SEMIGROUPS WITH LENGTH MORPHISMS

Bryan James Saunders

A Thesis Submitted for the Degree of PhD
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B.J. SAUNDERS

A thesis submitted for the degree of Doctor of Philosophy

of the University of St. Andrews

Department of Mathematical Science

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ABSTRACT

The class of metrical semigroups is defined as the set consisting of those semigroups which can be homomorphically mapped into the semigroup of natural numbers (without zero) under addition.

The finitely generated members of this class are characterised and the infinitely generated case is discussed. A semigroup is called locally metrical if every finitely generated subsemigroup is metrical.

The classical Green's relations are trivial on any metrical semigroup. Generalisations $\mathcal{H}^+, \mathcal{L}^+$ and $\mathcal{R}^+$ of the Green's relations are defined and it is shown thatfor any cancellative metrical semigroup, S, $\mathcal{H}^+$ is " as big as possible " if and only if S is isomorphic to a special type of semidirect product of $\mathbb{N}$ and a group.

Lyndon's characterisation of free groups by length functions is discussed andalink between length functions, metrical semigroups and semigroups embeddable into free semigroups is investigated.

Next the maximal locally metrical ideal of a semigroup is discussed, and the class of t-compressible semigroups is defined as the set consisting of those semigroups that have no locally metrical ideal. The class of t-compressible semigroups is seen to contain the classes of regular and simple semigroups.

Finally it is shown that a large class of semigroups can be decomposed into a chain of locally metrical ideals together with a t-compressible semigroup.

DECLARATION

I Bryan James Saunders hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree or professional qualification.

7th March 1988

DECLARATION

I was admitted to the Faculty of Science of the University of St. Andrews under Ordinance General No 12 in October 1985 and as a candidate for the degree of Ph.D. in October 1986.

7th March 1988

CERTIFICATE

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the Degree of Ph.D.

Signature of Supervisor ....

Date ...7 March 1988....

DEDICATION

To Clare, Matthew and Penny

PREFACE

I would like to thank the Science and Engineering Research Council for their financial support over the past three years. My thanks are also due to Professor John Howie, who not only introduced me to semigroup theory four years ago, but has also supervised me over the last three years.

NOTATION

The following commonly used symbols appear in the text

Symbol	Description
$\mathbf{N}$	The set of Natural numbers
$\mathbf{N}^o$	The set of Natural numbers with zero adjoined
$\mathbf{Q}$, $\mathbf{Q}^+$, $\mathbf{R}$, $\mathbf{R}^+$ and $\mathbf{R}^+_o$	The sets of rational, strictly positive rational,real,strictly positive real and non-negative real numbers

(Where the above sets appear as semigroups, the binary operation of addition is assumed unless stated otherwise)

X^+	The free semigroup on the set X, consistingofwords $x_1 \ldots x_n$ inthe elementsof X and binary operationof concatination, i.e. $(x_1 \ldots x_n).(y_1 \ldots y_m) = x_1 \ldots x_n y_1 \ldots y_m$
X^*	The free monoid on the set X, which is isomorphicto $X^+$with an identity adjoined
X^c	The free commutative semigroup on the set X
$(X^c)^1$	The free commutative monoid on the set X

$\|\cdot\|_x$	The morphism on X^+, X^*, X^c and $(X^c)^1$ that maps an element z to the number of occurrences of the letter x in z
$\|\cdot\|$	$\sum_{x \in X} \|\cdot\|_x$
FG(X)	The free group on the set X

The following symbols are either less common than the above, or are introduced for the first time in the text.

Symbol	Description	First appearance
C(S)	The compressible part of the semigroup S.	86
D(G)	The divisible hull of the group G.	35
D(S)	The divisible hull of the semigroup S.	36
L(S)	The locally metrical part of the semigroup S.	86
Q(S)	The group of quotients of the commutative semigroup S.	11
$S^{(n)}$	The subset of a commutative semigroup S given by $S^{(n)} = \{ s^n : s \in S \}$.	12
S_n	The subset of a metrical semigroup (S,f) given by $S_n = \{ s \in S : f(s) = n \}$.	4
χ_S	The smallest commutative congruence on the semigroup S.	18

CONTENTS

INTRODUCTION

A study of some of the major texts on algebraic semigroup theory (for example Clifford and Preston, [3], Howie, [10], and Petrich, [15]) reveals a subject dominated by the notions of regularity, simplicity and the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$. While the theory has made much progress in this direction, a certain amount of generality has been lost. This is well illustrated by the fact that for some of the most natural semigroups, for example $(\mathbf{N},+)$, X^+ and X^c, these notions and relations are either trivial or uninteresting.

On the other hand it is these kinds of semigroup that have contributed applications to practical problems, particularly in the areas of codes and theoretical computer science. This branch of the theory is known as combinatorial semigroup theory (see for example Lallement,[12]).

Semigroups with length morphisms, and other related ideas to be introduced in the course of this study, can be seen as an attempt to bridge the gap between the algebraic and the combinatorial branches of the theory. Our methods are mainly algebraic, and some of our results are reminiscent of more classical results, particularly in the first part of section 4. The class of semigroups with length morphisms, however, does not include most of the semigroups normally associated with classical algebraic semigroup theory, but instead those usually associated with the combinatorial branch. More specifically we use the notion of length morphisms to try and tackle some interesting algebraic questions that arise out of the problems encountered in combinatorial semigroup theory, most importantly the problem of finding a characterisation of F-semigroups (those finitely generated semigroups that can be embedded into a free semigroup). This is dealt with in the second part of section 4.

An important area of semigroup theory that does not exhibit the above polarity is that of commutative semigroup theory. Here again the notions of regularity and simplicity, and the Green's relations, are less interesting. The result of this is that commutative semigroup theory has taken a path that lies closer to that I have taken in this thesis, and as a result many of our results are adaptations of commutative results to the generally more interesting non-commutative case.

The thesis falls into three parts. In sections 1,2 and 3 we introduce the notion of a length morphism on a semigroup and other related ideas. We then characterise some of the classes introduced.

In section 4 we investigate semigroups with length morphisms in more detail, and introduce generalisations of the Green's relations, $\mathcal{R}^+,\mathcal{L}^+,\mathcal{H}^+,\mathcal{D}^+$ and $\mathcal{J}^+$, that are non-trivial on such semigroups. Some links with the well known generalisations of the Green's relations due to Fountain are also discussed. We then give an example of the type of results that can be obtained by characterising those cancellative semigroups with length morphism in which the $\mathcal{H}^+$

classes are " as large as possible ". The second part of section 4 investigates links between semigroups with length morphisms and combinatorial semigroup theory.

Sections 5 and 6 introduce a decomposition of a large class of semigroups into those semigroups introduced in the first part. It is hoped that such a decomposition will help generalise results on semigroups with length morphisms to the larger class of semigroups.

Thus our first part can be considered as an introduction to the new notion, the second part as a justification for introducing it, and the third part as an insight into the possible potential of the subject to semigroup theory in general.

Semigroups with length morphisms

1. Definitions and preliminaries.

1.1 Metrical semigroups and metrical monoids.

A semigroup S is said to be a metrical semigroup if there exists a morphism $f: S \longrightarrow (\mathbf{N}, +)$.

Given a metrical semigroup (S,f) we define subsets S_n of S by

$$S_n = \{ s \in S : f(s) = n \}.$$

EXAMPLES 1.1.

a) The free semigroup X^+ on a set X is metrical. The relevant morphism is the 'length' function, $|.|$, given by

$$|x_1 x_2 \ldots x_n| = n.$$

We shall call this particular morphism the usual length morphism on X^+ and denote it either by $|.|$ or by u.

b) The free commutative semigroup X^C on a set X is metrical. The usual length morphism is defined in the same way as for the non-commutative case, and is again denoted by $|.|$ or by u.

c) Subsemigroups of the above are metrical.

Motivated by the first of these examples we call a morphism $f: S \longrightarrow (\mathbf{N}, +)$ a length morphism on S and call the pair (S,f) the metrical semigroup S with length morphism f.

Let $n \in \mathbf{N}$. Then if f is a length morphism on a semigroup S, so is the morphism $nf: S \longrightarrow \mathbf{N}$ given by

$$(nf)(s) = n.f(s).$$

Thus a metrical semigroup has an infinite number of length morphisms. We say that two length morphisms f and g are <u>equivalent</u> if there exist m,n in $\mathbf{N}$ such that nf = mg. A metrical semigroup can have two non-equivalent length morphisms: for example let X = {a,b}, then the maps

$$f: X \longrightarrow \mathbf{N} \text{ given by } f(a) = f(b) = 1$$

and $$g: X \longrightarrow \mathbf{N} \text{ given by } g(a) = 1,\ g(b) = 2$$

both extend uniquely to length morphisms on X^+. The resultant length morphisms are non equivalent.

Clearly if S is a metrical semigroup with length morphism f, and R is a semigroup such that there exists a morphism θ: R $\longrightarrow$ S, then R is a metrical semigroup, with length morphism $f \circ \theta$, which we call the <u>length morphism induced by θ and f</u>. In particular if θ is a morphism from R to either X^+ or X^c, we simply call the length morphism induced by θ and u the <u>length morphism induced by θ</u>.

A monoid M is called a <u>metrical monoid</u> if there exists a morphism g: M $\longrightarrow$ ($\mathbf{N}^o$,+) such that $g^{-1}(0) = \{ 1 \}$. For convenience we also refer to g as a length morphism, since it will be clear in context to what type of semigroup we are referring . A metrical monoid with length morphism g is denoted by (M,g). If (S,f) is a metrical semigroup, then S^1 is a metrical monoid with length morphism the unique extension $\tilde{f}$ of f obtained by letting $\tilde{f}(1) = 0$. For simplicity's sake we denote the metrical monoid S^1 with length morphism $\tilde{f}$ by (S^1,f).

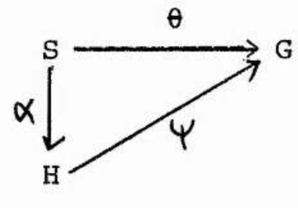
Given a metrical semigroup S (resp. a metrical monoid M) we can define a binary operation on mor(S,$\mathbf{N}$) (resp. mor(M,$\mathbf{N}^o$)) by

$$(f+g)(s) = f(s) + g(s).$$

Throughout the thesis we shall treat mor(S,$\mathbf{N}$) as a semigroup with this operation.

1.2. The free group on a semigroup. The group of right quotients of a semigroup.

Using the notation of [3] we define a free group on a semigroup S to be a pair (H,α), where H is a group and α is a morphism from S to H for which $\alpha(S)$ generates H, such that for any group G and any morphism θ: S $\longrightarrow$ G, there exists a morphism ψ: H $\longrightarrow$ G for which the diagram



commutes.

Clifford and Preston prove that given any semigroup S, such a pair exists ([3] construction 12.3), and that it is unique up to " equivalence ", where (H,α) is said to be equivalent to (H',β) if there exists an isomorphism γ from H to H′ such that $\beta = \gamma\circ\alpha$. ([3] lemma 12.1).

A semigroup, S, is group embeddable if there exists a group G and a monomorphism θ: S $\longrightarrow$ G. It is an easy consequence of the universal nature of the free group on a semigroup (see for example [3] theorem 12.4)that S is group embeddable if and only if the morphism α in the pair (H,α) is a monomorphism. When this is the case we can assume that S is a subsemigroup of H and that α is the canonical injection morphism.

If a semigroup S can be embedded in a group then it can be embedded in a group for which S is a set of group generators. Every element of such a group will be a finite product of elements of S and of inverses (in the group) of elements of S. For a group embeddable semigroup S we now construct the group of right quotients, which turns out to be a free group on S. Results will be stated without proof, since details can be found in [3] section 12.4.

Let S be a group embeddable semigroup. For a,b in S such that $Sa \cap Sb \neq \emptyset$ we define the right quotient

$$a/b = \{ (x,y) : xa = yb \}.$$

LEMMA 1.1. Let S be a group embeddable semigroup. Let a,b,c,d be elements of S. Then

$$a/b = c/d \text{ if and only if } a/b \cap c/d \neq \emptyset.$$

CONSTRUCTION. Let S be a group embeddable semigroup. Let Q be the set of right quotients of pairs of elements of S:

$$Q = \{ a/b : a,b \in S \ , \ Sa \cap Sb \neq \emptyset \}.$$

Let τ be the congruence on the free semigroup, Q^+, on Q generated by the set

$\{(xy,z) : x= a/b,\ y= b/c,\ z= a/c$ for some $a/b, b/c, a/c$ in $Q\}$.

Let G be the semigroup Q^+/τ . Then G is a group and if we define the mapping $\alpha: S \longrightarrow G$ by $\alpha(s) = [sx/x]$, then the pair

(G,α) is a free group on S.

The pair (G,α) will be called the group of right quotients of S.

By identifying the semigroup S with the subsemigroup $\alpha(S)$ of G, we can again consider the monomorphism α as the canonocal injection morphism. Instead of denoting the group of right quotients by a pair (G,α), we denote it simply by G. We shall do this without comment througout the thesis.

The next two results will be of use in section 4:

LEMMA 1.2. Let S be a metrical semigroup, and let f be a length morphism on S. Then f extends uniquely to a morphism from the group of quotients of S into $\mathbb{Z}$.

Proof. We define a map f_1 from Q into $\mathbb{Z}$ by

$$a/b \longrightarrow f(a) - f(b).$$

f_1 extends uniquely to a morphism f_2 from Q^+ into $\mathbb{Z}$. The resultant morphism f_2 is invarient under the basic transitions, and so itself induces a unique morphism $\tilde{f}$ from the group of right quotients of S into $\mathbb{Z}$.

It is easily seen that $\tilde{f}$ is an extension of f. It is unique because the set S is a set of group generators for the group of right quotients of S.

LEMMA 1.3. Let S be a group embeddable semigroup, and let G be the group of right quotients of S. Suppose that S' is a subsemigroup of G that contains S. The group of right quotients of S' is then equal to G.

Proof. Suppose that H is a group and that θ is a morphism from S' into H. Consider the restriction $\theta|_S$ of θ to S. Since G is a free group on S it follows that there exists a morphism $\psi: G \longrightarrow H$ for which the diagram

$$\begin{array}{ccc} S & \xrightarrow{\iota} & G \\ & {\scriptstyle \theta|_S} \searrow & \downarrow {\scriptstyle \psi} \\ & & H \end{array}$$

commutes.

S is a set of group generators for G, and so if $\theta(s) = \psi(s)$ for all s in S it follows that $\theta(s') = \psi(s')$ for all s' in S'. We deduce that the diagram

$$\begin{array}{ccc} S' & \xrightarrow{\iota} & G \\ & {\scriptstyle \theta} \searrow & \downarrow {\scriptstyle \psi} \\ & & H \end{array}$$

commutes,

and so G is a free group on S'. It follows that G is isomorphic to the group of right quotients of S'. Equality follows from our convention of specifying the canonical injection morphism when considering free groups on semigroups.

1.3. Semigroup Presentations.

Given a set X and words u_i, v_i ($i \in I$) in X^+ we follow the usual notation of defining the presentation

$$P(S) \equiv \langle X : u_i = v_i \quad , i \in I \rangle$$

to be of the semigroup $S = X^+/\rho$, where ρ is the smallest congruence on X^+ containing $\{ (u_i, v_i) : i \in I \}$.

We call an ordered pair (x,y) a basic P(S) transition if there exist words w,z in X^+ such that

$$x = wpz \ , \ y = wqz$$

where $(p,q) = (u_i, v_i)$ or (v_i, u_i) for some i in I. The pair (u,v) is then in ρ if and only if $u = v$ or there exist words $x_0, \ldots, x_n$ such that

$$x_0 = u \ , \ x_n = v$$

and

(x_i, x_{i+1}) $i = 0,..,n-1$ are basic P(S) transitions.

We define the <u>commutative presentation</u>

$$P(S) \equiv \langle X : u_i = v_i \ , \ i \in I \rangle$$

to be of the commutative semigroup $S = X^c/\rho$, where ρ is the smallest congruence on X^c containing $\{ (u_i, v_i) : i \in I \}$.

2. Finitely generated metrical semigroups.

2.1. The congruence χ and a characterisation of finitely generated metrical semigroups.

In this section we give a characterisation of finitely generated metrical semigroups. The characterisation in the commutative case is particularly simple (corollary 2.2). The non-commutative case is an adaptation of the commutative case that is less satisfactory, but nevertheless useful in later sections of the thesis (corollary 2.3).

The key result of this section is a theorem due to Grillet, theorem 2.1. The proof of theorem 2.1 requires some preliminary results, lemmas 2.1 and 2.2 and corollary 2.1.

DEFINITION 2.1. Let S be a cancellative commutative semigroup. Define the congruence π on $S \times S$ by $(a,b)\pi(c,d)$ if and only if $ad = bc$. Denote the semigroup $(S \times S)/\pi$ by Q(S). Q(S) is then an abelian group, with identity element $[(x,x)]_\pi$ ($= [(y,y)]_\pi$ for any y in S), called the quotient group of S, and S is embeddable into Q(S).

LEMMA 2.1. Let S be a finitely generated cancellative and power cancellative commutative semigroup. Then S is embeddable into a free abelian group.

Proof. Consider the group of quotients Q(S) of S. First notice that Q(S) is finitely generated, for if X generates S then the set $\{ [(xa,a)]_\pi : x \in X \}$ is a set of group generators for Q(S).

Suppose that $[(a,b)]_\pi^n = [(x,x)]_\pi$, the identity. Then $[(a^n,b^n)]_\pi = [(x,x)]_\pi$ and so $a^n x = b^n x$. The cancellativity of S implies that $a^n = b^n$, and then the power cancellativity of S implies that a = b. Therefore $[(a,b)]_\pi = [(a,a)]_\pi = [(x,x)]_\pi$. We deduce that Q(S) is torsion free.

It is well known (see for example [6]) that a finitely generated torsion free abelian group is isomorphic to a free abelian group, and so the result follows.

Given a commutative semigroup S we define the subsemigroup $S^{(n)}$ of S by

$$S^{(n)} = \{ s^n : s \in S \}.$$

Note that when S is a power cancellative commutative semigroup then $S^{(n)}$ is isomorphic to S.

LEMMA 2.2. Let S be a finitely generated subsemigroup of a free abelian group G not containing the identity element of G. Then either S is isomorphic to a free commutative semigroup or else there exists a subsemigroup $\overline{S}$ of G and a positive integer k such that

a) The rank of $\overline{S}$ is less than that of S,

b) $S^{(k)}$ is contained in $\overline{S}$,

c) $\overline{S}$ does not contain the identity element of G.

Proof. Suppose that the rank of S is n, and let A be a set of size n that generates S. If S is isomorphic to a free commutative semigroup then the result follows immediately. If S is not isomorphic to a free commutative semigroup then there exists a non-trivial relation between elements of A. By cancellation we may suppose that we have a non-trivial relation

$$(*) \qquad s_1^{n_1} s_2^{n_2} \ldots s_p^{n_p} = r_1^{m_1} r_2^{m_2} \ldots r_q^{m_q}$$

such that the r_i and the s_j are all distinct elements of A, and n_i, m_j are strictly positive integers. Suppose further that p is minimal over the set of all such non-trivial relations. We immediately have that $p > 0$ since S does not contain the identity. Now define $\overline{S}$ to be the subsemigroup of G generated by

$$\{r_1^{m_1} . s_2^{-n_2} .. s_p^{-n_p} \; ; \; r_2, \ldots, r_q \; , s_2, \ldots, s_p \; ; \; t_1, \ldots, t_k \; \}$$

where the t_i are the elements in A not appearing in the relation *. It is clear that $\overline{S}$ is of rank less than that of S, and so $\overline{S}$ satisfies a).

Clearly all the elements of A except for r_1 and s_1 are contained in $\overline{S}$. Now

$$r_1^{m_1} = (r_1^{m_1} . s_2^{-n_2} .. s_p^{-n_p}) \, s_2^{n_2} .. s_p^{n_p}$$

and

$$s_1^{n_1} = (r_1^{m_1} . s_2^{-n_2} .. s_p^{-n_p}) \, r_2^{m_2} .. r_q^{m_q} .$$

Therefore all the elements of the set $A^{(m_1 n_1)}$ are contained in $\overline{S}$, and thus the subsemigroup $S^{(m_1 n_1)}$ of G is contained in $\overline{S}$, and so letting $k = m_1 n_1$ we have that $\overline{S}$ satisfies b).

We now show that $\overline{S}$ satisfies c). Suppose, by way of contradiction, that $\overline{S}$ contains the identity. Then we have a relation

$$(r_1^{m_1}.s_2^{-n_2}..s_p^{-n_p})^w.r_2^{x_2}...r_q^{x_q}.s_2^{y_2}...s_p^{y_p}.t_1^{z_1}...t_k^{z_k} = 1$$

where w, the x_i, y_i and the z_i are all non-negative integers that are not all zero. If w is zero then we have that 1 is contained in S, giving us a contradiction, and if w is greater than zero we have the non-trivial relation

$$s_2^{wn_2}..s_p^{wn_p} = r_1^{wm_1}.r_2^{x_2}...r_q^{x_q}.s_2^{y_2}...s_p^{y_p}.t_1^{z_1}...t_k^{z_k}$$

which contradicts the minimality of p. We deduce that $\overline{S}$ does not contain the identity. This completes the proof of the lemma.

COROLLARY 2.1. <u>Let S be a finitely generated subsemigroup of a free abelian group G that does not contain the identity element of G. Then there exists a subsemigroup T of G and a positive integer k such that</u>

<u>a) $S^{(k)}$ is contained in T,</u>

<u>b) T is isomorphic to a free commutative semigroup.</u>

Proof. We can now define semigroups S_i as follows. Let $S_0 = S$, and for all $i > 0$ let $S_i = S_{i-1}$ if S_{i-1} is isomorphic to a free commutative semigroup, and $\overline{S}_{i-1}$ otherwise. Then by lemma 2.2 there exist $k_0, k_1, ... > 0$ such that $S_{i-1}^{(k_{i-1})}$ is contained in S_i.

Since the rank of S finite, and since either the rank of S_{i+1} is less than the rank of S_i or else $S_j = S_i$ for all $j \geqslant i$, we have that the sequence $S_0, S_1, \ldots$ of semigroups stabilises, and therefore that there exist a semigroup T (= S_t for some t) and an integer k (= $k_0 k_1 \ldots k_t$) such that $S^{(k)}$ is contained in T and T is isomorphic to a free commutative semigroup.

It can be shown that if G in corollary 2.1 is generated by only two elements, then a T can be found for which the value k = 1 satisfies the given conditions. It is not known whether this is the case in general.

The following lemma is almost identical to corollary 2.1 and will be of use in section 3.

LEMMA 2.3. Let G be a direct sum of copies of the group $(\mathbf{Q},+)$ and let S be a finitely generated subsemigroup of G that does not contain the identity. Then there exists a subsemigroup T of G such that T contains S and T is isomorphic to a free commutative semigroup.

Proof. Let $G = \bigoplus_{i \in I} \mathbf{Q}$. Then G can be considered as a vector space over $\mathbf{Q}$. It follows that there is an action of $\mathbf{Q}$ on G that corresponds to scalar multiplication. To avoid confusion we shall use bold type for elements of G and normal type for scalars. Let X be a basis for G and let A be a finite subset of G that generates S. Then for any element $\mathbf{s}$ in A we have

$$\mathbf{s} = (a_{i_1}/b_{i_1})\mathbf{x}_{i_1} + (a_{i_2}/b_{i_2})\mathbf{x}_{i_2} + \ldots + (a_{i_n}/b_{i_n})\mathbf{x}_{i_n}$$

where $i_j \in I$; $a_{i_j}, b_{i_j} \in \mathbf{Z}$ and $\mathbf{x}_{i_j} \in X$.

Let $b_{\mathbf{s}}$ be the lowest common multiple of the b_{i_j}.

Let b be the lowest common multiple of the $b_{\mathbf{s}}$ as $\mathbf{s}$ varies across A. Then we have that A is contained in the subgroup H of G generated by $Y = \{ (1/b)\mathbf{x} : \mathbf{x} \in X \}$.

Now X is a basis, and is therefore a linearly independent set. We deduce that no non-trivial relation can exist between the elements of Y. Thus Y is a set of free commutative group generators for H. We therefore deduce that H is a free commutative group containing S.

It follows from corollary 2.1 (using additive notation instead of multiplicative notation) that there exist an integer k and a subsemigroup T of H such that T is isomorphic to a free commutative semigroup and $\{ k\mathbf{s} : \mathbf{s} \in S \}$ is contained in T. We deduce that S is contained in the subsemigroup $(1/k)T$ ($= \{ (1/k)\mathbf{t} : \mathbf{t} \in T \}$), which is also isomorphic to a free commutative semigroup, completing the proof of the lemma.

THEOREM 2.1. ([8] theorem 2.2.). A finitely generated commutative semigroup is embeddable into a free commutative semigroup if and only if it is cancellative, power cancellative and does not have an identity element.

Proof. The direct implication is obvious. For the converse suppose that S is cancellative, power cancellative and does not contain an identity. From lemma 2.1 S is isomorphic to a subsemigroup of a free abelian group G. Since S does not contain the identity element it follows from corollary 2.1 that there exist a number n and a subsemigroup T of G such that $S^{(n)}$ is contained in T and T is isomorphic to a free commutative semigroup. S, being power cancellative, is isomorphic to $S^{(n)}$ and so we deduce that S is embeddable into a free commutative semigroup.

We now use theorem 2.1 to give us a characterisation of finitely generated commutative metrical semigroups.

COROLLARY 2.2. A finitely generated commutative semigroup, T, is metrical if and only if there do not exist $t, r \in T$ such that $tr = t$.

Proof. The direct part is obvious since if T has length morphism f, then $tr = t$ would imply that $f(r) = 0$, which is not possible.

Conversely suppose that there do not exist t,r in T with $tr = t$. Let $T' = T/\tau$ where τ is the congruence given by

$$a \tau b \iff \exists n \in \mathbf{N},\ c \in T \text{ such that } a^n c = b^n c.$$

T' is then finitely generated, commutative, cancellative and power cancellative. Furthermore T' contains no identity element, since

$$xe \, \tau \, x$$

$$\Rightarrow (xe)^n c = x^n c \qquad \text{for some } n \text{ in } \mathbf{N},\ c \text{ in } T$$

$$\Rightarrow (x^n c)e = x^n c.$$

Theorem 2.1 then gives us that T' is embeddable into a free commutative semigroup. Thus T', and therefore T, is metrical.

Given an arbitrary semigroup S define the congruence χ_S to be that generated by $\{ (sr,rs) : r,s \in S \}$. S/χ_S is then a commutative semigroup and is the largest commutative image of S in the sense that if $\alpha : S \longrightarrow T$ is a morphism, and T is a commutative semigroup then there exists $\beta : S/\chi_S \longrightarrow T$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & T \\ {\scriptstyle \natural}\downarrow & \nearrow_{\beta} & \\ S/\chi_S & & \end{array}$$

commutes.

Where there is no possible ambiguity we write simply χ.

LEMMA 2.4. <u>Let S be a semigroup, and let T be a commutative semigroup. Then</u>

$$\mathrm{mor}(S,T) \neq \emptyset \quad \Longleftrightarrow \quad \mathrm{mor}(S/\chi,T) \neq \emptyset.$$

<u>Furthermore, when this is the case,</u>

$$\varphi : \mathrm{mor}(S,T) \longrightarrow \mathrm{mor}(S/\chi,T)$$

<u>given by</u>

$$(\varphi(f))[s] = f(s)$$

<u>is a well defined isomorphism. In particular</u>

$$\mathrm{mor}(S,\mathbf{N}) \cong \mathrm{mor}(S/\chi,\mathbf{N}).$$

Proof. The proof is immediate from the fact that χ is the smallest commutative congruence on S.

Corollary 2.2 and lemma 2.4 now give us a characterisation for non-commutative finitely generated metrical semigroups:

COROLLARY 2.3. <u>Let S be a finitely generated semigroup. Then S is metrical if and only if</u> $\nexists\, r,s \in S$ <u>such that</u> $rs /_S r$.

DEFINITION 2.2. We associate with an arbitrary semigroup S the commutative, cancellative and power cancellative semigroup $S' = (S/\chi)/\tau$ where the congruence χ is defined above, and the congruence τ is that defined in the proof of corollary 2.2. We define the morphism θ to be the natural map from S to S'.

The following lemma will be of use later on:

LEMMA 2.5. Let S be a semigroup. Then

$$\mathrm{mor}(S,\mathbf{N}) \neq \emptyset \iff \mathrm{mor}(S',\mathbf{N}) \neq \emptyset.$$

Furthermore when this is the case we have that

$$\mathrm{mor}(S,\mathbf{N}) \cong \mathrm{mor}(S',\mathbf{N}).$$

Proof. The proof is immediate from the fact that the congruence $\theta^{-1}\circ\theta$ is the smallest commutative, cancellative and power cancellative congruence on S.

2.2. Finitely presented metrical semigroups

In this section we show how a finitely presented semigroup can be associated with an integer matrix. We then show how some classical algorithms from the theories of linear programming and linear diophantine equations enable us to say things about the existence of length morphisms on the semigroup concerned. In contrast to the previous section, the results are equally valid for commutative and non-commutative semigroups.

Suppose that S is a finitely presented semigroup. Let P(S) be the presentation

$$P(S) \equiv \langle a_1,..,a_n : r_1=s_1 ,..,r_m=s_m \rangle$$

where the r_i and the s_i are words in the a_j . Let

$$k_{ij} = |r_i|_{a_j} , \quad h_{ij} = |s_i|_{a_j},$$

$$x_{ij} = k_{ij} - h_{ij}.$$

Then we define the <u>relational</u> <u>matrix</u> RM(P(S)) of the presentation P(S) of S to be the n m matrix with entries x_{ij}.

A morphism f from the free semigroup $\{a_1,...,a_n\}^*$ into $\mathbf{N}$ induces a morphism $\overline{f}: S \longrightarrow \mathbf{N}$ if and only if $f(r_i) = f(s_i)$ for $i = 1,..,m$. Now

$$
\begin{aligned}
& f(r_i) - f(s_i) \\
&= f(\prod_{j=1}^{n} a_j^{k_{ij}}) - f(\prod_{j=1}^{n} a_j^{k_{ij}}) \\
&= \sum_{j=1}^{n} k_{ij}f(a_j) - \sum_{j=1}^{n} h_{ij}f(a_j) \\
&= \sum_{j=1}^{n} x_{ij}f(a_j).
\end{aligned}
$$

So f induces a morphism $\overline{f} : S \longrightarrow \mathbf{N}$ if and only if

$$RM(P(S))(f(a_1),...,f(a_n))^T = \mathbf{0}.$$

Thus we have

LEMMA 2.6. <u>Let P(S) be a finite presentation for S with generators $A = \{ a_1,..,a_n \}$. Then there is an isomorphism between mor(S,$\mathbf{N}$) and the semigroup of strictly positive solutions to the equation RM(P(S))$\mathbf{x} = \mathbf{0}$ (with binary operation of vector addition) given by</u>

$$\underline{RM(P(S))(m_1,\ldots,m_n)^T = \mathbf{0} \quad \Leftrightarrow \quad f_{\mathbf{m}} \in mor(S,\mathbf{N})}$$

where $f_{\mathbf{m}}$ is the unique extension to S of the map from A to $\mathbf{N}$ given by

$$\underline{f_{\mathbf{m}}(a_i) = m_i.}$$

Similarly we have

LEMMA 2.7. Let P(S) be a finite presentation for S with generators $A = \{ a_1,\ldots,a_n \}$. Then there is an isomorphism between $mor(S,\mathbf{N}^0)$ and the semigroup of non-negative solutions to the equation $RM(P(S))\mathbf{x} = \mathbf{0}$ given by

$$\underline{RM(P(S))(m_1,\ldots,m_n)^T = \mathbf{0} \quad \Leftrightarrow \quad f_{\mathbf{m}} \in mor(S,\mathbf{N}^0)}$$

where $f_{\mathbf{m}}$ is the unique extension to S of the map from A to $\mathbf{N}^0$ given by

$$\underline{f_{\mathbf{m}}(a_i) = m_i.}$$

DEFINITION 2.3 The general integer linear programming problem can be expressed as: Find integers

$$x_1 \geqslant 0,\ x_2 \geqslant 0\ ,\ldots\ x_n \geqslant 0$$

that minimise z when

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = z,$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned}$$

where the a_{ij}, b_i and c_j are given integer constants.

The general linear programming problem is written in terms of equality constraints. We will, however, be wanting to solve problems with inequality constraints. In these cases we convert the inequalities to equalities by the addition of a slack variable, for

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \geqslant b$$

if and only if

$$a_1x_1 + a_2x_2 + \dots + a_nx_n - x_{n+1} = b$$

for some $x_{n+1} \geqslant 0$

and

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \leqslant b$$

if and only if

$$a_1x_1 + a_2x_2 + \dots + a_nx_n + x_{n+1} = b$$

for some $x_{n+1} \geqslant 0$. Allowing $c_{n+1} = 0$ we obtain a general integer linear program with the required solution. A different slack variable is of course used for each inequality.

A general linear programming problem is always defined in terms of finding a minimal solution. We can, however, use a program to find whether any non trivial solution to the equations exist. This is equivalent to solving a set of linear diophantine equations. We do this by adding the constraint

$$x_1 + x_2 + \dots + x_n \geqslant 1$$

to our list of equations, and finding a solution that minimises

$$z = x_1 + x_2 + \dots + x_n.$$

There are various different algorithms for solving the general integer linear programming problem. Some of these can be found in [7]. We are not interested here in the actual algorithms, but just in the fact that they exist, and how they help determine length morphisms on semigroups.

THEOREM 2.2. Let S be a semigroup given by a finite presentation $P(S) \quad < x_1,..,x_n : R >$. Then S is metrical if and only if there exists a solution to the system of equations

$$\begin{aligned} RM(P(S))\, \mathbf{x} &= \mathbf{0} \\ x_1 &\geqslant 1 \\ x_2 &\geqslant 1 \\ &\vdots \\ x_n &\geqslant 1. \end{aligned}$$

Proof. The integer vector $(m_1,\dots,m_n)$ is a solution to the given system if and only if it is strictly positive, and if

$$RM(P(S))\ (m_1,..,m_n)^T = \mathbf{0}.$$

By lemma 2.6 this occurs if and only if the map $f_{\mathbf{m}}$ from S to $\mathbf{N}$ is a morphism.

COROLLARY 2.4 Suppose that the semigroup S is given by the finite presentation P(S). Then there exists an algorithm for determining whether or not S is metrical.

The following well known result due to Redei ensures that the algorithm refered to in corollary 2.4 can be used for any finitely generated commutative semigroup.

LEMMA 2.8. ([17] theorem 72). Any finitely generated commutative semigroup is finitely presented.

Before closing this section we briefly discuss the special case when a semigroup S is given in terms of a finite presentation P(S) that consists of a single relation. We already know that there exist algorithms for determining whether or not S is metrical. In general these algorithms will only give us one particular length morphism on S.

Elliott, in his paper " On linear homogeneous diophantine equations " from 1903 (reference [4]), describes an algorithm for determining a generating set for the non-negative solutions to a single diophantine equation. The algorithm is also described by MacMahon in the more easily found reference [14] . It is also known that such a set is always finite (lemma 2.9 below). Thus we have

THEOREM 2.3. Let S be a semigroup given by a finite presentation P(S) consisting of a single defining relation. Then $mor(S,\mathbf{N}^{o})$ is finitely generated and there exists an algorithm for determining a finite generating set for $mor(S,\mathbf{N}^{o})$.

Proof. This is a direct consequence of Elliott's algorithm and lemma 2.7.

In the case where P(S) has more than one relation we do not know of an algorithm for determining a generating set for $mor(S,\mathbf{N}^{o})$. We do, however, have

LEMMA 2.9 ([18] page 5). Given an $n \times m$ integer matrix A, the set $\{ \mathbf{x} \in (\mathbf{N}^{O})^{n} : A\mathbf{x} = \mathbf{0} \}$ is finitely generated.

and therefore

COROLLARY 2.5. Let S be a finitely generated semigroup. Then $mor(S,\mathbf{N}^{O})$ is finitely generated.

Proof. By lemma 2.4 $mor(S,\mathbf{N}^{O})$ is isomorphic to $mor(S/\chi,\mathbf{N}^{O})$. S/χ is a finitely generated commutative semigroup, and is therefore finitely presented. Thus by lemma 2.7 there exists an integer matrix A such that $mor(S,\mathbf{N}^{O})$ is isomorphic to $\{ \mathbf{x} : A\mathbf{x} = \mathbf{0} \}$, which is finitely generated by lemma 2.9.

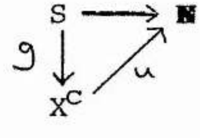
2.3. Length morphisms on semigroups not given in terms of generators and relations.

In section 2.2 we showed that if a semigroup S is given in terms of a finite presentation, then the problem of determining all the length morphisms on S can be reduced to a problem of linear algebra.

In section 2.1 we gave an algebraic characterisation of finitely generated metrical semigroups, but the characterisation did not give us any way of determining what the length morphisms on the given semigroup were. Thus we can determine algebraically whether or not $mor(S,\mathbf{N})$ is empty, but cannot give any interesting information about the structure of $mor(S,\mathbf{N})$ or its relationship to S itself.

In this section we begin to compensate for this deficiency by giving two results. The first tells us that length morphisms on finitely generated metrical semigroups " almost " always arise from an embedding of a commutative cancellative and power cancellative semigroup into a free commutative semigroup. The second gives a condition for two elements of a semigroup to be of equal length under all the length morphisms of the semigroup.

Given any finitely generated semigroup we have by lemma 2.5 that mor(S,**N**) is isomorphic to mor(S´,**N**), where S´ is given in definition 2.2. Thus any length morphism on a finitely generated semigroup S is induced by the morphism θ and a length morphism on an associated commutative, cancellative and power cancellative finitely generated semigroup S´Grillet´s theorem gives us that a commutative, cancellativeand power cancellative finitely generated semigroup is metrical if and only if it is embeddable into a free commutative semigroup. The natural question to ask is whether or not all length morphisms on such a semigroup are induced by such an embedding. In more formal terms given any commutative, cancellative and power cancellative finitely generated metrical semigroup (S,f) does there necessarily exist a morphism g: S $\longrightarrow$ X^C such that the diagram



commutes,

where u is the usual length morphism on X^C?

The following example shows that this is not the case:

EXAMPLE 2.1. Let S be the commutative semigroup with presentation

$$< a,b,c,d : ab=cd >.$$

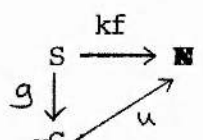
It is easily seen that S is power cancellative and cancellative and has a length morphism f given by

$$f(a)=f(b)=f(c)=f(d)=1.$$

However this length morphism cannot be the restriction of a usual length morphism on a free commutative semigroup, since in such a situation the only elements of length 1 are the basis elements, between which no non-trivial relation can exist.

We do have a slightly weaker result:

LEMMA 2.10. Given a finitely generated commutative, cancellative and power cancellative metrical semigroup (S,f) there exists an integer k and a monomorphism $g: S \longrightarrow X^C$ such that the diagram



commutes,

where u is the usual length morphism on X^C and kf is the morphism that maps s to k.f(s).

Proof. The proof of this result falls into two parts, the first corresponding to lemma 2.2, and the second corresponding to corollary 2.1. Note first that Q(S) is, by lemma 2.1, a free abelian group. The length morphism f extends uniquely to a morphism $f: Q(S) \longrightarrow \mathbb{Z}$. We now show that either S is free or else there exists a subsemigroup $\bar{S}$

of Q(S) and a positive integer z such that

a) The rank of $\bar{S}$ is less than that of S,

b) $S^{(z)}$ is contained in $\bar{S}$,

c) $f(\bar{S}) > 0$, and so $\bar{S}$ is metrical.

If Sis free then the result follows immediately. Suppose therefore that S is not free, and that A is a minimal generating set for s. There then exists a non-trivial relation

$$(*) \qquad s_1^{n_1} s_2^{n_2} \ldots s_p^{n_p} = r_1^{m_1} r_2^{m_2} \ldots r_q^{m_q}$$

such that the r_i and the s_j are all distinct elements of A, and n_i, m_j are strictly positive integers. Suppose further that p is minimal over the set of all such non-trivial relations. S is metrical and so $p \neq 0$. Let h be the integer

$$h = f(r_1^{m_1} r_2^{m_2} \ldots r_q^{m_q})$$

Now define $\bar{S}$ to be the subsemigroup of Q(S) generated by

$$\{ w^{f(r_1)} r_1^h, \ldots, w^{f(r_q)} r_q^h \ ; \ s_2, \ldots, s_p \ ; \ t_1, \ldots, t_k \}$$

$$\text{where} \quad w = s_2^{-n_2} \ldots s_p^{-n_p}$$

and the t_j are the elements of A not appearing in (*).

It is routine to show that $\bar{S}$ satisfies (a) and (c). $\bar{S}$ also satisfies (b) with $z = n_1 h$, the only difficulty being the proof that

$$s_1^{hn_1}$$

belongs to $\bar{S}$. This is indeed the case, since

$$s_1^{hn_1} = (r_1^{hm_1} r_2^{hm_2} \ldots r_q^{hm_q}) w^h$$

$$= (w^{f(r_1)} r_1 h)^{m_1} \ldots (w^{f(r_q)} r_q h)^{m_1}.$$

We now define subsemigroups S_i of Q(S) in precisely the same way as we did in the proof of corollary 2.1. Then we stabilise at a semigroup S_t that is isomorphic to a free commutative semigroup and which contains $S^{(k)}$ for some k in **N**, and for which f is strictly positive. Let X be a free generating set for S_t. Then the morphism $\alpha: S_t \longrightarrow X^c$ given by $\alpha(x) = x^{f(x)}$ is a monomorphism. Clearly $f|_S = u \circ \alpha$.

Let β be the morphism from S into S_t given by $\beta(s) = s^k$. The power cancellativity of S ensures that β is also a monomorphism. Thus $g = \alpha \circ \beta$ is a monomorphism. Furthermore given any s in S,

$$u(g(s)) = u(\alpha(\beta(s))) = u(\alpha(s^k)) = f(s^k) = kf(s).$$

We contrast lemma 2.10 with the non-commutative case in section 4.

The next result gives us a link between mor(S,**N**) and S.

LEMMA 2.11. Given a finitely generated semigroup S and two elements $r,s \in S$ then $f(r)=f(s)$ for all $f \in$ mor(S,**N**) if and only if $\theta(r)=\theta(s)$.

Proof. Suppose that $\theta(r)$ is not equal to $\theta(s)$. Let g embed S' into X^c. Then $g(\theta(r))$ is not equal to $g(\theta(s))$, and so there exists an x in X such that $|g(\theta(r))|_x$ is not equal to $|g(\theta(s))|_x$. If h is any element in mor(S,**N**) then either $h(r) \neq h(s)$ or else the map $h': S \longrightarrow$ **N** given by

$$h'(t) = h(t) + |g(\theta(t))|_x$$

is such that $h'(r) \neq h'(s)$. We therefore do not have that the lengths of r and s are the same for all length morphisms on S. The converse is easy since any morphism from S to $\mathbb{N}$ is invariant under $\theta^{-1} \circ \theta$.

3. Infinitely generated metrical semigroups. Rationally metrical, really metrical and locally metrical semigroups.

Corollaries 2.2 and 2.3 do not hold for infinitely generated semigroups. For example the semigroup ($\mathbf{Q}^+$,+) of strictly positive rational numbers under addition satisfies the conditions of corollary 2.2, but if

$$f: \mathbf{Q}^+ \longrightarrow \mathbf{N},$$

then

$$f(1)=n \Rightarrow f(1/2n)=1/2 ,$$

which is not possible. This prompts us to make some new definitions.

We say that a semigroup S is rationally metrical, or $\mathbf{Q}$-metrical if there exists a morphism q: $S \longrightarrow \mathbf{Q}^+$, is really-metrical, or $\mathbf{R}$-metrical, if there exists a morphism r: $S \longrightarrow \mathbf{R}^+$, and we say that a semigroup S is locally metrical if every finitely generated subsemigroup of S is metrical. In the finitely generated case we shall see that these concepts are all equivalent. In general we have that

$$\text{metrical} \Rightarrow \mathbf{Q}\text{-metrical} \Rightarrow \mathbf{R}\text{-metrical}$$

We have already seen that the first implication is strict. The second implication is also strict as the following example illustrates:

EXAMPLE 3.1. Let S be the semigroup $\mathbf{R}^+ \cap \mathbf{Q}[\sqrt{2}]$ under addition. Then S is $\mathbf{R}$-metrical since it is a subsemigroup of $\mathbf{R}^+$. Suppose by way of contradiction that S is $\mathbf{Q}$-metrical, and let f: $S \longrightarrow \mathbf{Q}^+$ be a morphism. Notice that for all a,b in $\mathbf{N}$ and x in S that

$$af(x) = f(ax) = f(b(a/b)x) = bf((a/b)x).$$

Thus for any q in $\mathbf{Q}^+$, we have $f(qx) = qf(x)$. In particular for all q in $\mathbf{Q}^+$ we have $f(q) = qf(1)$.

Consider $f(\sqrt{2})$. Now $f(\sqrt{2}) \neq f(1)\sqrt{2}$ since $f(1)\sqrt{2}$ is irrational. Suppose that $f(\sqrt{2}) < f(1)\sqrt{2}$. Then $f(1)\sqrt{2} - f(\sqrt{2})$ is an element of S. However

$$f(\ f(1)\sqrt{2} - f(\sqrt{2})\ ) \ = \ f(1)f(\sqrt{2}) - f(\sqrt{2})f(1) = 0$$

which is not possible. Similarly we get a contradiction if $f(\sqrt{2}) > f(1)\sqrt{2}$.

Notice that S is countable, so although we have that a finitely generated **Q**-metrical semigroup is metrical, we do not necesarily have that a countably generated **R**-metrical semigroup is **Q**-metrical.

We also have the implication

R-metrical => locally metrical

since if $f\colon S \longrightarrow \mathbf{R}^+$ is a morphism, and U is a finitely generated subsemigroup of S, then f(U) is a finitely generated subsemigroup of $\mathbf{R}^+$, which is commutative, cancellative, power cancellative and without identity. It follows from corollary 2.2 that f(U) is metrical, and therefore that U is metrical. Thus every finitely generated subsemigroup of S is metrical, and so S is locally metrical.

Since a finitely generated semigroup is metrical if and only if it is locally metrical, the set of implications

metrical => **Q**-metrical => **R**-metrical => locally metrical

implies that these concepts are all equivalent in the finitely generated case.

The following example shows that the implication

$\mathbf{R}$-metrical => locally metrical

is strict.

EXAMPLE 3.2. Let S be the commutative semigroup with presentation $< a,b,x_1,x_2,.. : a^n x_{n+1} = bx_n \ ; n \in \mathbf{N} >$. Let S_i be the subsemigroup of S generated by $\{ a,b,x_1,..,x_i \}$. Let $f_i : S \longrightarrow \mathbf{Z}$ be given by

$$f_i(a)=1, \ f_i(b)=i \ , \ f_i(x_n) = ni - \frac{n(n-1)}{2} + 1.$$

Then f_i is well defined since it is invariant under the basic transitions of the presentation of S, i.e.

$$f_i(a^n x_{n+1}) = f_i(bx_n).$$

Furthermore $f_i(S_i) > 0$. We deduce that each S_i is metrical, and since any finitely generated subsemigroup of S is contained in S_j for some j, we deduce that S is locally metrical. S is however not $\mathbf{R}$-metrical. Suppose by way of contradiction that $r: S \longrightarrow \mathbf{R}^+$ is a morphism, then suppose that $r(a)=w$, $r(b)=y$, $r(x_1)=z$. To ensure that r is well defined we must have that

$$r(x_{i+1}) = r(x_i) + r(b) - ir(a) \quad \text{for all } i > 0$$

from which we deduce

$$r(x_n) = z + (n-1)y - \frac{(n(n-1))}{2}w \quad \text{for all } n > 0$$

which is less than 0 for sufficiently large n. Thus S is a semigroup that is locally metrical but not $\mathbf{R}$-metrical.

3.1 Locally metrical semigroups.

In section 2 we defined the congruence χ on an arbitrary semigroup. We then gave a characterisation of finitely generated metrical semigroups in corollary 2.3. As mentioned at the beginning of this section, the result does not hold in the infinitely generated case. We do, however, have the following weaker result:

THEOREM 3.1. A semigroup S is locally metrical if and only if there do not exist elements r and s in S such that rs χ r.

Proof. Suppose that S is locally metrical but, by way of contradiction, that there exists $r,s \in S$ such that rs χ r. Then there exists x_i, y_i, u_i, v_i ($i=1,..,n$) such that:

$$rs=x_1u_1v_1y_1\ ;\ x_iv_iu_iy_i = x_{i+1}u_{i+1}v_{i+1}y_{i+1} \quad i=1,..,n-1;$$
$$x_nv_nu_ny_n = r.$$

Now let $U = < x_i,y_i,u_i,v_i,s \ (i=1,..,n) >$. U is finitely generated, and so since S is locally metrical, U must be metrical; but this is not possible since if $f: U \longrightarrow$ ■ is a morphism then $f(rs) = f(x_1u_1v_1y_1) = f(x_1v_1u_1y_1) =...=f(r)$ and so $f(s)=0$, which is not possible.

Conversely suppose that there do not exist r and s in S such that rs χ r. Let U be a finitely generated subsemigroup of S. Since $\chi_U \subseteq \chi_S \cap (U \times U)$ we have that there do not exist r and s in U such that rs χ_U r, and so U is metrical by corollary 2.3.

Example 3.2 gave us a locally metrical semigroup that is not **R**-metrical, and therefore not **Q**-metrical. We do, however, have a weaker, but very useful , result in lemma 3.3. First we discuss some results in abelian group theory, details of which can be found in Fuchs' book [6].

An abelian group (D,.) is said to be divisible if for all a in D and n in $\mathbb{N}$ there exists x in D such that $x^n = a$. The divisible hull D(G) of an abelian group G is the unique (up to isomorphism) divisible group such that G is embeddable into D(G) in such a way that the image of G is not contained in any divisible subgroup of D(G). Fuchs shows that given any abelian group G the divisible group D(G) exists, and is isomorphic to a direct sum

$$\left(\bigoplus_{i \in I} (\mathbb{Q},+) \right) \oplus C \qquad (*)$$

where C is a periodic group.

LEMMA 3.2. Suppose that G is a torsion free abelian group. The periodic group C in * is then the trivial group. Thus D(G) is isomorphic to a direct sum of copies of the group $(\mathbb{Q},+)$.

Proof. Let f be the required embedding of G into D(G). Then for any d in D(G)\1 there exists n in $\mathbb{N}$ such that $d^n \in f(G\backslash 1)$ (otherwise the subgroup of D(G) consisting of all elements that satisfy this condition, together with the identity element, would be a proper divisible subgroup of D(G) containing G). Suppose by way of contradiction that c is an element of D(G)\1, but that $c^m = 1$. Now $c \in D(G)\backslash 1$ and so there exist g in G\1 and n in $\mathbb{N}$ such that $c^n = f(g)$. Thus $f(g)^m = 1$, and so $f(g^m) = 1$. Now f embeds G into D(G) and so

$g^m = 1$. This is a contradiction since G is torsion free. Thus D(G) contains no non-trivial periodic elements and so the group C in * must be trivial.

Given a cancellative commutative semigroup S we define the divisible hull D(S) of S to be the divisible hull of the quotient group Q(S) of S given in definition 2.1. S is then embeddable into D(S). The divisible rank of S is the size of the set I in *. We then have the following corollary to lemma 3.2.

COROLLARY 3.1. A cancellative and power cancellative commutative semigroup S is embeddable into a direct sum of copies of the group $(\mathbf{Q},+)$.

Proof. We know that S is embeddable into D(S) = D(Q(S)). Since S is power cancellative we have that Q(S) is torsion free (see the proof of lemma 2.1). D(Q(S)) is therefore isomorphic to a direct sum $\oplus$ $(\mathbf{Q},+)$.

LEMMA 3.3. Let S be a locally metrical semigroup. Then if U is a finitely generated subsemigroup of S there exists a morphism $q\colon S \longrightarrow \mathbf{Q}$ such that $q(U) \subseteq \mathbf{N}$.

Proof. Suppose that S is locally metrical. Then the semigroup S' defined in section 2 is commutative, cancellative and power cancellative and does not contain an identity element; and there exists a morphism θ from S onto S'.

It follows from corollary 3.1 that S' is contained in a direct sum of copies of the group $(\mathbf{Q},+)$. Denote this direct sum by V. Then V can be considered as a vector space over **Q**.

Let U be a finitely generated subsemigroup of S. Then $\theta(U)$ is a finitely generated subsemigroup of V, and so it follows from lemma 2.3 that there exists a subsemigroup T of V that contains $\theta(U)$, and is isomorphic to a free commutative semigroup. Let u be the usual length morphism on T.

Let **Q**T be the set $\{ r\mathbf{t} : r \in \mathbf{Q} , \mathbf{t} \in T \}$. Then **Q**T is a subspace of V, and u extends to the morphism $u' : \mathbf{Q}T \longrightarrow \mathbf{Q}$ given by $u'(r\mathbf{t}) = r.u(\mathbf{t})$.

Since u' is a linear functional on a subspace of V it can be extended to a linear functional, q, on the whole of V.

The composition $q \circ \theta$ is then a morphism from S to **Q**, and $q \circ \theta(U) = q(T) = u(T)$, which is contained in **N**, as required.

The following lemma, closely related to lemma 3.3, is due to Hamilton, Nordahl and Tamura:

LEMMA 3.4 ([9] Theorem 4.1) Let S be a cancellative commutative semigroup of finite divisible rank that does not contain an identity. Then there exists a non-trivial morphism from S into the semigroup of non-negative rational numbers.

COROLLARY 3.2. Suppose that S is a locally metrical semigroup such that S' is of finite divisible rank. Then there exists a non-trivial morphism from S into the semigroup of non-negative rational numbers.

Kobayashi gives an example in [11] of a semigroup of infinite divisible rank for which there is no such morphism.

3.2. **R**-metrical and **Q**-metrical semigroups

We end this section with a brief discussion of **R**-metrical and **Q**-metrical semigroups.

The following two theorems concern characterisations of **R**-metrical cancellative commutative semigroups.

Tamura has shown ([19], theorem 3.1 and [20] theorem 2.1.) the following:

THEOREM 3.2. A cancellative commutative semigroup is **R**-metrical if and only if it is embeddable into a cancellative commutative archimedean semigroup without idempotent.

The next theorem is due to Kobayashi ([11]).

THEOREM 3.3. A cancellative commutative semigroup S of finite divisible rank is **R**-metrical if and only if it satisfies the condition "B" that for any a and b in S there exists an n in **N** such that for all m in **N**, a^{nm} is not a divisor of b^m.

Notice that the semigroup of example 3.2 is locally metrical, cancellative and of finite divisible rank, but is not **R**-metrical, and so condition "B" is less general than the condition of local metricality.

Theorem 3.3 does not hold for cancellative commutative semigroups of infinite divisible rank. For example take the semigroup

$$(\mathbf{R}_o^+)^{\mathbf{N}}\backslash\mathbf{0}.$$

This semigroup satisfies condition "B" since for any $\mathbf{x},\mathbf{y}$ in $(\mathbf{R}_o^+)^{\mathbf{N}}\backslash\mathbf{0}$ such that $\mathbf{x}|\mathbf{y}$ we must have that $x_i \leqslant y_i$ for all i in $\mathbf{N}$. Let $\mathbf{a},\mathbf{b}$ be elements of $(\mathbf{R}_o^+)^{\mathbf{N}}\backslash\mathbf{0}$. Then the a_i and the b_i are not all zero, and so it is possible to find n,j in $\mathbf{N}$ such that $na_j > b_j$, and so $mna_j > mb_j$ for all m in $\mathbf{N}$, and so we do not have $\mathbf{a}^{nm} \mid \mathbf{b}^m$ for any m in $\mathbf{N}$.

This semigroup is not, however, $\mathbf{R}$-metrical. Suppose, by way of contradiction that the exists a morphism

$$f:\ (\mathbf{R}_o^+)^{\mathbf{N}}\backslash\mathbf{0} \longrightarrow \mathbf{R}^+.$$

Let $a_i = f(0,0,..,0,1,0,....)$ (where 1 is in the i^{th} position).

Let $b,c \in \mathbf{N}$. Then

$$\begin{aligned} c.f(0,0,..,0,b/c,0,...) &= f(c.(0,0,...,0,b/c,0,..)) \\ &= f(0,0,...,0,b,0,..) \\ &= b.f(0,0,..,0,1,0,..) \end{aligned}$$

and so for any q in $\mathbf{Q}^+$ we have that

$$f(0,0,..,0,q,0,...) = q.a_i$$

For every i in $\mathbf{N}$ choose q_i in $\mathbf{Q}^+$ such that $q_i a_i > 1$. Then let

$$\mathbf{x} = (\ q_1,q_2,...\ )$$

Then for all n in $\mathbf{N}$

$$(\ q_1,..,q_n,0,0,..\ ) \mid \mathbf{x}\ ,$$

and therefore

$$f(\ q_1,..,q_n,0,0,..\ ) < f(\mathbf{x}).$$

This is not possible, since for all n in $\mathbb{N}$

$$f(q_1,..,q_n,0,0,..) = f(q_1,0,0,..) +..+ f(0,..0,q_n,0,..)$$
$$> n$$

and so we would have $f(\mathbf{x}) > n$ for all n in $\mathbb{N}$.

It appears to be difficult to characterise $\mathbb{Q}$-metrical semigroups,and indeed metrical semigroups, in the infinitely generated case.

4. Some results concerning metrical semigroups.

4.1. $\mathcal{L}^+$, $\mathcal{R}^+$ and the semigroup (G,N,α).

The well known Green's relations $\mathcal{L}$ and $\mathcal{R}$ on a semigroup S defined by

$$a \mathcal{L} b \iff S^1a = S^1b$$

$$a \mathcal{R} b \iff aS^1 = bS^1$$

are trivial whenever S is metrical. For example suppose that (S,f) is a metrical semigroup and that $a \mathcal{L} b$. Then $S^1a = S^1b$ and so there exist u,v in S^1 such that ua = b and vb = a. Let (S^1,f) be the metrical monoid obtained from S by the addition of an identity element, as described in section 1. Then we have that f(u) = f(v) = 0, and therefore that u = v = 1, and so a = b.

We can, however, define generalisations of $\mathcal{L}$ and $\mathcal{R}$ that are not necessarily trivial on a metrical semigroup. We define the relations $\mathcal{L}^+$ and $\mathcal{R}^+$ on a semigroup S by

$$a \mathcal{L}^+ b \iff Sa = Sb$$

$$a \mathcal{R}^+ b \iff aS = bS.$$

First notice that these relations are indeed generalisations of the Green's relations, since, for example, if $a \mathcal{L} b$ in a semigroup S, then either a = b, in which case we must have $a \mathcal{L}^+ b$, or else there exist u,v in S such that

$$ua = b\ ,\ vb = a.$$

Now clearly $Sa \subseteq S^1a$. The opposite inclusion also holds, since $a = vua \in Sa$. Thus $Sa = S^1a$. Similarly $Sb = S^1b$. We deduce that Sa = Sb, and so a $\mathcal{L}^+$ b.

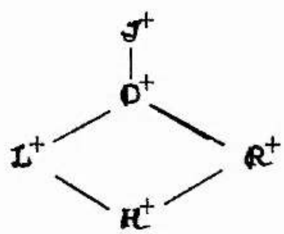
Consider the null semigroup S given by rs = 0 for all r,s in S. Then $\mathcal{L}$ and $\mathcal{R}$ are both the identity relation, while $\mathcal{L}^+$ and $\mathcal{R}^+$ are both the universal relation. Thus $\mathcal{L}^+$ and $\mathcal{R}^+$ are non-trivial generalisations of $\mathcal{L}$ and $\mathcal{R}$.

We also define $\mathcal{H}^+$ as $\mathcal{L}^+ \cap \mathcal{R}^+$, and $\mathcal{J}^+$ by

$$a \ \mathcal{J}^+ \ b \iff SaS = SbS,$$

and $\mathcal{D}^+$ as the smallest equivalence relation on S that contains both $\mathcal{L}^+$ and $\mathcal{R}^+$.

As in the original Green case we have the Hasse diagram



Unfortunately we need not have that $\mathcal{L}^+ \circ \mathcal{R}^+ = \mathcal{R}^+ \circ \mathcal{L}^+$. For example take the semigroup S with elements {a,b,c,x,y,z} and binary operation given by the table

	a	b	c	x	y	z
a	x	x	y	z	z	z
b	x	y	x	z	z	z
c	x	x	x	z	z	z
x	z	z	z	z	z	z
y	z	z	z	z	z	z
z	z	z	z	z	z	z

S is then indeed a semigroup, since S^3 = { z }, and so the operation is associative. Now aS = bS and Sb = Sc, and so we have that a ($\mathcal{R}^+ \circ \mathcal{L}^+$) c . We do not, however, have that a ($\mathcal{L}^+ \circ \mathcal{R}^+$) c, since the $\mathcal{L}^+$-class of a is { a } and the

$\mathcal{R}^+$-class of c is { c } and these two sets have an empty intersection.

We shall take this opportunity to discuss another generalisation of the Green's relations due to Fountain (see, for example, [5]). We define the relation $\mathcal{L}^*$ by a $\mathcal{L}^*$ b in S if and only if there exists a semigroup R and a monomorphism θ: S $\longrightarrow$ R such that $\theta(a)\ \mathcal{L}\ \theta(b)$ in R. The relation $\mathcal{R}^*$ is similarly defined. It is well known (see, for example, [5] lemma 1.1) that

a $\mathcal{L}^*$ b if and only if for all x,y in S^1, ax = ay <=> bx = by.

Similarly

a $\mathcal{R}^*$ b if and only if for all x,y in S^1, xa = ya <=> xb = yb.

Fountain gives an example of a semigroup in which

$$\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$$

(see [5] example 1.11).

The two generalisations are distinct. For example in the semigroup $\mathbf{N}$ we have $\mathcal{L}^* = \mathcal{R}^* = \mathbf{N}\ \mathbf{N}$, while $\mathcal{L}^+ = \mathcal{R}^+ = id$.

A semigroup S is said to be <u>left</u> <u>reductive</u> if given any x,y in S

$$zx = zy \text{ for all } z \text{ in } S \Rightarrow x = y.$$

<u>Right</u> <u>reductive</u> is defined dually, and a semigroup is said to be reductive if it is both right and left reductive. The following result links the relations $\mathcal{L}^+$ and $\mathcal{L}^*$ in the class of left reductive semigroups.

PROPOSITION 4.1. Let S be a left reductive semigroup. Then for all a,b in S

$$a \mathcal{L}^+ b \Rightarrow a \mathcal{L}^* b.$$

Proof. Let S be a left reductive semigroup. Then we can embed S into the full transformation semigroup $\mathcal{T}(S)$ by mapping the element s of S to the transformation ρ_s, where

$$r\rho_s = rs.$$

The left reducibility of S ensures that this map is 1-1. The associativity of S ensures that this map is a morphism.

It is well known (see for example [10] exercise 2.10) that in a full transformation group

$$\alpha \mathcal{L} \beta \iff \mathrm{Im}(\alpha) = \mathrm{Im}(\beta).$$

Let x,y in S be such that $x \mathcal{L}^+ y$. Then Sx = Sy, and so $\mathrm{Im}(\rho_x) = \mathrm{Im}(\rho_y)$, which then implies that $\rho_x \mathcal{L} \rho_y$. We deduce that $x \mathcal{L}^* y$.

The above result does not hold in general. For example let S be the semigroup consisting of elements { a,b,p,z } and with binary operation given by the table

	a	b	p	z
a	z	p	z	z
b	p	z	z	z
p	z	z	z	z
z	z	z	z	z

S is indeed a semigroup, since $S^3 = \{ z \}$. Now Sa = Sb and so $a \mathcal{L}^+ b$. We do not, however, have that $a \mathcal{L}^* b$, since aa = az, but ba $\neq$ bz.

Finally notice that in a cancellative semigroup $\mathcal{L}^*$ and

$\mathcal{R}^*$ are universal. This need not be the case for $\mathcal{L}^+$ and $\mathcal{R}^+$ (for example in the semigroup $\mathbf{N}$ we have $\mathcal{L}^+ = \mathcal{R}^+ = id.$). Thus $\mathcal{L}^+$ and $\mathcal{R}^+$ are generally more interesting in the class of cancellative semigroups. For the rest of the section we will be concentrating on cancellative metrical semigroups.

We now investigate our new relations for the class of metrical semigroups.

LEMMA 4.1. <u>Let (S,f) be a metrical semigroup. Then for all a,b in S, a $\mathcal{J}^+$ b => f(a) = f(b).</u>

Proof. Let a $\mathcal{J}^+$ b. Suppose by way of contradiction that f(a) > f(b). Let x be an element of S of minimal length. Then there exist y,z in S such that xax = ybz, from which we deduce that 2f(x) + f(a) = f(y) + f(z) + f(b). Since f(a) > f(b) this implies that f(y) + f(z) < 2f(x), contradicting the minimality of the length of x.

COROLLARY 4.1. <u>Let (S,f) be a metrical semigroup. Then for all a,b in S, a $\mathcal{L}^+$ b or a $\mathcal{R}^+$ b or a $\mathcal{H}^+$ b => f(a) = f(b).</u>

We call a metrical semigroup, (S,f), <u>special</u> if $\mathcal{H}^+ = f^{-1} \circ f$.

We dedicate the rest of section 4.1 to finding a characterisation of special cancellative metrical semigroups.

Recall from section 1 that the subset S_n of a metrical semigroup (S,f) is defined by

$$S_n = \{ s \in S : f(s) = n \}.$$

LEMMA 4.2. _Suppose that (S,f) is a cancellative special metrical semigroup. Then for all s in S and for all n in f(S) we have_

$$sS_n = S_n s = S_{n+f(s)}.$$

Proof. Clearly sS_n and $S_n s$ are contained in $S_{n+f(s)}$ for all s in S and all n in f(S). For the converse let s be in S, n in f(S) and t in $S_{n+f(s)}$.

Let r be an element of S_n. Then f(rs) = f(t) and thus, by the speciality of (S,f),

$$rs \,\mathcal{R}^+\, t.$$

Let u be an arbitrary element of S. Then there exists v in S such that

$$rsu = tv.$$

Now f(su) = f(sv) and so

$$su \,\mathcal{L}^+\, sv$$

and so there exists p in S such that

$$rsu = psv.$$

We deduce that

$$psv = tv$$

and so by cancellation

$$t = ps \in S_n s.$$

Similarly $t \in sS_n$.

Using the terminology of [3], a semigroup S is said to be _left reversible_ if for any a,b in S

$$aS \cap bS \neq \emptyset.$$

LEMMA 4.3. _Let (S,f) be a special metrical semigroup. Then S is left reversible._

Proof. Let s,r be elements of S. Then there exist p and q in $\mathbb{N}$ such that $f(s^p) = f(r^q)$. It then follows from the speciality of (S,f) that $s^pS = r^qS$, and so there exist u and v in S such that $s^p u = r^q v$, and so $s(s^{p-1}u) = r(r^{q-1}v)$.

The following result is well known (see for example [3] pp 300 - 302).

LEMMA 4.4. A cancellative left reversible semigroup is embeddable into a group.

COROLLARY 4.2. Let (S,f) be a cancellative special metrical semigroup. Then S is group embeddable.

The following technical lemma is necessary later on.

LEMMA 4.5. Let (S,f) be a cancellative special metrical semigroup. Then there exists a special cancellative metrical semigroup (S′,f′) such that $f'(S') = hcf\{f(S)\}.\mathbf{N}$ and $S = \{ x \in S' : f'(x) \in f(S) \}$.

Proof. By corollary 4.2 there exists a group G and a monomorphism $\theta: S \longrightarrow G$. For convenience we shall suppose that θ is the inclusion map, by identifying s and $\theta(s)$. Consider the subset H of G given by

$$H = \{ rs^{-1} : r,s \in S \}.$$

H is then a subgroup of G; for if $r,s,u,v \in S$ then

$$rs^{-1}(uv^{-1})^{-1}$$

$$= rs^{-1}vu^{-1} = rs^{-1}vx(ux)^{-1}$$

where x is an element of S chosen to ensure that $f(vx) = f(s) + k$ for some k in f(S). It then follows from lemma 4.2 that there exists w in S_k such that vx = sw. thus

$$rs^{-1}(uv^{-1})^{-1} = rw(ux)^{-1}$$

and so H is a subgroup of G.

Now f extends uniquely to a morphism $\tilde{f}: H \longrightarrow \mathbf{Z}$ by the rule $\tilde{f}(rs^{-1}) = f(r) - f(s)$. Clearly $\tilde{f}(H) \subseteq \text{hcf}\{f(S)\}.\mathbb{Z}$. Now the well known Euler's algorithm tells us that there exist $n_1,..,n_k,m_1,...,m_t \in \mathbb{N}$ and $x_1,...,x_n,y_1,...,y_k \in f(S)$ such that

$$\text{hcf}\{f(S)\} = n_1x_1 + .. + n_kx_k - (m_1y_1 + .. + m_ty_t).$$

$f(S)$ is a subsemigroup of $\mathbb{N}$, and so $(n_1x_1 + .. + n_kx_k)$ and $(m_1y_1 + .. + m_ty_t)$ are both elements of $f(S)$. We deduce that $\text{hcf}\{(f(S)\} \in \tilde{f}(H)$, and so, since H is a group, that $\text{hcf}\{f(S)\}.\mathbb{Z} \subseteq \tilde{f}(H)$. Thus $\tilde{f}(H) = \text{hcf}\{f(S)\}.\mathbb{Z}$.

We now define S' by

$$S' = \{ rs^{-1} \in H : f(r) > f(s) \}$$

and define the length function f' on S' to be the restriction to S' of $\tilde{f}$.

Clearly if s is an element of S then $s = s^2s^{-1}$ and so

$$S \subseteq \{ x \in S' : f'(x) \in f(S) \}.$$

Furthermore if $x \in S'$ is such that $f'(x) = k \in f(S)$ then

$$x = rs^{-1}$$

where $f(r) = f(s) + k$. It then follows from lemma 4.2 that there exists w in S_k such that $r = ws$, and thus

$$x = wss^{-1} = w \in S.$$

We deduce that $S = \{ s \in S' : f'(s) \in f(S) \}$, as required.

We now introduce a class of semigroups that arise as a semidirect product of a group G and $\mathbb{N}$. Ultimately we shall characterise special cancellative metrical semigroups in terms of semigroups from this class. First we investigate some properties of the class, culminating in theorem 4.1, which tells us when two members of the class are isomorphic.

DEFINITION 4.1. Let G be a group, let α be an automorphism of G. We then define the semigroup $(G,\mathbf{N},\alpha)$ to be the semidirect product of G and $\mathbf{N}$ given by $(G,\mathbf{N},\alpha) = (G \ltimes \mathbf{N}, *)$ with $(g,n)*(h,m) = (g\alpha^n(h), m+n)$.

Note that the semigroup $(G,\mathbf{N},\alpha)$ is cancellative.

We define $(G,\mathbf{Z},\alpha)$ in the same way, but allowing n and m to vary across $\mathbf{Z}$.

We denote the subset $\{ (g,n) : g \in G \}$ by (G,n). Notice (G,0) is a group that is isomorphic to G.

LEMMA 4.6. <u>$(G,\mathbf{N},\alpha)$ is metrical. Furthermore the only length morphisms on $(G,\mathbf{N},\alpha)$ are the projection pr_2 onto $\mathbf{N}$ and its scalar multiples.</u>

Proof. Clearly the projection onto $\mathbf{N}$ is a length morphism. We now show that for any length morphism f on $(G,\mathbf{N},\alpha)$, $f(g,n) = f(h,m)$ if and only if $n = m$, from which the result follows.

Let $f:(G,\mathbf{N},\alpha) \longrightarrow \mathbf{N}$ be a morphism. Suppose by way of contradiction that there exist (g,n) and (h,m) in $(G,\mathbf{N},\alpha)$ such that $n > m$ but $f(g,n) = f(h,m)$. Then

$$(g,n) = (h,m)*(\alpha^{-m}(h^{-1}g),n-m)$$

and so $f(\alpha^{-m}(h^{-1}g),n-m) = 0$, which is not possible.

Conversely suppose by way of contradiction that there exist g,h in G and n in $\mathbf{N}$ such that $f(g,n) = p > q = f(h,n)$.

Then $f((h,n)^p) = pq = f((g,n)^q)$. Now there exist h' and g' in G such that $(h,n)^p = (h',pn)$ and $(g,n)^q = (g',qn)$, and so we have h' and g' such that $f(h',pn) = f(g',qn)$, which we have already shown is not possible.

LEMMA 4.7. <u>The metrical semigroup (($G,\mathbf{N},\alpha$), pr_2) is special.</u>

Proof. Let g,h be elements of G and n be in $\mathbf{N}$. Then given any (x,m) in ($G,\mathbf{N},\alpha$) we have

$$(g,n)*(x,m) = (g\alpha^n(x),n+m) = (h,n)*(\alpha^{-n}(h^{-1}g)x,m)$$

and

$$(x,m)*(g,n) = (x\alpha^m(g),n+m) = (x\alpha^m(gh^{-1}),m)*(h,n).$$

Thus (($G,\mathbf{N},\alpha$), pr_2) is special.

LEMMA 4.8. <u>($G,\mathbf{Z},\alpha$) a group and ($G,\mathbf{N},\alpha$) is a set of group generators for ($G,\mathbf{Z},\alpha$).</u>

Proof. First notice that ($G,\mathbf{Z},\alpha$) is a group, since for any (g,z) in ($G,\mathbf{Z},\alpha$) we have

$$(1,0)*(g,z) = (g,z)*(1,0) = (g,z)$$

and

$$(g,z)*(\alpha^{-z}(g^{-1}),-z) = (\alpha^{-z}(g^{-1}),-z)*(g,z) = (1,0).$$

Next notice that ($G,\mathbf{N},\alpha$) generates ($G,\mathbf{Z},\alpha$), since if $n \geq 0$ and $g \in G$, then

$$(g,-n) = (g,1)*(1,-1-n) = (g,1)*(1,1+n)^{-1}.$$

LEMMA 4.9. <u>In the group ($G,\mathbf{Z},\alpha$), the subgroup (G,0) is equal to $(G,n)*(G,n)^{-1}$ for any n in $\mathbf{Z}$.</u>

Proof. Given any n in $\mathbf{Z}$ and any g,h in G, we have

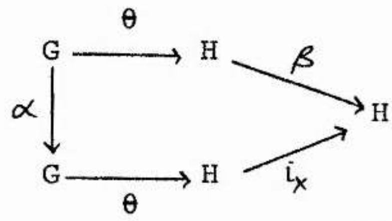
$$(g,n)*(h,n)^{-1} = (g,n)*(\alpha^{-n}(h^{-1}),-n) = (gh^{-1},0)$$

and so $(G,n)*(G,n)^{-1} \subseteq (G,0)$. Conversely given any g in G we have that

$$(g,0) = (g,n)*(1,n)^{-1}$$

and so $(G,0) \subseteq (G,n)*(G,n)^{-1}$.

THEOREM 4.1. *$(G, \mathbb{N}, \alpha)$ is isomorphic to $(H, \mathbb{N}, \beta)$ if and only if there exists an isomorphism $\theta: G \to H$ and an element x in H such that the diagram*



commutes,

where i_x is the inner automorphism of H defined by

$$i_x(h) = x^{-1}hx.$$

Proof. Suppose that $\psi: (G, \mathbb{N}, \alpha) \to (H, \mathbb{N}, \beta)$ is an isomorphism. Clearly $\psi((G,1)) = (H,1)$ since these are the sets of elements without proper divisors. ψ extends uniquely to an isomorphism $\psi': (G, \mathbb{Z}, \alpha) \to (H, \mathbb{Z}, \beta)$ by the rule $\psi'(g,-n) = \psi(g,1)*(\psi(1,1+n))^{-1}$ for n in $\mathbb{N}^0$.

By lemma 4.9 we have $(G,0) = (G,1)*(G,1)^{-1}$ and $(H,0) = (H,1)*(H,1)^{-1}$. We deduce that the map $(g,1)*(g',1)^{-1} \to \psi'((g,1)*(g',1)^{-1})$ is an isomorphism from (G,0) to (H,0), and therefore that G and H are isomorphic. Let θ be the isomorphism from G to H given by $\theta(g) = pr_1(\psi'(g,0))$.

Let x be such that $(x,1) = \psi(1,1)$.

Now given any g in G we have $(g,1) = (1,1)*(\alpha^{-1}(g),0)$.

Thus
$$\psi(g,1) = \psi'(g,1)$$
$$= \psi'(1,1)*\psi'(\alpha^{-1}(g),0) \qquad \text{since } \psi \text{ is a morphism}$$
$$= (x,1)*(\theta(\alpha^{-1}(g)),0)$$
$$= (x\beta(\theta(\alpha^{-1}(g))),1).$$

Again ψ' is a morphism, and so for any g in G we have

$$\psi'((g,0)*(1,1)) = \psi'(g,0)*\psi(1,1).$$

$$\text{Thus } \psi'((g,1)) = \psi'(g,0)*\psi'(1,1),$$

$$\text{i.e. } (x\beta(\theta(\alpha^{-1}(g))),1) = (\theta(g),0)*(x,1).$$

$$\text{Hence } x\beta(\theta(\alpha^{-1}(g)) = \theta(g)x \text{ for all g in G}$$

$$\text{and so } \beta\circ\theta = i_x\circ\theta\circ\alpha$$

Thus the diagram commutes.

Conversely suppose that there exist an element x of H and an isomorphism $\theta: G \longrightarrow H$ such that the diagram commutes.

Define a map ψ from $(G,\mathbf{N},\alpha)$ to $(H,\mathbf{N},\beta)$ by the rule

$$\psi(g,n) = (x\beta(x)\beta^2(x)\ldots\beta^{n-1}(x)\beta^n(\theta(\alpha^{-n}(g))),n).$$

It is clear that ψ is both 1-1 and onto, and it therefore remains to show that ψ is a morphism.

First we show that for all g in G and m in $\mathbf{N}$

$$(*) \quad x\beta(x)\ldots\beta^{m-1}(x)\beta^m(\theta(\alpha^{-m}(g))) = \theta(g)x\beta(x)\ldots\beta^{m-1}(x).$$

We show this by induction on m. That the result holds for m=1 follows directly from the fact that the diagram commutes. Suppose therefore that the result holds for $m \leqslant k$. Then

$$x\beta(x)\ldots\beta^k(x)\beta^{k+1}(\theta(\alpha^{-k-1}(g)))$$

$$= x\beta[x\beta(x)\ldots\beta^{k-1}(x)\beta^k(\theta(\alpha^{-k}(\alpha^{-1}(g))))]$$

$$= x\beta[\theta(\alpha^{-1}(g))x\beta(x)\ldots\beta^{k-1}(x)]$$

(by the induction hypothesis)

$$= x\beta[\theta(\alpha^{-1}(g))]\beta(x)\beta^2(x)\ldots\beta^k(x)$$

$$= \theta(g)x\beta(x)\beta^2(x)\ldots\beta^k(x)$$

(since the result holds for m = 1).

Thus the result holds for m = k+1 and so the inductive proof is complete.

We now show that the map ψ defined above is indeed a morphism. Let $(g,n),(h,m)$ be elements of $(G,\mathbf{N},\alpha)$. Then

$\psi(g,n)*\psi(h,m) = (z,n+m)$ where

$$z = x\beta(x)..\beta^{n-1}(x)\beta^{n}(\theta(\alpha^{-n}(g)))\beta^{n}[x\beta(x)..\beta^{m-1}(x)\beta^{m}(\theta(\alpha^{-m}(h)))]$$

$$= x\beta(x)..\beta^{n-1}(x)\beta^{n}[\theta(\alpha^{-n}(g))x\beta(x)..\beta^{m-1}(x)\beta^{m}(\theta(\alpha^{-m}(h)))]$$

$$= x\beta(x)..\beta^{n-1}(x)\beta^{n}[x\beta(x)..\beta^{m-1}(x)\beta^{m}(\theta(\alpha^{-m-n}(g)\alpha^{-m}(h)))]$$

by (*) and the fact that θ is a morphism

$$= x\beta(x)....\beta^{m+n-1}(x)\beta^{m+n}(\theta(\alpha^{-m-n}(g\alpha^{n}(h))))$$

Thus $(z,m+n) = \psi((g,n)*(h,m))$ as required.

Theorem 4.1 has the following corollaries:

COROLLARY 4.3. ($G,\mathbf{N},\alpha$) is isomorphic to $G\times\mathbf{N}$ if and only if α is inner.

Proof. $G\times\mathbf{N}$ is isomorphic to $(G,\mathbf{N},id)$. Thus by theorem 4.1, $(G,\mathbf{N},\alpha)$ is isomorphic to $G\times\mathbf{N}$ if and only if there exists an automorphism θ of G and an inner automorphism i of G such that

$$id\circ\theta = i\circ\theta\circ\alpha,$$

$$\text{i.e }\ \alpha = \theta^{-1}\circ i^{-1}\circ\theta.$$

The set of inner automorphisms of a group G forms a normal subgroup of the group of all automorphism of the group G, and so we deduce that such i and θ exist if and only if α is inner.

COROLLARY 4.4. ($G,\mathbf{N},\alpha$) is isomorphic to $G\times\mathbf{N}$ if and only if there exists z in G such that (z,1) is central (that is (z,1) commutes with every element of ($G,\mathbf{N},\alpha$).)

Proof. Suppose that ($G,\mathbf{N},\alpha$) is isomorphic to $G\times\mathbf{N}$. Then by corollary 4.3 there exists z in G such that for all g in G

$$\alpha(g) = z^{-1}gz.$$

Then for all (g,n) in (G,$\mathbf{N}$,α)

$$\begin{aligned} &(g,n)*(z,1) \\ &= (gz^{-n}zz^{n},n+1) \\ &= (gz,n+1) \\ &= (zz^{-1}gz,n+1) \\ &= (z,1)*(g,n) \end{aligned}$$

Conversely, if z in G is such that (z,1) is central, then in particular

$$(g,1)*(z,1) = (z,1)*(g,1) \text{ for all } g \text{ in } G;$$

that is $\quad g\alpha(z) = z\alpha(g)$ for all g in G.

Taking g = 1 we have that $z = \alpha(z)$, and then taking g as arbitrary we have that $\alpha(g) = z^{-1}gz$. Thus α is an inner automorphism, and so by corollary 4.3 (G,$\mathbf{N}$,α) is isomorphic to $G \times \mathbf{N}$.

We now generalise the class of semigroups of type (G,$\mathbf{N}$,α) to include semidirect products of groups and subsemigroups of $\mathbf{N}$.

DEFINITION 4.2. Let X be a subsemigroup of $\mathbf{N}$. Then for any group G and any automorphism α of G, we define the semigroup (G,X,α) to be the subsemigroup of (G,$\mathbf{N}$,α) consisting of elements { (g,n) : g in G, n in X }.

We are now in a position to state the main theorem of this section, which characterises cancellative special metrical semigroups. In the statement and proof of the theorem it is convenient to use the following notation : given a subset X of $\mathbb{N}$ and an element k of $\mathbb{N}$ that divides all the elements of X, we denote by X/k the set $\{ x/k : x \in X \}$.

THEOREM 4.2. Let (S,f) be a cancellative metrical semigroup and let $k = \mathrm{hcf}\{f(S)\}$. Then (S,f) is special if and only if there exists a group G and an automorphism α of G such that

$$(S,f) = ((G,f(S)/k,\alpha), k.pr_2).$$

Proof. The converse is similar to lemma 4.7 adapted for the case $X \neq \mathbb{N}$.

For the direct part suppose that (S,f) is a cancellative special metrical semigroup. By lemma 4.5 there exists a cancellative special metrical semigroup (S',f') such that $S = \{ x \in S' : f'(x) \in f(S) \}$ and $f'(S') = k\mathbb{N}$, where we recall that $k = \mathrm{hcf}\{f(S)\}$.

Fix an element t in S'_k.

Now in the proof of lemma 4.5 we showed that S' is contained in a group H such that

$$H = \{ rs^{-1} : r,s \in S' \} .$$

Let G be the subgroup of H defined by

$$G = \{ rs^{-1} : r,s \in S' , f'(r) = f'(s) \}.$$

Define an automorphism α on G by

$$\alpha(rs^{-1}) = t^{-1}rs^{-1}t.$$

α is indeed a map from G to G, since

$$t^{-1}rs^{-1}t = t^{-1}rw(sw)^{-1}t$$

where w is an element of S' chosen to ensure that $f'(rw) = f'(sw) > k = f'(t)$, and therefore, by lemma 4.2, that there exist r' and s' in $S'_{f'(rw)-k}$ such that

$$rw = tr' \ , \ sw = ts'$$

giving us

$$(rs^{-1}) = r'(s')^{-1} \in G.$$

Define a map θ from $(G, \mathbf{N}, \alpha)$ to S' by

$$\theta(rs^{-1}, n) = rs^{-1}t^n.$$

1) The range θ does lie inside S'.

This follows since if rs^{-1} is an element of G then

$$rs^{-1}t^n = rw(sw)^{-1}t^n$$

where w is an element of S' chosen to ensure that $f'(sw) > kn = f'(t^n)$. Then, by lemma 4.2, there exists s' in $S'_{f'(sw)-kn}$ such that

$$sw = t^n s'.$$

Now $f'(rw) = f'(sw)$ and so, again by lemma 4.2, there exists y in S'_{kn} such that $rw = ys'$. Thus

$$\theta(rs^{-1}, n) = rws'^{-1} = y \in S'.$$

2) θ is 1-1.

This follows since if $rs^{-1}t^n = pq^{-1}t^m$,

where $f'(r) = f'(s)$ and $f'(p) = f'(q)$, then

$$kn = f'(t^n rs^{-1}) = f'(t^m pq^{-1}) = km$$

and also we then have by cancellation of t^n that

$$rs^{-1} = pq^{-1}.$$

3) θ is onto.

Let s be an element of S'. Then $f'(s) = kn$ for some n in $\mathbf{N}$. Then

$$s = st^{-n}t^n = \theta(st^{-n}, n).$$

4) θ is a morphism.

Let $(rs^{-1},n),(pq^{-1},m)$ be elements of $(G,\mathbf{N},\alpha)$.

Then $\theta((rs^{-1},n)*(pq^{-1},m))$

$$= \theta(rs^{-1}t^{n}pq^{-1}t^{-n},n+m)$$

$$= rs^{-1}t^{n}pq^{-1}t^{-n}t^{n+m}$$

$$= (rs^{-1}t^{n})(pq^{-1}t^{m})$$

$$= \theta(rs^{-1},n)\theta(pq^{-1},m).$$

Thus $\theta: (G,\mathbf{N},\alpha) \rightarrowtail S'$ is an isomorphism.

It is clear that θ is a metrical isomorphism from $((G,\mathbf{N},\alpha),k.pr_2)$ to (S',f').

Finally

$$\theta(G,f(S)/k,\alpha) = \{ x \in S' : f'(x) \in f(S) \} = S$$

and so the restriction of θ to $(G,f(S)/k,\alpha)$ is a metrical isomorphism between $((G,f(S)/k,\alpha),k.pr_2)$ and (S,f) .

4.2. Length cancellative metricalsemigroups,F-semigroups and F-metrical semigroups.

One of the cornerstones of infinite group theory is the theorem of Nielsen and Schreier to the effect that any subgroup of a free group is itself free (see for example [13] theorem 7.3). In the category of semigroups there is no Nielsen-Schreier type result, as the following well known example illustrates:

EXAMPLE 4.1. Consider the subsemigroup S of the infinite monogenic semigroup $< x >$ consisting of elements $\{ x^2,x^3,x^4,\ldots \}$. The unique minimal generating set for S is the set $\{ x^2,x^3 \}$. However,

$$x^2 .x^2.x^2 = x^3.x^3$$

is a non-trivial relation between x^2 and x^3, and so S is not free.

This suggests two questions. The first asks when the subsemigroup of a free semigroup X^+ generated by a set A X^+ is isomorphic to the free semigroup on A. When this happens A is said to be a code. There exists an extensive theory of codes; see for example [12].

The second question asks what semigroups can arise as subsemigroups of free semigroups. This second question is that which we refer to as the <u>free semigroup embeddability problem</u>.

Budkina and Markov ([2]) define a semigroup to be an

F-semigroup if it is isomorphic to a finitely generated subsemigroup of a free semigroup. They then characterise those F-semigroups generated by three elements. Their results do not easily generalise to the general case. Notice that an F-semigroup is necessarily embeddable into a group, since it is embeddable into a free semigroup, which is in turn embeddable into a group. The problem of free semigroup embeddability is therefore closely related to that of group embeddability. Group embeddability has interested semigroup theorists since the very early days of the subject. For a discussion of the problem see either [3] chapter 12 or [1] chapter 2.

A good illustration of the role that length morphisms can play in this area of semigroup theory is given by the well known Levi's lemma. A semigroup, S, is said to be equidivisible if for all a,b,c,d in S, ab = cd => $\exists$ u$\in S^1$ st

either $\quad a = cu\ ,\ d = ub$

or $\quad c = au\ ,\ b = ud$.

LEMMA 4.10. (Levi's Lemma, see for example [12] corollary 5.1.6). A metrical semigroup is free if and only if it is equidivisible.

Motivated by Levi's lemma we now use length morphisms to investigate the problem of free semigroup embeddability. In section 4.2.1 we define the notions of metrical morphism and length cancellativity. In section 4.2.2 we introduce the notions of gauges and length functions in groups and state some technical lemmas to be used later. Finally in section 4.2.3 we see how the ideas of sections 4.2.1 and 4.2.2 link with each other and the free semigroup embeddability problem.

4.2.1 Metrical morphisms and length cancellativity.

A metrical morphism from a metrical semigroup (S,f) to a metrical semigroup (R,g) is a morphism $\theta: S \longrightarrow R$ such that $f = g \circ \theta$. We define metrical epimorphism/ monomorphism/ isomorphism in the obvious way, and say that a metrical semigroup (S,f) is metrically embeddable into a metrical semigroup (R,g) if there is a metrical monomorphism from (S,f) into (R,g).

A metrical semigroup (S,f) is said to be left length cancellative if for all a and b in S,

$$f(a)=f(b) \text{ and } aS \cap bS \neq \emptyset \Rightarrow a=b.$$

We define right length cancellativity dually, and say that a metrical semigroup (S,f) is length cancellative if it is both left and right length cancellative.

Notice that a cancellative metrical semigroup is length cancellative if and only if it is either left or right length cancellative. For example, suppose that (S,f) is a cancellative left length cancellative semigroup. Let a,b in S be such that f(a) = f(b) and there exists u,v in S such that ua = vb. Then f(u) = f(v), and so by left length cancellativity we have u = v. Thus ua = ub, and so a = b by cancellativity. We deduce that (S,f) is also right length cancellative.

A length cancellative metrical semigroup is in a sense the opposite of a special metrical semigroup, for if (S,f) is special then

$$f(a) = f(b) \Rightarrow aS = bS \ , \ Sa = Sb$$

while if (S,f) is length cancellative then (if $a \neq b$)

$$f(a) = f(b) \Rightarrow aS \cap bS = \emptyset \ , \ Sa \cap Sb = \emptyset.$$

4.2.2 Semigauges, gauges and length functions on groups.

The notion of a length function on a group was first introduced by Lyndon in 1963 ([13]), and used to prove the Nielsen-Schreier theorem. The idea was generalised to the notion of a semigauge on a group by Promislov in 1985 ([16]).

DEFINITION 4.3. Let G be a group. A **$\mathbb{Z}$**-semigauge, or semigauge, on G is a mapping $p: G \longrightarrow \mathbb{N}^0$ such that for all x,y in G

$$p(xy) \leqslant p(x) + p(y)$$

$$p(x^{-1}) = p(x)$$

$$p(1) = 0.$$

p is said to be a **$\mathbb{Z}$**-gauge, or gauge, if in addition

$$p(x) = 0 \Leftrightarrow x = 1.$$

DEFINITION 4.4. Let G be a group. Then a **$\mathbb{Z}$**-gauge p on G is said to be a length function on G if it satisfies

A1) $p(x^2) > p(x)$ for all x in $G\backslash 1$

A2) $p(x) + p(y) - p(xy) < p(x) + p(z) - p(xz)$

$$\Rightarrow \quad p(y) + p(z) - p(y^{-1}z) = p(x) + p(y) - p(xy)$$

THEOREM 4.3 ([13] Corollary 7.2) A group G is free if and only if there exists a length function p on G.

The Nielsen-Schreier theorem is a corollary to the above theorem, since if G is a group with length function p, and if H is a subgroup of G, then the restriction of p to H is a length function on H.

Lyndon also introduces the map from $G \times G$ to $\mathbf{N}^{0}$ given by $d(x,y) = 1/2(p(x) + p(y) - p(xy^{-1}))$. Notice that $d(x,y) = d(y,x)$.

With the new notation we can restate condition (A2) as

A2′) $d(x,y) < d(x,z) \Rightarrow d(y,z) = d(x,y)$.

Lemmas 4.11 and 4.12 are also from [13], and so we state them without proof.

LEMMA 4.11. ([13] proposition 2.2) Suppose that p is a length function on a group G. Then for all x,y,z in G

$$d(x,y^{-1}) + d(y,z^{-1}) \geqslant p(y)$$
$$\Rightarrow p(xyz) \leqslant p(x) - p(y) + p(z).$$

LEMMA 4.12. ([13] proposition 2.4 and condition (A1)) Suppose that p is a length function on a group G. Then p also satisfies

A3) $p(xy) + p(yx) \leqslant 2p(x) = 2p(y) \Rightarrow xy = 1$.

Suppose now that (S,f) is a group embeddable metrical semigroup. Let G be the free group on S. We say that a semigauge p on G is a semigauge extension of f if $p|_S = f$. We call a gauge on G that is a semigauge extension of f a gauge extension of f, and a length function on G that is a semigauge extension of f a length function extension of S.

4.2.3 F-metrical semigroups.

Recall that a semigroup is called an F-semigroup if it is finitely generated and is embeddable into a free semigroup. As mentioned above, a general characterisation of F-semigroups appears to be difficult. In this section we introduce the notion of an F-metrical semigroup. Although a characterisation of F-metrical semigroups is also difficult some promising progress is made, in which we link up the ideas F-semigroups, length cancellative metrical semigroups and gauges on groups.

We say that a metrical semigroup (S,f) is F-metrical if it is finitely generated and is metrically embeddable into a free semigroup with its usual length function. Clearly for (S,f) to be F-metrical we must have that S is an F-semigroup. We see in Example 4.2 that a metrical F-semigroup need not be F-metrical.

Let f be any length morphism on a free semigroup X^+.

Then the metrical semigroup (X^+,f) is F-metrical, with the required embedding map being the unique extension to X^+ of the map from X to X^+ given by $x \longrightarrow x^{f(x)}$. It follows that for a metrical semigroup (S,f) to be F-metrical, it is sufficient for it to be metrically embeddable into a free semigroup with arbitrary length function f.

LEMMA 4.13. <u>An F-metrical semigroup (S,f) is length cancellative.</u>

Proof. Let (S,f) be F-metrical. Then there exists a set X and a metrical monomorphism $\theta: (S,f) \longrightarrow (X^+,u)$, where u is the usual length morphism on X^+.

Suppose that a,b in S are such that

$$f(a) = f(b) \text{ and } aS \cap bS \neq \emptyset.$$

Then there exist x,y in S such that ax = by, and so

$$\theta(ax) = \theta(by)$$

i.e. $$\theta(a)\theta(x) = \theta(b)\theta(y).$$

Equidivisibility in X^+ (lemma 4.10) then implies that there exists w in X^* such that

either $$\theta(a) = \theta(b)w \qquad (*)$$

or $$\theta(b)= \theta(a)w \qquad (**).$$

θ is a metrical morphism, and so

$$u(\theta(a)) = f(a) = f(b) = u(\theta(b)).$$

Thus in either case (*) or (**) we have u(w) = 0 , which implies that w = 1.

We deduce that $\theta(a) = \theta(b)$, and since θ is monomorphic this implies that a = b. Thus (S,f) is left length cancellative.

In a similar way, (S,f) is right cancellative, and we therefore deduce that (S,f) is length cancellative.

Example 4.2 gives us a metrical semigroup (S,f) which is an F-semigroup (i.e. S is embeddable into a free semigroup), but which is not length cancellative, and therefore , from lemma 4.13, not F-metrical.

EXAMPLE 4.2. Let G be the free group on { a,b,c } and let (S,f) be the metrical subsemigroup of G generated by

$$\{ a,b,ac^{-1},cb \}$$

$$\text{with } f(a)=f(b)=f(ac^{-1})=f(cb) = 1.$$

f is well defined since it is the restriction to S of the morphism g from G to $\mathbb{Z}$ given by g(a)=g(b)=1, g(c)=0. Thus (S,f) is a metrical semigroup. In fact it is a metrical F-semigroup since S is contained in the subsemigroup of G generated by { ac^{-1},c,b } which is free.

(S,f), however, is not length cancellative, since $ab = (ac^{-1})(cb)$ and $f(a) = f(ac^{-1})$. Lemma 4.13 then gives us that (S,f) it is not F-metrical. Thus there exists a metrical F-semigroup (S,f) that is not F-metrical.

Example 4.2 contrasts with the commutative case. In lemma 2.10 we showed that if T is embeddable into a free commutative semigroup, and if f is a length morphism on T, then there exists a positive integer k such that (T,kf) is metrically embeddable into a free commutative semigroup.

Given an arbitrary metrical semigroup (R,g), and any positive integer t, it is easy to see that (R,g) is length cancellative if and only if (R,tg) is length cancellative. We deduce that the metrical semigroup (S,f) of example 4.2 is such that (S,tf) is never metrically embeddable into a free semigroup with usual length morphism, and therefore the above statement of lemma 2.10 does not hold in the non-commutative case.

We represent our findings so far by a diagram of implications. Given a finitely generated semigroup S and a length morphism f on S we have

$$\begin{array}{ccc} & (S,f) \text{ is F-metrical} & \\ \overset{(a)}{\swarrow\!\!\swarrow} & & \overset{(b)}{\searrow\!\!\searrow} \\ S \text{ is an F-semigroup} & \not\Longrightarrow & (S,f) \text{ is length cancellative} \end{array}$$

Example 4.2 shows that the implication (a) is strict. We shall show in example 4.3 that the implication (b) is strict. It is helpful first to prove two straightforward lemmas:

LEMMA 4.14. <u>Let w,x,y,z be elements of a free semigroup X^+ such that $w^2x^2 = y^2z^2$, $|w| = 2$ and $|y| = 3$. Then $wy = yw$.</u>

Proof. Equidivisibility in X^+ (lemma 4.10) and the fact that $|y| > |w|$ imply that there exists u in X^+ such that $y = wu$. Furthermore $|u| = 1$. The identity $w^2x^2 = y^2z^2$ then becomes

$$w^2x^2 = (wu)^2z^2.$$

Cancellation then gives

$$wx^2 = uwuz^2.$$

Equidivisibility in free semigroups and the fact that $|w| > |u|$ then imply that there exists v in X^+ such that $w = uv$. Furthermore $|v| = 1$. The identity $wx^2 = uwuz^2$ then becomes

$$uvx^2 = u^2vuz^2.$$

Cancellation then gives

$$vx^2 = uvuz^2.$$

Equidivisibility in free semigroups and the fact that $|u| = |v|$ then imply that $u = v$.

Thus $w = u^2$ and $y = u^3$, and so $wy = yw$ as required.

Suppose that a metrical semigroup (S,f) is given in terms of a presentation $P(S) \equiv < X \mid R >$. The length morphism f on S then induces a length morphism f' on X^+ given by the rule $f'(u) = f(u)$.

LEMMA 4.15. Let (S,f) be a metrical semigroup where S has presentation

$$P(S) \equiv < X \mid u_i = v_i \; : \; i \quad I >.$$

Then, for any $a,b \in X^+$ with

$$f'(a) = f'(b) < \min\{ f(u_i) : i \in I \},$$

$$a = b \text{ in } S \Rightarrow a = b \text{ in } X^+.$$

Proof. Suppose that $a = b$ in S. Then there exist words $x_0,..,x_n$ in X^+ such that

$$a = x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_n = b$$

is a sequence of basic P(S) transitions from a to b.

It follows that for any $1 \leqslant i \leqslant n$ either

$$x_{i-1} = x_i \text{ in } X^+$$

or else there exist w,z in X^* such that

either $\quad x_{i-1} = wu_jz$, $x_i = wv_jz$ for some j in I

or $\quad x_{i-1} = wv_jz$, $x_i = wu_jz$ for some j in I.

Since $f'(x_i) = f'(a) < \min\{ f(u_j) : j \in I \}$, we deduce that $x_{i-1} = x_i$ for $1 \leqslant i \leqslant n$, and so a = b in X^+ as required.

EXAMPLE 4.3. Let (S,f) be the metrical semigroup with presentation

$$S = \langle a,b,c,d : a^2b^2=c^2d^2 \rangle$$

and with f defined by

$$f(a)=2, f(b)=4, f(c)=3 \text{ and } f(d)=3.$$

It can be shown (see, for example, [17] chapter 2 corollary 1) that any semigroup given by a presentation with a single reduced relation (that is, an identity u = v in the generators in which u and v have no common left or right factor) is group embeddable. S is therefore group embeddable, and so S is definitely cancellative.

We now show that S is length cancellative. First notice that since the last letter of a^2b^2 is not the first letter of either a^2b^2 or c^2d^2, and the last letter of c^2d^2 is not the first letter of either a^2b^2 or c^2d^2, we must have that for any words u,v in $\{a,b,c,d\}^+$ such that u = v in S there exist words $x_0,\ldots,x_n$ in $\{a,b,c,d\}^*$ such that

$$u = x_0p_1\ldots x_{n-1}p_nx_n$$

$$v = x_0q_1\ldots x_{n-1}q_nx_n$$

where $(p_j,q_j) = (a^2b^2,c^2d^2)$ or (c^2d^2,a^2b^2) for $1 \leqslant j \leqslant n$.

Suppose, then, that there exist x,y,w,z in S such that xw = yz.

From the above, letting $u = xw$ and $v = yz$, we must have that

$$x = x_0p_1\ldots x_{i-1}x' \qquad \text{for some } 0 \leqslant i \leqslant n$$

$$y = x_0q_1\ldots x_{j-1}y' \qquad \text{for some } 0 \leqslant j \leqslant n$$

where $x' \mid p_ix_i$ and $y' \mid q_jx_j$ in $\{a,b,c,d\}^+$, and where (p_i,q_i) are as above.

It then follows that if $f(x) = f(y)$ then $i = j$ and so

$$f(x') = f(y') \text{ and } x' \mid p_ix_i \text{ and } y' \mid q_ix_i \text{ in } \{a,b,c,d\}^+.$$

With the given f this is only possible when

$$x' = p_ix_i' \ , \ y' = q_ix_i'$$

for some $x_i' \mid x_i$. We deduce that $x' = y'$ in S, and so $x = y$ in S. Thus S is left length cancellative, and therefore, being a cancellative semigroup, is length cancellative.

Suppose now, by way of contradiction, that (S,f) is F-metrical. Then S is contained in a free semigroup X^+ in such a way that $f(s) = |s|$ for all s in S, where $|.|$ is, as always, the usual length morphism on X^+. Then the identity $a^2b^2 = c^2d^2$ and the fact that $|a| = 2$ and $|c| = 3$ imply, by lemma 4.14, that $ac = ca$ in S. This is a contradiction, since $f(ac) = 5$, and so such a relation cannot exist by lemma 4.15.

We now try to link the notions of F-metricality and length cancellativity with the notion of gauges and length functions in groups. The first result is theorem 4.4, which illustrates the close link between F-metricality and length functions on free groups. To prove theorem 4.4 we need lemmas 4.16 and 4.17.

In lemmas 4.16 and 4.17 suppose that G is a free group and that p is a gauge on G satisfying conditions A1 and A2 of definition 4.4. Suppose further that S is a finitely generated subsemigroup of G such that the restriction of p to S is a length morphism on S. Let X be the unique minimal generating set for S.

LEMMA 4.16. $(S,p|_S)$ is length cancellative.

Proof. Suppose that there exist elements x,y, of S and s,r of S such that p(x) = p(y) and xs = yr. Then

$$\begin{aligned} & p(xs(ys)^{-1}) + p((ys)^{-1}xs) \\ = \quad & p(xy^{-1}) + p(s^{-1}y^{-1}yr) \\ = \quad & p(xy^{-1}) + p(s^{-1}r) \\ \leqslant \quad & p(x) + p(y) + p(r) + p(s) \quad \text{since p is a gauge} \\ = \quad & p(xs) + p(yr) \quad \text{since p is a morphism on S} \\ = \quad & 2p(xs) = 2p(ys). \end{aligned}$$

(A3) now gives us that xs = ys; and therefore x = y as required.

LEMMA 4.17. Suppose that there exist elements x,y in X and r,s in S such that p(x) > p(y) and xs = yr. Let S′ be the subsemigroup of G generated by $X\backslash x \cup \{y^{-1}x\}$. Then p restricted to S′ is a morphism into **N**.

Proof. First we will show that given $u_1,..,u_n$ in S^1,

$$p(u_1y^{-1}xu_2...u_{n-1}y^{-1}xu_n)$$
$$= p(u_1..u_n) + (n-1)(\ p(x)-p(y)\ ) .$$

To prove this we use induction on n.

Clearly the result holds for n=1.

Suppose now that the result holds for all $n < k$.

Given elements $v_1, .., v_k$ of S^1 let

$$v = v_1, \; z = v_2 y^{-1} x v_3 ... v_{k-1} y^{-1} x v_k.$$

Now

$$p(xz) = p(xv_2 y^{-1} x v_3 ... v_{k-1} y^{-1} x v_k)$$

$$= p(xv_2 v_3 ... v_k) + (k-2)(\, p(x) - p(y) \,)$$

by the induction hypothesis

$$= p(x) + p(v_2 v_3 ... v_k) + (k-2)(\, p(x) - p(y) \,)$$

since the restriction of p to S is a morphism

$$= p(x) + p(z)$$ again by the induction hypothesis.

Now

$$2d(vy^{-1}, (xs)^{-1}) = p(vy^{-1}) + p(xs) - p(vy^{-1}xs)$$

$$= p(vy^{-1}) + p(xs) - p(vr)$$

$$= p(vy^{-1}) + p(xs) - p(v) - p(xs) + p(y)$$

(since p is a morphism on S and so

$$p(vr) = p(v) + p(r)$$

$$= p(v) + p(yr) - p(y)$$

$$= p(v) + p(xs) - p(y) \quad)$$

$$= p(vy^{-1}) - p(v) + p(y)$$

$$\leqslant p(v) + p(y) - p(v) + p(y) = 2p(y)$$

and

$$2d((xz)^{-1}, (xs)^{-1}) = p(xz) + p(xs) - p(z^{-1}s)$$

$$= 2p(x) + d(z,s)$$

(since $p(xz) = p(x) + p(z)$, as shown earlier)

$$\geqslant 2p(x).$$

Next $p(x) > p(y)$, and it therefore follows that

$$d((xz)^{-1}, (xs)^{-1}) > d(vy^{-1}, (xs)^{-1})$$

and so from A2′ we have that

$$d(vy^{-1},(xz)^{-1}) = d(vy^{-1},(xs)^{-1})$$

which expanded gives

$$p(vy^{-1}) + p(xz) - p(vy^{-1}xz) = p(vy^{-1}) + p(xs) - p(vy^{-1}xs).$$

Thus $\quad p(z) - p(vy^{-1}xz) = p(s) - p(v) - p(r)$

and so $\quad p(vy^{-1}xz) = p(v) + p(z) + p(r) - p(s)$

which implies $\quad p(vy^{-1}xz) = p(v) + p(z) + p(x) - p(y).$

Recall that

$$p(z) = p(v_2..v_k) + (k-2)(\ p(x) - p(y)\ )$$

and so

$$p(v_1y^{-1}xv_2...v_{k-1}y^{-1}xv_k)$$
$$= p(vy^{-1}xz) = p(v_1..._k) + (k-1)(\ p(x) - p(y)\ )$$

as required. The induction is complete.

The result now follows, since if w,z are elements of S′ then there exist elements $a_1,..,a_n$ and $b_1,..,b_m$ of S^1 such that

$$w = a_1y^{-1}xa_2y^{-1}x...y^{-1}xa_n$$

and $\quad z = b_1y^{-1}xb_2y^{-1}x...y^{-1}xb_m$

and so

$$p(wz)$$
$$= p(a_1y^{-1}xa_2y^{-1}x...y^{-1}xa_nb_1y^{-1}xb_2y^{-1}x...y^{-1}xb_m)$$
$$= p(a_1...a_nb_1...b_m) + (n+m-2)(\ p(x) - p(y)\ )$$
$$= p(a_1..a_n) + (n-1)(p(x)-p(y)) + p(b_1..b_m) + (m-1)(p(x)-p(y))$$

$= p(w) + p(z)$. Thus $p|_{S'}$ is a morphism as required.

THEOREM 4.4. Suppose that G is a free group and that p is a gauge on G satisfying (A1) and (A2) of definition 4.4. Suppose that S is a finitely generated subsemigroup of G such that p restricted to S is a morphism into **N**. Then $(S, p|_S)$ is F-metrical.

Proof. Let X be the unique minimal finite generating set for S.

If there do not exist distinct elements x and y in X and elements r and s in S such that xs = yr then S is free and the result follows trivially.

If there do exist such elements then we have from lemma 4.16 that either $p(x) > p(y)$ or else $p(y) > p(x)$, since $p(x) = p(y)$ would imply, by length cancellativity, that $x = y$.

Suppose without loss of generality that $p(x) > p(y)$. Then from lemma 4.17 the subsemigroup S' of G generated by $X' = X\backslash x \cup \{y^{-1}x\}$ is such that p restricted to S' is a morphism into **N**. Furthermore S is contained in S'.

Finally notice that

$$p(X) = \sum_{x \in X} p(x) > \sum_{x \in X'} p(x') = p(X')$$

and so repetitions of the above procedure must, by the finiteness of $p(X)$, eventually terminate with a set Y such that Y generates a free semigroup containing S. Furthermore the restriction of p to $\langle Y \rangle$ is a length morphism, and so the inclusion

$$i : (S, p|_S) \longrightarrow (\langle Y \rangle, p|_{\langle Y \rangle})$$

is a metrical embedding of $(S, p|_S)$ into a free semigroup.

COROLLARY 4.5. A finitely generated metrical semigroup (S,f) is F-metrical if and only if S is group embeddable, and f has a length function extension on some group containing S.

Proof. Suppose that (S,f) is F-metrical. Then there exists a set X such that the inclusion

$$i: (S,f) \longrightarrow (X^+,|.|)$$

is a metrical monomorphism, i.e. $f(s) = |s|$ for all s in S.

X^+ is contained in the free group, FG(X), on the set X. Thus FG(X) is a group containing S.

Let p be the usual length function on FG(X) (that is, the map from FG(X) to $\mathbf{N}^0$ that maps an element g of FG(X) to the length of the shortest word on the alphabet $X \cup X^{-1} \cup \{1\}$ that " represents " g). The restriction of p to X^+ is then equal to $|.|$. In particular, the restriction of p to S is equal to f.

Conversely suppose that S is group embeddable and that H is a group that contains S. If f has a length function extension on H then H is, by theorem 4.3, a free group. It follows from theorem 4.4 that (S,f) is F-metrical.

In practice it may be difficult to find a length function extension for a length morphism on a metrical semigroup. Theorem 4.5 which follows is less tidy than corollary 4.5, but does give us a procedure for determining whether or not a finitely generated metrical semigroup (S,f) is F-metrical. It may be possible to develop the procedure into an algorithm. The theorem is, at any rate, interesting in so far as it gives us a better link between the notions of F-metricality and length cancellativity than that given in lemma 4.13.

First we introduce some specific semigauge extensions.

Let S be a group embeddable semigroup and let H be the group of right quotients of S (see section 1). As usual we identify S with its image under α. Thus any element h of H can be written in the form

$$h = x_1 y_1^{-1} \ldots x_n y_n^{-1}$$

where the x_i and the y_i are elements of S such that

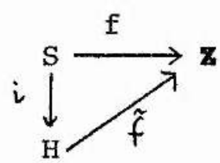
$$Sx_i \cap Sy_i \neq \emptyset.$$

DEFINITIONS 4.5. Let S be a group embeddable semigroup and let H be the group of right quotients of S. Let f be a morphism from S into $\mathbb{N}^o$. (Ultimately we shall only be interested in metrical semigroups and morphisms f into $\mathbb{N}$. For the moment it is convenient to define the following for any morphism into $\mathbb{N}^o$.)

We define the morphism $\tilde{f}$: H $\longrightarrow$ $\mathbb{Z}$ by

$$\tilde{f}(x_1 y_1^{-1} \ldots x_n y_n^{-1}) = f(x_1 \ldots x_n) - f(y_1 \ldots y_n).$$

$\tilde{f}$ is a well defined morphism. (It is in fact the unique morphism $\tilde{f}$: H $\longrightarrow$ $\mathbb{Z}$ the makes the diagram



commute.)

Notice that $\tilde{f}(s) = f(s)$ for all s in S.

We define the map p_f from H into $\mathbb{N}^o$ by

$p_f(h)$

$$= \min\{\sum_{i=1}^{n} |f(x_i) - f(y_i)| : h = \prod_{i=1}^{n} x_i y_i^{-1};\ x_i, y_i \in S; Sx_i \cap Sy_i \neq \emptyset\ \}.$$

It is an immediate consequence of the triangle inequality that

$$p_f(h) \geqslant |f(h)|$$

for all h in H.

We say that $h = \prod_{i=1}^{n} x_i y_i^{-1}$ is a <u>good representation of h with respect to S and p_f</u>, or simply a <u>good representation of h</u> if

1) $x_i, y_i \in S$

2) $Sx_i \cap Sy_i \neq \emptyset$

and 3) $p_f(h) = \sum_{i=1}^{n} |f(x_i) - f(y_i)|$.

LEMMA 4.18. <u>Let S be a group embeddable semigroup and let f be a morphism from S into $\mathbb{N}^0$. The map p_f from the group of right quotients, H, of S into $\mathbb{Z}$ is then is a semigauge and the restriction of p_f to S is equal to f.</u>

Proof. First we show that $p_f(h) = p_f(h^{-1})$.
Let h be an element of H. Then there exist $x_i, y_i \in S$ such that

$$h = x_1 y_1^{-1} \ldots x_n y_n^{-1}$$

is a good representation of h.

Now

$$h^{-1} = y_n x_n^{-1} \ldots y_1 x_1^{-1}$$

and so

$$\begin{aligned} &p_f(h^{-1}) \\ &= \min\{\sum_{i=1}^{k} |f(w_i) - f(z_i)| : h^{-1} = \prod_{i=1}^{k} w_i z_i^{-1};\ w_i, z_i \in S, Sw_i \cap Sz_i \neq \emptyset\}. \\ &\leqslant \sum_{i=1}^{n} |f(y_i) - f(x_i)| \\ &= p_f(h). \end{aligned}$$

Similarly $p_f(h) \leqslant p_f(h^{-1})$, and so $p_f(h^{-1}) = p_f(h)$ as required.

Next we show that for any h,g in H

$p_f(hg) \leqslant p_f(h) + p_f(g)$.

h and g are elements of H and so there exist elements $x_1,\ldots,x_n$; $y_1,\ldots,y_n$; $w_1,\ldots,w_m$; $z_1,\ldots,z_m$ in S such that

$$h = x_1y_1^{-1}\ldots x_ny_n^{-1}$$

and
$$g = w_1z_1^{-1}\ldots w_mz_m^{-1}$$

are good representations of h and g respectively.

Now $hg = x_1y_1^{-1}\ldots x_ny_n^{-1}w_1z_1^{-1}\ldots w_mz_m^{-1}$

and so $p_f(hg)$

$$= \min\{\sum_{i=1}^{k}|f(a_i)-f(b_i)| : hg = \prod_{i=1}^{k}a_ib_i^{-1};\ a_i,b_i\in S, Sa_i\cap Sb_i \neq \emptyset\}.$$

$$\leqslant \sum_{i=1}^{n}|f(x_i) - f(y_i)| + \sum_{j=1}^{m}|f(w_j) - f(z_j)|$$

$= \quad p_f(h) + p_f(g)$ as required.

Finally p_f is an extension of f, since for all s in S $s = (ss)s^{-1}$ and $Ss^2\cap Ss \neq \emptyset$, from which we deduce that

$$p_f(s) \leqslant |f(s^2) - f(s)| = f(s).$$

Furthermore, since for every s in S we have that $f(s) = |\tilde{f}(s)|$, and since for every h in H we have $|\tilde{f}(h)| \leqslant p_f(h)$, we deduce that $p_f(s) = f(s)$.

Thus p_f is a semigauge extension of f.

Lemma 4.19 deals specifically with metrical semigroups.

LEMMA 4.19. <u>Let (S,f) be a group embeddable metrical semigroup and let H be the group of right quotients of S. Then the semigauge p_f is a gauge if and only if (S,f) is length cancellative.</u>

Proof. Suppose that p_f is a gauge.

Let $a,b \in S$ be such that $f(a) = f(b)$ and $Sa \cap Sb \neq \emptyset$. Let $g = ab^{-1}$; then $p_f(g) = f(a) - f(b) = 0$. Thus $g = 1$, and so $a = b$. Thus (S,f) is right length cancellative, and, being cancellative, is therefore also left length cancellative.

Conversely suppose that (S,f) is length cancellative. Let g be an element of H such that $p_f(g) = 0$. Now there exist x_i, y_i $i = 1,..,k$ such that

$$g = \prod_{i=1}^{k} x_i y_i^{-1}$$

is a good representation of g. It follows that

$$\sum_{i=1}^{k} |f(x_i) - f(y_i)| = 0 \qquad (*)$$

and $$Sx_i \cap Sy_i \neq \emptyset. \qquad (**)$$

$(*)$ is only possible if $f(x_i) = f(y_i)$ $i = 1,..,k$ and so $(**)$ and length cancellativity imply $x_i = y_i$ $i = 1,..,k$. We deduce that $g = 1$.

Thus $p_f(g) = 0 \Rightarrow g = 1$, and so p_f is a gauge.

DEFINITION 4.6

Let S be a group embeddable semigroup without identity, let H be the group of right quotients of S and let f be a morphism from S into $\mathbf{N}^o$. We define semigauges p_i, semigroups $S_{(i)}$ and morphisms f_i from $S_{(i)}$ into $\mathbf{N}^o$ recursively as follows:

Let $S_{(0)} = S$ and let $f_0 = f$ and let $p_1 = p_f$.
For any k in $\mathbf{N}$ let p_k be the semigauge on H given by

$$p_k = p_{f_{k-1}} .$$

Then define the subsemigroup $S_{(k)}$ of H by

$$S_{(k)} = \{ h \in H\backslash 1 \; : \; p_k(h) = \tilde{f}(h) \}$$

(remark: $S_{(k)}$ is a subsemigroup of H containing S, and so we deduce from lemma 1.3 that H is also the group of right quotients of $S_{(k)}$)

Let f_k be the restriction to $S_{(k)}$ of $\tilde{f}$.

Notice that f_k is indeed a morphism since it is a restriction of the morphism $\tilde{f}$, and f_k does indeed map into $\mathbf{N}^0$ since it is a restriction of the map p_k.

Notice also that $S_{(k)}$ contains $S_{(k-1)}$ and that the restriction of f_k to $S_{(k-1)}$ is equal to f_{k-1}. It follows that the $\tilde{f}$ remains the same each time we apply the rules of definitions 4.5.

We are now in a position to state the final result of this section:

THEOREM 4.5. <u>Let (S,f) be a metrical semigroup. Then (S,f) is F-metrical if and only if</u>

a) <u>S is group embeddable</u>

<u>and for all n in $\mathbf{N}$</u>

b) <u>$f_n > 0$</u>

<u>(i.e. f_n is a length morphism on $S_{(n)}$)</u>

<u>and c) $(S_{(n)}, f_n)$ is length cancellative.</u>

Proof. Suppose that (S,f) is F-metrical. Then for some set X there exists a metrical monomorphism $\theta: (S,f) \longrightarrow (X^+, |.|)$. For convenience we shall suppose that θ is the inclusion map. Thus S is a subsemigroup of X^+, and for all s in S $f(s) = |s|$.

First notice that S is contained in the free group on the set X, and so S is certainly group embeddable.

Let H be the group of right quotients of S. From definitions 4.5 we have a morphism $\tilde{f}: H \longrightarrow \mathbb{Z}$ given by

$$f(r_1 s_1^{-1} \ldots r_m s_m^{-1})$$
$$= f(r_1 .. r_m) - f(s_1 .. s_m)$$
$$= |r_1 .. r_m| - |s_1 .. s_m|.$$

We now prove that $S_{(n)}$ is contained in X^+, and that for all x in $S_{(n)}$ we have $f_n(x) = |x|$. It will then follow that $f_n > 0$, and lemma 4.13 will ensure that $(S_{(n)}, f_n)$ is length cancellative, thus completing the direct part of the proof of the theorem.

We prove the result by induction on n. Since $S_{(0)} = S$, and $f_0 = f$, we clearly have that the result holds for $n = 0$.

Suppose now that the result holds for $n \leqslant k$.

Consider an element z of the semigroup $S_{(k+1)}$.

Notice first that lemma 1.3 implies that H is equal to the group of right quotients of $S_{(k)}$, since $S_{(k)}$ contains S.

Now z is an element of H and so there exists a good representation of z with respect to $S_{(k)}$ and p_{k+1},

$$z = x_1 y_1^{-1} \ldots x_t y_t^{-1}.$$

It follows from the definition of a good representation that

a) $x_i, y_i \in S_{(k)}$ for $i = 1,..,t$

b) $S_{(k)}x_i \cap S_{(k)}y_i \neq \emptyset$ for $i = 1,..,t$

and

c) $p_{k+1}(z) = \sum_{i=1}^{t} | f_k(x_i) - f_k(y_i) |$.

Now the induction hypothesis holds for $n = k$, and so $f_k(x_i) = |x_i|$ and $f_k(y_i) = |y_i|$ for $i = 1,..,t$. It follows therefore that

$$p_{k+1}(z) = \sum_{i=1}^{t} | \, |x_i| - |y_i| \, |.$$

Now z is an element of $S_{(k+1)}$, and so from definition 4.6,

$$p_{k+1}(z) = \tilde{f}(z),$$

i.e. $$\sum_{i=1}^{t} |x_i| - \sum_{i=1}^{t} |y_i| = \sum_{i=1}^{t} | \, |x_i| - |y_i| \, |$$

which is only possible if

$$|x_i| - |y_i| = | \, |x_i| - |y_i| \, | \text{ for } i = 1,..,t.$$

Thus

$$|x_i| \geqslant |y_i| \quad \text{for } i = 1,...,t.$$

If for any j we have $|x_j| = |y_j|$, then by the induction hypothesis $f_k(x_j) = f_k(y_j)$. The induction hypothesis also tells us that $(S_{(k)}, f_k)$ is length cancellative, and so (b) gives us that $x_j = y_j$. If this is the case then we can simply miss the pair $x_j y_j^{-1}$ from the good representation of z. Furthermore we cannot have $x_i = y_i$ for all $i = 1,..,t$ since $S_{(k+1)}$ is defined in such a way as to exclude the identity element.

Thus the element z is of the form

$$z = x_1 y_1^{-1} \dots x_t y_t^{-1}$$

where $x_i, y_i \in S_{(k)}$

$$S_{(k)}x_i \cap S_{(k)}y_i \neq \emptyset$$

and $|x_i| > |y_i|$.

Now by the induction hypothesis again, $S_{(k)}$ is contained in X^+. It therefore follows from equidivisibility in X^+ (lemma 4.10) that there exist elements $w_1,..,w_t$ such that

$$x_i = w_i y_i$$

Thus $z = w_1 \dots w_t \quad X^+$.

Finally

$$\begin{aligned} & f_{k+1}(z) \\ &= p_{k+1}(z) \\ &= \sum_{i=1}^{t} \big|\, |x_i| - |y_i| \,\big| \\ &= \sum_{i=1}^{t} |w_i| \\ &= |z| \end{aligned}$$

This holds for all z in $S_{(k+1)}$, and so we have $S_{(k+1)} \subseteq X^+$ and $f_{k+1} = |.|$, completing the inductive proof, and therefore completing the direct part of the thoerem.

Conversely suppose that S is group embeddable, that $f_n > 0$ for all n and that $(S_{(n)}, f_n)$ is length cancellative for all n. Again we let H be the group of right quotients of S.

Let R be the subsemigroup of H given by

$$R = \bigcup_{n=0}^{\infty} S_{(n)}.$$

Let r be an element of R. Then there exists k in $\mathbb{N}$ such that $r \in S_{(k)}$. We deduce that $\tilde{f}(r) = f_k(r)$, which is strictly positive. Thus $\tilde{f}$ induces a length morphism on R.

We now show that R is equidivisible. It will then follow from Levi's lemma (lemma 4.10) that R is a free semigroup.

Let a,b,c,d be elements of R such that

$$ab = cd.$$

We can suppose without loss of generality that $\tilde{f}(b) \geqslant \tilde{f}(d)$.

Now for sufficiently large m we have that

$$a,b,c,d \in S_{(m)}.$$

If $\tilde{f}(b) = \tilde{f}(d)$, then $f_m(b) = f_m(d)$, and so length cancellativity in $(S_{(m)}, f_m)$ implies that b = d. It follows from cancellativity that a = c, and so the conditions of equidivisibility are satisfied.

Suppose therefore that $\tilde{f}(b) > \tilde{f}(d)$.

Now consider the element $h = bd^{-1}$ of H.

Now $$b,d \in S_{(m)},$$

and $$S_{(m)}b \cap S_{(m)}d \neq \emptyset.$$

Also $|f_m(b) - f_m(d)| = |\tilde{f}(b) - \tilde{f}(d)| = \tilde{f}(bd^{-1})$

so

$$\begin{aligned} &p_{m+1}(h) \\ &= \min\{\sum_{i=1}^{k} |f(x_i) - f(y_i)| : h = \prod_{i=1}^{k} x_i y_i^{-1};\ x_i, y_i \in S_{(m)}, \\ &\qquad S_{(m)}x_i \cap S_{(k)}y_i \neq \emptyset\ \}. \\ &\leqslant \tilde{f}(bd^{-1}) = \tilde{f}(h). \end{aligned}$$

The opposite inequality always holds, and so we deduce that

$$p_{m+1}(h) = \tilde{f}(h),$$

and so $h \in S_{(m+1)}$.

Thus bd^{-1} is an element of R. Since $b = (bd^{-1})d$, and so $c = a(bd^{-1})$, we deduce that the conditions for equidivisibility are satisfied.

Both cases $\tilde{f}(b) = \tilde{f}(d)$ and $\tilde{f}(b) > \tilde{f}(d)$ having been considered, we deduce that R is equidivisible, and is therefore a free semigroup by Levi's lemma (lemma 4.10).

Finally notice that $\tilde{f}|_S = f$, and so (S,f) is metrically embedded into $(R,\tilde{f}|_R)$. Thus (S,f) is F-metrical.

REMARK. Notice that theorem 4.5 holds for infinitely generated metrical semigroups.

5. Metrical and locally metrical ideals.

5.1 The relation χ.

We start by introducing some convenient notation.

Recall that the congruence χ_S on a semigroup S is defined to be the smallest congruence containing

$$\{ (rs,sr) : r,s \;\; S \}.$$

(See page 18).

Suppose that S is a semigroup, and distinct elements u and v in S. Then $u \chi_S v$ if and only if there exists a family of elements $\{ x_i, y_i, u_i, v_i : i = 1,..,n \}$ in S^1 such that

$$u = x_1 u_1 v_1 y_1 \;;$$

$$x_i v_i u_i y_i = x_{i+1} u_{i+1} v_{i+1} y_{i+1} \quad i = 1,..,n-1 \;;$$

$$x_n v_n u_n y_n = v.$$

We call such a family an <u>ordered χ-linking family</u> or an <u>OLF</u> from u to v in S. We refer to the parameter n as the <u>order</u> of the OLF.

LEMMA 5.1. <u>Let $\{ x_i, y_i, u_i, v_i : i = 1,..,n \}$ be an OLF from u to v in S of minimal order. Then we have $u_i, v_i \neq 1$ for all $1 \leqslant i \leqslant n$.</u>

Proof. Suppose that $u_k = 1$. Suppose that $k = 1$, then

$$u = x_1 v_1 y_1 = x_2 u_2 v_2 y_2$$

and so $\{ x_i, y_i, u_i, v_i : i = 2,..,n \}$ is an OLF from u to v of order less than n, giving us a contradiction. Suppose therefore that $k > 1$. Then

$$x_{k-1} v_{k-1} u_{k-1} y_{k-1} = x_k v_k y_k = x_{k+1} u_{k+1} v_{k+1} y_{k+1} \quad (\text{ or } v \text{ if } k=n)$$

and so $\{ x_i, y_i, u_i, v_i : i = 1,..,k-1,k+1,..,n \}$ is an OLF from u to v of order less than n, which again gives us a contradiction. We deduce that $u_i \neq 1$, $i = 1,..,n$. Similarly $v_i \neq 1$, $i = 1,..,n$.

We say that an OLF is <u>heavy</u> if it is completely contained in S, i.e. if no x_i, y_i, u_i or v_i is equal to 1.

LEMMA 5.2. <u>Let u be an element of $S \backslash S^2$. Then $v \chi u \Rightarrow v = u$.</u>

Proof. Suppose that $v \chi u$ but that $v \neq u$. Then there exists a minimal OLF from u to v. Thus there exist elements x_1, y_1 in S^1 and u_1, v_1 in S such that $u = x_1 u_1 v_1 y_1$. This is a contradiction since u is an element of $S \backslash S^2$.

5.2. The locally metrical and compressible parts of a semigroup.

DEFINITION 5.1. Suppose that S is an arbitrary semigroup. Define the <u>locally metrical part, L(S) of S</u> by:

$$L(S) = \{ s \in S : \nexists r,t \in S \text{ such that } rst \chi r \}.$$

Define the <u>compressible part, C(S) of S</u> by $C(S) = S \backslash L(S)$.

LEMMA 5.3. <u>If S is an arbitrary semigroup then C(S) is a subsemigroup of S and L(S) is a locally metrical ideal of S.</u>

Proof. First, C(S) is a subsemigroup of S since if r,s are elements of C(S) then there exists w,x,y,z in S such that $wrx \chi w$ and $ysz \chi y$. Thus $wyrsxz \chi wrxysz \chi wy$, and so rs is contained in C(S). Next L(S) is an ideal of S; for suppose that r,s in S are such that rs is contained in C(S); then there exist x,y in S such that $xrsy \chi x$, and so r and s are both contained in C(S). Theorem 3.1 now completes the proof.

The following lemma will be of use later on:

LEMMA 5.4. Suppose that I is an ideal of a semigroup S, that (G,+) is an abelian group, and that $f: I \longrightarrow G$ is a morphism. Then f is uniquely extendable to a morphism $\tilde{f}: S \longrightarrow G$. Furthermore, if $G = \mathbb{Z}$ and $im(f) \subseteq \mathbb{N}$ then $im(\tilde{f}) \subseteq \mathbb{N}^0$.

Proof.Let x be an arbitrary element of I. Define

$$\tilde{f}(s) = f(xs) - f(x) \qquad (s \in S)$$

Note first that for all i in I, $\tilde{f}(i) = f(xi) - f(x) = f(i)$ (by the morphism property of $f:I \longrightarrow G$), and so $\tilde{f}$ is an extension of f.

Now $f(xs) = f(xsx) - f(x) = f(sx)$. It follows that $\tilde{f}$ is independent of the choice of x; for if y is another element of I then

$$\begin{aligned} & f(ys) - f(y) \\ &= f(ysx) - f(y) - f(x) \\ &= f(sx) - f(x) \\ &= f(xs) - f(x). \end{aligned}$$

We also have that $\tilde{f}$ is a morphism, since if s and r are elements of S then

$$\begin{aligned} \tilde{f}(sr) &= f(srx) - f(x) \\ &= f(xsrx) - 2f(x) \\ &= f(xs) - f(x) + f(rx) - f(x). \\ &= \tilde{f}(s) + \tilde{f}(r). \end{aligned}$$

$\tilde{f}$ is the unique extension of f since if g is an extension of f then for all s in S $g(xs) = g(x) + g(s)$ implies that $g(s) = g(xs) - g(x) = f(xs) - f(x) = \tilde{f}(s)$.

Suppose now that $G = \mathbf{Z}$ and that $\mathrm{im}(f) \subsetneq \mathbf{N}$. Suppose by way of contradiction that there exists an element $s \in S \setminus I$ for which $\tilde{f}(s) < 0$. Now for any element x in I we have that

$$\tilde{f}(s) = f(xs) - f(x),$$

and so we have that

$$(*) \qquad f(xs) = \tilde{f}(s) + f(x) \quad \text{for all } x \text{ in } I.$$

Let i be a specific element of I. Now for any $n \geqslant 2$ we have that is^{n-1} is an element of I. It follows from (*) that

$$f(is^n) = \tilde{f}(s) + f(is^{n-1}).$$

We deduce that

$$f(is^n) = f(i) + n.\tilde{f}(s).$$

This is the required contradiction, for we would then have that $f(is^n) < 0$ for sufficiently large n.

LEMMA 5.5. <u>If L is a locally metrical ideal of a semigroup S, then $L \subseteq L(S)$.</u>

Proof. Suppose by way of contradiction that there exists u in $L \cap C(S)$. Then there exists r,s in S such that $rus \,\chi_S\, r$. Now L is an ideal, and so us is in L. Thus there exists r in S and v (= us) in L such that $rv \,\chi_S\, r$. Now for any morphism $f: S \longrightarrow \mathbf{Q}$ we have that $f(v) = 0$, since $f(rv) = f(r)$. However L is locally metrical, and $\langle v \rangle$ is a finitely generated subsemigroup of L. Therefore by lemma 3.3 there exists a morphism $q: L \longrightarrow \mathbf{Q}$ such that $q(v) > 0$. We now have a contradiction since, by lemma 5.4, q extends uniquely to a morphism $\tilde{q} : S \longrightarrow \mathbf{Q}$ with $\tilde{q}(v) > 0$.

LEMMA 5.6. <u>If S is a finitely generated semigroup of rank</u>

n, then either L(S) is empty, or L(S) is metrical and C(S) is of rank less than n.

Proof. Let A be a finite generating set for S of minimal size. Let $A' = A \cap C(S)$ and $A'' = A \cap L(S)$. Then $C(S) = \langle A' \rangle$ and $L(S) = S^1 A'' S^1$. It follows that if L(S) is non empty then the rank of C(S) is less than the rank of S. It remains to show that L(S) is metrical.

Define a relation τ on $\langle A'' \rangle$ by $u \,\tau\, v$ if and only if there exist c,d in C(S) such that $uc \,\chi\, vd$. Then τ is a congruence and $\langle A'' \rangle/\tau$ is a finitely generated commutative semigroup. Now suppose that there exist u,v in $\langle A'' \rangle$ such that $uv \,\tau\, u$. Then there exist c,d in C(S) such that $uvc \,\chi\, ud$. Now since d is in C(S) we have that there exist r,s in S such that $rds \,\chi\, r$, and so $ruvcs \,\chi\, ruds \,\chi\, urds \,\chi\, ur \,\chi\, ru$, and so v is in C(S); but this is a contradiction. We deduce that there do not exist u,v in $\langle A'' \rangle$ such that $uv \,\tau\, u$, and so, by corollary 2.2, $\langle A'' \rangle/\tau$ is metrical.

Let $f: \langle A'' \rangle/\tau \longrightarrow \mathbf{N}$ be a morphism. Define $\tilde{f}: L(S) \longrightarrow \mathbf{N}$ as follows:

If u is in L(S) then $u = a_1 b_1 \ldots a_n b_n a_{n+1}$ where $n > 0$, $a_i \in \langle A' \rangle^1$ and $b_i \in \langle A'' \rangle$. Let $\tilde{f}(u) = f(\, [b_1 .. b_n]_\tau \,)$. $\tilde{f}$ is then well defined, since if $u = a_1' b_1' \ldots a_m' b_m' a_{m+1}'$, then $b_1 .. b_n \,\tau\, b_1' .. b_m'$. It is clear that $\tilde{f}$ is a morphism.

THEOREM 5.1. Let S be a finitely generated semigroup, and suppose that L is a locally metrical ideal of S. Then L is metrical.

Proof. By lemma 5.5 L is contained in L(S). By lemma 5.6 L(S) is metrical. We deduce that L is metrical.

The following result will be of use in section 6.

LEMMA 5.7. Let S be a semigroup and suppose that x,y in C(S) are such that $xy \chi_S x$. Then $xy \chi_{C(S)} x$.

Proof. $xy \chi x$ and so there exists an OLF $\{x_i, y_i, u_i, v_i\}$ from xy to x in S. Now if for any j we have x_j, y_j, u_j or v_j in L(S) then $x_j v_j u_j y_j$ is also in L(S), L(S) being an ideal of S^1. Since $C(S)^1$ is a subsemigroup of S^1, and since $x_j v_j u_j y_j = x_{j+1} u_{j+1} v_{j+1} y_{j+1}$ in S we have that either x_{j+1}, y_{j+1}, u_{j+1} or v_{j+1} is in L(S). Repeating the argument we deduce that x is in L(S), which is a contradiction. Thus x_i, y_i, u_i and v_i are in $C(S)^1$ for all i=1,..,n, and so our OLF is in C(S) and so $xy \chi_{C(S)} x$ as required.

Lemma 5.8 gives an alternative definition of C(S) which is sometimes more useful for calculating C(S) for specific semigroups.

LEMMA 5.8. Let S be a semigroup. Then an element t of S belongs to the compressible part of S if and only if there exist $a_1, a_2, \ldots a_n$ in S and r in S and a permutation σ of n elements such that $(a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)}) tr = a_1 a_2 \ldots a_n$.

Proof. Since χ is a commutative congruence we have that the condition is sufficient. Conversely suppose that t is in the compressible part of S. Then there exist s and r in S such that $str \chi s$. Thus there exists an OLF $\{ x_i, u_i, v_i, y_i \}$

from str to s in S. Notice that some but not all of the x_i, y_i, u_i, v_i may be 1. We then have

$$(x_{n-1}v_{n-1}u_{n-1}y_{n-1} \cdots x_1v_1u_1y_1x_nv_nu_ny_n)tr$$

$$= (x_{n-1}v_{n-1}u_{n-1}y_{n-1} \cdots x_1v_1u_1y_1)str$$

$$= (x_nu_nv_ny_n \cdots x_2u_2v_2y_2)x_1u_1v_1y_1.$$

6. Height, t-compressible semigroups and Ultimately Locally Metrical Semigroups.

In section 5 we defined the compressible and locally metrical parts of a semigroup. Clearly given a semigroup S we have that $C(L(S)) = \emptyset$ and that $L(L(S)) = L(S)$. In general, however, we do not have that $L(C(S)) = \emptyset$. For example consider the commutative semigroup T with presentation $< a,b,c : ab = a , bc = b >$. Then $L(T) = < a >$ and $C(T) = < b,c >$, but $L(C(T)) = < b >$.

We follow the convention of defining $C^n(S)$ for n in $\mathbb{N}^0$ by

$$C^0(S) = S \quad \text{and} \quad C^n(S) = C(C^{n-1}(S)) \text{ when } n > 0.$$

Given a semigroup S if we have for some $k(S) \geqslant 0$ that $C^{k(S)+1}(S) = C^{k(S)}(S)$ ($\neq C^{k(S)-1}(S)$ if $k(S) > 0$) we say S is of finite height and that the height of S is k(S). If no such k(S) exists then we say that S is of infinite height and define k(S) to be equal to ∞.

Lemma 5.6 ensures that any finitely generated semigroup is of finite height, and that the height of such a semigroup is less than or equal to its rank.

In section 6.1 we consider some basic properties of semigroups whose locally metrical part is empty.

In section 6.2 we consider ultimately locally metrical semigroups. Such semigroups are unions of locally metrical semigroups.

In section 6.3 we investigate ideals of semigroups with the properties described in sections 6.1 and 6.2.

Finally, in section 6.4, we investigate cartesian products of semigroups with the properties described in sections 6.1 and 6.2.

6.1 t-compressible semigroups.

If a semigroup is of height zero, that is if $C(S)=S$, then we say that S is totally compressible or t-compressible.

LEMMA 6.1. Let S be a semigroup, and let C be a t-compressible subsemigroup of S. Then $C \subseteq C(S)$.

Proof. If c is in C then there exist b,d in C such that $bcd\ \chi_C\ b$. Hence $bcd\ \chi_S\ b$ and so c is in $C(S)$.

LEMMA 6.2. Let S be a finitely generated semigroup. If S is t-compressible, then there exists an x in S such that $x^2\ \chi\ x$.

Proof. We define a relation π on S by the rule that $a\pi b$ if and only if there exists c such that $bac\ \chi\ b$. Note that if $a\pi b$ then $a\pi bb'$ for all b' in S, and that if we also have that $a'\pi b$, then $aa'\pi b$. Let $s_1,\ldots,s_n$ generate S. Then if S is t-compressible we have that there exist $x_1,\ldots,x_n$ such that $s_i\pi x_i$ for $i=1,\ldots,n$. Let $x = x_1.x_2\ldots x_n$. It follows from the above remarks that $s_i\pi x$ for $i=1,\ldots,n$, and since the s_i generate S it follows that $s\pi x$ for all s in S. In particular $x\pi x$, and so there exists y such that $xyx\ \chi\ x$, from which we deduce that $(xy)^2 \chi\, xy$.

Lemma 6.2 fails in the infinitely generated case, as the following example illustrates:

EXAMPLE 6.1. Consider the commutative semigroup, S, with presentation $< x_1, x_2, \ldots : x_i x_{i+1} = x_{i+1} ; i \text{ in } \mathbf{N} >$. This semigroup is t-compressible, since x_i is contained in C(S) for all i, and therefore S = C(S). The elements of S are simply powers of the x_i, none of which are idempotent.

LEMMA 6.3. <u>Any homomorphic image of a t-compressible semigroup is itself t-compressible.</u>

Proof. Let S be t-compressible and let f: S $\longrightarrow$ R be a surjective morphism. Since S is t-compressible then for every s in S there exist u and v such that $usv \chi_S u$. Hence for any $r = f(s)$ in R we have $f(u)rf(v) \chi_R f(u)$, and so r is in C(R).

LEMMA 6.4. <u>Any ideal of a t-compressible semigroup is t-compressible.</u>

Proof. Let S be t-compressible and let I be an ideal of S. We show later (lemma 6.8) that $L(I) = I \cap L(S)$. We deduce that L(I) is empty, and so I is t-compressible.

LEMMA 6.5. <u>Let S be a semigroup with a right/left/two-sided ideal I such that I is t-compressible. Then S is t-compressible.</u>

Proof. Let r be an element of S and let j be an element of I. Then either jr or rj is in I, so either jr or rj is in

C(S). Thus r is in C(S).

EXAMPLES 6.2. The following are examples of t-compressible semigroups.

a) A regular semigroup S is t-compressible, since for all elements a there exists an element x such that axa=a, from which we have $a \in C(S)$.

b) A simple semigroup, S, is t-compressible, since S has no proper non-trivial ideal and so we deduce that L(S) = S or $L(S) = \emptyset$. If L(S)=S then S would be locally metrical. This is not the case since there exist elements a,u,v in S such that uav=a.

c) The subsemigroup S of $Mat_{2x2}(\mathbf{Z})$ under matrix multiplication consisting of elements

$$\left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \quad a,b,c,d \in \mathbf{N} \right\}$$

is neither regular nor simple.

First S is not simple, for if we consider the element

$$\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$$

then it is routine to show that

$$S^1 \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} S^1$$

$$= \quad \left\{ \begin{pmatrix} 2a & 0 \\ 2b & 0 \end{pmatrix} : a,b \in \mathbf{N} \right\} \cup \left\{ \begin{pmatrix} 0 & 2a \\ 0 & 2b \end{pmatrix} : a,b \in \mathbf{N} \right\}$$

which does not contain the element $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Second S is not regular, for if we consider again the element

$$s = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$$

then it is routine to show that the set

$$\{ sxs : x \in S \}$$

$$= \left\{ \begin{pmatrix} 4a & 0 \\ 4a & 0 \end{pmatrix} : a \in \mathbf{N} \right\},$$

which does not contain s.

S is, however, t-compressible. To show that this is the case let s be an arbitrary element of S.

Suppose first that s is of the form $\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}$. Then $rs \chi r$ with $r = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}$ since

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} =$$

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix} \chi \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & d \end{pmatrix} =$$

$$\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \chi \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}.$$

Similarly if s is of the form $\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}$ then we can find r such that $rs \chi r$. Thus we have for all s in S that there exists an r such that $rss \chi rs \chi s$, and so $s \in C(S)$. We deduce that S is t-compressible.

REMARK. We see from the examples above that the class of t-compressible semigroups is a non-trivial extension of the class of regular or simple semigroups. It is worth noting that Fountain's generalisation of regular semigroups, abundant semigroups (see for example [5]), are not contained in the class of t-compressible semigroups. For example the semigroup ($\mathbf{N}$,.) is abundant, but not t-compressible.

6.2 Ultimately locally metrical and ultimately metrical semigroups.

We say that a semigroup S is ultimately locally metrical, or ULM if for all x in S there exists n in $\mathbb{N}^0$ such that x is contained in $L(C^n(S))$, that is if $S = \bigcup_{n=0}^{\infty} L(C^n(S))$. We say that S is ultimately metrical (respectively ultimately **R**-metrical, ultimately **Q**-metrical), or UM, if it is ultimately locally metrical and in addition $L(C^n(S))$ is metrical (**R**-metrical, **Q**-metrical) for every $n < k(S)$. Notice that theorem 5.1 and lemma 5.6 ensure that a finitely generated semigroup is ultimately metrical whenever it is ultimately locally metrical.

LEMMA 6.6. If a semigroup S is ULM, then it contains no t-compressible subsemigroup. Furthermore if S is of finite height then it is ULM if and only if it contains no t-compressible subsemigroup.

Proof. Suppose that S is ULM. If S contains a t-compressible subsemigroup U then by repeated applications of lemma 6.1 we have that $U \subseteq C^n(S)$ for all n in $\mathbb{N}$; but then for any u in U we have that there does not exist m such that u is contained in $L(C^m(S))$, contrary to assumption. Furthermore if S is of finite height then we have that $C^{k(S)+1}(S) = C^{k(S)}(S)$, and so $C^{k(S)}(S)$ is either empty, in which case S is ULM, or is a non-empty t-compressible subsemigroup of S.

COROLLARY 6.1. <u>A finitely generated semigroup is UM if and only if it contains no element x such that $x^2 \chi_S x$. In particular a finitely generated commutative semigroup is UM if and only if it is idempotent free.</u>

Proof. Suppose that S contains an element x such that $x^2 \chi_S x$. Clearly x is contained in C(S), and so by lemma 5.7 $x^2 \chi_{C(S)} x$. Repeating this argument k(S)-1 times we deduce that x is contained in $C^{k(S)}(S)$ and so $C^{k(S)}(S)$ is non-empty, and hence S is not ULM.

Conversely suppose that S does not contain an element x such that $x^2 \chi_S x$. Now $C^{k(S)}(S)$ is either a finitely generated t-compressible subsemigroup or is empty. The former is not possible since then by lemma 6.2 we have that there exists x in $C = C^{k(S)}(S)$ such that $x^2 \chi_C x$, and so S contains an element x such that $x^2 \chi_S x$, contrary to assumption. Thus $C^{k(S)}(S)$ is empty and so S is a finitely generated ULM semigroup, and is therefore UM.

EXAMPLE 6.3. Let S be the commutative semigroup with presentation $< z, x_1, x_2, \ldots : zx_i = x_i \;, x_i x_{i+1} = x_i \;;\; i \in \mathbb{N} >$.

Notice first that

$$L(S) = < x_1 >$$

and

$$C(S) = < z, x_2, x_3, \ldots >.$$

Repeating this observation n times we see that

$$L(C^n(S)) = < x_{n+1} >$$

and

$$C^{n+1}(S) = < z, x_{n+2}, x_{n+3}, \ldots >.$$

S is not ULM since there does not exist k in $\mathbf{N}^{0}$ such that z is contained in $L(C^{k}(S))$.

Finally notice that S does not contain a t-compressible subsemigroup. Suppose by way of contradiction that C is a t-compressible subsemigroup of S. Then repeated applications of lemma 6.1 imply that $C \subseteq C^{n}(S)$ for all n in $\mathbf{N}$. We deduce that $C \subsetneq \langle z \rangle$. This is a contradiction, since $\langle z \rangle$ is isomorphic to the infinite monogenic semigroup, which is metrical, and so contains no t-compressible subsemigroup.

Thus the second part of lemma 6.6 does not hold in the infinite height case.

LEMMA 6.7. Let S be an ULM semigroup. Then every subsemigroup R of S is also ULM. Furthermore if S is of finite height then R is also of finite height and $k(R) \leqslant k(S)$.

Proof. Suppose that R is a subsemigroup of S. Then if x is contained in C(R) we have that there exist u,v in R such that uxv χ_R u. Hence uxv χ_S u and so $x \in C(S)$. Repeated applications of this argument give us that $C^{n}(R) \subseteq C^{n}(S) \cap R$ for all n in $\mathbf{N}$. From this we deduce that R is ultimately locally metrical and that if S is of finite height then so is R, with $k(R) \leqslant k(S)$.

The following example shows that a subsemigroup of a UM semigroup need not be UM.

EXAMPLE 6.4. Consider the commutative semigroup T with commutative presentation

$$P(T) \equiv \langle a_1, a_2, \ldots ; x, z : xz = x , a_j = a_{j+1}z ; j = 1,2,.. \rangle.$$

Let S be the subsemigroup of T generated by

$$X = \{ z ; a_j (j=1,2.,..)\}.$$

First notice that S is isomorphic to the semigroup S´ with commutative presentation

$$P'(S') \equiv \langle z, a_1, a_2, \ldots : a_j = a_{j+1}z \rangle.$$

This follows for if u and v are elements of X^+ such that u = v in T then there exist $u_1, \ldots, u_n$ in $(\{x\} \cup X)^+$ such that

$$u = u_1 \to u_2 \to \ldots \to u_n = v$$

is a sequence of elementary P(T) transitions. Since u is a word in X, and therefore does not contain an occurrence of the element x, and since the only P(T) transition that involves the element x is the transition $xz \longleftrightarrow x$, which does not alter the number of occurrences of the element x in a word, we deduce that none of the u_i contain an occurrence of the element x. Thus all transitions $u_i \to u_{i+1}$ must be of the form $u_i = a_j w$, $u_{i+1} = a_{j+1}zw$ (or the reverse), and so we have a sequence of elementary P´(S´) transitions from u to v. Clearly if u is connected to v by a sequence of P´(S´) transitions, then u is connected to v by a sequence of P(T) transitions.We deduce that S is isomorphic to S´, thus S has commutative presentation

$$P'(S) \equiv \langle z, a_1, a_2, \ldots : a_j = a_{j+1}z \rangle.$$

Let $f: X^+ \longrightarrow \mathbb{Z}$ be given by

$$f(w) = \sum_{i=1}^{\infty} i|w|_{a_i} - |w|_z$$

and let $g: X^+ \longrightarrow \mathbb{Z}$ be given by

$$g(w) = \sum_{i=1}^{\infty} (i-1)|w|_{a} - |w|_z.$$

i

Both f and g are well defined morphisms on S, since $f(a_j) = f(a_{j+1}z)$ and $g(a_j) = g(a_{j+1}z)$.

Suppose by way of contradiction that $C(S) \neq \emptyset$. Let s be an element of C(S). Then there exist r,t in S such that

$$rst \ \chi_S \ r.$$

Now the morphisms f and g are invariant under the congruence χ_S, and so we deduce that

$$f(rst) = f(r) \quad \text{and} \quad g(rst) = g(r),$$

and so

$$f(st) = g(st) = 0.$$

This gives us the required contradiction since if w is an element of X^+ then

$$f(w) = g(w)$$

$$\Rightarrow \sum_{i=1}^{\infty} i|w|_{a_i} - |w|_z \quad = \sum_{i=1}^{\infty} (i-1)|w|_{a_i} - |w|_z$$

$$\Rightarrow \sum_{i=1}^{\infty} |w|_{a_i} = 0$$

$$\Rightarrow \quad w = z^n \qquad \text{for some } n \text{ in } \mathbb{N}$$

and so $f(w) = g(w) = 0$ implies that $w = 1$, which is not an element of X^+. We therefore deduce that there is no element s of S such that $f(s) = g(s) = 0$.

Thus C(S) is empty, and so S is locally metrical.

S is not, however, metrical, since if $h: S \longrightarrow \mathbb{N}$ is a morphism then for h to be well defined we must have that $h(a_{n+1}) = h(a_1) - nh(z)$, which is less than zero for sufficiently large n, giving us a contradiction.

Finally we show that T is UM. To show this first notice that

$$L(T) = \langle \{ a_1 z^k : k \in \mathbf{N} \} \cup \{ x, a_1, a_2, .. \} \rangle.$$

This is metrical, for if we let f: $T \longrightarrow \mathbf{Z}$ be the morphism given by $f(z) = 0$, $f(x) = 1$, $f(a_i) = 1$ for $i \in \mathbf{N}$, which is well defined, then the restriction of f to L(T) is a length morphism on L(T). Next notice that $C(T) = \langle z \rangle$, which is isomorphic to $(\mathbf{N},+)$, and is therefore metrical.

S is therefore a subsemigroup of a UM semigroup that is not itself UM.

EXAMPLES 6.5. We now give some examples of ultimately locally metrical semigroups.

a) Any finitely generated commutative idempotent free semigroup is ultimately metrical and of finite height (corollary 6.1).

b) Consider the semigroup S of 2×2 matrices over $\mathbf{R}$ consisting of elements $\{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a \geq 1, b > 0 \}$.

The set $L = \{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} : a > 1, b > 0 \}$ is then an ideal of S that is $\mathbf{R}$-metrical, with the morphism f into $\mathbf{R}$ given by

$$f(\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}) = \log_e(a).$$

In fact $L = L(S)$, since given an element $s = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ of $S \backslash L$ we have that $sr = rs^2$ where $r = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, and so $s \in C(S)$.

We have $C(S) = \{\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} : b > 0 \}$, which is $\mathbf{R}$-metrical, with the morphism g into $\mathbf{R}$ given by

$$g(\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}) = b.$$

We therefore have that S is an ultimately $\mathbf{R}$-metrical semigroup of height 2.

6.3 More results concerning ideals.

In most of the following, results for metricality will also hold for $\mathbf{Q}$ and $\mathbf{R}$-metricality.

LEMMA 6.8. Let S be a semigroup and I be an ideal of S. Then for all n in $\mathbf{N}^0$, $L(C^n(I)) = L(C^n(S)) \cap I$ and $C^n(I) = C^n(S) \cap I$.

Proof. First note that $L(S) \cap I$ is a locally metrical ideal of I and so is contained in L(I) by lemma 5.5. To show the reverse inclusion we show that L(I) is an ideal of S. Let x be in L(I) and let s be in S. Then sx is contained in I, since I is an ideal. In fact sx is contained in L(I), for if sx were in C(I) we would have that there exist i,j in I such that $i(sx)j \chi_I i$. Now $i(sx)j \chi_I i \Rightarrow ij(sx) \chi_I i \Rightarrow i(js)x \chi_I i \Rightarrow x \in C(I)$, contrary to assumption. We deduce that sx is in L(I), and it can be shown similarly that xs is in L(I). Thus L(I) is a locally metrical ideal of S , and is therefore contained in L(S). It also follows that $C(I) = C(S) \cap I$. Thus C(I) is an ideal of C(S) and we can repeat the above argument indefinitely to deduce that $L(C^n(I)) = L(C^n(S)) \cap I$ and $C^n(I) = C^n(S) \cap I$ for all n.

We saw above (Example 6.4) that a subsemigroup of an ultimately metrical semigroup, while it must be ultimately locally metrical, need not be ultimately metrical. By contrast we have

COROLLARY 6.2. Let S be an ultimately metrical semigroup, and let I be an ideal of S. Then I is ultimately metrical.

Proof. That I is ultimately locally metrical follows from lemma 6.7. Furthermore $L(C^n(I)) = L(C^n(S)) \cap I$, and so $L(C^n(I))$ is either empty or metrical for all n.

The next result is a slightly stronger version of corollary 6.2.

COROLLARY 6.3. Let S be a semigroup with ideal I. If I is not ultimately locally metrical, then neither is S and the height of S is equal to the height of I. Furthermore, if this is the case then for all n in $\mathbb{N}$, $L(C^n(S))$ is metrical if and only if $L(C^n((I))$ is metrical.

Proof. Suppose that I is not ULM. It follows from lemma 6.7 that S is not ULM. Suppose that I is of infinite height. Then by lemma 6.7 we have that S is of infinite height.

Suppose that I is of finite height. Notice first that

(*) $C^n(S) = \{ s \in C^{n-1}(S) : (\exists i \in C^{n-1}(I)) \ si \in C^n(I) \}$

since if $s \in C^{n-1}(S)$ and if i is an element of $C^{n-1}(I)$ (and therefore, by lemma 6.8, an element of $C^{n-1}(S)$), then

$si \in C^n(I)$

implies that $si \in C^n(S)$ by lemma 6.8

and so $s \in C^n(S)$,

and conversely if s is an element of $C^n(S)$ then taking any element j in $C^n(I)$ we have that $sj \in C^n(S) \cap I$, and so by lemma 6.8 again, $sj \in C^n(I)$. (The fact that I is not ULM ensures that $C^n(I)$ is not empty, and so j can always be found .)

We deduce that if $C^k(I) = C^{k+1}(I)$, then $C^k(S) =$

$C^{k+1}(S)$, and therefore that $k(S) \leqslant k(I)$. Since we always have $k(I) \leqslant k(S)$, we deduce that $k(S) = k(I)$, completing the proof of the first part of the corollary.

Now suppose that $L(C^n(I))$ is metrical for some n in $\mathbb{N}^0$.

Let $f: L(C^n(I)) \longrightarrow \mathbb{N}$ be a morphism.

Now (*) gives us that

$$L(C^n(S)) = \{ s \in C^n(S) : (\forall i \in C^n(I)) \ si \in L(C^n(I)) \}.$$

$L(C^n(I))$ is an ideal of $C^n(S)$, by lemma 6.8, and so we can extend f to a morphism $\tilde{f}: C^n(S) \longrightarrow \mathbb{N}^0$ in the usual way (see lemma 5.4).

Next notice that given any $i \in C^{n+1}(I)$ there exist r,s in $C^n(S)$ such that $ris \ \chi_{C^n(S)} \ r$, and so we must have that $\tilde{f}(i) = 0$.

Now for any s in $L(C^n(S))$ and any i in $C^{n+1}(I)$ we have that $si \in C^n(I)$, since $C^n(I)$ is an ideal of $C^n(S)$. Furthermore, we must have that $si \in L(C^n(I))$, since were si in $C^{n+1}(I)$ we would have, by (*), that $s \in C^{n+1}(S)$, which is not the case.

Finally notice that

$$\begin{aligned} & \tilde{f}(s) \\ = \ & \tilde{f}(si) - \tilde{f}(i) \\ = \ & \tilde{f}(si) \\ = \ & f(si) > 0. \end{aligned}$$

We deduce that the restriction of $\tilde{f}$ to $L(C^n(S))$ is strictly positive, and therefore that $L(C^n(S))$ is metrical, as required.

Conversely if $L(C^n(S))$ is metrical, then so is $L(C^n(I))$, since it is contained in $L(C^n(S))$.

Clearly a semigroup with an ultimately metrical ideal need not be ultimately metrical nor even ultimately locally metrical; for given any semigroup S disjoint from $\mathbf{N}$ we can have a multiplication * on $S \cup \mathbf{N}$ given by the multiplication on S, addition on $\mathbf{N}$, and the rule $s*n = n*s = n$ for all n in $\mathbf{N}$ and s in S. Then $\mathbf{N}$ is a metrical ideal of $S \cup \mathbf{N}$, but $C(S \cup \mathbf{N}) = S$ is entirely arbitrary.

LEMMA 6.9. Let S be a semigroup with ideal I. Then $x \in C(S)$ if and only if there exists z in S and w in I such that $wxz \, \chi_I \, w$.

Proof. The "if" part is obvious from the definition of C(S) and the fact that $\chi_I \subseteq \chi_S$.

For the other part suppose that x is in C(S). Then there exist y and z in S such that $yxz \, \chi_S \, y$, and so there exists an OLF $\{ x_i, u_i, v_i, y_i \quad i=1,..,n \}$ from yxz to y in S. Now let j be an arbitrary element of I. We show that $(j^4y)xz \, \chi_I \, j^4y$, completing the proof of the lemma.

First notice that

$$(j^4y)xz = j^4x_1u_1v_1y_1$$

$$\chi_I \quad j^2x_1u_1v_1y_1j^2 \, ,$$

since j^2, $j^2x_iu_iv_iy_i \in I$.

Similarly

$$j^2 x_n v_n u_n y_n j^2 \ \chi_I \ j^4 y$$

Then note that for $i = 1,\ldots,n$

$$j^2 x_i u_i v_i y_i j^2 \ \chi_I \ j x_i u_i j j v_i y_i j$$

$(\text{since } j, jx_iu_i, v_iy_ij \in I)$

$$\chi_I \ \ j x_i j v_i u_i j y_i j$$

$(\text{since } u_ij \text{ and } jv_i \in I)$

$$\chi_I \ \ j^2 x_i v_i u_i y_i j^2$$

$(\text{since } j, jx_i, y_ij \in I)$

$$= j^2 x_{i+1} u_{i+1} v_{i+1} y_{i+1} j^2$$

(since $\{ x_j, u_j, v_j, y_j \}$ is an OLF).

The required result now follows.

6.4. Products of semigroups.

In this section we give some basic lemmas concerning metricality and cartesian products. Our results concern products of two semigroups, but can be extended to the case of a finite cartesian product.

LEMMA 6.10. Let A and B be semigroups. Then $(a,b)\ \chi_{A\times B}\ (c,d)$ only if $a\ \chi_A\ c$ and $b\ \chi_B\ d$.

Proof. Suppose that $(a,b)\ \chi_{A\times B}\ (c,d)$. Notice that $(A\times B)^1$ is a submonoid of $A^1 \times B^1$, and so there exists an OLF from (a,b) to (c,d) in $A^1 \times B^1$, i.e. there exist

$$(x_i,x_i'),(y_i,y_i'),(u_i,u_i'),(v_i,v_i') \ \text{ in } A^1\times B^1 \quad i = 1,\ldots,n$$

such that

$$(x_1,x_1')(u_1,u_1')(v_1,v_1')(y_1,y_1') = (a,b) \;;$$

$$(x_i,x_i')(v_i,v_i')(u_i,u_i')(y_i,y_i')$$
$$= (x_{i+1},x_{i+1}')(u_{i+1},u_{i+1}')(v_{i+1},v_{i+1}')(y_{i+1},y_{i+1}')$$
$$i = 1,\ldots,n-1 \;;$$

and $(x_n,x_n')(v_n,v_n')(u_n,u_n')(y_n,y_n') = (c,d)$.

It follows that

$$a = x_1u_1v_1y_1 \;;$$
$$x_iv_iu_iy_i = x_{i+1}u_{i+1}v_{i+1}y_{i+1} \quad i = 1,..,n-1 \;;$$
$$x_nv_nu_ny_n = c \,,$$

and

$$b = x_1'u_1'v_1'y_1' \;;$$
$$x_i'v_i'u_i'y_i' = x_{i+1}'u_{i+1}'v_{i+1}'y_{i+1}' \quad i = 1,..,n-1 \;;$$
$$x_n'v_n'u_n'y_n' = d \,.$$

Thus we have an OLF from a to c in A and an OLF from b to d in B, and so $a \chi_A c$ and $b \chi_B d$ as required.

The converse of lemma 6.10 does not hold, as the following example illustrates:

EXAMPLE 6.6. Let A be the free semigroup $\{a\}^+$ and let B be the free semigroup $\{c,d\}^+$. Then clearly $a \chi_A a$, and $cd \chi_B dc$. Now (a,cd) and (a,dc) have no proper divisors in $A \times B$, and therefore by lemma 5.2 are not $\chi_{A\times B}$ related.

LEMMA 6.11. <u>Let A and B be semigroups. Then $C(A\times B)=C(A)\times C(B)$.</u>

Proof. Suppose that $(a,b)\in C(A\ B)$. Then there exist (c,d) and (e,f) in $A\times B$ such that $(c,d)(a,b)(e,f) \chi_{A\times B} (c,d)$. We deduce from lemma 6.10 that $cae \chi_A c$ and $dbf \chi_B d$. Thus $(a,b)\in C(A)\times C(B)$.

For the converse suppose that $a \in C(A)$ and $b \in C(B)$. Thus there exist c,e in A and d,f in B such that $cae \, \chi_A \, c$ and $dbf \, \chi_B \, d$. Let $\{ x_i, y_i, u_i, v_i : l = 1,..,m \}$ be an OLF from cae to c in A of minimal order. Then the OLF

$$\{ c^2 x_i, y_i c, u_i, v_i \}$$

is a heavy OLF from c^3aec to c^4 in A (recall from page 86 that an OLF in a semigroup S is heavy if it is completely contained in S). Thus we have a heavy OLF

$$\{ x_i', y_i', u_i', v_i' : i = 1,..,n \}$$

from c^3aec to c^4. Similarly we have a heavy OLF

$$\{ p_j, q_j, r_j, s_j : j = 1,..,m \}$$

from d^3bfd to d^4. We can suppose that $m = n$, since if, for example, m is less than n then we extend our OLF from d^3bfd to d^4 to one of order n by the addition of the OLF $\{ d,d,d,d : k = m+1,.,n \}$. We then have the (heavy) OLF

$$\{ (x_i', p_i), (y_i', q_i), (u_i', r_i), (v_i', s_i) \; i = 1,..,n \}$$

from (c^3aec, d^3bfd) to (c^4, d^4) in $A \times B$.

Since

$$\begin{aligned} &(c^3aec, d^3bfd) \\ &= (c^3ae, d^3bf)(c,d) \\ &\chi_{A \times B} \; (c,d)(c^3ae, d^3bf) \\ &= (c^4ae, d^4bf) \\ &= (c^4, d^4)(a,b)(e,f) \end{aligned}$$

we deduce that $(c^4,d^4)(a,b)(e,f) \, \chi_{A \times B} \, (c^4,d^4)$, and so (a,b) is in $C(A \times B)$.

COROLLARY 6.4. Let A,B be semigroups. Then $A \times B$ is locally metrical if and only if A or B is locally metrical.

Proof. $A \times B$ is locally metrical if and only if $C(A \times B)$ is empty; that is, if and only if either $C(A)$ or $C(B)$ is empty.

We also have

LEMMA 6.12. Let A,B be semigroups. Then $A \times B$ is metrical if and only if A or B is metrical.

Proof. The "if" part isobvious.

For the other part suppose that there exists a morphism $f\colon A \times B \longrightarrow \mathbf{N}$. Notice first that for all a,c in A and b,d in B

$$(a,b^2)(c,b) = (a,b)(c,b^2)$$

and

$$(a^2,b)(a,d) = (a,b)(a^2,d)$$

and therefore

$$\left.\begin{aligned} f(a,b^2) - f(a,b) &= f(c,b^2) - f(c,b) \\ &\text{and} \\ f(a^2,b) - f(a,b) &= f(a^2,d) - f(a,d) \end{aligned}\right\} \quad (*)\ .$$

Now for any a in A and b in B and $n > 1$ we have

$$(a,b^n)(a,b) = (a,b^{n-1})(a,b^2)$$

and so

$$f(a,b^n) = f(a,b^{n-1}) + f(a,b^2) - f(a,b)$$

which gives us that

$$f(a,b^n) = f(a,b) + (n-1)(\ f(a,b^2) - f(a,b)\ ).$$

Since $f > 0$ we deduce that $f(a,b^2) \geqslant f(a,b)$.

Similarly

$$f(a^2,b) \geqslant f(a,b) \qquad (**)$$

for all $a \in A$, $b \in B$.

Suppose that there exists a in A such that for all b in B,

$$f(a,b^2) > f(a,b).$$

We then define $\tilde{f}: B \longrightarrow \mathbb{N}$ by

$$\tilde{f}(b) = f(a,b^2) - f(a,b).$$

This is a morphism since

$$\tilde{f}(bc) = f(a,bcbc) - f(a,bc)$$

$$= f(a^4,bcbc) - f(a^4,bc) \quad \text{from } (*)$$

$$= f(a^2,b^2) - f(a^2,b) + f(a^2,c^2) - f(a^2,c)$$

$$= \tilde{f}(b) + \tilde{f}(c)$$

and so B is metrical. Similarly if for a given b in B we have for all a in A that

$$f(a^2,b) > f(a,b) \qquad (***)$$

then A is metrical.

Finally suppose that B is not metrical. Then for all a in A there exists b in B such that

$$f(a,b^2) = f(a,b).$$

Then by (*) we have that for all x in A,

$$f(x,b^2) = f(x,b).$$

Now if A is not metrical then from (**) and (***) there must exist x in A such that

$$f(x^2,b^2) = f(x,b^2)$$

and we therefore deduce that

$$f((x,b)(x,b)) = f(x,b)$$

which is not possible.

Thus if $A \times B$ is metrical we must have that either A or B is metrical.

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