IDEMPOTENTS, NILPOTENTS, RANK AND ORDER IN FINITE TRANSFORMATION SEMIGROUPS

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A thesis submitted for the degree of Doctor of Philosophy of the University of St. Andrews

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DECLARATION

I declare that the accompanying thesis has been composed by myself and that it is record of my own work. No part of the thesis has been accepted in any previous application for a higher degree.

Goje Uba Garba

DECLARATION

I declare that I was admitted in March 1989 under Court Ordinance General Number 12 as a full-time research student in the Department of Mathematical and Computational Sciences.

Goje Uba Garba

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ABSTRACT

Let E, E_1 denote, respectively, the set of singular idempotents in T_n (the semigroup of all full transformations on a finite set $X_n = \{1, \ldots, n\}$) and the set of idempotents of defect 1. For a singular element α in T_n, let $k(\alpha), k_1(\alpha)$ be defined by the properties

$$\alpha \in E^{k(\alpha)}, \qquad \alpha \notin E^{k(\alpha)-1},$$

$$\alpha \in E_1^{k_1(\alpha)}, \qquad \alpha \notin E_1^{k_1(\alpha)-1}.$$

In this Thesis, we obtain results analogous to those of Iwahori (1977), Howie (1980), Saito (1989) and Howie, Lusk and McFadden (1990) concerning the values of $k(\alpha)$ and $k_1(\alpha)$ for the partial transformation semigroup P_n. The analogue of Howie and McFadden's (1990) result on the rank of the semigroup $K(n,r) = \{\alpha \in T_n : |\operatorname{im} \alpha| \leq r, 2 \leq r \leq n-1\}$ is also obtained.

The nilpotent-generated subsemigroup of P_n was characterised by Sullivan in 1987. In this work, we have obtained its depth and rank.

Nilpotents in IO_n and PO_n (the semigroup of all partial one-one order-preserving maps, and all partial order-preserving maps) are studied. A characterisation of their nilpotent-generated subsemigroups is obtained. So also are their depth and rank. We have also characterised their nilpotent-generated subsemigroup for the infinite set $X = \{1, 2, \ldots\}$.

The rank of the semigroup $L(n,r) = \{\alpha \in S : |\operatorname{im} \alpha| \leq r, 1 \leq r \leq n-2\}$ is investigated for $S = O_n, PO_n, SPO_n$ and I_n (where O_n is the semigroup of all order-preserving full transformations, SPO_n the semigroup of all strictly partial order-preserving maps, and I_n the semigroup of one-one partial transformation).

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INTRODUCTION

Since the paper by Howie **[16]** in 1966, establishing that every singular selfmap of a finite set $X_n = \{1, \ldots, n\}$ is expressible as a product (i. e. a composition) of idempotent selfmaps, there have been many articles on this topic. We draw attention in particular to Iwahori **[23]**, Howie **[19]**, Saito **[25]**, Howie, Lusk and McFadden **[20]** and Howie and McFadden **[21]**.

All of these papers are concerned with the full transformation semigroup T_n consisting of all maps $\alpha : X_n \to X_n$. Evseev and Podran **[8]** considered the larger semigroup P_n consisting of all partial maps from X_n into itself and showed that there too all elements other than permutations of X_n are expressible as products of idempotents. However, the considerable developments that have taken place concerning total transformations have not as yet been matched by corresponding progress in the partial case.

In fact it turns out to be relatively easy to prove results of P_n corresponding to those for T_n. By a result of Vagner **[29]**, quoted in Clifford and Preston **[3]**, there is an isomorphism between P_n and a subsemigroup P_n^* of the full transformation semigroup U_n consisting of all maps $\alpha : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ for which $0\alpha = 0$. This isomorphism proves to be a powerful tool in translating results on T_n to very similar results concerning P_n.

Let SP_n be the semigroup of all strictly partial transformations on the set X_n. An element α in P_n is said to have *projection characteristic* (k, r) or to belong to $[k, r]$ if $|\operatorname{dom} \alpha| = k$ and $|\operatorname{im} \alpha| = r$. Every element $\alpha \in [n-1, n-1]$ has domain $X_n \backslash \{i\}$ and image $X_n \backslash \{j\}$ for some i, j in X_n. Hence there is a unique element $\bar{\alpha}$ in $[n, n]$ associated with α, defined by

$$i\bar{\alpha} = j, \qquad x\bar{\alpha} = x\alpha \quad \text{otherwise},$$

and called the *completion* of α. In **[12]** Gomes and Howie proved that if n is even the subsemigroup SI_n of P_n consisting of all strictly partial one-one transformations is nilpotent generated. For n odd they showed that the nilpotents in SI_n generate $SI_n \backslash W_{n-1}$ where W_{n-1} consists of all $\alpha \in [n-1, n-1]$ whose completions are odd permutations. They also proved that the depth of $\langle N \rangle$, denoted by $\Delta(\langle N \rangle)$ (where N is the set of all nilpotent elements in SI_n), which is the unique k for which

$$\langle N \rangle = N \cup N^2 \cup \cdots \cup N^k \text{ and } \langle N \rangle \neq N \cup N^2 \cup \cdots \cup N^{k-1},$$

to be equal to 3 or 2 according to n is even or odd.

Simultaneously and independently, Sullivan **[26]** investigated the corresponding question for SP_n, the subsemigroup of P_n consisting of all elements that are stictly partial, where the answer turns out to be similar: If N is the set of nilpotents in SP_n then

$$\langle N \rangle = \begin{cases} SP_n & \text{if } n \text{ is even,} \\ SP_n \backslash W_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

In **[13]** Gomes and Howie raised the question of the rank of the semigroup Sing_n (the semigroup of all singular selfmap on X_n) and that of SI_n. They showed that both the rank and the idempotent rank of $Sing_n$ are equal to $\frac{1}{2}n(n-1)$. On the other hand they showed that SI_n has rank $n+1$, and if n is even its nilpotent rank is also $n+1$. For n odd they showed that both the rank and the nilpotent rank of $SI_n \backslash W_{n-1}$ are equal to $n+1$. Their result for Sing_n was later generalised by Howie and McFadden **[21]**, who showed that the rank and the idempotent rank of $K(n,r) = \{\alpha \in \mathrm{Sing}_n : |\mathrm{im}\,\alpha| \leq r, 1 \leq r \leq n-1\}$ are both equal to $S(n,r)$, the Stirling number of the second kind.

In another paper **[14]**, Gomes and Howie investigated the rank of the semigroup O_n, PO_n and SPO_n (the semigroup of order-preserving full transformations, order-preserving partial transformations and order-preserving strictly partial transformations on X_n). They showed that the rank of O_n is n, that of PO_n is $2n-1$ and SPO_n has rank $2n-2$. The idempotent rank of O_n is $2n-2$ (O_n was proved to be idempotent-generated by Howie [17]), PO_n is idempotent-generated and its idempotent rank is $3n-2$. The semigroup SPO_n is not idempotent-generated and so the question of its idempotent rank does not arise.

Let E, E_1 denote, respectively, the set of singular idempotents in T_n and the set of idempotents of defect 1. For a singular element α in T_n, let $k(\alpha)$, $k_1(\alpha)$ be defined by the properties

$$\alpha \in E^{k(\alpha)}, \qquad \alpha \notin E^{k(\alpha)-1},$$

$$\alpha \in E_1^{k_1(\alpha)}, \qquad \alpha \notin E_1^{k_1(\alpha)-1}.$$

In Chapter 1, we show how the results of Iwahori [23], Howie [19], Saito [25] and Howie, Lusk and McFadden [20] concerning the values of $k(\alpha)$ and $k_1(\alpha)$ can be modified to deal with the partial case.

For $2 \leq r \leq n-1$, let

$$K(n,r) = \{\alpha \in T_n : |\operatorname{im} \alpha| \leq r\}.$$

Then $K(n,r)$ is a semigroup, whose rank was shown by Howie and McFadden [21] to be the Stirling number $S(n,r)$ of the second kind. We show a very similar result about the partial transformation semigroup P_n.

In Chapter 2, we extend the result of Gomes and Howie [13] on the depth, rank and nilpotent rank of the nilpotent-generated subsemigroup of SI_n to the partial transformation semigroup P_n. We have also generalised the idea of a rank in line with Howie and McFadden to the nilpotent-generated subsemigroup.

Let N be the set of all nilpotent elements in IO_n (the semigroup of all partial one-one order-preserving transformations on X_n). In Chapter 3, a description of the subsemigroup of IO_n generated by the set N is given. The set $\{\alpha \in IO_n : |\operatorname{im} \alpha| \leq r\}$ is shown to be contained in $\langle N \rangle$ if and only if $r \leq n/2$. The depth of $\langle N \rangle$ is shown to be equal to 3 for all $n \geq 3$. The rank and the nilpotent rank of $\{\alpha \in IO_n : |\operatorname{im} \alpha| \leq r \text{ and } r \leq n/2\}$ are shown to be both equal to $\binom{n}{r} - 1$, and for the set $\{\alpha \in IO_n : |\operatorname{im} \alpha| \leq r, \alpha \in \langle N \rangle \text{ and } n/2 < r \leq n-2\}$ they are shown to be both equal to $\binom{n}{r} - \binom{r-1}{n-r} - 1$.

In Chapter 4, we give a description of the subsemigroup of PO_n generated by the set M of its nilpotent elements. The set $\{\alpha \in PO_n : |\operatorname{im} \alpha| \leq r\}$ is shown to be contained in $\langle M \rangle$ if and only if $r \leq n/2$ and $|X \setminus \operatorname{dom} \alpha| \geq r$. The depth of $\langle M \rangle$ is shown to be equal to 3 for all $n \geq 3$. The rank of the set $\{\alpha \in PO_n : |\operatorname{im} \alpha| \leq n-2 \text{ and } \alpha \in \langle M \rangle\}$ is shown to be equal to $6(n-2)$, and its nilpotent rank to be equal to $7n - 15$.

The results of Gomes and Howie on the rank of O_n, PO_n, SPO_n and SI_n are generalised in Chapter 5. If we let

$$L(n, r) = \{\alpha \in S : |\operatorname{im} \alpha| \leq r \text{ and } r \leq n-2\}.$$

Then we have shown that for $S = O_n$, the rank and the idempotent rank of $L(n, r)$ are both equal to $\binom{n}{r}$. For $S = PO_n$ they are shown to be both equal to $\sum_{k=r}^{n} \binom{n}{k}\binom{k-1}{r-1}$. For $S = SPO_n$ they are shown to be both equal to $\sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}$. The rank and the nilpotent rank of $L(n, r)$ are also shown to be both equal to $\binom{n}{r} + 1$.

Related questions have been considered in recent years for different types of finite semigroups of transformations. For example Erdos [7], Dawlings [5, 6] and Ballantine [1] studied the semigroup of endomorphisms of a finite dimensional vector space over a field F. Howie and Schein [22] studied the semigroup of forgetful endomorphisms of a finite Boolean algebra. Umar [27, 28] considered the semigroup of order-decreasing finite transformations for both full and partial one-one transformations on the set X_n.

CHAPTER ONE

IDEMPOTENTS IN PARTIAL TRANSFORMATION SEMIGROUPS

1. Products of idempotents

Let E be the set of all idempotents of $P_n \backslash [n, n]$ and E_1 the set of all idempotents of $P_n \backslash [n, n]$ with projection characteristic $(n, n-1)$ or $(n-1, n-1)$. It has been shown in [8] that E, and even E_1 generates $P_n \backslash [n, n]$. We denote by U_n the semigroup of all full transformations of X_0, where $X_0 = X_n \cup \{0\}$.

Now for convenience we record the result of Vagner [29] (also to be found in [3, p254]).

Theorem 1.1.1 *For each α in P_n, define the transformation α^* of X_0 by*

$$x\alpha^* = \begin{cases} x\alpha & \text{if } x \in \text{dom } \alpha, \\ 0 & \text{if } x \notin \text{dom } \alpha. \end{cases}$$

Then α^* belongs to the subsemigroup P_n^* of U_n consisting of all those transformations of X_0 leaving 0 fixed. Conversely if $\beta \in P_n^*$, then its restriction to X_n,

$$\beta | X_n = \beta \cap (X_n \times X_n)$$

is a partial transformation of X_n. The domain of $\beta | X_n$ is the set of all x in X_n for which $x\beta \neq 0$. Then the mapping $\alpha \to \alpha^*$ and $\beta \to \beta | X_n$ are mutually inverse isomorphisms of P_n onto P_n^* and vice–versa. ■

Following Howie and McFadden **[21]**, we shall denote $|\operatorname{im}\alpha|$ by $h(\alpha)$ and refer to it as the *height* of α. Then an idempotent η of height n in P_n^* corresponds in this isomorphism either to an idempotent in $[n, n-1]$ (if $0\eta^{-1} = \{0\}$) or to an idempotent in $[n-1, n-1]$ (if $|0\eta^{-1}| = 2$). So for example

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}^* \qquad (\text{in } [3,2]),$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}^* \qquad (\text{in } [2,2]). \qquad \blacksquare$$

Since E_1 is the set of idempotents in $[n, n-1] \cup [n-1, n-1]$, we define E_1^* to be the set of idempotents in P_n^* of height n.

Now let α be an element of $P_n \backslash [n, n]$. Then α^* is a singular element of U_n with the property that $0\alpha^* = 0$. By the method of Howie **[19]** α^* can be expressed as a product $\xi_1 \xi_2 \cdots \xi_m$ of idempotents in U_n and the method ensures that $0\xi_i = 0$ for all i. Hence $\xi_i = \epsilon_i^*$ for some idempotent ϵ_i in $P_n \backslash [n, n]$. Moreover, if ξ_i is an idempotent of height n in U_n then ϵ_i has projection characteristic (n, n-1) or (n-1, n-1) as remarked earlier. So every singular element of P_n^* is a product of idempotents from E_1^*.

Associated with each α in P_n is a digraph $\Gamma(\alpha)$ whose vertices are $1, 2, \cdots, n$ and where (i, j) is a (directed) edge if and only if $i\alpha = j$. A connected component Ω of $\Gamma(\alpha)$ has a *core* $k(\Omega)$ defined by the property that $x(\in \Omega)$ is in $k(\Omega)$ if and only if there exists $p > 0$ such that $x\alpha^p = x$. In the terminology of **[19]**, Ω is called *standard* if $1 < |k(\Omega)| < |\Omega|$, *acyclic* if $1 = |k(\Omega)| < |\Omega|$, *cyclic* if $1 < |k(\Omega)| = |\Omega|$, and *singleton* if $1 = |k(\Omega)| = |\Omega|$. On the other hand it is clear that a connected component Ω of $\Gamma(\alpha)$ has no core if and only if there exists a positive integer $p > 0$ such that $x\alpha^p = 0$ for all $x \in \Omega$. We shall refer to such Ω as *terminal*. For example if

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 2 & 6 & 6 & 8 & 7 & 9 & 11 \end{pmatrix}$$

then $\Gamma(\alpha)$ is as shown in Fig1 (see Appendix), and the components, reading from left to right are respectively standard, acyclic, cyclic, singleton, and terminal.

Definition 1.1.2 For an element $\alpha \in P_n$ we define the *gravity* of α, denoted by $g(\alpha)$, as $n + c(\alpha) - f(\alpha)$. (Here $c(\alpha)$ is the number of cyclic components of $\Gamma(\alpha)$ and $f(\alpha)$ the number of fixed points of α). The *defect* of α, denoted by $d(\alpha)$, is defined as $|X \backslash \mathrm{im}\, \alpha|$.

The following lemma is obvious:

Lemma 1.1.3 $c(\alpha^*) = c(\alpha), \;\; f(\alpha^*) = f(\alpha) + 1.$ ■

As a consequence we have

$$g(\alpha^*) = (n+1) + c(\alpha^*) - f(\alpha^*) = (n+1) + c(\alpha) - (f(\alpha) + 1)$$
$$= n + c(\alpha) - f(\alpha) = g(\alpha).$$

Lemma 1.1.4 $d(\alpha^*) = d(\alpha)$.

Proof.

$$d(\alpha^*) = |X_0 \backslash \mathrm{im}\, \alpha^*| = |(X \cup \{0\}) \backslash (\mathrm{im}\, \alpha \cup \{0\})|$$
$$= |X \backslash \mathrm{im}\, \alpha| = d(\alpha)$$

. ■

We have the following result from **[19 & 23]**.

Theorem 1.1.5 **[19,** Theorem 3.1] *Let* Sing_n *denote the semigroup of singular mappings from* X_n *into* X_n*, and let* F *denote the set of idempotents of height* $n-1$ *in*

Sing_n. *For each* α *in* Sing_n *the least* k *for which* $\alpha \in F^k$ *is* $k = g(\alpha)$, *where* $g(\alpha)$ *is the gravity of* α. ■

Theorem 1.1.6 *For all* α *in* $P_n \backslash [n, n]$,

$$\alpha \in E_1^{g(\alpha)}, \quad \alpha \notin E_1^{g(\alpha)}.$$

Proof. Since $\alpha \notin [n, n]$, α^* is a singular element of P_n^*. By Theorem 1.1.5 and the remark made on page 6, α^* is expressible as a product $\epsilon_1^* \epsilon_2^* \cdots \epsilon_k^*$ of idempotents in P_n^* of height n, where $k = g(\alpha^*)$. Hence by the isomorphism

$$\alpha = \epsilon_1 \epsilon_2 \cdots \epsilon_k$$

and $k \geq g(\alpha)$.

Suppose by way of contradiction that $\alpha = \epsilon_1 \cdots \epsilon_l$, with $\epsilon_1, \cdots, \epsilon_l \in E_1$ and $l < g(\alpha)$. Then $\alpha^* = \epsilon_1^* \cdots \epsilon_l^*$, with $\epsilon_1^*, \cdots, \epsilon_l^* \in E_1^*$. This is a contradiction. ■

Theorem 1.1.7 [25, Theorem9] *Let* Sing_n *be the semigroup of all singular mappings from* X_n *into* X_n, *and let* F *be the set of idempotents of* Sing_n. *For* α *in* Sing_n, *let* $k(\alpha)$ *be the unique positive integer for which* $\alpha \in F^{k(\alpha)}, \alpha \notin F^{k(\alpha)-1}$ *and* $g(\alpha)$ *the gravity of* α *and* $d(\alpha)$ *the defect of* α. *Then* $k(\alpha) = \lceil g(\alpha)/d(\alpha) \rceil$ *or* $\lceil g(\alpha)/d(\alpha) \rceil + 1$, *where* $\lceil x \rceil$ *for any real number* x *denotes the least integer* m *for which* $m \geq x$. ■

Theorem 1.1.8 *For all* $\alpha \in P_n \backslash [n, n]$, *let* $k(\alpha)$ *be defined by the property that*

$$\alpha \in E^{k(\alpha)}, \qquad \alpha \notin E^{k(\alpha)-1}.$$

Then $k(\alpha) = \lceil g(\alpha)/d(\alpha) \rceil$ *or* $\lceil g(\alpha)/d(\alpha) \rceil + 1$.

Proof. Since $\alpha \notin [n, n]$, α^* is a singular element of P_n^*. By Theorem 1.1.7 and the remark made on page 6, α^* is expressible as a product $\epsilon_1^* \cdots \epsilon_k^*$ of idempotents in E^*, where $k = \lceil g(\alpha^*)/d(\alpha^*) \rceil$ or $\lceil g(\alpha^*)/d(\alpha^*) \rceil + 1$. Hence by the isomorphism

$$\alpha = \epsilon_1 \cdots \epsilon_k,$$

and $k = \lceil g(\alpha^*)/d(\alpha^*) \rceil$ or $\lceil g(\alpha^*)/d(\alpha^*) \rceil + 1$, i.e

$$k = \lceil g(\alpha)/d(\alpha) \rceil \text{ or } \lceil g(\alpha)/d(\alpha) \rceil + 1.$$

■

Theorem 1.1.9 [20, Theorem 2.1] *Let $\alpha \in \mathrm{Sing}_n$ and $p \geq 1$. If*

$$g(\alpha) \leq (p-1)d(\alpha) + 1$$

then $\alpha \in F^p$. ■

Theorem 1.1.10 *Let $\alpha \in P_n \backslash [n, n]$ and $p \geq 1$. If*

$$g(\alpha) \leq (p-1)d(\alpha) + 1$$

then $\alpha \in E^p$.

Proof. The element α^* is singular and $g(\alpha^*) = g(\alpha)$, $d(\alpha^*) = d(\alpha)$. Thus if $g(\alpha) \leq (p-1)d(\alpha) + 1$, then equivalently we have $g(\alpha^*) \leq (p-1)d(\alpha^*) + 1$. Hence from Theorem 1.1.9 $\alpha^* \in (E^*)^p$, and from the isomorphism we have $\alpha \in E^p$. ■

Theorem 1.1.11 *Let $\alpha \in P_n \backslash [n, n]$ and $d(\alpha) = d$. If $n - d$ is even, then*

$$g(\alpha) \leq \frac{1}{2}(3n - d).$$

If $n-d$ is odd, then

$$g(\alpha) \leq \frac{1}{2}(3n-d-1).$$

Proof. Let $\alpha \in P_n \backslash [n, n]$ and let $d(\alpha) = d$. Suppose first that $n-d$ is even. Then $(n+1)-d$ is odd. However, every α^* must contain at least one fixed point, namely 0, and so, arguing as in the proof of [**20**, Theorem 2.2], we see that the unique way of obtaining an element of P_n^* of defect d and maximum gravity is by forming $(n-d)/2$ 2–cycles and mapping the remaining $d+1$ elements to 0. This gives

$$g(\alpha) = g(\alpha^*) = (n+1) + (n-d)/2 - 1 = (3n-d)/2.$$

Suppose now that $n-d$ is odd. Again α^* has at least one fixed point. This time, as in [**20**], there are several ways of achieving an element of defect d and maximum gravity. One way is to form $(n-d-1)/2$ 2–cycles, and to choose z outside these 2–cycles, mapping z to 0 and all other elements to z. No other device will produce an element of greater gravity. The gravity we obtain in this way is

$$(n+1) + (n-d-1)/2 - 1 = (3n-d-1)/2.$$ ■

Theorem 1.1.12 *Let $\alpha \in P_n \backslash [n, n]$ and let $p \geq 3$. If $h(\alpha)$ is even and*

$$h(\alpha) \leq \frac{1}{2p-1}((2p-4)n+2),$$

then $\alpha \in E^p$. If $h(\alpha)$ is odd and

$$h(\alpha) \leq \frac{1}{2p-1}((2p-4)n+3),$$

then $\alpha \in E^p$.

Proof. If $h(\alpha)(= n-d)$ is even and if

$$h(\alpha) \leq \frac{1}{2p-1}((2p-4)n+2),$$

then

$$(2p-1)(n-d) \leq (2p-4)n+2$$

which after rearrangement gives

$$3n-d \leq 2(p-1)d+2.$$

Hence by Theorem 1.1.10 and 1.1.11

$$g(\alpha) \leq \frac{1}{2}(3n-d) \leq (p-1)d+1,$$

and so $\alpha \in E^p$.

The case where $h(\alpha)$ is odd is similar. ∎

2. Elements having maximum gravity and small gravity

In this section, we shall determine the number of elements in $P_n \backslash [n, n]$ that have maximum gravity and formulae for the number of elements of small gravity.

From Theorem1.1.11 (proof), those elements in P_n^* having maximum gravity are to be found in the $\mathcal{D}$-class D_n of U_n. As in [20], a typical $\mathcal{R}$-class within this $\mathcal{D}$-class denoted by $R(\{i,j\})(0 \leq i,j \leq n, i \neq j)$; it consists of all elements α for which $k(\alpha)$ is the equivalence whose only non-singleton class is $\{i,j\}$. A typical $\mathcal{L}$-class within D_n may be denoted by $L(i)(0 \leq i \leq n)$; it consists of all α for which $\operatorname{im}\alpha = X_n \backslash \{i\}$. There are $\binom{n+1}{2}$ $\mathcal{R}$-classes, and each $\mathcal{H}$-class contains $n!$ elements. Those elements in P_n^* are to be found in all the $\binom{n+1}{2}$ $\mathcal{R}$-classes and n of the $\mathcal{L}$-classes (since $L(0)$ cannot have 0 as a fixed point). Also from [20], a $\mathcal{H}$-class $H(\{i,j\},k)$ is a group $\mathcal{H}$-class if and only if $k \in \{i,j\}$. It therefore follows that for the group $\mathcal{H}$-class $H(\{i,j\},k)$ to contain an element in P_n^*, k must be different from zero. Hence there are n group $\mathcal{H}$-classes with no element from P_n^* and $n(n+1) - n(= n^2)$ group $\mathcal{H}$-classes having elements from P_n^*. Similarly, the $\mathcal{L}$-class $L(0)$ contains no non-group $\mathcal{H}$-class having elements from P_n^*. The number of non-group $\mathcal{H}$-classes in $L(0)$ is $\binom{n+1}{2} - n(= \frac{n(n-1)}{2})$. Thus the total number of non-group $\mathcal{H}$-classes having elements from P_n^* is $\frac{n^2(n-1)}{2}$.

As in the proof of Theorem 1.1.11 we shall find elements of maximum gravity by having as many cyclic components as possible and as few fixed points as possible. This will frequently involve constructing a number p of 2-cycles from $2p$ elements of X_n, and it may be useful to note now that the number of ways of doing this is $(2p)!/(p!2^p)$.

Lemma 1.2.1 *Let $n = 2m$ be even, with $m \geq 2$. Then the number of elements of P_n^* with maximum gravity (equal to $3m-1$) in a single group $\mathcal{H}$-class is*

$$\frac{(2m-1)!}{3 \cdot 2^{m-2}(m-2)!}$$

and the number in a non-group $\mathcal{H}$-class is

$$\frac{(2m-2)!}{2^{m-1}(m-1)!}.$$

The total number is

$$\frac{(2m+1)!}{3 \cdot 2^{m-1}(m-1)!}.$$

Proof. Consider first a typical group $\mathcal{H}$-class $H(\{i,j\},i)$ in P_n^* (where $i = 0$). Within this $\mathcal{H}$-class the only configuration that leads to maximum gravity is as shown in figure 2 (see Appendix), so the number of elements of maximum gravity in this class is

$$\frac{(2m-1)!}{3 \cdot 2^{m-2}(m-2)!}.$$

Now consider a typical non-group $\mathcal{H}$-class $H(\{i,j\},k)$ in P_n^*, where $k \neq i,j$. Within this $\mathcal{H}$-class the only configuration that leads to maximum gravity is as shown in figure 3 (see Appendix). So the number of elements of maximum gravity in this class is

$$\frac{(2m-2)!}{2^{m-1}(m-1)!}.$$

Notice that once i is fixed, so is the group $\mathcal{H}$-class $H(\{i,0\},i)$. So there are $2m$ group $\mathcal{H}$-classes having elements from P_n^* with maximum gravity. On the other hand, an element α^* whose configuration is given by figure 3 belongs to the non-group $\mathcal{H}$-class $H(\{i,0\},k)$. So there are $2m(2m-1)$ non-group $\mathcal{H}$-classes having elements from P_n^* with maximum gravity, arising from the choise of i and k. Hence the total number of elements in P_n^* with maximum gravity is

$$\frac{(2m+1)!}{3\cdot 2^{m-1}(m-1)!}.$$

Lemma 1.2.2 *Let $n=2m+1$ be odd, with $m\geq 0$. Then the elements of P_n^* with maximum gravity (equal to $3m+1$) are all group elements, and the number in any group $\mathcal{H}$-class is*

$$\frac{(2m)!}{2^m m!}.$$

The total number of elements in P_n^* with maximum gravity is

$$\frac{(2m+1)!}{2^m m!}.$$

Proof. Here there is only one configuration giving maximum number of gravity (see Fig 4 in the Appendix).

All elements having this configuration are group elements. The number of them in any $\mathcal{H}$-class containing them is

$$\frac{(2m)!}{2^m m!}.$$

The total number of them in P_n^* is

$$\frac{(2m)!(2m+1)}{2^m m!} = \frac{(2m+1)!}{2^m m!}.$$

■

Theorem 1.2.3 *Let $\alpha \in P_n \backslash [n, n]$. If $n = 2m$ with $m \geq 2$, then the total number of elements in $P_n \backslash [n, n]$ with maximum gravity (equal to $3m - 1$) is*

$$\frac{(2m+1)!}{3 \cdot 2^{m-1}(m-1)!}.$$

If $n = 2m + 1$ with $m \geq 0$, then the total number of elements in $P_n \backslash [n, n]$ with maximum gravity (equal to $3m + 1$) is

$$\frac{(2m+1)!}{2^m m!}.$$

Proof. The results follows from Lemmas 1.2.1 and 1.2.2 by the isomorphism. ■

We now look at the singular elements of P_n^* and determine those of them with minimum gravity .

Let $N^*(n+1, h, g)$ denote the number of elements $\alpha^* \in P_n^*$ such that $h(\alpha^*) = h$ and $g(\alpha^*) = g$.

Lemma 1.2.4 *For all $n \geq 1$,*

$$N^*(n+1, n, 1) = n^2.$$

Proof. The only configuration that leads to gravity 1 is as shown in Fig 4 (see Appendix).

The number of elements corresponding to this configuration is

$$n \cdot n = n^2.$$

■

Lemma 1.2.5 *For all $n \geq 2$,*

$$N^*(n+1, n, 2) = n(n-1)^2$$

Proof. For $g(\alpha^*)$ to equal 2 we require to have $f(\alpha^*) - c(\alpha^*) = (n+1) - 2 = n-1$. If there were n fixed points and one cycle the total number of elements in X_0 would have to be at least $n+2$, which is not possible. So α^* must have $n-1$ fixed points and no cycles. Since $h(\alpha^*) = n$, the only possible configuration is as shown in Fig 6 (see Appendix). The number of elements corresponding to this configuration is

$$n(n-1)(n-1) = n(n-1)^2.$$

■

Lemma 1.2.6 *For all $n \geq 3$,*

$$N^*(n+1, n, 3) = n(n-1)^2(n-2).$$

For all $n \geq 4$,

$$N^*(n+1, n, 4) = \frac{1}{2}n(n-1)(n-2)(2n^2 - 7n + 4).$$

Proof. Let $h(\alpha^*) = n$ and $g(\alpha^*) = 3$, so that $f(\alpha^*) - c(\alpha^*) = n-2$. If we suppose there are $n-1$ fixed points and one cycle, then this would mean that all $n+1$ elements of X_0 lie in $\mathrm{im}(\alpha^*)$, and this cannot occur. There must therefore be $n-2$ fixed points and no cycles. The configurations which give this are shown in Fig 7 (see Appendix).

The number of elements with these configuration is

$$n(n-1)(n-2) + n(n-1)(n-2)(n-2) = n(n-1)^2(n-2).$$

Now let $h(\alpha^*) = n$ and $g(\alpha^*) = 4$, so that $f(\alpha^*) - c(\alpha^*) = n-3$. Here we have $n-3$ fixed points and no cycles or $n-2$ fixed points and one cycle. The possible configurations are shown in Fig 8 (see Appendix).

The total number of elements with these configurations is

$$\begin{aligned} &n(n-1)(n-2)(n-3) + n(n-1)(n-2)(n-3) \\ &+ n(n-1)(n-2)(n-3)(n-3) + n(n-1)(n-2)(n-2)/2 \\ &= \frac{1}{2}n(n-1)(n-2)(2n^2-7n+4). \end{aligned}$$

Note that an element $\alpha^* \in P_n^*$ of height $r+1$ corresponds to an element $\alpha \in [k, r]$ where $r \leq k \leq n$. ■

Theorem 1.2.7 *Let $\alpha \in P_n \backslash [n, n]$ and let $N(p, r, g)$ be the number of elements in $P_n \backslash [n, n]$ for which $\alpha \in [p, r]$ and $g(\alpha) = g$. Also let $T(r, g)$ be the total number of elements in $P_n \backslash [n, n]$ for which $g(\alpha) = g$ and $h(\alpha) = r$. Then:*

(a) *For all $n \geq 1$,*

$$N(n, n-1, 1) = n(n-1);\quad N(n-1, n-1, 1) = n \text{ and } T(n-1, 1) = n^2.$$

(b) *For all $n \geq 2$,*

$$N(n, n-1, 2) = n(n-1)(n-2);\quad N(n-1, n-1, 2) = n(n-1)$$
$$\text{and } T(n-1, 2) = n(n-1)^2.$$

(c) (i) *For all* $n \geq 3$,

$$N(n, n-1, 3) = n(n-1)(n-2)^2; \;\; N(n-1, n-1, 3) = n(n-1)(n-2)$$
$$\text{and } T(n-1, 3) = n(n-1)^2(n-2).$$

(ii) *For all* $n \geq 4$,

$$N(n, n-1, 4) = n(n-1)(n-2)(n-3)(2n-3)/2;$$
$$N(n-1, n-1, 4) = n(n-1)(n-2)(2n-5)/2$$
$$\text{and } T(n-1, 4) = n(n-1)(n-2)(2n^2 - 7n + 4)/2.$$

Proof. The result (a), (b) and (c) follows from Lemma 1.2.4, 1.2.5 and 1.2.6 respectively, and the configurations given in the Lemmas as well as the isomorphism. ■

Let us now look at the bottom end of P_n^* rather than at the top. If $h(\alpha^*) = 1$ then α^* is necessarily idempotent. There is one fixed point 0 and there are no cycles; hence

$$g(\alpha^*) = (n+1) - 1 = n \text{ and } N^*(n+1, 1, n) = 1. \tag{1.2.8}$$

Lemma 1.2.9 *Let* $\alpha^* \in P_n^*$ *with* $h(\alpha^*) = 2$. *Then* $g(\alpha^*) = n$ *or* $n-1$.

Proof. Since $0 \in \operatorname{im} \alpha^*$ and is a fixed point, there cannot be any cycle, for every cycle contributes two elements to the image and this will mean $h(\alpha^*) = 3$. Hence $g(\alpha^*) = n$ (one fixed point 0) or $n-1$ (two fixed points, 0 and another element). We conclude that $N^*(n+1, 2, g) = 0$ except when $g = n$ or $n-1$. ■

Lemma 1.2.10 *Let* $n \geq 2$. *Then*

$$N^*(n+1, 2, n) = (2^{n-1} - 1)n; \;\; N^*(n+1, 2, n-1) = 2^{n-1} n.$$

Proof. Suppose first that $h(\alpha^*) = 2$, $g(\alpha^*) = n$. There cannot be more than one component of α^*, since two acyclic or singleton components would imply that there were two fixed points, and the existence of a standard or acyclic component in addition to the component containing the fixed point 0 would immediately give $h(\alpha^*) \geq 3$. So there is just one component, containing a fixed point 0 and exactly one other element p in im α^* as shown in Fig 9 (see Appendix). The choice of p can be made in n ways. The remaining $n-1$ elements map either to p or to 0, but cannot all map to 0, since that would give $h(\alpha^*) = 1$. So the total number is $n(2^{n-1}-1)$.

Next, suppose that $h(\alpha^*) = 2, g(\alpha^*) = n-1$. There are two fixed points 0 and p (another element different from 0), the remaining elements map directly to one or other of them. There are n ways in which we can choose the element p and 2^{n-1} ways of assigning the other elements, making $2^{n-1}n$ ways in all. ■

Theorem 1.2.11 *Let $\alpha \in P_n \backslash [n, n]$ and define $N(p,r,g)$ and $T(r,g)$ as in Theorem 1.2.7. Then:*

(a) *For all $n \geq 1$,*

$$N(0,0,n) = 1 \quad \text{and} \quad T(0,n) = 1,$$

i.e. the empty transformation is expressible as a product of n idempotents.

(b) *For all $n \geq 2$,*

$$N(k,1,n) = n\binom{n-1}{k}; \quad k = 1,2,\cdots,n-1$$
$$\text{and} \quad T(1,n) = n(2^{n-2}-1).$$

(c) *For all $n \geq 2$,*

$$N(k,1,n-1) = n\binom{n-1}{k-1}; \quad k = 1,2,\cdots,n-1$$
$$\text{and} \quad T(1,n-1) = 2^{n-2}n.$$

Proof. The results (a), (b) and (c) follow from Lemmas 1.2.8, 1.2.9 and 1.2.10 respectively, and by the isomorphism. ■

3. The idempotent rank

If S is a semigroup, we say that a subset T of S is a generating set for S, and write $\langle T \rangle = S$, if every element of S is expressible as a product of elements of T. Then we define

$$\operatorname{rank}(S) = \min\{|T| : \langle T \rangle = S\}.$$ ■

Lemma 1.3.1 *P_n^* is a regular subsemigroup of U_n.*

Proof. If $\alpha \in P_n^*$, define ξ to be any element of U_n mapping each element of im α to one of its inverse images and in particular mapping 0 to 0, and every element in $X_0 \setminus \operatorname{im} \alpha$ to some arbitrary fixed element of X_0. Then $\xi \in P_n^*$ and $\alpha\xi\alpha = \alpha$; thus P_n^* is regular. ■

By [**18**, Proposition 2.4.5] we have

$$\mathcal{L}(P_n^*) = \mathcal{L}(U_n) \cap (P_n^* \times P_n^*) \text{ and } \mathcal{R}(P_n^*) = \mathcal{R}(U_n) \cap (P_n^* \times P_n^*).$$

If $\alpha, \beta \in P_n^*$ are of the same height, then there exists $\gamma \in U_n$ such that

$$\operatorname{im} \alpha = \operatorname{im} \gamma, \quad \ker \gamma = \ker \beta.$$

Now $0\alpha = 0$ and so $0 \in \operatorname{im} \alpha = \operatorname{im} \gamma$. We may have $\gamma \notin P_n^*$, but we now show that we can choose $\delta \in P_n^*$ with $\operatorname{im} \alpha = \operatorname{im} \delta$, $\ker \delta = \ker \beta$. Let $0\delta^{-1} = 0\beta^{-1}$.

There remain $r-1$ members of $X_0/\ker\beta$, and $r-1$ non-zero members of im α. Arrange for δ to map the non-zero (ker β)-classes to the non-zero members of im α. Then ker δ = ker β, im α = im δ. So in fact

$$\mathcal{J}(P_n^*) = \mathcal{J}(U_n) \cap (P_n^* \times P_n^*). \tag{1.3.2}$$

A typical $\mathcal{J}$-class of U_n (consisting of elements of height $r(1 \leq r \leq n+1)$) has $S(n+1,r)$ $\mathcal{R}$-classes (where $S(n+1,r)$ is the Stirling number of the second kind), $\binom{n+1}{r}$ $\mathcal{L}$-classes, and each $\mathcal{L}$-class corresponds to a subset of U_n of cardinality r. Not every $\mathcal{L}$-class intersects P_n^*. In fact an $\mathcal{L}$-class intersects P_n^* if and only if its corresponding subset contains 0. So there are $\binom{n}{r-1}$ $\mathcal{L}$-classes containing 0, and $\binom{n}{r}$ $\mathcal{L}$-classes not containing 0. (Observe as a check that

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}.)$$

By contrast, every $\mathcal{R}$-class intersects P_n^*, since for a given equivalence ρ on U_n with r classes, we can choose an α such that $(0\rho)\alpha = 0$ and all other classes map in an arbitrary way. If we choose an $\mathcal{H}$-class $H = (\rho, A)$ (where ρ is an equivalence on U_n with r classes and A a subset of U_n of cardinality r) we have $H \cap P_n^* = \emptyset$ if $0 \notin A$ and $|H \cap P_n^*| = (r-1)!$ if $0 \in A$.

For $1 \leq r \leq n+1$, let

$$K^*(n+1,r) = \{\alpha \in P_n^* : |\text{im }\alpha| \leq r\}. \qquad \blacksquare$$

Each $K^*(n+1,r)$ is a two-sided ideal of P_n^*, and from (1.3.2), we have that $K^*(n+1,r)/K^*(n+1,r-1)$ is a principal factor of P_n^*, which we denote by PF_r^*. It is a completely 0-simple semigroup whose non-zero elements may be thought of as the elements of P_n^* of height r precisely. The product of two elements of PF_r^* is 0 whenever their product in P_n^* is of height strictly less than r. Thus the number of non-zero $\mathcal{L}$-classes in PF_r^* is $\binom{n}{r-1}$ and the number of non-zero $\mathcal{R}$-classes is $S(n+1,r)$.

The next two Lemmas are from [21].

Lemma 1.3.3 [**21** Lemma 1] *Let x, y be non-zero elements in a completely 0-simple semigroup. Then $xy \neq 0$ if and only if $L_x \cap R_y$ contains an idempotent, and in this case $xy \in R_x \cap Ly$.* ■

Lemma 1.3.4 [**21** Lemma 2] *In the principal factor PF_r^* consider the $\mathcal{H}$-class (ρ, A) determined by an equivalence ρ on X_0 and a subset A of X_0 such that $|X_0/\rho| = |A| = r$. Then (ρ, A) contains an idempotent if and only if the elements of A form a transversal of the ρ-classes.* ■

Lemma 1.3.5 *For all n, and for all r such that $2 \leq r \leq n$,*

$$\operatorname{rank}(PF_r^*) \geq S(n+1, r).$$

Proof. This is a consequence of Lemma 1.3.3. If $T = \{\alpha_1, \cdots, \alpha_k\}$ is a generating set for PF_r^* and ξ is a non-zero element of PF_r^* then for ξ to be a product of elements in T it is necessarily that at least one element of T be $\mathcal{R}$-equivalent to ξ. Thus T must cover the $\mathcal{R}$-classes of PF_r^* and so $|T| \geq S(n+1, r)$. ■

Note that T must cover the $\mathcal{L}$-classes of PF_r^* also; but $\binom{n+1}{r} \leq S(n+1, r)$ when $r \geq 2$ and so this gives a weaker conclusion.

Lemma 1.3.6 *Let α be an element in P_n^* of height r, where $r \leq n-1$. Then there exists β, γ in P_n^* of height $r+1$ such that $\alpha = \beta\gamma$.*

Proof. Let

$$\alpha = \begin{pmatrix} A_0 & A_1 & \dots & A_{r-1} \\ 0 & b_1 & \dots & b_{r-1} \end{pmatrix}.$$

We shall consider the two cases $A_0 = \{0\}$ and $A_0 \neq \{0\}$ separately.

If $A_0 = \{0\}$, then $\delta = \alpha|X$ is a full transformation of X of height $r-1$, and therefore expressible as a product $\delta = \delta_1\delta_2$ of two full transformations of X of height r by [**21**, Lemma 4]. Define β and γ by

$$x\beta = \begin{cases} x\delta_1 & \text{if } x \in X, \\ 0 & \text{if } x = 0, \end{cases} \quad \text{and} \quad x\gamma = \begin{cases} x\delta_2 & \text{if } x \in X, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $\beta, \gamma \in P_n^*$ and each is of height $r+1$. Moreover, $\alpha = \beta\gamma$.

If $A_0 \neq \{0\}$, suppose that $a_0 \in A_0 \backslash \{0\}$ and let $a_i \in A_i$ $(i = 1, \cdots, r-1)$. Since $|\{0, a_0, a_1, \cdots, a_{r-1}\}| = r+1 \leq n$, we may choose c in $X_0 \backslash \{0, a_0, a_1, \cdots, a_{r-1}\}$. Define

$$\beta = \begin{pmatrix} A_0\backslash\{a_0\} & A_1 & \dots & A_{r-1} & a_0 \\ 0 & a_1 & \dots & a_{r-1} & c \end{pmatrix}.$$

Let $Z = X_0 \backslash \{0, c, a_1, \cdots, a_{r-1}\}$; since $a_0 \in Z$, we have $Z \neq \emptyset$. Let

$$\gamma = \begin{pmatrix} \{0, c\} & a_1 & \dots & a_{r-1} & Z \\ 0 & b_1 & \dots & b_{r-1} & d \end{pmatrix},$$

where $d \in X_0 \backslash \{0, b_1, \cdots, b_{r-1}\}$. (The element d exists since $|\{0, b_1, b_2, \cdots, b_{r-1}\}| = r \leq n-1$.) Then it is easy to verify that $\beta, \gamma \in P_n^*$, are of height $r+1$ and $\alpha = \beta\gamma$. ■

As a consequence of this Lemma, a set of elements of height r generates PF_r^* if and only if it generates $K^*(n+1, r)$.

Lemma 1.3.7 *Every element α in P_n^* of height r is expressible as a product of idempotents of height r. That is, PF_r^* is generated by its own idempotents.*

Proof. Let α be an element in P_n^* of height r. Then α is expressible as a product of idempotents $\xi_1, \xi_2, \cdots, \xi_m$ in P_n^*, where each $\xi_i (i = 1, \cdots, m)$ is of height greater than or equal to r. By the method in [**8**], each idempotent ξ_i can be replaced by an idempotent ϵ_i in U_n of height r (where $\epsilon_i = \xi_i$ if ξ_i is of height r) to give $\alpha = \epsilon_1 \cdots \epsilon_m$. But the method ensures that each fixed point of ξ_i remains fixed by ϵ_i. Thus $\epsilon \in P_n^*$ for all $i = 1, 2, \cdots, m$. ■

Theorem 1.3.8 *The semigroup $K^*(n+1,r)$ is of rank $S(n+1,r)$ for $r \geq 2$.*

The proof depends on the following Lemma, which is the P_n^* version of Lemma 6 in **[21]**:

Lemma 1.3.9 *Let $A_1, \cdots, A_m$ (where $m = \binom{n}{r-1}$ and $r \geq 3$) be a list of the 0-subsets of X_0 with cardinality r (where '0-subset' means a subset of X_0 containing 0). Suppose that there exist distinct equivalences $\pi_1, \cdots, \pi_m$ of weight r with the property that A_{i-1}, A_i are both transversals of π_i ($i = 2, \cdots, m$) and A_m, A_1 are both transversals of π_1. Then each $\mathcal{H}$-class (π_i, A_i) in P_n^* contains an idempotent ϵ_i in P_n^*, and there exist idempotents $\epsilon_{m+1}, \cdots, \epsilon_p$ (where $p = S(n+1,r)$) in P_n^* such that $\{\epsilon_1, \cdots, \epsilon_p\}$ is a set of generators for $K^*(n+1,r)$.* ■

Proof. Notice first that the product $\epsilon_{i-1}\epsilon_i$ $(i = 2, \cdots, m)$ is an element of height r, since we have a configuration

$$\begin{matrix} \epsilon_{i-1} & * \\ \circ & \epsilon_i \end{matrix}$$

in which the $\mathcal{H}$-class labelled $\circ$ contains an idempotent. Moreover, the element $\epsilon_{i-1}\epsilon_i$ is in the position $*$ by Lemma 1.3.3. By the same token the product $\epsilon_m\epsilon_1$ is of height r, and $\epsilon_m \mathcal{R} \epsilon_m\epsilon_1 \mathcal{L} \epsilon_1$.

Choose the idempotents $\epsilon_{m+1}, \cdots, \epsilon_p$ from PF_r^* so that $\epsilon_1, \cdots, \epsilon_p$ covers all the $\mathcal{R}$-classes in PF_r^*. Then if η is an arbitrary idempotent in PF_r^* there exists a unique $i \in \{1, \cdots, p\}$ such that $\eta \mathcal{R} \epsilon_i$ and a unique $j \in \{1, \cdots, m\}$ such that $\eta \mathcal{L} \epsilon_j$.

$$\begin{matrix} \epsilon_k & & \\ \circ & \epsilon_{k+1} & \\ \vdots & \ddots & \\ & \cdots & \epsilon_j \\ \vdots & & \vdots \\ \epsilon_i & \cdots & \eta \end{matrix}$$

Moreover, there is a unique $k \in \{1, \cdots, m\}$ such that $\epsilon_i \mathcal{L} \epsilon_k$. If $k = j$ then $\eta = \epsilon_i$ and there is nothing to prove. If $k \leq j$ then

$$\epsilon_i\epsilon_{k+1}\epsilon_{k+2}\cdots\epsilon_j$$

is of height r and belongs to the same group $\mathcal{H}$-class as η. Hence for some $q \geq 1$

$$\eta = (\epsilon_i \epsilon_{k+1} \cdots \epsilon_j)^q.$$

If $k > j$ then

$$\epsilon_i \epsilon_{k+1} \cdots \epsilon_m \epsilon_1 \cdots \epsilon_j$$

is of height r and is in the same group $\mathcal{H}$-class as η. Hence again for some $q \geq 1$

$$\eta = (\epsilon_i \epsilon_{k+1} \cdots \epsilon_m \epsilon_1 \cdots \epsilon_j)^q.$$

We have shown that every idempotent in PF_r^* can be expressed as a product of the $p = S(n+1, r)$ idempotents $\epsilon_1, \cdots, \epsilon_p$. Since PF_r^* is generated by its idempotents (Lemma 1.3.7) we conclude that PF_r^* is generated by the idempotents $\epsilon_1, \cdots, \epsilon_p$. Then by Lemma 1.3.6 it follows that $\langle \epsilon_1, \cdots, \epsilon_p \rangle = K^*(n+1, r)$. ■

To prove that the listing of images and kernel equivalences postulated in the statement of Lemma 1.3.9 can be carried out, let $n \geq 3$ and $2 \leq r \leq n-1$, and consider the following Proposition which is the P_n^* version of the equivalent Proposition in [21]:

Proposition1.3.10 *There is a way of listing the* 0*-subsets of* X_0 *of cardinality* r *as* $A_1, \cdots, A_m$ *(with* $m = \binom{n}{r-1}$*,* $A_1 = \{0, 1, \cdots, r-1\}$*,* $A_m = \{0, n-r+2, \cdots, n\}$*) so that there exist equivalences* $\pi_1, \cdots, \pi_m$ *of weight* r *with the property that* A_{i-1}, A_i *are transversals of* π_i *(*$i = 2, \cdots, m$*) and* A_m*,* A_1 *are transversals of* π_1*.*

All we have to do here is to consider $\{0\}$ as a singleton class for each π_i ($i = 1, \cdots, m$) and the rest of the π_i-classes as in [21]. To exemplify the process, we now consider the following example from [21].

Let $n = 5$ and consider the set $\{1, 2, 3, 4, 5\}$ first. The list given in [21] of all the subsets of cardinality 2 with the equivalences is as follows:

$$A_1' = \{1, 2\} \quad \pi_1' = 2\ 4/1\ 3\ 5,$$

$$A'_2 = \{1,3\} \quad \pi'_2 = 1/2\ 3\ 4\ 5,$$
$$A'_3 = \{2,3\} \quad \pi'_3 = 1\ 2\ 4\ 5/3,$$
$$A'_4 = \{1,4\} \quad \pi'_4 = 1\ 2/3\ 4\ 5,$$
$$A'_5 = \{2,4\} \quad \pi'_5 = 1\ 2\ 3/4\ 5,$$
$$A'_6 = \{3,4\} \quad \pi'_6 = 1\ 4\ 5/2\ 3,$$
$$A'_7 = \{1,5\} \quad \pi'_7 = 1\ 4/2\ 3\ 5,$$
$$A'_8 = \{2,5\} \quad \pi'_8 = 1\ 2\ 4/3\ 5,$$
$$A'_9 = \{3,5\} \quad \pi'_9 = 1\ 5/2\ 3\ 4,$$
$$A'_{10} = \{4,5\} \quad \pi'_{10} = 1\ 3\ 4/2\ 5.$$

For the set $\{0,1,2,3,4,5\}$, we have the 0-subsets of cardinality 3 and the equivalences as follows:

$$A_1 = \{0,1,2\} \quad \pi_1 = 0/2\ 4/1\ 3\ 5,$$
$$A_2 = \{0,1,3\} \quad \pi_2 = 0/1/2\ 3\ 4\ 5,$$
$$A_3 = \{0,2,3\} \quad \pi_3 = 0/1\ 2\ 4\ 5/3,$$
$$A_4 = \{0,1,4\} \quad \pi_4 = 0/1\ 2/3\ 4\ 5,$$
$$A_5 = \{0,2,4\} \quad \pi_5 = 0/1\ 2\ 3/4\ 5,$$
$$A_6 = \{0,3,4\} \quad \pi_6 = 0/1\ 4\ 5/2\ 3,$$
$$A_7 = \{0,1,5\} \quad \pi_7 = 0/1\ 4/2\ 3\ 5,$$
$$A_8 = \{0,2,5\} \quad \pi_8 = 0/1\ 2\ 4/3\ 5,$$
$$A_9 = \{0,3,5\} \quad \pi_9 = 0/1\ 5/2\ 3\ 4,$$
$$A_{10} = \{0,4,5\} \quad \pi_{10} = 0/1\ 3\ 4/2\ 5.$$

Theorem 1.3.11 *Let $K(n,r) = \{\alpha \in P_n : |\operatorname{im} \alpha| \leq r\}$. Then the rank of $K(n,r)$ is equal to its idempotent rank, and is $S(n+1, r+1)$, for $1 \leq r \leq n-1$.*

Proof. In the isomorphism between P_n and P_n^*, elements of height r in P_n correspond to elements of height $r+1$ in P_n^*. It therefore follows that the image of $K(n, r)$ under this isomorphism is $K^*(n+1, r+1)$, and the result follows from Theorem 1.3.8. ■

CHAPTER TWO

NILPOTENTS IN PARTIAL TRANSFORMATION SEMIGROUPS

1. The depth of the nilpotent-generated subsemigroups

An element α of P_n is called nilpotent if $\alpha^k = 0$ for some positive integer $k \geq 1$. Let α be an element in P_n, and suppose that it is expressible as a product of nilpotents. It can be deduced from Lemma 3 in [26] and Theorem 4.2 in [12] that α is expressible as a product of at least four or five nilpotents according as n is odd or even. We now show that for all $n \geq 4$ (odd or even) the number of nilpotents required is at least three, and that this number is best possible.

The following Lemma follows from Remark 3.16 and Lemma 4.1 in [12]

Lemma 2.1.1 *Every element α of height r ($r \leq n-2$) in SI_n is expressible as a product of two nilpotents in SI_n of the same height.* ∎

However, elements of height $n-1$ in SI_n prove more difficult to handle. The key result in [12] is:

Theorem 2.1.2 [**12** Theorem 4.2] *For $n \geq 3$ let SI_n be the inverse semigroup of all proper subpermutations of $X_n = \{1, \cdots, n\}$, and let N be the set of all nilpotents in SI_n. Let $\Delta(\langle N \rangle)$ be the unique k such that*

$$\langle N \rangle = N \cup N^2 \cup \cdots \cup N^k \neq N \cup N^2 \cup \cdots \cup N^{k-1}.$$

Then $\Delta(\langle N\rangle) = 2$ *or* 3 *according as* n *is odd or even.* ■

For the more general semigroup P_n we now prove the following result:

Theorem 2.1.3 *Let* SP_n *be the semigroup of all strictly partial maps on the set* $X_n = \{1, \cdots, n\}$, *and let* N *be the set of all nilpotents in* SP_n. *Let* $\Delta(\langle N\rangle)$ *be the unique* k *such that*

$$\langle N\rangle = N \cup N^2 \cup \cdots \cup N^k \neq N \cup N^2 \cup \cdots \cup N^{k-1}.$$

Then $\Delta(\langle N\rangle) = 3$ *for all* $n \geq 4$.

Proof. Suppose first that $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \in [k, r]$, $r \leq n-2$ and $r < k \leq n-1$. Then there exists an element $x \in X_n \setminus \mathrm{dom}\ \alpha$ such that

$$\alpha = \beta\gamma,$$

with

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{r-1} & A_r \\ a_2 & a_3 & \dots & a_r & x \end{pmatrix}, \quad \gamma = \begin{pmatrix} a_2 & a_3 & \dots & a_r & x \\ b_1 & b_2 & \dots & b_{r-1} & b_r \end{pmatrix},$$

where $a_i \in A_i$ for all i. Clearly β is nilpotent in SP_n, and by Lemma 2.1.1 γ can be expressed as a product of two nilpotents. Thus α is expressible as a product of three nilpotents.

Suppose now that $\alpha \in [r, r]$. Then it follows from Theorem 2.1.2 that α can be expressed as a product of 2 or 3 nilpotents according as n is odd or even.

It now remains to show that three is best possible in all cases. It is clear from Theorem 2.1.2 that three is best possible when n is even. To show that three is also best possible when n is odd, we consider a particular element in $\langle N\rangle$ and show that it cannot be expressed as a product of two nilpotents. Incidentally, for this particular element the result is true irrespective of whether n is odd or even.

Let

$$\alpha = \begin{pmatrix} \{1,2\} & 3 & 4 & \dots & n-1 \\ n & 3 & 4 & \dots & n-1 \end{pmatrix} \in [n-1, n-2],$$

and suppose that α is expressible as a product of two nilpotents, say

$$\alpha = \lambda_1\lambda_2 . \tag{2.1.4}$$

Then we must have λ_1 or $\lambda_2 \in [n-1, n-2]$. If $\lambda_1 \notin [n-1, n-2]$ then necessarily $\lambda_1 \in [n-1, n-1]$. So suppose first that $\lambda_1 \in [n-1, n-1]$; then the only possible configurations of λ_1 are as shown in Fig 10 (see Appendix), where

$$\{x_1, \cdots, x_s, y_1, \cdots, y_t, z_1, \cdots, z_u\} = \{3, \cdots, n-1\}.$$

Note that if $u = 0$ then $2\lambda_1 = n$ or $1\lambda_1 = n$, and since $\{1, 2\}\alpha = n$ we must have $n\lambda_2 = n$. Thus λ_2 is non-nilpotent. So we will suppose that $u > 0$. But then since $z_i\alpha = z_i$ for $1 \leq i \leq u$ we must have λ_2 equal to

$$\begin{pmatrix} \{y_1, z_1\} & z_2 & \dots & z_u & n & x_2 & \dots & x_s & 1 & y_2 & \dots & y_t & 2 \\ n & z_1 & \dots & z_{u-1} & z_u & x_1 & \dots & x_{s-1} & x_s & y_1 & \dots & y_{t-1} & y_t \end{pmatrix}$$

by considering configuration (a) (which is sufficient). From this it follows that $z_i\lambda_2^k \neq 0$ for all k, again resulting in λ_2 being non-nilpotent. Hence in the expression (2.1.4) $\lambda_1 \notin [n-1, n-1]$.

Suppose now that $\lambda_1 \in [n-1, n-2]$ then we may assume without loss of generality that $\lambda_2 \in [n-2, n-2]$. The only possible configuration of λ_1 is as shown in Fig 11 (see Appendix), where as in the earlier case

$$\{x_1, \cdots, x_s, y_1, \cdots, y_t, z_1, \cdots, z_u\} = \{3, \cdots, n-1\}.$$

We note here too if $u = 0$ then $\{1, 2\}\lambda_1 = n$, and since $\{1, 2\}\alpha = n$ we must have $n\lambda_2 = n$, forcing λ_2 to be non-nilpotent. So we will suppose that $u > 0$. But then since $z_i\alpha = z_i$ we must have

$$\lambda_2 = \begin{pmatrix} z_1 & z_2 & \dots & z_u & n & x_2 & \dots & x_s & 1 & y_2 & \dots & y_t & 2 \\ n & z_1 & \dots & z_{u-1} & z_u & x_1 & \dots & x_{s-1} & x_s & y_1 & \dots & y_{t-1} & y_t \end{pmatrix},$$

from which it follows that $z_i\lambda_2^k \neq 0$ for all k. Again we conclude that λ_2 is not nilpotent.

The conclusion is that in all cases λ_2 cannot be nilpotent, and hence α is not expressible as a product of two nilpotents. This completes the proof of the Theorem.■

2. The nilpotent rank

In this section we shall show that SP_n has rank $n+2$, and if n is even its nilpotent rank is also $n+2$. The rank and the nilpotent rank of $SP_n \backslash W_{n-1}$ are also shown to be both equal to $n+2$. We have also generalised the idea of a rank in line with Howie and McFadden **[21]** to the nilpotent-generated subsemigroup.

Lemma 2.2.1 *SP_n is a regular subsemigroup of P_n.*

Proof. If $\alpha \in SP_n$, define η to be any element of P_n mapping each element of im α to one of its inverse image. Then $\eta \in SP_n$ and $\alpha\eta\alpha = \alpha$; thus SP_n is regular. ■

By **[18**, Proposition 2.4.5], we have

$$\mathcal{L}(SP_n) = \mathcal{L}(P_n) \cap (SP_n \times SP_n) \text{ and } \mathcal{R}(SP_n) = \mathcal{R}(P_n) \cap (SP_n \times SP_n).$$

If $\alpha, \beta \in SP_n$ are of the same height, then there exists $\gamma \in P_n$ such that im α = im γ, ker γ = ker β. We may have $\gamma \notin SP_n$, but we can choose $\delta \in SP_n$ with im α = im δ, ker δ = ker β by arranging for δ to map (ker β)-classes to im α. So in fact

$$\mathcal{J}(SP_n) = \mathcal{J}(P_n) \cap (SP_n \times SP_n).$$

The subsemigroup SP_n has n $\mathcal{J}$-classes, namely $J_{n-1}, J_{n-2}, \cdots, J_0$ (where J_0 consists of the empty map). For each r in $\{1, 2, \cdots, n-1\}$,

$$J_r = \bigcup_{k=1}^{n-1} [k, r].$$

Lemma 2.2.2 *For each $\mathcal{J}$-class J_r in SP_n, where $r \leq n-2$, we have*

$$J_r \subseteq (J_{r+1})^2.$$

Proof. Suppose first that $\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \in [k, r]$ where $r \leq k \leq n-2$. Then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r & x \\ a_1 & a_2 & \dots & a_r & x \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & a_r & y \\ b_1 & b_2 & \dots & b_r & z \end{pmatrix},$$

a product of two elements in J_{r+1}, where $x, y \in X_n \backslash \text{dom}\, \alpha$, $z \in X_n \backslash \text{im}\, \alpha$ with $x \neq y$ and $a_i \in A_i$ for all i.

Suppose now that $\alpha \in [n-1, r]$. We may suppose that A_1 contains more than one element. If $a_1, a_1' \in A_1$, then

$$\alpha = \begin{pmatrix} A_1 \backslash \{a_1'\} & a_1' & A_2 & \dots & A_r \\ a_1 & a_1' & a_2 & \dots & a_r \end{pmatrix} \begin{pmatrix} \{a_1, a_1'\} & a_2 & \dots & a_r & x \\ b_1 & b_2 & \dots & b_r & y \end{pmatrix}$$

a product of two elements in J_{r+1}, where $x \in X_n \backslash \text{dom}\, \alpha$, $y \in X_n \backslash \text{im}\, \alpha$ and $a_i \in A_i$ for all i. ■

Lemma 2.2.3 *For all $r \leq n-2$,*

$$[r, r] \subseteq ([r+1, r+1])^2.$$

Proof. Suppose that $\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \in [r, r]$. Then

$$\alpha = \begin{pmatrix} a_1 & \dots & a_r & x \\ a_1 & \dots & a_r & x \end{pmatrix} \begin{pmatrix} a_1 & \dots & a_r & y \\ b_1 & \dots & b_r & z \end{pmatrix}$$

where $x, y \in X_n \backslash \text{dom}\, \alpha$, $z \in X_n \backslash \text{im}\, \alpha$ with $x \neq y$. ■

Lemma 2.2.4 *Every element $\alpha \in SP_n$ of height r ($r \leq n-2$) is expressible as a product of nilpotents of the same height in SP_n.*

Proof. We first note that any idempotent in SP_n of height r is expressible as a product of nilpotents of the same height in SP_n. If $a_i \in A_i$ for $i = 1, \cdots, r$, and $b \notin \bigcup\{A_i : i = 1, \cdots, r\}$, where $A_1, \cdots, A_r$ are pairwise disjoint subsets of X_n, then

$$\begin{pmatrix} A_1 & \dots & A_r \\ a_1 & \dots & a_r \end{pmatrix} = \begin{pmatrix} A_1 & \dots & A_{r-1} & A_r \\ a_2 & \dots & a_r & b \end{pmatrix} \begin{pmatrix} a_2 & \dots & a_r & b \\ a_1 & \dots & a_{r-1} & a_r \end{pmatrix}.$$

Now suppose that $\alpha \in SP_n$, then

$$\alpha = \begin{pmatrix} A_1 & \dots & A_r \\ x_1 & \dots & x_r \end{pmatrix} = \begin{pmatrix} A_1 & \dots & A_r \\ a_1 & \dots & a_r \end{pmatrix} \begin{pmatrix} a_1 & \dots & a_r \\ x_1 & \dots & x_r \end{pmatrix}.$$

The result now follows from Lemma 2.1.1. ■

Before considering the next result, we would like to clarify the notion of rank in an inverse semigroup and in a semigroup that is not necessarily inverse. By the rank of an inverse semigroup S we shall mean the cardinality of any subset A of minimal order in S such that $\langle A \cup V(A) \rangle = S$, where $V(A)$ is the set of inverses of elements in A. On the other hand, the rank of the semigroup S is simply the cardinality of any subset B of minimal order in S such that $\langle B \rangle = S$. If the subset A (or B) consists of nilpotents, the rank is called nilpotent rank. We shall sometimes want to distinguish between the rank of an inverse semigroup S as an inverse semigroup and its rank as a semigroup.

The next result is from **[13]**.

Theorem 2.2.5 [**13** Theorem 3.3] *Let $B = B(G, \{1, \cdots, n\})$ be a Brandt semigroup, where G is a group of rank r, (The trivial group is deemed to have rank 0.) and $r > 1$. Then the rank of B (as an inverse semigroup) is $r + n - 1$.* ■

Remark 2.2.6 It is remarked in the proof of Theorem 3.3 in **[13]** that the generating set

$$A = \{(1, g_1, 1), \dots, (1, g_r, 1), (1, e, 2), (2, e, 3), \dots, (n-1, e, n)\}$$

(where e is the identity of G and $\{g_1, \ldots, g_r\}$ is a generating set of G) of the Brandt semigroup $B(G, \{1, \ldots, n\})$ is not unique. For in the first place, the set $\{g_1, \ldots, g_r\}$ may be replaced by any generating set of G with cardinality r. Also, the elements

$$(1, e, 2), \ldots, (n-1, e, n)$$

cover $n-1$ $\mathcal{H}$-classes and may be replaced by other elements from these same $\mathcal{H}$-classes, or by their inverses. In fact this is far from being the limit of the variability of A, as shown in the second part of the proof. ■

Proposition 2.2.7 *Let $B = B(G, \{1, \cdots, n\})$ be a Brandt semigroup, where G is a finite group of rank $r (r \geq 1)$ and $n > 1$. Then the rank of B(as a semigroup) is $r + n - 1$.*

Proof. By Theorem 2.2.5 the rank of B as an inverse semigroup is $r + n - 1$. But the rank of B as a semigroup is potentially greater than its rank as an inverse semigroup. For if A is a generating set for B as a semigroup and $|A| = s$, then certainly A together with its inverses generates B, and so $s \geq r + n - 1$.

It now remains to show that we can select a generating set for B consisting of $r + n - 1$ elements. Let A be the set

$$\{(1, g_1, 1), \cdots, (1, g_{r-1}, 1), (1, g_r, 2), (2, e, 3), \cdots, (n-1, e, n), (n, e, 1)\}$$

where e is the identity of G and $\{g_1, \cdots, g_r\}$ is a generating set for G. We first show that $(1, g_r, 1)$ and $(1, e, 2)$ belong to $\langle A \rangle$. First,

$$(1, g_r, 1) = (1, g_r, 2)(2, e, 3) \cdots (n-1, e, n)(n, e, 1) \in \langle A \rangle.$$

Observe now that

$$(1, g_r^2, 2) = (1, g_r, 2)(2, e, 3) \cdots (n-1, e, n)(n, e, 1)(1, g_r, 2)$$

and

$$(1,g_r^3,2) = (1,g_r^2,2)(2,e,3)\cdots(n-1,e,n)(n,e,1)(1,g_r,2).$$

Continuing in this way, we see that $(1,g_r^s,2) \in \langle A\rangle$ for $s = 0,1,2,\cdots$. If t is the least integer for which $g_r^t = e$ then

$$(1,e,2) = (1,g_r^t,2) \in \langle A\rangle.$$

Now let (i,g,j) be an arbitrary element in B. Then

$$(i,g,j) = (i,e,i+1)\cdots(n-1,e,n)(n,e,1)(1,g,1)(1,e,2)\cdots(j-1,e,j)$$

and it is clear that $(1,g,1)$ can be expressed as a product of the elements $(1,g_1,1)$, $\cdots$, $(1,g_r,1)$. Hence

$$\langle A\rangle = B.$$

Since $|A| = r+n-1$ the proof is complete. ■

As remarked in [13], the principal factor $PF_{n-1} = SP_n/(J_{n-2}\cup\cdots\cup J_0)$ is a Brandt semigroup, where PF_{n-1} may be thought of in the usual way as $J_{n-1}\cup\{0\}$, and the product in PF_{n-1} of two elements of J_{n-1} is the product in SP_n if this lies in J_{n-1} and is 0 otherwise. The Brandt semigroup PF_{n-1} has the structure $B(G,I)$, where $G = S_{n-1}$, the symmetric group on $n-1$ symbols, and $I = \{1,\ldots,n\}$. (See [24], section II.3.)

Let A be an irredundant set of generators of SI_n. Since SI_n is generated by the elements in J_{n-1}, we may choose to regard A as a subset of PF_{n-1}. The conclusion (as in [13]) is that *A generates SI_n if and only if it generates PF_{n-1}*.

The following Proposition now follows:

Proposition 2.2.8 *Let SI_n be the inverse semigroup of all strictly partial one-one maps on X_n, where $n \geq 3$. Then the rank of SI_n(as a semigroup) is $n+1$.* ■

Proposition 2.2.9 *Let $n \geq 4$ be even. Then the nilpotent rank of SI_n(as a semigroup) is $n+1$.*

Proof. Define $H_{i,j}$ to consist of all elements α for which $\operatorname{dom}\alpha = X_n \backslash \{i\}$ and $\operatorname{im}\alpha = X_n \backslash \{j\}$. For $i = 4, \cdots, n-1$ define a mapping $\xi_i \in H_{i,n}$ by

$$\xi_i = \begin{cases} \begin{pmatrix} 1 & 2 & \dots & i-1 & i+1 & \dots & n-i+1 & n-i+2 & n-i+3 & n-i+4 & \dots & n \\ i & i+1 & \dots & 2i-2 & 2i-1 & \dots & n-1 & 2 & 1 & 3 & \dots & i-1 \end{pmatrix} & \text{if } i \leq \frac{n}{2}, \\ \begin{pmatrix} 1 & 2 & \dots & (n/2)-1 & (n/2) & (n/2)+2 & (n/2)+3 & \dots & n \\ (n/2)+1 & (n/2)+2 & \dots & n-1 & 2 & 1 & 3 & \dots & (n/2) \end{pmatrix} & \text{if } i = \frac{n}{2}+1, \\ \begin{pmatrix} 1 & 2 & \dots & n-i & n-i+1 & n-i+2 & n-i+3 & \dots & i-1 & i+1 & \dots & n \\ i & i+1 & \dots & n-1 & 2 & 1 & 3 & \dots & 2i-n-1 & 2i-n & \dots & i-1 \end{pmatrix} & \text{if } i \geq \frac{n}{2}+2, \end{cases}$$

and

$$\xi_1 = \begin{pmatrix} 2 & 3 & \dots & n \\ 1 & 2 & \dots & n-1 \end{pmatrix}, \ \xi_2 = \begin{pmatrix} 1 & 3 & 4 & \dots & n-1 & n \\ 1 & 3 & 4 & \dots & n-1 & 2 \end{pmatrix},$$

$$\xi_3 = \begin{pmatrix} 1 & 2 & 4 & \dots & n-2 & n-1 & n \\ 3 & 4 & 5 & \dots & n-1 & 2 & 1 \end{pmatrix}, \ \xi_n = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 \\ 2 & 1 & 3 & \dots & n-1 \end{pmatrix}.$$

Then it is easy to verify that the mapping

$$\phi : B(S_{n-1}, \{1, \cdots, n\}) \to Q_{n-1}$$

defined by

$$(i, \eta, j)\phi = \xi_i \eta \xi_j^{-1}$$

is an isomorphism, where S_{n-1} is the symmetric group on X_{n-1}, and Q_{n-1} is the principal factor $SI_n/(J_{n-2} \cup \cdots \cup J_0)$.

From Proposition 2.2.7, the set

$$A = \{(1, g_1, 1), (1, g_2, 2), (2, e, 3), \cdots, (n-1, e, n), (n, e, 1)\},$$

where $g_1 = (1\ 2\ 3 \cdots n-1), g_2 = (1\ 2)$ and e the identity permutation in S_{n-1} generates $B(S_{n-1}, \{1, \cdots, n\})$. Thus $A\phi$ generates Q_{n-1} and hence SI_n. From [12] we borrow the notation $||a_1 a_2 \cdots a_n||$ for the nilpotent α with domain $X_n \backslash \{a_n\}$ and image $X_n \backslash \{a_1\}$ for which $a_i\alpha = a_{i+1}$ $(i = 1, \cdots, n-1)$. Then it is easy to verify that

$$A\phi = \{\beta, \alpha_1, \alpha_2, \cdots, \alpha_n\}$$

where

$$\beta = \begin{pmatrix} 2 & 3 & \dots & n-1 & n \\ 3 & 4 & \dots & n & 2 \end{pmatrix},\ \alpha_n = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 \\ 3 & 2 & 4 & \dots & n \end{pmatrix},$$

$$\alpha_1 = ||2\ n\,n-1 \cdots 3\ 1||,\ \alpha_{n-1} = ||n\,n-2 \cdots 1\ n-1||$$

and

$$\alpha_i = ||i+1\ i-1\ i-2 \cdots 1\ n\,n-1 \cdots i+2\ i|| \text{ for } i = 2, \cdots, n-2.$$

$\xi_1 g_1 \xi_1^{-1} = \beta$, $\xi_1 g_2 \xi_2^{-1} = \alpha_1$, $\xi_2\xi_3^{-1} = \alpha_2$, and $\xi_n\xi_1^{-1} = \alpha_n$ are straightforward to verify. If $4 < i < \frac{n}{2} - 1$ we have $\xi_i\xi_{i+1}^{-1}$ to be

$$\begin{aligned} &\begin{pmatrix} 1 & \dots & i-1 & i+1 & \dots & s+1 & s+2 & s+3 & s+4 & \dots & n \\ i & \dots & 2i-2 & 2i-1 & \dots & n-1 & 2 & 1 & 3 & \dots & i-1 \end{pmatrix} \\ &\circ \begin{pmatrix} i+1 & \dots & 2i & 2i+1 & \dots & n-1 & 2 & 1 & 3 & \dots & i \\ 1 & \dots & i & i+2 & \dots & s & s+1 & s+2 & s+3 & \dots & n \end{pmatrix} \\ &= ||i+1\ i-1 \cdots 4\ 3\ 2\ 1\ n\,n-1 \cdots i+3\ i+2\ i|| = \alpha_i \end{aligned}$$

where $s = n - i$. If $i = \frac{n}{2}$ then $\xi_{\frac{n}{2}}\xi_{\frac{n}{2}+1}^{-1}$ is

$$\begin{aligned} &\begin{pmatrix} 1 & 2 & \dots & \frac{n}{2}-1 & \frac{n}{2}+1 & \frac{n}{2}+2 & \frac{n}{2}+3 & \frac{n}{2}+4 & \dots & n \\ \frac{n}{2} & \frac{n}{2}+1 & \dots & n-2 & n-1 & 2 & 1 & 3 & \dots & \frac{n}{2}-1 \end{pmatrix} \\ &\circ \begin{pmatrix} \frac{n}{2}+1 & \frac{n}{2}+2 & \dots & n-1 & 2 & 1 & 3 & \dots & \frac{n}{2} \\ 1 & 2 & \dots & \frac{n}{2}-1 & \frac{n}{2} & \frac{n}{2}+2 & \frac{n}{2}+3 & \dots & n \end{pmatrix} \\ &= ||\frac{n}{2}+1\ \frac{n}{2}-1\ \frac{n}{2}-2 \cdots 2\ 1\ n\,n-1 \cdots \frac{n}{2}+2\ \frac{n}{2}|| = \alpha_{\frac{n}{2}}. \end{aligned}$$

If $i = \frac{n}{2} + 1$ then $\xi_{\frac{n}{2}+1}\xi_{\frac{n}{2}+2}^{-1}$ is

$$\begin{aligned} &\begin{pmatrix} 1 & 2 & \dots & \frac{n}{2}-1 & \frac{n}{2} & \frac{n}{2}+2 & \frac{n}{2}+3 & \frac{n}{2}+4 & \dots & n \\ \frac{n}{2}+1 & \frac{n}{2}+2 & \dots & n-1 & 2 & 1 & 3 & 4 & \dots & \frac{n}{2} \end{pmatrix} \\ &\circ \begin{pmatrix} \frac{n}{2}+2 & \frac{n}{2}+3 & \dots & n-1 & 2 & 1 & 3 & 4 & \dots & \frac{n}{2}+1 \\ 1 & 2 & \dots & \frac{n}{2}-2 & \frac{n}{2}-1 & \frac{n}{2} & \frac{n}{2}+1 & \frac{n}{2}+3 & \dots & n \end{pmatrix} \\ &= ||\frac{n}{2}+2\ \frac{n}{2}\ \frac{n}{2}-1 \cdots 2\ 1\ n\,n-1 \cdots \frac{n}{2}+3\ \frac{n}{2}+1|| = \alpha_{\frac{n}{2}+1}. \end{aligned}$$

If $i > \frac{n}{2} + 1$ then $\xi_i \xi_{i+1}^{-1}$ is

$$\begin{pmatrix} 1 & \ldots & s & s+1 & s+2 & s+3 & \ldots & i-1 & i+1 & \ldots & n \\ i & \ldots & n-1 & 2 & 1 & 3 & \ldots & 2i-n-1 & 2i-n & \ldots & i-1 \end{pmatrix} \circ$$
$$\begin{pmatrix} i+1 & \ldots & n-1 & 2 & 1 & 3 & \ldots & 2i-n+1 & 2i-n+2 & \ldots & i \\ 1 & \ldots & s-1 & s & s+1 & s+2 & \ldots & i & i+2 & \ldots & n \end{pmatrix}$$
$$= \|i+1\ i-1 \cdots 4\ 3\ 2\ 1\ n n-1 \ldots i+2\ i\|$$

where $s = n - i$.

Now, let

$$\delta_1 = \|1\ 4\ 6 \cdots n-2\ n 3\ 5 \cdots n-1\ 2\|, \ \delta_2 = \|2\ n-1\ n-2\ n-3 \cdots 3\ 1\ n\|$$

then

$$\alpha_1 \delta_1 = \beta \text{ and } \delta_2 \delta_1 = \alpha_n.$$

Hence the $n+1$ nilpotents

$$\alpha_1, \cdots, \alpha_{n-1}, \delta_1, \delta_2$$

generate SI_n. ■

The next two Lemmas are from **[12]**.

Lemma 2.2.10 [12, Lemma 3.10] *Let n be odd and let α be an element in SI_n of height $n-1$. Then α is expressible as a product of nilpotents in SI_n if and only if its completion $\bar{\alpha}$ is an even permutation of X_n.* ■

Lemma 2.2.11 [12, Lemma 3.15] *Every element α of height $n-2$ in SI_n is expressible as a product of two nilpotents of height $n-1$ in SI_n.* ■

Proposition 2.2.12 *Let N' be the set of all nilpotents in SI_n, where $n \geq 5$ is odd. Then the rank of $\langle N' \rangle$(as a semigroup) is equal to its nilpotent rank and is $n+1$.*

Proof. Let $k = (n+1)/2$. For $i = 4, \cdots, n-1$ define a mapping $\lambda_i \in H_{i,n}$ by

$$\lambda_i = \begin{cases} \begin{pmatrix} 1 & \dots & i-1 & i+1 & \dots & n-i+1 & n-i+2 & n-i+3 & n-i+4 & n-i+5 & \dots & n \\ i & \dots & 2i-2 & 2i-1 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & i-1 \end{pmatrix} & \text{if } 4 \le i < k, \\ \begin{pmatrix} 1 & 2 & \dots & k-1 & k+1 & k+2 & k+3 & k+4 & \dots & n \\ k & k+1 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & k-1 \end{pmatrix} & \text{if } i = k, \\ \begin{pmatrix} 1 & 2 & \dots & k-2 & k-1 & k & k+2 & k+3 & \dots & n \\ k+1 & k+2 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & k \end{pmatrix} & \text{if } i = k+1, \\ \begin{pmatrix} 1 & \dots & n-i & n-i+1 & n-i+2 & n-i+3 & n-i+4 & \dots & i-1 & i+1 & \dots & n \\ i & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & 2i-n-1 & 2i-n & \dots & i-1 \end{pmatrix} & \text{if } i \ge k+2, \end{cases}$$

and

$$\lambda_n = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 \\ 2 & 3 & 1 & 4 & \dots & n-1 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 1 & 3 & 4 & \dots & n-1 & n \\ 3 & 1 & 4 & \dots & n-1 & 2 \end{pmatrix},$$

$$\lambda_1 = \begin{pmatrix} 2 & 3 & \dots & n \\ 1 & 2 & \dots & n-1 \end{pmatrix}, \ \lambda_3 = \begin{pmatrix} 1 & 2 & 4 & \dots & n-2 & n-1 & n \\ 1 & 4 & 5 & \dots & n-1 & 2 & 3 \end{pmatrix}.$$

Then the mapping

$$\psi : B(A_{n-1}, \{1, \cdots, n\}) \to Q_{n-1} \backslash W_{n-1}$$

defined by

$$(i, \mu, j)\psi = \lambda_i \mu \lambda_j^{-1}$$

is an isomorphism, where A_{n-1} is the alternating group on X_{n-1}. For if we let

$$\bar{\lambda_i} = \lambda_i \cup (i, n),$$

then the total number of inversions in $\bar{\lambda_i}$ is

$$\begin{cases} \text{n-1} & \text{if } i = 1, \\ \text{2n-4} & \text{if } i = 2, \\ \text{3n-11} & \text{if } i = 3, \\ \text{i(n-i)+2} & \text{if } i \ge 4. \end{cases} \qquad (2.2.13)$$

(See for example [2] pp 60 – 61.)

To exemplify how (2.2.13) can be obtained we consider the following example:

Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}.$$

Then,

the number of inversions in 5 3 4 2 1 is 4 ,

the number of inversions in 3 4 2 1 is 2 ,

the number of inversions in 4 2 1 is 2 ,

the number of inversions in 2 1 is 1 .

Hence the total number of inversions in α is $4+2+2+1=9$. (From [2], the number of inversions in $c_1\ c_2\ c_3\ c_4\ c_5$ say, is the number of those c_i's that are less than c_1).

All the numbers given by (2.2.13) are clearly even. Thus $\bar{\lambda}_i$ is an even permutation for all i. Hence $\lambda_i \in Q_{n-1} \setminus W_{n-1}$ and so the mapping ψ is well defined. It is easy to verify that ψ is a bijective homomorphism.

From Coxeter and Moser ([4], section 6.3) we find that A_{n-1} is of rank 2 (provided $n \geq 5$ and is odd), being generated by

$$(1\ 2)(3 \cdots n-1) \text{ and } (1\ 2\ 3).$$

From Proposition 2.2.7 the set

$$A = \{(1, g_1, 1), (1, g_2, 2), (2, e, 3), \cdots, (n-1, e, n), (n, e, 1)\}$$

where $g_1 = (1\ 2)(3 \cdots n-1)$, $g_2 = (1\ 2\ 3)$ and e the identity permutation in A_{n-1} generates $B(A_{n-1}, \{1, \cdots, n\})$. Thus $A\psi$ generates $Q_{n-1} \setminus W_{n-1}$ and hence $\langle N' \rangle$.

It is easy to verify that

$$A\psi = \{\beta, \alpha_1, \cdots, \alpha_n\}$$

where

$$\beta = \begin{pmatrix} 2 & 3 & 4 & \dots & n-1 & n \\ 3 & 2 & 5 & \dots & n & 4 \end{pmatrix},$$

$$\alpha_1 = ||2\ n\,n-1 \cdots 3\ 1||, \quad \alpha_n = ||1\ 3\ 2\ 4\ 5 \cdots n||,$$

and

$$\alpha_i = ||i+1\ i-1 \cdots 1\ n\, n-1 \cdots i+2\ i|| \ \text{ for } \ i = 2, \cdots, n-1.$$

$\beta = \lambda_1 g_1 \lambda_1{}^{-1}$, $\alpha_1 = \lambda_1 g_2 \lambda_2{}^{-1}$, $\alpha_2 = \lambda_2\lambda_3{}^{-1}$, $\alpha_3 = \lambda_3\lambda_4{}^{-1}$ and $\alpha_n = \lambda_n\lambda_1{}^{-1}$ are straightforward to verify. If $4 \le i < \frac{n+1}{2} - 1$ then $\lambda_i\lambda_{i+1}^{-1}$ is

$$\begin{pmatrix} 1 & \dots & i-1 & i+1 & \dots & s+1 & s+2 & s+3 & s+4 & s+5 & \dots & n \\ i & \dots & 2i-2 & 2i-1 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & i-1 \end{pmatrix} \circ$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & i & i+1 & i+2 & \dots & 2i & 2i+1 & \dots & n-1 \\ s+3 & s+1 & s+2 & s+4 & \dots & n & 1 & 2 & \dots & i & i+2 & \dots & n-i \end{pmatrix}$$
$$= ||i+1\ i-1\ i-2 \cdots 2\ 1\ n\, n-1 \cdots i+2\ i|| = \alpha_i$$

where $s = n - i$. If $i = \frac{n+1}{2} - 1$ then putting $\frac{n+1}{2} = k$ we have $\lambda_{k-1}\lambda_k^{-1}$ to be

$$\begin{pmatrix} 1 & \dots & k-2 & k & k+1 & k+2 & k+3 & k+4 & k+5 & \dots & n \\ k-1 & \dots & n-3 & n-2 & n-1 & 2 & 3 & 1 & 4 & \dots & k-2 \end{pmatrix}$$
$$\circ \begin{pmatrix} k & k+1 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & k-1 \\ 1 & 2 & \dots & k-1 & k+1 & k+2 & k+3 & k+4 & \dots & n \end{pmatrix}$$
$$= ||k\ k-2 \cdots 4\ 3\ 2\ 1\ n\, n-1 \cdots k+1\ k-1|| = \alpha_{k-1}.$$

If $i = \frac{n+1}{2} (= k)$ we have $\lambda_k\lambda_{k+1}$ to be

$$\begin{pmatrix} 1 & 2 & \dots & k-1 & k+1 & k+2 & k+3 & k+4 & \dots & n \\ k & k+1 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & k-1 \end{pmatrix}$$
$$\circ \begin{pmatrix} k+1 & k+2 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & k \\ 1 & 2 & \dots & k-2 & k-1 & k & k+2 & k+3 & \dots & n \end{pmatrix}$$
$$= ||k+1\ k-1 \cdots 3\ 2\ 1\ n\, n-1 \cdots k+2\ k|| = \alpha_k.$$

If $i = \frac{n+1}{2} + 1 (= k+1)$ then $\lambda_{k+1}\lambda_{k+2}$ is

$$\begin{pmatrix} 1 & 2 & \dots & k-2 & k-1 & k & k+2 & k+3 & \dots & n \\ k+1 & k+2 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & k \end{pmatrix} \circ$$
$$\begin{pmatrix} k+2 & \dots & n-1 & 2 & 3 & 1 & 4 & 5 & \dots & k+1 \\ 1 & \dots & k-3 & k-2 & k-1 & k & k+1 & k+3 & \dots & n \end{pmatrix}$$
$$= ||k+2\ k \cdots 2\ 1\ n\, n-1 \cdots k+3\ k+1|| = \alpha_{k+1}.$$

If $i \geq \frac{n+1}{2} + 2$ then $\lambda_i \lambda_{i+1}^{-1}$ is

$$\begin{pmatrix} 1 & \dots & s & s+1 & s+2 & s+3 & s+4 & \dots & i-1 & i+1 & \dots & n \\ i & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & 2i-n-1 & 2i-n & \dots & i-1 \end{pmatrix}$$
$$\circ \begin{pmatrix} i+1 & \dots & n-1 & 2 & 3 & 1 & 4 & \dots & 2i-n+1 & 2i-n+2 & \dots & i \\ 1 & \dots & s-1 & s & s+1 & s+2 & s+3 & \dots & i & i+2 & \dots & n \end{pmatrix}$$
$$= \| i+1 \; i-1 \; i-2 \cdots 4 \; 3 \; 2 \; 1 \; n \; n-1 \cdots i+2 \; i \| = \alpha_i$$

where $s = n - i$.

Now, let $\delta_1 = \| n \; 1 \; 3 \; 4 \; 5 \cdots n-1 \; 2 \|$; then

$$\beta = \alpha_1 \delta_1 \alpha_n .$$

Hence the $n+1$ nilpotents

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \delta_1$$

generate $\langle N' \rangle$. ■

Theorem 2.2.14 *The rank of SP_n is $n+2$.*

Proof. We begin by showing that every generating set G of SP_n must contain at least $n+2$ elements. The top $\mathcal{J}$-class is $[n-1, n-1]$, and since this consists entirely of one-one maps it does not generate SP_n. From Lemmas 2.2.2 and 2.2.3 we have

$$SP_n = \langle [n-1, n-1] \cup [n-1, n-2] \rangle.$$

It is clear that in generating the elements of $[n-1, n-1]$ only elements of $[n-1, n-1]$ may be used, and by Proposition 2.2.8 at least $n+1$ elements are needed to generate $[n-1, n-1]$. Thus

$$|G \cap [n-1, n-1]| \geq n+1.$$

In generating the elements of $[n-1, n-2]$ at least one of the elements must be from $[n-1, n-2]$. That is

$$|G| \geq (n+1) + 1 = n+2.$$

To generate $[n-1, n-2]$ we now show that only one element from $[n-1, n-2]$ is needed. Let $\alpha \in [n-1, n-2]$ be given by

$$\alpha = \begin{pmatrix} \{a_1, a_2\} & a_3 & \dots & a_{n-1} \\ b_1 & b_3 & \dots & b_{n-1} \end{pmatrix}.$$

Then

$$\alpha = \gamma_1 \gamma_2 \gamma_3 ,$$

where

$$\gamma_1 = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} \\ 1 & 2 & \dots & n-1 \end{pmatrix}, \; \gamma_2 = \begin{pmatrix} \{1,2\} & 3 & \dots & n-1 \\ 3 & 4 & \dots & n \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 3 & 4 & \dots & n & 1 \\ b_1 & b_3 & \dots & b_{n-1} & b_2 \end{pmatrix}$$

and $b_2 \in X_n \backslash \text{im}\, \alpha$. It is clear that $\gamma_1, \gamma_3 \in [n-1, n-1]$ and γ_2 is a fixed element in $[n-1, n-2]$. This completes the proof of the Theorem. ■

Theorem 2.2.15 *Let $n \geq 4$ be even. Then the nilpotent rank of SP_n is $n+2$.*

Proof. From Proposition 2.2.9 and the proof of Theorem 2.2.14, the $n+2$ nilpotents

$$\alpha_1, \cdots, \alpha_{n-1}, \delta_1, \delta_2, \gamma_2$$

generate SP_n, where $\alpha_1, \cdots, \alpha_{n-1}, \delta_1, \delta_2$ are as defined in Proposition 2.2.9 and γ_2 as in Theorem 2.2.14. ■

Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ c_1 & c_2 & c_3 & \dots & c_n \end{pmatrix}$$

be a permutation on X_n, and define another permutation β on X_n by

$$1\beta = 2\alpha, \; 2\beta = 1\alpha \text{ and } x\beta = x\alpha \text{ otherwise.}$$

Thus

$$\beta = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ c_2 & c_1 & c_3 & \dots & c_n \end{pmatrix} = (1\ 2)\alpha.$$

So α is even if and only if β is odd and vice versa.

Theorem 2.2.16 *Let $n \geq 5$ be odd. Then the rank and the nilpotent rank of $SP_n \backslash W_{n-1}$ are both equal to $n+2$.*

Proof. $[n-1, n-1] \backslash W_{n-1}$ consists of one-one maps, so it does not generate $SP_n \backslash W_{n-1}$, as remarked in the proof of Theorem 2.2.14. From Lemmas 2.2.2 and 2.2.11

$$SP_n \backslash W_{n-1} = \langle ([n-1, n-1] \backslash W_{n-1}) \cup [n-1, n-2] \rangle.$$

From Proposition 2.2.12, at least $n+1$ elements are needed to generate $[n-1, n-1] \backslash W_{n-1}$. Moreover the $n+1$ elements may as well be all nilpotents.

As remarked in the proof of Theorem 2.2.15, to generate $[n-1, n-2]$, at least one of the elements must be from $[n-1, n-2]$. Thus if G is a set of generators of $SP_n \backslash W_{n-1}$, then

$$|G| \geq n+2.$$

It now remains to prove that every element $\alpha \in [n-1, n-2]$ is expressible as a product of nilpotents in $[n-1, n-1]$ and a fixed nilpotent from $[n-1, n-2]$. So let $\alpha \in [n-1, n-2]$ be

$$\begin{pmatrix} \{a_1, a_2\} & a_3 & \dots & a_{n-1} \\ b_1 & b_3 & \dots & b_{n-1} \end{pmatrix}.$$

Then α can be expressed as $\gamma_1 \beta \delta$, or alternatively as $\gamma_2 \beta \delta$, where

$$\gamma_1 = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} \\ 1 & 2 & 3 & \dots & n-1 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} \\ 2 & 1 & 3 & \dots & n-1 \end{pmatrix},$$

$$\beta = \begin{pmatrix} \{1,2\} & 3 & 4 & \dots & n-1 \\ 3 & 4 & 5 & \dots & n \end{pmatrix}, \ \delta = \begin{pmatrix} 3 & 4 & 5 & \dots & n \\ b_1 & b_3 & b_4 & \dots & b_{n-1} \end{pmatrix}.$$

Here β is a fixed nilpotent in $[n-1, n-2]$, δ is an element in $[n-2, n-2]$ and by Lemma 2.2.11 is expressible as a product of two nilpotents in $[n-1, n-1]$. By the argument preceding the statement of the Theorem either the completion of γ_1 or that of γ_2 is even, and hence by Lemma 2.2.10 either γ_1 or γ_2 is expressible in terms of nilpotents in $[n-1, n-1]$. Hence the result. ■

The following result is of independent interest. Here $\{e\}$ denotes the trivial group.

Proposition 2.2.17 *The rank of $B(\{e\}, \{1, \cdots, n\})$ (as a semigroup) is n.*

Proof. The set of n elements

$$A = \{(1, e, 2), (2, e, 3), \cdots, (n-1, e, n), (n, e, 1)\}$$

generates $B(\{e\}, \{1, \cdots, n\})$. For if (i, e, j) is an arbitrary element in $B(\{e\}, \{1, \cdots, n\})$ then

$$(i, e, j) = (i, e, i+1) \cdots (n-1, e, n)(n, e, 1)(1, e, 2) \cdots (j-1, e, j).$$

Since any generating set must cover the $\mathcal{R}$-classes(as well as the $\mathcal{L}$-classes) by Lemma 1.3.5, and since the number of $\mathcal{R}$-classes in $B(\{e\}, \{1, \cdots, n\})$ is n, no set of fewer than n elements can generate $B(\{e\}, \{1, \cdots, n\})$. Hence the result follows. ■

We remark here that the rank of $B(\{e\}, \{1, \cdots, n\})$ as an inverse semigroup is $n-1$. This follows from Theorem 2.2.5 and the fact that $\{e\}$ has rank 0.

Let

$$K(n, r) = \{\alpha \in SP_n : |\text{im } \alpha| \leq r \text{ and } 2 \leq r \leq n-2\}.$$

By Lemmas 2.2.2 and 2.2.4, $K(n, r)$ is generated by the nilpotents in SP_n of height r.

The number of $\mathcal{L}$-classes in the $\mathcal{J}$-class J_r of $K(n, r)$ is the number of image sets in X_n of cardinality r, namely $\binom{n}{r}$. The number of $\mathcal{R}$-classes in the $\mathcal{J}$-class J_r of

$K(n,r)$ is the number of equivalence relations ρ on each of the subsets A of cardinality k (where $n-1 \geq k \geq r$) for which $|A/\rho| = r$, and this number is

$$\begin{aligned}\sum_{k=r}^{n-1} \binom{n}{k} S(k,r) &= \sum_{k=r}^{n} \binom{n}{k} S(k,r) - S(n,r) \\ &= S(n+1, r+1) - S(n,r) \\ &= (r+1) S(n, r+1).\end{aligned}$$

where $S(k,l)$ is the Stirling number of the second kind.

$K(n,r)$ has $r+1$ $\mathcal{J}$-classes, namely $J_r, J_{r+1}, \cdots, J_0$ (where J_0 consists of the empty map). For each t, $1 \leq t \leq r$

$$J_t = \bigcup_{k=t}^{n-1} [k,t].$$

If we define

$$\operatorname{rank}(K(n,r)) = \min\{|T| : \langle T \rangle = K(n,r)\},$$

then we have:

Lemma 2.2.18 *For all n, and for all r such that $2 \leq r \leq n-2$,*

$$\operatorname{rank}(K(n,r)) \geq (r+1) S(n, r+1).$$

Proof. The result follows from the fact that every generating set of $K(n,r)$ must cover both the $\mathcal{R}$-classes and the $\mathcal{L}$-classes in J_r. ■

Let $r \in \{1, \cdots, n-1\}$ and let $a_1, \cdots, a_{r+1}$ be distinct elements of X_n. The element

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ a_2 & a_3 & \ldots & a_{r+1} \end{pmatrix}$$

(mapping a_i to a_{i+1} $(i = 1, \cdots, r)$) is clearly nilpotent of index $r + 1$. Let us write it as $||a_1\ a_2 \cdots a_{r+1}||$ and call it a *primitive* nilpotent in SI_n. Then we have the following result from **[12]**:

Theorem 2.2.19 [**12** Theorem 2.8] *Every non-zero nilpotent α in SI_n is a disjoint union*

$$\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_k$$

of primitive nilpotents.

Proof. Certainly dom $\alpha \neq (\text{dom}\,\alpha)\alpha = \text{im}\,\alpha$ by [**12** Lemma 2.1]. Let $a_1 \in$ dom $\alpha \backslash$ im α and consider the sequence

$$a_1, a_2 = a_1\alpha, a_3 = a_2\alpha, \cdots.$$

The sequence terminates when we reach an $a_{r+1} = a_r\alpha$ such that $a_{r+1} \notin$ dom α. There can be no repetitions in the sequence: if $a_i = a_{i+j} = a_i\alpha^j (j > 0)$ then the non-empty set $\{a_i, \cdots, a_{i+j-1}\}$ is invariant under α, which is impossible by [**12** Lemma 2.1]. Hence the sequence must terminate in the way described.

If $r = h(\alpha)$ then α is the primitive nilpotent $||a_1\ a_2 \cdots a_{r+1}||$. Otherwise α is a disjoint union

$$\alpha = ||a_1\ a_2 \cdots a_{r+1}|| \cup \beta.$$

Since

$$\alpha^m = ||a_1\ a_2 \cdots a_{r+1}||^m \cup \beta^m$$

for $m = 1, 2, \cdots$ it follows that β is nilpotent. But $h(\beta) < h(\alpha)$ and so we may suppose inductively that β is a disjoint union of primitive nilpotents. Hence we may express α as $\alpha_1 \cup \cdots \cup \alpha_k$ as required. ■

Theorem 2.2.20 *The rank of the semigroup $K(n, r)$ is equal to its nilpotent rank and is $(r + 1)S(n, r + 1)$.*

The proof depends on the following Lemma:

Lemma 2.2.21 *Suppose that we can arrange the subsets $A_1, \cdots, A_m$ (where $m = \binom{n}{r}$ and $n-2 \geq r \geq 2$) of X_n of cardinality r in such a way that $|A_i \cap A_{i-1}| = r-1$ for $i = 1, \cdots, m-1$ and $|A_m \cap A_1| = r-1$. Then there exists nilpotents $\alpha_1, \cdots, \alpha_p$ (where $p = (r+1)S(n, r+1)$) such that $\{\alpha_1, \cdots, \alpha_p\}$ is a set of generators for $K(n, r)$.* ■

Proof. Notice first that every element $\alpha \in [k, r]$, $r < k \leq n-1$ is expressible as a product of a nilpotent in its own $\mathcal{R}$-class and an element in $[r, r]$. For

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_{r-1} & A_r \\ a_2 & a_3 & \dots & a_r & x \end{pmatrix} \begin{pmatrix} a_2 & a_3 & \dots & a_r & x \\ b_1 & b_2 & \dots & b_{r-1} & b_r \end{pmatrix}$$

where $\alpha = \begin{pmatrix} A_1 & \dots & A_r \\ b_1 & \dots & b_r \end{pmatrix}$, $a_i \in A_i$ for all i and $x \in X_n \setminus \operatorname{dom} \alpha$.

In the arrangement of our subsets $A_1, \cdots, A_m$ we shall assume that $A_1 = \{n-r+1, n-r+2, \cdots, n\}$, $A_2 = \{n-r, \cdots, n-1\}$ and $A_m = \{1, n-r+2, \cdots, n\}$. We shall also represent any two adjacent subsets A_i, A_{i+1} by the two subsets $\{x_1, \cdots, x_{r-1}, y\}$ and $\{x_1, \cdots, x_{r-1}, z\}$, where $z, y \neq x_i$ for any i and $z \neq y$. Define H_{A_i,A_j} to consist of all elements $\alpha \in [r, r]$ for which $\operatorname{dom} \alpha = A_i$ and $\operatorname{im} \alpha = A_j$. For $i = 1, \cdots, m$ define a mapping $\xi_i \in H_{A_i,A_m}$ as follows:

$$\xi_1 = \begin{pmatrix} n-r+1 & n-r+2 & \dots & n \\ 1 & n-r+2 & \dots & n \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} n-r & n-r+1 & n-r+2 & \dots & n-1 \\ n-r+2 & 1 & n-r+3 & \dots & n \end{pmatrix},$$

and for $i = 3, \cdots, m$ if

$$\xi_{i-1} = \begin{pmatrix} x_1 & x_2 & \dots & x_{r-1} & y \\ t_1 & t_2 & \dots & t_{r-1} & t_r \end{pmatrix}$$

define

$$\xi_i = \begin{pmatrix} x_1 & x_2 & \dots & x_{r-1} & z \\ t_2 & t_3 & \dots & t_r & t_1 \end{pmatrix}.$$

Then it is easy to see that the mapping

$$\phi : B(S_r, \{1, \cdots, m\}) \to Q_r$$

defined by $(i, \eta, j)\phi = \xi_i \eta \xi_j^{-1}$ is an isomorphism. Here S_r is the symmetric group on $\{1, n-r+2, \cdots, n\}$, Q_r is the principal factor $[r, r]/([r-1, r-1] \cup \cdots \cup [0, 0])$.

From Proposition 2.2.7, the set

$$T = \{(1, g_1, 1), (1, g_2, 2), (2, e, 3), \cdots, (m-1, e, m), (m, e, 1)\},$$

where $g_1 = (1\ n-r+2\ \cdots n)$, $g_2 = (1\ n-r+2)$ and e the identity permutation in S_r generates $B(S_r, \{1, \cdots, m\})$. Thus $T\phi$ generates Q_r and hence $[r, r]$. If we now define

$$\alpha_1 = \xi_1 g_2 \xi_2^{-1}, \quad \alpha_i = \xi_i \xi_{i+1}^{-1} \text{ for } i = 2, \cdots, m-1$$

and

$$\beta = \xi_m \xi_1^{-1}, \qquad \delta = \xi_1 g_1 \xi_1^{-1}$$

we obtain a generating set $\{\beta, \delta, \alpha_1, \cdots, \alpha_{m-1}\}$ of $[r, r]$, where

$$\alpha_1 = ||n\, n-1 \cdots n-r+1\ n-r||$$

$$\alpha_i = ||y\ x_{r-1} \cdots x_1\ z|| \text{ for } i = 2, \cdots, m-1$$

are all nilpotents.

$$\delta = \begin{pmatrix} n-r+1 & n-r+2 & \dots & n-1 & n \\ n-r+2 & n-r+3 & \dots & n & n-r+1 \end{pmatrix}$$

which is clearly non-nilpotent. However if r is odd we have

$$\delta = \alpha_1 \lambda_1 \tag{2.2.22}$$

where

$$\lambda_1 = ||n-r\ n-r+2 \cdots n-1\ n-r+1\ n-r+3 \cdots n-2\ n||.$$

If r is even, and is of the form $4q+2 (q \geq 0)$, then

$$\delta = \alpha_1 \eta_1 \eta_2 \tag{2.2.23}$$

where

$$\eta_1 = ||n-r+1\ n-r+5 \cdots n-1\ n-r+3\ n-r+7 \cdots n-3\ n-r$$
$$n-r+4 \cdots n-2\ 1|| \cup ||n-r+2\ n-r+6 \cdots n||$$

and

$$\eta_2 = ||n-r\ n-1\ n-3 \cdots n-r+3\ n-r+1|| \cup ||1\ n n-2\ n-4 \cdots n-r+2||.$$

If r is even and of the form $4q (q \geq 1)$ then

$$\delta = \alpha_1 \psi_1 \psi_2 \tag{2.2.24}$$

where

$$\psi_1 = ||n-r+1\ n-r\ n-r+3\ n-r+2\ n-r+5\ n-r+4 \cdots n-1\ n-2\ 1||$$

and

$$\psi_2 = ||n-r\ n-r+3\ n-r+2\ n-r+5 \cdots n-2\ n-r+1|| \cup ||1\ n||.$$

Next, β may or may not be nilpotent. However if β is non-nilpotent, then by Lemma 2.1.1 it is expressible as a product of two nilpotents in $[r, r]$, say

$$\beta = \zeta_1 \zeta_2. \tag{2.2.25}$$

It is clear that $\beta \mathcal{R} \zeta_1$ and $\beta \mathcal{L} \zeta_2$, that is $R_{\zeta_1} = A_m$ and $L_{\zeta_2} = A_1$.

We now define $\lambda_1', \eta_1', \eta_2', \psi_1', \psi_2'$ and ζ_2' as follows:

$$\lambda_1' = \lambda_1 \cup (1, n),$$

$$\eta_1' = \eta_1 \cup (n, 1), \qquad \eta_2' = \eta_2 \cup (n - r + 1, n),$$

$$\psi_1' = \psi_1 \cup (n, n-2), \qquad \psi_2' = \psi_2 \cup (n, n - r + 1).$$

Before we define ζ_2', we note that from Theorem 2.2.19 ζ_2 can be expressed as a disjoint union of k primitive nilpotents, say

$$\zeta_2 = \mu_1 \cup \mu_2 \cup \cdots \cup \mu_k.$$

If $k \geq 2$, then assume

$$\mu_1 = ||x_1 \cdots x_s|| \quad \text{and} \quad \mu_2 = ||y_1 \cdots y_t||$$

and define ζ_2' as

$$x\zeta_2' = x\zeta_2 \quad \text{if} \quad x \in \text{dom}\ \zeta_2$$

and

$$x_s\zeta_2' = y_t.$$

On the other hand if $k = 1$ then $|\,\text{dom}\ \zeta_2 \cup \text{im}\ \zeta_2\,| = r + 1$, and since $r \leq n - 2$ we have $X_n \backslash (\text{dom}\ \zeta_2 \cup \text{im}\ \zeta_2)$ to be non-empty. Then define ζ_2' as

$$\zeta_2' = \zeta_2 \cup (x, n - r + 1)$$

where $x \in X_n \backslash (\text{dom}\ \zeta_2 \cup \text{im}\ \zeta_2)$.

Note that $\lambda_1', \eta_1', \eta_2', \psi_1', \psi_2'$ and ζ_2' are distinct, and belong to $[r + 1, r]$. If we now replace $\lambda_1, \eta_1, \eta_2, \psi_1$ and ψ_2 by $\lambda_1', \eta_1', \eta_2', \psi_1'$ and ψ_2' respectively in equations 2.2.22 - 2.2.24, then is easy to that the equations remain unaltered. Since β, ζ_1, ζ_2 are all one-one and of the same height, we must have

$$\text{dom}\ \beta = \text{dom}\ \zeta_1, \ \ \text{im}\ \zeta_1 = \text{dom}\ \zeta_2,$$

and since $x_s, x \notin \text{dom}\ \zeta_2 = \text{im}\ \zeta_1$ we conclude that

$$\zeta_1\zeta_2 = \zeta_1\zeta_2'.$$

Now, if β is nilpotent then $K(n, r)$ is generated by

$$\{\beta, \lambda_1', \alpha_1, \cdots, \alpha_{p-2}\}, \quad \{\beta, \eta_1', \eta_2', \alpha_1, \cdots, \alpha_{p-3}\}$$

or

$$\{\beta, \psi_1', \psi_2', \alpha_1, \cdots, \alpha_{p-3}\}$$

according to whether r is odd, even and of the form $4q + 2$ $(q \geq 0)$ or even and of the form $4q$ $(q \geq 1)$, and $\alpha_m, \cdots, \alpha_{p-k}$ $(k = 2, 3)$ are chosen arbitrarily to cover all the $\mathcal{R}$-classes in J_r.

If β is non-nilpotent, then $K(n, r)$ is generated by

$$\{\zeta_1, \zeta_2', \lambda_1', \alpha_1, \cdots, \alpha_{p-3}\}, \quad \{\zeta_1, \zeta_2', \eta_1', \eta_2', \alpha_1, \cdots, \alpha_{p-4}\}$$

or

$$\{\zeta_1, \zeta_2', \psi_1', \psi_2', \alpha_1, \cdots, \alpha_{p-4}\}$$

according to whether r is odd, even and of the form $4q + 2$ $(q \geq 0)$ or even and of the form $4q$ $(q \geq 1)$, and $\alpha_m, \cdots, \alpha_{p-k}$ $(k = 3, 4)$ are chosen arbitrarily to cover all the $\mathcal{R}$-classes in J_r. ■

It remains to prove that the listing of the subsets of X_n of cardinality r as postulated in the statement of Lemma 2.2.21 can actually be carried out. Let $n \geq 4$ and $2 \leq r \leq n-2$, and consider the following Proposition:

$\mathbf{P(n,r)}$: *There is a way of listing the subsets of X_n of cardinality r as $A_1, A_2, \cdots, A_m$ (with $m = \binom{n}{r}$), $A_1 = \{n-r+1, \cdots, n\}$, $A_2 = \{n-r, \cdots, n-1\}$, $A_m = \{1, n-r+2, \cdots, n\}$) such that $|A_i \cap A_{i+1}| = r-1$ for $i = 1, \cdots, m-1$ and $|A_m \cap A_1| = r-1$.*

We shall prove this by a double induction on n and r, the key step being a kind of Pascal's Triangle implication.

$$\mathbf{P(n-1, r-1)} \text{ and } \mathbf{P(n-1, r)} \Rightarrow \mathbf{P(n, r)}.$$

First, however, we anchor the induction with two Lemmas:

Lemma 2.2.26 $\mathbf{P(n,2)}$ *holds for every* $n \geq 4$.

Proof. Consider the following arrangement of the subsets of X_n of cardinality 2.

$$\begin{array}{ccccc} \{1,2\}, & \{1,3\}, & \ldots, & \{1,n-1\}, & \{1,n\} \\ & \{2,3\}, & \ldots, & \{2,n-1\}, & \{2,n\} \\ & & \ddots & \vdots & \vdots \\ & & & \{n-2,n-1\}, & \{n-2,n\} \\ & & & & \{n-1,n\} \end{array}$$

If we denote the first row by R_1, second row by R_2, etc., then we note that the first entry in R_i is $\{i, i+1\}$ and the last entry is $\{i, n\}$. Thus the number of elements in R_i is $n-i$, and the total number of subsets in all the rows is

$$\sum_{i=1}^{n-1}(n-i) = \frac{n}{2}(n-1) = \binom{n}{2}.$$

Hence above is a complete list of the subsets of X_n of cardinality 2.

Note that for any two subsets A_s, A_r in R_i, $A_s \cap A_r = \{i\}$, and the intersection of the last entry in R_{i+1} with the first entry in R_i is $\{i+1\}$. Hence the following arrangement satisfies $\mathbf{P(n,2)}$:

$$R_{n-1}, R_{n-2}, \cdots, R_{i+1}, R_i, \cdots, R_2, R_1.$$

That is, the list begins with all the subsets in R_{n-1}, followed by the subsets in R_{n-2}, followed by the subsets in R_{n-3}, and so on , until R_1 is reached. ■

Lemma 2.2.27 $\mathbf{P(n, n-2)}$ *holds for every* $n \geq 4$.

Proof. Note that $\mathbf{P(4,2)}$ follows from Lemma 2.2.26. So we will assume that $n \geq 5$. Let R_i' be the list of the complements of the subsets in R_i arranged in the same order as in R_i. Let $(R_i')^{-1}$ be R_i' arranged in the reverse order. For example

$$R_{n-2} = \{n-2, n-1\},\ \{n-2, n\},$$

$$R'_{n-2} = \{1,\cdots,n-3,n\},\ \{1,\cdots,n-3,n-1\},$$

$$(R'_{n-2})^{-1} = \{1,\cdots,n-3,n-1\},\ \{1,\cdots,n-3,n\}.$$

Let $T = \{1,3\},\{1,4\},\cdots,\{1,n-1\}$ and T' be its complement, i.e. $T' = R'_1\backslash(\{1,2\}' \cup \{1,n\}')$. It is clear that for any two subsets A'_s, A'_r in R'_i we have $|A'_s \cap A'_r| = n-3$, and the intersection of the last subset in R'_{i+1} and the first subset in R'_i also contains $n-3$ elements. We also have $n-3$ elements in the intersection of the last subset in R'_3 with the first subset in $(R'_2)^{-1}$, and the same number of elements in the intersection of the last subset in T' with the subset in R'_{n-1}. We now have the following arrangement satisfying $\mathbf{P(n, n-2)}$:

$$A'_1, A'_2, T', R'_{n-1}, R'_{n-2}, \cdots, R'_3, (R'_2)^{-1},$$

where $A'_1 = \{1,2\}'$ and $A'_2 = \{1,n\}'$. ■

Lemma 2.2.28 *Let* $n \geq 6$ *and* $3 \leq r \leq n-3$. *Then* $\mathbf{P(n-1,r-1)}$ *and* $\mathbf{P(n-1,r)}$ *together imply* $\mathbf{P(n,r)}$.

Proof. From the assumption $\mathbf{P(n-1,r)}$ we have a list $A_1,\cdots,A_m$ (where $m = \binom{n-1}{r}$) of the subsets of X_{n-1} with cardinality r such that $|A_i \cap A_{i+1}| = r-1$ for $i = 1,\cdots,m-1$, and

$$A_1 = \{n-r,\cdots,n-1\},\quad A_2 = \{n-r-1,\cdots,n-2\},$$

$$A_m = \{1,n-r+1,\cdots,n-1\}.$$

From the assumption $\mathbf{P(n-1,r-1)}$ we have a list $B_1,\cdots,B_t$ (where $t = \binom{n-1}{r-1}$) of subsets of X_{n-1} of cardinality $r-1$ such that $|B_i \cap B_{i+1}| = r-2$ for $i = 1,\cdots,r-1$, and

$$B_1 = \{n-r+1,\cdots,n-1\},\quad B_2 = \{n-r,\cdots,n-2\},$$

$$B_t = \{1,n-r+2,\cdots,n-1\}.$$

Let $B_i' = B_i \cup \{n\}$. Then

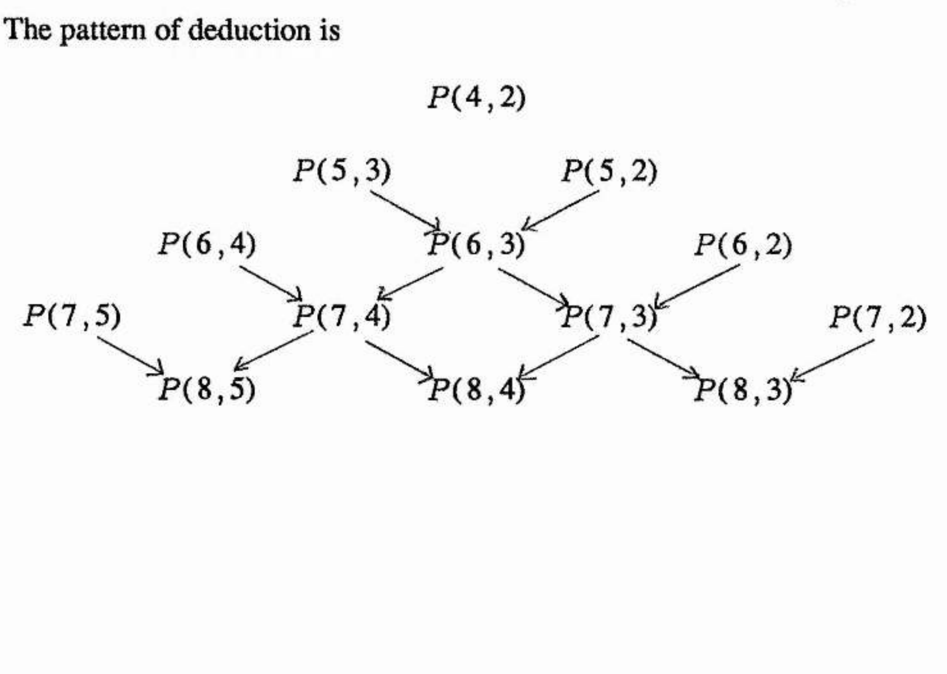
$$A_1, \cdots, A_m, B_1', \cdots, B_t'$$

is a complete list of the subsets of X_n of cardinality r. (Notice that $t + m = \binom{n}{r}$.) Now, arrange the above subsets as follows:

$$B_1', A_1, A_m, \cdots, A_2, B_2', \cdots, B_t'.$$

Then it is easy to verify that this arrangement satisfies $\mathbf{P(n,r)}$. Hence the induction is complete and we may deduce that $\mathbf{P(n,r)}$ is true for all $n \geq 4$ and all r such that $2 \leq r \leq n-2$. ■

The pattern of deduction is



CHAPTER THREE

NILPOTENTS IN SEMIGROUPS OF PARTIAL ONE-ONE ORDER-PRESERVING MAPPINGS

If a transformation semigroup S contains zero, then it contains nilpotents, and so it is natural to ask for a description of the subsemigroup of S generated by all the nilpotents of S. Gomes and Howie [**12**], and Sullivan [**26**] independently initiated this study by considering I_n, the symmetric inverse semigroup and P_n, the semigroup of all partial maps on X_n respectively. In this chapter we shall consider the inverse semigroup

$$IO_n = \{\alpha \in I_n : x \leq y \text{ implies } x\alpha \leq y\alpha\}$$

consisting of all partial one-one order-preserving maps. We shall investigate its nilpotent-generated subsemigroup, and its depth and rank properties.

1. The nilpotent generated subsemigroup

By [**18** Proposition 2.4.5] we have

$$\mathcal{L}(IO_n) = \mathcal{L}(I_n) \cap (IO_n \times IO_n) \text{ and } \mathcal{R}(IO_n) = \mathcal{R}(I_n) \cap (IO_n \times IO_n).$$

If we let $\alpha, \beta \in IO_n$ be of height r, and suppose that

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}, \ \beta = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ d_1 & d_2 & \dots & d_r \end{pmatrix},$$

where $a_1 < a_2 < \cdots < a_r$, $b_1 < b_2 < \cdots < b_r$, $c_1 < c_2 < \cdots < c_r$, $d_1 < d_2 < \cdots < d_r$, then we have im α = im γ, dom β = dom γ, where

$$\gamma = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

is clearly an element in IO_n. Thus

$$\mathcal{J}(IO_n) = \mathcal{J}(I_n) \cap (IO_n \times IO_n).$$

The semigroup IO_n has $n+1$ $\mathcal{J}$-classes $J_0, J_1, \ldots, J_n$ where J_r ($r = 0, \ldots, n$) consists of all elements of height r, that is, of all elements α for which

$$|\,\mathrm{dom}\ \alpha| = |\,\mathrm{im}\ \alpha| = r.$$

Each $\mathcal{H}$-class consists of only one element. Thus

$$|J_r| = \binom{n}{r}^2$$

and

$$|IO_n| = \sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}.$$

The first step in our investigation is to give a characterization of nilpotent elements. However it is convenient at this stage to prove the more general result concerning PO_n.

Lemma 3.1.1 *An element α in PO_n is nilpotent if and only if for all $x \in$ dom α, $x\alpha \neq x$.*

Proof. If $\alpha = 0$ (the empty map) the result is trivial. We may therefore suppose that dom $\alpha \neq \emptyset$. It is clear that if $x\alpha = x$ for some $x \in$ dom α, then α cannot be nilpotent. For we will have

$$x = x\alpha = x\alpha^2 = \cdots.$$

Conversely, suppose that $x\alpha \neq x$ for all $x \in \text{dom}\,\alpha$. We first show that if $\text{dom}\,\alpha^k \neq \emptyset$ $(k \geq 2)$ then $x\alpha^k \neq x$ for all $x \in \text{dom}\,\alpha^k$. (Note that if $\text{dom}\,\alpha^k = 0$ for some k then α is nilpotent.) Let $x \in \text{dom}\,\alpha^k$ then $x \in \text{dom}\,\alpha^t$ for all t such that $1 \leq t \leq k$. In particular $x \in \text{dom}\,\alpha$, and thus $x\alpha \neq x$. We therefore have $x\alpha < x$ or $x\alpha > x$. By the order-preserving property we have $x\alpha^k < x$ or $x\alpha^k > x$. Thus $x\alpha^k \neq x$.

Denote an element $\alpha \in PO_n$ by

$$\begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

where $r = |\,\text{im}\,\alpha|$. Now, if $b_r \in \text{dom}\,\alpha$ then $b_r < x_r$ (the $\min\{x : x \in A_r\}$), and by the order-preserving property we must have $\text{im}\,\alpha \cap A_r = \emptyset$. Thus $b_r \notin \text{im}\,\alpha^2$, and so $\text{im}\,\alpha^2 \subset \text{im}\,\alpha$ (properly). If $b_r \notin \text{dom}\,\alpha$ then $A_r \not\subseteq \text{dom}\,\alpha^2$, and since the blocks in $\text{dom}\,\alpha^2$ are union of blocks in $\text{dom}\,\alpha$, we can again conclude that $\text{im}\,\alpha^2 \subset \text{im}\,\alpha$.

If we now denote by s the cardinality of $\text{im}\,\alpha^2$, then α^2 can be written as

$$\begin{pmatrix} A_1' & A_2' & \dots & A_s' \\ b_1' & b_2' & \dots & b_s' \end{pmatrix}.$$

Since $x\alpha^2 \neq x$ for all $x \in \text{dom}\,\alpha^2$, repeating the same argument as above to α^2 we obtain $\text{im}\,\alpha^3 \subset \text{im}\,\alpha^2$. If this process is to continue we will obtain a strict descent

$$\text{im}\,\alpha \supset \text{im}\,\alpha^2 \supset \text{im}\,\alpha^3 \supset \cdots$$

and since $|\,\text{im}\,\alpha|$ is finite there exists m $(2 \leq m \leq |\,\text{im}\,\alpha|)$ such that $\text{im}\,\alpha^m = \emptyset$, that is such that $\alpha^m = 0$. ■

Let

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \tag{3.1.2}$$

be an element in IO_n, and suppose that

$$\alpha = n_1 n_2 \cdots n_k, \tag{3.1.3}$$

is a product of k nilpotents. Each n_i must be of height at least r. If any n_i is of height greater than r we can replace it by one of height exactly r simply by removing the redundant elements. Accordingly, we may assume that in the product (3.1.5) each n_i has height precisely r. Thus dom α = dom n_1. If

$$n_1 = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix}$$

we must have $c_i \neq a_i$ for all i. Hence in expressing α as a product of nilpotents, we shall first seek a set $\{c_1, c_2, \dots, c_r\}$ for which $c_i < c_{i+1}$ and $a_i \neq c_i$. Such a set may or may not exist. For example, if $n = 5$ and dom $\alpha = \{1, 3, 5\}$ then no such set exists. (One can verify this by looking at all the 10 subsets of cardinality 3.)

We shall say that α has an *upper jump* of length k [a *lower jump* of length k] if there exists an i such that

$$a_{i+1} = a_i + k + 1 \;\; [b_{i+1} = b_i + k + 1].$$

If $a_1 = k + 1$ $[b_1 = k + 1]$ and $k \geq 1$ we shall also say that α has an upper jump of length k [a lower jump of length k]. For example,

$$\alpha = \begin{pmatrix} 1 & 3 & 6 & 7 \\ 2 & 5 & 6 & 8 \end{pmatrix}$$

has upper jumps of length 1 and 2 (between 1& 3, and between 3& 6 respectively), lower jumps of length 1, 2 and 1 (before 2, between 2& 5 and between 6& 8 respectively). We shall refer to the sum of the lengths of upper jumps of α as the *total upper jump* of α, and denote it by $j^*(\alpha)$. Likewise, the sum of lengths of lower jumps of α is the *total lower jump* of α, and we denote it by $j_*(\alpha)$. In the example above $j^*(\alpha) = 3$ and $j_*(\alpha) = 4$. It is easy to verify that if α is given by (3.1.4), then

Lemma 3.1.4 $j^*(\alpha) = a_r - r, \;\; j_*(\alpha) = b_r - r.$

Theorem 3.1.5 *For* $n \geq 2$. *Let*

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

be an element of J_r where $r < n$. Then α is not a product of nilpotents if and only if α satisfies one or both of the following:

(i) $a_1 = 1, a_r = n$ *and all upper jumps are of length 1,*

(ii) $b_1 = 1, b_r = n$ *and all lower jumps are of length 1.*

Proof. Suppose that α satisfies neither (i) nor (ii). We shall consider four cases separately.

Case 1. $a_1 \neq 1, b_1 \neq 1$. For this case we have $a_i, b_i > i$ for all i, and so

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ 1 & 2 & \dots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

a product of two nilpotents.

Case 2. $a_1 = 1, b_1 \neq 1$. We must look for a set $A = \{c_1, \dots, c_r\}$ where $c_i < c_{i+1}$ for $i = 1, \dots, r-1$ and $a_i \neq c_i$ for all i. It is clear that if such a set exists then we have $c_i > i$ for all i, and so

$$\begin{aligned} \alpha &= \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ 1 & 2 & \dots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & r \\ b_1 & b_r & \dots & b_r \end{pmatrix} \\ &= n_1 n_2 n_3 \in N^3. \end{aligned}$$

To find the required set A, consider first the case where $a_r \neq n$, when we may define

$$c_i = a_i + 1 \quad (i = 1, \dots, r).$$

In the case where $a_r = n$, we have by assumption that α has at least one upper jump of length greater than one. Suppose that the first upper jump of length greater than one occurs between a_k and a_{k+1}. Then define

$$c_i = \begin{cases} a_i + 1 & \text{if } i \leq k, \\ a_i - 1 & \text{if } i > k. \end{cases}$$

Note that $c_{k+1} = a_{k+1} - 1 \geq (a_k + 3) - 1 = a_k + 2 \geq c_k$. Hence $c_i < c_{i+1}$ for all i and so n_1 is a nilpotent element in IO_n.

Case 3. $a_1 \neq 1, b_1 = 1$. Here we have that α^{-1} is as in Case 2. So we shall choose the set

$$A = \{c_1, c_2, \ldots, c_r\}$$

as in Case 2 to obtain

$$\begin{aligned}\alpha^{-1} &= \begin{pmatrix} b_1 & b_2 & \ldots & b_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix}\begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ 1 & 2 & \ldots & r \end{pmatrix}\begin{pmatrix} 1 & 2 & \ldots & r \\ a_1 & a_2 & \ldots & a_r \end{pmatrix} \\ &= n_1 n_2 n_3,\end{aligned}$$

from which $\alpha = n_3^{-1} n_2^{-1} n_1^{-1}$, a product of three nilpotents.

Case 4. $a_1 = 1$, $b_1 = 1$. We choose two sets $A = \{c_1, c_2, \ldots, c_r\}$ and $B = \{d_1, d_2, \ldots, d_r\}$ such that $a_i \neq c_i$ and $b_i \neq d_i$ for all i. If $r > 1$ we also have $c_i < c_{i+1}$ and $d_i < d_{i+1}$ for all $i \in \{1, \ldots, r-1\}$. Then

$$\alpha = n_1 n_2 n_3 n_4,$$

where

$$\begin{aligned} n_1 &= \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix}, \quad n_2 = \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ 1 & 2 & \ldots & r \end{pmatrix}, \\ n_3 &= \begin{pmatrix} 1 & 2 & \ldots & r \\ d_1 & d_2 & \ldots & d_r \end{pmatrix}, \quad n_4 = \begin{pmatrix} d_1 & d_2 & \ldots & d_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}. \end{aligned}$$

The details are similar to those of Case 2.

Conversely, suppose that α satisfies condition (i) and that α is expressible as a product

$$\alpha = n_1 n_2 \cdots n_k$$

of k nilpotents with

$$n_1 = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix}.$$

We first show by induction that $c_i \geq a_i + 1$ for all i. The result is clearly true for $i = 1$. So suppose that it is true for $i \leq k$ and that $c_{k+1} \leq a_{k+1} - 1$. Then since all the upper jumps of α are of length 1, we have $a_{k+1} \leq a_k + 2$. Thus

$$c_{k+1} \leq a_{k+1} - 1 \leq a_k + 1 \leq c_k.$$

This is impossible, so $c_i \geq a_i + 1$ for all i. In particular we have $c_r \geq a_r + 1 = n + 1$, and so c_r does not exist. Hence α is not a product of nilpotents.

A similar argument holds if α satisfies (ii). ■

Corollary 3.1.6 *Let α be as in Theorem 3.1.5. Then α is a product of nilpotents if and only if it satisfies any of the following:*

(i) $a_1 \neq 1, b_1 \neq 1$;

(ii) $a_1 = 1, b_1 \neq 1$ *and* $a_r \neq n$;

(iii) $a_1 = 1$, $b_1 \neq 1$ *and* $a_r = n$ *with α having at least one upper jump of length greater than 1;*

(iv) $a_1 \neq 1, b_1 = 1$ *and* $b_r \neq n$;

(v) $a_1 \neq 1, b_1 = 1$ *and* $b_r = n$ *with α having at least one lower jump of length greater than 1;*

(vi) $a_1 = 1, b_1 = 1$ *and* $a_r \neq n, b_r \neq n$;

(vii) $a_1 = 1, b_1 = 1$ *and* $a_r = n, b_r \neq n$ *with α having at least one upper jump of length greater than 1;*

(viii) $a_1 = 1, b_1 = 1$ *and* $a_r \neq n, b_r = n$ *with α having at least one lower jump of length greater than 1;*

(ix) $a_1 = 1, b_1 = 1$ *and* $a_r = n, b_r = n$ *with α having an upper jump and a lower jump of length greater than 1.* ■

If α satisfies condition $(j)(j = \text{i}, \ldots, \text{ix})$ we shall say that α is of type (j).

Theorem 3.1.7 *The set $\{\alpha \in IO_n : |\operatorname{im} \alpha| \leq r\}$ is contained in $\langle N \rangle$ if and only if $r \leq n/2$.*

Proof. Let α be as in Theorem 3.1.5 with $|\operatorname{im} \alpha| = r$. Suppose that α is not in $\langle N \rangle$. Then by Theorem 3.1.7 we have either

(i) $a_1 = 1, a_r = n$, and all upper jumps of α are of length 1; or

(ii) $b_1 = 1, b_r = n$, and all lower jumps of α are of length 1.

It will be sufficient to consider the case (i). Here we have $j^*(\alpha) \leq r - 1$ and

$$n = a_r = r + j^*(\alpha),$$

from which it follows that

$$j^*(\alpha) = n - r.$$

This gives a contradiction if and only if $r \leq n/2$. Thus if $\alpha \notin \langle N \rangle$, then $|\operatorname{im} \alpha| \geq n/2 + 1$. So if $r \leq n/2$, then $|\operatorname{im} \alpha| \leq r$ implies $\alpha \in \langle N \rangle$.

To complete the proof of the theorem, we now show that if $r > n/2$, then there exists $\alpha \in IO_n$ such that $\alpha \notin \langle N \rangle$.

Consider the element α for which $|\operatorname{im} \alpha| = r \geq n/2 + 1$ and $X_n \setminus \operatorname{dom} \alpha = X_n \setminus \operatorname{im} \alpha = \{2, 4, \ldots, 2s\}$, where $s = n - r$. Then $\operatorname{dom} \alpha = \operatorname{im} \alpha$, and since $r \geq n/2 + 1$ we have

$$2s = 2(n-r) \leq 2n - (n+2) = n - 2.$$

From which we can conclude that $n \in \operatorname{dom} \alpha = \operatorname{im} \alpha$, and thus $a_r = b_r = n$. It is clear that $a_1 = b_1 = 1$ and that all upper and lower jumps of α are of length 1. Hence α satisfies the conditions in Theorem 3.1.5. So α is not a product of nilpotents. ■

2. The depth of the nilpotent-generated subsemigroup

Let α be as in Theorem 3.1.5 and suppose that it is expressible as a product of nilpotents. By the proof of Theorem 3.1.5 we can express α as a product of at most four nilpotents, with elements having $a_1 = 1, b_1 = 1$ expressible as a product of exactly four nilpotents. We now show that even such elements can be expressed as a product of two or three nilpotents. In other words the depth of the nilpotent-generated subsemigroup of IO_n is at most three.

Proposition 3.2.1 *Let α be as in Theorem 3.1.5, with $a_1 = 1$, $b_1 = 1$. Then α is expressible as a product of two or three nilpotents.*

Proof. By Corollary 3.1.6, α is of type (vi), (vii), (viii) or (ix). If α is of (vi), define

$$c_i = \max\{a_i, b_i\} + 1 \qquad (i = 1, \dots, r).$$

Then $c_i < c_{i+1}$ for all i with $c_r \le n$ and $c_i \ne a_i, b_i$. Thus

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

a product of two nilpotents.

If α is of type (vii), then suppose that the first upper jump of length greater than 1 occurs between a_k and a_{k+1}. Define

$$c_i = \begin{cases} a_i + 1 & \text{if } 1 \le i \le k, \\ a_i - 1 & \text{if } i > k, \end{cases}$$

and

$$d_i = \max\{c_i, b_i\} + 1.$$

Note that $c_{k+1} = a_{k+1} - 1 \ge a_k + 3 - 1 > a_k + 1 = c_k$ and thus $c_i < c_{i+1}$ for all $i = 1, \dots, r-1$ with $c_r = n - 1$. Also $\max\{c_i, b_i\} < \max\{c_{i+1}, b_{i+1}\}$, and so $d_i < d_{i+1}$ for all $i = 1, \dots, r-1$, with $d_r = n$. It is clear that $a_i \ne c_i$, $c_i \ne d_i$, $d_i \ne b_i$ for all i and thus

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ d_1 & d_2 & \dots & d_r \end{pmatrix} \begin{pmatrix} d_1 & d_2 & \dots & d_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

a product of three nilpotents.

If α is of type (viii), then α^{-1} is of type (vii) and thus α is a product of three nilpotents.

If α is of type (ix), let $t_i = a_i - i$, $s_i = b_i - i$ and suppose that α has its first upper jump of length greater than one between a_k and a_{k+1}, and its first lower jump of length greater than one between b_l and b_{l+1}. We shall consider three cases separately, namely $t_k = s_l$, $t_k < s_l$ and $t_k > s_l$.

Case1. $t_k = s_l$. Without loss of generality we will suppose that $k \le l$. We then define

$$c_i = \begin{cases} \max\{a_i, b_i\} + 1 & \text{if } 1 \le i \le k, \\ i + t_k + 1 & \text{if } i > k. \end{cases}$$

Observe that $a_{k+j} \ge a_k + j + 2$ and that $c_{k+j} = a_k + j + 1$ for $j = 1, \ldots, r-k$, that $b_i < i + s_l + 1 = c_i$ for $k+1 \le i \le l$, that $b_{l+j} \ge b_l + j + 2$ and that $c_{l+j} = (l+j) + s_l + 1$. It therefore follows that $c_i \ne a_i, b_i$ for all i. Also, if $\max\{a_k, b_k\} = a_k$, then $c_{k+1} = a_k + 2 > a_k + 1 = \max\{a_k, b_k\} + 1 = c_k$, and if $\max\{a_k, b_k\} = b_k$, then $c_{k+1} \ge b_{k+1} + 1 > b_k + 1 = \max\{a_k, b_k\} + 1 = c_k$. In conclusion we have $c_i < c_{i+1}$ for all $i = 2, \ldots, r-1$ and

$$\alpha = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix},$$

a product of two nilpotents.

Case2. $t_k < s_l$. Define

$$c_i = \begin{cases} i + t_i + 1 & \text{if } 1 \le i \le k, \\ i + t_k + 1 & \text{if } i > k, \end{cases}$$

and

$$d_i = \begin{cases} \max\{c_i, b_i\} + 1 & \text{if } 1 \le i \le l, \\ b_i - 1 & \text{if } i > l. \end{cases}$$

Note that $b_{l+j} \ge b_l + j + 2 = s_l + l + j + 2 > t_k + l + j + 2 = c_{l+j} + 1$. Thus $d_{l+j} = b_{l+j} - 1 > c_{l+j}$ for $j = 1, 2, \ldots, r-l$ and so $d_i \ne c_i$ for all i. It is clear that $d_i \ne b_i$ for all i. Now,

$$d_{l+1} = b_{l+1} - 1 \ge b_l + 2 > b_l + 1 = \max\{c_l, b_l\} + 1 = d_l$$

(since $c_l = l + t_k + 1 \leq l + s_l = b_l$). We therefore have $d_i < d_{i+1}$ for $i = 1, 2, \ldots, r-1$. Also since $a_i = i + t_i$ for all i, we have $a_i < c_i$ for $1 \leq i \leq k$, that $a_{k+j} \geq a_k + j + 2$ and that $c_{k+j} = a_k + j + 1$. Thus $a_i \neq c_i$ for all i. It is clear from the definition of c_i that $c_i < c_{i+1}$ for all i. Hence

$$\alpha = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ d_1 & d_2 & \ldots & d_r \end{pmatrix} \begin{pmatrix} d_1 & d_2 & \ldots & d_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix},$$

a product of three nilpotents.

Case3. $t_k > s_l$. Here α^{-1} falls into Case 2 and so is expressible as a product of three nilpotents. Hence α is also expressible as a product of three nilpotent elements. This completes the proof of the theorem. ■

Lemma 3.2.2 *For $n \geq 4$, the element*

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & n-3 & n \\ 1 & 2 & 3 & \ldots & n-3 & n-1 \end{pmatrix}$$

cannot be expressed as a product of fewer than three nilpotents.

Proof. The only image set for which

$$\begin{pmatrix} 1 & 2 & \ldots & n-3 & n \\ c_1 & c_2 & \ldots & c_{n-3} & c_{n-2} \end{pmatrix}$$

is nilpotent is the set $\{2, \ldots, n-2, n-1\}$, and the only domain for which

$$\begin{pmatrix} d_1 & d_2 & \ldots & d_{n-3} & d_{n-2} \\ 1 & 2 & \ldots & n-3 & n-1 \end{pmatrix}$$

is nilpotent are the sets $\{2, \ldots, n-2, n\}$, $\{2, 4, 5, \ldots, n-1, n\}$ and $\{3, \ldots, n-1, n\}$. Since neither of the domains is equal to the image set $\{2, \ldots, n-2, n-1\}$, it follows that α cannot be expressed as a product of two nilpotents. ■

The following Theorem now follows from Proposition 3.2.1 and Lemma 3.2.2:

Theorem 3.2.3 *Let N be the set of all nilpotents in IO_n, $\langle N\rangle$ the subsemigroup of IO_n generated by the nilpotent elements, and $\Delta(\langle N\rangle)$ the unique k for which*

$$\langle N\rangle = N \cup N^2 \cup \cdots \cup N^k, \quad \langle N\rangle \neq N \cup N^2 \cup \cdots \cup N^{k-1}.$$

Then $\Delta(\langle N\rangle) = 3$ for all $n \geq 4$. ■

3. The nilpotent rank

For a subset A of an inverse semigroup S, the inverse subsemigroup $\langle A\rangle$ generated by A is the smallest inverse subsemigroup of S containing A. It consists of all finite products of elements of A and their inverses. By the *rank* of S we shall mean the cardinality of any subset A of minimal order in S such that $\langle A\rangle = S$. The cardinality of the smallest subset A *consisting of nilpotents* for which $\langle A\rangle = S$ is called the *nilpotent rank* of S. In this section we show that if $r \leq n/2$ then the rank and the nilpotent rank of

$$L(n,r) = \{\alpha \in IO_n : |\operatorname{im}\alpha| \leq r\}$$

are both equal to

$$\binom{n}{r} - 1.$$

If $n/2 < r \leq n-2$ then the rank and the nilpotent rank of

$$M(n,r) = \{\alpha \in IO_n : |\operatorname{im}\alpha| \leq r \text{ and } \alpha \in \langle N\rangle\}$$

are both equal to

$$\binom{n}{r} - \binom{r-1}{n-r} - 1.$$

Proposition 3.3.1 *For $n \geq 4$ and $r \leq n-3$, each $\alpha \in \langle N\rangle \cap J_r$ can be written as a product of elements in $\langle N\rangle \cap J_{r+1}$.*

Proof. If α is of type (i), then let $c = \max\{x : x \in X_n \backslash \operatorname{dom} \alpha\}$ and $d = \max\{x : x \in X_n \backslash \operatorname{im} \alpha\}$. Suppose also that c is between a_i and a_{i+1}, and d is between b_j and b_{j+1}. We then distinguish three cases.

Case1. $i = j$. Let

$$\beta = \begin{pmatrix} a_1 & a_2 & \dots & a_i & c & a_{i+1} & \dots & a_r \\ 1 & 2 & \dots & i & i+1 & i+3 & \dots & r+2 \end{pmatrix},$$

$$\delta = \begin{pmatrix} 1 & 2 & \dots & i & i+2 & i+3 & \dots & r+2 \\ b_1 & b_2 & \dots & b_i & d & b_{i+1} & \dots & b_r \end{pmatrix}.$$

Then $\alpha = \beta\delta$.

Case2. $i < j$. Let

$$\beta = \begin{pmatrix} a_1 & \dots & a_i & c & a_{i+1} & \dots & a_j & a_{j+1} & \dots & a_r \\ 1 & \dots & i & i+1 & i+2 & \dots & j+1 & j+3 & \dots & r+2 \end{pmatrix},$$

$$\delta = \begin{pmatrix} 1 & \dots & i & i+2 & \dots & j+1 & j+2 & j+3 & \dots & r+2 \\ b_1 & \dots & b_i & b_{i+1} & \dots & b_j & d & b_{j+1} & \dots & b_r \end{pmatrix}.$$

Then $\alpha = \beta\delta$.

Case3. $i > j$. Let

$$\beta = \begin{pmatrix} a_1 & \dots & a_j & a_{j+1} & \dots & a_i & c & a_{i+1} & \dots & a_r \\ 1 & \dots & j & j+2 & \dots & i+1 & i+2 & i+3 & \dots & r+2 \end{pmatrix},$$

$$\delta = \begin{pmatrix} 1 & \dots & j & j+1 & j+2 & \dots & i+1 & i+3 & \dots & r+2 \\ b_1 & \dots & b_j & d & b_{j+1} & \dots & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

Then $\alpha = \beta\delta$.

Note that $r + 2 < n$ by assumption; hence β is of type (iv) and δ is of type (ii) in all cases.

If α is of type (ii), then let $c = \min\{x : x \in X \backslash \operatorname{dom} \alpha\}$ and $d = \max\{x : x \in X \backslash \operatorname{im} \alpha\}$. If we distinguish the three cases $i = j$, $i < j$ and $i > j$, with β and δ as above, then $\alpha = \beta\delta$, where β is now of type (vi) and δ is of type (ii).

If α is of type (iii), choose c in such a way that the property of having at least one upper jump of length greater than 1 is maintained. (Notice that, since $r \leq n - 3$

, α must have at least one jump of length 3 or two jumps of length 2 each.) Let $d = \max\{x : x \in X\backslash \text{im } \alpha\}$. Distinguishing the three cases $i = j$, $i < j$ and $i > j$, with β and δ as above, we have $\alpha = \beta\delta$ where β is of type (vii) and δ is of type (ii).

If α is of type (iv), α^{-1} is of type (ii) and so the result follows.

If α is of type (v), α^{-1} is of type (iii) and so the result follows also.

If α is of type (vi), let $c = \min\{x : x \in X\backslash \text{dom}\alpha\}$ and $d = \min\{x : x \in X\backslash \text{im}\alpha\}$ Distinguishing the three cases $i = j$, $i < j$ and $i > j$, with β , δ as above, we have $\alpha = \beta\delta$, where β is now of type (vi) and δ is of type (vi) also.

If α is of type (vii), choose c in such a way that the property of having at least one upper jump of length greater than 1 is maintained, and $d = \min\{x : x \in X\backslash \text{im } \alpha\}$. Again distinguishing the three cases $i = j$, $i < j$ and $i > j$, with β , δ as above, we have $\alpha = \beta\delta$, where β is of type (vii) and δ is of type (vi).

If α is of type (viii), α^{-1} is of type (vii) and the result follows.

If α is of type (ix), choose c and d in such a way that the property of having at least one upper jump of length greater than 1, and that of having one lower jump of length greater than 1 are maintained. Distinguishing the three cases $i = j$, $i < j$ and $i > j$, with β , δ as above, we have $\alpha = \beta\delta$, where β is now of type (vii) and δ is of type (viii). ■

Proposition 3.3.2 *Let* $\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$. *Then*

$$\langle N \cap J_{n-1} \rangle = \{\alpha \in IO_n : a_{i+1} = a_i + 1, b_{i+1} = b_i + 1 \text{ for all } i\}.$$

Proof. Let

$$A = \{\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \in IO_n : a_{i+1} = a_i + 1, b_{i+1} = b_i + 1 \text{ for all } i\} \cup \emptyset.$$

Then A is a subsemigroup of IO_n. For if

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_1 & c_2 & \dots & c_s \\ d_1 & d_2 & \dots & d_s \end{pmatrix}$$

are any two elements in A, then

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}\begin{pmatrix} c_1 & c_2 & \dots & c_s \\ d_1 & d_2 & \dots & d_s \end{pmatrix} = \begin{pmatrix} a_i & a_{i+1} & \dots & a_{i+t} \\ d_j & d_{j+1} & \dots & d_{j+t} \end{pmatrix},$$

which is an element in A, where $1 \le i \le r$, $1 \le j \le s$ and $0 \le t \le \min\{r-1, s-1\}$.

By Theorem 3.1.5, the only elements in $\langle N \rangle \cap J_{n-1}$ are the following:

$$\eta = \begin{pmatrix} 1 & 2 & \dots & n-1 \\ 2 & 3 & \dots & n \end{pmatrix}, \quad \eta^{-1}, \quad \eta\eta^{-1}, \quad \text{and} \quad \eta^{-1}\eta.$$

It is clear that all these elements are in A. Thus

$$\langle N \cap J_{n-1} \rangle \subseteq A,$$

since A is a subsemigroup.

Now, if $r, s \in \{1, 2, \dots, n\}$, then

$$\eta^r = \begin{pmatrix} 1 & 2 & \dots & n-r \\ r+1 & r+2 & \dots & n \end{pmatrix},$$

$$(\eta^{-1})^r = \begin{pmatrix} r+1 & r+2 & \dots & n \\ 1 & 2 & \dots & n-r \end{pmatrix} = (\eta^r)^{-1}$$

and

$$(\eta^r)(\eta^{-1})^s = \begin{cases} \begin{pmatrix} 1 & 2 & \dots & n-r \\ r-s+1 & r-s+2 & \dots & n-s \end{pmatrix} & \text{if } s < r, \\ \begin{pmatrix} s-r+1 & s-r+2 & \dots & n-r \\ 1 & 2 & \dots & n-s \end{pmatrix} & \text{if } s > r, \\ \begin{pmatrix} 1 & 2 & \dots & n-r \\ 1 & 2 & \dots & n-r \end{pmatrix} & \text{if } s = r. \end{cases}$$

We now show that $A \subseteq \langle N \cap J_{n-1} \rangle$. For this purpose we express an element α in A as follows:

$$\alpha = \begin{pmatrix} i+1 & i+2 & \dots & i+k \\ j+1 & j+2 & \dots & j+k \end{pmatrix}$$

where $0 \le i, j \le n-1$ and $k = |\operatorname{im} \alpha|$. Then

$$\alpha = \begin{pmatrix} i+1 & i+2 & \dots & i+k \\ 1 & 2 & \dots & k \end{pmatrix}\begin{pmatrix} 1 & 2 & \dots & k \\ j+1 & j+2 & \dots & j+k \end{pmatrix}$$
$$= [\eta^{n-i-k}(\eta^{-1})^{n-k}][\eta^{n-k}(\eta^{-1})^{n-j-k}].$$

■

Note that rank (A) = nilrank $(A) = 1$, since $A = \langle \eta \rangle$.

Let

$$L(n, r) = \{\alpha \in IO_n : h(\alpha) \leq r\}$$

and

$$P_r = L(n, r)/L(n, r-1).$$

P_r may be thought of in the usual way as $J_r \cup \{0\}$, where the product in P_r of two elements of J_r is the product in $L(n, r)$ if this lies in J_r and is 0 otherwise. It has $\binom{n}{r}$ $\mathcal{R}$-classes corresponding to the $\binom{n}{r}$ possible domains of cardinality r, and $\binom{n}{r}$ $\mathcal{L}$-classes corresponding to the $\binom{n}{r}$ possible images. It is a Brandt semigroup isomorphic to $B(\{e\}, \{1, \ldots, \binom{n}{r}\})$, where $\{e\}$ is a group consisting of the identity element only. By Theorem 2.2.5 it has rank equal to $\binom{n}{r} - 1$ (since the group $\{e\}$ has rank 0).

Theorem 3.3.3 *For $r \leq n/2$ and $n \geq 3$,*

$$\text{rank}\,\big(L(n, r)\big) = \text{nilrank}\,\big(L(n, r)\big) = \binom{n}{r} - 1.$$

Proof. All that is required is to select a generating set of $L(n, r)$ consisting of $\binom{n}{r} - 1$ nilpotents. Let $A_1, A_2, \ldots, A_m$ (with $m = \binom{n}{r}$) be a list of the subsets of $X_n = \{1, 2, \ldots, n\}$ of cardinality r. Let $\alpha_{i,j}$ be the single element in the $\mathcal{H}$-class H_{A_i, A_j} whose domain is A_i and image is A_j. Arrange to have $A_1, A_2, \ldots, A_k$ (with $k = \binom{n-1}{r-1}$) to be the subsets of cardinality r containing 1. In particular let $A_1 = \{1, 2, \ldots, r\}$. Let $A_{k+1} = \{2, 3, \ldots, r+1\}$. Then it is clear that the elements

$$\alpha_{k+1,1}, \alpha_{k+2,1}, \ldots, \alpha_{m,1}$$

are all nilpotents. By Remark 2.2.5, the elements

$$\alpha_{k+1,1}, \alpha_{k+2,1}, \ldots, \alpha_{m,1}, \alpha_{k+1,2}, \alpha_{k+1,3}, \ldots, \alpha_{k+1,k}$$

generate P_r. But

$$\alpha_{k+1,2}, \ldots, \alpha_{k+1,k}$$

may not necessarily be all nilpotents. However, if we suppose that

$$\alpha_{k+1,i} = \begin{pmatrix} 2 & 3 & 4 & \ldots & r+1 \\ 1 & b_2 & b_3 & \ldots & b_r \end{pmatrix} \quad (i = 2, 3, \ldots, k),$$

then if $b_r \neq n$ we have

$$\alpha_{k+1,i} = \alpha_{k+1,1}\alpha_{j,1}^{-1}\beta_i,$$

where

$$\alpha_{j,1} = \begin{pmatrix} b_2 & b_3 & \ldots & b_r & n \\ 1 & 2 & \ldots & r-1 & r \end{pmatrix}$$

and

$$\beta_i = \begin{pmatrix} b_2 & b_3 & \ldots & b_r & n \\ 1 & b_2 & \ldots & b_{r-1} & b_r \end{pmatrix}$$

with j in $\{k+1, k+2, \ldots, m\}$. If $b_r = n$ then $\alpha_{k+1,i}$ must have a lower jump of length greater than 1. If this jump occurs between b_l and b_{l+1}, then

$$\alpha_{k+1,i} = \alpha_{k+1,1}\alpha_{p,1}^{-1}\beta_i,$$

where

$$\alpha_{p,1} = \begin{pmatrix} b_2 & \ldots & b_l & b_l+1 & b_l+2 & b_{l+1} & \ldots & b_{r-1} \\ 1 & \ldots & l-1 & l & l+1 & l+2 & \ldots & r \end{pmatrix}$$

and

$$\beta_i = \begin{pmatrix} b_2 & b_3 & \ldots & b_l & b_l+1 & b_l+2 & b_{l+1} & \ldots & b_{r-1} \\ 1 & b_2 & \ldots & b_{l-1} & b_l & b_{l+1} & b_{l+2} & \ldots & b_r \end{pmatrix}$$

with p in $\{k+1, k+2, \ldots, m\}$. Note that for each $i = 2, \ldots, k$, the element β_i is nilpotent and is distinct from all of the nilpotents $\alpha_{k+1,1}, \ldots, \alpha_{m,1}$. Thus the $\binom{n}{r} - 1$ nilpotents

$$\alpha_{k+1,1}, \ldots, \alpha_{m,1}, \beta_2, \ldots, \beta_k$$

generate P_r and hence $L(n, r)$. ■

Lemma 3.3.4 *Let s, k be two positive integers with $s \leq k/2$. Then the number of ways of choosing s numbers from $\{1, 2, \ldots, k\}$ so that no two consecutive numbers are chosen is*

$$\binom{k-s+1}{s}.$$

Proof. Let us represent a given choice of numbers by a sequence of length k of 0's and 1's in which 1 occurs precisely s times, with 0 indicating that a number is not chosen and 1 indicating that it is. The condition that consecutive numbers not be chosen translates to a ban on consecutive 1's. We place the $k - s$ 0's first. Then the s 1's can be placed into the $k - s - 1$ places between 0's or at the end or beginning, making $k - s + 1$ possible locations in all. There are $\binom{k-s+1}{s}$ ways to select s places to receive 1's, and so there are $\binom{k-s+1}{s}$ different sequences in which no two 1's are consecutive. Hence the proof. ■

Let $n/2 < r \leq n - 2$ and let

$$A_1, A_2, \ldots, A_s, A_{s+1}, \ldots, A_{s+t}, A_{s+t+1}, \ldots, A_{s+t+u}, A_{s+t+u+1}, \ldots, A_m$$

be a list of the subsets of $X_n = \{1, 2, \cdots, n\}$ of cardinality r, where

$m = \binom{n}{r}$

s = the number of 1-subsets not containing n,

t = the number of 1-subsets containing n with at least two consecutive numbers missing,

u = the number of subsets not containing 1.

The number of subsets containing 1 and n with no two consecutive numbers missing is

$$m - (s + t + u).$$

Consider the complements, which are subsets of $\{2,\cdots,n-1\}$ of cardinality $n-r$ and containing no two consecutive numbers. By Lemma 3.3.4 this number is

$$\binom{(n-2)-(n-r)+1}{n-r} = \binom{r-1}{n-r}.$$

From Theorem 3.1.5, the subsets $A_{s+t+u+1},\ldots,A_m$ characterise those $\mathcal{R}$-classes and $\mathcal{L}$-classes in J_r that do not contain nilpotents or elements that are expressible as products of nilpotents. On the other hand, the subsets $A_1,\ldots,A_{s+t+u}$ characterise all the $\mathcal{R}$-classes and $\mathcal{L}$-classes in J_r containing nilpotents or elements that are expressible as products of nilpotents.

Let

$$M(n,r) = \{\alpha \in IO_n : |\text{im}\,\alpha| \leq r \text{ and } \alpha \in \langle N \rangle\}.$$

We have , for $r \leq n/2$,

$$M(n,r) = L(n,r).$$

Call B a *full* subset of $X_n = \{1,2,\ldots,n\}$ if $1 \in B$, $n \in B$ and B has no 'gap' of size more than 1, in the sense that no two consecutive numbers in $\{2,\ldots,n-1\}$ belong to $X_n \backslash B$.

A subset B of cardinality r is full if and only if its complement B' is a subset of $\{2,\ldots,n-1\}$ of cardinality $n-r$ containing no two consecutive numbers. So the number of full subsets of X_n is equal to

$$\binom{(n-2)-(n-r)+1}{n-r} = \binom{r-1}{n-r}.$$

An element

$$\alpha = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$$

belongs to $M(n,r)$ if and only if neither of $\{a_1,a_2,\ldots,a_r\}$, $\{b_1,b_2,\ldots,b_r\}$ is full. So

$$|M(n,r)| = \sum_{r=0}^{n-1}\left[\binom{n}{r} - \binom{r-1}{n-r}\right]^2$$

with

$$\binom{r-1}{n-r} = 0 \text{ if } r \leq n/2\,.$$

Let W_r be the inverse subsemigroup of P_r generated by the nilpotents. Then for $r \leq n-2$,

$$W_r = M(n,r)/M(n,r-1)\,.$$

It is a Brandt semigroup isomorphic to $B(\{e\},\{1,2,\ldots,l\})$ where

$$l = \binom{n}{r} - \binom{r-1}{n-r}.$$

By Theorem 2.2.5 its rank is $l-1$.

Theorem 3.3.5 *For $n/2 < r \leq n-1$*

$$\textit{rank}\,\big(M(n,r)\big) = \textit{nilrank}\,\big(M(n,r)\big) = \binom{n}{r} - \binom{r-1}{n-r} - 1\,.$$

Proof. Let $A_1 = \{1,2,\ldots,r\}$ and $A_{s+t+1} = \{2,3,\ldots,r+1\}$. As in Theorem 3.3.3, the elements

$$\alpha_{s+t+1,1}\,,\cdots,\alpha_{s+t+u,1}\,,\alpha_{s+t+1,2}\,,\cdots,\alpha_{s+t+1,s+t}$$

generate W_r. But

$$\alpha_{s+t+1,2}\,,\cdots,\alpha_{s+t+1,s+t}$$

may not necessarily be all nilpotents. However if we suppose that

$$\alpha_{s+t+1,i} = \begin{pmatrix} 2 & 3 & \ldots & r+1 \\ 1 & b_2 & \ldots & b_r \end{pmatrix} \quad (i = 2,\ldots,s+t)\,,$$

then $\alpha_{s+t+1,i} = \alpha_{s+t+1,1}\alpha_{j,1}^{-1}\beta_i$ where $j \in \{s+t+1, s+t+2,\cdots,s+t+u\}$ and β_i is as in Theorem 3.3.3. ■

CHAPTER FOUR

NILPOTENTS IN SEMIGROUPS OF PARTIAL ORDER-PRESERVING TRANSFORMATIONS

In the last chapter we considered the semigroup IO_n of all partial one-one order-preserving maps on the set $X_n = \{1, \ldots, n\}$. In this chapter we shall consider the larger semigroup PO_n of all partial order-preserving transformations on the set $X_n = \{1, \ldots, n\}$. We shall investigate its nilpotent-generated subsemigroup, and its depth and rank properties. We shall also characterise the nilpotent-generated subsemigroup for PO_ω, the semigroup of all partial order-preserving transformations of the infinite set $X = \{1, 2, \ldots\}$.

1. The nilpotent-generated subsemigroup

Recall the following result from Chapter 3.

Lemma 4.1.1 *An element α in PO_n is nilpotent if and only if for all $x \in \text{im } \alpha \cap \text{dom } \alpha$, $x\alpha \neq x$.* ■

We will denote an element α in PO_n by

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$$

where for each $a_i \in A_i$, $a_i < a_{i+1}$ $(i = 1, \ldots, r)$ and $b_1 < b_2 < \cdots < b_r$. Let $x_i = \min\{x : x \in A_i\}$ and $y_i = \max\{x : x \in A_i\}$. For $i = 1, \ldots, r$, let $S_i = \{x \in$

$X_n : x_i \leq x \leq y_i\}$, and for $i = 1, \ldots, r-1$, $T_i = \{x \in X_n : y_i < x < x_{i+1}\}$. Let $T_0 = \{x \in X_n : x < x_1\}$ and $T_r = \{x \in X_n : x > y_r\}$.

Theorem 4.1.2 *An element α in PO_n is not a product of nilpotents if and only if α satisfies one or both of the following:*

(i) $1 \in A_1$, $n \in A_r$ *and for all* i $A_i = S_i$ *and* $|T_i| \leq 1$,

(ii) $b_1 = 1$, $b_r = n$ *and all lower jumps of* α *are of length* 1.

Proof. Suppose that α does not satisfy condition (i) and (ii). We distinguish four cases.

Case 1. $1 \notin A_1, b_1 \neq 1$. Here

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ 1 & 2 & \ldots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$$

a product of two nilpotents.

Case 2. $1 \in A_1, b_1 \neq 1$.

(a) if $n \notin A_r$, then

$$\alpha = n_1 n_2 n_3$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \ldots & A_{r-1} & A_r \\ n-r+1 & \ldots & n-1 & n \end{pmatrix},$$

$$n_2 = \begin{pmatrix} n-r+1 & \ldots & n-1 & n \\ 1 & \ldots & r-1 & r \end{pmatrix} \text{ and } n_3 = \begin{pmatrix} 1 & \ldots & r-1 & r \\ b_1 & \ldots & b_{r-1} & b_r \end{pmatrix}.$$

(b) $n \in A_r$ and $A_i \neq S_i$ for some i. Then there exists $c \in S_i \backslash A_i$ such that $x_i < c < y_i$ and

$$\alpha = n_1 n_2 n_3,$$

where

$$n_1 = \begin{pmatrix} A_1 & \ldots & A_{i-2} & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \ldots & A_r \\ x_2 & \ldots & x_{i-1} & x_i & c & y_i & y_{i+1} & \ldots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} x_2 & \ldots & x_i & c & y_i & \ldots & y_{r-1} \\ 1 & \ldots & i-1 & i & i+1 & \ldots & r \end{pmatrix}$$

and

$$n_3 = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & r \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

(c) $n \in A_r$ and $|T_i| \geq 2$ for some i. Then there exists $c, d \in T_i$ with $c < d$, and

$$\alpha = n_1 n_2 n_3,$$

where

$$n_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ y_2 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} y_2 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r \end{pmatrix}$$

and

$$n_3 = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & r \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

Case 3. $1 \notin A_1, b_1 = 1$.

(a) $b_r \neq n$. Define

$$c_i = b_i + 1,$$

then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ 1 & 2 & \dots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

a product of three nilpotents.

(b) $b_r = n$. Then α must have at least one lower jump of length greater than 1. We may suppose that the first lower jump of length greater than 1 occurs between b_k and b_{k+1}. Define

$$c_i = \begin{cases} b_i + 1 & \text{if } i \leq k, \\ b_i - 1 & \text{if } i > k. \end{cases}$$

Note that $c_{k+1} = b_{k+1} - 1 \geq (b_k + 3) - 1 = b_k + 2 > c_k$. Hence $c_i < c_{i+1}$ for all i, and

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ 1 & 2 & \dots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

a product of three nilpotents.

Case 4. $1 \in A_1, b_1 = 1$.

(a) $n \notin A_r, b_r \neq n$. Define

$$c_i = \max\{y_i, b_i\} + 1$$

for all i, then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

a product of two nilpotents.

(b) $n \notin A_r, b_r = n$. Then α must have at least one lower jump of length greater than 1. We may suppose that the first lower jump of length greater than 1 occurs between b_k and b_{k+1}. Define

$$c_i = \begin{cases} b_i + 1 & \text{if } 1 \le i \le k, \\ b_i - 1 & \text{if } i > k. \end{cases}$$

Then

$$\alpha = \begin{pmatrix} A_1 & \dots & A_r \\ n-r+1 & \dots & n \end{pmatrix} \begin{pmatrix} n-r+1 & \dots & n \\ c_1 & \dots & c_r \end{pmatrix} \begin{pmatrix} c_1 & \dots & c_r \\ b_1 & \dots & b_r \end{pmatrix}$$

a product of three nilpotents.

(c) $n \in A_r, b_r \neq n$.

(i) $A_i \neq S_i$ for some i. Then there exists c in $S_i \backslash A_i$ such that $x_i < c < y_i$, and

$$\alpha = n_1 n_2 n_3$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & \dots & A_r \\ x_2 & \dots & x_i & c & y_i & \dots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} x_2 & \dots & x_i & c & y_i & \dots & y_{r-1} \\ c_1 & \dots & c_{i-1} & c_i & c_{i+1} & \dots & c_r \end{pmatrix}, \quad n_3 = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

and

$$c_j = \begin{cases} \max\{x_{j+1}, b_j\} + 1 & \text{if } 1 \le j \le i-1, \\ \max\{c, b_j\} + 1 & \text{if } j = i, \\ \max\{y_{j-1}, b_j\} + 1 & \text{if } j > i. \end{cases}$$

(ii) $|T_i| \geq 2$ for some i. Then there exists $c, d \in T_i$ with $c < d$ and

$$\alpha = n_1 n_2 n_3$$

where

$$n_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ x_2 & \dots & x_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} x_2 & \dots & x_i & c & d & y_{i+1} & \dots & y_{r-1} \\ c_1 & \dots & c_{i-1} & c_i & c_{i+1} & c_{i+2} & \dots & c_r \end{pmatrix},$$

$$n_3 = \begin{pmatrix} c_1 & \dots & c_{i-1} & c_i & c_{i+1} & \dots & c_r \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}$$

and

$$c_j = \begin{cases} \max\{x_{j+1}, b_j\} + 1 & \text{if } 1 \le j \le i-1, \\ \max\{c, b_j\} + 1 & \text{if } j = i, \\ \max\{d, b_j\} + 1 & \text{if } j = i+1, \\ \max\{y_{j-1}, b_j\} + 1 & \text{if } j > i+1. \end{cases}$$

(d) $n \in A_r, b_r = n$. Then α has at least one lower jump of length greater than 1. We may assume that the first lower jump of length greater than 1 occurs between b_k and b_{k+1}. Define

$$c_j = \begin{cases} b_j + 1 & \text{if } 1 \le j \le k, \\ b_j - 1 & \text{if } j > k. \end{cases}$$

Then

$$\alpha = n_1 n_2 n_3 n_4$$

where

$$n_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ x_2 & \dots & x_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} x_2 & \dots & x_i & c & d & y_{i+1} & \dots & y_{r-1} \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r \end{pmatrix},$$

$$n_3 = \begin{pmatrix} 1 & 2 & \dots & r \\ c_1 & c_2 & \dots & c_r \end{pmatrix}, \quad n_4 = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

$c \in S_i \backslash A_i$ and $d = y_i$ if $A_i \neq S_i$ for some i, or $d \in T_i$ if $|T_i| \geq 2$ for some i (with $c < d$).

Conversely, suppose that α satisfies condition (i) and that α is expressible as a product

$$\alpha = n_1 n_2 \cdots n_k$$

of k nilpotents with

$$n_1 = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix}.$$

We must first show by induction that $c_i > y_i$ for all i. The result is clearly true for $i = 1$. So suppose that it is true for all $i \leq k$ and that $c_{k+1} < y_{k+1}$. Then since $A_{k+1} = S_{k+1}$ we must have $c_{k+1} < x_{k+1}$. Thus

$$y_k < c_k < c_{k+1} < x_{k+1}.$$

But this will mean $|T_k| \geq 2$, which is a contradiction. So $c_i > y_i$ for all i. In particular we have $c_r > y_r = n$, and so c_r does not exist. Hence α is not a product of nilpotents.

Suppose that α satisfies (ii) and α is expressible as a product

$$\alpha = n_1 n_2 \cdots n_k$$

of k nilpotents. We may then assume that

$$n_k = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

where $\{c_1, \dots, c_r\} = \text{im } n_{k-1}$. We will begin by showing inductively that $c_i \geq b_i + 1$ for all i. The result is clearly true for $i = 1$. So suppose that it is true for all $i \leq k$ and that $c_{k+1} \leq b_{k+1} - 1$. Then since all the lower jumps of α are of length 1, we have $b_{k+1} \leq b_k + 2$. Thus

$$c_{k+1} \leq b_{k+1} - 1 \leq b_k + 1 \leq c_k.$$

This is impossible. So $c_i \geq b_i + 1$ for all i. In particular we have $c_r \geq b_r + 1 = n + 1$, and so c_r does not exist. Hence α is not a product of nilpotents. ■

Note that Theorem 4.1.2 is analogous to Theorem 3.1.5. To make this point clear, we now restate Theorem 3.1.5.

Theorem 3.1.5 *For* $n \geq 2$. *Let*

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

be an element of J_r *where* $r < n$. *Then* α *is not a product of nilpotents if and only if* α *satisfies one or both of the following:*

(i) $a_1 = 1, a_r = n$ *and all upper jumps are of length 1,*

(ii) $b_1 = 1, b_r = n$ *and all lower jumps are of length 1.* ∎

By analogy with Theorem 3.1.7, we have:

Theorem 4.1.3 *The set*

$$A = \{\alpha \in PO_n : |\operatorname{im} \alpha| \leq r \text{ and } |X_n \backslash \operatorname{dom} \alpha| \geq r\}$$

is contained in $\langle N \rangle$ *if and only if* $r \leq \frac{1}{2}n$.

Proof. Let $\alpha \in A$, and suppose that $r \leq \frac{1}{2}n$. Then by Theorem 4.1.2, to show that $\alpha \in \langle N \rangle$ we are required to prove the following:

(i) If $1 \in A_1, n \in A_r$, then for some i it is the case that $A_i \neq S_i$ or $|T_i| \geq 2$.

(ii) If $b_1 = 1, b_r = n$, then α has a lower jump of length greater than 1.

So suppose by way of contradiction that $1 \in A_1, n \in A_r$ and that there exists no i for which $A_i \neq S_i$ or $|T_i| \geq 2$. Then

$$X_n \backslash \operatorname{dom} \alpha = \bigcup_{i=1}^{r-1} T_i,$$

and

$$r \leq |X_n \backslash \operatorname{dom} \alpha| = \bigcup_{i-1}^{r-1} |T_i| \leq r - 1.$$

This is a contradiction; thus α satisfies (i).

Again suppose that $b_1 = 1, b_r = n$ and that all lower jumps of α are of length 1. Then

$$j_*(\alpha) \leq r - 1.$$

Also

$$n = b_r = r + j_*(\alpha)$$

and so

$$j_*(\alpha) = n - r \geq r$$

(since $r \leq \frac{1}{2}n$). This is also a contradiction; thus α satisfies (ii).

To complete the proof of the theorem, we now show (as in Theorem 3.1.7) that if $r > n/2$, then there exists $\alpha \in A$ such that $\alpha \notin \langle N \rangle$.

Consider the element α for which $|\operatorname{im} \alpha| = r \geq n/2 + 1$ and $X_n \setminus \operatorname{im} \alpha = \{2, 4, \ldots, 2s\}$, where $s = n - r$. Then we have

$$2s = 2(n - r) \leq 2n - (n + 2) = n - 2.$$

From which we can conclude that $n \in \operatorname{im} \alpha$, and thus $b_r = n$. It is clear that $b_1 = 1$ and that all lower jumps of α are of length 1. Hence α satisfies condition (ii) in Theorem 4.1.2. So α is not a product of nilpotents. ■

2. The depth of the nilpotent-generated subsemigroup

By the proof of Theorem 4.1.2 we can express α in $\langle N \rangle$ as a product of at most four nilpotents, with elements having $1 \in A_1, n \in A_r, b_1 = 1, b_r = n$ expressible as a product of exactly four nilpotents. As in Section 3.2 we now show that even such elements can be expressed as a product of two or three nilpotents.

Proposition 4.2.1 *Let α in $\langle N \rangle$ be such that $1 \in A_1$, $n \in A_r$, $b_1 = 1$ and $b_r = n$. Then α is expressible as a product of at most three nilpotents.*

Proof. By Theorem 4.1.2 there exists i for which $A_i \neq S_i$ or $|T_i| \geq 2$, and α has a lower jump of length greater than 1. We will assume that the first lower jump of length greater than 1 occurs between b_k and b_{k+1}.

Let $c \in S_i \backslash A_i$ or $c = \min\{x : x \in T_i\}$, and $d \in T_i$ with $d \neq c$. We first show inductively that

$$c - i + j > y_j \quad \text{if} \quad 1 \leq j \leq i - 1$$

and

$$c - i + j < x_j \quad \text{if} \quad j > i.$$

The results are true respectively for $j = i - 1$ and $j = i + 1$, since

$$y_{i-1} < x_i \leq c - 1 \text{ and } c + 1 \leq (y_i \text{ or } d) < x_{i+1}.$$

Suppose that they are true (respectively) for $j = s \leq i - 1$ and $j = t > i + 1$; that is, $y_s < c - i + s$ and $x_t > c - i + t$. Then

$$y_{s-1} \leq y_s - 1 < c - i + s - 1 \text{ and } c - i + t + 1 < x_t + 1 \leq x_{t+1},$$

as required. Next we show that

$$b_k - k + j + 1 > b_j \text{ if } 1 \leq j \leq k$$

and

$$b_k - k + j + 1 < b_j \text{ if } j > k.$$

For $j = k$ and $k + 1$ we have

$$b_k + 1 > b_k \text{ and } b_k + 2 < b_{k+1}.$$

So suppose that the results are true for $j = s \leq k$ and $j = t \geq k+1$, that is $b_k - k + s + 1 > b_s$ and $b_k - k + t + 1 < b_t$. Then

$$b_k - k + s > b_s - 1 \geq b_{s-1} \text{ and } b_k - k + t + 2 < b_t + 1 \leq b_{t+1}.$$

We now distinguish two cases.

Case 1. $c - i + k = b_k + 1$. Then $c - i + j = b_k - k + j + 1$ for all $j = 1, \ldots, r$ and

$$\alpha = n_1 n_2,$$

a product of two nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & \ldots & A_k & A_{k+1} & \ldots & A_r \\ b_k - k + 2 & \ldots & b_k + 1 & b_k + 2 & \ldots & b_k - k + r - 1 \end{pmatrix},$$

and

$$n_2 = \begin{pmatrix} b_k - k + 2 & \ldots & b_k + 1 & b_k + 2 & \ldots & b_k - k + r - 1 \\ b_1 & \ldots & b_k & b_{k+1} & \ldots & b_r \end{pmatrix}.$$

Case 2. $c - i + k \neq b_k + 1$. Then $c - i + j \neq b_k - k + j + 1$ for all $j = 1, \ldots, r$ and

$$\alpha = n_1 n_2 n_3$$

where

$$n_1 = \begin{pmatrix} A_1 & \ldots & A_k & A_{k+1} & \ldots & A_r \\ c - i + 1 & \ldots & c - i + k & c - i + k + 1 & \ldots & c - i + r \end{pmatrix},$$

$$n_2 = \begin{pmatrix} c - i + 1 & \ldots & c - i + k & c - i + k + 1 & \ldots & c - i + r \\ b_k - k + 2 & \ldots & b_k + 1 & b_k + 2 & \ldots & b_k - k + r + 1 \end{pmatrix}$$

and

$$n_3 = \begin{pmatrix} b_k - k + 2 & \ldots & b_k + 1 & b_k + 2 & \ldots & b_k - k + r + 1 \\ b_1 & \ldots & b_k & b_{k+1} & \ldots & b_r \end{pmatrix}.$$

■

The following Theorem now follows from Proposition 4.2.1 and Theorem 3.2.3:

Theorem 4.2.2 *Let N be the set of all nilpotents in PO_n, $\langle N\rangle$ the subsemigroup of PO_n generated by the nilpotent elements, and $\Delta(\langle N\rangle)$ the unique k for which*

$$\langle N\rangle = N \cup N^2 \cup \cdots \cup N^k, \quad \langle N\rangle \neq N \cup N^2 \cup \cdots \cup N^{k-1}.$$

Then $\Delta(\langle N\rangle) = 3$ for all $n \geq 3$. ■

3. The nilpotent rank

Before considering the next result, we remark that if α is a total map, then α is not expressible as a product of nilpotents. This follows from the fact that α satisfies condition (i) in Theorem 4.1.2.

Lemma 4.3.1 *Every element $\alpha \in \langle N\rangle \cap J_r$, $r \leq n-3$ is expressible as a product of elements in $\langle N\rangle \cap J_{r+1}$.*

Proof. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

be an element in $\langle N\rangle \cap J_r$, $r \leq n-3$. From Proposition 3.3.1 if $\alpha \in \langle N\rangle \cap [r,r]$ then α can be expressed as a product of two elements in $\langle N\rangle \cap [r+1, r+1]$. We will therefore assume that $\alpha \in \langle N\rangle \cap [k,r]$, $r+1 \leq k \leq n-1$.

Now, if $|T_i| \geq 2$ for some $i = 1, \dots, r-1$ (or $|T_0| \geq 1$ or $|T_r| \geq 1$) then

$$\alpha = \gamma_1\gamma_2\gamma_3,$$

where

$$\gamma_1 = \begin{pmatrix} A_1 & \dots & A_{j-1} & x_j & A_j\backslash\{x_j\} & A_{j+1} & \dots & A_r \\ 1 & \dots & j-1 & j & j+1 & j+2 & \dots & r+1 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 1 & \dots & j-1 & \{j,j+1\} & j+2 & \dots & r+1 & r+2 \\ 2 & \dots & j & j+2 & j+3 & \dots & r+2 & r+3 \end{pmatrix},$$

$$\gamma_3 = \begin{pmatrix} 2 & \dots & j & j+2 & j+3 & \dots & r+2 \\ b_1 & \dots & b_{j-1} & b_j & b_{j+1} & \dots & b_r \end{pmatrix},$$

and is assumed that $|A_j| \geq 2$, $x_j = \min\{x : x \in A_j\}$. Observe that γ_2 and γ_3 belong to $\langle N \rangle$ by Theorem 3.1.5, and since $|T_i| \geq 2$ for some i, we have γ_1 to be an element in $\langle N \rangle$ also, by Theorem 4.1.2.

If $A_i \neq S_i$ for some i and $k < n-1$, then we may assume that there exists $x \in X_n \backslash \operatorname{dom} \alpha$ such that $y_j < x < y_{j+1}$ for some j, where $y_t = \max\{x : x \in A_t\}$. Here we have

$$\alpha = \beta_1 \beta_2,$$

where

$$\beta_1 = \begin{pmatrix} A_1 & \dots & A_j & x & A_{j+1} & \dots & A_r \\ 1 & \dots & j & j+1 & j+3 & \dots & r+2 \end{pmatrix},$$

$$\beta_2 = \begin{pmatrix} 1 & \dots & j & j+3 & \dots & r+2 \\ b_1 & \dots & b_j & b_{j+1} & \dots & b_r \end{pmatrix}.$$

Observe here too, that β_2 belongs to $\langle N \rangle$, and since $A_i \neq S_i$ for some i, we have β_1 to be an element in $\langle N \rangle$ also, by Theorem 4.1.2.

If $A_i \neq S_i$ for some i and $k = n-1$, then it is clear that $|A_i| \geq 2$. If $|A_i| = 2$ then there exists another block, say A_k, such that $|A_k| \geq 2$ (since $r \leq n-3$), and

$$\alpha = \delta_1 \delta_2 \delta_3,$$

where

$$\delta_1 = \begin{pmatrix} A_1 & \dots & A_{k-1} & x_k & A_k \backslash \{x_k\} & A_{k+1} & \dots & A_r \\ 1 & \dots & k-1 & k & k+1 & k+2 & \dots & r+1 \end{pmatrix},$$

$$\delta_2 = \begin{pmatrix} 1 & \dots & k-1 & \{k,k+1\} & k+2 & \dots & r+2 \\ 2 & \dots & k & k+2 & k+3 & \dots & r+3 \end{pmatrix}$$

and

$$\delta_3 = \begin{pmatrix} 2 & \dots & k & k+2 & k+3 & \dots & r+2 \\ b_1 & \dots & b_{k-1} & b_k & b_{k+1} & \dots & b_r \end{pmatrix}.$$

If $|A_i| > 2$ then there exists $a_i \in A_i$ and $s_i \in S_i$ such that either $x_i < a_i < s_i < y_i$ or $x_i < s_i < a_i < y_i$. If $x_i < a_i < s_i < y_i$ then

$$\alpha = \lambda_1 \lambda_2 \lambda_3 ,$$

where

$$\lambda_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & x_i & A_i \backslash \{x_i\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r+1 \end{pmatrix},$$

$$\lambda_2 = \begin{pmatrix} 1 & \dots & i-1 & \{i, i+1\} & i+2 & \dots & r+2 \\ 2 & \dots & i & i+2 & i+3 & \dots & r+3 \end{pmatrix}$$

and

$$\lambda_3 = \begin{pmatrix} 2 & \dots & i & i+2 & i+3 & \dots & r+2 \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

If $x_i < s_i < a_i < y_i$ then

$$\alpha = \tau_1 \tau_2 \tau_3 ,$$

where

$$\tau_1 = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i \backslash \{y_i\} & y_i & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & r+1 \end{pmatrix},$$

$$\tau_2 = \begin{pmatrix} 1 & \dots & i-1 & \{i, i+1\} & i+2 & \dots & r+2 \\ 2 & \dots & i & i+2 & i+3 & \dots & r+3 \end{pmatrix}$$

and

$$\tau_3 = \begin{pmatrix} 2 & \dots & i & i+2 & i+3 & \dots & r+2 \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

Note that by the same argument as in the other cases, $\lambda_1, \lambda_2, \lambda_3$ and τ_1, τ_2, τ_3 are all elements in $\langle N \rangle$. ■

Let N_1 and N_2 be the set of all nilpotent elements in PO_n of height $n-1$ and $n-2$ respectively. Then, since all the elements in N_1 are one-one maps, we have by

Proposition 3.3.2 that N_1 does not generate $\langle N \rangle$. However, by Lemma 4.3.1 above we do have

$$\langle N_2 \rangle = \langle N \rangle \backslash J_{n-1}.$$

Our aim here is to determine the rank and the nilpotent rank of $\langle N_2 \rangle$.

Recall from Chapter 3 Section 3 that the number of $\mathcal{R}$-classes and that of $\mathcal{L}$-classes containing nilpotents, or elements that are expressible as products of nilpotents in a $\mathcal{J}$-class, J_r of IO_n, where $n/2 < r \leq n-2$, are both equal to

$$\binom{n}{r} - \binom{r-1}{n-r}.$$

It therefore follows that the number of $\mathcal{R}$-classes in $\langle N_2 \rangle \cap [n-2, n-2]$ is equal to the number of $\mathcal{L}$-classes in $\langle N_2 \rangle \cap J_{n-2}$ and is

$$\binom{n}{n-2} - \binom{n-3}{2} = 3(n-2).$$

By Theorem 4.1.2 any convex equivalence of weight $n-2$ on the subset $\{1, \ldots, n-1\}$ or $\{2, \ldots, n\}$ determines an $\mathcal{R}$-class in $\langle N_2 \rangle \cap [n-1, n-2]$. Thus the number of $\mathcal{R}$-classes in $\langle N_2 \rangle \cap [n-1, n-2]$ determined by these convex equivalences is $2(n-2)$. On the other hand any convex equivalence of weight $n-2$ on a subset containing 1 and n represents an $\mathcal{R}$-class in $\langle N_2 \rangle \cap [n-1, n-2]$ if and only if i and $i+2$ belong to the same equivalence class for some i in $\{1, \ldots, n-2\}$. Thus the number of such convex equivalences is $n-2$. Hence the number of $\mathcal{R}$-classes in $\langle N_2 \rangle \cap [n-1, n-2]$ is $3(n-2)$. We therefore have $6(n-2)$ as the number of $\mathcal{R}$-classes in $\langle N_2 \rangle \cap J_{n-2}$.

We now show that every element $\alpha \in \langle N_2 \rangle \cap [n-1, n-2]$ is expressible in terms of a fixed element in its own $\mathcal{R}$-class and an element in $\langle N_2 \rangle \cap [n-2, n-2]$. More generally we shall show:

Lemma 4.3.2 *Every element $\alpha \in \langle N_2 \rangle \cap [k, r]$, $r < k \leq n-1$ is expressible as a product of a fixed nilpotent in $\langle N_2 \rangle \cap [k, r]$ and an element in $\langle N_2 \rangle \cap [r, r]$.*

Proof. Let $\alpha \in \langle N_2 \rangle \cap [k, r]$, and suppose that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}.$$

We shall distinguish four cases.

Case 1. $1 \notin A_1$. Then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ 1 & 2 & \dots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}.$$

Case 2. $n \notin A_r$. Then

$$\alpha = \beta\gamma$$

where

$$\beta = \begin{pmatrix} A_1 & A_2 & \dots & A_{r-1} & A_r \\ n-r+1 & n-r+2 & \dots & n-1 & n \end{pmatrix},$$

$$\gamma = \begin{pmatrix} n-r+1 & n-r+2 & \dots & n-1 & n \\ b_1 & b_2 & \dots & b_{r-1} & b_r \end{pmatrix}.$$

Case 3. $1 \in A_1$, $n \in A_r$ and $A_i \neq S_i$ for some i. Let c be a fixed element in $S_i \backslash A_i$. Then

$$\alpha = \lambda\mu$$

where

$$\lambda = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & \dots & A_r \\ x_2 & \dots & x_i & c & y_i & \dots & y_{r-1} \end{pmatrix},$$

$$\mu = \begin{pmatrix} x_2 & \dots & x_i & c & y_i & \dots & y_{r-1} \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_r \end{pmatrix}.$$

Case 4. $1 \in A_1$, $n \in A_r$, $A_i = S_i$ for all i and $|T_i| \geq 2$ for some i. Let c, d be two fixed elements in T_i with $c < d$. Then

$$\alpha = \zeta\xi$$

where

$$\zeta = \begin{pmatrix} A_1 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ y_2 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$\xi = \begin{pmatrix} y_2 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & b_{i+2} & \dots & b_r \end{pmatrix}.$$

■

Theorem 4.3.3 $\operatorname{rank}(\langle N_2 \rangle) = 6(n-2)$.

Proof. Since $\langle N_2 \rangle \cap J_{n-2}$ has $6(n-2)$ $\mathcal{R}$-classes we have

$$\operatorname{rank}(\langle N_2 \rangle) \geq 6(n-2).$$

By Proposition 2.2.7, $[n-2, n-2] \cap \langle N_2 \rangle$ is generated by a set of $3(n-2)$ elements. If we now choose a set of $3(n-2)$ elements to cover the $\mathcal{R}$-classes in $[n-1, n-2]$ as in Lemma 4.3.2, we obtain a generating set of $\langle N_2 \rangle$ consisting of $6(n-2)$ elements. Hence the proof. ■

Lemma 4.3.4 *Every $\mathcal{L}$-class in J_{n-2} represented by the set*

$$\{1, 2, \dots, i-1, i+2, \dots, n\}$$

for $i = 2, \dots, n-2$ contains a single nilpotent. Thus there are $n-3$ $\mathcal{L}$-classes in J_{n-2} containing only one nilpotent.

Proof. Let α be an element whose $\mathcal{L}$-class is $\{1, \dots, i-1, i+2, \dots, n\}$. Then the only domain for which α is nilpotent is that represented by the set $\{2, \dots, n-1\}$. ■

Theorem 4.3.5 $\operatorname{nilrank}(\langle N_2 \rangle) = 7n - 15$.

Proof. Since any generating set of $\langle N_2 \rangle$ must cover the $\mathcal{L}$-classes in $\langle N_2 \rangle \cap J_{n-2}$, the $n-3$ nilpotents whose image set is $\{1, \dots, i-1, i+2, \dots, n\}$ for $i = 2, \dots, n-2$ must be contained in a generating set consisting of only nilpotent elements (see Lemma 4.3.4). By the same Lemma 4.3.4 (proof) all the $n-3$ nilpotents belong to the same $\mathcal{R}$-class, determined by the set $\{2, \dots, n-1\}$. For the generating set to cover all the

$\mathcal{R}$-classes we must now choose $6(n-2)-1$ nilpotents from the remaining $\mathcal{R}$-classes, making a total of $7n-16$ nilpotents. However the $7n-16$ nilpotents cannot generate $\langle N_2 \rangle$. For if α is an element in the same $\mathcal{R}$-class as the $n-3$ nilpotents (that is the $\mathcal{R}$-class represented by the set $\{2, \ldots, n-1\}$) and suppose that

$$\alpha = n_1 n_2 \cdots n_k$$

is the decomposition of α in terms of nilpotents from the chosen $7n-16$ nilpotents, then we must have

$$n_1 = \begin{pmatrix} 2 & 3 & \ldots & i & i+1 & \ldots & n-1 \\ 1 & 2 & \ldots & i-1 & i+2 & \ldots & n \end{pmatrix},$$

$$n_2 = \begin{pmatrix} 1 & 2 & \ldots & i-1 & i+2 & \ldots & n \\ 2 & 3 & \ldots & i & i+1 & \ldots & n-1 \end{pmatrix}$$

and

$$n_3 = \begin{pmatrix} 2 & 3 & \ldots & j & j+1 & \ldots & n-1 \\ 1 & 2 & \ldots & j-1 & j+2 & \ldots & n \end{pmatrix}$$

for some $i, j = 2, \ldots, n-2$. But then $n_1 n_2$ is a left identity for n_3, and so

$$\alpha = n_3 n_4 \cdots n_k.$$

By the same reasoning we must also have

$$n_4 = \begin{pmatrix} 1 & 2 & \ldots & j-1 & j+2 & \ldots & n \\ 2 & 3 & \ldots & j & j+1 & \ldots & n-1 \end{pmatrix}$$

and

$$n_5 = \begin{pmatrix} 2 & 3 & \ldots & l & l+1 & \ldots & n-1 \\ 1 & 2 & \ldots & l-1 & l+2 & \ldots & n \end{pmatrix}.$$

But again $n_3 n_4$ is then a left identity for n_5, and

$$\alpha = n_5 \cdots n_k.$$

Continuing this way we obtain

$$\alpha = \begin{cases} n_k & \text{if } k \text{ is odd,} \\ \begin{pmatrix} 2 & 3 & \ldots & n-1 \\ 2 & 3 & \ldots & n-1 \end{pmatrix} & \text{if } k \text{ is even.} \end{cases}$$

Thus if α is not any of the $n-3$ nilpotents in its $\mathcal{R}$-class, the left identity in the $\mathcal{R}$-class, then α cannot be expressed as a product of nilpotents from the chosen $7n-16$ nilpotents. We therefore have

$$\text{nilrank}\,(\langle N_2\rangle) \geq 7n-15\,.$$

We now show that we can choose $7n-15$ nilpotents in N_2 that can generate $\langle N_2\rangle$. Denote by $A_{i,j}$ the subset $X_n\backslash\{i,j\}$ of cardinality $n-2$, and by $\alpha^{i,j}_{s,t}$ the element whose domain is $A_{i,j}$ and image $A_{s,t}$. Then arrange the $3(n-2)$ subsets of X_n of cardinality $n-2$, representing the $\mathcal{L}$- and the $\mathcal{R}$-classes in $\langle N_2\rangle \cap [n-2\,,n-2]$ as follows:

$$A_{2,n}, A_{1,3}\,, A_{3,n}, \ldots, A_{1,i}, A_{i,n}, \ldots, A_{1,n-1}\,, A_{n-1,n}\,,$$
$$A_{1,n}, A_{2,3}\,, A_{3,4}\,, \ldots, A_{n-2,n-1}\,, A_{1,2}\,.$$

By Proposition 2.2.7, $\langle N_2\rangle \cap [n-2\,,n-2]$ is generated by the set

$$B = \{\alpha^{2,n}_{1,3}, \alpha^{1,3}_{3,n}, \alpha^{3,n}_{1,4}, \ldots, \alpha^{1,i}_{i,n}, \alpha^{i,n}_{1,i+1}, \ldots, \alpha^{1,n-1}_{n-1,n}, \alpha^{n-1,n}_{1,n}\,,$$
$$\alpha^{1,n}_{2,3}, \alpha^{2,3}_{3,4}, \ldots, \alpha^{n-3,n-2}_{n-2,n-1}\,, \alpha^{n-2,n-1}_{1,2}\,, \alpha^{1,2}_{2,n}\}.$$

It is easy to see that $\alpha^{1,i}_{i,n}, \alpha^{i,n}_{1,i+1}$ (for $i = 3,\ldots,n-1$), $\alpha^{2,n}_{1,3}, \alpha^{1,n}_{2,3}$ and $\alpha^{1,2}_{2,n}$ are all nilpotents. It is also not difficult to see that

$$\alpha^{2,3}_{3,4}, \ldots, \alpha^{n-3,n-2}_{n-2,n-1}\,, \alpha^{n-2,n-1}_{1,2} \tag{4.3.6}$$

are all non-nilpotent. In fact n is fixed by all of these elements. Let us denote by B' the set of all nilpotent elements in B. Let T be the set of $4(n-2)-1$ elements given by

$$T = B' \cup \{\alpha^{1,n}_{3,4}, \ldots, \alpha^{1,n}_{n-2,n-1}\,, \alpha^{1,n}_{1,2}, \alpha^{2,3}_{1,n}, \ldots, \alpha^{n-2,n-1}_{1,n}\}.$$

It is easy here too, to see that all the elements in T are nilpotents. Next we observe that the non-nilpotent elements in B, given by (4.3.6) are expressible as products of elements in T. In fact we have

$$\alpha^{i,i+1}_{i+1,i+2} = \alpha^{i,i+1}_{1,n}\,\alpha^{1,n}_{i+1,i+2} \quad \text{for} \quad i = 2,\ldots,n-3$$

and

$$\alpha_{1,2}^{n-2,n-1} = \alpha_{1,n}^{n-2,n-1} \alpha_{1,2}^{1,n}.$$

Thus

$$\langle B \rangle = \langle T \rangle.$$

If we now choose a set H of $3(n-2)$ nilpotents to cover the $\mathcal{R}$-classes in $\langle N_2 \rangle \cap [n-1, n-2]$ as in Lemma 4.3.2 we obtain a generating set $H \cup T$ of $\langle N_2 \rangle$ consisting of nilpotent elements. Since $|H \cup T| = 7n - 15$ the proof is complete. ■

4. The infinite case

Let $X = \{1, 2, \ldots\}$ be the set of all natural numbers. Denote by PO_ω the set of all partial order-preserving maps on X, and an element α in PO_ω by

$$\begin{pmatrix} A_i \\ b_i \end{pmatrix}$$

where i belongs to some index set I, $b_i < b_j$ for all $i < j$ and if $a_i \in A_i$ then $a_i < a_j$ for all $i < j$. Observe that im α is finite if and only if dom α is finite or $|y\alpha^{-1}| < \infty$ ($y \in \text{im } \alpha$) except for at most one y in im α, namely $\max\{y : y \in \text{im } \alpha\}$. We will write such an element α as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix},$$

where $r = |\text{im } \alpha|$. As usual, let $x_i = \min\{x : x \in A_i\}$ for $i = 1, \ldots, r$, $y_i = \max\{x : x \in A_i\}$ for $i = 1, \ldots, r-1$. (Note that y_r does not exist.) Let $S_i = \{x \in X : x_i \leq x \leq y_i\}$ for $i = 1, \ldots, r-1$, $S_r = \{x \in X : x \geq x_r\}$, $T_i = \{x \in X : y_i < x < x_{i+1}\}$ for $i = 1, \ldots, r-1$ and $T_0 = \{x \in X : x < x_1\}$.

Let N be the set of all nilpotents in PO_ω and $\langle N\rangle$ the subsemigroup of PO_ω generated by N. Define the *gap* and the *defect* of an element α in PO_ω by

$$\text{gap } \alpha = X \setminus \text{dom } \alpha, \quad \text{def } \alpha = X \setminus \text{im } \alpha,$$

and

$$U = \{\alpha \in PO_\omega : \exists y \in \text{im } \alpha \text{ for which } |y\alpha^{-1}| = |X|,$$
$$1 \leq |\text{ gap } \alpha| < |\text{ im } \alpha|, 1 \in A_1, S_i = A_i \text{ and } |T_i| \leq 1 \text{ for all } i\}.$$

Then

Lemma 4.4.1 $\langle N\rangle \cap U = \emptyset$.

Proof. Let $\alpha \in U$. Then α can be written as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix},$$

where $|A_r| = |X|$. Suppose that α is a product of k nilpotents, say

$$\alpha = n_1 n_2 \cdots n_k,$$

with

$$n_1 = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ c_1 & c_2 & \dots & c_r \end{pmatrix},$$

where $c_1 < c_2 < \cdots < c_r$. We first show by induction that $c_i > y_i$ for all i, $1 \leq i \leq r-1$. The result is clearly true for $i = 1$. So suppose that it is true for all $i \leq k$ and that $c_{k+1} < y_{k+1}$. Then, since $A_{k+1} = S_{k+1}$ we must have $c_{k+1} < x_{k+1}$. Thus

$$y_k < c_k < c_{k+1} < x_{k+1},$$

which means that $|T_k| \geq 2$. This is a contradiction. So $c_i > y_i$ for all i, $1 \leq i \leq r-1$. But since $A_r = S_r$ we must have $c_r < x_r$ and so $y_{r-1} < c_{r-1} < c_r < x_r$, that is

$|T_{r-1}| \geq 2$. This again is a contradiction. Hence α cannot be a product of nilpotents. ■

Observe that if $|\text{gap } \alpha| \geq |\text{im } \alpha|$ then $|T_i| \geq 2$ for some i.

Lemma 4.4.2 *Let* $\alpha \in PO_\omega$ *and* $y \in \text{im } \alpha$ *be such that* $|y\alpha^{-1}| = |X|$. *Then* α *is nilpotent if and only if* $x\alpha \neq x$ *for all* $x \in \text{dom } \alpha$.

Proof. An element α satisfying this Lemma can be written as

$$\begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

where $b_r = y$, $|A_r| = |X|$ and $r = |\text{im } \alpha|$. The proof of Lemma 3.1.1 applies to this case also. ■

The following Lemma is from **[26]**.

Lemma 4.4.4 **[26**, Lemma 12] *If* $\alpha \in \langle N \rangle$ *then* $|\text{def } \alpha| = |X|$ *and either* $|\text{gap } \alpha| = |X|$ *or there exists* $y \in \text{im } \alpha$ *such that* $|y\alpha^{-1}| = |X|$. ■

It is clear that if $\text{dom } \alpha \cap \text{im } \alpha = \emptyset$, then α is nilpotent (of index 2).

Theorem 4.4.5 $\alpha \in \langle N \rangle$ *if and only if*

(i) $\text{gap } \alpha \neq \emptyset$,

(ii) $|\text{def } \alpha| = |X|$ *and*

(iii) *either* $|\text{gap } \alpha| = |X|$ *or there exists* $y \in \text{im } \alpha$ *such that* $|y\alpha^{-1}| = |X|$ *and* $\alpha \notin U$.

Proof. Suppose that $|\text{def } \alpha| = |\text{gap } \alpha| = |X|$. We shall consider two cases separately, namely $|X \setminus (\text{im } \alpha \cup \text{dom } \alpha)| = |X|$ and $|X \setminus (\text{im } \alpha \cup \text{dom } \alpha)| < |X|$.

If $|X \setminus (\text{im } \alpha \cup \text{dom } \alpha)| = |X|$, then choose c_i in $X \setminus (\text{im } \alpha \cup \text{dom } \alpha)$ such that $|\{c_i\}| = |\text{im } \alpha|$ with $c_i < c_j$ for all $i < j$. Then

$$\alpha = \begin{pmatrix} A_i \\ c_i \end{pmatrix} \begin{pmatrix} c_i \\ b_i \end{pmatrix},$$

a product of two nilpotents in PO_ω.

If $|X\backslash(\text{im }\alpha \cup \text{dom }\alpha)| < |X|$, then $|\text{im }\alpha| = |\text{im }\alpha \cap (X\backslash \text{dom }\alpha)| = |X|$ and so there exists $c_i \in \text{im }\alpha \cap (X\backslash \text{dom }\alpha)$ and $d_i \in X\backslash \text{im }\alpha$ with $c_i < c_j$, $d_i < d_j$ for all $i < j$, such that

$$\alpha = \begin{pmatrix} A_i \\ c_i \end{pmatrix} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \begin{pmatrix} d_i \\ b_i \end{pmatrix},$$

a product of three nilpotents in PO_ω.

Now, suppose $1 \leq |\text{gap }\alpha| < |X|$ and there exists $y \in \text{im }\alpha$ with $|y\alpha^{-1}| = |X|$ and $\alpha \notin U$. Here we shall distinguish three cases.

Case1. $1 \notin A_1$. It is clear that in this case $i \notin A_i$ for all i. Choose $c_i \in X\backslash(\{1, \ldots, r\} \cup \text{im }\alpha)$ with $c_i < c_{i+1}$ for $i = 1, \ldots, r$. Then

$$\alpha = \begin{pmatrix} A_1 & A_2 & \ldots & A_r \\ 1 & 2 & \ldots & r \end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & r \\ c_1 & c_2 & \ldots & c_r \end{pmatrix} \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix},$$

a product of three nilpotents.

Case2. $S_i \neq A_i$ for some i. Then there exists $c \in S_i\backslash A_i$. Choose $c_i \in X\backslash(\{x_2, \ldots, x_i, y_i, \ldots, y_{r-1}, c\} \cup \text{im }\alpha)$ with $c_i < c_{i+1}$ for $i = 1, \ldots, r$. Then

$$\alpha = n_1 n_2 n_3,$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & A_2 & \ldots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \ldots & A_r \\ x_2 & x_3 & \ldots & x_i & c & y_i & y_{i+1} & \ldots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} x_2 & x_3 & \ldots & x_i & c & y_i & y_{i+1} & \ldots & y_{r-1} \\ c_1 & c_2 & \ldots & c_{i-1} & c_i & c_{i+1} & c_{i+2} & \ldots & c_r \end{pmatrix}$$

and

$$n_3 = \begin{pmatrix} c_1 & c_2 & \ldots & c_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}.$$

Case3. $|T_i| \geq 2$ for some i. Then there exist $c, d \in T_i$. Choose $c_i \in X\backslash(\{y_2, \ldots, y_{r-1}, c, d\} \cup \text{im }\alpha)$ with $c_i < c_{i+1}$ for $i = 1, \ldots, r$. Then

$$\alpha = n_1 n_2 n_3,$$

a product of three nilpotents, where

$$n_1 = \begin{pmatrix} A_1 & A_2 & \dots & A_{i-1} & A_i & A_{i+1} & A_{i+2} & \dots & A_r \\ y_2 & y_3 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \end{pmatrix},$$

$$n_2 = \begin{pmatrix} y_2 & y_3 & \dots & y_i & c & d & y_{i+1} & \dots & y_{r-1} \\ c_1 & c_2 & \dots & c_{i-1} & c_i & c_{i+1} & c_{i+2} & \dots & c_r \end{pmatrix}$$

and

$$n_3 = \begin{pmatrix} c_1 & c_2 & \dots & c_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}.$$

■

The following result is now immediate:

Corollary 4.4.6 *Let IO_ω be the semigroup of all one-one partial order-preserving maps on X. Then an element α in IO_ω is a product of nilpotents in IO_ω if and only if*

$$|X \setminus \text{dom } \alpha| = |X \setminus \text{im } \alpha| = |X|.$$

Moreover, α can be expressed as a product of three or fewer nilpotents, each with index 2. ■

Lemma 4.4.7 *For $n \geq 4$, the element*

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & n-3 & A \\ 1 & 2 & \dots & n-3 & n-1 \end{pmatrix},$$

where $A = \{n, n+1, \dots\}$ cannot be expressed as a product of fewer than three nilpotents.

Proof. The only image set for which

$$\begin{pmatrix} 1 & 2 & \dots & n-3 & A \\ c_1 & c_2 & \dots & c_{n-3} & c_{n-2} \end{pmatrix}$$

is nilpotent is $\{2, 3, \ldots, n-1\}$. But

$$\begin{pmatrix} 2 & 3 & \ldots & n-2 & n-1 \\ 1 & 2 & \ldots & n-3 & n-1 \end{pmatrix}$$

is not nilpotent, whence the result. ∎

We now have the following result:

Theorem 4.4.8 *If we let $\Delta(\langle N \rangle)$ be the least k for which*

$$\langle N \rangle = N \cup N^2 \cup \cdots \cup N^k, \quad \langle N \rangle \neq N \cup N^2 \cup \cdots \cup N^{k-1}.$$

Then $\Delta(\langle N \rangle) = 3$. ∎

CHAPTER FIVE

RANK PROPERTIES

The questions of the ranks of O_n, PO_n and SPO_n, and the idempotent ranks of O_n and PO_n were considered by Gomes and Howie in [**14**]. In another paper [**13**], Gomes and Howie considered the rank and the nilpotent rank of the subsemigroup of I_n generated by the nilpotent elements. We now generalise these questions (in line with Howie and McFadden [**21**]) by asking for the rank, idempotent rank and the nilpotent rank of the semigroup

$$K(n,r) = \{\alpha \in S : |\operatorname{im}\alpha| \leq r \text{ and } r \leq n-1\}$$

as the case may be, where S is O_n, PO_n, SPO_n or the subsemigroup of I_n generated by the nilpotents.

1. Order-preserving full transformations

We begin this section with the result of Howie [**17**].

Theorem 5.1.1 [**17**, Theorem 1.1] *If X_n is a finite totally ordered set, then every element of the semigroup O_n of order-preserving mappings of X_n into itself is expressible as a product of idempotents in O_n.* ■

Lemma 5.1.2 *Every element α in J_r $(r \leq n-2)$ is expressible as a product of elements in J_{r+1}.*

Proof. Let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}.$$

Then at least one block, say A_i, contains more than one element. Let $c = \min\{a_i : a_i \in A_i\}$. Suppose that $\{b_1, b_2, \dots, b_r\}$ has a gap in position j, and let y be such that $b_{j-1} < y < b_j$. We distinguish three cases.

Case 1. $i = j - 1$. Let

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & c & A_i \backslash \{c\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+3 & \dots & r+2 \end{pmatrix}$$

and

$$\delta = \begin{pmatrix} 1 & \dots & i-1 & \{i, i+1\} & i+2 & i+3 & \dots & r+1 & A' \\ b_1 & \dots & b_{i-1} & b_i & y & b_{i+1} & \dots & b_{r-1} & b_r \end{pmatrix}$$

where $A' = X_n \backslash \{1, 2, \dots, r+1\}$. Then $\beta, \delta \in J_{r+1}$ and $\alpha = \beta\delta$.

Case 2. $i < j - 1$. Suppose here that β and δ are given by

$$\begin{pmatrix} A_1 & \dots & A_{i-1} & c & A_i \backslash \{c\} & A_{i+1} & \dots & A_{j-1} & A_j & \dots & A_r \\ 1 & \dots & i-1 & i & i+1 & i+2 & \dots & j & j+2 & \dots & r+2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & \dots & i-1 & Y & i+2 & \dots & j & j+1 & j+2 & \dots & r+1 & A' \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_{j-1} & y & b_j & \dots & b_{r-1} & b_r \end{pmatrix}$$

respectively, where $Y = \{i, i+1\}$. Then $\beta, \delta \in J_{r+1}$ and $\alpha = \beta\delta$.

Case 3. $i = j$. Let

$$\beta = \begin{pmatrix} A_1 & \dots & A_{i-1} & c & A_i \backslash \{c\} & A_{i+1} & \dots & A_r \\ 1 & \dots & i-1 & i+1 & i+2 & i+3 & \dots & r+2 \end{pmatrix}$$

and

$$\delta = \begin{pmatrix} 1 & \dots & i-1 & i & \{i+1, i+2\} & i+3 & \dots & r+1 & A' \\ b_1 & \dots & b_{i-1} & y & b_i & b_{i+1} & \dots & b_{r-1} & b_r \end{pmatrix},$$

where $A' = X_n \backslash \{1, \dots, r+1\}$. Then $\beta, \delta \in J_{r+1}$ and $\alpha = \beta\delta$.

Case 4. $i > j$. Suppose β and δ are given by

$$\begin{pmatrix} A_1 & \dots & A_{j-1} & A_j & \dots & A_{i-1} & c & B & A_{i+1} & \dots & A_r \\ 1 & \dots & j-1 & j+1 & \dots & i & i+1 & i+2 & i+3 & \dots & r+2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & \dots & j-1 & j & j+1 & \dots & i & Z & i+3 & \dots & r+1 & A' \\ b_1 & \dots & b_{j-1} & y & b_j & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_{r-1} & b_r \end{pmatrix}$$

respectively, where $B = A_i \backslash \{c\}$ and $Z = \{i+1, i+2\}$. Then $\beta, \delta \in J_{r+1}$ and $\alpha = \beta\delta$. Hence the proof. ■

The following Lemma is from **[15]**:

Lemma 5.1.3 [**15**, Lemma 1] *Let S be a regular semigroup with set E of idempotents, and let*

$$a = \epsilon_1 \epsilon_2 \cdots \epsilon_n \in E^n.$$

Then there exist idempotents $f_1, f_2, \dots, f_n$ in the $\mathcal{J}$-class containing a such that

$$a = f_1 f_2 \cdots f_n.$$

■

For $r \leq n-2$, let

$$K(n,r) = \{\alpha \in O_n : |\operatorname{im} \alpha| \leq r\}.$$

Then by Lemma 5.1.2

$$K(n,r) = \langle J_r \rangle.$$

If we let E_r be the set of idempotents in J_r, then by Lemma 5.1.3, and Theorem 5.1.1,

$$J_r \subseteq \langle E_r \rangle.$$

Thus

$$K(n,r) = \langle E_r \rangle.$$

By [**18**, Proposition 2.4.5 and Exercise 2.10] we have that in O_n

$$\alpha \,\mathcal{L}\, \beta \text{ if and only if } \operatorname{im} \alpha = \operatorname{im} \beta,$$

$$\alpha \,\mathcal{R}\, \beta \text{ if and only if } \ker \alpha = \ker \beta,$$

$$\alpha \,\mathcal{J}\, \beta \text{ if and only if } |\operatorname{im} \alpha| = |\operatorname{im} \beta|.$$

Thus O_n is the union of $\mathcal{J}$-classes

$$J_1, J_2, \ldots, J_{n-1},$$

where

$$J_r = \{\alpha \in O_n : |\operatorname{im} \alpha| = r\}.$$

The semigroup O_n is *aperiodic* (i.e., has trivial $\mathcal{H}$-classes); for once we fix im α and ker α there is precisely one order-preserving map having the given image and kernel. It is easy to see that the (ker α)-classes are *convex* subsets C of X_n, in the sense that

$$x, y \in C \text{ and } x \le z \le y \Rightarrow z \in C.$$

In this section, as well as the next, we shall refer to an equivalence ρ on the set X_n as *convex* if its classes are convex subsets of X_n, and we shall say that ρ is of weight r if $|X_n/\rho| = r$. A convex equivalence of weight r is determined by the insertion of $r-1$ 'boundaries'in the $n-1$ spaces between $1, 2, \ldots, n$. (Thus, for example, the convex equivalence of weight 3 on X_6 whose classes are $\{1,2,3\}$, $\{4\}$ and $\{5,6\}$ is determined by inserting two boundaries, between 3 and 4 and between 4 and 5.) We deduce that the number of convex equivalences of weight r on X_n is

$$\binom{n-1}{r-1}.$$

Thus J_r has $\binom{n-1}{r-1}$ $\mathcal{R}$-classes corresponding to the $\binom{n-1}{r-1}$ convex equivalences of weight r on X_n, and $\binom{n}{r}$ $\mathcal{L}$-classes corresponding to the $\binom{n}{r}$ subsets of X_n of cardinality r.

From Lemma 1.3.3 we deduce that the rank of $K(n,r)$ must be at least as large as the number of $\mathcal{L}$-classes in J_r. Thus we have:

Lemma 5.1.4 $\operatorname{rank}\left(K(n,r)\right) \geq \binom{n}{r}$. ■

We now show:

Theorem 5.1.5 *For $2 \leq r \leq n-2$, we have*

$$\operatorname{rank}\left(K(n,r)\right) = \operatorname{idrank}\left(K(n,r)\right) = \binom{n}{r}.$$

The proof depends on a Lemma very similar to Lemma 6 in [21].

Lemma 5.1.6 *Let $\pi_1, \pi_2, \ldots, \pi_m$ (where $m = \binom{n-1}{r-1}, r \geq 3$) be a list of the convex equivalences of weight r on X_n. Suppose that there exist distinct subsets $A_1, A_2, \ldots, A_m$ of cardinality r of X_n with the property that A_i is a transversal of π_{i-1}, π_i ($i = 2, \ldots, m$) and A_1 is a transversal of π_1, π_m. Then each $\mathcal{H}$-class (π_i, A_i) consists of an idempotent ϵ_i, and there exist idempotents $\epsilon_{m+1}, \ldots, \epsilon_p$ (where $p = \binom{n}{r}$) such that $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_p\}$ is a set of generators for $K(n,r)$.*

Notice that the product $\epsilon_i\epsilon_{i-1}$ ($i = 2, \ldots, m$) is an element of height r, since we have a configuration

ϵ_{i-1}	$\circ$
$*$	ϵ_i

in which the $\mathcal{H}$-class labelled $\circ$ consists of an idempotent. Moreover, the element $\epsilon_i\epsilon_{i-1}$ is in the position * by Lemma 1.3.3. By the same token the product $\epsilon_1\epsilon_m$ is of height r, and $\epsilon_m \,\mathcal{L}\, \epsilon_1\epsilon_m \,\mathcal{R}\, \epsilon_1$.

Choose the idempotents $\epsilon_{m+1}, \ldots, \epsilon_p$ so that $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ covers all the $\mathcal{L}$-classes in J_r. Then if η is an arbitrary idempotent in J_r there exists a unique $i \in \{1, \ldots, p\}$ such that $\eta \, \mathcal{L} \, \epsilon_i$, and a unique $j \in \{1, \ldots, m\}$ such that $\eta \, \mathcal{R} \, \epsilon_j$.

$$\begin{array}{ccccc} \epsilon_k & \cdots & & \cdots & \epsilon_i \\ & & \vdots & & \vdots \\ & \epsilon_{j-1} & \circ & \cdots & \\ & & \epsilon_j & \cdots & \eta \end{array}$$

Moreover, there is a unique $k \in \{1, \ldots, m\}$ such that $\epsilon_i \, \mathcal{R} \, \epsilon_k$. (If $i \in \{1, \ldots, m\}$ then of course $k = i$.) If $k = j$ then $\eta = \epsilon_i$ and there is nothing to prove. If $k < j$ then

$$\eta = \epsilon_j \epsilon_{j-1} \cdots \epsilon_{k+1} \epsilon_i .$$

If $k > j$ then

$$\eta = \epsilon_j \cdots \epsilon_1 \epsilon_m \cdots \epsilon_{k+1} \epsilon_i .$$

We have shown that every idempotent in J_r can be expressed as a product of the $p = \binom{n}{r}$ idempotents, $\epsilon_1, \ldots, \epsilon_p$. Hence

$$K(n, r) = \langle \epsilon_1, \epsilon_2, \ldots, \epsilon_p \rangle .$$

■

It remains to prove that the listing of convex equivalences and images postulated in the statement of Lemma 5.1.6 can actually be carried out. Let $n \geq 4$ and $2 \leq r \leq n - 2$, and consider the Proposition:

P(n,r): *There is a way of listing the convex equivalences of weight r as $\pi_1, \pi_2, \ldots, \pi_m$ (with $m = \binom{n-1}{r-1}$ and π_1 having $\{r, r+1, \ldots, n\}$ as the only non-singleton class, π_2 having $\{r-1, r\}$ and $\{r+1, \ldots, n\}$ as the only non-singleton classes, π_m having $\{r-1, \ldots, n-1\}$ as the only non-singleton class) so that there exist subsets $A_1, \ldots, A_m$ of X_n of cardinality r with the property that A_i is a transversal of π_{i-1}, π_i $(i = 2, \ldots, m)$ and A_1 is a transversal of π_1, π_m.*

The approach to the proof is similar to that of Chapter 2 Section 2, i.e., we show

$$\mathbf{P(n-1,r-1)}\ \text{ and }\ \mathbf{P(n-1,r)} \Rightarrow \mathbf{P(n,r)}.$$

First, however, we anchor the induction with two Lemmas.

Lemma 5.1.7 $\mathbf{P(n,2)}$ *holds for every* $n \geq 4$.

Proof. Consider the list $\pi_1, \ldots, \pi_{n-1}$ of convex equivalences of weight 2 on X_n, where

$$\pi_i = 1\ 2\ 3 \cdots i/i+1 \cdots n.$$

Let

$$A_1 = \{1, n\},\ \ A_2 = \{1, 3\}\ \text{ and }\ A_i = \{i-1, n\}$$

for $i = 3, \ldots, n-1$. Then it is easy to verify that $\pi_1, \pi_2, \ldots, \pi_{n-1}$ and $A_1, A_2, \ldots, A_{n-1}$ have the required property. ■

Lemma 5.1.8 $\mathbf{P(n,n-2)}$ *holds for every* $n \geq 4$.

Proof. The proof is by induction. We shall show that for $k \geq 4$,

$$\mathbf{P(k,k-2)} \Rightarrow \mathbf{P(k+2,k)}.$$

For $k = 4$ the result follows from Lemma 5.1.7, and for $k = 5$ we have the list of the six convex equivalences and the six subsets as follows:

$$1/2/3\ 4\ 5 \qquad \{1,2,5\},$$

$$1/2\ 3/4\ 5 \qquad \{1,2,4\},$$

$$1\ 2/3/4\ 5 \qquad \{1,3,4\},$$

$$1\ 2/3\ 4/5 \qquad \{1,3,5\},$$

$$1\,2\,3/4/5 \qquad \{2,4,5\},$$

$$1/2\,3\,4/5 \qquad \{1,4,5\}.$$

Suppose inductively that $\mathbf{P(k,k-2)}$ holds ($k \geq 4$). Thus we have a list $\pi_1, \pi_2, \ldots, \pi_m$ (with $m = \binom{k-1}{k-3}$) of convex equivalences of weight $k-2$ on X_k, and a list $A_1, A_2, \ldots, A_m$ of subsets of X_k of cardinality $k-2$ such that A_i is a transversal of π_{i-1}, π_i ($i = 2, \ldots, m$) and A_1 is a transversal of π_1, π_m. We may also assume that

π_1 has $\{k-2, k-1, k\}$ as the only non-singleton class,

π_2 has $\{k-3, k-2\}$ and $\{k-1, k\}$ as the only non-singleton class,

π_m has $\{k-3, k-2, k-1\}$ as the only non-singleton class,

$A_2 = X_k \backslash \{k-2, k\}$.

Let $\sigma_1, \ldots, \sigma_k$ be the list of convex equivalences of weight k on X_{k+1}, where

σ_i has $\{k-i+1, k-i+2\}$ as the only non-singleton class.

(Thus in particular σ_1, σ_2 and σ_k have $\{k, k+1\}$, $\{k-1, k\}$ and $\{1, 2\}$ as the only non-singleton classes respectively.) Let $\tau_1, \tau_2, \ldots, \tau_{k-1}$ be the list of convex equivalences of weight $k-1$ on X_k, where

τ_i has $\{k-i, k-i+1\}$ as the only non-singleton class.

(In particular each of τ_1, τ_2 and τ_{k-1} has $\{k-1, k\}$, $\{k-2, k-1\}$ and $\{1, 2\}$ as the only non-singleton class respectively.) Define the convex equivalences

$\pi_i' = \pi_i \cup \{(k+1, k+1)\} \cup \{(k+2, k+2)\}$, for $i = 1, \ldots, m$,

$\sigma_i' = \sigma_i$ with $k+2$ adjoined to the class containing $k+1$, $i = 1, \ldots, k$,

$\tau_i' = \tau_i$ with $k+1$ adjoined to the class containing k, and $k+2$ as a singleton class, for $i = 1, \ldots, k-1$.

Then arrange them as follows:

$$\sigma'_1, \ldots, \sigma'_k, \tau'_{k-1}, \ldots, \tau'_2, \pi'_2, \ldots, \pi'_m, \pi'_1, \tau'_1. \tag{5.1.9}$$

Notice that these convex equivalences are all distinct, and (5.1.9) is a complete list of the convex equivalences of weight k on X_{k+2}, since

$$\begin{aligned} m + k + k - 1 &= \binom{k-1}{k-3} + 2k + 1 \\ &= \frac{1}{2}(k-1)(k-2) + 2k - 1 \\ &= \frac{1}{2}k(k+1) \\ &= \binom{k+1}{k-1}. \end{aligned}$$

We now define the subsets

$$\begin{aligned} A'_i &= A_i \cup \{k+1, k+2\} \quad \text{for} \quad i = 1, \ldots, m, \\ B_i &= X_{k+2} \backslash \{k - i + 2, k + 2\} \quad \text{for} \quad i = 2, \ldots, k, \\ C_i &= X_{k+2} \backslash \{k - i + 1, k + 1\} \quad \text{for} \quad i = 1, 3, \ldots, k, \\ D &= X_{k+2} \backslash \{k - 1, k\}. \end{aligned}$$

It follows from the hypothesis that A'_i is a transversal of π'_{i-1}, π'_i for $i = 3, \ldots, m$ and that A'_1 is a transversal of π'_m, π'_1. It is also not difficult to verify that, for $i = 2, \ldots, k$, B_i is a transversal of σ'_{i-1}, σ'_i; for $i = 3, \ldots, k-1$, C_i is a transversal of τ'_{i-1}, τ'_i; C_1 is a transversal of σ'_1, τ'_1; C_k is a transversal of σ'_k, τ'_{k-1}; A'_2 is a transversal of τ'_2, π'_2; and finally D is a transversal of π'_1, τ'_1. It therefore remains to show that the subsets

$$C_1, B_2, \ldots, B_k, C_k, C_{k-1}, \ldots, C_3, A'_2, A'_3, \ldots, A'_m, A'_1, D \tag{5.1.10}$$

are all distinct. It is clear that the A''s B's and C's are all distinct. (Since the A''s contain $k+1$ and $k+2$, the B's contain $k+1$ but not $k+2$, while the C's contain

$k+2$ but not $k+1$.) Also D is distinct from the B's and the C's (since the later must not contain $k+1$or $k+2$). Note that the $\mathcal{L}$-classes characterised by $D\backslash\{k+1,k+2\} = X_k\backslash\{k-1,k\}$ contain only one idempotent namely

$$\begin{pmatrix} 1 & 2 & \dots & k-3 & \{k-2,k-1,k\} \\ 1 & 2 & \dots & k-3 & k-2 \end{pmatrix}.$$

Hence $D\backslash\{k+1,k+2\}$ is not one of the A's, and consequently D is distinct from the A''s. So all the subsets in (5.1.10) are distinct. ∎

Lemma 5.1.11 *Let* $n \geq 6$ *and* $3 \leq r \leq n-3$. *Then* $\mathbf{P(n-1,r-1)}$ *and* $\mathbf{P(n-1,r)}$ *together imply* $\mathbf{P(n,r)}$.

Proof. From the assumption $\mathbf{P(n-1,r)}$ we have a list $\sigma_1,\sigma_2,\dots,\sigma_m$ (where $m = \binom{n-2}{r-1}$) of convex equivalences of weight r on X_{n-1} and a list $A_1,\dots,A_m$ of distinct subsets of X_{n-1} of cardinality r such that A_i is a transversal of σ_{i-1},σ_i ($i = 2,\dots,m$) and A_1 is a transversal of σ_m,σ_1. We may also assume that

σ_1 has $\{r,\dots,n-1\}$ as the only non-singleton class,

σ_2 has $\{r-1,r\}$ and $\{r+1,\dots,n-1\}$ as the only non-singleton classes,

σ_m has $\{r-1,\dots,n-2\}$ as the only non-singleton class,

$A_2 = \{1,2,\dots,r-1,r+1\}$.

From the assumption $\mathbf{P(n-1,r-1)}$ we have a list $\tau_1,\dots,\tau_t$ (where $t = \binom{n-2}{r-2}$) of convex equivalences of weight $r-1$ on X_{n-1} and a list $B_1,\dots,B_t$ of distinct subsets of cardinality $r-1$ on X_{n-1} such that B_i is a transversal of τ_{i-1},τ_i ($i = 2,\dots,t$) and B_1 is a transversal of τ_t,τ_1. We may also assume that

τ_1 has $\{r-1,\dots,n-1\}$ as the only non-singleton class,

τ_2 has $\{r-2,r-1\}$ and $\{r,\dots,n-1\}$ as the only non-singleton classes,

τ_t has $\{r-2,\dots,n-2\}$ as the only non-singleton class,

$B_2 = \{1,2,\dots,r-2,r\}$.

Now, for $i = 1, \ldots, m$ let

$$\sigma'_i = \sigma_i \text{ with } n \text{ adjoined to the class containing } n-1,$$

for $j = 1, \ldots, t$ let

$$\tau'_j = \tau_j \cup \{(n, n)\}.$$

Then arrange the convex equivalences as follows:

$$\sigma'_1, \ldots, \sigma'_m, \tau'_2, \ldots, \tau'_t, \tau'_1. \qquad (5.1.12)$$

Note that $m + t = \binom{n-1}{r-1}$. Hence above is a complete list of all the convex equivalences of weight r on X_n. Next we define

$$A = \{1, 2, \ldots, r-1, n\},$$
$$B'_i = B_i \cup \{n\} \text{ for } i = 1, \ldots, t$$

and arrange the subsets as follows:

$$A, A_2, A_3, \ldots, A_m, B'_2, \ldots, B'_t, B'_1. \qquad (5.1.13)$$

Then A_i is a transversal of σ'_{i-1}, σ'_i ($i = 2, \ldots, m$); B'_i is a transversal of τ'_{i-1}, τ'_i ($i = 3, \ldots, t$); B'_1 is a transversal of τ'_t, τ'_1; A is a transversal of σ'_1, τ'_1 and B'_2 is a transversal of σ'_m, τ'_2.

It is clear that $A_2, \ldots, A_m, B'_1, \ldots, B'_t$ are all distinct subsets of X_n of cardinality r, and A is distinct from $A_2, \ldots, A_m$. If $A = B'_i$ for some $i = 1, \ldots, t$ then

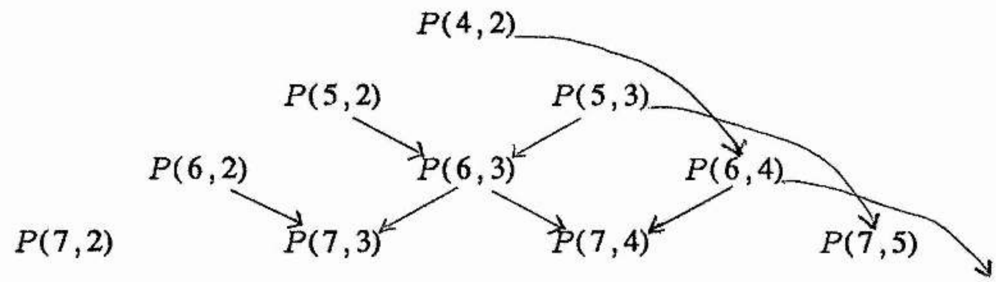
$$A \backslash \{n\} = B_i = \{1, 2, \ldots, r-1\}.$$

But the $\mathcal{L}$-class characterised by $\{1, 2, \ldots, r-1\}$ has only one idempotent, namely

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & r-2 & A' \\ 1 & 2 & 3 & \ldots & r-2 & r-1 \end{pmatrix}$$

where $A' = X_n \backslash \{1, 2, \ldots, r-2\}$. This is contrary to the hypothesis that the $\mathcal{L}$-class characterised by B_i must contain at least two idempotents. Hence all the subsets are distinct. Thus the induction step is complete, and we may deduce that $\mathbf{P(n, r)}$ is true for all $n \geq 4$ and all r such that $2 \leq r \leq n-2$.

The pattern of deduction is



2. The order-preserving partial transformation semigroup

The $\mathcal{J}$-class

$$J_r = \{\alpha \in PO_n : |\text{im}\,\alpha| = r\}$$

is the union of the sets $[k, r]$, where $r \leq k \leq n$. The number of $\mathcal{L}$-classes in J_r is the number of image sets in X_n of cardinality r, namely $\binom{n}{r}$. The number of $\mathcal{R}$-classes in J_r is the number of convex equivalences of weight r on all the subsets of X_n of cardinality k, where $r \leq k \leq n$. This number is

$$\sum_{k=r}^{n} \binom{n}{k} \binom{k-1}{r-1}.$$

Next we have the following result from [**14**]:

Lemma 5.2.1 *For* $r = 1, 2, \ldots, n-2$,

$$[r, r] \subseteq \left([r+1, r+1]\right)^2.$$

Lemma 5.2.2 $J_r \subseteq (J_{r+1})^2$ *for* $1 \leq r \leq n-3$.

Proof. Let α in J_r be in $[k, r]$, $2 \leq r \leq k \leq n$. If $k = r$, the result follows from Lemma 5.2.1. If $k > r$, then the proof of Lemma 5.1.2 applies equally to this case by adjusting A' to $\{r+2\}$ in δ. ■

Let

$$K'(n,r) = \{\alpha \in PO_n : |\operatorname{im}\alpha| \leq r\},$$

then from Lemma 1.3.3 we deduce that the rank of $K'(n,r)$ must be at least as large as the number of $\mathcal{R}$-classes in J_r. Thus we have:

Lemma 5.2.3 *For* $1 \leq r \leq n-2$,

$$\operatorname{rank}\left(K'(n,r)\right) \geq \sum_{k=r}^{n} \binom{n}{k}\binom{k-1}{r-1}.$$

■

Theorem 5.2.4 *For* $1 \leq r \leq n-2$,

$$\operatorname{rank}\left(K'(n,r)\right) = \operatorname{idrank}\left(K'(n,r)\right) = \sum_{k=r}^{n} \binom{n}{k}\binom{k-1}{r-1}.$$

The proof follows the same basic strategy as that of Theorem 5.1.5. It depends on the following Lemma.

Lemma 5.2.5 *Let* $A_1, \ldots, A_m$ *(where* $m = \binom{n}{r}$ *and* $r \geq 2$*) be a list of subsets of* X_n *with cardinality* r. *Suppose that there exist distinct convex equivalences* $\pi_1, \ldots, \pi_m$ *of weight* r *on* X_n *with the property that* A_{i-1}, A_i *are both transversals* *of π_i ($i = 2, \ldots, m$) and A_m, A_1 are both transversals of π_1. Then each $\mathcal{H}$-class (π_i, A_i) consists of an idempotent ϵ_i, and there exist idempotents $\epsilon_{m+1}, \ldots, \epsilon_p$ (where $p = \sum_{k=r}^{n} \binom{n}{k}\binom{k-1}{r-1}$) such that $\{\epsilon_1, \ldots, \epsilon_p\}$ is a set of generators for $K'(n,r)$.*

Notice that $\epsilon_{i-1}\epsilon_i$ ($i = 2, \ldots, m$) is an element of rank r, so also is $\epsilon_m\epsilon_1$, and $\epsilon_m \mathcal{R}\, \epsilon_m\epsilon_1 \,\mathcal{L}\, \epsilon_1$.

Choose the idempotents $\epsilon_{m+1}, \ldots, \epsilon_p$ so that $\epsilon_1, \ldots, \epsilon_p$ covers all the $\mathcal{R}$-classes in J_r. Then if η is an arbitrary idempotent in J_r there exists a unique $i \in \{1, \ldots, p\}$ such that $\eta \,\mathcal{R}\, \epsilon_i$, and a unique $j \in \{1, \ldots, m\}$ such that $\eta \,\mathcal{L}\, \epsilon_j$. Moreover, there is a unique $k \in \{1, \ldots, m\}$ such that $\epsilon_i \,\mathcal{L}\, \epsilon_k$. (If $i \in \{1, \ldots, m\}$ then of course $k = i$.) If $k = j$ then $\eta = \epsilon_i$ and there is nothing to prove. If $k < j$ then

$$\eta = \epsilon_i\epsilon_{k+1}\epsilon_{k+2}\cdots\epsilon_j.$$

If $k > j$ then

$$\eta = \epsilon_i\epsilon_{k+1}\cdots\epsilon_m\epsilon_1\cdots\epsilon_j.$$

■

Note that in O_n, the number of $\mathcal{L}$-classes in any $\mathcal{J}$-class exceeds the number of $\mathcal{R}$-classes, in PO_n the number of $\mathcal{L}$-classes in a $\mathcal{J}$-class is smaller than the number of $\mathcal{R}$-classes. This accounts for the difference of the strategies in Lemmas 5.1.6 and 5.2.5.

It remains to prove that the listing of images and convex equivalences postulated in the statement of Lemma 5.2.5 can actually be carried out. Let $n \geq 4$ and $2 \leq r \leq n-2$, and consider the Proposition.

P(n,r): *There is a way of listing the subsets of X_n of cardinality r as $A_1, \ldots, A_m$ (with $m = \binom{n}{r}$), $A_1 = \{1,2,\ldots,r\}$, $A_2 = \{1,2,\ldots,r-1,r+1\}$, $A_m = \{1,2,\ldots,r-1,n\}$) so that there exist distinct convex equivalences $\pi_1, \ldots, \pi_m$ of weight r with the* *property that A_{i-1}, A_i are both transversals of π_i ($i = 2, \ldots, m$) and A_m, A_1 are both transversals of π_1.*

The proof is by double induction on n and r, the key step being a kind of Pascal's triangular implication.

$$\mathbf{P(n-1, r-1)} \text{ and } \mathbf{P(n-1, r)} \Rightarrow \mathbf{P(n, r)}.$$

First, however, we anchor the induction with two Lemmas.

Lemma 5.2.6 $\mathbf{P(n,2)}$ *holds for every* $n \geq 4$.

Proof. The proof is by induction. For $n = 4$ we have the list of 6 subsets and 6 equivalences as follows:

$$\begin{array}{ll} \{1,2\} & 1/2\ 4, \\ \{1,3\} & 1/2\ 3, \\ \{2,3\} & 1\ 2/3, \\ \{2,4\} & 2/3\ 4, \\ \{3,4\} & 2\ 3/4, \\ \{1,4\} & 1\ 3/4. \end{array}$$

Suppose inductively that $\mathbf{P(n-1,2)}$ holds ($n \geq 5$). Thus we have a list $A_1, \ldots, A_t$ (where $t = \binom{n-1}{2}$) of subsets of X_{n-1} of cardinality 2, and a list $\pi_1, \ldots, \pi_t$ of distinct convex equivalences of weight 2 such that for $i = 2, \ldots, t$ the sets A_{i-1}, A_i are both transversals of π_i and A_t, A_1 are both transversals of π_1. Suppose moreover that $A_1 = \{1,2\}$, $A_2 = \{1,3\}$ and $A_t = \{1, n-1\}$. Let

$$B_i = \{i, n\}$$

for $i = 1, \ldots, n-1$, and define

$$\begin{aligned}
\pi_1' &= \pi_1 \text{ with } n-1 \text{ being replaced by } n, \\
\sigma_1 &= 1\ 2/n-1\ n, \\
\sigma_i &= i\ i+1/n \text{ for } i = 2, \ldots, n-2, \\
\sigma_{n-1} &= 1\ n-1/n.
\end{aligned}$$

Arrange the subsets and the convex equivalences as follows:

$$\begin{array}{l}
A_1, A_2, \ldots, A_t, B_2, B_3, \ldots, B_{n-1}, B_1 \\
\pi_1', \pi_2, \ldots, \pi_t, \sigma_1, \sigma_2, \ldots, \sigma_{n-2}, \sigma_{n-1}.
\end{array}$$

Then, it is easy to verify that the subsets and the convex equivalences as arranged above satisfy $\mathbf{P(n,2)}$. Notice that these subsets are all the subsets of X_n of cardinality 2, and the convex equivalences are all distinct. ■

Lemma 5.2.7 $\mathbf{P(n, n-2)}$ *holds for every* $n \geq 4$.

Proof. We shall show that for $k \geq 4$, $\mathbf{P(k, k-2)} \Rightarrow \mathbf{P(k+2, k)}$. But first we show that $\mathbf{P(4,2)}$ and $\mathbf{P(5,3)}$ are true.

For $n = 4$, the result follows from Lemma 5.2.6. For $n = 5$, we have the list of 10 subsets and 10 equivalences as follows:

$$\begin{array}{ll}
\{1,2,3\} & 1/2/3\ 5, \\
\{1,2,4\} & 1/2/3\ 4, \\
\{1,3,4\} & 1/2\ 3/4, \\
\{2,3,4\} & 1\ 2/3/4, \\
\{2,3,5\} & 2/3/4\ 5,
\end{array}$$

$$\{2,4,5\} \qquad 2/3\,4/5,$$

$$\{3,4,5\} \qquad 2\,3/4/5,$$

$$\{1,4,5\} \qquad 1\,3/4/5,$$

$$\{1,3,5\} \qquad 1/3\,4/5,$$

$$\{1,2,5\} \qquad 1/2\,3/5.$$

Suppose inductively that $\mathbf{P(k,k-2)}$ holds ($k \geq 4$). Thus we have a list $A_1,\ldots,A_m$ (where $m = \binom{k}{k-2}$) of subsets of X_k of cardinality $k-2$, and a list $\pi_1,\ldots,\pi_m$ of distinct convex equivalences of weight $k-2$ such that for $i = 2,\ldots,m$ the sets A_{i-1}, A_i are both transversals of π_i and A_m, A_1 are both transversals of π_1. We may also assume that

$$A_1 = \{1,2,\ldots,k-2\}, \quad A_2 = \{1,2,\ldots,k-3,k-1\}$$

and

$$A_m = \{1,2,\ldots,k-3,k\}.$$

Let $B_1,\ldots,B_{k+1}$ be the list of subsets of X_{k+1} of cardinality k, where $B_i = X_{k+1}\backslash\{k+2-i\}$. (Thus in particular $B_1 = \{1,2,\ldots,k\}$ and $B_{k+1} = \{2,3,\ldots,k+1\}$.) Let $C_1,\ldots,C_k$ be the list of subsets of X_k of cardinality $k-1$, where $C_i = X_k\backslash\{k+1-i\}$. (In particular $C_1 = \{1,2,\ldots,k-1\}$ and $C_k = \{2,3,\ldots,k\}$.) Define

$$A'_i = A_i \cup \{k+1,k+2\} \text{ for } i = 1,\ldots,m,$$

$$C'_i = C_i \cup \{k+2\} \qquad \text{for } i = 1,\ldots,k.$$

Notice that the subsets $A'_1,\ldots,A'_m, B_1,\ldots,B_{k+1}, C'_1,\ldots,C'_k$ are all distinct, and form a complete list of subsets of X_{k+2} of cardinality k, since

$$m + k + (k+1) = \binom{k}{k-2} + 2k + 1$$

$$= \frac{1}{2}k(k-1) + 2k + 1$$
$$= \frac{1}{2}(k+2)(k+1)$$
$$= \binom{k+2}{k}.$$

Denote by $|i,j|$ the convex equivalence of weight $n-1$ on a set T of n elements, where $\{i,j\}$ is the only non-singleton class. Then define

$$\sigma_i = |k+2-i, k+3-i| \text{ on } X_{k+1} \qquad \text{for } i = 2, \ldots, k+1,$$
$$\tau_i = |k+1-i, k+2-i| \text{ on } X_k \cup \{k+2\} \quad \text{for } i = 2, 4, \ldots, k-1,$$
$$\pi_i' = \pi_i \cup \{(k+1, k+1)\} \cup \{(k+2, k+2)\} \quad \text{for } i = 1, 3, \ldots, m,$$
$$\delta_1 = |k, k+2| \text{ on } X_k \cup \{k+2\},$$
$$\delta_2 = |k+1, k+2| \text{ on } X_{k+2} \backslash \{1\},$$
$$\delta_3 = |k, k+1| \text{ on } X_{k+2} \backslash \{k-2\},$$
$$\delta_4 = |k, k+1| \text{ on } X_{k+2} \backslash \{k-1\}.$$

Now, arrange the subsets and the convex equivalences as follows:

$$\begin{array}{l} B_1, B_2, \ldots, B_{k+1}, C_k', C_{k-1}', \ldots, C_3', A_2', A_3', \ldots, A_m', A_1', C_2', C_1' \\ \delta_1, \sigma_2, \ldots, \sigma_{k+1}, \delta_2, \tau_k, \ldots, \tau_4, \delta_3, \pi_3', \ldots, \pi_m', \pi_1', \delta_4, \tau_2. \end{array}$$

With this arrangement it is easy to verify that the subsets and the convex equivalences satisfy $\mathbf{P(k+2,k)}$.

Since an $\mathcal{R}$-class characterised by a convex equivalence of weight $n-1$ on a set of n elements contains only two idempotents, the convex equivalences above are unique, and therefore distinct. ■

Lemma 5.2.8 *Let $n \geq 5$ and $3 \leq r \leq n-3$. Then $\mathbf{P(n-1, r-1)}$ and $\mathbf{P(n-1, r)}$ together imply $\mathbf{P(n, r)}$.*

Proof. From the assumption $\mathbf{P(n-1,r)}$ we have a list $A_1, \ldots, A_m$ (where $m = \binom{n-1}{r}$) of the subsets of X_{n-1} with cardinality r and a list $\sigma_1, \ldots, \sigma_m$ of distinct convex equivalences of weight r such that A_{i-1}, A_i ($i = 2, \ldots, m$) are transversals of σ_i, and A_1, A_m are transversals of σ_1. We may also assume that

$$A_1 = \{1, 2, \ldots, r\}, \quad A_2 = \{1, \ldots, r-1, r+1\}, \quad A_m = \{1, \ldots, r-1, n-1\}$$

and σ_2 has $\{r, r+1\}$ as the only non-singleton class.

From the assumption $\mathbf{P(n-1,r-1)}$ we have a list $B_1, \ldots, B_t$ (where $t = \binom{n-1}{r-1}$) of subsets of X_{n-1} of cardinality $r-1$, and a list $\tau_1, \ldots, \tau_t$ of distinct convex equivalences of weight $r-1$ such that B_{j-1}, B_j ($j = 2, \ldots, t$) are transversals of τ_j, and B_1, B_t are transversals of τ_1. We may also assume that

$$B_1 = \{1, 2, \ldots, r-1\}, \quad B_2 = \{1, \ldots, r-2, r\}, \quad B_t = \{1, \ldots, r-2, n-1\}$$

and that τ_2 has $\{r-1, r\}$ as the only non-singleton class.

Let

$$B_i' = B_i \cup \{n\}.$$

Then $A_1, \ldots, A_m, B_1', \ldots, B_t'$ is a complete list of the subsets of X_n of cardinality r. (Notice that $m + t = \binom{n}{r}$.) Define

$$\sigma_1' = \sigma_1 \text{ with } n-1 \text{ replaced by } n,$$

$$\tau_i' = \tau_i \cup \{(n,n)\} \text{ for } i = 1, 3, \ldots, t,$$

$$\tau_2' = \tau_2 \cup \{(n-1, n)\}.$$

Then $\sigma_2, \ldots, \sigma_m, \tau_1', \ldots, \tau_t'$ are all distinct (since the σ's do not contain n, while the τ''s contain n). Also σ_1' is distinct from all of them, since σ_1' contains r and n in the same equivalence class.

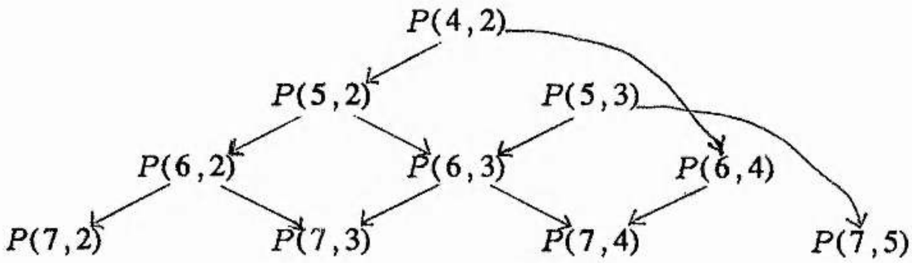
Arrange the subsets and the convex equivalences as follows:

$$A_1, A_2, \ldots, A_m, B_2', \ldots, B_t', B_1'$$

$$\sigma_1', \sigma_2, \ldots, \sigma_m, \tau_2', \ldots, \tau_t', \tau_1'.$$

With this arrangement it is easy to verify that the convex equivalences and the subsets satisfy **P(n,r)**. ∎

The pattern of deduction here is



Remark 5.2.9 Observe that in Lemmas 5.2.6, 5.2.7, and 5.2.8 (proof) all the convex equivalences used have only one non-singleton class, except for τ_2' in Lemma 5.2.8 which has two. In all cases the non-singleton class (or classes) contained only two elements, and since $n \geq 4$, $r = 2$ in 5.2.6, $n \geq 4$, $r = n-2$ in 5.2.7 and $n \geq 5$, $r \leq n-3$ in 5.2.8 the convex equivalences are all partial. Thus in the generating set $\{\epsilon_1, \ldots, \epsilon_p\}$ of Lemma 5.2.5, $\epsilon_1, \ldots, \epsilon_m$ need not be full idempotents.

We shall find this useful in the next section.

3. Strictly partial order-preserving transformations

The $\mathcal{J}$-class

$$J_r = \{\alpha \in SPO_n : |\operatorname{im} \alpha| = r\}$$

is the union of $[k, r]$, where $r \leq k \leq n-1$. The number of $\mathcal{L}$-classes in J_r is $\binom{n}{r}$, while that of $\mathcal{R}$-classes is

$$\sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}.$$

Lemma 5.3.1 *For $1 \leq r \leq n-3$, we have*

$$J_r \subseteq (J_{r+1})^2.$$

Proof. The proof of Lemma 5.1.2 applies to this case also, by adjusting A' to $\{r+2\}$ in δ. ■

Lemma 5.3.2 *Let E_2 be the set of all idempotents in J_{n-2}. Then*

$$J_{n-2} \subseteq \langle E_2 \rangle.$$

Proof. Notice that $J_{n-2} = [n-1, n-2] \cup [n-2, n-2]$. We shall first consider an element $\alpha \in [n-2, n-2]$. Let dom $\alpha = X_n \backslash \{i, j\}$ and assume that $i < j$, and im $\alpha = X_n \backslash \{k, l\}$ with $k < l$. Let ϵ be the partial identity on dom α. We now distinguish several cases.

Case 1. $i = k$. (a) $j < l$. Let $A = \text{dom}\,\alpha \cup \{j\}$. For $s = 1, \ldots, l-j$ define the idempotents ϵ_s on A by

$$\{j+s-1, j+s\}\epsilon_s = j+s-1$$

and

$$x\epsilon_s = x$$

for all $x \in A \backslash \{j+s-1, j+s\}$. Then

$$\alpha = \epsilon\epsilon_1\epsilon_2 \cdots \epsilon_{l-j}.$$

(b)$j > l$. If an element α satisfies this subcase, then its inverse satisfies (a), and the result follows.

(c) $j = l$. Here α is an idempotent.

Case 2. $j = l$. (a) $i < k$. Let $A = \text{dom}\,\alpha \cup \{i\}$. For $s = 1, \ldots, k-i$, define ϵ_s by

$$\{i+s-1, i+s\}\epsilon_s = i+s-1$$

and

$$x\epsilon_s = x$$

for all $x \in A \backslash \{i+s-1, i+s\}$. Then

$$\alpha = \epsilon\epsilon_1 \cdots \epsilon_{k-i}.$$

(b) $i > k$. Here α^{-1} satisfies (a) above, and therefore the result follows.

Case 3. (a) $i < k < j < l$. Let $A = \text{dom}\,\alpha \cup \{i\}$, $B = X_n \backslash \{k\}$. For $s = 1, \ldots, k-i$ and $t = 1, \ldots, l-j$ define ϵ_s and η_t as follows:

$$\{i+s-1, i+s\}\epsilon_s = i+s-1$$

and

$$x\epsilon_s = x$$

for all $x \in A \backslash \{i+s-1, i+s\}$.

$$\{j+t-1, j+t\}\eta_t = j+t-1$$

and

$$x\eta_t = x$$

for all $x \in B \backslash \{j+t-1, j+t\}$. Then

$$\alpha = \epsilon\epsilon_1 \cdots \epsilon_{k-i}\eta_1 \cdots \eta_{l-j}.$$

(b) $k < i < l < j$. Here α^{-1} is of type (a) above.

Case 4. (a) $i < k < l < j$. Let $A = \operatorname{dom}\alpha \cup \{i\}$, $B = X_n \backslash \{k\}$. For $s = 1, \ldots, k - i$ and $t = 1, \ldots, j - l$ define ϵ_s and η_t as follows:

$$\{i + s - 1, i + s\}\epsilon_s = i + s - 1$$

and

$$x\epsilon_s = x$$

for all $x \in A \backslash \{i + s - 1, i + s\}$,

$$\{l + t - 1, l + t\}\eta_t = l + t$$

and

$$x\eta_t = x$$

for all $x \in B \backslash \{l + t - 1, l + t\}$. Then

$$\alpha = \epsilon\epsilon_1 \cdots \epsilon_{k-i}\eta_1 \cdots \eta_{j-l}.$$

(b) $k < i < j < l$. Here α^{-1} is of type (a) above.

Case 5. (a) $i < j < k < l$. Let $A = \operatorname{dom}\alpha \cup \{i\}$. For $s = 1, \ldots, j - i - 1$, $t = 1, \ldots, k - j$ and $u = 1, \ldots, l - k - 1$ define ϵ_s, η_t and δ_u as follows:

$$\{i + s - 1, i + s\}\epsilon_s = i + s - 1$$

and

$$x\epsilon_s = x$$

for all $x \in A \backslash \{i + s - 1, i + s\}$,

$$\{j - 1, j + 1\}\epsilon_{j-i} = j - 1$$

and

$$x\epsilon_{j-i} = x$$

for all $x \in A \backslash \{j-1, j+1\}$.

$$\{j+t-1, j+t+1\}\eta_t = j+t-1$$

and

$$x\eta_t = x$$

for all $x \in B_t \backslash \{j+t-1, j+t+1\}$, where $B_t = X_n \backslash \{j+t\}$.

$$\{k+u, k+u+1\}\delta_u = k+u$$

and

$$x\delta_u = x$$

for all $x \in B_{k-j}$. Then

$$\alpha = \epsilon\epsilon_1 \cdots \epsilon_{j-i}\eta_1 \cdots \eta_{k-j}\delta_1 \cdots \delta_{l-k-1}.$$

(b) $k < l < i < j$. Here α^{-1} satisfies (a) above.

Case 6. $i < j = k < l$. Let $A = \operatorname{dom} \alpha \cup \{i\}$. For $s = 1, \ldots, j-i-1$ and $t = 2, \ldots, l-j$ define ϵ_s, η_1 and η_t as follows:

$$\{i+s-1, i+s\}\epsilon_s = i+s-1$$

and

$$x\epsilon_s = x$$

for all $x \in A \backslash \{i+s-1, i+s\}$,

$$\{j-1, j+1\}\eta_1 = j-1$$

and

$$x\eta_1 = x$$

for all $x \in A\backslash\{j-1, j+1\}$,

$$\{j+t-1, j+t\}\eta_t = j+t-1$$

and

$$x\eta_t = x$$

for all $x \in A\backslash\{j+t-1, j+t\}$. Then

$$\alpha = \epsilon\epsilon_1 \cdots \epsilon_{j-i-1}\eta_1 \cdots \eta_{l-j}.$$

Now, if $\alpha \in [\, n-1\,, n-2]$ then it can be expressed as follows:

$$\begin{pmatrix} a_1 & \dots & a_{i-1} & \{a_i, a_{i+1}\} & a_{i+2} & \dots & a_{n-1} \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_{n-2} \end{pmatrix}.$$

But then

$$\alpha = \epsilon\beta,$$

where

$$\epsilon = \begin{pmatrix} a_1 & \dots & a_{i-1} & \{a_i, a_{i+1}\} & a_{i+2} & \dots & a_{n-1} \\ a_1 & \dots & a_{i-1} & a_i & a_{i+2} & \dots & a_{n-1} \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} a_1 & \dots & a_{i-1} & a_i & a_{i+2} & \dots & a_{n-1} \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \dots & b_{n-2} \end{pmatrix}.$$

Note that ϵ is an idempotent, and that $\beta \in [\, n-2\,, n-2]$. Hence α is expressible in terms of idempotents in E_2. ∎

Let

$$K''(n,r) = \{\alpha \in SPO_n : |\operatorname{im}\alpha| \leq r\}.$$

By Lemmas 5.3.1 and 5.1.3, every element in J_r $(r \leq n-2)$ is expressible in terms of idempotents in J_r. Hence $K''(n,r)$ is generated by the idempotents in J_r.

Theorem 5.3.3 *For $1 \leq r \leq n-2$ we have*

$$\operatorname{rank}\left(K''(n,r)\right) = \operatorname{idrank}\left(K''(n,r)\right) = \sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}.$$

Proof. The reason for choosing $\epsilon_1, \ldots, \epsilon_m$ in the generating set $\{\epsilon_1, \ldots, \epsilon_p\}$ for $K'(n,r)$ to be non-full idempotents (see Remark 5.2.9) is to make the corresponding result for $K''(n,r)$ much easier to deduce, since we may choose the same idempotents $\epsilon_1, \ldots, \epsilon_m$ and $\epsilon_{m+1}, \ldots, \epsilon_q$ (where $q = \sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}$) from the remaining $\mathcal{R}$-classes to obtain the generating set $\{\epsilon_1, \ldots, \epsilon_q\}$ of $K''(n,r)$. ■

4. One-one partial transformations

The rank and the nilpotent rank of the subsemigroup of I_n generated by the nilpotent elements were considered by Gomes and Howie **[13]**, where they showed that both the rank and the nilpotent rank of this subsemigroup (as an inverse semigroup) are equal to $n+1$. We now generalise this problem by considering the rank and the nilpotent rank of

$$L(n,r) = \{\alpha \in I_n : |\operatorname{im}\alpha| \leq r \text{ and } r \leq n-2\}.$$

Lemma 5.4.1 *For all $r \leq n-2$, we have*

$$J_r \subseteq (N \cap J_r)^2.$$

Proof. The result follows from Remark 3.16 in **[12]** that

$$J_{n-2} \subseteq (N \cap J_{n-2})^2$$

and Lemma 4.1 in the same paper, that if

$$J_r \subseteq (N \cap J_r)^k \text{ then } J_{r-1} \subseteq (N \cap J_{r-1})^k$$

for $2 \leq r \leq n-1$. ■

Denote by P_r the principal factor $L(n,r)/L(n,r-1)$. Then P_r has $\binom{n}{r}$ $\mathcal{R}$-classes corresponding to the $\binom{n}{r}$ possible domains of cardinality r, and $\binom{n}{r}$ $\mathcal{L}$-classes corresponding to the $\binom{n}{r}$ possible images. It is a Brandt semigroup isomorphic to $B(S_r, \{1,\ldots,m\})$, where S_r is the symmetric group on r-symbol and $m = \binom{n}{r}$, and so by Theorem 2.2.4 it has rank (as an inverse semigroup) equal to

$$\text{gprank}(S_r) + \binom{n}{r} - 1$$

where gprank(S_r) is the group rank of S_r.

Theorem 5.4.2

$$\text{rank}\big(L(n,r)\big) = \text{nilrank}\big(L(n,r)\big) = \text{gprank}(S_r) + \binom{n}{r} - 1.$$

Proof. All that is required is to select a generating set of $L(n,r)$ consisting of gprank$(S_r) + \binom{n}{r} - 1$ nilpotents.

Let $A_1, A_2, \ldots, A_m$ be a list of the subsets of X_n of cardinality r. Let H_{A_i,A_j} denote the $\mathcal{H}$-class in J_r consisting of all the elements whose domain is A_i and image A_j $(i,j = 1,2,\ldots,m)$.

Suppose $A_1 = \{1,2,\ldots,r\}$. Then the $\mathcal{H}$-class H_{A_1,A_1} is the symmetric group on $\{1,2,\ldots,r\}$, and if $r \geq 3$ then it is generated by the elements σ, τ where

$$\sigma = (1\ 2), \qquad \tau = (1\ 2\ \cdots r).$$

We now show that each of σ, τ can be expressed as a product of nilpotents. For this purpose, we will suppose that $A_2 = \{2,\ldots,r,r+1\}$, $A_3 = \{1,\ldots,r-1,r+1\}$ and $A_4 = \{2,\ldots,r-1,r+1,r+2\}$.

The proof depends on whether r is odd or even. For r odd we have

$$\sigma = \alpha_2^{-1}\beta\alpha_3 \quad \text{and} \quad \tau = \gamma_2^{-1}\alpha_2$$

where

$$\begin{aligned}
\alpha_2 &= \|r+1\ r\ r-1 \cdots 2\ 1\| \in H_{A_2,A_1},\\
\beta &= \|r\ r-2\ r-4 \cdots 3\ r+1\ r-1 \cdots 4\ 2\ 1\| \in H_{A_2,A_3},\\
\alpha_3 &= \|r+1\ 1\ 2 \cdots r\| \in H_{A_3,A_1},\\
\gamma_2 &= \|r+1\ r-1 \cdots 2\ r\ r-2 \cdots 3\ 1\| \in H_{A_2,A_1}.
\end{aligned}$$

If for this case we now choose a nilpotent $\alpha_i \in H_{A_i,A_1}$ for $i = 4, \ldots, m$ in an arbitrary way, we see that

$$\sigma, \tau, \alpha_2, \ldots, \alpha_m \in \langle \alpha_2, \ldots, \alpha_m, \beta, \gamma_2 \rangle.$$

By Remark 2.2.5 the elements $\sigma, \tau, \alpha_2, \ldots, \alpha_m$ generate P_r. It follows that P_r, and hence also $L(n, r)$ is generated by the $m + 1$ nilpotents $\alpha_2, \ldots, \alpha_m, \beta, \gamma_2$ provided r is odd.

For r even we have

$$\sigma = \alpha_3^{-1}\beta\alpha_4 \quad \text{and} \quad \tau = \gamma_4^{-1}\alpha_4$$

where

$$\begin{aligned}
\alpha_3 &= \|r+1\ 2\ 3 \cdots r-2\ r-1\ 1\ r\| \in H_{A_3,A_1},\\
\beta &= \|1\ r-2\ r+1\ 3\ 2\ 5\ 4 \cdots r-5\ r-6\ r-3\ r-4\ r-1\ r+2\| \in H_{A_3,A_4},\\
\alpha_4 &= \|r+2\ 2\ 4 \cdots r\| \cup \|r+1\ r-1 \cdots 3\ 1\| \in H_{A_4,A_1},\\
\gamma_4 &= \|r+1\ r-1\ r-4\ r-3\ r-6\ r-5 \cdots 9\ 6\ 7\ 4\ 5\ 2\ 3\ r\| \cup \|r+2\ 1\| \in H_{A_4,A_1}.
\end{aligned}$$

In this case P_r and hence $L(n, r)$ is generated by the $m+1$ nilpotents $\alpha_2, \ldots, \alpha_m, \beta, \gamma_4$, where $\alpha_i \in H_{A_i,A_1}$ are chosen arbitrarily for $i = 2, 5, 6, \ldots, m$.

It now remains to show that the result is true for $r = 2$ and 1.

If $r = 2$, S_2 is cyclic and thus has only one generator. For this case we will suppose that $A_1 = \{1, 2\}$ and $A_m = \{n-1, n\}$. The $\mathcal{H}$-class H_{A_1,A_1} is the symmetric group on A_1 and is generated by

$$\sigma = (1\ 2).$$

Now,

$$\sigma = \gamma_m^{-1}\alpha_m$$

where

$$\alpha_m = ||n-1\ 2|| \cup ||n\,1|| \in H_{A_m,A_1},$$
$$\gamma_m = ||n-1\ 1|| \cup ||n\,2|| \in H_{A_m,A_1}.$$

So if we choose nilpotents $\alpha_2, \ldots, \alpha_{m-1}$ as in the above cases we see that $\alpha_2, \ldots, \alpha_m, \gamma_m$ generate $L(n, r)$. Thus $L(n, r)$ has rank $1 + m - 1 = m$.

If $r = 1$, the symmetric group S_1 has zero generators, and it is easy to verify that the following $n - 1$ nilpotents generate $L(n, r)$:

$$||2\ 1||,\ \ ||3\ 1||,\ \ ||4\ 1||, \ldots, ||n\,1||.$$

■

APPENDIX

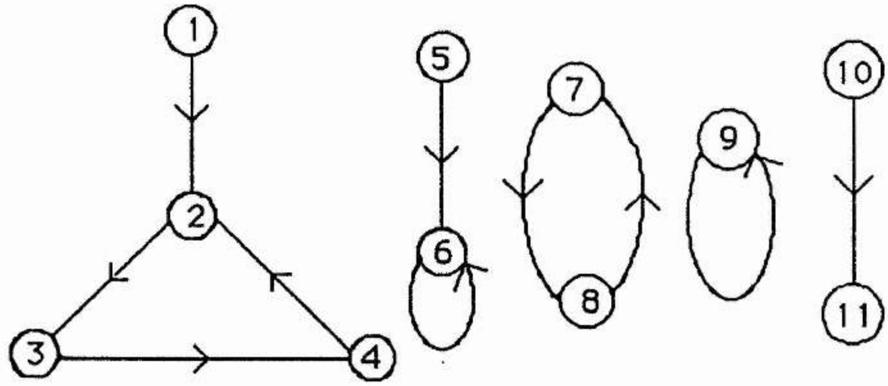


Fig 1. Components of α.

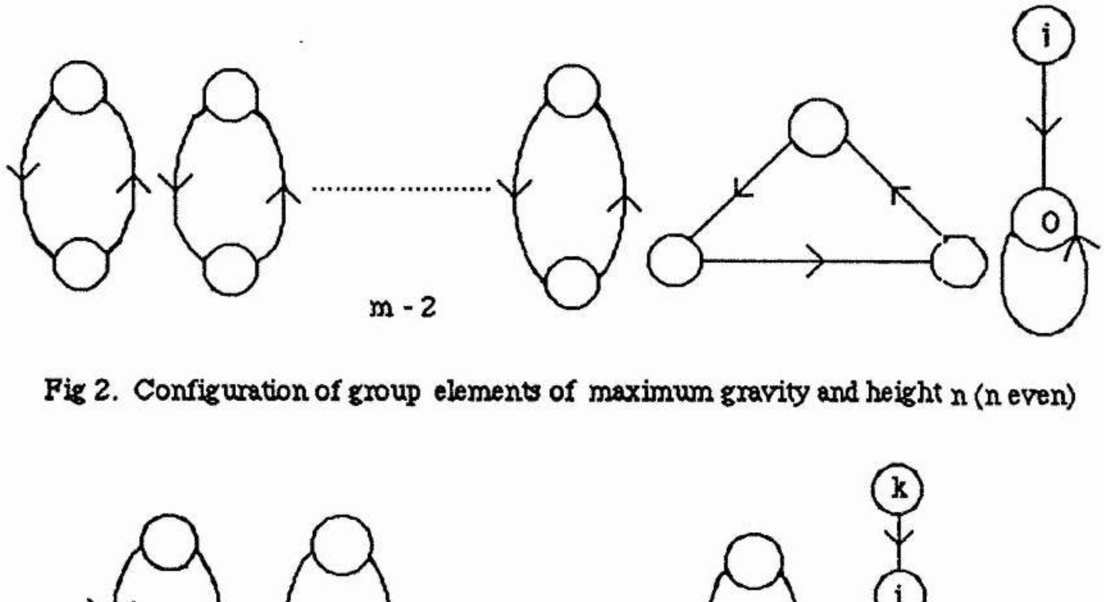


Fig 2. Configuration of group elements of maximum gravity and height n (n even)

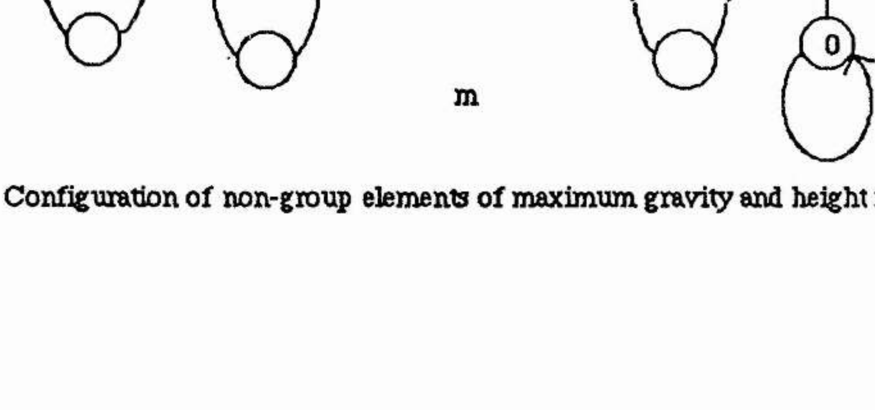


Fig 3. Configuration of non-group elements of maximum gravity and height n (n odd)

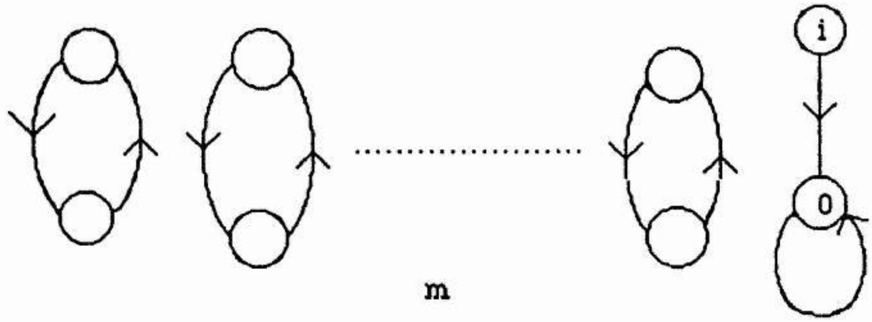


Fig 4 Configuration of group elements of maximum gravity and height n (n odd)

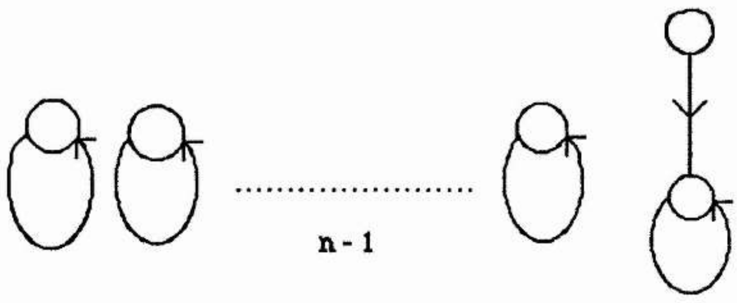


Fig 5. Configuration of elements with gravity 1 and height n in P_n^*.

Fig 6. Configuration of elements with gravity 2 and height n in P_n^*.

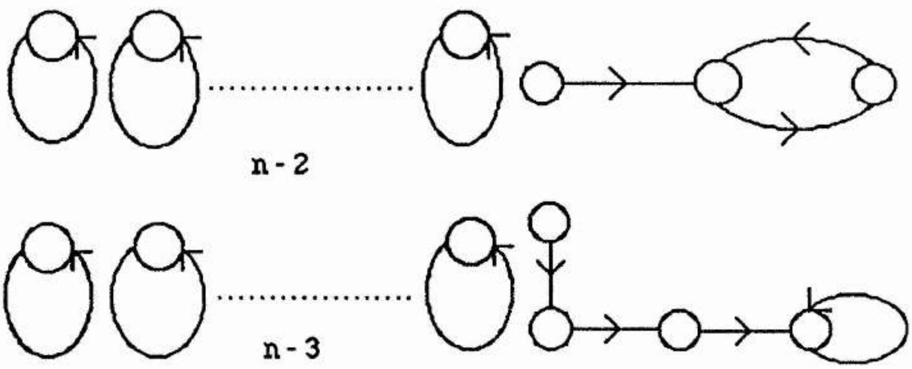


Fig 7. Configuration of elements with gravity 3 and height n in P_n^*.

Fig 9. One fixed point 0.

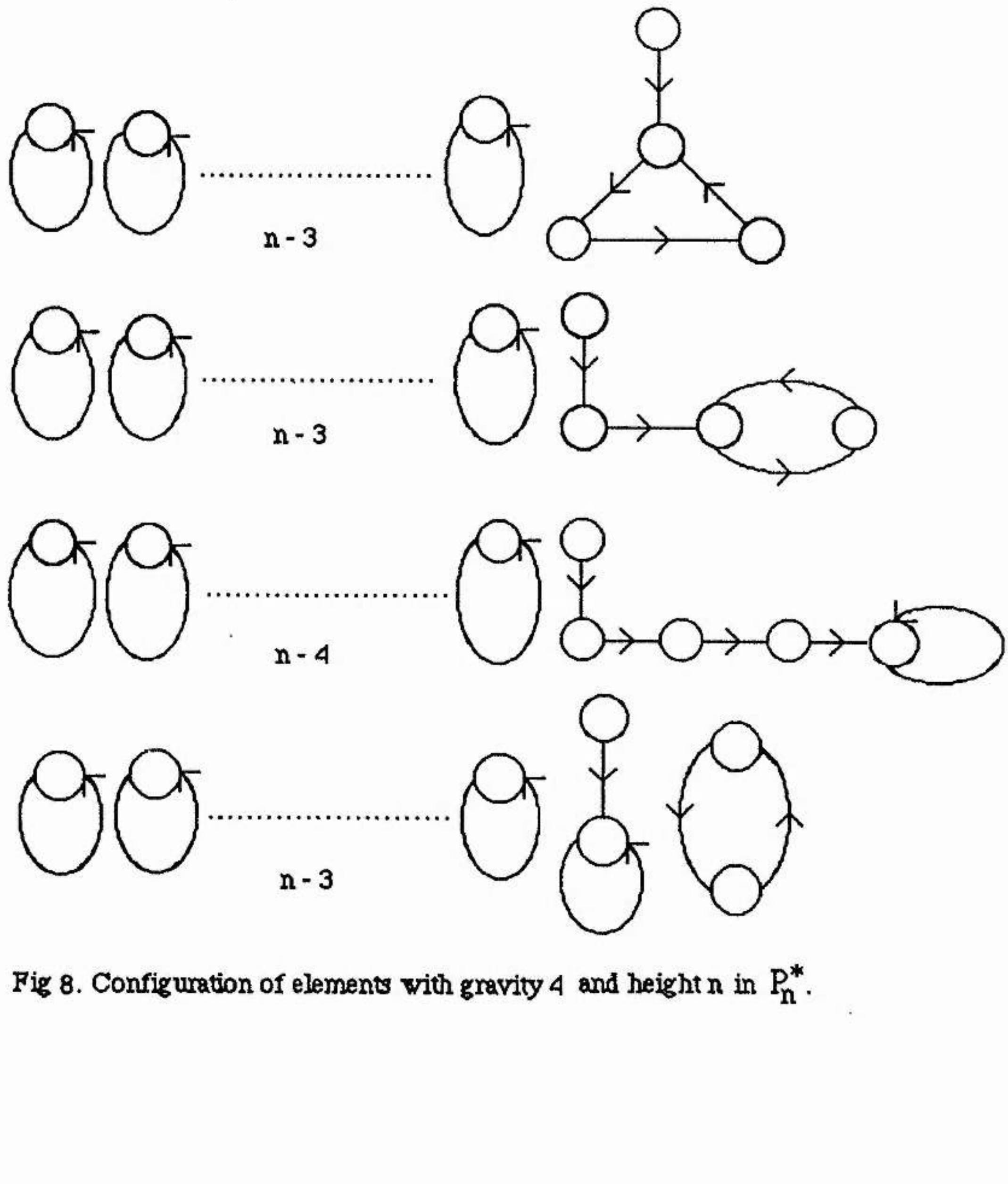


Fig 8. Configuration of elements with gravity 4 and height n in P_n^*.

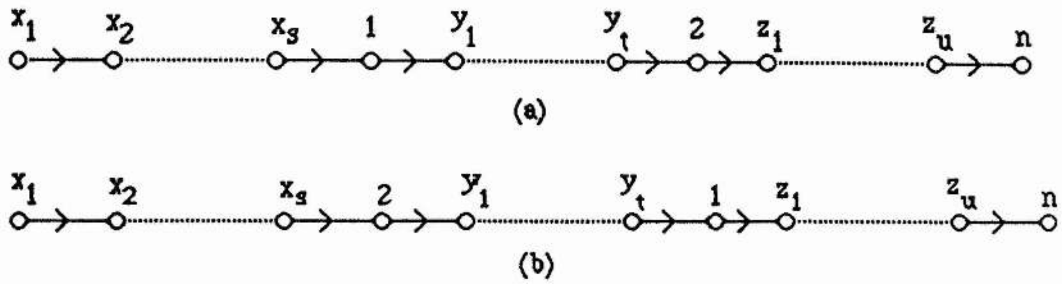


Fig 10. Configurations of λ_1 with λ_1 in [n-1, n-1].

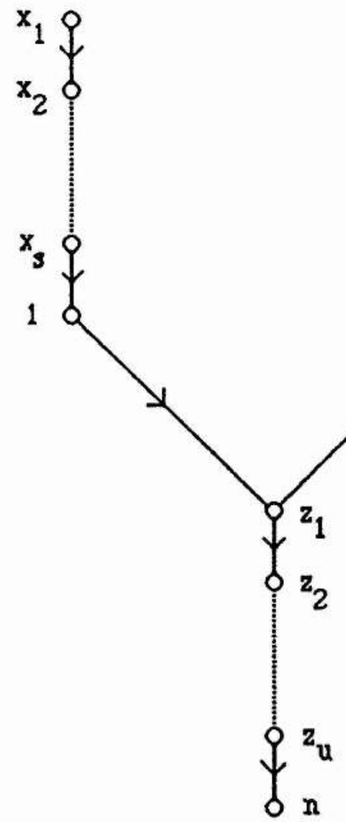


Fig 11. Configuration of λ_1 with λ_1 in [n-1, n-2].

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