_2 & a_1 \end{array}$$

Using the fact that $I^o = X \setminus g^{-1}(I) = X \setminus g(I)$ we obtain the corresponding MS-algebras as follows :

		0 1 a b c d e f g h i j k l α β γ	
$(L_{DF_6}; g_1)$	:	1 0 l k j i h g f e d c b a α γ β	**K**
$(L_{DF_6}; g_2)$	:	1 0 k l j i g h e f d c a b α β γ	**M**

Now we turn our attention to the fixed points of the de Morgan algebras on $L_{DF_{2n}}$. By Theorem 1.5 in Chapter 1, the fixed points of $(L_X, {}^o)$ are those ideals of X that are distinguished under g, in the sense that $g(I) \subseteq X \setminus I$ and $g(X \setminus I) \subseteq I$. In the case under consideration, the mapping g is surjective and so we can consider those ideals I such that $g(I) = X \setminus I$ and $g(X \setminus I) = I$.

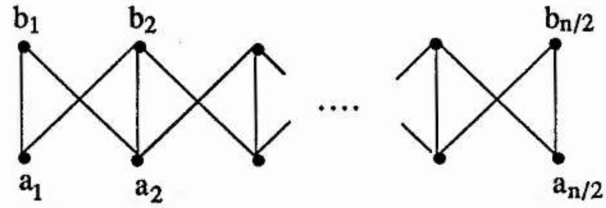
Theorem 2.1.3 $|\, \mathrm{Fix} L_{(DF_{2n};\, g_1)} \,| = 1.$

Proof Let I be an ideal of DF_{2n}. If there exists some $b_i \in I$ then $a_i = g_1(b_i) \notin X \setminus I$, so $g_1(I) \neq X \setminus I$ which shows I is not a distinguished ideal under g_1. Consequently, with respect to the mapping g_1 the only distinguished ideal is $I = \{a_1, a_2, ..., a_n\}$. ◊

Theorem 2.1.4 $|\, \mathrm{Fix} L_{(DF_{2n};\, g_2)} \,| = \begin{cases} J_{n/2} & \text{if n is even,} \\ J_{(n+1)/2} & \text{if n is odd.} \end{cases}$

Proof (a) Consider first the case where n is even. Since g_2 can be regarded as a rotation through 180^o, the fixed points are those ideals of DF_{2n} that contain half of the a_i and have a 'skew-symmetric profile'. To be more precise, consider the subset A of DF_{2n} described by

$A \simeq DF_n$:



Let $A' = DF_{2n} \setminus A$ and for every ideal I of A let $I_* = A' \setminus g_2(I)$. Then we have $I_* = g_2(A \setminus I)$ and $g_2(I_*) = A \setminus I$. We now show that, if I is an ideal of A such that $a_{n/2} \in I$ and $b_{n/2} \notin I$, then $I \cup I_*$ is a distinguished ideal of DF_{2n} under g_2.

Since $b_{n/2} \notin I$, we see that I is an ideal of DF_{2n}. Suppose now that $a \in I \cup I_*$ and $x \le a$. If $a \in I$ then clearly $x \in I \cup I_*$. If $a \in I_*$, then since $b_{n/2+1} = g_2(a_{n/2}) \notin g_2(A \setminus I) = I_*$ we must have $x \neq a_{n/2}$, whence $x \in A' = g_2(A)$ and so $g_2(x) \in A$. Since $g_2(a) \in g_2(I_*) = A \setminus I$, we have $g_2(a) \notin I$ whence it follows from the fact that $g_2(x) \ge g_2(a)$ that $g_2(x) \in A \setminus I$ and that $x \in g_2(A \setminus I) = I_*$. Consequently, $I \cup I_*$ is an ideal of DF_{2n}.

We now show that $I \cup I_*$ is distinguished. Clearly, $g_2(I \cup I_*) = g_2(I) \cup g_2(I_*) = g_2(I) \cup (A \setminus I)$. Now let $x \in g_2(I \cup I_*)$. Then we have either $x \in g_2(I)$ or $x \in A \setminus I$. For the latter, clearly $x \notin I \cup I_*$; for the former we have $x \notin I$ and $g_2(x) \notin A \setminus I$ so $x \notin g_2(A \setminus I) = I_*$. It follows that $x \notin I \cup I_*$ and that $g_2(I \cup I_*) \subseteq DF_{2n} \setminus (I \cup I_*)$. Similarly, $g_2(DF_{2n} \setminus (I \cup I_*)) \subseteq I \cup I_*$.

We shall now show that every distinguished ideal of DF_{2n} under g_2 is of the form $I \cup I_*$ where I is an ideal of A such that $a_{n/2} \in I$ and $b_{n/2} \notin I$. For this purpose, observe that if K is a distingushed ideal of DF_{2n} under g_2 then necessarily $a_{n/2} \in K$ and $b_{n/2} \notin K$; for $a_{n/2} \notin K$ gives $b_{n/2+1} = g_2(a_{n/2}) \in K$ whence the contradiction $a_{n/2} \in K$, and $b_{n/2} \in K$ gives $a_{n/2+1} = g_2(b_{n/2}) \notin K$ which contradicts $b_{n/2} \in K$. Also,

$$\begin{aligned}(K \cap A)_* &= A' \setminus g_2(K \cap A)\\ &= (A' \setminus g_2(K)) \cup (A' \setminus g_2(A))\\ &= A' \setminus g_2(K).\end{aligned}$$

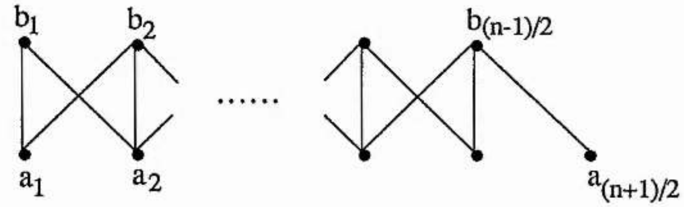
If now $x \in K$ then either $x \in K \cap A$ or $x \in K \cap A'$, the latter giving $x \in A' \setminus g_2(K)$, and hence $x \in (K \cap A) \cup (K \cap A)_*$. Conversely, if $x \in (K \cap A) \cup (K \cap A)_*$ then either $x \in (K \cap A)$ or $x \in (K \cap A)_* = A' \setminus g_2(K)$. In both cases we have $x \in K$. We conclude that

$$K = (K \cap A) \cup (K \cap A)_*.$$

It follows from these observations that the number of fixed points of DF_{2n} (when n is even) is precisely the number of ideals of A that contain $a_{n/2}$ but not $b_{n/2}$. By the Corollary 2 of Theorem 2.1.2 this is $J_{n/2}$.

(b) Consider now the case where n is odd. Here we consider the subset B of DF_{2n} described by

B:



Clearly, B is a distinguished ideal of DF_{2n} under g_2 . Let $B' = g_2(B)$ and for every ideal I of B let $I_* = B' \setminus g_2(I)$. Using a similar argument to that in (a), we can show that every distinguished ideal of DF_{2n} under g_2 is of the form $I \cup I_*$ where I is an ideal of B that contains $a_{(n+1)/2}$. Thus the number of fixed points of DF_{2n} (when n is odd) is precisely the number of ideals of B that contain $a_{(n+1)/2}$. Clearly, this is

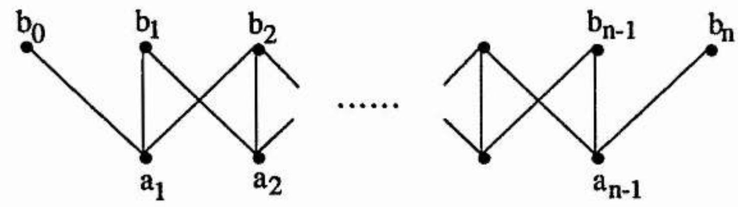
$$\#(DF_n;\ a_{(n+1)/2}, \bar{b}_{(n+1)/2})$$

which, by Corollary 2 of Theorem 2.1.2, is $J_{(n+1)/2}$. ◊

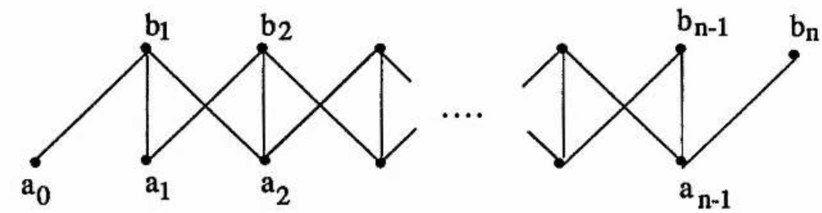
2.2 Extended double fences

Definition By *extended double fences* we shall mean ordered sets of the form

E_1DF_{2n} :



E_2DF_{2n} :



Clearly, on $X \simeq E_1DF_{2n}$ there is only one dual closure f with a self-dual image, namely that given by

$$f(b_o) = a_1, \quad f(b_n) = a_{n-1},$$

$$f(a_i) = a_i, \quad f(b_i) = b_i \quad (i = 1, 2, \dots, n\text{-}1).$$

Since there are only two dual isomorphisms on Im f, namely a reflection in the horizontal and a rotation through 180^o, there are therefore two antitone mappings g_i : $X \to X$ such that $g_i^2 \leq id_X$, namely that given by

(1) $g_1(b_o) = b_1, \quad g_1(b_n) = b_{n-1},$

$g_1(a_i) = b_i, \quad g_1(b_i) = a_i \quad (i = 1, 2, \dots, n\text{-}1);$

(2) $g_2(b_o) = b_{n-1}, \quad g_2(b_n) = b_1,$

$g_2(a_i) = b_{n-i}, \quad g_2(b_i) = a_{n-i} \quad (i = 1, 2, \dots, n\text{-}1).$

Since g_1, g_2 are not surjective, the corresponding **MS**-algebras are not de Morgan algebras. We can say more, $L(E_1DF_{2n}; g_1) \in \mathbf{M} \vee \mathbf{K}_2 \vee \mathbf{K}_3$. This follows from Theorem 1.7 of Chapter 1: for all $x \in X \setminus \{b_0, b_n\}$, $x = g_1^2(x)$ and $g_1^2(b_0) = a_1 < g(b_0) = b_1$, $g^2(b_n) = a_{n-1} < g(b_n) = b_{n-1}$.

Theorem 2.2.1 $|L_{E_1DF_{2n}}| = \frac{1}{2}(J_{n+2} + 1)$.

Proof For each n let $\alpha_n = \#(E_1DF_{2n})$ and $\beta_n = \#(E_1DF_{2n}; \bar{b}_n)$. Then we have

$$\begin{aligned}\alpha_n &= \#(E_1DF_{2n}; b_n) + \#(E_1DF_{2n}; \bar{b}_n)\\ &= \#(E_1DF_{2n} \setminus \{a_{n-1}, b_n\}) + \beta_n\\ &= \#(E_1DF_{2n-2}) + \beta_n\end{aligned}$$

which gives

$$(1) \quad \alpha_n = \alpha_{n-1} + \beta_n.$$

But we have

$$\begin{aligned}\beta_n &= \#(E_1DF_{2n}; \bar{b}_n) = \#(E_1DF_{2n} \setminus \{b_n\})\\ &= \#(DF_{2n} \setminus \{a_1\})\\ &= \#(DF_{2n} \setminus \{a_n\}).\end{aligned}$$

Since two dually isomorphic posets have the same number of ideals, so we have

$$\#(DF_{2n} \setminus \{a_n\}) = \#(DF_{2n} \setminus \{b_n\}),$$

and so

$$\begin{aligned}\beta_n &= \#(DF_{2n} \setminus \{b_n\}) = \#(DF_{2n}; \bar{b}_n)\\ &= \#(DF_{2n}) - \#(DF_{2n}; b_n)\\ &= J_{n+1} - \frac{1}{2}(J_{n-1} + J_n)\\ &= \frac{1}{2}(2J_{n+1} - J_{n-1} - J_n)\\ &= \frac{1}{2}(J_{n+1} + J_n),\end{aligned}$$

whence (1) can be written in the form

$$\alpha_n = \alpha_{n-1} + \frac{1}{2}(J_{n+1} + J_n).$$

We deduce from this that

$$\alpha_n = \alpha_2 + \frac{1}{2} J_3 + J_4 + \dots + J_n + \frac{1}{2} J_{n+1}.$$

Since $\alpha_2 = 9 = \frac{1}{2} + J_0 + J_1 + J_2 + \frac{1}{2} J_3$, we then have

$$\begin{aligned} \alpha_n &= \frac{1}{2} + \sum_{i=0}^{n} J_i + \frac{1}{2} J_{n+1} \\ &= \frac{1}{2} + \frac{1}{2}(J_{n+1} + J_n) + \frac{1}{2} J_{n+1} \\ &= \frac{1}{2}(J_{n+2} + 1). \quad \Diamond \end{aligned}$$

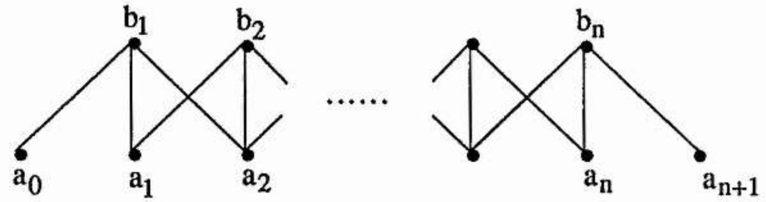
Corollary 1 $\#(DF_{2n};\ b_1, b_n) = \frac{1}{2}(J_{n-1} + 1)$.

Proof For n = 1, 2, 3 the result follows by direct computation. For $n \geq 4$ we have

$$\begin{aligned} \#(DF_{2n};\ b_1, b_n) &= \#(DF_{2n} \setminus \{a_1, a_2, b_1, a_{n-1}, a_n, b_n\}) \\ &= \#(E_1 DF_{2n-6}) \\ &= \frac{1}{2}(J_{n-1} + 1). \quad \Diamond \end{aligned}$$

Corollary 2 The number of ideals of an ordered set of the form

M_{2n+2} :

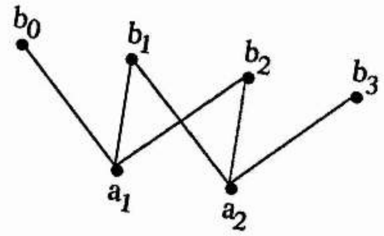


is given by

$$\#(M_{2n+2}) = \frac{1}{2}(J_{n+3} + 1).$$

Proof If two posets are dually isomorphic then they have the same number of ideals. The result therefore follows by Theorem 2.2.1. $\Diamond$

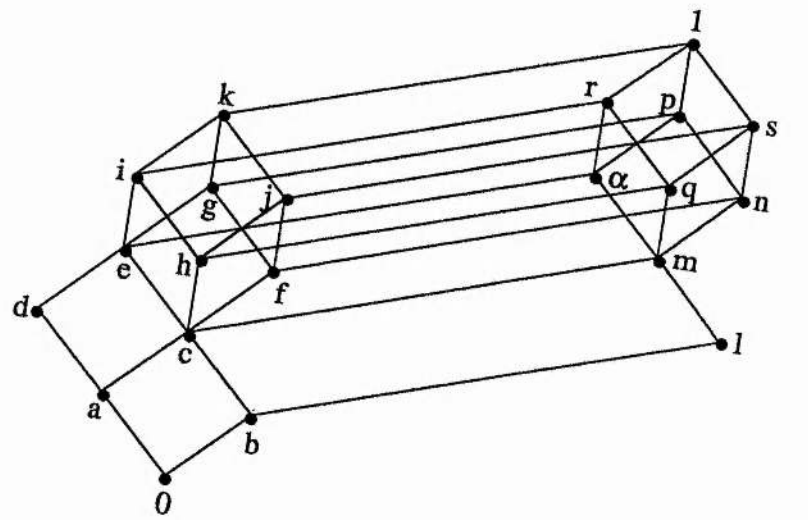
By way of illustration, the Hasse diagram of E_1DF_6 is



The mappings g_1 and g_2 are given by

$$
\begin{array}{rl}
x : & a_1 \; a_2 \; b_0 \; b_1 \; b_2 \; b_3 \\
g_1(x) : & b_1 \; b_2 \; b_1 \; a_1 \; a_2 \; b_2 \\
g_2(x) : & b_2 \; b_1 \; b_2 \; a_2 \; a_1 \; b_1
\end{array}
$$

By Theorem 2.2.1 we see that $|E_1DF_6| = \frac{1}{2}(J_5 + 1) = \frac{1}{2}(41+1) = 21$ elements.
Its underlying lattice (the order ideals of E_1DF_6) is



The corresponding **MS**-algebras as follows:

$$
\begin{array}{rl}
x : & 0\;1\;a\;b\;c\;d\;e\;f\;g\;h\;i\;j\;k\;l\;m\;n\;p\;q\;r\;s\;\alpha \\
L(E_1DF_6; g_1) : & 1\;0\;p\;r\;\alpha\;p\;\alpha\;d\;d\;l\;1\;0\;0\;r\;\alpha\;d\;d\;l\;1\;0\;\alpha \quad \mathbf{M} \vee \mathbf{K}_2 \vee \mathbf{K}_3 \\
L(E_1DF_6; g_2) : & 1\;0\;r\;p\;\alpha\;r\;\alpha\;l\;1\;d\;d\;0\;0\;p\;\alpha\;l\;1\;d\;d\;0\;\alpha \quad \mathbf{M}_1
\end{array}
$$

We now consider the number of fixed points of **MS**-algebras on $L_{E_1DF_{2n}}$. We have the following results.

Theorem 2.2.2 $|\,Fix(E_1DF_{2n}\,;\,g_1\,| = 1.$

Proof Let I be a distinguished ideal of E_1DF_{2n} under g. Then by the definition of distinguished ideal, $g_1(I) \subseteq E_1DF_{2n} \setminus I$ and $g_1(E_1DF_{2n} \setminus I) \subseteq I$. If there was some $a_i \notin I$ then $b_i = g_1(a_i) \in g_1(E_1DF_{2n} \setminus I) \subseteq I$ which gives the contradiction $a_i \in I$. Hence we have

$$\{\, a_1, a_2, ..., a_n \} \subseteq I.$$

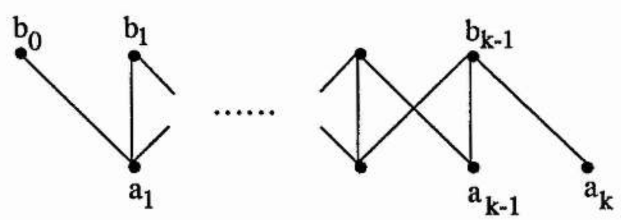
If there was some $b_i \in I$ for $i \neq 0$, n then $a_i \in I$ which contradicts $a_i = g_1(b_i) \in g_1(I) \subseteq E_1DF_{2n} \setminus I$. Whence $b_i \notin I$ for all $i \neq 0$, n. If now $b_0 \notin I$ then $b_1 = g_1(b_0) \in g_1(E_1DF_{2n} \setminus I\,) \subseteq I$ which contradicts the fact that $b_1 \notin I$. Whence $b_0 \in I$, and similarly we can argue that $b_n \in I$.

It follows from these observations above that, under the mapping g_1, the only distinguished ideal is $I = \{b_0, b_n, a_1, a_2, ..., a_{n-1}\}$. ◊

Theorem 2.2.3 $|\,FixL_{(E_1DF_{2n};\, g_2)}\,| = \begin{cases} J_{n/2} & \text{if n is even,} \\ J_{(n-1)/2} & \text{if n is odd.} \end{cases}$

Proof Consider first the case where n is even, say n = 2k. Let the subset A of E_1DF_{2n} described by

A :



Clearly, A is distinguished ideal of E_1DF_{2n} under g_2. Let $A' = g_2(A)$ and for every ideal I of A let $I_* = A' \setminus g_2(I)$. Observe that if K is a distinguished ideal then necessarily $a_{n/2} \in K$ and either $a_1, b_0 \in K$ or $a_1, b_0 \notin K$; for $a_{n/2} \notin K$ gives $b_{n/2} \in K$ whence the contradiction $a_{n/2} \in K$, and $a_1 \in K$ gives $b_{n-1} = g_2(a_1) \notin K$ and then $b_0 \in K$, and $a_1 \notin K$ gives $b_0 \notin K$. Arguing as in the proof of Theorem 2.1.4, we see that the distinguished ideals of E_1DF_{2n} are of the form $I \cup I_*$ where I is an ideal of A that contains $a_{n/2}$, and satisfies the condition that

$$\text{either } a_1, b_0 \in I \text{ or } a_1, b_0 \notin I.$$

If α_k denotes the number of such ideals then we have

$$\alpha_k = \#(A; a_k) - \#(A; a_k, a_1, \bar{b}_0).$$

Now

$$\begin{aligned}\#(A; a_k) &= \#(A \setminus \{a_k\}) = \#(DF_{2k+2}; b_{k+1}) \\ &= \frac{1}{2}(J_k + J_{k+1}) \text{ (by Corollary 1 of Theorem 2.1.2)}\end{aligned}$$

and

$$\begin{aligned}\#(A; a_k, a_1, \bar{b}_0) &= \#(A \setminus \{a_k, b_0\}; a_1) \\ &= \#(DF_{2k-2}; a_1) \\ &= \#(DF_{2k-2}; a_{k-1}) \\ &= \#(DF_{2k-2}; b_{k-1}) + \#(DF_{2k-2}; a_{k-1}, \bar{b}_{k-1}) \\ &= \frac{1}{2}(J_{k-2} + J_{k-1}) + J_{k-1} \text{ (by Corollaries 1, 2 of Theorem 2.1.2)} \\ &= \frac{1}{2}(J_k + J_{k-1}).\end{aligned}$$

It follows that

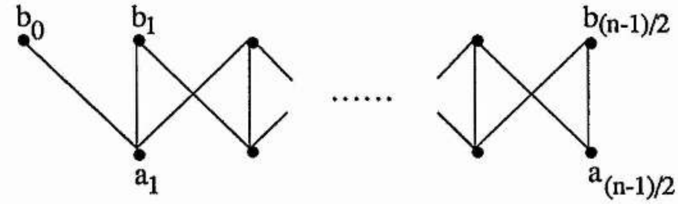
$$\begin{aligned}\alpha_k &= \frac{1}{2}(J_k + J_{k+1}) - \frac{1}{2}(J_k + J_{k-1}) \\ &= J_k\end{aligned}$$

that is

$$|\, \mathrm{FixL}(E_1DF_{2n}; g_2) \,| = J_{n/2} \text{ (when n is even).}$$

(b) Consider the case where n is odd. Let the subset B of E_1DF_{2n} described by

B :



Using a similar argument to that in (a), we can show that every distinguished ideal of E_1DF_{2n} under g_2 is of the form $I \cup I_*$ where I is an ideal of B that contains $a_{(n-1)/2}$ but not $b_{(n-1)/2}$ and satisfies the condition that

$$\text{either } a_1, b_0 \in I \text{ or } a_1, b_0 \notin I.$$

If β_k, where $k = \frac{n-1}{2}$, denotes the number of such ideals then we have

$$\alpha_k = \#(B; a_k, \bar{b}_k) - \#(B; a_k, \bar{b}_k, a_1, \bar{b}_0).$$

Now

$$\begin{aligned}\#(B; a_k, \bar{b}_k) = \#(B\backslash\{b_k\}; a_k) &= \#(B\backslash\{a_k, b_k\}) \\ &= \#(DF_{2k+2}; b_{k+1}) \\ &= \tfrac{1}{2}(J_k + J_{k+1}) \text{ (by Corollary 1 of Theorem 2.1.2)}\end{aligned}$$

and

$$\begin{aligned}\#(B; a_k, \bar{b}_k, a_1, \bar{b}_0) = \#(B\backslash\{b_0\}; a_k, \bar{b}_k, a_1) &= \#(DF_{2k-2}; a_1) \\ &= \#(DF_{2k-2}; a_{k-1}) \\ &= \#(DF_{2k}; b_k) \\ &= \tfrac{1}{2}(J_{k-1} + J_k).\end{aligned}$$

It follows that $\beta_k = \frac{1}{2}(J_k + J_{k+1}) - \frac{1}{2}(J_{k-1} + J_k) = J_k$. We therefore have

$$|\, \mathrm{FixL}_{(E_1DF_{2n};\, g_2)} \,| = J_{(n-1)/2} \text{ (when n is odd).} \quad \Diamond$$

We now turn our attention to E_2DF_{2n}. There is clearly only one dual closure f with a self-dual image on E_2DF_{2n}, namely f = id; and there is only one dual isomorphism g on Im f, namely that described geometrically by a rotation through 180^o, given by

$$g(a_i) = b_{n-i}, \qquad g(b_i) = a_{n-i}.$$

The corresponding **MS**-algebra belongs properly to the class **M** of de Morgan algebras.

Theorem 2.2.4 $|L_{E_2DF_{2n}}| = \frac{1}{2}(J_{n+2} - 1)$.

Proof Observe first that

$$\begin{aligned}\#(E_2DF_{2n}) &= \#(E_2DF_{2n};\, a_o) + \#(E_2DF_{2n};\, \bar{a}_o) \\ &= \#(E_2DF_{2n}\setminus a_o) + \#(E_2DF_{2n-2}) \\ &= \#(DF_{2n+2};\, b_{n+1}) + \#(E_2DF_{2n-2}) \\ &= \frac{1}{2}(J_n + J_{n+1}) + \#(E_2DF_{2n-2})\,.\end{aligned}$$

the final equality following by Corollary 1 to Theorem 2.1.2. Thus, if we let $u_i = \#(E_2DF_{2i})$ and $v_i = \frac{1}{2}(J_i + J_{i+1})$, we have

$$u_i = v_i + u_{i-1}.$$

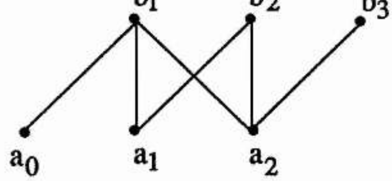
Writing this in the form $v_i = u_i - u_{i-1}$ and summing over i, we obtain

$$u_n - u_o = v_1 + v_2 + \ldots + v_n.$$

Since clearly $u_o = 1 = v_o$, this becomes, using Theorem 2.1.1,

$$\begin{aligned}u_n &= v_o + v_1 + v_2 + \ldots + v_n \\ &= \frac{1}{2}\sum_{i=0}^{n}(J_i + J_{i+1}) \\ &= \frac{1}{2}\sum_{i=0}^{n}J_i + \frac{1}{2}\sum_{i=0}^{n}J_{i+1} \\ &= \frac{1}{4}[(J_n + J_{n+1})] + \frac{1}{4}[(J_{n+1} + J_{n+2})] - \frac{1}{2} \\ &= \frac{1}{4}(J_n + 2J_{n+1} + J_{n+2}) - \frac{1}{2} \\ &= \frac{1}{2}(J_{n+2} - 1)\,. \quad \Diamond\end{aligned}$$

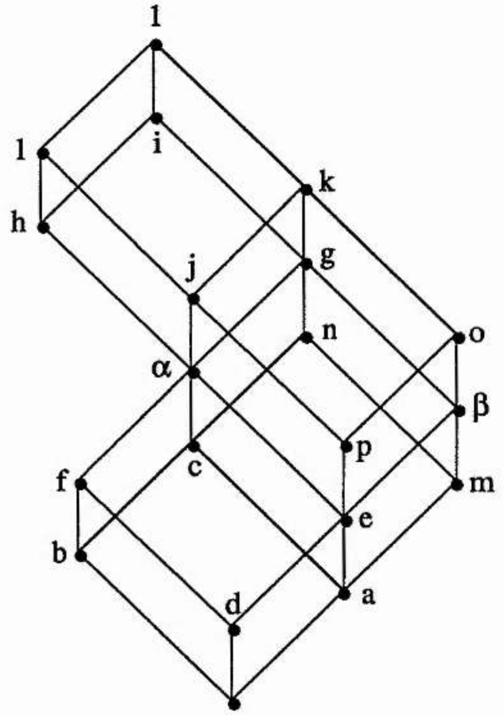
By way of illustration, consider the extended fence E_2DF_6 :



By Theorem 2.2.4 we have

$$|L_{E_2DF_6}| = \frac{1}{2}(J_5 - 1) = 20.$$

The Hasse diagram of $L_{E_2DF_6}$ is



and the corresponding de Morgan algebra is

$$x \;:\; 0\,1\,a\,b\,c\,d\,e\,f\;\,g\,h\,i\;\,j\,k\,l\,m\,n\;\;o\,p\,\alpha\,\beta$$

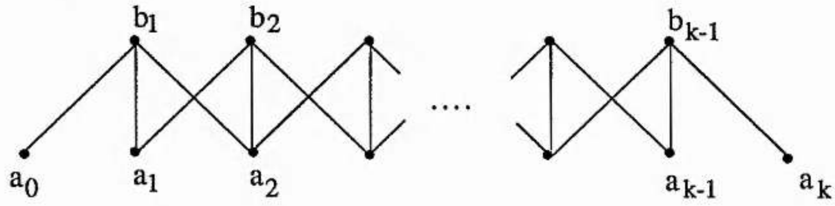
$$x^o :\; 1\,0\,k\,l\;\,j\;\,i\,g\,h\;\,e\,f\,d\,c\,a\,b\,o\,p\,m\,n\,\alpha\,\beta$$

As for the fixed points, we have the following result.

Theorem 2.2.5 $|\,\mathrm{FixL}(E_2DF_{2n};\, g)\,| = \begin{cases} \frac{1}{2}(J_{n/2} + J_{n/2+1}) & \text{if n is even,} \\ \frac{1}{2}(J_{(n-1)/2} + J_{(n+1)/2}) & \text{if n is odd.} \end{cases}$

Proof (a) Consider first the case where n is even, say $n = 2k$. Let A be the subset of E_2DF_{2n} described by

A :



Observe that $|A| = n$, and A is a distinguished of E_2DF_{2n} under g. Arguing as in the proof of Theorem 2.1.4, we see that the distinguished ideals of E_2DF_{2n} are of the form $I \cup I_*$ where I is an ideal of A that contains a_k, and $I_* = (E_2DF_{2n} \setminus A) \setminus g(I)$. If γ_k denotes the number of such ideals then we have

$$\gamma_k = \#(A \setminus \{a_k\}).$$

Considering those ideals of $A \setminus \{a_k\}$ that contain a_o, and those that do not, we have

$$\begin{aligned} \gamma_k &= \#(DF_{2(k-1)}) + \gamma_{k-1} \\ &= J_k + \gamma_{k-1} \quad (\text{ by Theorem 2.1.2}). \end{aligned}$$

Since this holds for each value of γ_k. we deduce that

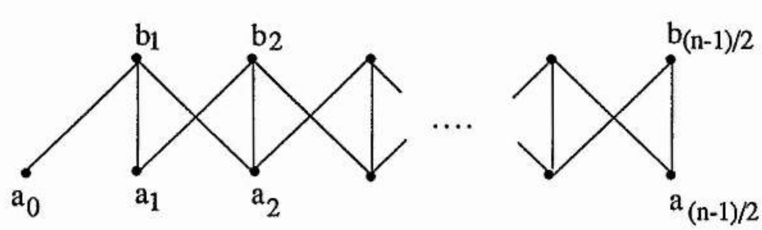
$$\gamma_k - \gamma_o = J_1 + J_2 + \ldots + J_k.$$

Since clearly $\gamma_o = 1 = J_o$ it follows by Theorem 2.1.1 that

$$\gamma_k = \sum_{i=0}^{k} J_i = \frac{1}{2}(J_k + J_{k+1}).$$

(b) Consider now the case where n is odd. Let B be the subset of E_2DF_{2n} described by

B :



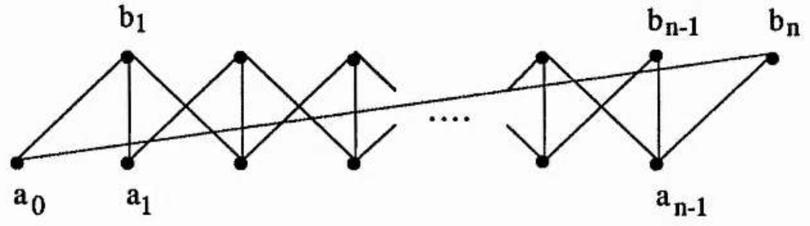
Arguing as in the proof of Theorem 2.1.4, we see that the number of fixed points of LE_2DF_{2n} in this case is the number of ideals of B that contain $a_{(n-1)/2}$ but not $b_{(n-1)/2}$. Thus, using Theorem 2.2.4, we see that

$$
\begin{aligned}
|\operatorname{Fix} L(E_2DF_{2n};g)| &= \#(E_2DF_{n-1};\ \bar{b}_{(n-1)/2}) \\
&= \#(E_2DF_{n-1}) - \#(E_2DF_{n-1};\ b_{(n-1)/2}) \\
&= \#(E_2DF_{n-1}) - \#(E_2DF_{n-3}) \\
&= \frac{1}{2}\,(J_{(n+1)/2} - 1) - \frac{1}{2}\,(J_{(n-1)/2} - 1) \\
&= \frac{1}{2}\,(J_{(n+1)/2} + J_{(n-1)/2})\,. \qquad \Diamond
\end{aligned}
$$

2.3 Half tiaras

Definition By a *half tiara* we shall mean an ordered set of the form

HT_{2n} :



On HT_{2n} there is clearly only one dual closure f with a self-dual image, namely f = id. There is only one dual isomorphism g on Im f, namely that described geometrically by a rotation 180^o, given by

$$g(a_i) = b_{n-i}, \quad g(b_i) = a_{n-i}.$$

The corresponding **MS**-algebra belongs properly to the class **M** of de Morgan algebras.

Theorem 2.3.1 $|L_{HT_{2n}}| = J_{n+1}$.

Proof We obtain HT_{2n} from E_2DF_{2n} by linking a_o and b_n. Consider the effect of adding to E_2DF_{2n} the link $a_o — b_n$. Clearly, this reduces the number of ideals. More precisely, in so doing we suppress all the ideals of E_2DF_{2n} that contain b_n but not a_o. Observe first that

$$\begin{aligned}\#(E_2DF_{2n}; b_n, \bar{a}_o) &= \#(E_2DF_{2n} \setminus \{a_o, b_1\}; b_n) \\ &= \#(E_2DF_{2n} \setminus \{a_o, b_1, a_{n-1}, b_n\}) \\ &= \#(E_2DF_{2n-4}).\end{aligned}$$

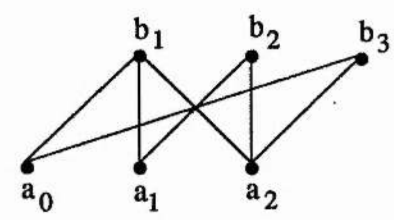
It now follows that

$$\begin{aligned}\#(HT_{2n}) &= \#(E_2DF_{2n}) - \#(E_2DF_{2n}; b_n, \bar{a}_o) \\ &= \#(E_2DF_{2n}) - \#(E_2DF_{2n-4})\end{aligned}$$

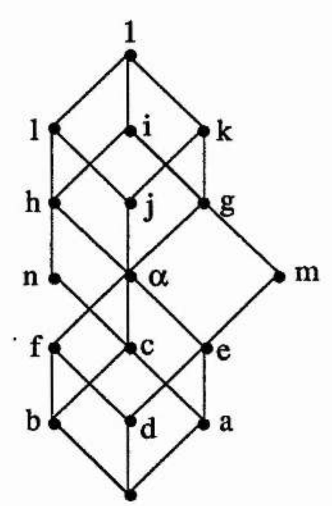
$$= \frac{1}{2}(J_{n+2} - 1) - \frac{1}{2}(J_n - 1) \quad \text{(by Theorem 2.2.4)}$$
$$= \frac{1}{2}(2\,J_{n+1} + J_n - 1) - \frac{1}{2}(J_n - 1)$$
$$= J_{n+1}. \quad \Diamond$$

By way of illustration, consider the half tiara

HT_6 :



By Theorem 2.3.1, we have $|\,L_{HT_6}\,| = J_4 = 17$. The Hasse diagram of L_{HT_6}



and the corresponding de Morgan algebra is

x : 0 1 a b c d e f g h i j k l m n α

x°: 1 0 i k g l h j c e a f b d n m α

As for the fixed points, we have the following result.

Theorem 2.3.2 $|\mathrm{FixL}(HT_{2n}; g)| = \begin{cases} J_{n/2} & \text{if n is even,} \\ J_{(n-1)/2} & \text{if n is odd.} \end{cases}$

Proof Observe first that $HT_{2n} \setminus \{a_o, b_n\}$ is isomorphic to DF_{2n-2} and is closed under g.

If I is a distinguished ideal of HT_{2n} under g. Then I must contain a_o but not b_n. In fact, since $g^2 = id$ so we have that $g(I) = X \setminus I$ and $g(X \setminus I) = X$. If $a_o \notin I$ then it gives $b_n = g(a_o) \notin g(I) = X \setminus I$ and $b_n \in I$ whence the contradiction $a_o \in I$; if $b_n \in I$ then it gives $a_o = g(b_n) \in g(I) = X \setminus I$ and $a_o \notin I$ which contradicts $b_n \in I$. This shows that if I is a distinguished ideal of T_{2n} under g then I must contain a_o but not b_n. Consequently, $I \setminus \{a_o\}$ is a distinguished ideal of $HT_{2n} \setminus \{a_o, b_n\}$ under g, equivalently, $I \setminus \{a_o\}$ is a distinguished ideal of DF_{2n-2} under g.

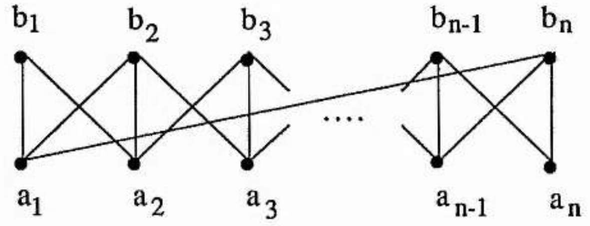
Conversely, if J is a distinguished ideal of DF_{2n-2} under g, equivalently, J is a distinguished ideal of $HT_{2n} \setminus \{a_o, b_n\}$ under g, then $J \cup \{a_o\}$ is a distinguished ideal of HT_{2n} under g. This correspondence of distinguished ideals is clearly a bijection. So we deduce by Theorem 2.1.4 that

$$|\mathrm{FixL}(HT_{2n}; g)| = |\mathrm{FixL}(DF_{2n-2}; g)| = \begin{cases} J_{n/2} & \text{if n is even,} \\ J_{(n-1)/2} & \text{if n is odd.} \end{cases} \qquad \Diamond$$

2.4 Tiaras

Definition By a *tiara* we shall mean an ordered set of the form

T_{2n} :



On the T_{2n} there is clearly one dual closure f with a self-dual image, namely f = id. There is only one dual isomorphism on Im f = T_{2n}, namely that described geometrically by a rotation through 180°, given by

$$g(a_i) = b_{n-i+1}, \quad g(b_i) = a_{n-i+1}.$$

Since $g^2 = id$, the corresponding **MS**-algebra ($L_{T_{2n}}$; g) is de Morgan algebra.

Theorem 2.4.1 $|L_{T_{2n}}| = 2J_n + \frac{1}{2}(J_{n-1} + 1)$.

Proof We obtain T_{2n} from HT_{2n-2} by adding two elements b_1 and a_n with relations $a_1 < b_1$, $a_2 < b_1$ and $a_n < b_{n-1}$, $a_n < b_n$. As this suggests, we can see that

$$\#(T_{2n}) = \#(HT_{2n-2}) + \#(T_{2n}; b_1) + \#(T_{2n} \setminus \{b_n, b_{n-1}, b_1, a_n\}).$$

Now

$$\begin{aligned}\#(T_{2n}; b_1) &= \#(T_{2n} \setminus \{b_1, a_1, a_2\}) \\ &= \#(DF_{2n} \setminus \{b_1, a_1, a_2\}) \\ &= \#(DF_{2n}; b_1) \\ &= \frac{1}{2}(J_{n-1} + J_n) \quad \text{(by the Corollary 1 to Theorem 2.1.2)}\end{aligned}$$

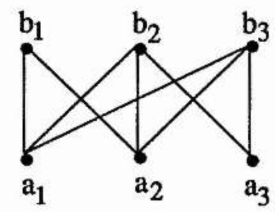
and $\#(T_{2n} \setminus \{b_n, b_{n-1}, b_1, a_n\}) = \#(M_{2n-4})$.

By Theorem 2.3.1 and the Corollary 2 to Theorem 2.2.1 it follows that

$$
\begin{aligned}
|L_{T_{2n}}| &= |L_{HT_{2n-2}}| + \#(DF_{2n}; b_1) + \#(M_{2n-4}) \\
&= J_n + \frac{1}{2}(J_{n-1} + J_n) + \frac{1}{2}(J_n + 1) \\
&= 2J_n + \frac{1}{2}(J_{n-1} + 1). \quad \Diamond
\end{aligned}
$$

By way of illustration, the tiara T_6 is

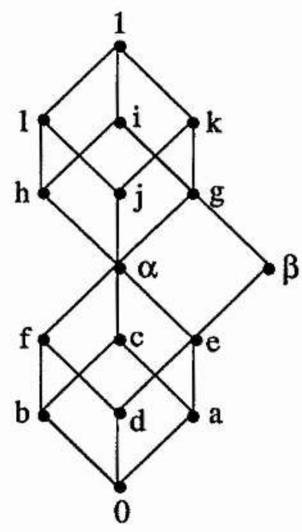
T_6:



By Theorem 2.4.1 we have

$$|L_{T_6}| = 2J_3 + \frac{1}{2}(J_2 + 1) = 16.$$

The underlying lattice is



and the corresponding de Morgan algebra is

x :	0	1	a	b	c	d	e	f	g	h	i	j	k	l	α	β
x^o :	1	0	k	l	j	i	g	h	e	f	d	c	a	b	α	β M

As for the fixed points, we have the following result.

Theorem 2.4.2 $|\operatorname{Fix} L(T_{2n}; g)| = \begin{cases} \frac{1}{2}(J_{(n-2)/2} + J_{n/2}) & \text{if n is even,} \\ \frac{1}{2}(J_{(n-1)/2} + J_{(n+1)/2}) & \text{if n is odd.} \end{cases}$

Proof Observe first that $T_{2n} \setminus \{a_1, b_n\}$ is isomorphic to E_2DF_{2n-2} and is closed under g.

Let I is a distinguished ideal of T_{2n} then I must contain a_1 but not b_n. In fact, if $a_1 \notin I$ then we have $b_n = g(a_1) \in I$ whence the contradiction $a_1 \in I$, and if $b_n \in I$, we have $a_1 = g(b_n) \notin I$ which contradicts $b_n \in I$. Consequently, $I \setminus \{a_1\}$ is a distinguished ideal of $T_{2n} \setminus \{a_1, b_n\}$.

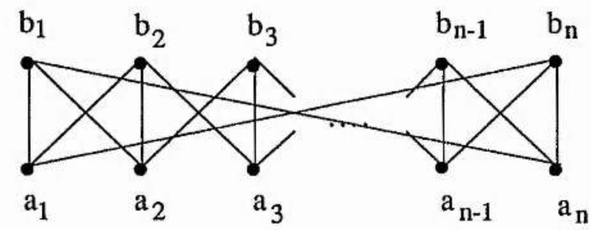
Conversely, if J is a distinguished ideal of $T_{2n} \setminus \{a_1, b_n\}$, then $J \cup \{a_1\}$ is a distinguished ideal of T_{2n}. This correspondence of distinguished ideals is clearly a bijection, so we deduce by Theorem 2.2.5 that

$$|\operatorname{Fix} L(T_{2n}; g)| = |\operatorname{Fix} L(E_2DF_{2n-2}; g)| = \begin{cases} \frac{1}{2}(J_{(n-2)/2} + J_{n/2}) & \text{if n is even,} \\ \frac{1}{2}(J_{(n-1)/2} + J_{(n+1)/2}) & \text{if n is odd.} \end{cases} \qquad \Diamond$$

2.5 Double crowns

Definition By a *double crown* we shall mean an ordered set of the form

DC_{2n} :



On the double crown DC_{2n} there is only one dual closure f with a self-dual image, namely f = id. All antitone maps g on DC_{2n} such that $g^2 \leq id$ are then such that $g^2 = id$, and give rise to de Morgan algebras. For every value of n there are the following:

(a) the horizontal reflection g_1 given by

$$g_1(a_i) = b_i, \quad g_1(b_i) = a_i;$$

(b) the rotation g_2 given by

$$g_2(a_i) = b_{n-i+1}, \quad g_2(b_i) = a_{n-i+1}.$$

For odd $n \geq 5$ there are the only possibilities. The case n = 3 is anomalous and will be illustrated below. For n even, however, there is also

(c) the slide-refection k given by

$$k(a_i) = b_{i+\frac{n}{2}}, \quad k(b_i) = a_{i+\frac{n}{2}}$$

the subscripts being reduced modulo n.

In the case n = 2, the slide-reflection k coincides with the rotation g_2.

Theorem 2.5.1 $|L_{DC_{2n}}| = 2J_n + 1.$

Proof We obtain DC_{2n} from T_{2n} by linking a_n with b_1. Consider first the effect of adding to T_{2n} the link a_n—b_1. Clearly, this reduces the number of ideals. More precisely, in so doing we suppress all the ideals of T_{2n} that contain b_1 but not a_n. So we have

$$|L_{DC_{2n}}| = |L_{T_{2n}}| - \#(T_{2n} ; \bar{a}_n, b_1).$$

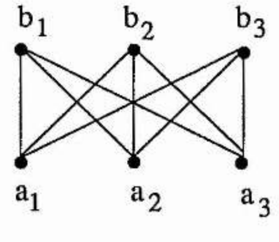
Since

$$\#(T_{2n} ; \bar{a}_n, b_1) = \#(T_{2n} \setminus \{b_{n-1}, b_n, a_n\}, b_1)$$
$$= \#(T_{2n} \setminus \{a_1, a_2, a_n, b_1, b_{n-1}, b_n\})$$

and since all the ideals of $T_{2n} \setminus \{a_1, a_2, a_n, b_1, b_{n-1}, b_n\}$ are equivalent to all the ideals of E_2DF_{2n-6}. It follows, from Theorem 2.4.1 and Theorem 2.2.4, therefore, that

$$\begin{aligned} |L_{DC_{2n}}| &= |L_{T_{2n}}| - \#(T_{2n}; \bar{a}_n, b_1) \\ &= |L_{T_{2n}}| - |L_{E_2DF_{2n-6}}| \\ &= 2J_n + \frac{1}{2}(J_{n-1} + 1) - \frac{1}{2}(J_{n-1} - 1) \\ &= 2J_n + 1. \quad \Diamond \end{aligned}$$

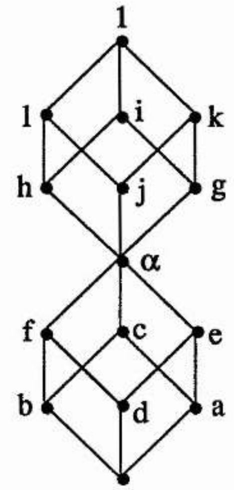
By way of illustration, consider the double crown DC_6:



Since, by symmetry, we can independently permute the a_i and the b_i to obtain the same diagram, there are six antitone mappings g on DC_6 such that $g^2 = id$. All of those produce Kleene algebras that are isomorphic. By Theorem 2.5.1 we have

$$|L_{DC_6}| = 2J_3 + 1 = 15,$$

the underlying lattice being



The six (isomorphic) Kleene algebras are described as follows:

0	1	a	b	c	d	e	f	g	h	i	j	k	l	α
1	0	i	k	g	l	h	j	c	e	a	f	b	d	α
1	0	i	l	h	k	g	j	e	c	a	f	d	b	α
1	0	k	i	g	l	j	h	c	f	b	e	a	d	α
1	0	l	i	h	k	j	g	f	c	b	e	d	a	α
1	0	k	l	j	i	g	h	e	f	d	c	a	b	α
1	0	l	k	j	i	h	g	f	e	d	c	b	a	α

Theorem 2.5.2 $|\,\mathrm{FixL}(DC_{2n};\, g_1)\,| = 1.$

Proof Similar to the proof of Theorem 2.1.3 we have that, under the mapping g_1, the only distinguished ideal of DC_{2n} is

$$I=\{a_1, a_2, ..., a_n\}. \quad \Diamond$$

Theorem 2.5.3 $|\,\mathrm{FixL}(DC_{2n};\, g_2)\,| = \begin{cases} J_{(n-2)/2} & \text{if n is even,} \\ J_{(n-1)/2} & \text{if n is odd.} \end{cases}$

Proof Observe first that $DC_{2n} \setminus \{a_1, a_n, b_1, b_n\}$ is isomorphic to DF_{2n-4} and is closed under g_2.

If I is distinguished ideal of DC_{2n} then I must contain a_1 and a_n but neither b_1 nor b_n. Consequently, $I \setminus \{a_1, a_n\}$ is a distinguished ideal of $DC_{2n} \setminus \{a_1, a_n, b_1, b_n\}$.

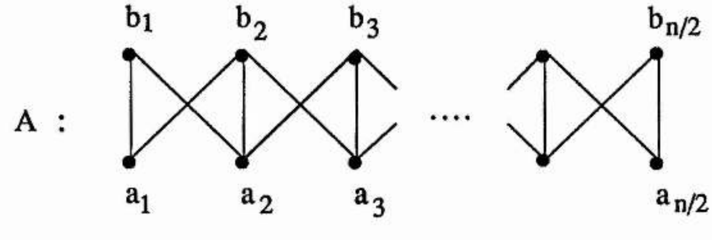
Conversely, if J is a distinguished ideal of $DC_{2n} \setminus \{a_1, a_n, b_1, b_n\}$ then $J \cup \{a_1, a_n\}$ is a distinguished ideal of DC_{2n}. Now this correspondence between the distinguished ideals of DC_{2n} and those of $DC_{2n} \setminus \{a_1, a_n, b_1, b_n\}$ is clearly a bijection, so we deduce by Theorem 2.1.4 that

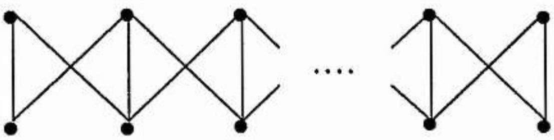
$$\begin{aligned} |\operatorname{Fix} L_{(DC_{2n};\, g_2)}| &= |\operatorname{Fix} L_{(DC_{2n} \setminus \{a_1, a_n, b_1, b_n\};\, g_2)}| \\ &= |\operatorname{Fix} L_{(DF_{2(n-2)};\, g_2)}| \\ &= \begin{cases} J_{(n-2)/2} & \text{if n is even,} \\ J_{(n-1)/2} & \text{if n is odd.} \end{cases} \qquad \Diamond \end{aligned}$$

We now determine the number of fixed points of $L_{(DC_{2n};\, k)}$.

Theorem 2.5.4 For n even, $|\operatorname{Fix} L_{(DC_{2n};\, k)}| = 2\, J_{n/2} - 1$.

Proof Consider the subsets A, B of DC_{2n} given by



B : 

Observe that $B = k(A) = DC_{2n} \setminus A$. For every ideal I of A let $I* = B \setminus k(I)$. Note that if J is a distinguished ideal of DC_{2n}, then

$$b_1 \in J \;\Rightarrow\; a_{n/2+1} = k(b_1) \notin J \;\Rightarrow\; b_{n/2} \notin J.$$

Arguing as in the proof of Theorem 2.1.4 we can show, using the geometric nature of k, that a subset J of DC_{2n} is a distinguished ideal under k if and only if it is of the $I \cup I*$ where I is an ideal of A that does not contain both b_1 and $b_{n/2}$, and $I*$ does not contain both $b_{n/2+1}$ and b_n. The latter condition is equivalent to $a_1 \in I$ and $a_{n/2} \in I$. It follows that the number of fixed points of $L_{(DC_{2n};\, k)}$ is

$$t = \#(DF_n) - \#(DF_n;\ b_1, b_{n/2}) - \#(DF_n;\ \bar{a}_1; \bar{a}_{n/2}).$$

Using Theorem 2.1.2, the Corollaries 1, 2 of Theorem 2.2.1, we deduce that

$$\begin{aligned} t &= \#(DF_n) - \#(DF_n;\ b_1, b_{n/2}) - \#(DF_n;\ \bar{a}_1; a_{n/2}) \\ &= \#(DF_n) - \#(DF_n;\ b_1, b_{n/2}) - \#(M_{2(n/2-3)}) \\ &= J_{n/2+1} - \frac{1}{2}\,(J_{n/2-1} + 1) - \frac{1}{2}\,(J_{n/2-1} + 1) \\ &= J_{n/2+1} - J_{n/2-1} - 1 \\ &= 2\,J_{n/2} - 1. \quad \Diamond \end{aligned}$$

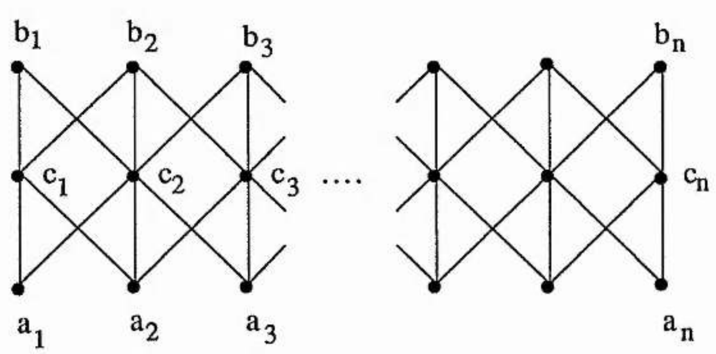
2.6 Tall double fences

We shall consider only one X whose height is greater than 1. This will suffice to illustrate the increasing complexity of the combinatorial arguments required. The X that we choose for this involves another sequence also defined recursively by a second order difference equation, namely the sequence $(r_n)_{n\geq 0}$ given by

$$r_0 = 1, r_1 = 2, \ (\forall\ n \geq 2)\ r_n = 3r_{n-1} - r_{n-2}.$$

Definition By a *tall double fence* we shall mean an ordered set of the form

TDF_{3n} :



We first determine the size of $|L_{TDF_{3n}}|$.

Theorem 2.6.1 $|L_{TDF_{3n}}| = 2\, r_n$.

Proof We show first that $\#(TDF_{3n}; c_n) = \#(TDF_{3n}; \bar{c}_n)$.

Observe that

$$\#(TDF_{3n}; c_n) = \#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n}; b_n)$$

$$= \#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n-3}; c_{n-1});$$

$$\#(TDF_{3n}; \bar{c}_n) = \#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n-3}; \bar{b}_{n-1}, \bar{c}_{n-1})$$

$$= \#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n-3}; \bar{c}_{n-1}).$$

So

$$\#(TDF_{3n}; c_n) - \#(TDF_{3n}; \bar{c}_n) = \#(TDF_{3n}; c_{n-1}) - \#(TDF_{3n}; \bar{c}_{n-1}).$$

Continuing this recursion, we obtain

$$\#(TDF_{3n}; c_n) - \#(TDF_{3n}; \bar{c}_n) = \#(TDF_{3n}; c_2) - \#(TDF_{3n}; \bar{c}_2).$$

A direct computation gives that

$$\#(TDF_6; c_2) = \#(TDF_6; \bar{c}_2) = 5.$$

So we obtain

$$\#(TDF_{3n}; c_n) = \#(TDF_{3n}; \bar{c}_n)$$

Observe now that

$$\#(TDF_{3n}; b_n) = \#(TDF_{3n-3}; c_{n-1})$$
$$= \#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n-3}; c_{n-1}, \bar{b}_{n-1}).$$

and $\#(TDF_{3n}; \bar{b}_n) = \#(TDF_{3n}; \bar{b}_n, c_n) + \#(TDF_{3n}; \bar{c}_n)$

$$= [\#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1})] +$$
$$[\#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n}; \bar{c}_{n-1})]$$
$$= 2\,\#(TDF_{3n-3}) - 2\,\#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n-3}; \bar{c}_{n-1}).$$

Then we have

$$\#(TDF_{3n}) = \#(TDF_{3n}; b_n) + \#(TDF_{3n}; \bar{b}_n)$$
$$= 2\,\#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n-3}; c_{n-1}, \bar{b}_{n-1}) +$$
$$\#(TDF_{3n-3}; \bar{c}_{n-1}).$$

Now

$$\#(TDF_{3n-3}; \bar{b}_{n-1}) = \#(TDF_{3n-3}; c_{n-1}, \bar{b}_{n-1}) + \#(TDF_{3n-3}; \bar{c}_{n-1}),$$

so

$$\#(TDF_{3n}) = 2\,\#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1}) + \#(TDF_{3n-3}; \bar{b}_{n-1})$$
$$= 2\,\#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1}) + [\#(TDF_{3n-3}) - \#(TDF_{3n-3}; b_{n-1})]$$
$$= 3\,\#(TDF_{3n-3}) - 2\,\#(TDF_{3n-3}; b_{n-1}).$$

Since

$$\#(TDF_{3n}; b_n) = \#(TDF_{3n-3}; c_{n-1}),$$

we have

$$\#(TDF_{3n-3}) = \#(TDF_{3n-3}; c_{n-1}) + \#(TDF_{3n-3}; \bar{c}_{n-1})$$

$$= 2\,\#(TDF_{3n-3}; c_{n-1})$$

$$= 2\,\#(TDF_{3n}; b_n).$$

We therefore have

$$\#(TDF_{3n}) = 3\,\#(TDF_{3n-3}) - \#(TDF_{3n-6}).$$

If now we let $r_n = \frac{1}{2}\,\#(TDF_{3n})$, then we obtain $r_n = 3r_{n-1} - r_{n-2}$ with $r_o = 1$, $r_1 = \frac{1}{2}\#(TDF_3) = 2$ and $\#(TDF_{3n}) = 2r_n$. ◊

Corollary 1 $\#(TDF_{3n}; b_n) = r_{n-1}$. ◊

Corollary 2 $\#(TDF_{3n}; a_n, \bar{b}_n) = 2(r_n - r_{n-1})$.

Proof Note first that

$$\#(TDF_{3n}; a_n, \bar{b}_n) = \#(TDF_{3n}; c_n, \bar{b}_n) + \#(TDF_{3n}; a_n, \bar{c}_n)\,.$$

Now

$$\#(TDF_{3n}) = \#(TDF_{3n}; b_n) + \#(TDF_{3n}; \bar{b}_n)$$

$$= \#(TDF_{3n}; b_n) + \#(TDF_{3n}; c_n, \bar{b}_n) + \#(TDF_{3n}; \bar{c}_n),$$

so

$$\#(TDF_{3n}; c_n, \bar{b}_n) = \#(TDF_{3n}) - \#(TDF_{3n}; b_n) - \#(TDF_{3n}; \bar{c}_n)$$

$$= \#(TDF_{3n}) - \#(TDF_{3n}; b_n) - \#(TDF_{3n}; a_n, \bar{c}_n) - \#(TDF_{3n}; \bar{a}_n, \bar{c}_n\,).$$

But

$$\#(TDF_{3n}; \bar{a}_n, \bar{c}_n) = \#(TDF_{3n}; \bar{a}_n)$$

$$= \#(TDF_{3n-3}) - \#(TDF_{3n-3}; c_{n-1})$$

$$= \#(TDF_{3n-3}) - \#(TDF_{3n}; b_n).$$

So

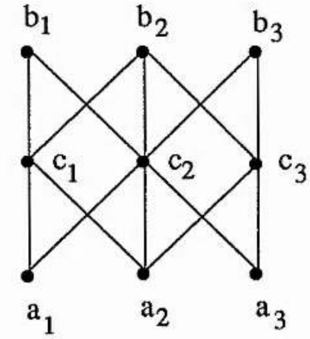
$$\#(TDF_{3n};\ c_n, \bar{b}_n) = \#(TDF_{3n}) - \#(TDF_{3n-3}) - \#(TDF_{3n};\ a_n, \bar{c}_n)\ .$$

It follows that

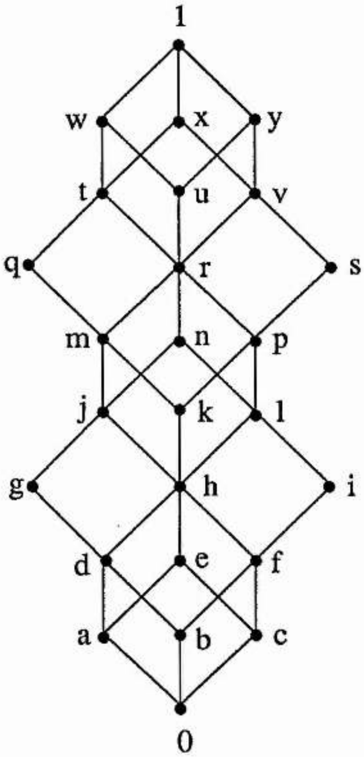
$$\begin{aligned}\#(TDF_{3n};\ a_n, \bar{b}_n) &= \#(TDF_{3n};\ c_n, \bar{b}_n) + \#(TDF_{3n};\ a_n, \bar{c}_n)\\ &= \#(TDF_{3n}) - \#(TDF_{3n-3})\\ &= 2(r_n - r_{n-1}).\quad \Diamond\end{aligned}$$

By way of illustration, consider the tall double fence

TDF_9 :



By Theorem 2.6.1 we have $|L_{TDF_9}| = 2\ r_3 = 26$. The Hasse diagram of L_{TDF_9} is



where $a = (a_1)^{\downarrow}$, $b = (a_2)^{\downarrow}$, $c = (a_3)^{\downarrow}$, $g = (c_1)^{\downarrow}$, $i = (c_3)^{\downarrow}$, $k = (c_2)^{\downarrow}$, $q = (b_1)^{\downarrow}$, $s = (b_3)^{\downarrow}$, $u = (b_2)^{\downarrow}$.

On TDF_{3n} there is clearly only one dual closure f with a self-dual image, namely $f = id$. It is readily seen by [13, Theorem 1.1] that, all antitone mappings g on Im f with $g^2 \leq id$ are such that $g^2 = id$, and so all corresponding **MS**-algebras are de Morgan algebras. There are only two such mappings, namely g and k given by

$$g(a_i) = b_{n-i+1}, \quad g(b_i) = a_{n-i+1}, \quad g(c_i) = c_{n-i+1};$$

$$k(a_i) = b_i, \quad k(b_i) = a_i, \quad k(c_i) = c_i.$$

These can be considered as, respectively, a reflection in the horizontal and a rotation through 180^o.

As far as k is concerned, every c_i is fixed by k, by Theorem 1.6 of Chapter 1, the corresponding **MS**-algebra $L_{(TDF_{3n};\, k)}$ is fixed point free, and it is a Kleene algebra. To see this, it suffices to observe that, for every $x \in X$, $k^2(x) = x$ and x is comparable with k(x) [Chapter 1, Theorem 1.7].

As for g, there are two cases to be considered:

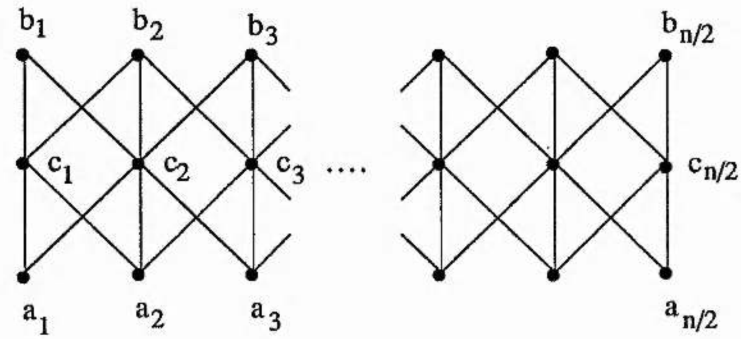
(a) Consider first the case when n is odd. In this case, the element $c_{(n+1)/2}$ is a fixed point of g. By Theorem 1.6 of Chapter 1, the corresponding **MS**-algebra is fixed point free. If $n \neq 1$ then we can see from Theorem 1.7 of Chapter 1, that $L_{(TDF_{3n};\, g)}$ is not a Kleene algebra, since $g(c_1) = c_n \parallel c_1 = g^2(c_1)$.

(b) Consider now the case when n is even. In this case g has no fixed points, so $L_{(TDF_{3n};\, g)}$ has fixed points. In the following result we shall see that $L_{(TDF_{3n};\, g)}$ has more than one fixed point, and therefore belongs properly to **M**.

Theorem 2.6.2 If n is even, then $|\operatorname{Fix} L_{(TDF_{3n};\, g)}| = 2\,(r_{n/2} - r_{n/2-1})$.

Proof Since g can be regarded as a rotation through 180^{o}, the fixed points of L_X are those ideals of X which contain half of the c_i and have a 'skew-symmetric profile'. To be more explicit, consider the subset A of TDF_{3n} described by

$A = TDF_{3n/2}$:



Let $A' = TDF_{3n} \setminus A$,and for every ideal I of A, let $I_* = A' \setminus g(I)$. Note that if J is a distinguished ideal of TDF_{3n} under g then

$$a_{n/2} \in J \text{ and } b_{n/2} \notin J.$$

Arguing as in the proof of Theorem 2.1.4 we can show, using the geometric nature of g, that a subset J of TDF_{3n} is a distinguished ideal under g if and only if it is of the form $I \cup I_*$ where I is an ideal of A which contains $a_{n/2}$ and does not $b_{n/2}$. It follows that the number of fixed points of TDF_{3n} (when n is even) is precisely the number of ideals of A that contain $a_{n/2}$ but not $b_{n/2}$. By Corollary 2 to Theorem 2.6.1, this is $2(r_{n/2} - r_{n/2-1})$. ◊

CHAPTER 3

CONGRUENCE LATTICES

In this chapter we shall describe some properties of congruence lattices of Ockham algebras. Our discussion here is based on the relations Φ_i ($i = 0, 1, 2, ...$) on an Ockham algebra $(L; f)$ which are defined by

$$(x, y) \in \Phi_i \iff f^i(x) = f^i(y) .$$

It is clear that, for each i, $\Phi_i \in \mathrm{Con}\, L$ and $\Phi_i \leq \Phi_{i+1}$. Note that $\Phi_0 = \omega$. Since Con L is a complete distributive lattice, $\bigvee_{i\geq 0} \Phi_i \in \mathrm{Con}\, L$. We denote this congruence by Φ_ω.

Theorem 3.1 If $(L; f)$ is an Ockham algebra then $(x, y) \in \Phi_\omega$ if and only if $f^n(x) = f^n(y)$ for some positive integer n (depending on x and y). Moreover, if L is non-trivial then $\Phi_\omega < \iota$.

Proof If $(x, y) \in \Phi_\omega$ then there exist elements $t_0, ..., t_k$ and congruences $\Phi_{i_1}, ..., \Phi_{i_k}$ such that

$$x = t_0 \, \Phi_{i_1} \, t_1 \, \Phi_{i_2} \, t_2 \, ... \, t_{k-1} \, \Phi_{i_k} \, t_k = y.$$

Denote the greatest of these Φ_i by Φ_n, i.e., $\Phi_n = \bigvee_{j=1}^{k} \Phi_{i_j}$.

Then we have $x \, \Phi_n \, y$, i.e., $f^n(x) = f^n(y)$.
Conversely, if $f^n(x) = f^n(y)$ then clearly $(x, y) \in \Phi_\omega$. Finally, if L is non-trivial then, since $f^n(0) \neq f^n(1)$, for all n we have $\Phi_\omega \neq \iota$. ◊

If an Ockham algebra $(L; f)$ belongs to a Berman class then there is a smallest Berman class to which it belongs. We denote this by $\mathbf{B}(L)$.

Theorem 3.2 If $(L; f) \in \mathbf{K}_{p,q}$ with $\mathbf{B}(L) = \mathbf{K}_{p,q}$. Then

$$\Phi_0 = \omega < \Phi_1 < \Phi_2 < \; ... < \Phi_q = \Phi_{q+1} = \; ... \; = \Phi_\omega$$

and Con L has length at least q+1.

Proof Observe that

$$\begin{aligned}(x,y) \in \Phi_{q+1} &\Leftrightarrow f^{q+1}(x) = f^{q+1}(y) \\ &\Leftrightarrow f^q(x) = f^{2p-1}[f^{q+1}(x)] = f^{2p-1}[f^{q+1}(y)] = f^q(y) \\ &\Leftrightarrow (x,y) \in \Phi_q .\end{aligned}$$

It follows that $\Phi_q = \Phi_{q+1}$. If now $(x, y) \in \Phi_{q+r}$ where $r \geq 1$ then $f^{q+r}(x) = f^{q+r}(y)$ gives

$$(f^{r-1}(x) , f^{r-1}(y)) \in \Phi_{q+1} = \Phi_q ,$$

and so $f^{q+r-1}(x) = f^{q+r-1}(y)$, i.e., $(x, y) \in \Phi_{q+r-1}$. Thus we see that $\Phi_q = \Phi_{q+1} = \dots$

Consequently,

$$\Phi_\omega = \bigvee_{i\geq 0} \Phi_i = \Phi_q .$$

Suppose now, by way of obtaining a contradiction, that for some n with $1< n \leq q$ we have $\Phi_{n-1} = \Phi_n$.

For $x, y \in L$, if $(x, y) \in \Phi_{n+1}$, then $f^{n+1}(x) = f^{n+1}(y)$ and so $(f(x), f(y)) \in \Phi_n = \Phi_{n-1}$. So we have $\Phi_{n+1} = \Phi_n$ and continuing with this process we obtain

$$(*) \qquad \Phi_{n-1} = \Phi_n = \dots = \Phi_q = \Phi_{q+1} = \dots .$$

But, $L \notin \mathbf{K}_{p,q-1}$ and so there exists $x \in L$ such that

$$f^{q-1}(x) \neq f^{2p+q-1}(x).$$

So $(x, f^{2p}(x)) \notin \Phi_{q-1}$. But $f^q(x) = f^{2p+q}(x)$, i.e., $(x, f^{2p}(x)) \in \Phi_q$ whence $\Phi_{q-1} < \Phi_q$ which contradicts (*). This then completes the proof of the theorem. ◊

Corollary For an Ockham algebra (L; f), the following statements are equivalent:

(1) $(\forall\, i \geq 1) \quad \Phi_i = \omega$;

(2) $\Phi_1 = \omega$;

(3) f is injective.

Moreover, if (L; f) belongs to some Berman class $\mathbf{K}_{p,q}$, then each of the above is equivalent to $(L; f) \in \mathbf{K}_{p,0}$.

Proof The equivalence of (1), (2), (3) is clear. As for the final statement, suppose that for every $x \in L$, we have $f^{2p+q}(x) = f^q(x)$. Then, by (3), we have $f^{2p+q-1}(x) =$

$f^{q-1}(x)$, and so on, whence eventually, $f^{2p}(x) = x$, and then $L \in \mathbf{K}_{p,0}$. Conversely, suppose that $(L; f) \in \mathbf{K}_{p,0}$ and that $\Phi_1 \neq \omega$. Then there exist $x, y \in L$ such that $x \neq y$ and $f(x) = f(y)$. This implies that $f^{2p}(x) = f^{2p}(y)$, and hence the contradiction $x = y$. ◊

Note that the hypothesis that L belongs to a Berman class is necessary in the above. As the following example shows, it is possible in general for f to be injective with $L \notin \mathbf{K}_{p,0}$ for any p.

Example 3.1 Let L be the infinite chain

$$0 < x_1 < x_3 < x_{2n+1} < \ldots < x_{2n} < \ldots < x_2 < x_0 < 1.$$

Define f by

$$f(0) = 1,\ f(1) = 0,\quad (i = 0, 1, 2, \ldots)\ f(x_i) = x_{i+1}.$$

Then (L; f) is an Ockham algebra on which f is injective. For all $m \neq n$ and all i we have $f^m(x_i) \neq f^n(x_i)$ and so L does not belong to any Berman class.

For every Ockham algebra (L; f) it is clear that

$$\{0,1\} \leq \ldots \leq f^{i+1}(L) \leq f^i(L) \leq \ldots \leq f(L) \leq f^0(L) = L.$$

where in this context $\leq$ means ' is a subalgebra of '. It is easy to verify that there are Ockham algebra isomorphisms $L/\Phi_i \simeq f^i(L)$ when i is even, and $L/\Phi_i \simeq (f^i(L))^d$ when i is odd. Moreover, the following result is clear.

Theorem 3.3 If $(L; f) \in \mathbf{K}_{p,q}$, then, for $n \leq q$, $L/\Phi_n \in \mathbf{K}_{p,q-n}$. ◊

Theorem 3.4 Let (L; f) belong to a Berman class. If $\mathbf{B}(L) = \mathbf{K}_{p,q}$ then we have the mutually equivalent chains

$$(1)\ L \supset f(L) \supset f^2(L) \supset \ \dots\ \supset f^q(L) = f^{q+1}(L) = \ \dots\ ;$$

$$(2)\ \Phi_0 < \Phi_1 < \ \dots\ < \Phi_q = \Phi_{q+1} = \ \dots\ .$$

Conversely, each of these chains implies that $\mathbf{B}(L) = \mathbf{K}_{p,q}$ for some $p \geq 1$.

Proof From $f^q(x) = f^{2p+q}(x) = f^{q+1}[f^{2p-1}(x)] \in f^{q+1}(L)$ it follows that $f^q(L) \subseteq f^{q+1}(L)$,whence we have

$$f^q(L) = f^{q+1}(L) = \ \dots\ .$$

Suppose now, by way of obtaining a contradiction, that for some $n < q$ we have $f^{n-1}(L) = f^n(L)$. Then

$$f^q(L) = f^{q-n}[f^n(L)] = f^{q-n}[f^{n-1}(L)] = f^{q-1}(L)\ .$$

But since $\mathbf{B}(L) = \mathbf{K}_{p,q}$ we have $L \notin \mathbf{K}_{p,q-1}$ and so there exists $x \in L$ such that $f^{q-1}(x) \neq f^{2p+q-1}(x)$. Now $f^{q-1}(x) \in f^{q-1}(L) = f^q(L)$ gives $f^{q-1}(x) = f^q(y)$ for some $y \in L$, whence $f^q(y) = f^{q-1}(x) \neq f^{2p+q-1}(x) = f^{2p+q}(y)$; and this contradicts the fact that $L \in \mathbf{K}_{p,q}$. This then establishes the chain (1). The other chain is obtained immediately from Theorem 3.2.

For the converse, suppose that $\mathbf{B}(L) = \mathbf{K}_{p,n}$. If $n > q$ then from the chain (1) we have $f^n(L) = f^{n+1}(L)$; and if $n < q$ we have $f^{n-1}(L) \supset f^n(L)$. Thus we require $n = q$. ◊

Corollary If $\mathbf{B}(L) = \mathbf{K}_{p,q}$ then $\mathbf{B}(f^i(L)) = \mathbf{K}_{p,q-i}$. ◊

We now turn our attention to the congruence lattices of an Ockham algebra (L; f). We first establish the following results. In what follows we shall use the symbol $\prec$ to mean ' is covered by ' .

Theorem 3.5 Let (L; f) be an Ockham algebra. If $a, b \in L$ are such that $a < b$ and $f(a) = f(b)$, then $\theta(a, b) \in$ Con L has a complement in $[\omega, \Phi_1]$.

Proof By Theorem 1.9 of Chapter 1 we have

$$\theta(a, b) = \bigvee_{n\geq 0} \theta_{lat}(f^n(a), f^n(b)).$$

It follows that $\theta(a, b) = \theta_{lat}(a, b) \in$ Con L and clearly, $\theta(a, b) \leq \Phi_1$. Let now $\alpha = \theta(a, b)$. Since α is a principal lattice congruence, it has a complement β in $\mathrm{Con}_{lat}L$, namely $\beta = \theta_{lat}(0, a) \vee \theta_{lat}(b, 1)$. Consider the lattice congruence $\alpha' = \beta \wedge \Phi_1$. Since every lattice congruence contained in Φ_1 is a congruence, we have $\alpha' \in$ Con L.

Now

$$\alpha \vee \alpha' = \theta(a, b) \vee (\beta \wedge \Phi_1) = (\theta_{lat}(a, b) \vee \beta) \wedge (\theta_{lat}(a, b) \vee \Phi_1)$$
$$= \iota \wedge \Phi_1 = \Phi_1.$$

and

$$\alpha \wedge \alpha' = \theta(a, b) \wedge (\beta \wedge \Phi_1) = \omega \wedge \Phi_1 = \omega.$$

It follows that α' is the complement of α in $[\omega, \Phi_1]$. ◊

Theorem 3.6 Let (L; f) be an Ockham algebra. If $a, b \in L$ are such $a \prec b$ and $f(a) = f(b)$, then $\theta(a, b)$ is an atom of Con L.

Proof It is clear from the proof in Theorem 3.5 that $\theta(a, b) = \theta_{lat}(a, b) \leq \Phi_1$.

Suppose now that $\omega \leq \phi < \theta(a, b)$. Then $\phi = \phi \wedge \theta(a, b) = \phi \wedge \theta_{lat}(a, b)$. So, if $(x, y) \in \phi$ then $(x, y) \in \phi \wedge \theta_{lat}(a, b)$. Thus we have

$$(*) \quad x \wedge a = y \wedge a, \quad x \vee b = y \vee b, \quad (x, y) \in \phi.$$

Writing $s = (x \vee a) \wedge b$ and $t = (y \vee a) \wedge b$ we see that $(s, t) \in \phi$; and, since $a \prec b$ by the hypothesis, we have $\{s, t\} \subseteq \{a, b\}$. Now if $s \neq t$ then one of s, t must be a and the other b, whence $(a, b) \in \phi$. This gives the contradiction $\theta(a, b) \leq \phi$. Hence we must have $s = t$, i.e. $(x \vee a) \wedge b = (y \vee a) \wedge b$. But from $(*)$ we have $x \vee a \vee b = y \vee a \vee b$; so, by the distributivity of L, $x \vee a = y \vee a$. Again by $(*)$ and the distributivity of L we obtain $x = y$, and hence $\phi = \omega$. ◊

For an Ockham algebra $(L; f)$, let $\alpha \in \text{Con } L$. An α - class $[a]\alpha$ will be called *locally finite* if, whenever $x, y \in [a]\alpha$ with $x < y$, the interval $[x, y]$ has only finitely many elements in L. We now show the following result.

Theorem 3.7 Let $(L; f)$ be an Ockham algebra. If $a, b \in L$ are such that

$$a \prec b, \ (a, b) \notin \Phi_n, \ (a, b) \in \Phi_{n+1},$$

then $\Phi_n \vee \theta(a, b)$ is an atom of $[\Phi_n, \Phi_{n+1}]$. Moreover, if every Φ_{n+1}-class is locally finite, then every atom of $[\Phi_n, \Phi_{n+1}]$ is of this form.

Proof Since $f^{n+1}(a) = f^{n+1}(b)$ we have $(f(a), f(b)) \in \Phi_n$, so $\theta(f(a), f(b)) \leq \Phi_n$ and consequently

$$(1) \ \Phi_n \vee \theta(a, b) = \Phi_n \vee \theta_{lat}(a, b) \vee \theta(f(a), f(b)) = \Phi_n \vee \theta_{lat}(a, b).$$

Clearly, we have $\Phi_n < \Phi_n \vee \theta(a, b) \leq \Phi_{n+1}$. Suppose that $\alpha \in \text{Con } L$ is such that $\Phi_n \leq \alpha < \Phi_n \vee \theta(a, b)$. Then by an argument as in Theorem 3.6 we have

$$(2) \ \alpha \wedge \theta_{lat}(a, b) = \omega.$$

It now follows from (1) and (2) that

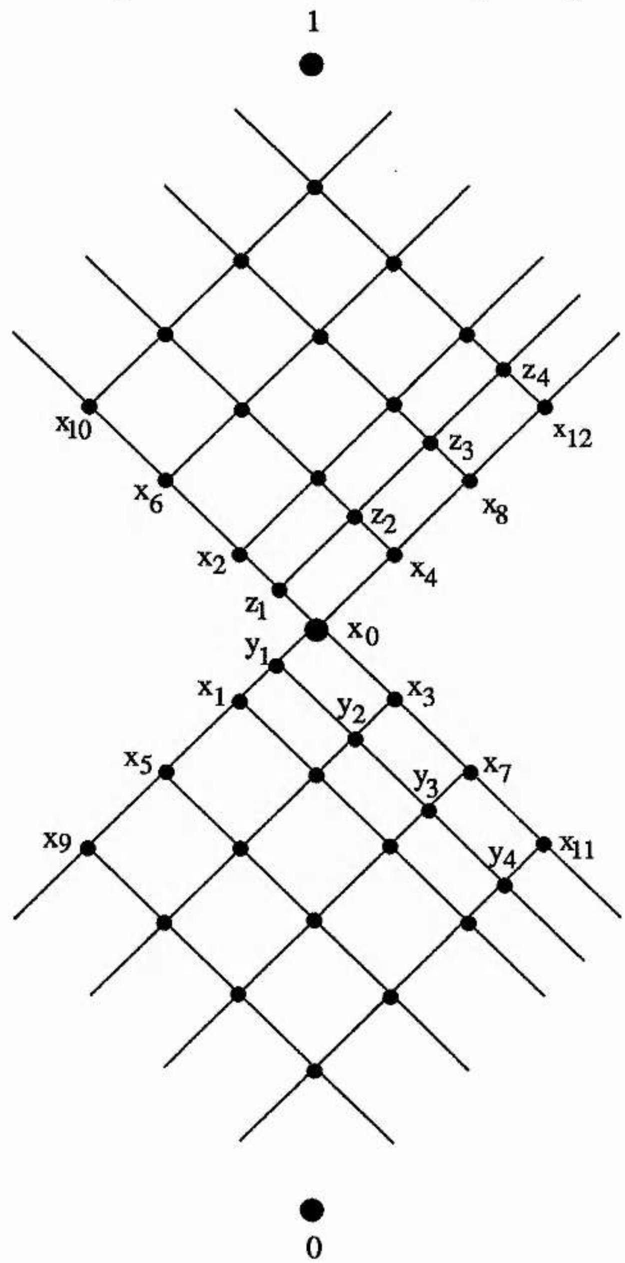
$$\alpha = \alpha \wedge (\Phi_n \vee \theta(a, b)) = \alpha \wedge (\Phi_n \vee \theta_{lat}(a, b)) = \alpha \wedge \Phi_n$$

and therefore $\alpha \leq \Phi_n$, whence $\alpha = \Phi_n$. Hence $\Phi_n \vee \theta(a, b)$ is an atom of $[\Phi_n, \Phi_{n+1}]$.

Finally, let ϕ be an atom of $[\Phi_n, \Phi_{n+1}]$. Then there exists $a, b \in L$ such that $a < b$, $(a, b) \notin \Phi_n$, and $(a, b) \in \phi \leq \Phi_{n+1}$. If every Φ_{n+1}-class is locally finite, there exist

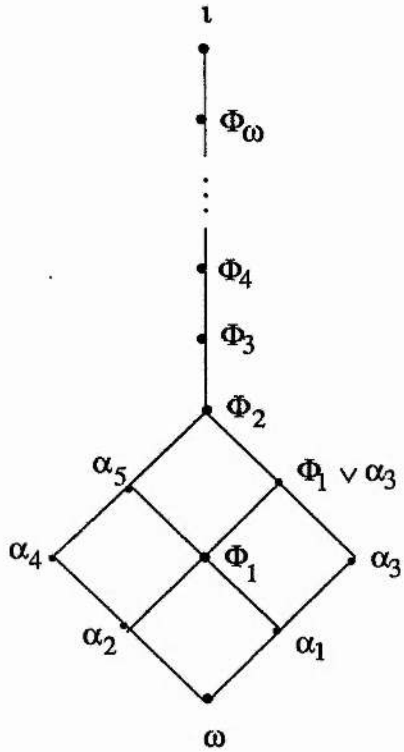
$p, q \in [a, b]$ such that $p \prec q$, $(p, q) \notin \Phi_n$, $(p, q) \in \phi$. For such p, q we have $\Phi_n \prec \Phi_n \vee \theta(p, q) \leq \phi$ whence $\phi = \Phi_n \vee \theta(p, q)$. ◊

Example 3.2 [The sink] Consider the ordered set L given by



and made into an Ockham algebra by defining $f(0) = 1$, $f(1) = 0$, and $f(x_i) = x_{i-1}$, $f(x_0) = x_0$, $f(z_i) = y_i$, $f(y_1) = f(x_0) = x_0$, $f(y_2) = f(x_3)$, $f(y_3) = f(x_7)$, ... , and extending to the whole of L.

Observe that the Φ_ω-classes are $\{0\}$, $\{1\}$, $L\setminus\{0,1\}$, and are locally finite. It is easy to see that Con L is as follows:

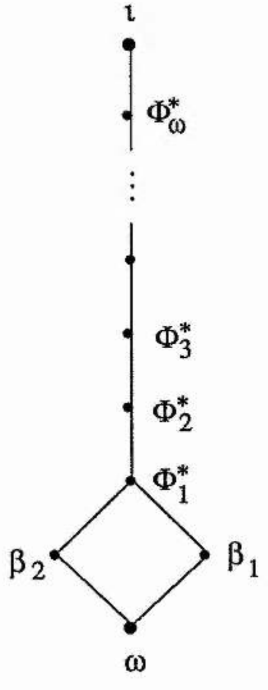


where $\Phi_1 = \theta(x_1, x_0)$, $\Phi_2 = \theta(x_1, x_2) = \theta(x_0, x_2)$, $\Phi_3 = \theta(x_1 \wedge x_3, x_2) = \theta(x_3, x_0)$, $\Phi_4 = \theta(x_1 \wedge x_3, x_2 \vee x_4) = \theta(x_0, x_4)$, ... ; $\alpha_1 = \theta(x_1, y_1)$, $\alpha_2 = \theta(y_1, x_0)$, $\alpha_3 = \theta(z_1, x_2)$, $\alpha_4 = \theta(y_1, z_1) = \theta(x_0, z_1)$, $\alpha_5 = \theta(x_1, z_1)$.

Note that $[\omega, \Phi_1]$ and $[\Phi_1, \Phi_2]$ are boolean lattices, and every interval $[\Phi_i, \Phi_{i+1}]$ $(i = 2, 3, ...)$ is a 2-element chain.

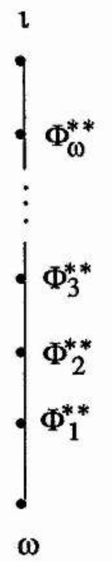
Note, by way of illustrating Theorem 3.5, that $f(y_1) = f(x_0)$ with $y_1 < x_0$. So $\theta(y_1, x_0)$ has a complement in $[\omega, \Phi_1]$, namely $\theta(x_1, y_1)$. Also since $y_1 \prec x_0$, Theorem 3.7 shown that $\theta(y_1, x_0)$ is an atom of Con L.

Example 3.3 Let (L; f) be the Ockham algebra in Example 3.2, and let $A = f(L)$. Then $A = L\backslash\{z_i;\ i \geq 1\}$, and the $\Phi_\omega|_{f(L)}$- classes are : $\{0\}$, $\{1\}$, $A\backslash\{0,1\} = f(L)\backslash\{0,1\}$ $= L\backslash(\{0,1\} \cup \{z_i;\ i \geq 1\})$, and are locally finite. It is easy to see that Con A = Con f(L) $\simeq$ Con $L/\Phi_1 \simeq [\Phi_1, \iota]$ in Con L, and is as follows:



where $\beta_1 = \theta_A(x_1, y_1)$, $\beta_2 = \theta_A(y_1, x_0)$, $\Phi^*_1 = \Phi_1|_A = \theta_A(x_1, x_0)$, $\Phi^*_2 = \Phi_2|_A =$ $\theta_A(x_1, x_2) = \theta_A(x_0, x_2)$, $\Phi^*_3 = \Phi_3|_A = \theta_A(x_1 \wedge x_3, x_2) = \theta_A(x_3, x_0)$, ..., $\Phi^*_\omega =$ $\Phi_\omega|_A$.

Example 3.4 Let (L; f) be the Ockham algebra in Example 3.2, and let $B = f^2(L)$. Then $B = f^2(L) \simeq L \setminus (\{y_i;\ i \geq 1\} \cup \{z_i;\ i \geq 1\})$, and $\Phi_\omega|_{f^2(L)}$-classes are $\{0\}$, $\{1\}$, $L \setminus (\{y_i;\ i \geq 1\} \cup \{z_i;\ i \geq 1\})$, and are locally finite. It is readily seen that Con B = Con $f^2(L) \simeq$ Con $L/\Phi_2 \simeq [\Phi_2, \iota]$, and that Con B is as follows:



where $\Phi_1^{**} = \Phi_1|_{f^2(L)} = \theta_B(x_3, x_0)$, $\Phi_2^{**} = \Phi_2|_{f^2(L)} = \theta_B(x_0, x_4)$, ..., $\Phi_\omega^{**} = \Phi_\omega|_{f^2(L)}$.

We now recall [6]

(1) the infinite distributive law (ID) : $x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i)$;

(2) that a lattice is *atomistic* if every non-zero element is the join of a family of atoms.

Theorem 3.8 Let L be a complete distributive lattice satisfying the infinite distributive law (ID). If an interval $[\alpha, \beta]$ of L is atomistic then it is boolean.

Proof Let $\{ x_i \mid i \in I\}$ be the set of atoms of $[\alpha, \beta]$. Then for $\alpha < \gamma < \beta$ we have

$$\gamma = \gamma \wedge \beta = \gamma \wedge \bigvee_{i\in I} x_i = \bigvee_{i\in I} (\gamma \wedge x_i).$$

Let $I_1^{(\gamma)} = \{ i \in I \mid x_i \leq \gamma \}$ and $I_2^{(\gamma)} = \{ i \in I \mid x_i \not\leq \gamma \}$. Then $\{I_1^{(\gamma)}, I_2^{(\gamma)}\}$ is a partition of I with $\gamma = \bigvee_{i\in I_1^{(\gamma)}} x_i$. Moreover,

$$(*) \quad (\forall\, i \in I_1^{(\gamma)})\ (\forall\, j \in I_2^{(\gamma)}) \qquad x_i \wedge x_j = x_i \wedge \gamma \wedge x_j = x_i \wedge \alpha = \alpha.$$

Now let $\delta = \bigvee_{j\in I_2^{(\gamma)}} x_j$. Then clearly $\gamma \vee \delta = \beta$. Also, by $(*)$ and (ID),

$$\gamma \wedge \delta = \bigvee_{i\in I_1^{(\gamma)}} x_i \wedge \bigvee_{j\in I_2^{(\gamma)}} x_j = \bigvee_{i\in I_1^{(\gamma)}} \Big(x_i \wedge \bigvee_{j\in I_2^{(\gamma)}} x_j\Big) = \alpha.$$

Hence δ is the complement of γ in $[\alpha, \beta]$. ◊

We now establish the following result.

Theorem 3.9 Let L be an Ockham algebra in which the Φ_ω-classes are locally finite. Then every non-trivial interval $[\Phi_n, \Phi_{n+1}]$ of Con L is a complete atomic boolean lattice.

Proof For every $\theta \in$ Con L we have $\theta = \bigvee\{\theta(a, b)\ ;\ (a, b) \in \theta\}$. Thus, if $\Phi_n \leq \theta$ then we have

$$\theta = \Phi_n \vee \bigvee\{\theta(a, b)\ ;\ (a, b) \notin \Phi_n,\ (a, b) \in \theta\}.$$

If now $\theta \in [\Phi_n, \Phi_{n+1}]$ then, since the Φ_{n+1}-classes are locally finite, we have

$$\theta = \Phi_n \vee \bigvee\{\theta(p, q) ;\ (p, q) \notin \Phi_n,\ (p, q) \in \theta, p \prec q\}.$$

Now for such p, q we have, by Theorem 3.7, that $\Phi_n \vee \theta(p, q)$ is an atom of $[\Phi_n, \Phi_{n+1}]$. Consequently, $[\Phi_n, \Phi_{n+1}]$ is atomistic and the result follows by Theorem 3.8. ◊

Corollary 1 Let L be finite. Then Con L contains the vertical sum

$$[\omega, \Phi_1] \mathbin{\bar{\oplus}} [\Phi_1, \Phi_2] \mathbin{\bar{\oplus}} \cdots \mathbin{\bar{\oplus}} [\Phi_{q-1}, \Phi_q] \mathbin{\bar{\oplus}} [\Phi_q, \iota]$$

where q is such that $\mathbf{B}(L) = \mathbf{K}_{p,q}$ and each summand is boolean.

Proof Observe first that $[\Phi_q, \iota] \simeq \mathrm{Con}\, L/\Phi_q \simeq \mathrm{Con}\, f^q(L)$ and that $f^q(L) \in \mathbf{K}_{p,0}$, so it follows from Theorem 1.10 in Chapter 1 that $[\Phi_q, \iota]$ is a boolean lattice. It is also readily seen from Theorem 3.9 that, for each n, $[\Phi_{n-1}, \Phi_n]$ is a boolean lattice. ◊

Corollary 2 Let (L; f) be a finite Ockham algebra. If f is injective, then Con L is boolean.

Proof Since, if f is injective then

$$\omega = \Phi_1 = \ldots = \Phi_q = \Phi_\omega.$$

The result follows immediately by Corollary 1. ◊

Example 3.5 Consider the chain C given by

$$0 < x_1 < x_2 < \ \ldots \ < \alpha < \ \ldots \ < y_3 < y_2 < y_1 < 1$$

and made into an Ockham algebra by defining

$$f(0) = 1,\ f(1) = 0,\ \ (\forall\, i)\ f(x_i) = f(y_i) = f(\alpha) = \alpha.$$

Here $\Phi_\omega = \Phi_1$ and has classes $\{0\}$, $\{1\}$, $C\setminus\{0,1\}$. The Φ_ω-class $C\setminus\{0,1\}$ is not locally finite.

Consider now the partition

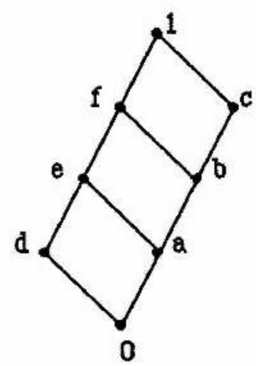
$$\{0\},\ \{x_i\,;\ i \geq 1\},\ \{\alpha\},\ \{y_i\,;\ i \geq 1\},\ \{1\}.$$

This defines a congruence in $[\omega, \Phi_1]$ which has no complement in $[\omega, \Phi_1]$. So in this case

$$\mathrm{Con}\, L \simeq [\omega, \Phi_1]\ \bar{\oplus}\ \{\iota\}$$

with $[\omega, \Phi_1]$ is not a boolean lattice. ◊

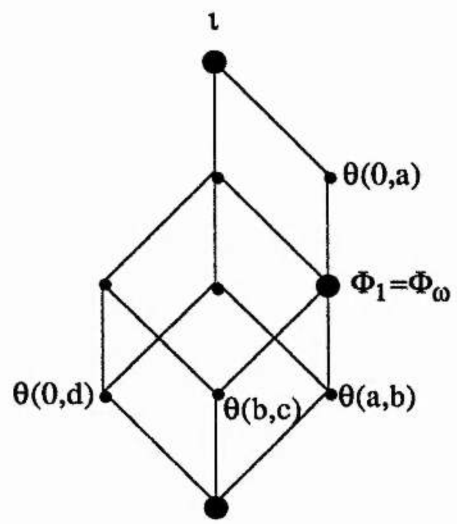
Example 3.6 Let the ordered set L be given by



and made into an Ockham algebra by defining

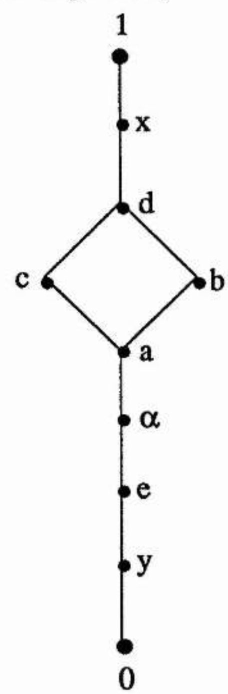
x :	0	1	a	b	c	d	e	f
$x^\sim$:	1	0	d	d	d	c	0	0

It is readily seen that $L \in \mathbf{MS}$ and that $\Phi_1 = \Phi_2 = \ldots = \Phi_\omega$. Con L is as follows:



where $\Phi_1 = \Phi_\omega = \theta(a, c)$, $\theta(0, d) \vee \theta(a, b) = \theta(a, f)$, $\theta(0, d) \vee \theta(b, c) = \theta(b, 1)$, $\theta(0, d) \vee \theta(a, b) \vee \theta(b, c) = \theta(a, 1)$, $\theta(0, a) = \theta(0, c)$. Clearly, $[0, \Phi_1]$ and $[\Phi_1, \iota]$ are boolean lattices, and Con L contains the vertical sum $[0, \Phi_1] \bar{\oplus} [\Phi_1, \iota]$.

Example 3.7 Consider the lattice L given by

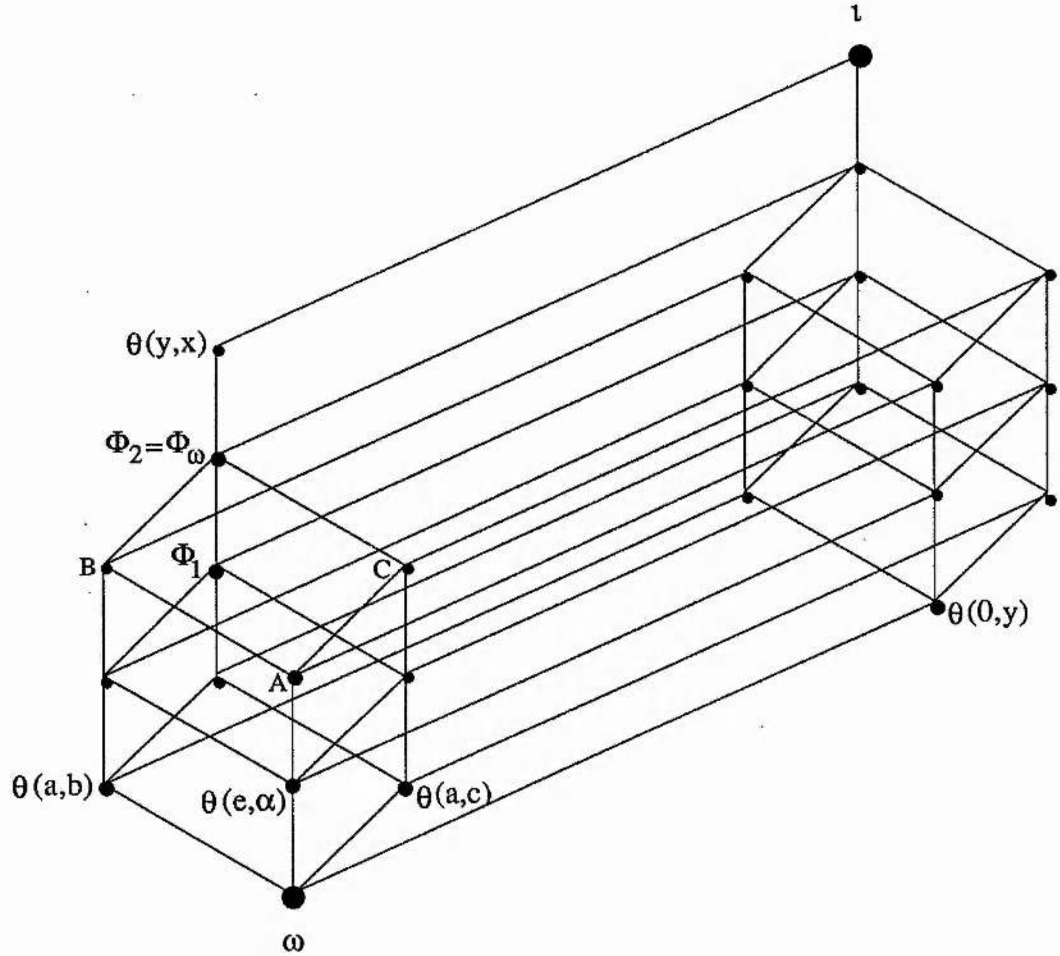


and made into an Ockham algebra by defining

$$f(0) = 1,\ f(1) = 0,\ f(x) = y,\ f(y) = x;$$

$$f(a) = f(b) = f(c) = f(d) = e,\ f(e) = f(\alpha) = \alpha.$$

It is clear that $\mathbf{B}(L) = \mathbf{K}_{1,2}$ and $\Phi_2 = \Phi_\omega$. The congruence lattice of L is as follows:



where $A = \theta(e, a) = \theta(\alpha, a)$, $B = \theta(e, b) = \theta(\alpha, b)$, $C = \theta(e, c) = \theta(\alpha, c)$, $\Phi_2 = \Phi_\omega = \theta(e, d) = \theta(\alpha, d)$, $\theta(a, b) \vee \theta(a, c) = \theta(a, d)$.

Here $[\omega, \Phi_1] \simeq \mathbf{2}^3$, $[\Phi_1, \Phi_2{=}\Phi_\omega] \simeq \mathbf{2}$, $[\Phi_\omega, \iota] \simeq \mathbf{2}^2$.

CHAPTER 4

SUBDIRECTLY IRREDUCIBLE OCKHAM ALGEBRAS

In this chapter, a generalised variety $\mathbf{K}_\omega$ of Ockham algebras that contains all varieties $\mathbf{K}_{p,q}$ is introduced. We shall show that $L \in \mathbf{K}_\omega$ is subdirectly irreducible if and only if its lattice of congruences reduces to the chain

$$\omega = \Phi_0 \preceq \Phi_1 \preceq \Phi_2 \preceq \ldots \preceq \Phi_\omega \prec \iota.$$

Here the symbol $\preceq$ means 'is covered by or is equal to'.

4.1 Weakly subdirectly irreducible Ockham algebras

An algebra is said to be *weakly subdirectly irreducible* if the intersection of two non-trivial principal congruences is non-trivial. More precisely, for a weakly subdirectly irreducible algebra L, if $\theta(a, b), \theta(c, d) \in \mathrm{Con}\, L$ with $a < b$ and $c < d$, and if $\theta(a, b) \wedge \theta(c, d) = \omega$ then either $\theta(a, b) = \omega$ or $\theta(c, d) = \omega$. Every subdirectly irreducible algebra is therefore clearly weakly subdirectly irreducible.

For an Ockham algebra (L; f), we consider the subset T(L) of L consisting of those elements x of L for which there is a smallest even positive integer $m_x = 2n_x$ such that $f^{m_x}(x) = x$. Clearly, $T(L) \neq \varnothing$ since it contains 0 and 1. If now $x, y \in T(L)$ let t = l.c.m.$[n_x, n_y]$. Then we have $f^{2t}(x \vee y) = f^{2t}(x) \vee f^{2t}(y) = x \vee y$, and similarly $f^{2t}(x \wedge y) = x \wedge y$. Since $x \in T(L)$ clearly implies $f(x) \in T(L)$, it follows that T(L) is a subalgebra of L.

For $i \geq 1$ define

$$T_i(L) = \{x \in L \mid f^i(x) = x\}.$$

Then $T_1(L)$ is the set of fixed points (possible empty) and we have the chain

$$C(L) = \{0,1\} \cup T_1(L) \subseteq T_2(L) \subseteq T_4(L) \subseteq \ldots \subseteq T_{2n}(L) \subseteq \ldots \subseteq T(L).$$

Note that $T_{2n}(L)$ is the largest $\mathbf{K}_{n,0}$ - subalgebra in L.

Example 4.1.1 Let $G = (\mathbf{2}^{\mathbf{N}}; f)$ where f is given by

$$f(X) = \{m \in \mathbf{N} \mid m + 1 \notin X\}.$$

Then G is an Ockham algebra. Using the fact that

$$f^2(X) = \{x - 2 \mid x \in X\}$$

we have, for $n \geq 1$,

$$f^{2n}(2n\mathbf{N} + 1) = 2n\mathbf{N} + 1, \ \ f^{2k}(2n\mathbf{N} + 1) \neq 2n\mathbf{N} + 1 \text{ (for } k < n).$$

It follows that $2nN + 1$ belongs to $T_{2n}(G)$ but does not to $T_{2k}(G)$ for any $k < n$. Consequently, we have the chain

$$C(G) = \{\emptyset, N, 2N, 2N + 1\} = T_2(G) \subset T_4(G) \subset \dots \subset T_{2n}(G) \subset \dots \subset T(G).$$

The following result is obtained by adapting a proof of Theorem 1.13 of Chapter 1 (see [24]), for the class $\mathbf{K}_{n,0}$.

Theorem 4.1.1 Let an Ockham algebra $(L; f)$ be such that $T_2(L) = C(L)$. If $a, b \in T(L)$ with $a < b$, then $\theta(a, b) = \iota$.

Proof For every $x \in T(L)$ let m_x be the least even positive integer such that $f^{m_x}(x) = x$ and let $n_x = \frac{1}{2} m_x$. Consider the elements

$$\alpha(x) = \bigwedge_{i=0}^{n_x-1} f^{2i}(x), \quad \beta(x) = \bigvee_{i=0}^{n_x-1} f^{2i}(x).$$

Observe that $f^2(\alpha(x)) = \alpha(x)$ and $f^2(\beta(x)) = \beta(x)$, so that $\alpha(x), \beta(x) \in T_2(L) = C(L)$.

Now let $a, b \in T(L)$ be such that $a < b$. Consider the sublattice M that is generated by

$$\{f^{2i}(a), f^{2j}(b) \mid 0 \le i \le n_a-1,\ 0 \le j \le n_b-1\}.$$

Clearly, M is finite with smallest element $\alpha(a)$ and greatest element $\beta(b)$. Let p be an atom of M and consider the interval $B = [\alpha(a), \beta(p)]$ in M. Since every atom of M is of the form $\bigwedge_{i \ne j} f^{2i}(a)$ for some j, it follows that $f^2(p)$ is also an atom of M. Consequently, B is boolean; for it is a finite distributive lattice whose greatest element is a join of atoms.

Observe that $\alpha(a) < \beta(p)$ and so, since both belong to C(L), we have that $\alpha(a)$ is either 0 or a fixed point, and $\beta(p)$ is either 1 or a fixed point.

Clearly, $a \wedge \beta(p)$ and $b \wedge \beta(p)$ belong to B, and

$$(a \wedge \beta(p), b \wedge \beta(p)) \in \theta(a, b).$$

If $a \wedge \beta(p) < b \wedge \beta(p)$, let c be an atom of B with $c \not\leq a \wedge \beta(p)$ and $c \leq b \wedge \beta(p)$. Then we have

$$(\alpha(a), c) = (a \wedge \beta(p) \wedge c, b \wedge \beta(p) \wedge c) \in \theta(a, b).$$

It follows that $(\alpha(a), \beta(c)) \in \theta(a, b)$. Since $\alpha(a), \beta(c) \in C(L)$ with $\alpha(a) < \beta(c)$ we deduce that $(0, 1) \in \theta(a, b)$ and therefore $\theta(a, b) = \iota$.

If now $a \wedge \beta(p) = b \wedge \beta(p)$ let $a_1 = a \vee \beta(p) < b \vee \beta(p) = b_1$. Then clearly we have $(a_1, b_1) \in \theta(a, b)$. Moreover, we cannot have $\beta(p) = 1$, so $\beta(p)$ must be a fixed point. It then follows that $\beta(b) = 1$; for otherwise $\beta(b) = \beta(p)$ gives the contradiction

$$a = a \wedge \beta(b) = a \wedge \beta(p) = b \wedge \beta(p) = b \wedge \beta(b) = b.$$

Considering therefore the interval $[\beta(p), 1]$ in M and a coatom q such that $q \geq a \vee \beta(p)$ and $q \not\geq b \vee \beta(p)$, we see in dual manner that again $\theta(a, b) = \iota$. ◊

From Theorem 4.1.1 we can obtain the following an important result.

Theorem 4.1.2 For an Ockham algebra L the following statements are equivalent:

(1) The subalgebra T(L) is simple;

(2) Every subalgebra $T_{2i}(L)$ is simple;

(3) $T_2(L) = C(L)$;

(4) All de Morgan subalgebras of L are simple.

Proof (1) ⇒ (2): This is clear since, by the congruence extension property [Chapter 1, Theorem 1.8], every subalgebra of a simple algebra is simple.

(2) ⇒ (3): If (2) holds then particularly $T_2(L)$ is simple. But $T_2(L) \in \mathbf{K}_{1,0} = \mathbf{M}$, and since there are only three non-isomorphic simple de Morgan algebras we see immediately that we must have $T_2(L) = C(L)$.

(3) $\Rightarrow$ (1): If (3) holds then by Theorem 4.1.1 every non-trivial principal congruence on T(L) coincides with ι. Since every congruence is the supremum of the principal congruences that it contains, it follows that T(L) is simple.

(3)$\Leftrightarrow$ (4): This follows from the fact that $T_2(L)$ is the largest de Morgan subalgebra of L. $\Diamond$

Corollary Let (L; f) be an Ockham algebra. If there exists some $i \in \mathbf{N}$ such that $\overline{T_{2i-1}(L)} = \{0, 1\} \cup T_{2i-1}(L) = T_{2i}(L)$ then T(L) is simple.

Proof It suffices to show that $C(L) = T_2(L)$. Let $x \in T_2(L)$ then clearly we have $f^{2i}(x) = x$ and $x \in T_{2i}(L) = \overline{T_{2i-1}(L)}$. If $x \in \{0, 1\}$ clearly $x \in C(L)$. If $x \notin \{0, 1\}$ then $f^{2i-1}(x) = x$ and then $f(x) = f(f^{2i-1}(x)) = f^{2i}(x) = x$ in this case x is a fixed point. Hence $T_2(L) \subseteq C(L)$ and therefore $C(L) = T_2(L)$. $\Diamond$

Theorem 4.1.3 If an Ockham algebra (L; f) is weakly subdirectly irreducible, then $T_2(L) = C(L)$. Moreover, f has at most two fixed points.

Proof Let $x \in T_2(L)$ and let $y = f(x)$. Then $f(y) = f^2(x) = x$.

Suppose now that $\{x, y\} \neq \{0, 1\}$. Then $0 < x < 1$ and $0 < y < 1$. If $x \wedge y = 0$ then $x \vee y = f(y) \vee f(x) = f(y \wedge x) = f(0) = 1$ and then we have, by Theorem 1.9 in Chapter 1, that

$$\theta(0, x) = \theta_{lat}(0, x) \vee \theta_{lat}(y, 1) \text{ and } \theta(0, y) = \theta_{lat}(0, y) \vee \theta_{lat}(x, 1).$$

So we have

$$\begin{aligned}\theta(0, x) \wedge \theta(0, y) &= [\theta_{lat}(0, x) \wedge \theta_{lat}(0, y)] \vee [\theta_{lat}(0, x) \wedge \theta_{lat}(x, 1)] \\ &\quad \vee [\theta_{lat}(y, 1) \wedge \theta_{lat}(0, y)] \vee [\theta_{lat}(y, 1) \wedge \theta_{lat}(x, 1)] \\ &= \omega\end{aligned}$$

a contradiction. Hence we must have $x \wedge y > 0$. Then we have

$$\begin{aligned}\theta(0, x \wedge y) \wedge \theta(x \wedge y, x \vee y) &= [\theta_{lat}(0, x \wedge y) \vee \theta_{lat}(x \vee y, 1)] \wedge \theta_{lat}(x \wedge y, x \vee y) \\ &= [\theta_{lat}(0, x \wedge y) \wedge \theta_{lat}(x \wedge y, x \vee y)] \\ &\quad \vee [\theta_{lat}(x \vee y, 1) \wedge \theta_{lat}(x \wedge y, x \vee y)] \\ &= \omega.\end{aligned}$$

By the definition of weakly subdirectly irreducible, we deduce from this that $x \wedge y = x \vee y$ whence $x = y$, i.e., $x \in C(L)$.

Now if $f(x) = x$ and $f(y) = y$ with $x \neq y$, then $f(x \wedge y) = x \vee y$ and $f(x \vee y) = x \wedge y$. Then $x \wedge y = 0$ and $x \vee y = 1$. Hence fixed points of f are complementary. So by distributivity there are at most two such. ◊

Theorem 4.1.4 If an Ockham algebra (L; f) is weakly subdirectly irreducible, then every Φ_1-class contains at most two elements. Moreover, if $a, b \in L$ are such that $a < b$ and $(a, b) \in \Phi_1$ then $\theta(a, b)$ is an atom of Con L.

Proof Suppose that a Φ_1-class contains at least three elements. Then it contains a three-element chain $a < b < c$ with $f(a) = f(b) = f(c)$. Then $\theta(a, b) = \theta_{lat}(a, b)$ and $\theta(b, c) = \theta_{lat}(b, c)$, whence we have the contradiction

$$\theta(a, b) \wedge \theta(b, c) = \theta_{lat}(a, b) \wedge \theta_{lat}(b, c) = \omega .$$

If now $(a, b) \in \Phi_1$ with $a < b$ then, by the above, we have $a \prec b$. It follows that $\theta_{lat}(a, b)$ is an atom of $\mathrm{Con}_{lat}\, L$, whence $\theta(a, b)$ is an atom of Con L. ◊

4.2 The generalised variety K_ω

It is well known that every finite Ockham algebra belongs to some Berman class. This is no longer true for an infinite Ockham algebra, so it is natural to consider classes that contain all the Berman classes $K_{p,q}$. In the following we introduce such a subclass of $\mathbf{O}$ denoted by K_ω.

Definition 4.2.1 The subclass K_ω of Ockham algebra is defined by

$$(L; f) \in K_\omega \iff (\forall x \in L)\ (\exists m \geq 1, n \geq 0)\quad f^{m+n}(x) = f^n(x).$$

By its very definition, K_ω is closed under the formation of subalgebras, and homomorphic images. However, it is not closed under the formation of arbitrary direct products, as can be seen by taking an algebra $L_q \in K_{p,q}$ for each $q \geq 0$ and considering the algebra

$$L_0 \times L_1 \times L_2 \times \ldots \ .$$

Nevertheless, the following result enables us to claim that K_ω is closed under finite direct products.

Theorem 4.2.1 Let $(L; f)$ be an Ockham algebra and let $x_1, x_2 \in L$ be such that there are natural numbers $m_1 \geq 1, m_2 \geq 1$ and $n_1 \geq 0, n_2 \geq 0$ with

$$f^{m_1+n_1}(x_1) = f^{n_1}(x_1) \quad \text{and} \quad f^{m_2+n_2}(x_2) = f^{n_2}(x_2).$$

Then there are natural numbers $m \geq 1, n \geq 0$ such that

$$f^{m+n}(x_1) = f^n(x_1) \quad \text{and} \quad f^{m+n}(x_2) = f^n(x_2).$$

Proof Let $m = \text{l.c.m.}\{m_1, m_2\}$ and $n = \text{l.c.m.}\{n_1, n_2\}$. Then $m = m_1 r \geq 1$, $n = n_1 s \geq 0$ and so

$$
\begin{aligned}
f^{m+n}(x_1) = f^{m_1 r+n_1 s}(x_1) &= f^{m_1(r-1)+n_1(s-1)}[f^{m_1+n_1}(x_1)] \\
&= f^{m_1(r-1)+n_1(s-1)}[f^{n_1}(x_1)] \\
&= f^{m_1(r-2)+n_1(s-1)}[f^{m_1+n_1}(x_1)] \\
&= f^{m_1(r-2)+n_1(s-1)}[f^{n_1}(x_1)] \\
&= \dots \\
&= f^{n_1(s-1)}[f^{n_1}(x_1)] \\
&= f^{n}(x_1).
\end{aligned}
$$

Similarly, $f^{m+n}(x_2) = f^n(x_2)$. ◊

As shown by Berman [Chapter 1, Theorem 1.1], each $\mathbf{K}_{p,q}$ is *locally finite* in sense that every finitely generated algebra in $\mathbf{K}_{p,q}$ is finite. Using Theorem 4.2.1, we can show that $\mathbf{K}_\omega$ enjoys the same property.

Theorem 4.2.2 $\mathbf{K}_\omega$ is locally finite. Moreover, if $L \in \mathbf{K}_\omega$ then every finitely generated subalgebra of L belongs to some Berman class.

Proof Suppose that $L \in \mathbf{K}_\omega$ is O–generated by $\{x_1, \dots, x_k\}$. Then there are natural numbers $m_i \geq 1, n_i \geq 0$ such that

$$(i = 1, \dots, k) \quad f^{m_i+n_i}(x_i) = f^{n_i}(x_i) .$$

By Theorem 4.2.1 and induction, there exist $m \geq 1, n \geq 0$ such that

$$(i = 1, \dots, k) \quad f^{m+n}(x_i) = f^n(x_i) \ [= f^{2m+n}(x_i)] .$$

It follows that L belongs to $\mathbf{K}_{m,n}$. The result now follows by Theorem 1.1 of Chapter 1. ◊

If $(L; f) \in \mathbf{K}_\omega$ and $(L; f) \notin \mathbf{K}_{p,q}$ for any p,q then we shall say that (L; f) belongs properly to $\mathbf{K}_\omega$. Here are examples of such algebras.

Example 4.2.1 Consider the infinite chain L

$$0 < \dots < a_{-n} < a_{-(n-1)} < \dots < a_{-1} < a_0 < a_1 < \dots < a_{n-1} < a_n < \dots < 1$$

made into an Ockham algebra by defining

$$f(0) = 1,\ f(1) = 0,\ f(a_0) = a_0,\ (\forall k \geq 1)\ f(a_k) = a_{-k+1},\ f(a_{-k}) = a_k .$$

Clearly, $(L; f)$ belongs properly to $\mathbf{K}_\omega$, and $T(L) = C(L) = \{0, 1, a_0\}$.

Example 4.2.2 The sink (see Example 3.2 in Chapter 3) belongs properly to $\mathbf{K}_\omega$ and $T(L) = C(L) = \{0, 1, x_0\}$.

Theorem 4.2.3 If $L \in \mathbf{K}_\omega$, then the following statements are equivalent.

(1) L is weakly subdirectly irreducible;

(2) L is subdirectly irreducible.

Proof (1) $\Rightarrow$ (2): Since $L \in \mathbf{K}_\omega$, for every $x \in L$ we have $f^{m+n}(x) = f^n(x)$ for some $m \geq 1$, $n \geq 0$. If $\Phi_1 = \omega$ then f is injective and we obtain $x = f^m(x)$ whence $x \in T(L)$. Thus $L = T(L)$ and it follows by Theorem 4.1.3 and Theorem 4.1.2 that L is simple, hence subdirectly irreducible. If, on the other hand, $\Phi_1 \neq \omega$ then by Theorem 4.1.4 the interval $[\omega, \Phi_1]$ of Con L contains an atom $\theta(a, b)$. If now $\varphi \in$ Con L is such that $\varphi \neq \omega$ then, since φ is the supremum of the non-trivial principal congruences which it contains, that is

$$\varphi = \bigvee \{ \theta(x, y) \mid (x, y) \in \varphi,\ x < y \}$$

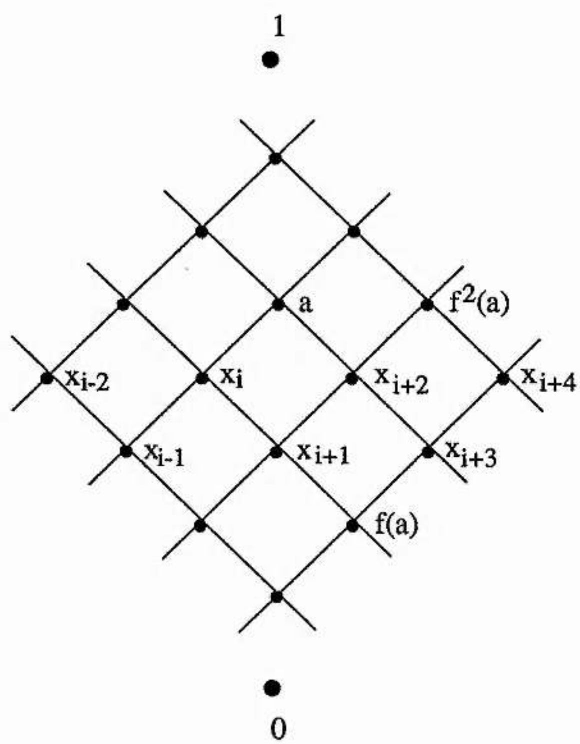
and since Con L satisfies the infinite distributive law $\beta \wedge \bigvee_{i \in I} \alpha_i = \bigvee_{i \in I} (\beta \wedge \alpha_i)$, it follows by the hypothesis that L is weakly subdirectly irreducible that $\theta(a, b) \wedge \varphi \neq \omega$. Since $\theta(a, b)$ is an atom in Con L it follows that $\theta(a, b) \leq \varphi$, whence $\theta(a, b)$ is the smallest non-trivial congruence on L and so L is subdirectly irreducible.

(2) $\Rightarrow$ (1): This is clear. ◊

Corollary If an Ockhan algebra L is weakly subdirectly irreducible but not subdirectly irreducible then necessarily $L \notin \mathbf{K}_\omega$ and f is injective.

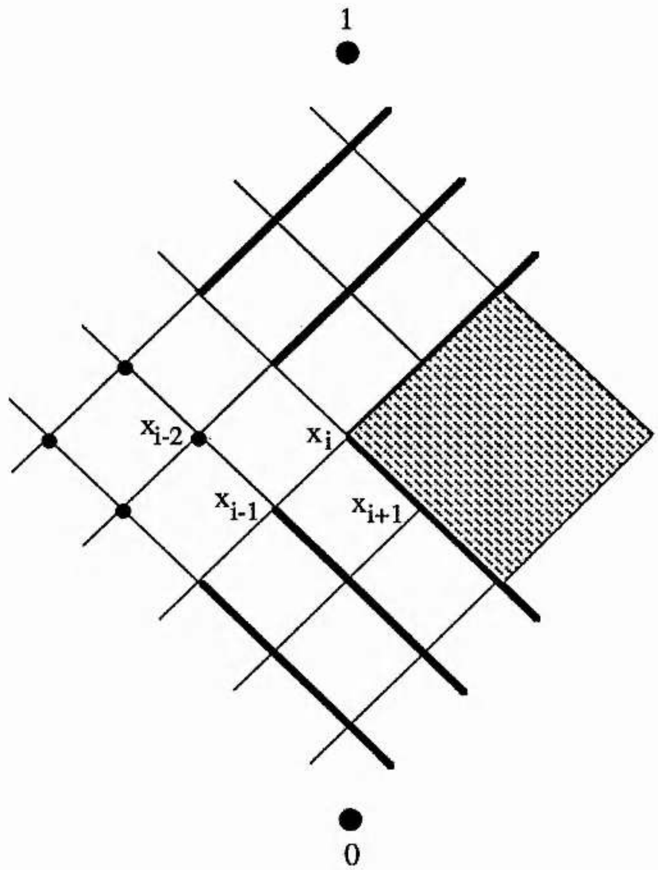
Proof That $L \notin \mathbf{K}_\omega$ follows from Theorem 4.2.3. Suppose that f were not injective. Then by Theorem 4.1.4 the interval $[\omega, \Phi_1]$ would contain an atom $\theta(a, b)$. As shown above, this implies that L is subdirectly irreducible, a contradiction. ◊

Example 4.2.3 [The pineapple] Consider the ordered set L given by



and made into an Ockham algebra by defining $f(0) = 1$, $f(1) = 0$, and $f(x_i) = x_{i+1}$ for each i, and extending to the whole of L.

Observe that f is injective and that $(L; f) \notin \mathbf{K}_\omega$. It can readily be verified that the classes modulo the congruence $\theta(x_{i+1}, x_i)$ are as in the diagram



Since every congruence is union of principal congruences, it is readily seen that Con L is the chain

$$\omega \prec \dots \prec \theta(x_{i+2}, x_{i+1}) \prec \theta(x_{i+1}, x_i) \prec \dots \prec \Psi \prec \iota.$$

where Ψ has classes $\{0\}$, $\{1\}$, $L\backslash\{0,1\}$. Thus L is weakly subdirectly irreducible but not subdirectly irreducible. Here also we have $T(L) = C(L) = \{0,1\}$.

Example 4.2.4 Let (B; f) be the Ockham algebra in Example 3.4 in Chapter 3. We see that $B \in \mathbf{K}_\omega$ and Con B is the chain

$$\omega \prec \Phi_1^{**} \prec \Phi_2^{**} \prec \dots \prec \Phi_\omega^{**} \prec \iota$$

where $\Phi_i^{**} = \Phi_i|_{f^2(L)}$ and $\Phi_\omega^{**} = \Phi_\omega|_{f^2(L)}$. Hence B is subdirectly irreducible. Here we have $T(B) = C(B) = \{0, 1, x_0\}$.

We shall now proceed to charcterise the (weakly) subdirectly irreducible algebras in $\mathbf{K}_\omega$. For this purpose we require the following results.

Theorem 4.2.4 Let A be an algebra that belongs to a class that has the congruence extension property. If A is subdirectly irreducible with monolith α then every subalgebra B of A for which $\alpha|_B \neq \omega$ is also subdirectly irreducible, with monolith $\alpha|_B$.

Proof Let $\alpha^* = \alpha|_B$. Every congruence φ^* on B with $\varphi^* \neq \omega$ extend to a congruence φ on A such that $\varphi \neq \omega$, and therefore $\varphi \geq \alpha$. It follows that $\varphi^* \geq \alpha^* \neq \omega$, and hence that B is subdirectly irreducible with monolith α^*. ◊

Theorem 4.2.5 If an Ockham algebra (L; f) is subdirectly irreducible. Then $\omega \preceq \Phi_1$. Moreover, if $\Phi_1 \neq \omega$ then Φ_1 is the monolith of L.

Proof Suppose that $\omega \neq \Phi_1$. Since every subdirectly irreducible Ockham algebra is weakly subdirectly irreducible, by Theorem 4.1.4, every Φ_1-class has at most two elements. By the definition of subdirectly irreducible that L has monolith α. Then $\omega \prec \alpha \preceq \Phi_1$ and so α has a two-element class, say {a, b} with a < b. Since every lattice congruence contained in Φ_1 is a congruence, it follows that

$$\alpha = \theta(a, b) = \theta_{lat}(a, b).$$

Since α is a principal lattice congruence it has a complement $\beta = \theta_{lat}(0, a) \vee \theta_{lat}(b, 1)$ in Con_{lat} L. Now $\beta \wedge \Phi_1$ is a lattice congruence contained in Φ_1 and so $\beta \wedge \Phi_1$ is a congruence. Since L is subdirectly irreducible it follows that either $\beta \wedge \Phi_1 \geq \alpha$ or $\beta \wedge \Phi_1 = \omega$. The former is excluded since it gives $\alpha = \beta \wedge \Phi_1 \wedge \alpha = \beta \wedge \alpha$, whence the contradiction $\alpha \leq \beta$. Thus $\beta \wedge \Phi_1 = \omega$. But $\iota = \beta \vee \alpha$ and $\Phi_1 \geq \alpha$ give $\iota = \beta \vee \Phi_1$. Hence β is the complement of Φ_1 in Con_{lat} L and therefore $\Phi_1 = \alpha$. ◊

We now establish a characterisation of the (weakly) subdirectly irreducible algebras in $\mathbf{K}_\omega$. We have the following result.

Theorem 4.2.6 If $L \in \mathbf{K}_\omega$, then L is subdirectly irreducible if and only if Con L reduces to the chain

$$\omega = \Phi_0 \leq \Phi_1 \leq \Phi_2 \leq \dots \leq \Phi_\omega \prec \iota.$$

More precisely

(1) If L belongs to a Berman class and $\mathbf{B}(L) = \mathbf{K}_{p,q}$ then L is subdirectly irreducible if and only if Con L reduces to the finite chain

$$\omega = \Phi_0 \prec \Phi_1 \prec \Phi_2 \prec \dots \prec \Phi_q = \Phi_\omega \prec \iota.$$

(2) If L belongs properly to $\mathbf{K}_\omega$ then L is subdirectly irreducible if and only if Con L reduces to the infinite chain

$$\omega = \Phi_0 \prec \Phi_1 \prec \Phi_2 \prec \dots \prec \Phi_\omega \prec \iota.$$

Proof $\Rightarrow$: Suppose that L is subdirectly irreducible. We show first that $\Phi_\omega \prec \iota$. For this purpose, we note that if $a, b \in L$ are such that $a < b$ and $(a, b) \notin \Phi_\omega$ then $\theta(a, b) = \iota$. To see this, observe that if $(a, b) \notin \Phi_\omega$ then $f^n(a) \neq f^n(b)$ for all $n \in N$. Now since $L \in \mathbf{K}_\omega$ we see by Theorem 4.2.1 that there exist $m \geq 1$, $n \geq 0$ such that

$$f^{m+n}(a) = f^n(a) \text{ and } f^{m+n}(b) = f^n(b).$$

If $n = 0$ then $a, b \in T(L)$ and it follows by Theorem 4.1.3 and Theorem 4.1.1 that $\theta(a, b) = \iota$. Let now $n > 0$ and let $c = f^n(a)$ and $d = f^n(b)$ so that $c, d \in T(L)$. We have also that $\theta(c, d) = \iota$. Consequently,

$$\theta(a, b) = \bigvee_{k=0}^{n-1} \theta_{lat}(f^k(a), f^k(b)) \vee \theta(c, d) = \iota.$$

Suppose then that $\varphi \in$ Con L is such that $\varphi \neq \iota$. Since $\varphi = \bigvee_{(x,y)\in\varphi} \theta(x, y)$ it follows from the above observation that

$$(x, y) \in \varphi \Rightarrow \theta(x, y) \neq \iota \Rightarrow (x, y) \in \Phi_\omega.$$

Thus $\varphi \le \Phi_\omega$ and so $\Phi_\omega \prec \iota$, and Con L = $[\omega, \Phi_\omega] \oplus \{\iota\}$.

We now show that the subalgebra f(L) is also subdirectly irreducible. In fact, suppose first that $\Phi_1 = \omega$. Then we have

$$\text{Con } f(L) \simeq \text{Con } L/\Phi_1 = \text{Con } L$$

whence f(L) is subdirectly irreducible. Suppose now that $\Phi_1 \neq \omega$ and let Φ_1^* be the restriction of Φ_1 to f(L). Since

$$(f(x), f(y)) \in \Phi_1^* \Leftrightarrow f^2(x) = f^2(y) \Leftrightarrow (x, y) \in \Phi_2$$

it follows that $\Phi_1^* = \omega \Leftrightarrow \Phi_1 = \Phi_2 = ... = \Phi_\omega \prec \iota$.

Thus, if $\Phi_1^* = \omega$ then Con L reduces to the three-element chain

$$\omega \prec \Phi_1 = \Phi_2 = ... = \Phi_\omega \prec \iota.$$

It follows by the congruence extension property that f(L) is also subdirectly irreducible. If , on the other hand, $\Phi_1^* \neq \omega$ then the same conclusion follows by Theorem 4.2.4, and by Con f(L) $\simeq$ Con L/Φ_1, the monolith of f(L) is Φ_1^*.

In conclusion, f(L) is subdirectly irreducible, whence so are all $f^n(L)$ since for every i we have that $f^i(L) = f(f^{i-1}(L))$. We thus have

$$\text{Con } L = \{\omega\} \oplus [\Phi_1, \iota] \text{ with } [\Phi_1, \iota] \simeq \text{Con } L/\Phi_1 \simeq \text{Con } f(L).$$

Similarly, Con L = $\{\omega\} \oplus \{\Phi_1\} \oplus [\Phi_2, \iota]$ with $[\Phi_2, \iota] \simeq$ Con L/$\Phi_2 \simeq$ Con $f^2(L)$.

We conclude from this that if L belongs to a Berman class and $\mathbf{B}(L) = \mathbf{K}_{p,q}$ then, by Theorem 3.2 of Chapter 3, Con L is the finite chain

$$\omega = \Phi_0 \prec \Phi_1 \prec \Phi_2 \prec \ \dots \ \prec \Phi_q \prec \iota.$$

If, on the other hand, L belongs properly to $\mathbf{K}_\omega$ then since there are infinitely many Φ_i, with Φ_{i+1} covering Φ_i and $\Phi_\omega = \bigvee_{i=0}^{\infty} \Phi_i$, we conclude that Con L is the infinite chain

$$\omega = \Phi_0 \prec \Phi_1 \prec \Phi_2 \prec \ \dots \ \prec \Phi_\omega \prec \iota.$$

$\Leftarrow$: This is clear. ◊

Corollary 1 If $L \in \mathbf{K}_{\omega}$ is subdirectly irreducible, so is every subalgebra of L. ◊

Corollary 2 Let $L \in \mathbf{O}$ be subdirectly irreducible. If $\Phi_{\omega} \neq \Phi_n$, for every n. Then, for every positive integer m, $f^m(L)$ is subdirectly irreducible.

Proof This is precisely as in Theorem 4.2.6. ◊

Corollary 3 Let $L \in \mathbf{O}$ be subdirectly irreducible. If f is injective, then $f^n(L)$ is subdirectly irreducible, for every n.

Proof This is precisely as in Theorem 4.2.6. ◊

Corollary 4 If $L \in \mathbf{K}_{p,q}$ then the following statements are equivalent:

(1) L is simple:

(2) L is subdirectly irreducible and f is a bijection.

Proof Subdirectly irreducibles in $\mathbf{K}_{p,q}$ are finite [Chapter 1, Theorem 1.12]. ◊

Corollary 5 If $L \in \mathbf{K}_{p,q}$ is simple then $L \in \mathbf{K}_{p,0}$.

Proof By Corollary 4, $L = f^q(L) \in \mathbf{K}_{p,0}$. ◊

In seeking to extend Theorem 4.2.6 to a general subdirectly irreducible Ockham algebra, it is natural to consider the congruence Φ_{ω}. However, here we have a difficulty : this congruence is not in general maximal. We illustrate this in the following example (adapted from an example of Goldberg [21])

Example 4.2.5 Let 2N+1 denote the set of odd positive integers and let $B_1 = \mathbf{P}(2N+1)$ be its power set. Let $B_2 = \{2N+1 \cup X \mid X \in \mathbf{P}(2N)\}$ and define $G = B_1 \bar{\oplus} B_2$.

Pictorially, this is as in **Figure 1**, in which $\overline{2n}$ means $2N+1 \cup \{2n\}$.

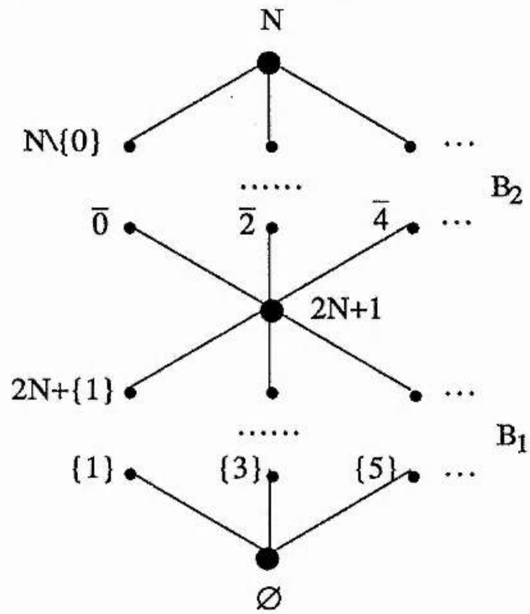


Figure 1

For every $X \in G$ define

$$f(X) = \{k \in N \mid k+1 \notin X\}.$$

Then (G; f) is an Ockham algebra. The effect of f can be seen in the following table.

X	f(X)	$f^2(X)$	$f^3(X)$
∅	N	∅	N
{1}	N\{0}	∅	N
{3}	N\{2}	{1}	N\{0}
⋮			
{2n+1}	N\{2n}	{2n-1}	N\{2n-2}
{1, 3}	N\{0, 2}	{1}	N\{0}
⋮			

Since B_1, B_2 are complete atomic boolean lattices it is readily seen that every non-trivial congruence on G identifies $N\backslash\{0\}$ and N. Consequently, G is subdirectly irreducible with smallest non-trivial congruence

$$\theta(N\backslash\{0\}, N) = \Phi_1.$$

The other congruences Φ_i are described by

$$\Phi_2 = \theta(\varnothing, \{1\}), \quad \Phi_3 = \theta(N\backslash\{0,2\}, N), \quad \Phi_4 = \theta(\varnothing, \{1, 3\}), \quad \dots \; .$$

Let us now describe the congruence Φ_ω on G. For this purpose, we shall say that $X, Y \in G$ *differ finitely* if $(X \cup Y) \setminus (X \cap Y)$ is finite. Then we have that

$$(A, B) \in \Phi_\omega \Leftrightarrow \text{ A, B differ finitely.}$$

To see this, suppose first that $(A, B) \in \Phi_\omega$. There are three case to consider

(1) $A, B \in B_1$.

If $A = \{a_i \mid i \geq 1\}$ and $B = \{b_i \mid i \geq 1\}$ with $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for every i, then for some i and j we have

$$\{a_i - 2k,\ a_{i+1} - 2k, \dots\} = f^{2k}(A) = f^{2k}(B) = \{b_i - 2k,\ b_{i+1} - 2k, \dots\}.$$

It follows that $a_i = bj$ and $a_{i+m} = b_{j+m}$ for all $m \geq 1$. Thus A, B differ finitely.

(2) $A, B \in B_2$.

This is similar to (1).

(3) $A \in B_1,\ B \in B_2$.

Here $(A, B) \in \Phi_\omega$ implies that $(A, 2N+1) \in \Phi_\omega$ and $(B, 2N+1) \in \Phi_\omega$ whence, by (1) and (2), A and B differ finitely.

For the converse,we observe first that if F is a finite subset of G then necessarily $F \subset 2N+1$. If $F = \{x_1, \dots, x_n\}$ with $x_i < x_{i+1}$ for each i then

$$f(F) = N\backslash\{x_1-1, x_2 - 1, \dots, x_n-1\},$$

$$f^2(F) = \begin{cases} \{x_1-1, x_2-1, \dots, x_n-1\} & \text{if } x_1 \geq 3, \\ \{x_2-2, x_3-2, \dots, x_n-2\} & \text{if } x_1 = 1 \end{cases}$$

and so on. It follows that for some k we have $f^{2k}(F) = \varnothing$. Consequently, if F_1, F_2 are finite subsets of G then $(F_1, F_2) \in \Phi_\omega$.

Suppose now that $A, B \in B_1$ differ finitely. Then we have

$$A = F_1 \cup (A \cap B), \quad B = F_2 \cup (A \cap B)$$

where $F_1, F_2 \in G$ are differ finitely. Since $(F_1, F_2) \in \Phi_\omega$ it follows that $(A, B) \in \Phi_\omega$. If $A, B \in B_2$ differ finitely let $2k$ be the biggest even integer in $(A \cup B) \setminus (A \cap B)$. Then clearly $f^{2k}(A) = f^{2k}(B)$ and so $(A, B) \in \Phi_\omega$. If $A \in B_1$, and $B \in B_2$ differ finitely then $A, 2N+1$ differ finitely and $B, 2N+1$ differ finitely, whence $(A, 2N+1) \in \Phi_\omega$ and $(B, 2N+1) \in \Phi_\omega$, so that $(A, B) \in \Phi_\omega$.

It follows from the above description of Φ_ω that G/Φ_ω is of the form $A_1 \bar{\oplus} A_2$ where A_1 and A_2 are atomless boolean lattices. We shall now show that G/Φ_ω is not subdirectly irreducible.

Suppose, by way of obtaining of a contradiction, that the quotient algebra $(G/\Phi_\omega; \hat{f})$, where $\hat{f}[X] = [f(X)]$, is subdirectly irreducible. Then there exists $\alpha \in \text{Con } G/\Phi_\omega$ such that $\omega \neq \alpha \leq \theta$ for all $\theta \in \text{Con } G/\Phi_\omega$ with $\theta \neq \omega$. Suppose that $A, B \in B_1$ are such $([A], [B]) \in \alpha$ with $[A] < [B]$. (Note that we can restrict attention to $A, B \in B_1$ since, from the way that f is defined, $\hat{f}$ is injective and so if $X, Y \in B_2$ are such that $([X], [Y]) \in \alpha$ with $[X] < [Y]$ then $(\hat{f}[Y], \hat{f}[X]) \in \alpha$ with $\hat{f}[Y] < \hat{f}[X]$.) Then A, $A \cap B$ differ finitely whereas A, B do not; in other words, A' denoting the complement of A in B_1, we have that $A' \cap B$ is infinite. Since $([\varnothing], [A'] \wedge [B]) \in \alpha$ we see that $([\varnothing], [Y]) \in \alpha$ for every Y with $[\varnothing] < [Y] < [A'] \cap [B]$; and since α is an atom, $\theta([\varnothing], [Y]) = \alpha$.

Now choose Y such that its elements are 'far enough apart'. More precisely, if

$$A' \cap B = \{x_1, x_2, ..., x_n, ...\}$$

with $x_i < x_{i+1}$ for each i, let $Y = \{y_i \mid i \geq 1\}$ be the subset of $A' \cap B$ formed as follows: let $y_1 = x_1, y_2 = x_2$ and

$$y_3 = \min\{x_i \in A' \cap B \mid x_i - y_2 \geq y_2 - y_1\};$$

$$y_4 = \min\{x_i \in A' \cap B \mid x_i - y_3 \geq y_3 - y_1\};$$

$$\vdots$$

$$y_k = \min\{x_i \in A' \cap B \mid x_i - y_{k-1} \geq y_{k-1} - y_1\};$$

$$\vdots$$

Thus, for every $y_i \in Y$ we have $y_{i+1} - y_i \geq y_i - y_1$. In other words, the distance from y_i to y_{i+1} is at least the distance from y_1 to y_i.

Suppose now that $n < m$. Then $z \in f^{2m}(Y) \cap f^{2n}(Y)$ if and only if, for some i and j, $z = y_i - 2m = y_j - 2n$. Consider the equation

$$y_i - y_j = 2(m - n).$$

If $j \neq 1$ then there is at most one pair y_i, y_j that satisfy this equation; and if $j = 1$ there are at most two pairs (namely when the situation $y_i - y_j = 2(m - n) = y_{i+1} - y_i$ occurs). It follows that $f^{2m}(Y) \cap f^{2n}(Y)$ is finite, and hence that

$$(m \neq n) \quad [f^{2m}(Y)] \wedge [f^{2n}(Y)] = [\varnothing].$$

Thus the subalgebra $<[Y]>$ of G/Φ_ω has atoms $[Y], [f^2(Y)], [f^4(Y)], \ldots$ and, since $\hat{f}$ is injective, coatoms $[f(Y)], [f^3(Y)], [f^5(Y)], \ldots$ Observe that

$$\bigvee_{n\leq k} [f^{2n}(Y)] \neq [2N+1] \neq \bigwedge_{n\leq k} [f^{2n+1}(Y)],$$

so that the fixed point $[2N+1]$ of G/Φ_ω does not belong to $<[Y]>$. Also, every element in the 'lower part ' of $<[Y]>$ can be expressed uniquely as a join of atoms, and every element in the 'upper part ' of $<[Y]>$ can be expressed uniquely as a meet of coatoms.

Denoting principal congruences in the subalgebra $<[Y]>$ by $\theta^*([H], [K])$, consider $\theta^*([\varnothing], [Y])$. This identifies all the atoms of $<[Y]>$, and likewise all the coatoms. Thus $\theta^*([\varnothing], [Y])$ has two classes (namely, the upper and lower parts of $<[Y]>$ and so is maximal in Con$<[Y]>$. Since, for any $[X] \in <[Y]>$, the congruence $\theta^*([\varnothing], [Y])$ identifies $[\varnothing], [f^2(X)]$ we have the chain

$$\theta^*([\varnothing], [Y]) \geq \theta^*([\varnothing], [f^2(Y)]) \geq \theta^*([\varnothing], [f^4(Y)]) \geq \ldots .$$

In fact, each of these inequalities is strict. For example, that $\theta^*([\varnothing], [Y]) > \theta^*([\varnothing], [f^2(Y)])$ follows from the observation that $\theta^*([\varnothing], [f^2(Y)])$ has four classes; those in the lower part of $<[Y]>$ are

(a) the lower part of $<[f^2(Y)]>$;

(b) $\{X \in <[Y]> \mid [Y] \leq [X] < [2N+1]\}$,

(so that (b) is the complement of (a) in lower part of $<[Y]>$). For the next inequality, consider in a similar way the restrictions of $\theta^*([\varnothing], [f^2(Y)])$ and $\theta^*([\varnothing], [f^4(Y)])$ to the subalgebra $<[f^2(Y)]>$.

It follows from the above observations that Con$<[Y]>$ contains the infinite descending chain

$$\theta^*([\varnothing], [Y]) > \theta^*([\varnothing], [f^2(Y)]) > \theta^*([\varnothing], [f^4(Y)]) > \dots .$$

By the congruence extension property [Chapter 1,Theorem 1.8], it follows that Con G/Φ_ω contains the infinite descending chain

$$\theta([\varnothing], [Y]) > \theta([\varnothing], [f^2(Y)]) > \theta([\varnothing], [f^4(Y)]) > \dots .$$

This contradicts the fact that $\alpha = \theta([\varnothing], [Y])$ is an atom in Con G/Φ_ω . Hence G/Φ_ω is not subdirectly irreducible.

It follows from these observations that Φ_ω is not maximal in Con L. For a subdirectly irreducible algebra L that does not belong to $\mathbf{K}_\omega$ the description of the interval $[\Phi_\omega, \iota]$ is still an open question.

4.3 Totally ordered subdirectly irreducible Ockham algebras

We now turn our attention to infinite totally ordered subdirectly irreducible Ockham algebras.

Theorem 4.3.1 Let L be a totally ordered Ockham algebra. If L is weakly subdirectly irreducible but not simple then one Φ_1-class has two elements and all other Φ_1-classes are singletons.

Proof Bearing Theorem 4.1.4, suppose that L has Φ_1-classes $\{a, b\}$ and $\{c, d\}$ with $a < b$ and $c < d$. Since L is totally ordered by the hypothesis, we can assume without loss of generality that $a < c$. So we have that

$$(*) \quad a < b < c < d.$$

In fact, since $[a]\Phi_1 \neq [c]\Phi_1$, so $b \neq c$. If $b > c$, then $a < c < b$ and then $f(a) \geq f(c) \geq f(b)$ whence a contradiction $f(a) = f(b) = f(c) = f(d)$, thus $(*)$ holds. So we have

$$\theta(a, b) \wedge \theta(c, d) = \theta_{lat}(a, b) \wedge \theta_{lat}(c, d) = \omega$$

which contradicts the fact that L is weakly subdirectly irreducible. ◊

Theorem 4.3.2 The only simple Ockham algebras that are totally ordered are the two-element boolean algebra and the three-element Kleene algebra..

Proof If $(L; f)$ is simple then $\Phi_1 = \omega$ and f is injective. If L had at least four elements then the equivalence relation of which the classes are $\{0\}$, $\{1\}$ and $L\setminus\{0,1\}$ is a non-trivial congruence on L, and L would not be simple. ◊

Theorem 4.3.3 Let L be a totally ordered subdirectly irreducible Ockham algebra. Then, for every $x \in L$ there is a positive integer n such that $f^n(x) \in C(L)$.

Proof If $x \in C(L)$ then $f^n(x) \in C(L)$ for every n. So suppose that $x \notin C(L)$. Then, by Theorem 4.3.2 , L is not simple; and by Theorem 4.3.1, precisely one Φ_1-class has two elements and all other Φ_1-classes are singletons. The subalgebra generated by x is

$$S = \{0, 1, x, f(x), f^2(x), ... \}.$$

Since $x \notin C(L)$ we have $x \neq f^2(x)$. There are then four possibilities for S:

(1) if $x > f(x)$ and $x < f^2(x)$ then S is the subchain

$$0 \leq ... \leq f^5(x) \leq f^3(x) \leq f(x) < x < f^2(x) \leq f^4(x) \leq ... \leq 1;$$

(2) if $x > f(x)$ and $x > f^2(x)$ then S is the subchain

$$0 \leq f(x) \leq f^3(x) \leq ... \leq f^4(x) \leq f^2(x) < x < 1;$$

(3) if $x < f(x)$ and $x > f^2(x)$ then S is the chain dual to (1);

(4) if $x < f(x)$ and $x < f^2(x)$ then S is the chain dual to (2).

Suppose that S is the chain (1). If the only two-element Φ_1-class is $\{y, z\}$ with $y \prec z$ then there are five possibilities:

(a) $... \leq f^{2k-2}(x) \leq y \prec z \leq f^{2k}(x) \leq ...$;

(b) $... \leq f^{2k+1}(x) \leq y \prec z \leq f^{2k-1}(x) \leq ...$;

(c) $... \leq f(x) \leq y \prec z \leq x \leq ...$;

(d) $0 \leq y \prec z \leq ... \leq f^{2k+1}(x) \leq ...$ for all k;

(e) $... \leq f^{2k}(x) \leq ... \leq y \prec z \leq 1$ for all k.

In case (a), if $f^{2k}(x) \neq f^{2k+2}(x)$ then $\alpha = \theta(f^{2k}(x), f^{2k+2}(x))$ separates y and z, whence we have the contradiction $\alpha \wedge \Phi_1 = \omega$. Thus $f^{2k}(x) = f^{2k+2}(x)$ and therefore $f^{2k}(x) \in C(L)$. Similar argument hold if S is the chain (b) or the chain (c). If S is the chain (d) the equivalence relation β whose classes are $\{0\}$, $\{y\}$, $\{z\}$, $\{1\}$, $L\backslash\{0, y, z, 1\}$ is a non-trivial congruence such that $\beta \wedge \Phi_1 = \omega$, which is not possible; and a similar situation obtains if S is the chain (e). Similar arguments hold when S is any of the subchains (2), (3), (4). ◊

Corollary 1 If L is a totally ordered subdirectly irreducible Ockham algebra then L $\in \mathbf{K}_\omega$. ◊

Corollary 2 If L is a totally ordered Ockham algebra then L is subdirectly irreducible if and only if its congruence lattice Con L reduces to the chain

$$\omega = \Phi_0 \leq \Phi_1 \leq \Phi_2 \leq \dots \leq \Phi_\omega \prec \iota.$$

Proof $\Rightarrow$: If L is subdirectly irreducible, then the result follows from Corollary 1 and Theorem 4.2.6.

$\Leftarrow$: This is clear. ◊

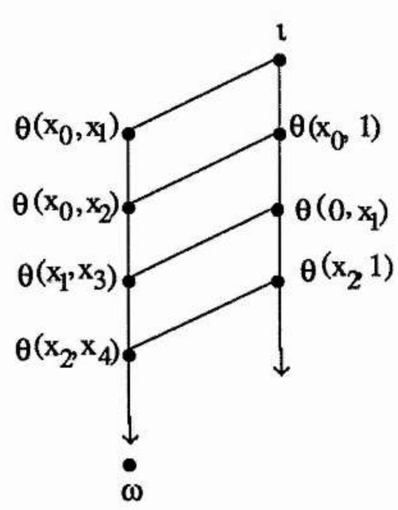
Example 4.3.1 [The see-saw] Let L be the infinite chain

$$0 < \dots < x_3 < x_1 < x_0 < x_2 < x_4 < \dots < 1.$$

Define f by

$$f(0) = 1,\ f(1) = 0, \quad (i = 0, 1, 2, \dots)\ f(x_i) = x_{i+1}.$$

Then (L; f) is an Ockham algebra on which f is injective. For all m, n ($m \neq n$) we have $f^m(x_i) \neq f^n(x_i)$ and so L $\notin \mathbf{K}_\omega$. It is readily seen that Con L is the lattice

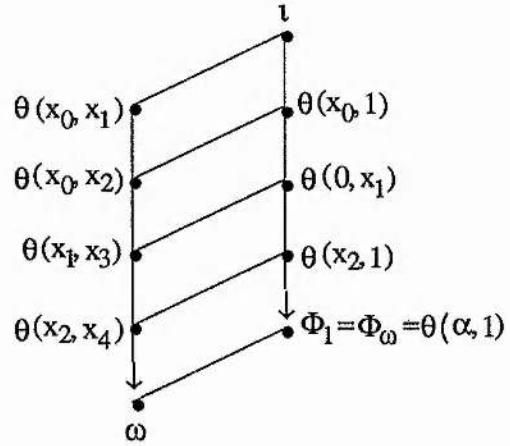


Thus L is weakly subdirectly irreducible but not subdirectly irreducible. Clearly, $\Phi_\omega = \omega$ is not maximal, and $T(L) = C(L) = \{0, 1\}$.

Example 4.3.2 Add to the see-saw a coatom α to obtain the chain

$$0 < \ldots < x_3 < x_1 < x_0 < x_2 < x_4 < \ldots < \alpha < 1,$$

and define $f(\alpha) = 0$. Then Con L becomes



Here L is not weakly subdirectly irreducible, and $T(L) = C(L) = \{0, 1\}$.

4.4 Ockham algebras generated by finite subdirectly irreducible algebras

As shown by Goldberg [20], in every Berman class $\mathbf{K}_{p,q}$ there is a greatest subdirectly irreducible algebra. Specifically, for $m > n \geq 0$ let m_n be the Ockham space consisting of the discretely ordered set $Z_m = \{0, 1, ..., m-1\}$ together with the mapping $g: Z_m \to Z_m$ defined by

$$(0 \leq k < m-1) \quad g(k) = k + 1 \quad \text{and} \quad g(m - 1) = n.$$

Then any order on m_n with respect to which g is antitone gives rise to the dual space of a subdirectly irreducible Ockham algebra; and conversely all dual spaces of finite subdirectly irreducible Ockham algebras arise in this way. If $L_{m,n}$ denotes the dual algebra obtained by using the discrete order then in the Berman class $\mathbf{K}_{p,q}$ the algebra $L_{2p+q,q}$ is the greatest subdirectly irreducible algebra; in particular, in $\mathbf{K}_{p,0}$ the algebra $L_{2p,0}$ is the greatest simple algebra .

Theorem 4.4.1 If $A \in \mathbf{K}_{p_1,q_1}$ and $B \in \mathbf{K}_{p_2,q_2}$ are subdirectly irreducible then the Ockham algebra [A, B] generated by A and B is in $\mathbf{K}_{p_1,q_1} \vee \mathbf{K}_{p_2,q_2}$ and is also subdirectly irreducible.

Proof Let $\mathbf{K}_{p,q} = \mathbf{K}_{p_1,q_1} \vee \mathbf{K}_{p_2,q_2}$. Then clearly A, B belong to $\mathbf{K}_{p,q}$ and so are subalgebras of $L_{2p+q,q}$. It follows that [A, B] is a subalgebra of $L_{2p+q,q}$ whence [A, B] is also subdirectly irreducible. ◊

Corollary If A, B are simple then so is [A, B].

Proof Take $q = q_1 = q_2 = 0$ in the above. ◊

In what follows we shall generalise Theorem 4.4.1 to an arbitrary family of finite subdirectly irreducible Ockham algebras.

Given an ascending chain

$$(L_1; f_1) \leq (L_2; f_2) \leq (L_3; f_3) \leq \dots$$

of Ockham algebras (in which $\leq$ means 'is a subalgebra of '), it is clear that under the following operations $(\bigcup_{i\geq 1} L_i ; f)$ is an Ockham algebra: given $x, y \in \bigcup_{i\geq 1} L_i$ there is a smallest j such that $x = x_j \in L_j$ and $y = y_j \in L_j$ so take $x \wedge y = x_j \wedge y_j$, $x \vee y = x_j \vee y_j$; likewise, given $x \in \bigcup_{i\geq 1} L_i$ there is a smallest i such that $x = x_i \in L_i$ so take $f(x) = f_i(x_i)$. Moreover, this is the Ockham algebra generated by the chain.

Theorem 4.4.2 Let $(A_i)_{i\in I}$ be a family of finite subdirectly irreducible Ockham algebras. Then the Ockham algebra L generated by this family belongs to $\mathbf{K}_\omega$ and is also subdirectly irreducible. If each A_i is simple then so is L.

Proof Being finite and subdirectly irreducible, every A_i belongs to a Berman class. Since there are countably many Berman classes, each containing finitely many subdirectly irreducible algebras, I is necessarily countable. Define recursively

$$L_1 = A_1 \quad \text{and} \quad (i \geq 2) \quad L_i = [L_{i-1}, A_i].$$

By Theorem 4.4.1, every L_i is subdirectly irreducible. Clearly, we have the chain

$$L_1 \leq L_2 \leq L_3 \leq \dots ,$$

and $L = \bigcup_{i\geq 1} L_i$ is the subalgebra generated by $(A_i)_{i\in I}$.

For every $x \in L$ we have $x \in L_i$ for some i. Then $f_i^{2p_i+q_i}(x) = f_i^{q_i}(x)$ where p_i, q_i are such that $B(L_i) = \mathbf{K}_{p_i,q_i}$. It follows that $L \in \mathbf{K}_\omega$.

Suppose now that every A_i is simple. Then, by the Corollary to Theorem 4.4.1, every L_i is simple. If now $\alpha \in \text{Con}\, L$ is such that $\omega \leq \alpha < \iota$, let $a, b \in L$ be such that $(a, b) \in \alpha$. We have $a, b \in L_i$ for some i. Now we cannot have $\alpha|_{L_i} = \iota|_{L_i}$; for

$0, 1 \in L_i$ and $(0, 1) \in \iota|_{L_i}$ would give $(0, 1) \in \alpha|_{L_i}$ and then $(0, 1) \in \alpha$ whence the contradiction $\alpha = \iota$. Since L_i is simple we must therefore have $\alpha|_{L_i} = \omega|_{L_i}$ whence $(a, b) \in \omega|_{L_i}$ and $a = b$. Hence $\alpha = \omega$ and L is simple.

Suppose now that not every A_i is simple. Then there is a smallest k such that $L_1, L_2, ..., L_k$ are simple and $L_{k+1}, L_{k+2}, ...$ are subdirectly irreducible but not simple. The congruence Φ_1 on L is then such that $\Phi_1|_{L_i} = \omega$ for $i \leq k$ and $\Phi_1|_{L_i} \neq \omega$ for $i \geq k+1$. Now let $\theta \in \text{Con } L$ be such that $\theta \neq \omega$. Let j be the smallest index such that $\theta|_{L_j} \neq \omega$. Then necessarily $j \leq k + 1$. In fact, if $j > k + 1$ then we have, since $\Phi_1|_{L_t}$ is the monolith of L_t for $t \geq k + 1$,

$$(1) \quad \omega \prec \Phi_1|_{L_j} \leq \theta|_{L_j}$$

and, by the definition of j,

$$(2) \quad \omega = \theta|_{L_{j-1}} \prec \Phi_1|_{L_{j-1}}.$$

By (2) there exist $x, y \in L_{j-1}$ with $x < y$ such that $(x, y) \in \Phi_1$. Since $x, y \in L_j$ we then have by (1) that $(x, y) \in \theta$ whence, by (2), the contradiction $x = y$.

Now if $j \leq k$ we have, since $L_1, L_2, ..., L_k$ are simple,

$$\begin{cases} \omega = \Phi_1|_{L_i} = \theta|_{L_i} & \text{if } i \leq j-1, \\ \omega = \Phi_1|_{L_i} \prec \iota|_{L_i} = \theta|_{L_i} & \text{if } j \leq i \leq k, \end{cases}$$

whereas if $j = k + 1$ we have

$$\begin{cases} \omega = \Phi_1|_{L_i} = \theta|_{L_i} & \text{if } i \leq j-1 = k, \\ \omega \prec \Phi_1|_{L_i} \leq \theta|_{L_i} & \text{if } i = j = k+1. \end{cases}$$

It follows from this that for $1 \leq i \leq k + 1$ we have $\Phi_1|_{L_i} \leq \theta|_{L_i}$. Since this clearly holds for all $i > k + 1$, it therefore holds for all values of i. Consequently, if $x, y \in L$ are such that $(x, y) \in \Phi_1$ then, since $x, y \in L_i$ for some i, we have $(x, y) \in \theta|_{L_i}$ whence $(x, y) \in \theta$. Thus $\Phi_1 \leq \theta$ and so L is subdirectly irreducible. ◊

Example 4.4.1 Consider the infinite chain L

$$0 < ... < a_{-n} < a_{-(n-1)} < ... < a_{-1} < a_0 < a_1 < ... < a_{n-1} < a_n < ... < 1$$

made into an Ockham algebra by defining

$$f(0) = 1,\ f(1) = 0,\ f(a_0) = a_0,\ (\forall k \geq 1)\ f(a_k) = a_{-k+1}\ ,\ f(a_{-k}) = a_k\ .$$

Now, for n = 1, 2, ... consider the finite chain L_n as follows

$$0 < a_{-n} < a_{-(n-1)} < ... < a_{-1} < a_0 < a_1 < ... < a_{n-1} < a_n < 1$$

Clearly, $(L_n; f)$ is a subalgebra of $(L; f)$ satisfying

$$L_1 \subseteq L_2 \subsetneq ... \subseteq L_n \subsetneq ... \text{ and } L = \cup_{i\geq 1} L_i\ .$$

Observe that $L_n \in \mathbf{K}_{1,2n-1}$ and that $\theta(a_0, a_1)$ is the smallest non-trivial congruence of Con L_n for all $n \geq 2$. Hence every L_n is subdirectly irreducible and then so is L.

Example 4.4.2 Let $G = (2^N; f)$ be the Ockham algebra in Example 4.1.1. Let $A_i = [2^i N+1]$. Since, for $n \geq 1$, $f^{2n}(2nN+1) = 2nN+1$, $f^{2k}(2nN+1) \neq 2nN+1$(for $k < n$), we see that $A_1 \subset A_2 \subset ... \subset A_i \subset ...$. Here we have $A_i = [2^i N+1] \simeq \mathbf{2}^i \bar{\oplus}\ \mathbf{2}^i$. So $A_i \in \mathbf{K}_{2^i,0}$ is simple, and so $\cup_{i\geq 1} A_i$ is simple.

CHAPTER 5

FINITE SIMPLE OCKHAM ALGEBRAS

An algebra L is called *simple* if Con L reduces to the 2-element chain $\{\omega, \iota\}$. It is known [Theorem 1.12 in Chapter 1 and Corollary 5 of Theorem 4.2.6 in Chapter 4] that if $L \in \mathbf{K}_{p,q}$ is simple, then L is finite and $L \in \mathbf{K}_{p,0}$. In this chapter we shall describe the structure of finite simple Ockham algebras.

The investigations of this chapter are based on the following results.

Theorem 5.1 Let L be a simple Ockham algebra. If $a \in L$ is such that a and $f^2(a)$ are comparable, then $a \in C(L)$.

Proof We assume that $f^2(a) \geq a$. Consider the relation θ_a defined by

$$(x, y) \in \theta_a \quad \Leftrightarrow \quad x \wedge a = y \wedge a \text{ and } x \vee f(a) = y \vee f(a).$$

Let $(x, y) \in \theta_a$; then $f(x) \wedge f^2(a) = f(y) \wedge f^2(a)$. Since $f^2(a) \geq a$, it follows that $f(x) \wedge a = f(y) \wedge a$, and so we can see that $\theta_a \in \text{Con } L$ with

$$\theta_a = \theta_{lat}(a, 1) \wedge \theta_{lat}(0, f(a)) = \theta_{lat}(a \wedge f(a), f(a)).$$

Since L is simple by the hypothesis, it follows that either $\theta_a = \iota$ or $\theta_a = \omega$. Now when $\theta_a = \iota$, then $(0, 1) \in \theta_a$ and then $0 \wedge a = 1 \wedge a$ whence $a = 0 \in C(L)$. When $\theta_a = \omega$, we have $a \wedge f(a) = f(a)$ whence $f(a) \leq a$. So $f(a) \leq a \leq f^2(a)$, $f^2(a) \geq f(a) \geq f^3(a)$ and so on, and we have the subalgebra chain

$$0 \leq \ldots \leq f^{2n+1}(a) \leq \ldots \leq f(a) \leq a \leq f^2(a) \leq \ldots \leq f^{2n}(a) \leq \ldots \leq 1.$$

Since every subalgebra of a simple Ockham algebra is also simple, it follows, by Theorem 4.3.2 in Chapter 4, that $a \in C(L)$.

A similar argument holds if $f^2(a) \leq a$. ◊

Theorem 5.2 Let L be a simple Ockham algebra. If $a \notin C(L)$, then $a \wedge f(a)$, $a \vee f(a) \notin C(L)$.

Proof We show first that $a \vee f(a) \notin C(L)$. Suppose, by way of obtaining a contradiction, that $a \vee f(a) \in C(L)$. We consider the following two cases:

(a) If $a \vee f(a) = 1$.

Consider the principal congruence $\theta_{lat}(0, a)$. For $(x, y) \in \theta_{lat}(0, a)$ we have $x \vee a = y \vee a$. Then $f(x) \wedge f(a) = f(y) \wedge f(a)$ and $(f(x) \wedge f(a)) \vee a = (f(y) \wedge f(a)) \vee a$. Since

$a \vee f(a) = 1$, we have that $f(x) \vee a = f(y) \vee a$. Consequently, $(f(x), f(y)) \in \theta_{lat}(0, a)$. Hence $\theta_{lat}(0, a) \in$ Con L. Since L is simple and $a \neq 0$, it follows that $\theta_{lat}(0, a) = \iota$, and so $(0, 1) \in \theta_{lat}(0, a)$. We therefore have that $0 \vee a = 1 \vee a$, whence the contradiction $a = 1$.

(b) If $a \vee f(a)$ is a fixed point.

In this case, we have $f(a) \wedge f^2(a) = a \vee f(a)$. So $f(a) \geq a$ and $f^2(a) \geq f(a)$, and so $f^2(a) = f(a)$. Since L is simple then f is injective, it follows that we must have $f(a) = a$, a contradiction.

Finally , by the above observations we have $f(a) \vee f^2(a) \notin C(L)$, for $f(a) \notin C(L)$. It follows $a \wedge f(a) \notin C(L)$. ◊

Theorem 5.3 Let $(L; f) \in \mathbf{K}_{p,0}$ be finite. Then f takes atoms to coatoms, and conversely.

Proof Let a be an atom of L and let $f(a) \leq y < 1$. Since L is finite and f injective, hence f is surjective, there exists $z \in L$ such that $y = f(z)$ and so $f(a) \leq f(z) < 1$. Since f is a dual automorphism we deduce that $a \geq z > 0$. Since a is an atom, it follows that $a = z$ whence $y = f(a)$ and so $f(a)$ is a coatom. Dually, if b is a coatom then $f(b)$ is an atom. ◊

Corollary Let $(L; f) \in \mathbf{K}_{p,0}$ be finite. If $a \in L$ is an atom then so is $f^2(a)$. ◊

Every finite simple Ockham algebra belongs to $\mathbf{K}_{i,0}$ for some i [Corollary 5 of Theorem 4.2.6 in Chapter 4]. The following example is of considerable interest.

Example 5.1 Consider the Ockham algebra (G; f) of Example 4.2.3 of Chapter 4.

Since G is subdirectly irreducible, it follows, by Theorem 1.11 of Chapter 1 and Theorem 4.1.2 of Chapter 4, that T(G) is a simple subalgebra. For each $i \geq 1$ consider $E_i = 2i\,N + 1$. It is readily seen that $f^{2i}(E_i) = E_i$ and so each $E_i \in T(G)$. Since T(G) is simple, so is each $\langle E_i \rangle$. We thus see that T(G) is an infinite simple algebra. Moreover, by Corollary 5 of Theorem 4.2.6 of Chapter 4, we can see that $\langle E_i \rangle \in \mathbf{K}_{i,0}$ which shows that the infinite simple algebra T(G) contains finite simple subalgebras in every permissible Berman class.

We now shall be concerned with finite simple Ockham algebras. The following examples, as we shall see, are of fundamental importance.

Example 5.2 Let L be the boolean lattice $\mathbf{2}^n$ with atoms $a_1, \ldots, a_n$. In order to make L into an Ockham algebra, it suffices to define $f(0) = 1$ and to specify $f(a_i)$ for each a_i; for, every $x \neq 0$ can be expressed uniquely in the form $x = \bigvee_{i \in I} a_i$ where I is a non-empty subset of $\{1, 2, \ldots, n\}$ and we can extend the definition of f by defining $f(x) = \bigwedge_{i \in I} f(a_i)$, thereby obtaining a dual endomorphism. Consider in this way the dual endomorphism f obtained by defining

$$f(a_i) = a'_{i+1}$$

where a'_{i+1} is the complement of a_i in $\mathbf{2}^n$, the subscripts being reduces modulo n where necessary. In particular, we have

$$f^2(a_i) = f(a'_{i+1}) = f(\bigvee_{j \neq i+1} a_j) = \bigwedge_{j \neq i+1} f(a_j) = \bigwedge_{j \neq i+1} a'_{j+1} = a_{i+2} .$$

It follows that if n is odd then f^2 induces the atom cycle

$$a_1 \to a_3 \to a_5 \to \cdots \to a_n \to a_2 \to a_4 \to \cdots \to a_{n-1} \to a_1 .$$

whereas if n is even then f^2 induces the two atom cycles

$$a_1 \to a_3 \to a_5 \to \cdots \to a_{n-1} \to a_1 ; \quad a_2 \to a_4 \to a_6 \to \cdots \to a_n \to a_2 .$$

(a) If n is odd.

Suppose that $\theta \in \text{Con}(\mathbf{2}^n; f)$ is such that $\theta \neq \omega$; and let $(x, y) \in \theta$ with $x < y$. Then there is an atom $a_k \leq y$ and $a_k \nleq x$. Consequently

$$(0, a_k) = (a_k \wedge x, a_k \wedge y) \in \theta$$

and therefore

$$(0, a_{k+2}) = (0, f^2(a_k)) \in \theta.$$

Since the atoms form a single cycle under f^2 it follows that $(0, a_i) \in \theta$ for every atom a_i. Hence

$$(0, 1) = (0, \bigvee_{i=1}^{n} a_i) \in \theta$$

and so $\theta = \iota$. Thus $(\mathbf{2}^n; f)$ is simple. Note that in this case there are no fixed points; for if α were a fixed point and a_i is an atom with $a_i \leq \alpha$ then $a_{i+2} = f^2(a_i) \leq \alpha$, so all the atoms would be contained in α, which is not possible.

(b) If n is even.

In this case let $\alpha = a_1 \vee a_3 \vee \ldots \vee a_{n-1}$ and $\beta = a_2 \vee a_4 \vee \ldots \vee a_n$. Then $\alpha \wedge \beta = 0$ and $\alpha \vee \beta = 1$, and so

$$\alpha = \beta' = \bigwedge_{i \in I} a'_{2i} = \bigwedge_{i \in I} f(a_{2i-1}) = f(\bigvee_{i \in I} a_{2i-1}) = f(\alpha).$$

Thus α is a fixed point; and similarly so is β. Arguing as in case (a), and using the fact that in this case there are two atom cycles under f^2, we see that

$$\text{either } (0, \alpha) = (0, \bigvee_{i \in I} a_{2i}) \in \theta, \quad \text{or} \quad (0, \beta) = (0, \bigvee_{i \in I} a_{2i+1}) \in \theta.$$

In either case we deduce that $(0, 1) \in \theta$, whence $\theta = \iota$ and again $(\mathbf{2}^n; f)$ is simple.

Example 5.3 Let $L = \mathbf{2}^n \,\bar{\oplus}\, \mathbf{2}^n$ be the vertical sum of two copies of $\mathbf{2}^n$. Let the atoms be $a_1, a_2, \ldots, a_n$ and let the coatoms of L be $b_1, b_2, \ldots, b_n$. Then we can make L into an Ockham algebra by defining $f(0) = 1$, $f(1) = 0$ and, with reduction modulo n where appropriate,

$$f(a_i) = b_i, \quad f(b_i) = a_{i+1}.$$

We extend f to a dual endomorphism by defining

$$f(\bigvee_{i\in I} a_i) = \bigwedge_{i\in I} b_i, \quad f(\bigwedge_{i\in I} b_i) = \bigvee_{i\in I} a_{i+1}.$$

Observe that $(2^n \mathbin{\overline{\oplus}} 2^n ; f)$ has a single fixed point, namely

$$\alpha = \bigvee_{i=1}^{n} a_i = \bigwedge_{i=1}^{n} b_i .$$

Suppose that $\theta \in \mathrm{Con}\,(2^n \mathbin{\overline{\oplus}} 2^n ; f)$ is such that $\theta \neq \omega$; and let $(x, y) \in \theta$ with $x < y$. We consider the following three cases:

(1) $x, y \in [0, \alpha]$.

In this case there exists an atom a_k such that $a_k \leq y$ and $a_k \not\leq x$. Consequently,

$$(0, a_k) = (a_k \wedge x, a_k \wedge y) \in \theta$$

and then

$$(0, a_{k+1}) = (0, f^2(a_k)) \in \theta.$$

Since the atoms form a single cycle under f^2 it follows that $(0, a_i) \in \theta$ for every atom a_i. We therefore have

$$(0, \alpha) = (0, \bigvee_{i=1}^{n} a_i) \in \theta.$$

It follows that $(0, 1) \in \theta$ and so $\theta = \iota$.

(2) $x < \alpha \leq y$.

This case as same as (1), there is an atom $a_k \not\leq x$ and $y > a_k$. Whence again $\theta = \iota$.

(3) $x, y \in [\alpha, 1]$.

In this case there exists a coatom b_k with $b_k > x$ and $b_k \not\geq y$. Consequently,

$$(b_k, 1) = (b_k \vee x, b_k \vee y) \in \theta$$

and therefore

$$(b_{k+1}, 1) = (f^2(b_k), 1) \in \theta.$$

Since the coatoms form a single cycle under f^2, we have that $(b_i, 1) \in \theta$ for every coatom b_i. Hence

$$(\alpha, 1) = (\bigwedge_{i=1}^{n} b_i, 1) \in \theta.$$

It follows again that $(0, 1) \in \theta$ and so $\theta = \iota$.

We therefore have from those observations above that $(2^n \bar{\oplus} 2^n; f)$ is simple.

Definition 5.1 If L is a finite Ockham algebra then a subalgebra A of L will be called a *full subalgebra* if A contains all the atoms of L.

We now establish the following result.

Theorem 5.4 Let L be a finite Ockham algebra. If L contains $(2^n \bar{\oplus} 2^n; f)$ as a full subalgebra then L is simple.

Proof Clearly, L has unique fixed point α; and if a is any atom of L then we have from Theorem 5.3 that every $f^{2i}(a)$ is atom and every $f^{2i+1}(a)$ is coatom. Consequently,

$$\alpha = a \vee f^2(a) \vee \dots \vee f^{2n-2}(a) = f(a) \wedge f^3(a) \wedge \dots \wedge f^{2n-1}(a).$$

Let $A = 2^n \bar{\oplus} 2^n$. Then we observe that, for $x \in L \setminus \{0,1\}$,

$$(*) \quad x \not\parallel \alpha \;\Rightarrow\; x \in A.$$

In fact, if $x \leq \alpha$ then

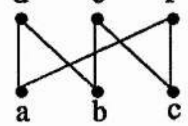
$$x = x \wedge \alpha = (x \wedge a) \vee (x \wedge f^2(a)) \vee \dots \vee (x \wedge f^{2n-2}(a))$$

where, for each i, $x \wedge f^{2i}(a)$ is either 0 or an atom (and not all are 0 since $x \neq 0$). Since A is a full subalgebra it follows that $x \in A$. A similar argument holds when $x \geq \alpha$.

Suppose now that $\theta \in \text{Con } L$ with $\theta \neq \omega$. Then there exist $x, y \in L$ with $x < y$ and $(x, y) \in \theta$. Since $x < y$, we have that either $x \wedge \alpha < y \wedge \alpha$ or $x \vee \alpha < y \vee \alpha$; and by $(*)$, $x \wedge \alpha$, $y \wedge \alpha$, $x \vee \alpha$, $y \vee \alpha \in A$. Then $(x \wedge \alpha, y \wedge \alpha) \in \theta$, $(x \vee \alpha, y \vee \alpha) \in \theta$

give $\theta|_A \neq \omega$ and so, since A is simple, $\theta|_A = \iota|_A$. Then $(0, 1) \in \theta|_A$ and so $(0, 1) \in \theta$. Consequently, $\theta = \iota$ and L is simple. ◊

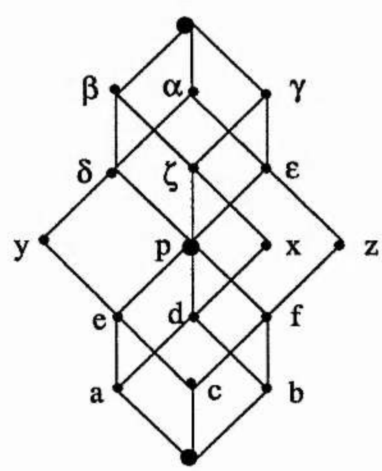
Example 5.4 Consider the Ockham space consisting of the crown $X = C_6$:



and the antitone mapping g given as follows:

x :	a b c d e f
g(x):	d e f b c a

It is easy to see that, for every $x \in X$, $g^{\omega}(\{x\}) = \{x, g(x), g^2(x), g^3(x), g^4(x), g^5(x)\} = X$. By [20, Corollary 2.4], the corresponding Ockham algebra is simple, its lattice reduct being the free distributive lattice on 3 generators (see [6], Page 33, Figure 8.):



t :	0 1 a b c d e f p α β γ δ ε ζ x y z
t^o:	1 0 α β γ δ ε ζ p b c a f d e y z x

It is clear that this simple algebra contains $\mathbf{2}^3 \mathbin{\bar{\oplus}} \mathbf{2}^3$ as a full subalgebra.

Our objective now is to show that the above types describe all finite simple Ockham algebras. We have the following result.

Theorem 5.5 Let (L ; f) be a finite simple Ockham algebra with n atoms. Then the structure of L is as follows:

(1) if L has no fixed points then n is odd and $L \simeq \mathbf{2}^n$;

(2) if L has two fixed points then n is even and $L \simeq \mathbf{2}^n$;

(3) if L has a unique fixed point then L contains $\mathbf{2}^n \mathbin{\bar{\oplus}} \mathbf{2}^n$ as a full subalgebra.

Proof Let $a \in L$ be an atom and let m be the smallest positive integer such that $f^{2m}(a) = a$. By Theorem 5.1, if a is neither 1 nor a fixed point then $m > 1$. By the corollary to Theorem 5.3 the elements $a, f^2(a), \dots, f^{2m-2}(a)$ are all atoms; and , by the hypothesis on m and the fact that f is injective, these atoms are all distinct. Consider the element

$$\alpha = a \vee f^2(a) \vee \dots \vee f^{2m-2}(a).$$

We have $f^2(\alpha) = \alpha$ and so $\alpha \in T_2(L) = C(L)$ [by Theorem 1.11 in Chapter 1]. Consequently, either $\alpha = 1$ or α is a fixed point. Since L has at most two fixed points [by Theorem 1.11 in Chapter 1], we consider the following cases:

(1) L has no fixed points.

In this case necessarily $\alpha = 1$ and so 1 is a join of the atoms $a, f^2(a), \dots, f^{2m-2}(a)$. It follows that $m = n$ and $L \simeq \mathbf{2}^n$. By Theorem 5.3 we therefore have the situation of Example 5.2 with n odd (no fixed points).

(2) L has two fixed points.

If L has two fixed points then, by Theorem 1.11 in Chapter 1, these are complementary in L. There must therefore exist an atom b that does not belong to the sequence a, $f^2(a), \ldots, f^{2m-2}(a)$. If p is the smallest positive integer such that $f^{2p}(b) = b$. Then the set of atoms of L is

$$\{a, f^2(a), \ldots, f^{2m-2}(a), b, f^2(b), \ldots, f^{2p-2}(b)\}$$

and the fixed points are $\alpha = \bigvee_{i=0}^{m-1} f^{2i}(a)$ and $\beta = \bigvee_{i=0}^{p-1} f^{2i}(b)$. Since α, β are complementary in L and since f is a dual automorphism it follows that

$$[0, \alpha] \simeq [\beta, 1] \overset{d}{\simeq} [0, \beta]$$

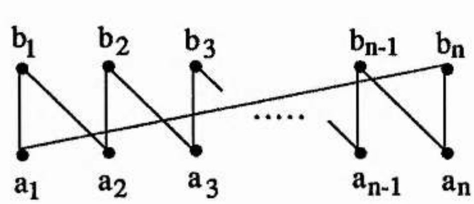
and hence that p = m. Consequently, n = 2m. Since $1 = \alpha \vee \beta$ is the join of all the atoms we have $L \simeq \mathbf{2}^n$. By Theorem 5.3 we therefore have the situation of Example 5.2 with n even (two fixed points).

(3) L has unique fixed point.

If L has precisely one fixed point α then $a, f^2(a), \ldots, f^{2m-2}(a)$ must be all the atoms of L, with $\alpha = \bigvee_{i=0}^{m-1} f^{2i}(a)$; for if b were an atom not in this cycle then for some k, $d = \bigvee_{i=0}^{k-1} f^{2i}(b)$ would also be a fixed point, which is not possible.

Thus the join of all the atoms is the fixed point α. It follows that $[0, \alpha] \simeq \mathbf{2}^n$. By Theorem 5.3 and the fact that f is a dual automorphism, it follows that L contains the algebra $\mathbf{2}^n \,\overline{\oplus}\, \mathbf{2}^n$ of Example 5.3 as a full subalgebra. ◊

The 'usual crown' $C_{2n,2}$, every vertex of which is of degree 2, is as follows:



with the antitone mapping g given as follows

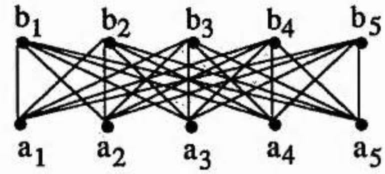
$$g(a_i) = b_i, \ g(b_i) = a_{i+1} (\text{mod } n) \ (i = 1, 2, \dots, n).$$

In the following examples of finite simple Ockham algebras we shall use ordered sets $C_{2n,k}$ which are particular extensions of $C_{2n,2}$ in which the degree of every vertex is k $\leq$ n. Since

$$\begin{aligned} g^{\omega}(\{a_i\}) &= \{a_i, g(a_i), \dots, g^{2(n-i)}(a_i), g^{2(n-i)+1}(a_i), \dots, g^{2n-2}(a_i), g^{2n-1}(a_i)\} \\ &= \{a_i, b_i, a_{i+1}, b_{i+1}, \dots, a_n, b_n, a_1, b_1, \dots, a_{i-1}, b_{i-1}\} \\ &= C_{2n,k} \\ g^{\omega}(\{b_i\}) &= \{b_i, g(b_i), \dots, g^{2(n-i)}(b_i), g^{2(n-i)+1}(b_i), \dots, g^{2n-2}(b_i), g^{2n-1}(b_i)\} \\ &= \{b_i, a_{i+1}, b_{i+1}, a_{i+2}, \dots, b_n, a_1, b_1, a_2, \dots, b_{i-1}, a_i\} \\ &= C_{2n,k} \end{aligned}$$

it follows from [20, Corollary 2.4] that the corresponding Ockham algebras are indeed simple.

Example 5.5 Consider the Ockham space $C_{10,5}$:



The corresponding Ockham algebra is as in Figure 1.

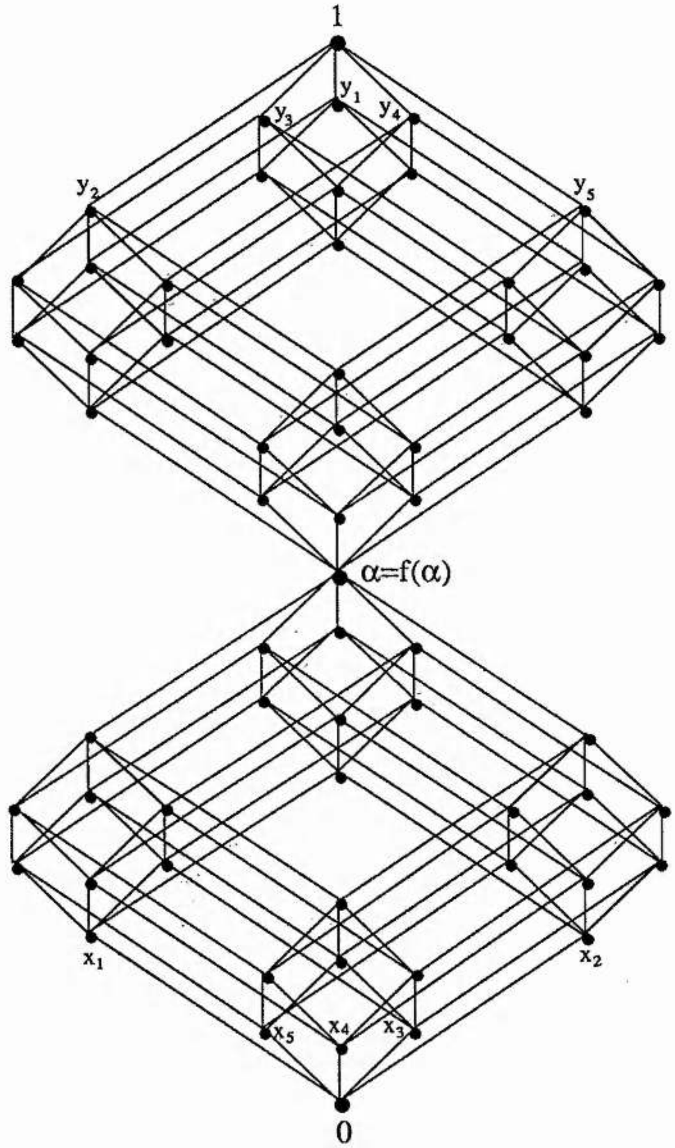
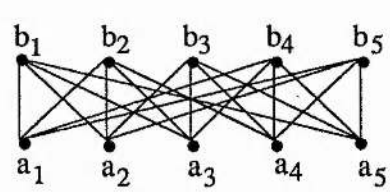


Figure 1.

where $f(x_i) = y_i$ and $f(y_i) = x_{i+1}$ (mod 5).

Example 5.6 Consider the Ockham space $C_{10,4}$:



The corresponding Ockham algebra is as in Figure 2. Note that it contains the algebra of Example 5.5 as a full subalgebra.

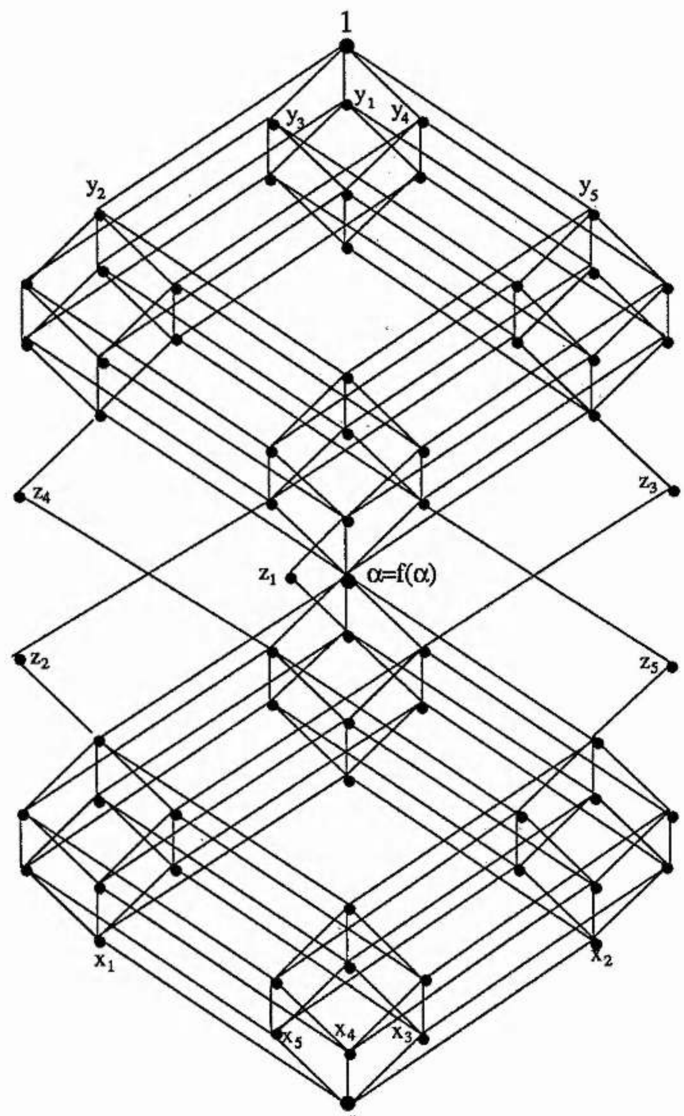
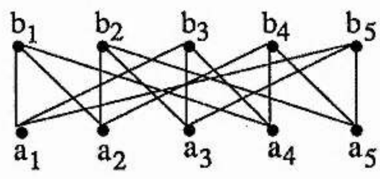


Figure 2

where $f(x_i) = y_i$, $f(y_i) = x_{i+1} (\text{mod } 5)$, $f(z_i) = z_{i+1}$ (mod 5) and $f(\alpha) = \alpha$.

Example 5.7 Consider the Ockham space $C_{10,3}$:



The corresponding Ockham algebra is as in Figure 3. It also contains the algebra of Example 5.5 as a full subalgebra.

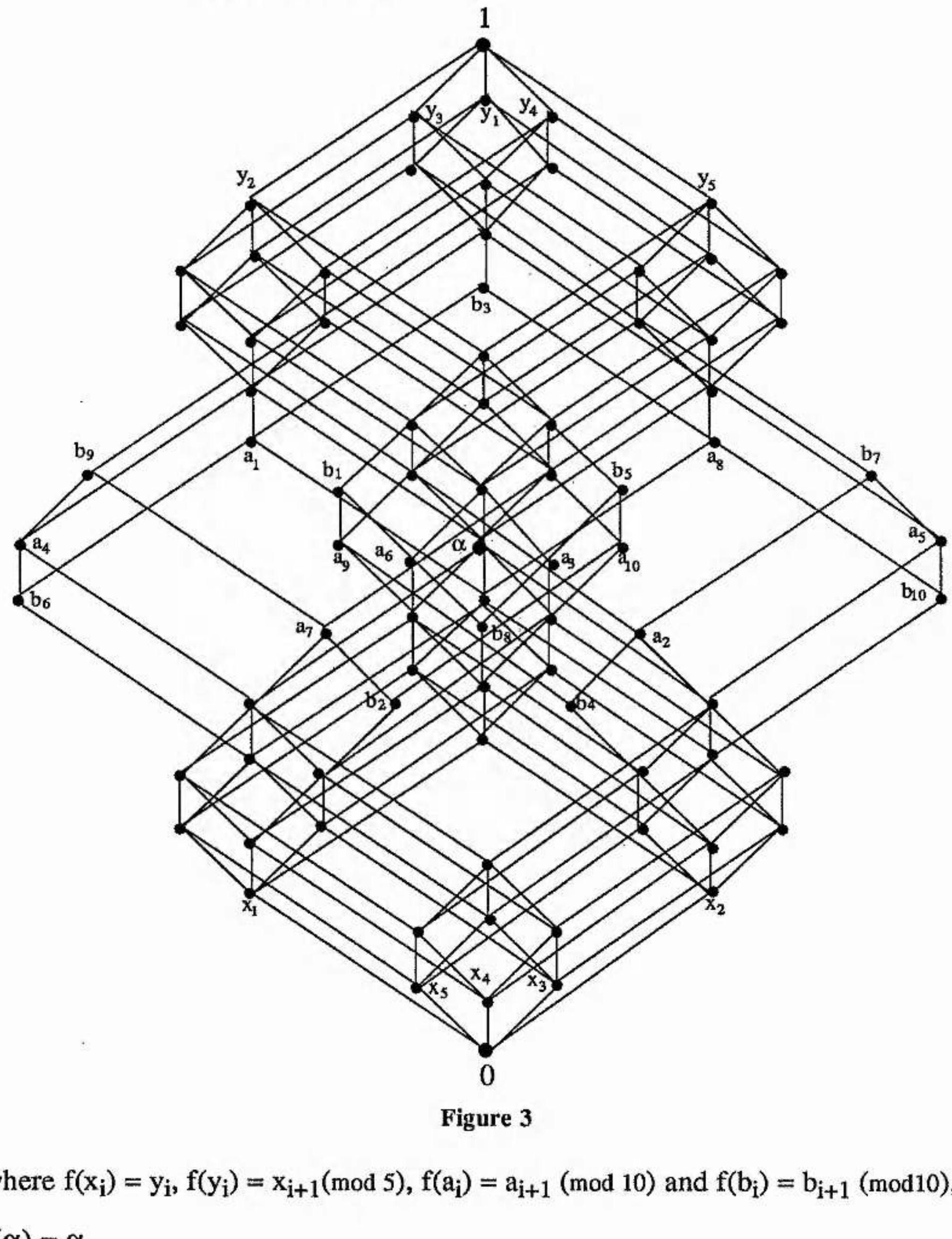
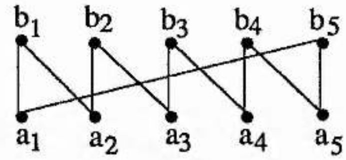


Figure 3

where $f(x_i) = y_i$, $f(y_i) = x_{i+1} (\text{mod } 5)$, $f(a_i) = a_{i+1} \text{ (mod 10)}$ and $f(b_i) = b_{i+1} \text{ (mod10)}$, and $f(\alpha) = \alpha$.

Example 5.8 Consider the Ockham space $C_{10,2}$:



The corresponding Ockham algebra is as in Figure 4. Again it contains the algebra of Example 5.5 as a full subalgebra.

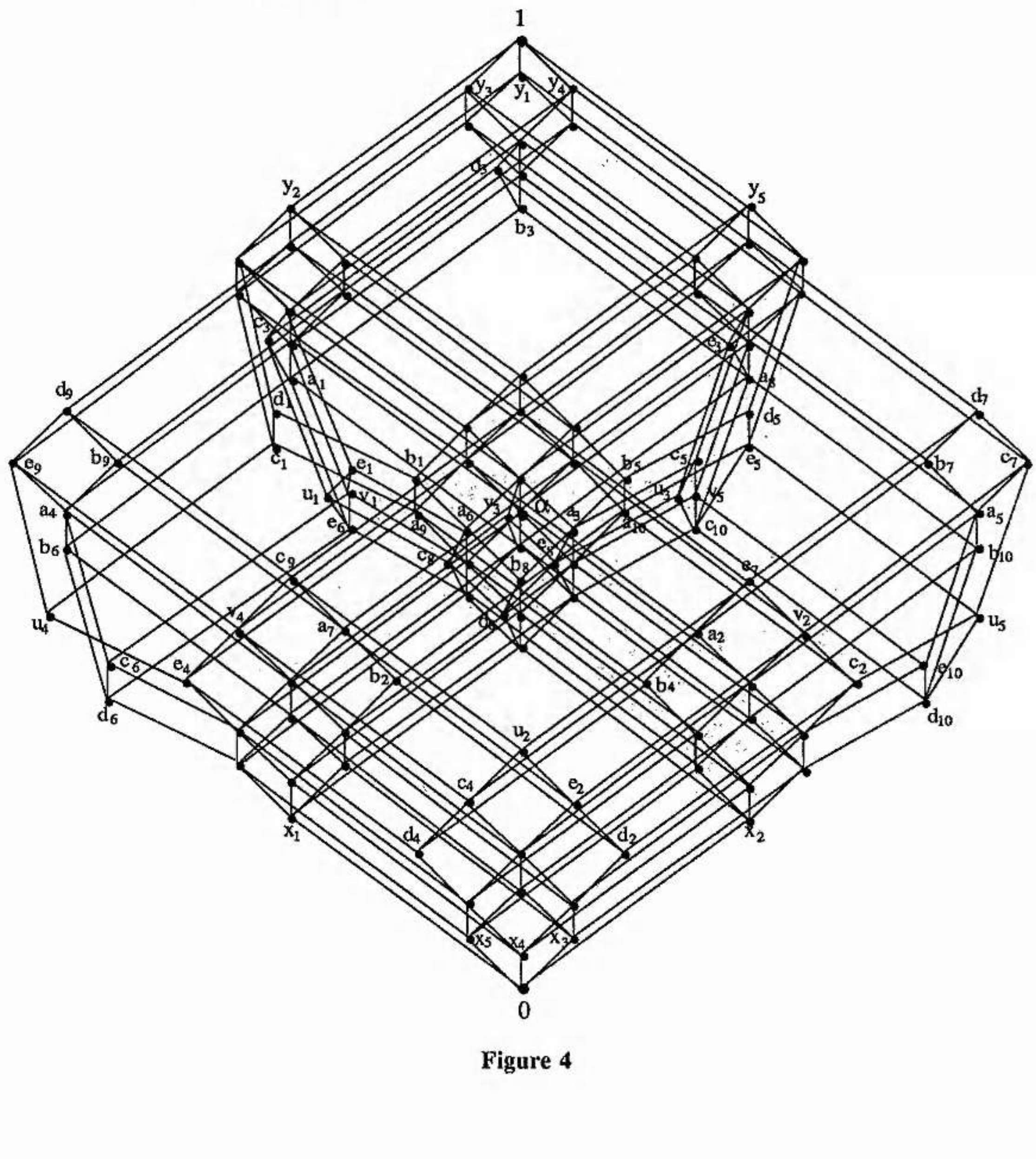


Figure 4

where $f(x_i) = y_i$, $f(y_i) = x_{i+1}$ (mod 5), $f(a_i) = a_{i+1}$ (mod 10), $f(b_i) = b_{i+1}$ (mod 10), $f(c_i) = c_{i+1}$(mod 10), $f(d_i) = d_{i+1}$(mod 10), $f(e_i) = e_{i+1}$(mod 10), $f(u_i) = u_{i+1}$, $f(v_i) = v_{i+1}$ (mod 5) and $f(\alpha) = \alpha$.

REFERENCES

[1]. C. J. Ash : Pseudovarieties, generalised varieties and similarly described classes, *Journal of Algebra,* **92**, 1985, 104 -115.

[2]. J. Berman : Distributive lattices with an additional unary operation, *Aequationes Math.,* **16**, 1977, 165 - 171.

[3]. J. Berman and P. Köhler : Cardinalities of finite distributive lattices, *Mitt. Math. Sem. Giessen,* **121**, 1976, 103 - 124.

[4]. R. Beazer : Injectives in some small varieties of Ockham algebras, *Glasgow Math. J.* **25**, 1984, 183 - 191.

[5]. R. Beazer : On some small varieties of Ockham algebras, *Glasgow Math. J.,* **25**, 1984, 175 - 181.

[6]. G. Birkhoff : Lattice theory, 3rd ed. Providence, R. I.: American Mathematical Society, 1967.

[7]. T. S. Blyth and J. C. Varlet : On a common abstraction of de Morgan algebras and Stone algebras, *Proc. Roy. Soc. Edinburgh Sect.* A **94**, 1983, 301 - 308.

[8]. T. S. Blyth and J. C. Varlet : Subvarieties of the class of **MS** - algebras, *Proc. Roy. Soc. Edinburgh Sect.* A **95**, 1983, 157 - 169.

[9]. T. S. Blyth and J. C. Varlet : Fixed points in **MS** - algebras, *Bull. Soc. Roy. Sci. Liège,* **53**, 1984, 3 - 8.

[10]. T. S. Blyth and J. C. Varlet : **MS** - algebras definable on a distributive lattice. *Bull. Soc. Roy. Sci. Liège* **54**, 1985, 167 - 182.

[11]. T. S. Blyth and J. C. Varlet : On the dual space of an **MS** - algebras, *Math. Pannonica,* **1/1**, 1990, 95 - 109.

[12]. T. S. Blyth, A. S. A. Noor, and J. C. Varlet : Ockham algebras with de Morgan skeletons, *Journal of Algebra,* **117**, 1988, 165 - 178.

[13]. T. S. Blyth, P. Goossens, and J. C. Varlet : **MS** - algebras arising from fences and crowns, *Contributions to General Algebra* **6**, *Vienna,* 1988, 31-48.

[14]. T. S. Blyth, Jie Fang, and J. C. Varlet : **MS** - algebras arising from hat racks, double fences and double crowns, *Bull. Soc. Roy. Sci. Liège* **58**, 1989, 63 - 84.

[15]. T. S. Blyth, Jie Fang, and J. C. Varlet : Subdirectly irreducible Ockham algebras, *Contributions to General Algebra* **7**, *Vienna,* 1991, 37-48.

[16]. R. Cignoli : Injective De Morgan and Kleene algebras, *Proc. Amer. Math. Soc.* **47**, 1975, 269 - 278.

[17]. J. Vazde Carvalho : The subvariety $\mathbf{K}_{2,0}$ of Ockham algebras, *Bull. Soc.Roy. Sci. Liège* **53**, 1984, 393 - 400.

[18]. B. A. Davey and H. A. Priestley : Introduction to Lattices and Order, *Cambridge Mathematical Textbooks,* 1990.

[19]. G. Grätzer : Lattice Theory, *Freeman and Co., San Francisco,* 1971.

[20]. M. S. Goldberg : Distributive Ockham algebras: free algebras and injectivity, *Bull. Austral. Math. Soc.,* **24**, 1981, 161 - 203.

[21]. M. S. Goldberg : Topological duality for distributive Ockham algebras, Studia Logica, **XLII,** 1983, 23 - 31.

[22]. J. A. Kalman : Lattices with involution. *Trans. Amer. Math. Soc.* **87**, 1958, 485 - 491.

[23]. H. A. Priestley : The construction of spaces dual to pseudocomplemented distributive lattices, *Quart. J. Math. Oxford,* **26**, 1975, 215 - 228.

[24]. H. A. Priestley : Ordered sets and duality for distributive lattices, *Annals of Discrete Math.,* **23**, 1984, 39 - 60.

[25]. M. Ramalho and M. Sequeira : On generalised **MS** - algebras, *Portugaliae Math.,* **44**, 1987, 315 - 328.

[26]. H. P. Sankappanavar : Distributive lattices with a dual endomorphism, Z. *Math. Logik Grundlag. Math.*, **31**, 1985, 385 - 392.

[27]. A. Urquhart : distributive lattices with a dual homomorphic operation, *Studia Logica,* **38**, 1979, 201 - 209.

[28]. J. Varlet : Congruences on de Morgan algebras, *Bull. Soc. Roy. Sci. Liège*, **50**, 1981, 332 - 343.