INFINITE TRANSFORMATION SEMIGROUPS

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INFINITE TRANSFORMATION SEMIGROUPS

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CONTENTS

DECLARATION

I declare that this thesis is my own composition, that the work of which it is a record has been carried out by me, and that it has not been submitted in any previous application for a Higher Degree.

This thesis describes results of research carried out in the Department of Pure Mathematics, United College of St. Salvator and St. Leonard, University of St. Andrews under the supervision of Prof. J. M. Howie since 1st October 1980.

Maria Paula Marques

CERTIFICATE

I hereby certify that Maria Paula Marques has spent eleven terms of research work under my supervision, has fulfilled the conditions of Ordinance Number 12 of St. Andrews University, and is qualified to submit the accompanying thesis in application for the degree of Doctor of Philosophy.

J. M. Howie

Supervisor

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for whom this thesis may be some
recompense for the mischief of more
than a quarter of a century.

SUMMARY

In this thesis some topics in the field of Infinite Transformation Semigroups are investigated.

In 1966 Howie considered the full transformation semigroup $\mathscr{T}(X)$ on an infinite set X of cardinality $\underset{\sim}{m}$. For each α in $\mathscr{T}(X)$ he defined *defect of* α = def α and *collapse of* $\alpha = C(\alpha)$ to be the sets $X \setminus X\alpha$ and $\{x \in X : (\exists y \in X, y \neq x)\ x\alpha = y\alpha\}$, respectively. Later, in 1981 he introduced the set

$$S_{\underset{\sim}{m}} = \{\alpha \in \mathscr{T}(X) : |\text{def } \alpha| = |C(\alpha)| = |\text{ran } \alpha| = \underset{\sim}{m},\ |y\alpha^{-1}| < \underset{\sim}{m},\ (\forall y \in \text{ran } \alpha)\},$$

which is a subsemigroup of $\mathscr{T}(X)$ provided the cardinal $\underset{\sim}{m}$ is *regular*. Taking $\underset{\sim}{m}$ to be a regular cardinal number, Howie proved that $S_{\underset{\sim}{m}}$ is then a bisimple, idempotent-generated semigroup of depth 4. Next he considered the congruence defined in $S_{\underset{\sim}{m}}$ by

$$\Delta_{\underset{\sim}{m}} = \{(\alpha,\beta) \in S_{\underset{\sim}{m}} \times S_{\underset{\sim}{m}} : \max(|D(\alpha,\beta)\alpha|, |D(\alpha,\beta)\beta|) < \underset{\sim}{m}\},$$

where $D(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\}$ and showed that $S^*_{\underset{\sim}{m}} = S_{\underset{\sim}{m}} / \Delta_{\underset{\sim}{m}}$ is a bisimple, congruence-free and idempotent-generated semigroup of depth 4.

In this thesis comparable results are obtained for the semigroup $P_{\underset{\sim}{m}}$ which is the top principal factor of the semigroup

$$Q_{\underset{\sim}{m}} = \{\alpha \in \mathscr{T}(X) : |\text{def } \alpha| = |C(\alpha)| = \underset{\sim}{m}\}.$$

Here it is no longer necessary to restrict to a regular cardinal $\underset{\sim}{m}$.

The set $S_{\underset{\sim}{m}}$ considered by Howie fails to be a subsemigroup of $\mathscr{T}(X)$ if $\underset{\sim}{m}$ is not regular. It is shown that in this case $\langle S_{\underset{\sim}{m}} \rangle = Q_{\underset{\sim}{m}}$.

In the case where $\underset{\sim}{m} = \aleph_0$ (a regular cardinal) it is shown that $\Delta_{\aleph_0}$ is the only proper congruence on $S_{\underset{\sim}{m}}$.

Within the symmetric inverse semigroup $\mathscr{I}(X)$, the Baer-Levi semigroup B of type $(\underset{\sim}{m},\underset{\sim}{m})$ on X is considered and a dual B^* found. The products BB^* and B^*B are investigated and the semigroup $K_{\underset{\sim}{m}} = \langle B^*B \rangle$ is described. The top principal factor of $K_{\underset{\sim}{m}}$ is denoted by $L_{\underset{\sim}{m}}$ and it is shown that $L_{\underset{\sim}{m}} = B^*B \cup \{0\}$. On the set $L_{\underset{\sim}{m}}$ a congruence $\delta_{\underset{\sim}{m}}$, closely analogous to the congruence $\Delta_{\underset{\sim}{m}}$ defined above, is considered, and it is shown that $L_{\underset{\sim}{m}} / \delta_{\underset{\sim}{m}}$ is a o-bisimple, inverse and nilpotent-generated semigroup.

Finally, two embedding theorems for inverse semigroups and semigroups in general are presented. The cardinalities of some of the semigroups introduced in this thesis are studied.

CHAPTER 1

INTRODUCTION AND BACKGROUND

According to Clifford and Preston [5] J. A. de Seguier, in 1904 [33] was probably the first person to use the term "*semigroup*" in a mathematical context. Though this was soon followed by L. E. Dickson [6], the first fundamental publication on Semigroup Theory was produced by A. K. Suschkewitsch [34], almost a quarter of a century later. Since then the interest in this field of abstract algebra has expanded with important results obtained by Rees [31], Clifford [4], Vagner [41], Preston [29], Dubreil [7] and others. The first book on Semigroups was by Ljapin [21]. Clifford and Preston vol I (1961) and vol II (1967) wrote a book on a much larger scale, collating the material published in the field up to that point. More recent books include those by Howie [12], Petrich [28] and Lallement [17], the last-named being devoted to the many recent applications of the theory.

Among the most obvious semigroups occurring in the "real world" is the semigroup of all mappings of a set into itself under the operation of composition of mappings. This is the semigroup analogue of the symmetric group on a set X and is indeed sometimes called the symmetric semigroup. More commonly, however, it is called the *full transformation semigroup on the set X* and is denoted by $\mathcal{T}(X)$. It has been studied by many authors, including Howie [11, 13, 15] and Munn [24]. In this thesis, some infinite transformation semigroups are studied.

In this introductory chapter a number of basic concepts and results on the full transformation semigroup are presented. Most of them will be indispensable for the remainder of this thesis. For elementary concepts and propositions as well as notation on Semigroup Theory see [5, 12].

Let X be an infinite set of cardinality $\underset{\sim}{m}$, and let $\mathcal{T}(X)$ be the full transformation semigroup on X. In 1966 Howie introduced the sets $S(\alpha)$, def $\alpha(= Z(\alpha))$ and $C(\alpha)$ as

$$S(\alpha) = \{x \in X : x\alpha \neq x\} \ ; \ \text{def}\ (\alpha) = X \setminus X\alpha\ ,$$

$$C(\alpha) = \bigcup\{t\alpha^{-1} : t \in X\alpha\ ,\ |t\alpha^{-1}| \geq \underset{\sim}{2}\}\ ,$$

and referred to the cardinals $|S(\alpha)|$, $|\text{def}\ \alpha|$ and $|C(\alpha)|$ as the *shift*, the *defect* and the *collapse of* α, respectively. In a more recent paper [15] some more precise terminology was introduced. For each infinite cardinal $\underset{\sim}{n}$ not exceeding $|X|$, a *balanced element of weight* $\underset{\sim}{n}$ is defined to be an element of $\mathcal{T}(X)$ for which

$$|S(\alpha)| = |\text{def}\ \alpha| = |C(\alpha)| = \underset{\sim}{n}\ .$$

In fact from the obvious remark that def $\alpha \subseteq S(\alpha)$ we may deduce that in the case where $\underset{\sim}{n} = \underset{\sim}{m}$ (the only case we shall be considering here) the condition $|S(\alpha)| = \underset{\sim}{m}$ is a consequence of the conditions $|\text{def}\ \alpha| = |C(\alpha)| = \underset{\sim}{m}$.

The set

$$\{\alpha \in \mathcal{T}(X) : |S(\alpha)| = |\text{def}\ \alpha| = |C(\alpha)| = \underset{\sim}{n}\} \qquad (1.1)$$

was denoted by $Q_{\underset{\sim}{n}}$. It is a regular subsemigroup of $\mathcal{T}(X)$.

Denoting the set of singular idempotents of $\mathcal{T}(X)$ by E, Howie [11] showed that the subsemigroup $\langle E\rangle$ generated in $\mathcal{T}(X)$ by E is given by

$$\langle E\rangle = F \cup \bigcup\{Q_{\underset{\sim}{n}} : \aleph_0 \leq \underset{\sim}{n} \leq \underset{\sim}{m}\}\ ,$$

where F is the subsemigroup of $\mathcal{T}(X)$ consisting of all elements of finite shift and finite non-zero defect. In [15] he showed that F and each $Q_{\underset{\sim}{n}}$ are generated by their idempotents

$$F = \langle E(F)\rangle\ , \quad Q_{\underset{\sim}{n}} = \langle E(Q_{\underset{\sim}{n}})\rangle\ .$$

They are therefore examples of *idempotent-generated semigroups*.

Let S be an idempotent-generated semigroup with set E of idempotents. Thus

$$E \subseteq E^2 \subseteq E^3 \subseteq \ldots\ldots \quad \text{and} \quad S = \langle E \rangle = \bigcup_{n=1}^{\infty} E^n .$$

If there exists a least k for which $E^k = S$ we say that S has *depth k*; otherwise, S has *infinite depth*. It is shown in [15] that F has infinite depth but that each Q_n ($\aleph_0 \leq n \leq m$) has depth 4.

Specialising to the case where $n = m$ (= $|X|$), Howie [16] describes the subset S_m of Q_m as

$$S_m = \{\alpha \in Q_m : |\operatorname{ran}\alpha| = m, \ |y\alpha^{-1}| < m \ (\forall y \in \operatorname{ran}\alpha)\}.$$

It is known [16] that S_m is a subsemigroup of Q_m provided the cardinal m is *regular*, i.e., if it has the property that $|\Lambda| < m$ and $m_\lambda < m$ for all $\lambda \in \Lambda$ together imply

$$\sum_{\lambda \in \Lambda} m_\lambda < m . \qquad (1.2)$$

(See [30] for this definition). We shall see this in more detail in Chapter 3.

In [16] Howie takes m to be a regular cardinal and shows that S_m is then a bisimple and idempotent-generated subsemigroup of Q_m of depth 4. Following Mal'cev [22], Howie considers the set

$$D(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\} ,$$

and the congruence

$$\Delta_m = \{(\alpha,\beta) \in S_m \times S_m : \max(|D(\alpha,\beta)\alpha|, |D(\alpha,\beta)\beta|) < m\} ,$$

and then shows that $S^*_m = S_m/\Delta_m$ is bisimple, idempotent-generated of depth 4, and congruence-free.

Since $S_{\underset{\sim}{m}}$ fails even to be a semigroup when $\underset{\sim}{m}$ is not regular, the question of whether or not it is possible to find a similar semigroup in the case of a general infinite cardinal arose naturally. The answer, although affirmative, was not straightforward. Within the semigroup $Q_{\underset{\sim}{m}}$ there is a top $\mathcal{J}$-class consisting of all α in $Q_{\underset{\sim}{m}}$ for which $|X\alpha| = \underset{\sim}{m}$ and an ideal

$$I_{\underset{\sim}{m}} = \{\alpha \in Q_{\underset{\sim}{m}} : |X\alpha| < \underset{\sim}{m}\} .$$

The principal factor

$$P_{\underset{\sim}{m}} = Q_{\underset{\sim}{m}}/I_{\underset{\sim}{m}}$$

turns out to have properties that to some extent mirror the properties of $S_{\underset{\sim}{m}}$. The object of Chapter 2 is to explore these properties. Also a o-bisimple, idempotent-generated and congruence-free semigroup $P^*_{\underset{\sim}{m}}$ is described ($\underset{\sim}{m}$ being a general infinite cardinal) and related with $S^*_{\underset{\sim}{m}}$.

Having found $P_{\underset{\sim}{m}}$ and therefore generalised [16] for the case of a general infinite cardinal, one question still remained - the problem of describing $\langle S_{\underset{\sim}{m}}\rangle$, the subsemigroup of $Q_{\underset{\sim}{m}}$ generated by the stable elements, for the case of a singular (i.e., non-regular) cardinal $\underset{\sim}{m}$. This problem is solved in the first part of Chapter 3.

In group theory congruences are determined provided one knows the normal subgroup which is the congruence class containing the identity. Similarly, in ring theory, congruences are determined if we know the ideal which is the congruence class containing the zero. Such a situation does not occur in semigroup theory and we are therefore forced to study congruences as such. Our purpose, in the second part of Chapter 3, is to study the congruences in $S_{\underset{\sim}{m}}$, where $\underset{\sim}{m}$ is a regular cardinal. The problem is completely solved for the case in which $\underset{\sim}{m} = \aleph_0$; but the question still remains unsolved for any other infinite regular cardinal.

Still inside the full transformation semigroup on X other semigroups were considered. In their paper (1932), R. Baer and F. Levi construct a right cancellative, right simple semigroup which is not a group. The semigroup they construct is the semigroup of all one-to-one mappings α of a countable set R into itself, with the property that $R\backslash R\alpha$ is not finite. More generally, following Clifford and Preston [5], if $\underset{\sim}{p},\underset{\sim}{q}$ are infinite cardinals such that $\underset{\sim}{p} \geq \underset{\sim}{q}$, we shall say that B is *a Baer-Levi semigroup of type* $(\underset{\sim}{p},\underset{\sim}{q})$ *on the set* A, if $|A| = \underset{\sim}{p}$ and if B is the semigroup of all one-to-one mappings α (combined under composition) of A into A, satisfying the property

$$|A \backslash A\alpha| = \underset{\sim}{q} \ .$$

In Chapter 4 we consider the Baer-Levi semigroup B of type $(\underset{\sim}{m},\underset{\sim}{m})$ on the set X and our aim is to dualize such a semigroup. Within the symmetric inverse semigroup $\mathscr{I}(X)$ (that is, the semigroup of all partial one-to-one mappings on X) there is a dual B^*. The products BB^* and B^*B are described. Particular interest is attached to the semigroup

$$K_{\underset{\sim}{m}} = \langle B^*B \rangle \ .$$

In a semigroup S with zero an element s is said to be *nilpotent* if $s^n = 0$ for some $n \geq 1$. If $s^n = 0$ but $s^{n-1} \neq 0$ we say that s is *nilpotent of index* n. It is shown that in the symmetric inverse semigroup $\mathscr{I}(X)$ the nilpotent elements of index 2 generate $K_{\underset{\sim}{m}}$. Also, a o-bisimple, inverse, congruence-free and nilpotent-generated semigroup is described.

Finally, in Chapter 5 two embedding theorems for inverse semigroups and semigroups in general are presented. Also, (section 4) a study of the cardinalities of some of the different semigroups introduced in this thesis is provided.

CHAPTER 2

A CONGRUENCE-FREE SEMIGROUP ASSOCIATED WITH AN INFINITE CARDINAL NUMBER

1. INTRODUCTION AND BACKGROUND

In this chapter the basic concepts and results are as presented in the first part of Chapter 1.

A preliminary objective of this section, Theorem 2.9, describes a o-bisimple, idempotent-generated semigroup of depth 4. There is a strong analogy with Howie's study of the semigroup of the stable elements [16].

In section 4 a congruence-free, idempotent-generated semigroup of depth 4 is obtained. Here again, the results are quite similar to the ones obtained by Howie [16, theorem 3.11].

2. PRELIMINARIES

Let X be a set with infinite cardinality $\underset{\sim}{m}$ and let $Q_{\underset{\sim}{m}}$ be the semigroup of the balanced elements as defined in (1.1).

LEMMA 2.1. *The set* $J_{\underset{\sim}{k}} = \{\alpha \in Q_{\underset{\sim}{m}} : |X\alpha| = \underset{\sim}{k}\}$ *is a* $\mathcal{J}$*-class in* $Q_{\underset{\sim}{m}}$ *for all* $\underset{\sim}{k} \leq \underset{\sim}{m}$.

Proof. Let $\alpha, \beta \in Q_{\underset{\sim}{m}}$ and suppose that $|X\alpha| = |X\beta|$. Then there is a bijection $\theta: X\alpha \to X\beta$. Let T be a cross-section of Ker β, that is, a set such that $|T \cap A| = \underset{\sim}{1}$ for every (Ker β)-class A. Then we shall show that $|X \setminus T| = \underset{\sim}{m}$. To see this, let

$$R = \{y \in X\beta : |y\beta^{-1}| \geq 2\},$$

giving

$$C(\beta) = \bigcup\{y\beta^{-1} : y \in R\}.$$

If $|R| = \underset{\sim}{m}$, then writing $T \cap y\beta^{-1} = \{t_y\}$ we have

$$X \setminus T = \bigcup\{y\beta^{-1} \setminus \{t_y\} : y \in R\}$$

and so $|X \setminus T| \geq |R| = \underset{\sim}{m}$. If $|R| < \underset{\sim}{m}$, let

$$R_f = \{y \in X\beta : 2 \leq |y\beta^{-1}| < \infty\}.$$

Then,

$$\left| \bigcup\{ y\beta^{-1} : y \in R_f \} \right| < \underset{\sim}{m} ;$$

hence, since $|C(\beta)| = \underset{\sim}{m}$ we must have

$$\left| \bigcup\{y\ \beta^{-1} : y \in R \setminus R_f\} \right| = \underset{\sim}{m}$$

i.e.,

$$\sum_{y \in R \setminus R_f} |y\beta^{-1}| = \underset{\sim}{m} .$$

But since $|y\beta^{-1}|$ is infinite for all y in $R \setminus R_f$, it follows that $|y\beta^{-1}| = |y\beta^{-1} \setminus \{t_y\}|$, $(y \in R \setminus R_f)$. Hence, since

$$X \setminus T \supseteq \bigcup\{y\beta^{-1} \setminus \{t_y\} : y \in R \setminus R_f\},$$

we obtain $|X \setminus T| = \underset{\sim}{m}$.

Now, define $\xi \in \mathscr{T}(X)$ as follows: for each $(\ker \alpha)$ -class A define

$$A\xi = A\alpha\theta\beta^{-1} \cap T.$$

Then, $\operatorname{Ker} \xi = \ker \alpha$ and so $|C(\xi)| = |C(\alpha)| = \underset{\sim}{m}$; also, $\operatorname{ran} \xi = T$ and so $|\operatorname{def} \xi| = |X \setminus T| = \underset{\sim}{m}$. So, ξ belongs to $Q_{\underset{\sim}{m}}$.

Next define η in $\mathscr{T}(X)$ by

$$\eta \mid X\beta = \theta^{-1} : X\beta \longrightarrow X\alpha,$$

and for all $x \in \operatorname{def} \beta$

$$x\eta = z,$$

where z is a fixed element of def α. Then $|C(\eta)| = |\text{def } \beta| = m$, $|\text{def } \eta| = |\text{def } \alpha \setminus \{z\}| = m$ and so η belongs to Q_m.

Finally, notice that, for each A in $X \setminus \text{Ker } \alpha$,

$$A\xi\beta\eta = (T \cap A\alpha\theta\beta^{-1})\beta\eta = A\alpha\theta\eta = A\alpha$$

and so $\xi\beta\eta = \alpha$.

Similarly we can find τ and ζ in Q_m such that $\beta = \tau\alpha\zeta$. Hence, we have shown that

$$|X\alpha| = |X\beta| \Rightarrow \alpha \mathscr{J} \beta \text{ in } Q_m .$$

Also, if α and β are two elements in Q_m such that $\alpha \mathscr{J} \beta$ in Q_m then $\alpha \mathscr{J} \beta$ in $\mathscr{T}(X)$ and so it follows from [12, Ex.II.10] that

$$|X\alpha| = |X\beta| = p ,$$

for some $p \leq m$. Hence $\alpha, \beta \in J_p$.

Lemma 2.1 is now proved.

The top $\mathscr{J}$-class in Q_m is $J_m = \{\alpha \in Q_m : |X\alpha| = m\}$, which is not a subsemigroup of Q_m, for it is possible to have α, β in Q_m such that $|X\alpha| = |X\beta| = m$ and $|X(\alpha\beta)| < m$. Suppose for instance that $X = Y \cup Z_1 \cup Z_2$ is a partition of X such that $|Y| = |Z_1| = |Z_2| = m$. Choose α in Q_m such that ran $\alpha = Y$. Now choose and fix a in Z_1 and let β map Y onto a and $Z_1 \cup Z_2$ onto Z_1 in a one-to-one manner. Then

$$|\text{ran } \beta| = |Z_1| = m.$$

Also $|C(\beta)| = |a\beta^{-1}| = |Y| = m$ and $|\text{def } \beta| = m$ since def $\beta \supseteq Z_2$. Hence $\beta \in J_m$ and it is obvious that

$$|X(\alpha\beta)| = |Y\beta| = |\{a\}| = 1 < m .$$

Consider now the ideal

$$I_m = \{\alpha \in Q_m : |X\alpha| < m\} .$$

The principal factor $P_m = Q_m / I_m$ is defined as

$$P_m = \{ \{\alpha\} : \alpha \in J_m \} \cup \{ I_m \}$$

and it is a semigroup. Because of its own structure (a semigroup of congruence-classes) it is certainly not a subsemigroup of Q_m, but we may think of it as

$$P_m = J_m \cup \{0\} , \qquad (2.2)$$

a $\mathcal{J}$-class with the zero element adjoined.

3. THE SEMIGROUP P_m

In this section we shall explore the properties of P_m.

Since P_m is a principal factor of the regular semigroup Q_m, we have

LEMMA 2.3 $\quad P_m$ *is regular.*

LEMMA 2.4 $\quad P_m$ *is o-simple.*

Proof. By [12, Theorem III.1.9] P_m is either o-simple or null. To show that P_m is not null it will certainly be sufficient to show that J_m contains an idempotent.

Since Q_m is regular, every element α in J_m has an inverse α' in Q_m. By [12, Theorem II.3.5] we have that $\alpha' \mathcal{D} \alpha$ and so, since $\mathcal{D} \subseteq \mathcal{J}$ it follows that $\alpha' \in J_m$. Since $\alpha\alpha' \mathcal{R} \alpha$ we have that $\alpha\alpha' \in J_m$. Hence $\alpha\alpha'$ is an idempotent in J_m as required.

LEMMA 2.5 $\quad P_m$ *is o-bisimple.*

Proof. Since Q_m is a regular subsemigroup of $\mathcal{T}(X)$, it follows [12, Proposition II.4.5] that if $\alpha, \beta \in J_m$ then $\alpha \mathcal{R} \beta$ in Q_m if and only if $\alpha \mathcal{R} \beta$ in $\mathcal{T}(X)$, i.e., if and only if $\ker \alpha = \ker \beta$ [12, Ex. II.10] .

Similarly, $\alpha \mathscr{L} \beta$ in $Q_{\underset{\sim}{m}}$ if and only if ran α = ran β. Since every element in a regular semigroup is $\mathscr{D}$-equivalent to an idempotent [12 Proportion II.3.2] the o-bisimplicity of $P_{\underset{\sim}{m}}$ will follow if we show that for every pair of idempotents ε, η in $J_{\underset{\sim}{m}}$ there exists $\alpha \in J_{\underset{\sim}{m}}$ such that $\varepsilon \mathscr{R} \alpha$ and $\alpha \mathscr{L} \eta$. Suppose that ε, η are idempotents in $J_{\underset{\sim}{m}}$. Then

$$| S(\varepsilon) | = |\text{def } \varepsilon| = | C(\varepsilon) | = | \text{ran } \varepsilon | = \underset{\sim}{m},$$

and

$$| S(\eta) | = |\text{def } \eta| = | C(\eta) | = | \text{ran } \eta | = \underset{\sim}{m}.$$

Since ε is an idempotent we also have $| X / \ker \varepsilon | = \underset{\sim}{m}$ for $|\text{ran } \varepsilon| = \underset{\sim}{m}$ and $a\varepsilon \overset{f}{\longmapsto} (a\varepsilon)\varepsilon^{-1}$ gives a one-to-one map from ran ε into $X / \text{Ker } \varepsilon$. Then, let θ be a bijection from $X / \text{Ker } \varepsilon$ onto ran η and define α in $\mathscr{T}(X)$ by

$$x\alpha = [\, x (\text{Ker } \varepsilon)\,]\theta.$$

It is obvious that ran α = ran η and that Ker α = Ker ε and so $\varepsilon \mathscr{R} \alpha$ and $\alpha \mathscr{L} \eta$. Notice now that $\alpha \in J_{\underset{\sim}{m}}$, since $| \text{def } \alpha | = | \text{def } \eta | = \underset{\sim}{m}$, $| C(\alpha) | = | C(\varepsilon) | = \underset{\sim}{m}$ and $| \text{ran } \alpha | = | \text{ran } \eta | = \underset{\sim}{m}$. Hence $P_{\underset{\sim}{m}}$ is o-bisimple.

LEMMA 2.6 $P_{\underset{\sim}{m}}$ *is an idempotent-generated semigroup of depth not exceeding* 4.

Proof. Since $Q_{\underset{\sim}{m}}$ is idempotent-generated of depth 4, it follows that for each α in $J_{\underset{\sim}{m}}$ there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ in $Q_{\underset{\sim}{m}}$ such that $\alpha = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$. From the general result that in any semigroup

$$J_{xy} \leq J_x \,,\; J_{xy} \leq J_y \,;$$

[12 Proposition II. 1.10], we deduce that

$$J_{\underset{\sim}{m}} = J_\alpha \leq J_{\varepsilon_i} \qquad (i = 1,2,3,4) \,,$$

and so $\varepsilon_i \in J_{\underset{\sim}{m}}$, since $J_{\underset{\sim}{m}}$ is the top $\mathscr{J}$-class. Hence the lemma follows.

To show that the depth $\Delta(P_{\underset{\sim}{m}})$ of $P_{\underset{\sim}{m}}$ is exactly 4 entails finding an element of $P_{\underset{\sim}{m}}$ that cannot be expressed as a product of three idempotents. To do this we need a preliminary lemma.

LEMMA 2.7. *Let* $\alpha \in J_{\underset{\sim}{m}}$. *If* $\alpha = \varepsilon_1\varepsilon_2\varepsilon_3$, *a product of three idempotents in* $J_{\underset{\sim}{m}}$, *then there exist two idempotents* η_1 *and* η_3 *in* $J_{\underset{\sim}{m}}$ *such that* $\operatorname{Ker} \eta_1 = \operatorname{Ker} \alpha$, $\operatorname{ran} \eta_3 = \operatorname{ran} \alpha$ *and* $\alpha = \eta_1\varepsilon_2\eta_3$.

Proof. By [15, Lemma 3.8] and its dual, we can find two idempotents η_1 and η_3 in $J_{\underset{\sim}{m}}$ such that $\alpha = \eta_1\varepsilon_2\eta_3$, $\eta_1 \mathscr{R} \alpha$ and $\eta_3 \mathscr{L} \alpha$. Hence, by [12, Ex.II.10], it follows that $\operatorname{Ker} \eta_1 = \operatorname{Ker} \alpha$ and $\operatorname{ran} \eta_3 = \operatorname{ran} \alpha$.

For reasons that will be apparent later, we shall find a whole collection of elements that cannot be expressed as a product of three idempotents.

LEMMA 2.8. *Let* R *be the subset of* $P_{\underset{\sim}{m}}$ *defined by the rule that* $\alpha \in R$ *if and only if the sets* $U = C(\alpha)$ *and* $V = X \setminus U$ *have the properties*

$$(R_1)\ \ |U \setminus \operatorname{ran} \alpha| < \underset{\sim}{m}\ ;\quad (R_2)\ \ |V \cap V\alpha| < \underset{\sim}{m}\ ,$$
$$(R_3)\ \ |U\alpha \cap V| = \underset{\sim}{m}\ .$$

Then no element of R *is expressible as a product of three idempotents.*

Proof. We show first that $R \neq \emptyset$. Since $|X| = \underset{\sim}{m}$ we may consider a partition of X into a disjoint union $X_1 \cup X_2 \cup X_3 \cup X_4$ such that

$$|X_1| = |X_2| = |X_3| = |X_4| = \underset{\sim}{m}\ .$$

Let $\theta : X_1 \cup X_2 \longrightarrow X_3 \cup X_4$, $\Phi : X_3 \longrightarrow X_4$ and $\Psi : X_3 \longrightarrow X_1$ be bijections. Define $\alpha : X \longrightarrow X$ by

$$x\alpha = x\theta \qquad (x \in X_1 \cup X_2)$$
$$x\alpha = (x\Phi)\alpha = x\Psi \qquad (x \in X_3)$$

Then, $U = C(\alpha) = X_3 \cup X_4$ and $V = X_1 \cup X_2$. Since $\operatorname{ran} \alpha = X_1 \cup X_3 \cup X_4$ it is clear that $\alpha \in J_{\underset{\sim}{m}}$. Furthermore, since $U \setminus \operatorname{ran} \alpha = \emptyset$, $V \cap V\alpha = \emptyset$ and

$U\alpha \cap V = X_1$, conditions (R_1), (R_2) and (R_3) are satisfied and so $\alpha \in R$. Thus $R \neq \emptyset$.

Now, take α in R and suppose, by way of contradiction, that $\alpha = \varepsilon_1\varepsilon_2\varepsilon_3$, where ε_1, ε_2, ε_3 are idempotents in J_m; by lemma 2.7 we may assume that Ker ε_1 = Ker α and that ran ε_3 = ran α. Take v in $U\alpha \cap V$ and let $U_v = v\alpha^{-1}$, i.e., $U_v\alpha = v$. Then U_v is a (Ker ε_1)-class since Ker α = Ker ε_1 and so maps by ε_1 to an element of itself, say $U_v\varepsilon_1 = u_v$. Consider now the element $z = u_v\varepsilon_2 = v\alpha^{-1}\varepsilon_1\varepsilon_2$. Either (i) $z \in V$ or (ii) $z \in U$.

In case (i) we have that $z \notin C(\alpha)$ and so $\{z\}$ is a singleton (Ker ε_1)-class. Hence, $z\varepsilon_1 = z$ and so we have

$$z\varepsilon_1\varepsilon_2 = z\varepsilon_2 = u_v\varepsilon_2^2 = u_v\varepsilon_2 = U_v\varepsilon_1\varepsilon_2;$$

thus, applying ε_3 to both sides, we get

$$z\alpha = z\varepsilon_1\varepsilon_2\varepsilon_3 = U_v\varepsilon_1\varepsilon_2\varepsilon_3 = U_v\alpha = v,$$

i.e., $z\alpha = v \in V \cap V\alpha$.

In case (ii) we have $z \notin$ ran α, for if $z \in$ ran α = ran ε_3, then $z\varepsilon_3 = z$. But

$$z\varepsilon_3 = u_v\varepsilon_2\varepsilon_3 = U_v\varepsilon_1\varepsilon_2\varepsilon_3 = U_v\alpha = v,$$

and so $v = z\varepsilon_3 = z$, which cannot happen since $V \cap U = \emptyset$. Thus $z \in U$ implies that $z \in U \setminus \text{ran}\,\alpha$.

Now define $\Psi : U\alpha \cap V \longrightarrow (V \cap V\alpha) \cup (U \setminus \text{ran}\,\alpha)$ as follows: for each $v \in U\alpha \cap V$,

$$v\Psi = v \quad \text{if } z = v\alpha^{-1}\varepsilon_1\varepsilon_2 \in V,$$

$$v\Psi = z \quad \text{otherwise.}$$

Notice that Ψ is one-to-one. For suppose that $v, v' \in U\alpha \cap V$ are such that $v\Psi = v'\Psi$. Hence, either both $v\Psi$ and $v'\Psi$ are in V or both $v\Psi$ and $v'\Psi$ are in U. In the former case $v = v\Psi = v'\Psi = v'$, while in the latter case $v\alpha^{-1}\varepsilon_1\varepsilon_2 = v'\alpha^{-1}\varepsilon_1\varepsilon_2$ from which it follows that $v\alpha^{-1}\varepsilon_1\varepsilon_2\varepsilon_3 = v'\alpha^{-1}\varepsilon_1\varepsilon_2\varepsilon_3$, i.e.

that $V = V'$. Hence Ψ is one-to-one and we have

$$|U\alpha \cap V| \leq |(V \cap V\alpha) \cup (U \setminus \operatorname{ran} \alpha)|$$
$$\leq |V \cap V\alpha| + |U \setminus \operatorname{ran} \alpha| < \underset{\sim}{m},$$

by (R_1) and (R_2). Since this contradicts (R_3') we conclude that α cannot be expressed as a product of three idempotents in $J_{\underset{\sim}{m}}$. We have proved

THEOREM 2.9. $P_{\underset{\sim}{m}}$ *is a* 0-*bisimple idempotent-generated semigroup of depth* 4.

4. A CONGRUENCE-FREE SEMIGROUP ASSOCIATED WITH $\underset{\sim}{m}$

We shall now recall that a semigroup S is called *congruence-free* if the only congruences on it are the identical congruence 1_S and the universal congruence $S \times S$. The semigroup $P_{\underset{\sim}{m}}$ is not congruence-free since Mal'cev's congruences [22] induce congruences on it. In more detail, if we define for $\alpha, \beta \in Q_{\underset{\sim}{m}}$

$$D(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\},$$

then for each $\underset{\sim}{n}$ such that $\aleph_0 \leq \underset{\sim}{n} \leq \underset{\sim}{m}$, we obtain a congruence on $Q_{\underset{\sim}{m}}$

$$\Delta_{\underset{\sim}{n}} = \{(\alpha,\beta) \in Q_{\underset{\sim}{m}} \times Q_{\underset{\sim}{m}} : \max(|D(\alpha,\beta)\alpha|, D(\alpha,\beta)\beta|) < \underset{\sim}{n}\}.$$

Notice that if $I^0_{\underset{\sim}{m}}$ is the Rees congruence on $Q_{\underset{\sim}{m}}$ whose quotient semigroup is $P_{\underset{\sim}{m}} = Q_{\underset{\sim}{m}}/I_{\underset{\sim}{m}}$ ($I_{\underset{\sim}{m}}$ being the ideal $\{\alpha \in Q_{\underset{\sim}{m}} : |\operatorname{ran} \alpha| < \underset{\sim}{m}$) then

$$I^0_{\underset{\sim}{m}} \subseteq \Delta_{\underset{\sim}{m}}.$$

Hence it follows from [12, Theorem I.5.6] that

$$\hat{\Delta}_{\underset{\sim}{m}} = \Delta_{\underset{\sim}{m}}/I^0_{\underset{\sim}{m}} = \{(\bar{\alpha},\bar{\beta}) \in P_{\underset{\sim}{m}} \times P_{\underset{\sim}{m}} : (\alpha,\beta) \in \Delta_{\underset{\sim}{m}}\}$$

is a congruence on $P_{\underset{\sim}{m}}$, where $\bar{\alpha}$ denotes the congruence class containing α. It is not difficult to see that

$$\hat{\Delta}_m = \{(\alpha,\beta) \in J_m \times J_m : (\alpha,\beta) \in \Delta_m\} \cup \{(0,0)\}. \tag{2.10}$$

In fact, if $\alpha,\beta \in P_{\underset{\sim}{m}}$ are such that $|\text{ran }\alpha| = \underset{\sim}{m}$ and $(\alpha,\beta) \in \Delta_{\underset{\sim}{m}}$ then

$$|D(\alpha,\beta)\alpha| < \underset{\sim}{m}; \quad |D(\alpha,\beta)\beta| < \underset{\sim}{m}.$$

Hence, since $|\text{ran }\alpha| = \underset{\sim}{m}$ and

$$\text{ran }\alpha = [D(\alpha,\beta)\alpha] \cup [X \setminus D(\alpha,\beta)]\alpha,$$

it follows that $|[X \setminus D(\alpha,\beta)]\alpha| = \underset{\sim}{m}$. Hence

$$\begin{aligned}\text{ran }\beta &= [D(\alpha,\beta)\alpha] \cup [X \setminus D(\alpha,\beta)]\beta \\ &= [D(\alpha,\beta)\beta] \cup [X \setminus D(\alpha,\beta)]\alpha\end{aligned}$$

gives $|\text{ran }\beta| = \underset{\sim}{m}$, i.e., $\beta \in J_{\underset{\sim}{m}}$. So if $\bar{\alpha},\bar{\beta}$ in $P_{\underset{\sim}{m}}$ are such that $(\alpha,\beta) \in \Delta_{\underset{\sim}{m}}$ then either both α,β are in $J_{\underset{\sim}{m}}$ or they are both in $I_{\underset{\sim}{m}}$.

The theorem we now state shows that $\hat{\Delta}_{\underset{\sim}{m}}$ (as defined in 2.10) is the unique maximum non-trivial congruence on $P_{\underset{\sim}{m}}$.

THEOREM 2.11. *Let* X *be a set with infinite cardinal* $\underset{\sim}{m}$ *and let* $P_{\underset{\sim}{m}} = J_{\underset{\sim}{m}} \cup \{0\}$ *be the semigroup defined in* (2.2). *Let* $\hat{\Delta}_{\underset{\sim}{m}}$ *be the congruence defined in* $P_{\underset{\sim}{m}}$ *by* (2.10) *and denote* $P_{\underset{\sim}{m}}/\hat{\Delta}_{\underset{\sim}{m}}$ *by* $P^*_{\underset{\sim}{m}}$. *Then* $P^*_{\underset{\sim}{m}}$ *is a congruence-free, o-bisimple, idempotent-generated semigroup of depth* 4.

Proof. Since $P_{\underset{\sim}{m}}$ is o-bisimple (Lemma 2.5) and idempotent-generated (Lemma 2.6) and since these properties are inherited by non-trivial homomorphic images, it follows that $P_{\underset{\sim}{m}}^{*}$ is a o-bisimple and idempotent-generated semigroup. From Lemma 2.6 it follows also that $\Delta(P_{\underset{\sim}{m}}^{*}) \leqslant 4$. We have to show now that $P_{\underset{\sim}{m}}^{*}$ is congruence-free and that $\Delta(P_{\underset{\sim}{m}}^{*}) = 4$.

It is known [39, 32] that a regular O-simple semigroup S is congruence-free if and only if the congruence

$$\sigma = \{(a,b) \in S \times S : (\forall s,t \in S^{1})\ sat = 0 \Leftrightarrow sbt = 0\}$$

is trivial. Applying this to $P_{\underset{\sim}{m}}^{*}$, we see that what we are required to show is that if $\alpha,\beta \in P_{\underset{\sim}{m}}$ are such that

$$(\forall\ \lambda,\nu \in P_{\underset{\sim}{m}}) \quad \lambda\,\alpha\,\nu = 0 \Longleftrightarrow \lambda\,\beta\,\nu = 0,$$

then $(\alpha,\beta) \in \Delta_{\underset{\sim}{m}}$.

Accordingly, let us suppose that α,β in $J_{\underset{\sim}{m}}$ are such that $(\alpha,\beta) \notin \Delta_{\underset{\sim}{m}}$. We shall find λ,ν in $J_{\underset{\sim}{m}}$ such that $|\operatorname{ran} \lambda\,\alpha\,\nu| = \underset{\sim}{m}$, $|\operatorname{ran} \lambda\,\beta\,\nu| < \underset{\sim}{m}$. We have that $\max(|D\alpha|, |D\beta|) = \underset{\sim}{m}$, where $D = D(\alpha,\beta)$, and so either $|D\alpha| = \underset{\sim}{m}$ or $|D\beta| = \underset{\sim}{m}$ (or both). Suppose, without loss of generality, that $|D\alpha| = \underset{\sim}{m}$ and consider the following Lemma, analogous to Lemma 2 in Lindsey and Madison [20] and to Lemma 3.12 in Howie [16]:

LEMMA 2.12. *If* $\alpha,\beta \in J_{\underset{\sim}{m}}$ *are such that* $(\alpha,\beta) \notin \Delta_{\underset{\sim}{m}}$ *and* $|D\alpha| = \underset{\sim}{m}$, *then there exists a non-empty subset* Y *of* D *such that* $Y\alpha \cap Y\beta = \emptyset$ *and* $\max(|Y\alpha|, |Y\beta|) = \underset{\sim}{m}$.

Proof. We have to consider two cases (i) $|D\beta| < \underset{\sim}{m}$ and (ii) $|D\beta| = \underset{\sim}{m}$. In case (i) we must have $|D\alpha \setminus D\beta| = \underset{\sim}{m}$. Consider then the set

$$Y = [(D\alpha \setminus D\beta)\alpha^{-1}] \cap D$$

and notice that $Y\alpha = D\alpha \setminus D\beta$. For it is obvious on one hand that $Y\alpha \subseteq (D\alpha \setminus D\beta)\,\alpha^{-1}\alpha = D\alpha \setminus D\beta$. On the other hand, if $x \in D\alpha \setminus D\beta$, then $x\alpha^{-1} \cap D \subseteq Y$ and so $(x\alpha^{-1} \cap D)\,\alpha \subseteq Y\alpha$; but $(x\alpha^{-1} \cap D)\alpha = x$ and so

$x \in Y\alpha$. Therefore, $D\alpha \setminus D\beta \subseteq Y\alpha$ and we have $Y\alpha = D\alpha \setminus D\beta$. Thus $|Y\alpha| = \underset{\sim}{m}$. Since $Y\beta \subseteq D\beta$, it follows also that $Y\alpha \cap Y\beta = \emptyset$. Hence the Lemma follows.

In case (ii) where $|D\beta| = \underset{\sim}{m}$, consider the set $\mathscr{C}$ of all subsets Z of D such that $Z\alpha \cap Z\beta = \emptyset$. Then $\mathscr{C} \neq \emptyset$ since it contains all singleton subsets of D. Also, if $\{C_\lambda : \lambda \in \Lambda\}$ is a tower in $\mathscr{C}$ and $C = \bigcup\{C_\lambda : \lambda \in \Lambda\}$ it is easily verified that $C \in \mathscr{C}$ and so, by Zorn's Lemma there exists a maximal subset Z of D such that $Z\alpha \cap Z\beta = \emptyset$. If one or other of $|Z\alpha|$, $|Z\beta|$ is equal to $\underset{\sim}{m}$ then Z is the set we require. So suppose that $|Z\alpha| < \underset{\sim}{m}$, $|Z\beta| < \underset{\sim}{m}$. Then $D \setminus Z \neq \emptyset$ for otherwise $D = Z$ and so $|Z\alpha| = |D\alpha| = \underset{\sim}{m}$. Also, for all d in $D \setminus Z$ the maximality of Z implies that

$$(Z \cup \{d\})\alpha \cap (Z \cup \{d\})\beta \neq \emptyset.$$

Hence, since $Z\alpha \cap Z\beta = \emptyset$ and $d\alpha \neq d\beta$, either $d\alpha \in Z\beta$ or $d\beta \in Z\alpha$. Let

$$D_1 = \{d \in D\setminus Z : d\beta \in Z\alpha\},$$
$$D_2 = \{d \in D\setminus Z : d\alpha \in Z\beta\};$$

Thus, $D \setminus Z = D_1 \cup D_2$ (not necessarily disjoint) and $D_1\beta \subseteq Z\alpha$, $D_2\alpha \subseteq Z\beta$. We have $D = D_1 \cup D_2 \cup Z$ and so $D\alpha = D_1\alpha \cup D_2\alpha \cup Z\alpha$. But $|D_2\alpha| \leqslant |Z\beta| < \underset{\sim}{m}$, $|Z\alpha| < \underset{\sim}{m}$ and $|D\alpha| = \underset{\sim}{m}$. Hence, $|D_1\alpha| = \underset{\sim}{m}$. We now have $|D_1\alpha| = \underset{\sim}{m}$ and $|D_1\beta| \leqslant |Z\alpha| < \underset{\sim}{m}$. So, $|D_1\alpha \setminus D_1\beta| = \underset{\sim}{m}$ and we can use the case (i) argument to find

$$Y = [(D_1\alpha \setminus D_1\beta)\alpha^{-1}] \cap D_1$$

such that $Y\alpha \cap Y\beta = \emptyset$ and $|Y\alpha| = \underset{\sim}{m}$. The lemma follows. Notice that the existence of Y does not contradict the maximality of Z since $Y \subseteq D_1$ and so $Y \cap Z = \emptyset$.

Let us now go back to the proof of Theorem 2.11. We were supposing that $|D\alpha| = \underset{\sim}{m}$. Consider then $Y \subseteq D$ such that $Y\alpha \cap Y\beta = \emptyset$ and

$\max(|Y\alpha|, |Y\beta|) = \underset{\sim}{m}$. Suppose that $|Y\alpha| = m$; then certainly $|Y| = \underset{\sim}{m}$. Let V and Z be two subsets of Y such that $|V| = |Z| = \underset{\sim}{m}$, $V \cap Z = \emptyset$ and $V \cup Z = Y$. Then $Y\alpha = V\alpha \cup Z\alpha$ and since $|Y\alpha| = \underset{\sim}{m}$ we have either $|V\alpha| = \underset{\sim}{m}$ or $|Z\alpha| = \underset{\sim}{m}$ (or both). Suppose that $|Y\alpha| = \underset{\sim}{m}$; let $\theta : Z \to V$ be a bijection, let v_o be an arbitrarily fixed element in V, and define $\lambda : X \longrightarrow X$ as follows:

$$\begin{aligned} v\lambda &= v & v &\in V \\ z\lambda &= z\theta & z &\in Z \\ w\lambda &= v_o & w &\in W = X \setminus Y. \end{aligned}$$

Then, ran $\lambda = V$ and so $|\text{ran } \lambda| = \underset{\sim}{m}$. Also, def $\lambda = S(\lambda) = Z \cup W$ giving $|\text{def } \lambda| = |S(\lambda)| = \underset{\sim}{m}$. Finally $v_o\lambda^{-1} = W \cup \{v_o, v_o\theta^{-1}\}$ and $v\lambda^{-1} = \{v, v\theta^{-1}\}$ ($v \in V$ and $v \neq v_o$). Therefore, $C(\lambda) = \bigcup_{v \in V} v\lambda^{-1} = X$ and so $|C(\lambda)| = \underset{\sim}{m}$. Thus $\lambda \in J_{\underset{\sim}{m}}$. Since ran $\lambda = V \subseteq Y$ and since $Y\alpha \cap Y\beta = \emptyset$, it follows that

$$\text{ran } \lambda\alpha \cap \text{ran } \lambda\beta = \emptyset .$$

We certainly have $|\text{ran } \lambda\alpha| = |V\alpha| = \underset{\sim}{m}$. If $|\text{ran } \lambda\beta| < \underset{\sim}{m}$ then our argument is complete, for we then have

$$\lambda\alpha 1 \neq 0 \quad \text{and } \lambda\beta 1 = 0,$$

in the semigroup $P_{\underset{\sim}{m}}$. Suppose therefore that $|\text{ran } \lambda\beta| = \underset{\sim}{m}$.

Now let x_o be a fixed element in ran $\lambda\beta$ and define $\nu : X \longrightarrow X$ by

$$\begin{aligned} x\nu &= x & (x &\in \text{ran } \lambda\alpha), \\ x\nu &= x_o & (x &\in \text{ran } \lambda\beta); \end{aligned}$$

if ran $\lambda\alpha \cup$ ran $\lambda\beta = X$, this defines ν completely; otherwise choose x_1 in $X \setminus (\text{ran } \lambda\alpha \cup \text{ran } \lambda\beta)$ and define

$$x\nu = x_1, \ (x \in X \setminus (\text{ran } \lambda\alpha \cup \text{ran } \lambda\beta)).$$

Then ran ν = ran $\lambda\alpha \cup \{x_o, x_1\}$ and so

$$|\text{ran } \nu| = |\text{ def } \nu| = \underset{\sim}{m}.$$

Also, $C(\nu) \supseteq$ ran $\lambda\beta$ and so $|C(\nu)| = \underset{\sim}{m}$ too. Thus $\nu \in J_{\underset{\sim}{m}}$. It is now clear that

$$|\text{ ran } \lambda\alpha\nu| = \underset{\sim}{m}, \quad |\text{ ran } \lambda\beta\nu| = \underset{\sim}{1}$$

and so in $P_{\underset{\sim}{m}}$ we have

$$\lambda\alpha\nu \neq 0, \quad \lambda\beta\nu = 0.$$

It follows that $P_{\underset{\sim}{m}}^*$ is congruence-free.

It remains to show that $\Delta(P_{\underset{\sim}{m}}^*) = 4$. If $\Delta(P_{\underset{\sim}{m}}^*) \leq 3$ then for all α^* in $P_{\underset{\sim}{m}}^*$ there exist idempotents $\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$ in $P_{\underset{\sim}{m}}^*$ such that $\alpha^* = \varepsilon_1^* \varepsilon_2^* \varepsilon_3^*$. Hence, by Lallement's lemma [12, Lemma II. 4.6] , for each α in $P_{\underset{\sim}{m}}$ there exists α^o in $P_{\underset{\sim}{m}}$ and idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3$ in $P_{\underset{\sim}{m}}$ such that $(\alpha, \alpha^o) \in \Delta_{\underset{\sim}{m}}$ and $\alpha^o = \varepsilon_1\varepsilon_2\varepsilon_3$. That this is not the case will follow from Lemma 2.8 and from Lemma 2.19. First we have

LEMMA 2.13. *Let* R *be the subset of* $P_{\underset{\sim}{m}}$ *defined in lemma* 2.8 *and let* $\alpha \in R$. *If* $\alpha^o \in P_{\underset{\sim}{m}}$ *is such that* $(\alpha, \alpha^o) \in \Delta_{\underset{\sim}{m}}$ *then*

$$\text{(i) } |U\alpha \cap U^o\alpha| = \underset{\sim}{m}, \quad \text{(ii) } |U\alpha \setminus U^o\alpha| < \underset{\sim}{m}.$$
$$\text{(iii) } |U^o\alpha \setminus U\alpha| < \underset{\sim}{m},$$

where $U = C(\alpha)$ *and* $U^o = C(\alpha^o)$.

Proof. If $\alpha \in R$ then $|U \setminus \text{ran } \alpha| < \underset{\sim}{m}$, $|V \cap V\alpha| < \underset{\sim}{m}$ and $|U\alpha \cap V| = \underset{\sim}{m}$, where $U = C(\alpha)$ and $V = X \setminus U$. Since $(\alpha, \alpha^o) \in \Delta_{\underset{\sim}{m}}$ we have that $|D\alpha| < \underset{\sim}{m}$ and $|D\alpha^o| < \underset{\sim}{m}$, where $D = D(\alpha, \alpha^o)$. We then have the Venn diagrams:

[See overleaf for diagram.]

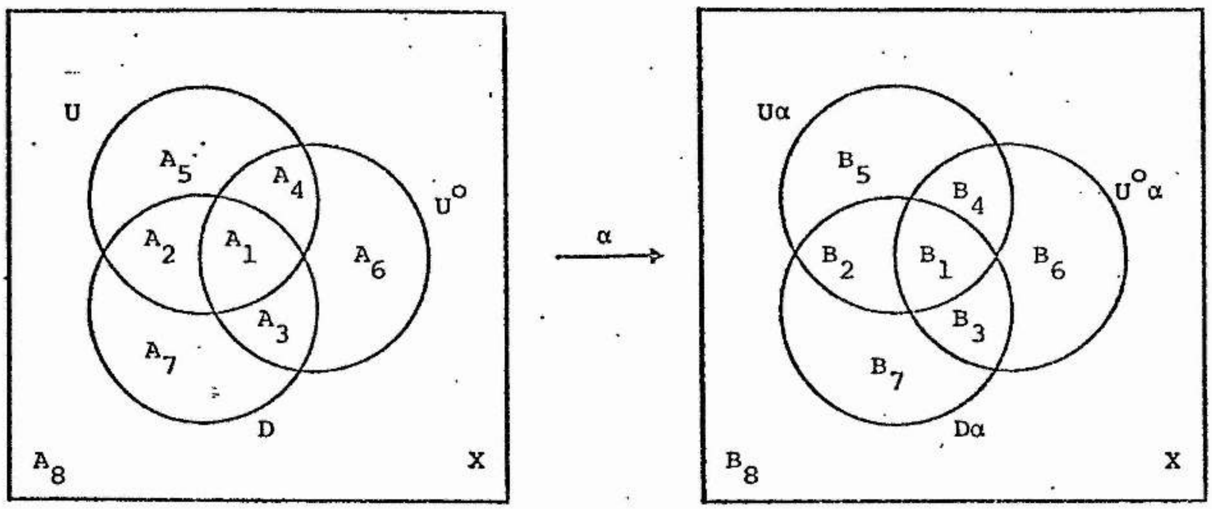


diagram (1)

To prove the lemma we require to investigate the cardinalities of certain of the sets B_i. First, since $| D\alpha | < \underset{\sim}{m}$, it follows that

$$| B_1 | , | B_2 | , | B_3 | , | B_7 | < \underset{\sim}{m} . \qquad (2.14)$$

Turning now to the set

$$B_5 = U\alpha \setminus (U^o\alpha \cup D\alpha) ,$$

we notice that for each x in B_5 there exists u in $U \setminus (U^o \cup D)$ such that $x = u\alpha$. Since $u \notin D$ we have $u\alpha = u\alpha^o$. Also, since $u \in U$, there exists $v \neq u$ in U such that $v\alpha = u\alpha = x$. Now, $v \notin D$, since $v\alpha = x \notin D\alpha$, and hence $v\alpha^o = v\alpha$. It follows that

$$u\alpha^o = u\alpha = v\alpha = v\alpha^o \quad , \quad (u \neq v)$$

and hence $u \in U^o$. This contradicts $u \in U \setminus (U^o \cup D)$ and so we must have

$$B_5 = \emptyset . \qquad (2.15)$$

Since $|U\alpha| = |B_1 \cup B_2 \cup B_4 \cup B_5| = \underset{\sim}{m}$ it now follows by (2.14) and (2.15) that

$$|B_4| = \underset{\sim}{m} \qquad (2.16)$$

and so certainly that

$$|U\alpha \cap U^{o}\alpha| = \underset{\sim}{m}.$$

Also, since $U\alpha \setminus U^{o}\alpha = B_2 \cup B_5$, we have

$$|U\alpha \setminus U^{o}\alpha| < \underset{\sim}{m}.$$

To prove the remaining assertion of the lemma we must consider B_6. Let $x \in B_6 = U^{o}\alpha \setminus (U\alpha \cup D\alpha)$. Then, arguing as for B_5 we see that there must exist u^{o} in $U^{o} \setminus (U \cup D)$ such that $x = u^{o}\alpha$. Since u^{o} is in $U^{o} = C(\alpha^{o})$ there exists $v^{o} \neq u^{o}$ in U^{o} such that $v^{o}\alpha^{o} = u^{o}\alpha^{o}$. Since $u^{o} \notin D$ we must have $u^{o}\alpha = u^{o}\alpha^{o}$. If we had $v^{o} \notin D$ then it would follow that

$$v^{o}\alpha = v^{o}\alpha^{o} = u^{o}\alpha^{o} = u^{o}\alpha$$

and hence that $u^{o} \in C(\alpha) = U$, contrary to hypothesis. Hence $v^{o} \in D$ and so

$$x = u^{o}\alpha = u^{o}\alpha^{o} = v^{o}\alpha^{o} \in D\alpha^{o}.$$

Thus $B_6 \subseteq D\alpha^{o}$ and so, from the assumption in the statement in the lemma that $(\alpha,\alpha^{o}) \in \Delta_{\underset{\sim}{m}}$ it follows that

$$|B_6| < \underset{\sim}{m}. \qquad (2.17)$$

It is now clear that

$$|U^{o}\alpha \setminus U\alpha| = |B_3 \cup B_6| < \underset{\sim}{m},$$

and the lemma is proved.

LEMMA 2.18. *Let* $\alpha \in R$ *(as defined in Lemma* 2.8) *and let* α^o *in* $P_{\underset{\sim}{m}}$ *be such that* $(\alpha, \alpha^o) \in \Delta_{\underset{\sim}{m}}$. *Then* $|A_i| = \underset{\sim}{m}$ $(i = 4, 8)$ *and* $|A_j| < \underset{\sim}{m}$ $(j = 2,3,5,6,7)$ *where* A_i $(i = 1,2,3,4,5,6,7,8)$ *are the subsets of* X *defined in the diagram* (1).

Proof. We show first that $|A_8| = \underset{\sim}{m}$. Since $\operatorname{def} \alpha \subseteq B_8$ and $\alpha \in J_{\underset{\sim}{m}}$ we certainly have that $|B_8| = \underset{\sim}{m}$. It is easy to see that $B_8\alpha^{-1} \subseteq A_8$; hence

$$|A_8| \geqslant |B_8\alpha^{-1}| \geqslant |B_8|$$

and so $|A_8| = \underset{\sim}{m}$.

To show that $|A_4| = \underset{\sim}{m}$ is a little harder. Certainly $|B_4| = \underset{\sim}{m}$ by (2.16) but it is not entirely obvious that $B_4\alpha^{-1} \subseteq A_4$. It is, however, true, and this is what we shall now show. Let

$$x \in B_4\alpha^{-1} = [(U\alpha \cap U^o\alpha) \setminus D\alpha]\alpha^{-1}.$$

Then

$$x\alpha \in (U\alpha \cap U^o\alpha) \setminus D\alpha ,$$

and so $x \notin D$. Also, since $x\alpha \in U\alpha$ there exists $u \in U$ such that $x\alpha = u\alpha$. Hence either $x = u \in U$, or $x \neq u$ in which case $x \in C(\alpha) = U$. Finally, since $x\alpha \in U^o\alpha$ there exists $u^o \in U^o$ such that $x\alpha = u^o\alpha$. As before, either $x = u^o \in U^o$ or $x \neq u^o$, in which case both x and u^o are in U. We have already observed that $x \notin D$. In fact, we also have $u^o \notin D$, since $u^o \in D$ would give $x\alpha = u^o\alpha \in D\alpha$ contrary to hypothesis. Hence,

$$x\alpha^o = x\alpha = u^o\alpha = u^o\alpha^o$$

and so $x \in C(\alpha^o) = U^o$. Thus

$$x \in (U \cap U^o) \setminus D = A_4 ,$$

giving $B_4\alpha^{-1} \subseteq A_4$. It now follows easily that $|A_4| = \underset{\sim}{m}$.

In considering A_j $(j = 2,3,5,6,7)$, notice first that $\alpha|V$ is one-to-one since $V = X \setminus C(\alpha)$. Hence the subset $A_3 \cup A_7 = V \cap D$ maps by α in a one-to-one manner into $D\alpha$. Since $|D\alpha| < \underset{\sim}{m}$ by $((\alpha,\alpha^o) \in \Delta\underset{\sim}{m})$ it follows that $|A_3 \cup A_7| < \underset{\sim}{m}$ and hence that

$$|A_3| < \underset{\sim}{m} \quad , \quad |A_7| < \underset{\sim}{m}.$$

Next, since $\alpha^o | V^o$ is one-to-one, the set $A_2 \cup A_7 = V^o \cap D$ maps by α^o in a one-to-one manner into $D\alpha^o$. Since $|D\alpha^o| < \underset{\sim}{m}$ it thus follows that

$$|A_2| < \underset{\sim}{m}.$$

Consider now the restriction of α to the set $A_6 = U^o \setminus (U \cup D)$. Since

$$A_6 \subseteq V = X \setminus U = X \setminus C(\alpha),$$

the map $\alpha|A_6$ is one-to-one. We now show that its image is contained in $B_6 = U^o\alpha \setminus (U\alpha \cup D\alpha)$. Let $x \in A_6$. Then $x \in U^o$ and so $x\alpha \in U^o\alpha$. On the other hand, if we had $x\alpha \in U\alpha$ then there would exist u in U such that $x\alpha = u\alpha$, and it would then follow either that $x = u \in U$ or that $x \neq u$, in which case $x \in C(\alpha) = U$. In any event $x \in U$, and since our assumption is that $x \in A_6 \subseteq X \setminus U$ we thus have a contradiction. Hence $x\alpha \notin U\alpha$.

Equally, $x\alpha \in D\alpha$ leads to a contradiction, for if $x\alpha = d\alpha$, with d in D, then either $x = d \in D$, which is contrary to assumption, or $x \neq d$ in which case $x \in C(\alpha) = U$, again contrary to assumption. Hence

$$x\alpha \in U^o\alpha \setminus (U\alpha \cup D\alpha) = B_6,$$

and so, by (2.17),

$$|A_6| = |A_6\alpha| \leq |B_6| < \underset{\sim}{m}.$$

Finally, we must consider $A_5 = U \setminus (U^o \cup D)$. If $x \in A_5$ then certainly $x\alpha \in U\alpha$. Also there exists $u \in U$ such that $u \neq x$ and $u\alpha = x\alpha$. In fact $u \in D$, for if $u \notin D$ then

$$x\alpha^{o} = x\alpha = u\alpha = u\alpha^{o},$$

giving $x \in U^{o}$, contrary to assumption. Hence $x\alpha = u\alpha \in D\alpha$, giving

$$A_5\alpha \subseteq U\alpha \cap D\alpha.$$

Moreover, $\alpha | A_5$ is one-to-one, since if $x, y \in A_5$ are such that $x\alpha = y\alpha$ then $x, y \notin D$ and so

$$x\alpha^{o} = x\alpha = y\alpha = y\alpha^{o},$$

which, if $x \neq y$, gives $x, y \in C(\alpha^{o}) = U^{o}$, contrary to assumption. It now follows that

$$| A_5 | = | A_5\alpha | \leq | U\alpha \cap D\alpha | \leq | D\alpha | < \underset{\sim}{m}.$$

Lemma 2.18 is now proved.

We can now prove a lemma which together with Lemma 2.8 will establish that $\Delta(P_{\underset{\sim}{m}}^{*}) > 3$.

LEMMA 2.19. *If* $\alpha \in R$ *(as defined in Lemma 2.8) and* $\alpha^{o} \in P_{\underset{\sim}{m}}$ *is such that* $(\alpha, \alpha^{o}) \in \Delta_{\underset{\sim}{m}}$, *then* $\alpha^{o} \in R$.

Proof. Suppose that α belongs to R so that

$$| U \setminus \operatorname{ran} \alpha | < \underset{\sim}{m}, \quad | V \cap V\alpha | < \underset{\sim}{m} \text{ and } | U\alpha \cap V | = \underset{\sim}{m}, \tag{2.20}$$

where $U = C(\alpha)$ and $V = X \setminus U$. We must show that

$$| U^{o} \setminus \operatorname{ran} \alpha^{o} | < \underset{\sim}{m}, \quad | V^{o} \cap V^{o}\alpha^{o} | \lessgtr \underset{\sim}{m} \text{ and } | U^{o}\alpha^{o} \cap V^{o} | = \underset{\sim}{m}.$$

Using a simplified notation in which A_{ij}, A_{ijk}, etc stand for $A_i \cup A_j$, $A_i \cup A_j \cup A_k$, etc, we can write

$$U \setminus \operatorname{ran} \alpha = A_{1245} \setminus \operatorname{ran} \alpha = (A_{14} \setminus \operatorname{ran} \alpha) \cup (A_{25} \setminus \operatorname{ran} \alpha).$$

Hence

$$| A_{14} \setminus \operatorname{ran} \alpha | < \underset{\sim}{m}. \tag{2.21}$$

Also

$$U^{o} \setminus \operatorname{ran} \alpha^{o} = A_{1346} \setminus \operatorname{ran} \alpha^{o}$$

$$= (A_{14} \setminus \operatorname{ran} \alpha^{o}) \cup (A_{36} \setminus \operatorname{ran} \alpha^{o}). \qquad (2.22)$$

By Lemma 2.18,

$$| A_{36} \setminus \operatorname{ran} \alpha^{o} | \leq | A_{36} | < \underset{\sim}{m}. \qquad (2.23)$$

Also

$$A_{14} \setminus \operatorname{ran} \alpha^{o} = A_{14} \setminus (A_{1237} \alpha^{o} \cup A_{4568} \alpha^{o})$$

$$= A_{14} \setminus (A_{1237} \alpha^{o} \cup A_{4568} \alpha)$$

$$\subseteq A_{14} \setminus A_{4568} \alpha. \qquad (2.24)$$

Now,

$$A_{14} \setminus \operatorname{ran} \alpha = A_{14} \setminus (A_{1237} \alpha \cup A_{4568} \alpha)$$

$$= (A_{14} \setminus A_{1237} \alpha) \cap (A_{14} \setminus A_{4568} \alpha).$$

Since by Lemma 2.18 we have $| A_{14} | = \underset{\sim}{m}$ and since $|A_{1237} \alpha | = | D\alpha | < \underset{\sim}{m}$, we must have

$$| A_{14} \setminus A_{1237} \alpha | = \underset{\sim}{m}.$$

This together with (2.21) implies that

$$| A_{14} \setminus A_{4568} \alpha | < \underset{\sim}{m}.$$

It thus follows from (2.24) that $| A_{14} \setminus \operatorname{ran} \alpha^{o} | < \underset{\sim}{m}$ and hence by (2.22) and (2.23) that

$$| U^{o} \setminus \operatorname{ran} \alpha^{o} | < \underset{\sim}{m}.$$

From the assumption (2.20) that $| V \cap V\alpha | < \underset{\sim}{m}$ and from the observation that $V = A_{3678}$ we deduce that

$$\Big| \bigcup_{i,j=3,6,7,8} A_i \cap A_j\alpha \Big| < \underset{\sim}{m} ,$$

and hence that

$$| A_i \cap A_j\alpha | < \underset{\sim}{m}. \quad (i,j=3,6,7,8) \qquad (2.25)$$

Observe now that

$$\begin{aligned} V^{o} \cap V^{o}\alpha^{o} &= A_{2578} \cap A_{2578}\,\alpha^{o} \\ &= A_{2578} \cap (A_{27}\,\alpha^{o} \cup A_{58}\alpha) \\ &=(A_{2578} \cap A_{27}\,\alpha^{o}) \cup (A_{2578} \cap A_{58}\,\alpha). \end{aligned} \tag{2.26}$$

Now

$$A_{2578} \cap A_{58}\;\alpha = \bigcup_{i=2,5,7,8} [(A_i \cap A_5\,\alpha) \cup (A_i \cap A_8\,\alpha)].$$

Since $| A_5 | < \underset{\sim}{m}$ by Lemma 2.18 we certainly have $| A_i \cap A_5\,\alpha | < \underset{\sim}{m}$ for $i = 2,5,7,8$. By the same lemma we have $| A_i \cap A_8\,\alpha | < \underset{\sim}{m}$ for $i = 2,5,7$ and so the cardinality of $A_{2578} \cap A_{58}\,\alpha$ hangs on the cardinality of $A_8 \cap A_8\,\alpha$. By formula (2.25) this too is less than $\underset{\sim}{m}$, and so we conclude that

$$| A_{2578} \cap A_{58}\,\alpha | < \underset{\sim}{m}$$

We turn now to the other component in the expression (2.26) for $V^{o} \cap V^{o}\alpha^{o}$. This is easier, since

$$A_{2578} \cap A_{27}\,\alpha^{o} \subseteq A_{27}\,\alpha^{o} = A_2\,\alpha^{o} \cup A_7\,\alpha^{o}.$$

Hence by Lemma 2.18,

$$| A_{2578} \cap A_{27}\,\alpha^{o} | \leqslant | A_2\,\alpha^{o} | + | A_7\,\alpha^{o} | < \underset{\sim}{m}.$$

It now follows from (2.26) that

$$| V^{o} \cap V^{o}\,\alpha^{o} | < \underset{\sim}{m}.$$

It remains to show that $| U^{o}\alpha^{o} \cap V^{o} | = \underset{\sim}{m}$. From the assumption (2.20) that $| U\alpha \cap V | = \underset{\sim}{m}$ we deduce that

$$| A_{1245}\alpha \cap A_{3678} | = \underset{\sim}{m}.$$

Now if we express the set $A_{1245}\alpha \cap A_{3678}$ as a union of sixteen sets of the form $A_i\alpha \cap A_j$ it is clear from lemma 2.18 that every $A_i\alpha \cap A_j$ with the exception of $A_4\alpha \cap A_8$ has cardinality less than $\underset{\sim}{m}$. Hence

$$|\, A_4\alpha \cap A_8 \,| = \underset{\sim}{m} \qquad (2.27)$$

It now follows (since $A_4 \subseteq X \setminus D$) that

$$\begin{aligned} A_4\alpha \cap A_8 &= A_4\, \alpha^{o} \cap A_8 \\ &\subseteq A_{1346}\alpha^{o} \cap A_{2578} = U^{o}\alpha^{o} \cap V^{o} \end{aligned}$$

Hence by (2.27)

$$|\, U^{o}\alpha^{o} \cap V^{o} \,| = \underset{\sim}{m}.$$

This completes the proof of lemma 2.19. Hence Theorem 2.11 is proved.

Notice that when $\underset{\sim}{m}$ is a regular cardinal number we have at least two congruence-free and idempotent-generated semigroups of depth 4, namely $S_{\underset{\sim}{m}}^{*}$ [16, Theorem 3.11] and $P_{\underset{\sim}{m}}^{*}$ as defined in Theorem 2.11. Moreover we have

PROPOSITION 2.28. *If $\underset{\sim}{m}$ is a regular cardinal number then $S_{\underset{\sim}{m}}^{*}$ is a subsemigroup of $P_{\underset{\sim}{m}}^{*}$.*

Proof. Let $\underset{\sim}{m}$ be a regular cardinal number. Let us recall that

$$S_{\underset{\sim}{m}} = \{\alpha \in Q_{\underset{\sim}{m}} : |\, y\alpha^{-1} \,| < \underset{\sim}{m} \quad (\forall y \in \operatorname{ran}\alpha)\},$$

and that

$$S_{\underset{\sim}{m}}^{*} = S_{\underset{\sim}{m}} / \Delta_{\underset{\sim}{m}},$$

where

$$\Delta_{\underset{\sim}{m}} = \{(\alpha,\beta) \in S_{\underset{\sim}{m}} \times S_{\underset{\sim}{m}} : \max(\,|\, D(\alpha,\beta)\alpha \,|\,,\ |\, D(\alpha,\beta)\beta \,|\,) < \underset{\sim}{m}\}.$$

Now let $\theta = \Delta_{\underset{\sim}{m}}^{\natural} : S_{\underset{\sim}{m}} \longrightarrow S_{\underset{\sim}{m}}^{*}$ and $\phi = \hat{\Delta}_{\underset{\sim}{m}}^{\natural} : P_{\underset{\sim}{m}} \longrightarrow P_{\underset{\sim}{m}}^{*}$ be epimorphisms and $\mu : S_{\underset{\sim}{m}} \longrightarrow P_{\underset{\sim}{m}}$ be the inclusion monomorphism. We then have the following diagram:

$$\begin{array}{ccc} S_{\underset{\sim}{m}} & \xrightarrow{\ \mu\ } & P_{\underset{\sim}{m}} \\ \theta \downarrow & & \downarrow \phi \\ S_{\underset{\sim}{m}}^{*} = S_{\underset{\sim}{m}}/\Delta_{\underset{\sim}{m}} & & P_{\underset{\sim}{m}}^{*} = P_{\underset{\sim}{m}}/\hat{\Delta}_{\underset{\sim}{m}} \end{array} \qquad \text{diagram (2)}$$

Now define $\Psi : S_{\underset{\sim}{m}}^{*} \longrightarrow P_{\underset{\sim}{m}}^{*}$ as follows: for each a in $S_{\underset{\sim}{m}}^{*}$ choose $b \in S_{\underset{\sim}{m}}$ such that $b\theta = a$. Then define

$$a\Psi = b\mu\phi.$$

Notice that Ψ is well defined for if b, b' in $S_{\underset{\sim}{m}}$ are such that $b \neq b'$ and $b\theta = b'\theta = a$ then $(b, b') \in \Delta_{\underset{\sim}{m}}$ and so

$$(b\mu)\phi = (b'\mu)\phi.$$

Also, Ψ is one-to-one. Suppose a, a' in $S_{\underset{\sim}{m}}^{*}$ are such that $a\Psi = a'\Psi$. Then,

$$a\Psi = (b\mu)\phi = (b'\mu)\phi = a'\Psi ,$$

where $b, b' \in S_{\underset{\sim}{m}}$ and $b\theta = a$ and $b'\theta = a'$. Then $(b\mu, b'\mu) \in \hat{\Delta}_{\underset{\sim}{m}}$ and so $(b, b') \in \Delta_{\underset{\sim}{m}}$. Thus $b\theta = b'\theta$, i.e.

$$a = a'.$$

Finally observe that since Ψ is a composition of two homomorphisms, μ and ϕ, Ψ is itself a homomorphism. Hence Ψ embeds $S_{\underset{\sim}{m}}^{*}$ into $P_{\underset{\sim}{m}}^{*}$ and diagram 2 can now be completed:

$$\begin{array}{ccc} S_{\underset{\sim}{m}} & \overset{\mu}{\longrightarrow} & P_{\underset{\sim}{m}} \\ \theta \downarrow & & \downarrow \phi \\ S_{\underset{\sim}{m}}^{*} & \underset{\Psi}{\longrightarrow} & P_{\underset{\sim}{m}}^{*} \end{array}$$

diagram (3)

It is obvious that this diagram commutes, i.e. that

$$\theta\Psi = \mu\phi.$$

Proposition 2.28 is now proved.

For the case in which $\underset{\sim}{m}$ is not a regular cardinal number the problem of relating $S_{\underset{\sim}{m}}^{*}$ and $P_{\underset{\sim}{m}}^{*}$ does not even arise for then $S_{\underset{\sim}{m}}$ fails to be a semigroup.

In the first part of Chapter 3 particular attention is given to this case and the semigroup generated by $S_{\underset{\sim}{m}}$ will then be described.

CHAPTER 3

FURTHER STUDIES ON THE SEMIGROUP OF THE STABLE ELEMENTS

1. INTRODUCTION

Let $\underset{\sim}{m}$ be an infinite cardinal number. Recall that $\underset{\sim}{m}$ is said to be *regular* if it has the property that $|\Lambda| < \underset{\sim}{m}$ and $|\underset{\sim}{m}_\lambda| < \underset{\sim}{m}$ for all $\lambda \in \Lambda$ together imply that

$$\sum_{\lambda \in \Lambda} \underset{\sim}{m}_\lambda < \underset{\sim}{m}.$$

As already mentioned in Chapter 1 (1.2) the set

$$S_{\underset{\sim}{m}} = \{\alpha \in Q_{\underset{\sim}{m}} : (\forall y \in \operatorname{ran}\alpha)\ |y\alpha^{-1}| < \underset{\sim}{m},\ |\operatorname{ran}\alpha| = \underset{\sim}{m}\} \qquad (3.1)$$

is a subsemigroup of $Q_{\underset{\sim}{m}}$ provided the cardinal $\underset{\sim}{m}$ is regular. In fact, if $\underset{\sim}{m}$ is not regular then there exists a set $\{B_i : i \in I\}$ of disjoint subsets of X such that $|B_i| < \underset{\sim}{m}$ (for all $i \in I$), $|I| < \underset{\sim}{m}$ and $|B| = \underset{\sim}{m}$, where

$$B = \bigcup_{i \in I} B_i.$$

We may also suppose that $|X \setminus B| = \underset{\sim}{m}$. If we define α to map $X \setminus B$ onto B in a one-to-one manner and, for each $i \in I$, to map the elements of B_i onto a single element of B_i then $\alpha \in S_{\underset{\sim}{m}}$. However, since $\operatorname{ran}\alpha^2 = B\alpha$ and $|B\alpha| = |I| < \underset{\sim}{m}$, the element α^2 does not belong to $S_{\underset{\sim}{m}}$ and therefore $S_{\underset{\sim}{m}}$ is not a semigroup.

It seemed to be sufficiently interesting to investigate the subsemigroup of $Q_{\underset{\sim}{m}}$ generated by $S_{\underset{\sim}{m}}$. The main result of the first part of this chapter, Theorem 3.22, describes this semigroup, which we shall denote by $\langle S_{\underset{\sim}{m}} \rangle$.

In the second and last section of this chapter we turn back to the case of a regular cardinal and the study of the congruences on the semigroup $S_{\underset{\sim}{m}}$ is started. Theorem 3.27 describes the lattice of congruences on $S_{\aleph_0}$. So far no results of sufficient completeness have been obtained describing the lattice of congruences on $S_{\underset{\sim}{m}}$ in the case where $\underset{\sim}{m}$ is an arbitrary infinite regular cardinal number.

2. THE SEMIGROUP $\langle S_{\underset{\sim}{m}} \rangle$

Let $\underset{\sim}{m}$ be an infinite non-regular cardinal and let $S_{\underset{\sim}{m}}$ be as defined in (3.1). In this section we shall prove that the set $S_{\underset{\sim}{m}}$ generates the whole semigroup $Q_{\underset{\sim}{m}}$.

Let α belong to $Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$. Four different situations can occur and need to be studied separately. The set

$$Y = \{y \in \operatorname{ran} \alpha : |\, y\alpha^{-1} \,| = \underset{\sim}{m}\} \tag{3.2}$$

is either non-empty or it is empty. In the latter case, since $\alpha \notin S_{\underset{\sim}{m}}$ we must have $|\operatorname{ran} \alpha| < \underset{\sim}{m}$. We shall consider first the case in which $Y \neq \emptyset$ and prove lemmas 3.4, 3.6 and 3.7.

LEMMA 3.3 *Let* ε *be an idempotent in* $\mathscr{T}(X)$, *where* $|\, X \,| = \underset{\sim}{m} \geq \aleph_0$. *Suppose that either*

$$|\, C(\varepsilon) \,| = \underset{\sim}{m} \quad \textit{or} \quad |\operatorname{def} \varepsilon| = \underset{\sim}{m}.$$

Then $\varepsilon \in Q_{\underset{\sim}{m}}$.

Proof. Let ε be an idempotent in $\mathscr{T}(X)$. If $\varepsilon \neq 1_X$ then

$$\varepsilon \in F \bigcup_{\aleph_0 \leq \underset{\sim}{n} \leq \underset{\sim}{m}} Q_{\underset{\sim}{n}},$$

(see chapter 1). If $\varepsilon \in F$ then $S(\varepsilon)$ is finite and so are $\operatorname{def} \varepsilon$ and $C(\varepsilon)$. Hence $\varepsilon \notin F$ and so

$$|\, C(\varepsilon) \,| = \underset{\sim}{m} \;\Rightarrow\; \varepsilon \in Q_{\underset{\sim}{m}},$$
$$|\operatorname{def} \varepsilon| = \underset{\sim}{m} \;\Rightarrow\; \varepsilon \in Q_{\underset{\sim}{m}}.$$

LEMMA 3.4 *Let* $\alpha \in Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$ *and let* $Y \neq \phi$ *be the set defined in (3.2). If* $|\, X \setminus \bigcup_{y \in Y} y\alpha^{-1} \,| = \underset{\sim}{m}$ *then there exist* ε *in* $E(S_{\underset{\sim}{m}})$ *(the set of idempotents of* $S_{\underset{\sim}{m}}$*) and* θ *in* $S_{\underset{\sim}{m}}$ *such that* $\alpha = \varepsilon\theta$.

Proof. Since $\underset{\sim}{m}$ is a non-regular cardinal, there exists a set $\{B_i : i \in I\}$ of disjoint subsets of X such that $|I| < \underset{\sim}{m}$, $|B_i| < \underset{\sim}{m}$ (for all $i \in I$) and $|B| = \underset{\sim}{m}$, where

$$B = \bigcup_{i \in I} B_i$$

We may assume, without loss of generality, that for all $i \in I$,

$$\underset{\sim}{2} \leqslant |B_i| < \underset{\sim}{m}.$$

For each $y \in Y$ let $f_y : B \longrightarrow y\alpha^{-1}$ be a bijection and write

$$B_i f_y = C_i^{(y)}. \tag{3.5}$$

Then for each $y \in Y$ and $i \in I$

$$\underset{\sim}{2} \leqslant |C_i^{(y)}| = |B_i| < \underset{\sim}{m}.$$

Moreover, the subsets $C_i^{(y)}$ $(i \in I,\ y \in Y)$ form a partition of $\bigcup_{y \in Y} y\alpha^{-1}$. If we denote by ρ the associated equivalence on X (all other ρ-classes being singleton) it is easy to see that the union of the non-singleton ρ-classes is given by

$$k(\rho) = \bigcup_{\substack{i \in I\\ y \in Y}} C_i^{(y)} = \bigcup_{y \in Y} y\alpha^{-1},$$

and so $|k(\rho)| = \underset{\sim}{m}$. Hence $\rho \in \mathscr{C}$, where

$$\mathscr{C} = \{\tau \in \mathscr{E}(X) : |k(\tau)| = \underset{\sim}{m},\ |x\tau| < m\ (\forall\, x \in X)\}$$

We can now define ε. For each $i \in I$ and for all $y \in Y$, choose $C_i^{(y)}\varepsilon$ in $C_i^{(y)}$. For x in $(X \setminus \bigcup_{y \in Y} y\alpha^{-1})$ write $x\varepsilon = x$.

It is obvious, on one hand, that this defines an idempotent ε of $\mathscr{T}(X)$ for which the non singleton (ker ε)-classes are the sets $C_i^{(y)}$ $(y \in Y,\ i \in I)$. The sets $\{x\}$ where $x \in X \setminus \bigcup_{y \in Y} y\alpha^{-1}$ are the singleton (ker ε)-classes. Hence it follows that

$$\ker \varepsilon = \rho \in \mathscr{C},$$

and that $\ker \varepsilon \subseteq \ker \alpha$; for if $a, b \in X$ are such that $a \neq b$ and $a\varepsilon = b\varepsilon$ then both a and b are in the same $(\ker \varepsilon)$-class, say $a, b \in C_i^{(y)}$ for some $i \in I$ and $y \in Y$. Since from (3.5)

$$C_i^{(y)} = B_i f_y \subseteq y\alpha^{-1},$$

we then have $a, b \in y\alpha^{-1}$, i.e., $a\alpha = b\alpha = y$. Also, since $(X \setminus \bigcup_{y \in Y} y\alpha^{-1}) \subseteq \operatorname{ran} \varepsilon$ and $| X \setminus \bigcup_{y \in Y} y\alpha^{-1} | = \underset{\sim}{m}$ it follows that $| \operatorname{ran} \varepsilon | = \underset{\sim}{m}$. Hence it follows from Lemma 3.3 that $\varepsilon \in S_{\underset{\sim}{m}}$.

We are now required to find θ in $S_{\underset{\sim}{m}}$ such that $\alpha = \varepsilon\theta$.

Since $\varepsilon \in S_{\underset{\sim}{m}}$ we have that $| \operatorname{def} \varepsilon | = \underset{\sim}{m}$ and so $\operatorname{def} \varepsilon$ can be partitioned into disjoint subsets of X, say X_r $(r \in R)$, such that $| X_r | = \underset{\sim}{2}$ for each $r \in R$ and $| R | = \underset{\sim}{m}$. Also $\alpha \in Q_{\underset{\sim}{m}}$ gives $| \operatorname{def} \alpha | = \underset{\sim}{m}$. Let

$$\operatorname{def} \alpha = U \cup V,$$

where $U \cap V = \emptyset$, $| U | = | V | = \underset{\sim}{m}$, and let $\xi : R \longrightarrow V$ be a bijection. Now define θ in $\mathscr{T}(X)$ by

$$x\theta = (x\varepsilon^{-1})\alpha \qquad \text{if } x \in \operatorname{ran} \varepsilon,$$

$$X_r\theta = r\xi \qquad \text{for } r \in R.$$

This gives a well defined mapping θ for if $x\varepsilon^{-1} = x'\varepsilon^{-1}$ then since $\ker \varepsilon \subseteq \ker \alpha$ it follows that

$$(x\varepsilon^{-1})\alpha = (x'\varepsilon^{-1})\alpha,$$

and so $x\theta = x'\theta$. It is also obvious that $\alpha = \varepsilon\theta$. We next show that $\theta \in S_{\underset{\sim}{m}}$.

Since $V \subseteq \operatorname{ran} \theta$ and $U \subseteq \operatorname{def} \theta$ it is clear that

$$| \operatorname{ran} \theta | = | \operatorname{def} \theta | = \underset{\sim}{m}.$$

It is not so easy to show that $| x\theta^{-1} | < \underset{\sim}{m}$, for all $x \in \operatorname{ran} \theta$. It is however true. First it is obvious that

$$| v\theta^{-1} | = \underset{\sim}{2} < \underset{\sim}{m},$$

for all $v \in V = \operatorname{ran} \theta \setminus \operatorname{ran} \alpha$. Now take $a \in \operatorname{ran} \alpha$. Then either $a = y \in Y$

(where Y is as defined in (3.2)) or $|a\alpha^{-1}| < \underset{\sim}{m}$. In the first case, it is not hard to see that the mapping $g: i \longmapsto C_i^{(y)}\varepsilon$ $(i \in I)$ gives a bijection from I onto $y\theta^{-1}$; for it follows from the definition of θ that

$$(C_i^{(y)}\ \varepsilon)\ \theta = [\ (C_i^{(y)}\ \varepsilon)\ \varepsilon^{-1}]\ \alpha = C_i^{(y)}\ \alpha = y,$$

and so g maps I into $y\theta^{-1}$. Also, if $C_i^{(y)}\varepsilon = C_j^{(y)}\varepsilon$ for i, j in I then, since $C_i^{(y)}$ and $C_j^{(y)}$ are non-singleton (ker ε)-classes we have

$$C_i^{(y)} = C_j^{(y)}$$

and so (3.5) gives $i = j$. Thus g is one-to-one. Notice finally that if $x \in X$ is such that $x\theta = y$ then since $y \in$ ran α it follows that $x \in$ ran ε, i.e.,

$$x\theta = (x\varepsilon^{-1})\ \alpha = y,$$

and so $x\varepsilon^{-1} \subseteq y\alpha^{-1}$. Hence $x\varepsilon^{-1} = C_{i_0}^{(y)}$ for some i_0 in I, i.e.,

$$x = C_{i_0}^{(y)}\ \varepsilon = g(i_0).$$

Thus $g : I \longrightarrow y\theta^{-1}$ is a bijection and so

$$|\ y\theta^{-1}\ | = |\ I\ | < \underset{\sim}{m}.$$

In the second case, that is, in the case in which $a \in$ ran $\alpha \cap$ ran θ is such that

$$|\ a\alpha^{-1}\ | < \underset{\sim}{m},$$

observe that $(a\theta^{-1})\ \varepsilon^{-1} = a\alpha^{-1}$ and hence $|\ a\theta^{-1}\ | \leqslant |a\alpha^{-1}| < \underset{\sim}{m}$. Hence $\theta \in S_{\underset{\sim}{m}}$.

LEMMA 3.6. *Let* $\alpha \in Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$ *and let* Y *be the set defined in* (3.2). *If* $|\ X \setminus \bigcup_{y \in Y} y\alpha^{-1}\ | < \underset{\sim}{m}$ *and* $|\ Y\ | = \underset{\sim}{m}$ *then* $\alpha = \varepsilon\theta$, *where* $\varepsilon \in E(S_{\underset{\sim}{m}})$ and $\theta \in S_{\underset{\sim}{m}}$.

Proof. In order to find ε in $E(S_{\underset{\sim}{m}})$ and θ in $S_{\underset{\sim}{m}}$ such that $\alpha = \varepsilon\theta$ we proceed exactly as in the proof of Lemma 3.4. Having defined ε, we then find that the argument given in that proof to show that $|$ ran $\varepsilon\ | = \underset{\sim}{m}$ fails since we now have

$$|\ X \setminus \bigcup_{y \in Y} y\alpha^{-1}\ | < \underset{\sim}{m}.$$

But we do have that $|\operatorname{ran}\varepsilon| = \underset{\sim}{m}$, for

$$|(\bigcup_{y \in Y} y\alpha^{-1})\varepsilon| = |\{c_i^{(y)} : i \in I, y \in Y\}|$$

$$= |I| . |Y|,$$

and since by hypothesis $|Y| = \underset{\sim}{m}$ it follows that $|I| . |Y| = \underset{\sim}{m}$, giving

$$|(\bigcup_{y \in Y} y\alpha^{-1})\varepsilon| = \underset{\sim}{m};$$

hence certainly $|\operatorname{ran}\varepsilon| = \underset{\sim}{m}$.

The mapping θ is defined as in the proof of Lemma 3.4. This completes the proof of Lemma 3.6.

Notice that Lemma 3.6 does not necessarily hold if $|Y| < \underset{\sim}{m}$. Consider, for instance, the following example. Let α be the constant map

$$X\alpha = x_o,$$

for some $x_o \in X$. Then clearly $\alpha \in Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$. Also, if Y is as defined in (3.2), we have

$$|X \setminus \bigcup_{y \in Y} y\alpha^{-1}| = \underset{\sim}{0} < \underset{\sim}{m},$$

and

$$|Y| = \underset{\sim}{1} < \underset{\sim}{m}.$$

Suppose now that $\alpha = \varepsilon\theta$ where $\theta \in S_{\underset{\sim}{m}}$. Since $|\operatorname{ran}\alpha| = \underset{\sim}{1}$ it follows that ran ε must be contained in a single (ker θ)-class and so, since $\theta \in S_{\underset{\sim}{m}}$ we must have

$$|\operatorname{ran}\varepsilon| < \underset{\sim}{m}.$$

Hence $\varepsilon \notin S_{\underset{\sim}{m}}$. Thus Lemma 3.6 is not satisfied for the case in which $|Y| < \underset{\sim}{m}$. We do however have a similar result for these elements.

LEMMA 3.7 *If α in $\underset{\sim}{Q}_m \setminus \underset{\sim}{S}_m$ is such that $\underset{\sim}{0} < |Y| < \underset{\sim}{m}$ and $|X \setminus \bigcup_{y \in Y} y\alpha^{-1}| < \underset{\sim}{m}$, where the set* Y *is as defined in* (3.2), *then $\alpha = \varepsilon_1\varepsilon_2\theta$ where $\varepsilon_1, \varepsilon_2$ are idempotents in $\underset{\sim}{S}_m$ and $\theta \in \underset{\sim}{S}_m$.*

Proof. Let $\alpha \in \underset{\sim}{Q}_m \setminus \underset{\sim}{S}_m$ and let

$$Y = \{y \in \text{ran } \alpha : |y\alpha^{-1}| = \underset{\sim}{m}\}.$$

Suppose that $\underset{\sim}{0} < |Y| < \underset{\sim}{m}$ and that

$$|X \setminus \bigcup_{y \in Y} y\alpha^{-1}| < \underset{\sim}{m}.$$

For each $y \in Y$ write

$$y\alpha^{-1} = \bigcup_{i \in I} P_i^{(y)} \tag{3.8}$$

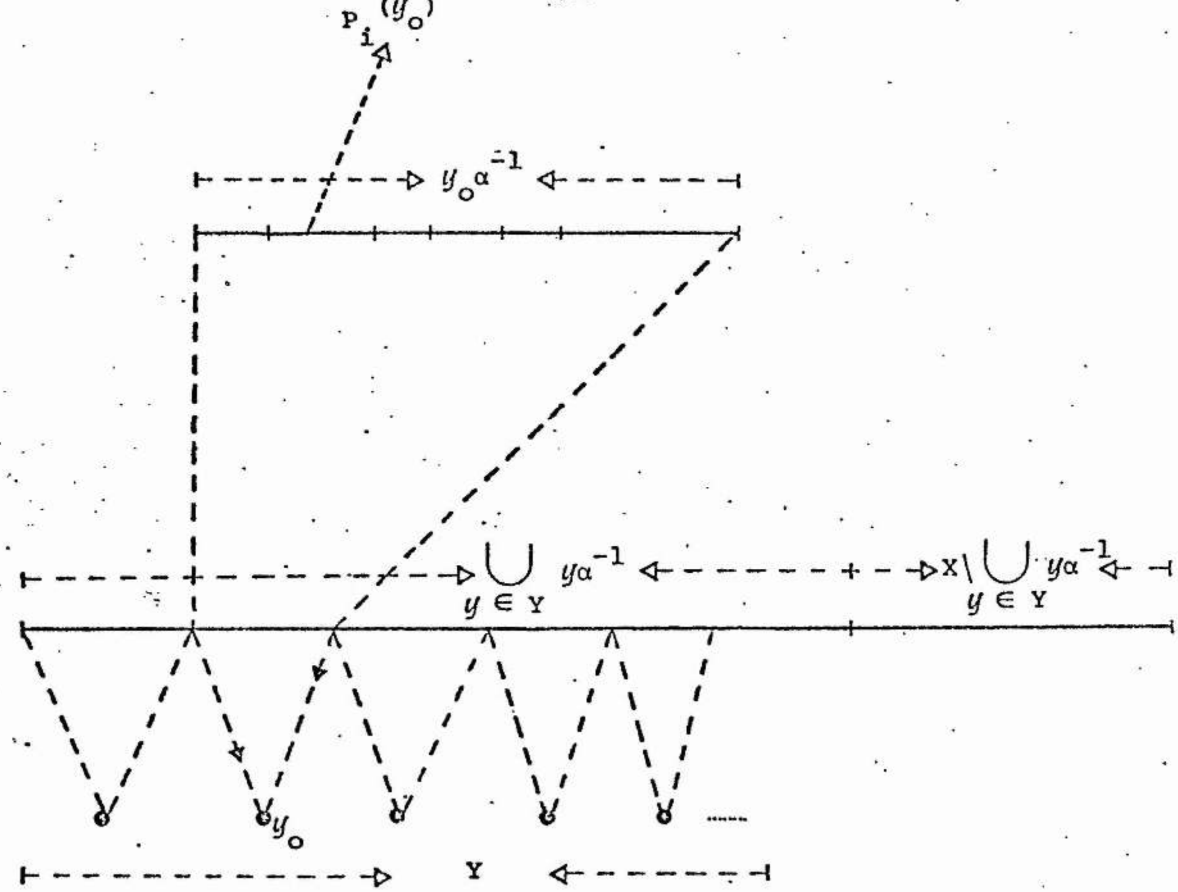
where $|I| = \underset{\sim}{m}$, $\underset{\sim}{2} \leq |P_i^{(y)}| < \underset{\sim}{m}$ (for all $i \in I$) and $P_i^{(y)} \cap P_{i'}^{(y)} = \emptyset$ if $i \neq i'$. It is then clear that the sets $P_i^{(y)}$ ($i \in I$, $y \in Y$) form a partition of $\bigcup_{y \in Y} y\alpha^{-1}$. If we denote by ρ the associated equivalence on X all the other ρ-classes being singleton, then $k(\rho)$ (the union of all the non-singleton ρ-classes) is such that

$$|k(\rho)| = |\bigcup_{\substack{i \in I \\ y \in Y}} P_i^{(y)}| = |\bigcup_{y \in Y} y\alpha^{-1}| = \underset{\sim}{m}.$$

Also, each ρ-class $x\rho$ is either singleton or has the same cardinality as $P_i^{(y)}$ for some $i \in I$ and $y \in Y$. Hence $\underset{\sim}{1} \leq |x\rho| < \underset{\sim}{m}$, for all $x \in X$.

We can now define ε_1. We have the following diagram:

[See overleaf for diagram.]



For each $i \in I$ and $y \in Y$ choose $P_i^{(y)}\varepsilon_1$ in $P_i^{(y)}$; if $x \notin \bigcup_{y \in Y} y\alpha^{-1}$ define $x\varepsilon_1 = x$. Hence ε_1 is clearly an idempotent such that $\ker \varepsilon_1 = \rho$, and so $|C(\varepsilon_1)| = \underset{\sim}{m}$ and $|x\varepsilon_1^{-1}| < \underset{\sim}{m}$ for all $x \in \operatorname{ran} \varepsilon_1$. Also, since the sets $P_i^{(y)}$ $(i \in I,\ y \in Y)$ form all the non-singleton $(\ker \varepsilon_1)$-classes, it follows from (3.8) that

$$\ker \varepsilon_1 \subseteq \ker \alpha.$$

Finally, it is obvious that

$$\begin{aligned} |\operatorname{ran} \varepsilon_1| &\geq |\{P_i^{(y)}\varepsilon_1 : i \in I,\ y \in Y\}| \\ &= |I| \cdot |Y| = \underset{\sim}{m} \cdot |Y| = \underset{\sim}{m}, \end{aligned}$$

giving $|\operatorname{ran} \varepsilon_1| = \underset{\sim}{m}$. Hence $\varepsilon_1 \in E(S_{\underset{\sim}{m}})$.

Now, since $\underset{\sim}{m}$ is a non-regular cardinal number, we can find a set $\{B_k : k \in K\}$ of disjoint subsets of X such that $\underset{\sim}{2} \leq |B_k| < \underset{\sim}{m}$, for all $k \in K$, $|K| < \underset{\sim}{m}$ and $|B| = \underset{\sim}{m}$, where

$$B = \bigcup_{k \in K} B_k .$$

For each $y \in Y$, let

$$p_i^{(y)} = P_i^{(y)} \varepsilon_1 \; ; \qquad (3.9)$$

then for each $y \in Y$ we have

$$|\{p_i^{(y)} : i \in I\}| = |I| = |B| = \underset{\sim}{m}.$$

Let $f_y : B \longrightarrow \{p_i^{(y)} : i \in I\}$ be a bijection and define

$$C_k^{(y)} = \{bf_y : b \in B_k\}.$$

Notice that for each $y \in Y$ we have that

$$\bigcup_{k \in K} C_k^{(y)} = \{p_i^{(y)} : i \in I\} ,$$

and so

$$\left| \bigcup_{k \in K} C_k^{(y)} \right| = |I| = \underset{\sim}{m}.$$

Since f_y is a bijection for all y in Y the sets $C_k^{(y)}$, $(y \in Y, k \in K)$ form a partition of $\bigcup_{y \in Y} \{p_i^{(y)} : i \in I\}$. Also,

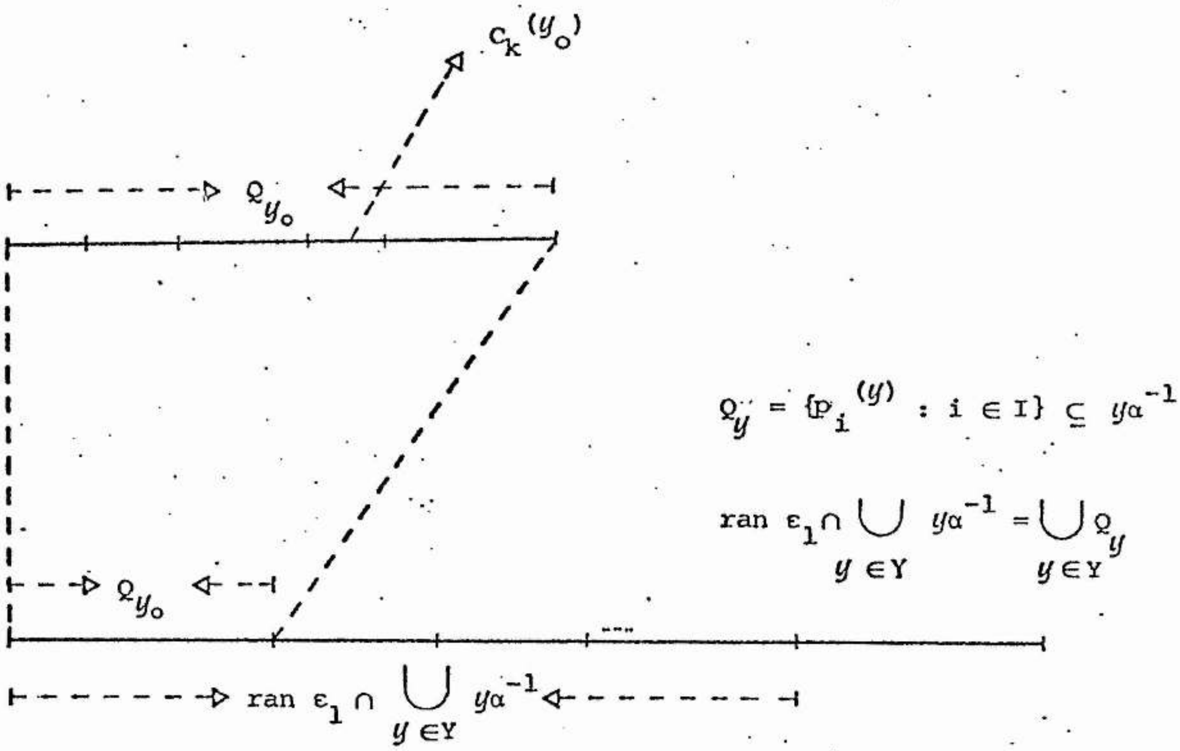
$$\underset{\sim}{2} \leqslant |C_k^{(y)}| = |B_k| < \underset{\sim}{m}, \qquad (3.10)$$

(for $y \in Y$ and $k \in K$) and clearly

$$\left| \bigcup_{\substack{y \in Y \\ k \in K}} C_k^{(y)} \right| = \underset{\sim}{m}. \qquad (3.11)$$

We now have the following diagram:

[See overleaf for diagram.]



Now $\varepsilon_1 \in S_{\underset{\sim}{m}}$ and so $|\text{def } \varepsilon_1| = \underset{\sim}{m}$. Let

$$\text{def } \varepsilon_1 = \bigcup_{j \in J} R_j,$$

where $|R_j| = \underset{\sim}{2}$ for all $j \in J$, $|J| = \underset{\sim}{m}$ and $R_i \cap R_j = \emptyset$, for all $i, j \in J$ and $i \neq j$. We define ε_2 as follows. For each $y \in Y$ and $k \in K$ choose $C_k^{(y)}\varepsilon_2$ in $C_k^{(y)}$ and for each $j \in J$ choose $R_j\varepsilon_2$ in R_j. This defines ε_2 for

$$\text{def } \varepsilon_1 \bigcup [(\bigcup_{y \in Y} y\alpha^{-1})\, \varepsilon_1] .$$

If $x \in (X \setminus \bigcup_{y \in Y} y\alpha^{-1})\, \varepsilon_1$ define $x\varepsilon_2 = x$. Clearly ε_2 is an idempotent for which the non-singleton $(\ker \varepsilon_2)$-classes are the sets $R_j (j \in J)$ and $C_k^{(y)}$ $(y \in Y, k \in K)$. Hence it follows from (3.10) and (3.11) that $|C(\varepsilon_2)| = \underset{\sim}{m}$ and so, by Lemma 3.3 we have that $\varepsilon_2 \in Q_{\underset{\sim}{m}}$. Also

$$\underset{\sim}{2} \leqslant |x\varepsilon_2^{-1}| < \underset{\sim}{m},$$

for all $x \in \text{ran } \varepsilon_2$. Finally, we have

$$|[(\text{def } \varepsilon_1)\, \varepsilon_2]| = |J| = \underset{\sim}{m},$$

giving $|\text{ ran } \varepsilon_2 \,| = \underset{\sim}{m}$. Hence $\varepsilon_2 \in \underset{\sim}{S_m}$.

We are now required to find θ in $\underset{\sim}{S_m}$ such that $\alpha = \varepsilon_1\varepsilon_2\theta$.

Notice first that ε_1 and ε_2 are both elements of the semigroup $\underset{\sim}{Q_m}$ and so, $\varepsilon_1\varepsilon_2 \in \underset{\sim}{Q_m}$ too. Then,

$$|\text{ def } (\varepsilon_1\varepsilon_2)\,| = \underset{\sim}{m}.$$

Let

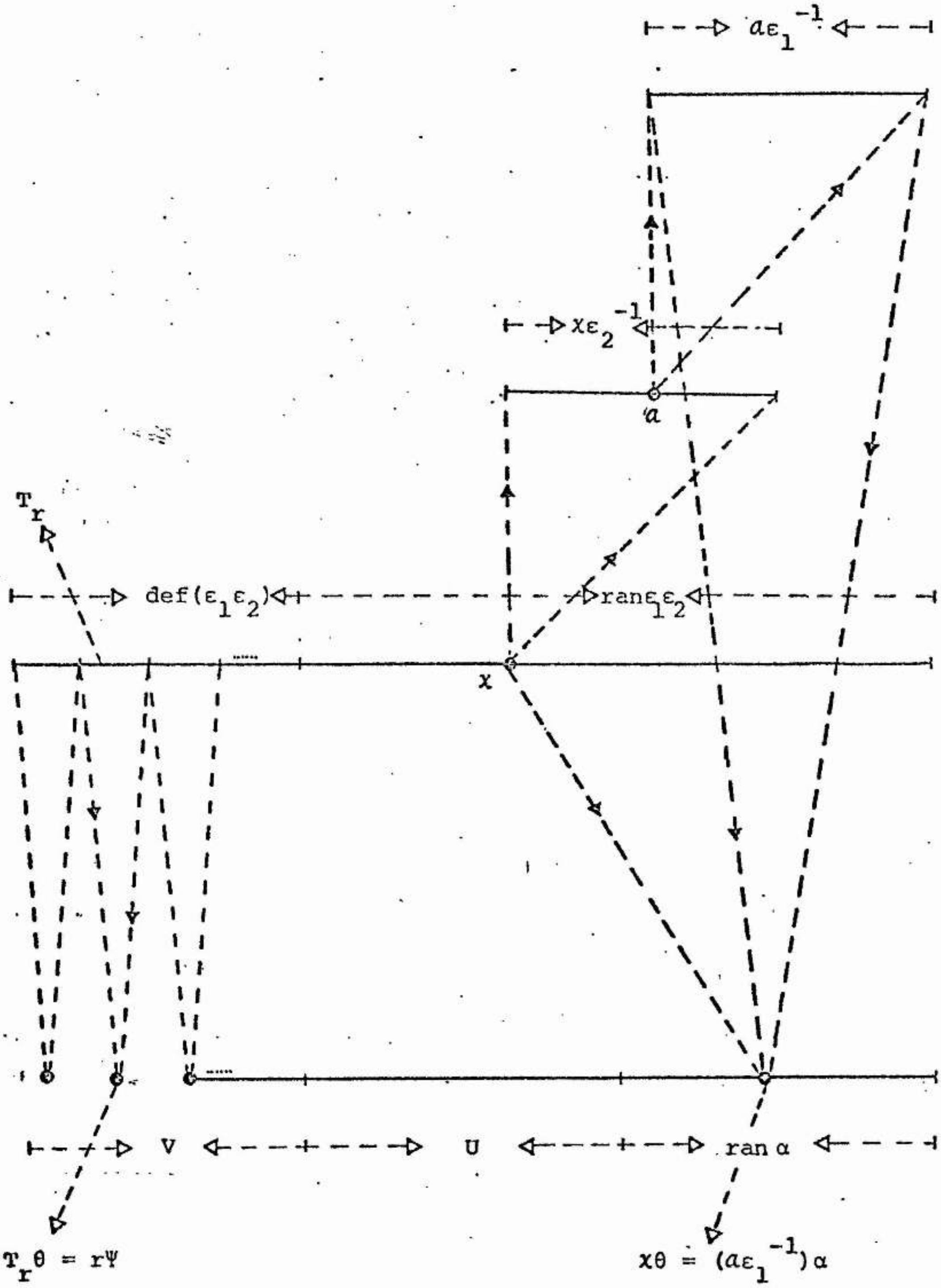
$$\text{def } (\varepsilon_1\varepsilon_2) = \bigcup_{r \in R} T_r,$$

where $|\,T_r\,| = \underset{\sim}{2}$ for all $r \in R$, and $|\,R\,| = \underset{\sim}{m}$. Consider also a partition of def α into disjoint subsetsU, V such that $|\,U\,| = |\,V\,| = \underset{\sim}{m}$, and let $\psi : R \longrightarrow V$ be a bijection. We define the required map θ as follows. If $x \in$ ran $(\varepsilon_1\varepsilon_2)$ then choose arbitrarily an element a in $x\varepsilon_2^{-1} \cap \text{ran } \varepsilon_1$ and write

$$x\theta = (a\varepsilon_1^{-1})\alpha.$$

Otherwise, write $T_r\theta = r\psi$ $(r \in R)$. We shall first show that θ is well defined.

[See overleaf for diagram.]



Observe first that if $x \in$ ran $(\varepsilon_1\varepsilon_2)$ then either $|\, x\varepsilon_2^{-1} \,| \geq 2$ or $|\, x\varepsilon_2^{-1} \,| = 1$. In the first case we have

$$x\varepsilon_2^{-1} = C_k^{(y)} = B_k\ f_y \subseteq \{p_i^{(y)} : i \in I\} ,$$

for some $y \in Y$ and $k \in K$. Then, since by (3.9)

$$p_i^{(y)} = P_i^{(y)}\varepsilon_1, \qquad (i \in I)$$

where $P_i^{(y)}$ $(i \in I)$ are non-singleton (ker ε_1)-classes, it follows that for all $a, b \in x\varepsilon_2^{-1} \cap \text{ran}\ \varepsilon_1$

$$(a\varepsilon_1^{-1})\ \alpha = P_i^{(y)}\ \alpha = y = P_j^{(y)}\ \alpha = (b\varepsilon_1^{-1})\ \alpha,$$

where $i, j \in I$.

If $|\ x\varepsilon_2^{-1}\ | = \underset{\sim}{1}$ it is easier, for the definition of ε_2 will then give

$$x\varepsilon_2^{-1} = \{x\},$$

where $x \in (X \setminus \bigcup_{y \in Y} y\alpha^{-1})\ \varepsilon_1$, that is, $|\ x\varepsilon_1^{-1}\ | = \underset{\sim}{1}$ and so the element a is uniquely determined. Hence θ is well defined. Also it is clear that $\alpha = \varepsilon_1\varepsilon_2\theta$.

It remains to show that $\theta \in S_{\underset{\sim}{m}}$.

Since $V \subseteq \text{ran}\ \theta$ and $U \subseteq \text{def}\ \theta$, it is obvious that

$$|\ \text{ran}\ \theta\ | = |\ \text{def}\ \theta\ | = \underset{\sim}{m}.$$

Also , $\bigcup_{r \in R} T_r \subseteq C(\theta)$ and so $|\ C(\theta)\ | = \underset{\sim}{m}$. Finally, we must show that

$$|\ x\theta^{-1}\ | < \underset{\sim}{m}, \tag{3.12}$$

for all $x \in \text{ran}\ \theta$. It is clear that for all $v \in V$

$$|\ v\theta^{-1}\ | = \underset{\sim}{2} < \underset{\sim}{m}.$$

It is not difficult either to see that (3.12) holds for the elements $a \in \text{ran}\theta \setminus V$. For if $z \in a\theta^{-1}$ and $a \notin V$ the definition of θ implies that $z \in (\text{ran}\ \varepsilon_1)\ \varepsilon_2$. Then either $|\ z\varepsilon_2^{-1}\ | = \underset{\sim}{1}$, in which case it follows from the definition of ε_2 that

$$z \in (X \setminus \bigcup_{y \in Y} y\alpha^{-1}), \tag{3.13}$$

or $|\ z\varepsilon_2^{-1}\ | \geq \underset{\sim}{2}$, which together with the fact that $z \in \text{ran}\ (\varepsilon_1\varepsilon_2)$ gives $z\ \varepsilon_2^{-1} = C_k^{(y)}$, for some k in K and y in Y. Thus

$$z = C_k^{(y)}\varepsilon_2. \tag{3.14}$$

Hence it follows from (3.13) and (3.14) that

$$a\theta^{-1} \subseteq (X \setminus \bigcup_{y \in Y} y\alpha^{-1}) \cup (\bigcup_{\substack{k \in K \\ y \in Y}} C_k^{(y)}\varepsilon_2). \qquad (3.15)$$

Since $|Y| < \underset{\sim}{m}$, $|K| < \underset{\sim}{m}$ and since all the sets $C_k^{(y)}\varepsilon_2$ are singleton, we have that

$$\left| \bigcup_{\substack{k \in K \\ y \in Y}} C_k^{(y)}\varepsilon_2 \right| = \left| \{C_k^{(y)}\varepsilon_2 : y \in Y,\ k \in K\} \right|$$

$$= |Y| \cdot |K| < \underset{\sim}{m}.$$

Also by hypothesis

$$\left| X \setminus \bigcup_{y \in Y} y\alpha^{-1} \right| < \underset{\sim}{m} ,$$

and so (3.15) gives

$$|a\theta^{-1}| < \underset{\sim}{m},$$

as required. Hence $\theta \in S_{\underset{\sim}{m}}$. This completes the proof of Lemma 3.7.

Lemmas 3.4, 3.6 and 3.7 give us all the information about elements α inside $Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$ for which the set

$$Y = \{y \in \text{ran } \alpha : |y\alpha^{-1}| = \underset{\sim}{m}\} ,$$

as defined before in (3.2), is non-empty. Finally, we must investigate what happens with the elements in $Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$ for which $Y = \emptyset$. For that we need two preliminary lemmas.

LEMMA 3.16. *Let* $\alpha \in Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$ *and let* Y *be the set defined in* (3.2). *Suppose that* $Y = \emptyset$. *Then the set*

$$D = \{y \in \text{ran } \alpha : |y\alpha^{-1}| \geq \aleph_0 \}$$

is not finite.

Proof. Let $\alpha \in Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$ and suppose that

$$Y = \{y \in \text{ran } \alpha : |y\alpha^{-1}| = \underset{\sim}{m} \} = \emptyset.$$

Then we must have that $|\operatorname{ran} \alpha| < \underset{\sim}{m}$. Now let

$$D = \{y \in \operatorname{ran} \alpha : \aleph_0 \leq |y\alpha^{-1}|\},$$

and notice that

$$X = \bigcup_{y \in \operatorname{ran} \alpha} y\alpha^{-1} = \Big(\bigcup_{y \in D} y\alpha^{-1}\Big) \cup \Big(\bigcup_{y \in \operatorname{ran}\alpha \setminus D} y\alpha^{-1}\Big).$$

If D were finite then we would have

$$\Big|\bigcup_{y \in D} y\alpha^{-1}\Big| < \underset{\sim}{m},$$

since $|y\alpha^{-1}| < \underset{\sim}{m}$ for all $y \in \operatorname{ran} \alpha$. Hence

$$\Big|\bigcup_{y \in \operatorname{ran}\alpha \setminus D} y\alpha^{-1}\Big| = \underset{\sim}{m},$$

which is not possible since each $y\alpha^{-1}$ is finite and since $|\operatorname{ran}\alpha \setminus D| \leq |\operatorname{ran} \alpha| < \underset{\sim}{m}$. Thus D is not finite. Moreover,

$$\Big|\bigcup_{y \in D} y\alpha^{-1}\Big| = \underset{\sim}{m}.$$

Let $\mathscr{E}(X)$ be the lattice of equivalences on X. If $\rho \in \mathscr{E}(X)$, denote by $k(\rho)$ the union of all the non-singleton ρ-classes. We have

LEMMA 3.17. *Let* $\alpha \in Q_{\underset{\sim}{m}} \setminus S_{\underset{\sim}{m}}$ *and let* Y *be the set defined in* (3.2). *Suppose that* $Y = \emptyset$. *Then there exists* $\rho \in \mathscr{E}(X)$ *such that*

(i) $\rho \subset \ker \alpha$ (ii) $|k(\rho)| = \underset{\sim}{m}$

(iii) $|\{x\rho : x \in X\}| = \underset{\sim}{m}$.

Proof. We have from lemma 3.16 that the set

$$D = \{y \in \operatorname{ran} \alpha : |y\alpha^{-1}| \geq \aleph_0\}$$

is such that $\aleph_0 \leq |D| < \underset{\sim}{m}$. Also,

$$\Big|\bigcup_{y \in D} y\alpha^{-1}\Big| = \underset{\sim}{m}. \tag{3.18}$$

Now, for each $y \in D$, consider the partition

$$y\alpha^{-1} = U_y \cup V_y, \tag{3.19}$$

where $|U_y| = |V_y| = |y\alpha^{-1}|$. Then it is obvious from (3.18) that

$$|\bigcup_{y \in D} U_y| = |\bigcup_{y \in D} V_y| = \underset{\sim}{m}. \qquad (3.20)$$

We next define ρ to be an element of $\mathscr{E}(X)$ whose non-singleton ρ-classes are the sets

$$U_y \qquad (y \in D),$$

the singleton ρ-classes being the sets $\{x\}$, where

$$x \in [\bigcup_{y \in D} V_y \cup (X \setminus \bigcup_{y \in D} y\alpha^{-1})].$$

Hence it is obvious from (3.19) that $\rho \subset \ker \alpha$. Also conditions (ii) and (iii) follow now directly from (3.20) since

$$k(\rho) = \bigcup_{y \in D} U_y$$

and

$$\{x\rho : x \in X\} \supseteq \bigcup_{y \in D} V_y.$$

The lemma is now proved.

LEMMA 3.21. *Let* $\alpha \in \underset{\sim}{Q}_m \setminus \underset{\sim}{S}_m$ *be such that the set* Y *as defined in* (3.2) *is empty. Then there exist an idempotent* ε *in* $\underset{\sim}{S}_m$ *and an element* θ *in* $\underset{\sim}{S}_m$ *such that* $\alpha = \varepsilon.\theta$.

Proof. Let $\alpha \in \underset{\sim}{Q}_m \setminus \underset{\sim}{S}_m$ and suppose that $Y = \emptyset$. By the previous lemma, we can find $\rho \in \mathscr{E}(X)$ such that

(i) $\rho \subset \ker \alpha$ $\qquad$ (ii) $|k(\rho)| = \underset{\sim}{m}$ and

(iii) $|\{x\rho : x \in X\}| = \underset{\sim}{m}$.

Thus let ε be an idempotent of $\mathscr{T}(X)$ such that $\ker \varepsilon = \rho$. Then

$$x\varepsilon = x \qquad \text{if } |x\rho| = 1$$

$$x\varepsilon \in x\rho \qquad \text{otherwise.}$$

It follows from (ii) that $| C(\varepsilon) | = \underset{\approx}{m}$ and $| y\varepsilon^{-1} | < \underset{\sim}{m}$, for all $y \in \text{ran } \varepsilon$. Since $| \text{ran } \varepsilon | = \underset{\sim}{m}$ by (iii), it now follows from lemma 3.3 that $\varepsilon \in E(S_{\underset{\sim}{m}})$.

Next we define θ. Since both α and ε are in $S_{\underset{\sim}{m}}$, we have

$$| \text{def } \alpha | = | \text{def } \varepsilon | = \underset{\sim}{m}.$$

Let

$$\text{def } \alpha = U \cup V$$

be a partition of def α such that $| U | = | V | = \underset{\sim}{m}$ and let

$$\text{def } \varepsilon = \bigcup_{k \in K} A_k,$$

where $| A_k | = \underset{\sim}{2}$, for all $k \in K$, $| K | = \underset{\sim}{m}$ and $A_k \cap A_j = \emptyset$, if $k \neq j$. Let $\psi : R \longrightarrow U$ be a bijection and define θ as follows

$$x\theta = (x\varepsilon^{-1}) \alpha \qquad \text{if } x \in \text{ran } \varepsilon,$$

$$A_k\theta = k\psi \qquad \text{if } k \in K.$$

Since ker $\varepsilon \subset$ ker α by (i), it follows that θ is well defined. Also, since ran $\theta \supset U$ and def $\theta \supseteq V$ we have

$$| \text{ran } \theta | = | \text{def } \theta | = \underset{\sim}{m}.$$

It is also clear that $| C(\theta) | = \underset{\sim}{m}$, for $C(\theta) \supseteq$ def ε. Finally, we show that

$$| x\theta^{-1} | < \underset{\sim}{m},$$

for all $x \in \text{ran } \theta$. If $u \in U$ then it is obvious that $| u\theta^{-1} | = \underset{\sim}{2} < \underset{\sim}{m}$. Now, if $x \in \text{ran } \theta \setminus U$ then

$$x = (a \varepsilon^{-1}) \alpha,$$

for some $a \in \text{ran } \varepsilon$. If $| x\theta^{-1} | = \underset{\sim}{m}$ then $| (x\theta^{-1}) \varepsilon^{-1} | = \underset{\sim}{m}$. But $(x\theta^{-1}) \varepsilon^{-1} = x (\varepsilon\theta)^{-1} = x\alpha^{-1}$ and so it would follow that

$$| x\alpha^{-1} | = \underset{\sim}{m},$$

which does not happen since $Y = \emptyset$. Hence $| x\theta^{-1} | < \underset{\sim}{m}$ for all $x \in \text{ran } \theta$. Clearly $\alpha = \varepsilon\theta$ and the lemma is now proved.

The theorem follows now from Lemmas 3.4, 3.6, 3.7 and 3.21.

THEOREM 3.22. *Let* m *be an infinite non-regular cardinal and denote by* $\langle S_m \rangle$ *the subsemigroup of* Q_m *generated by* S_m. *Then* $\langle S_m \rangle = Q_m$. *In fact*

$$Q_m = \langle S_m \rangle = [E(S_m)] \overset{2}{\cdot} S_m .$$

Proof. It remains to show that

$$\alpha \in S_m \Rightarrow \alpha \in [E(S_m)] \overset{2}{\cdot} S_m .$$

To see this let α' be an inverse of α in Q_m, and let $\varepsilon = \alpha\alpha'$. Then ε is an idempotent in Q_m. Also $\varepsilon \mathscr{R} \alpha$, giving $\ker \varepsilon = \ker \alpha$. Hence

$$|\operatorname{ran} \varepsilon| = |X/\ker \varepsilon| = |X/\ker \alpha| = |\operatorname{ran} \alpha| = m$$

and so we now have that $\varepsilon \in S_m$. Obviously

$$\alpha = \varepsilon\alpha,$$

and so $\alpha \in [E(S_m)] \overset{2}{\cdot} S_m$, as required.

3. <u>THE LATTICE OF CONGRUENCES ON</u> $S_{\aleph_0}$

Let X be an infinite set such that $|X| = \aleph_0$. Notice that for any $\alpha \in \mathscr{T}(X)$

$$X = \bigcup_{y \in \operatorname{ran} \alpha} y\alpha^{-1} ,$$

and so if $\alpha \in Q_{\aleph_0}$ (1.1) and $|y\alpha^{-1}| < \aleph_0$ for all $y \in \operatorname{ran} \alpha$, the regularity of $\aleph_0$ gives

$$|\operatorname{ran} \alpha| = \aleph_0 .$$

Hence it follows from (3.1) that

$$S_{\aleph_0} = \{\alpha \in Q_{\aleph_0} : |y\alpha^{-1}| < \aleph_0 \ (\forall y \in \operatorname{ran} \alpha)\}.$$

It also follows from the work of Mal'cev [22] and Howie [16] that the relation

$$\Delta_{\aleph_0} = \{ (\alpha,\beta) \in S_{\aleph_0} \times S_{\aleph_0} : \max(|D(\alpha,\beta)\alpha|, |D(\alpha,\beta)\beta|) < \aleph_0 \},$$

where

$$D(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\}$$

is a congruence on $S_{\aleph_0}$. In fact, since $\aleph_0$ is a regular cardinal, we have a simpler formula for $\Delta_{\aleph_0}$ as follows:

$$\Delta_{\aleph_0} = \{(\alpha,\beta) \in S_{\aleph_0} \times S_{\aleph_0} : |D(\alpha,\beta)| < \aleph_0\} ; \qquad (3.23)$$

for if $|D(\alpha,\beta)| < \aleph_0$ then certainly $|D(\alpha,\beta)\alpha|$ and $|D(\alpha,\beta)\beta|$ are less than $\aleph_0$; and conversely if $|D(\alpha,\beta)\alpha| < \aleph_0$ then

$$D(\alpha,\beta) \subseteq \bigcup_{y \in D(\alpha,\beta)\alpha} y\alpha^{-1} ,$$

and so has cardinality less than $\aleph_0$ since $\alpha \in S_{\aleph_0}$ and $\aleph_0$ is regular.

We shall show in this section that $\Delta_{\aleph_0}$ as defined in (3.23) is the only proper congruence on $S_{\aleph_0}$.

LEMMA 3.24. *Let* $\alpha,\beta \in S_{\aleph_0}$ *be such that* $1 \leq |D(\alpha,\beta)| < \aleph_0$ *and let* ρ *be a congruence on* $S_{\aleph_0}$ *containing* (α,β). *Then* $(\gamma,\delta) \in \rho$ *for all* $\gamma,\delta \in S_{\aleph_0}$ *such that* $|D(\gamma,\delta)| = 1$.

Proof. Let $D(\gamma,\delta) = \{x_o\}$. Then $\ker\gamma \cap \ker\delta$ has classes as follows:

(1) $\{x_o\}$;

(2) up to two finite classes in $x_o\gamma\gamma^{-1} \cup x_o\delta\delta^{-1}$;

(3) infinitely many finite classes that are both $\ker\gamma$- and $\ker\delta$-classes.

To see this, notice first that if $(x, x_o) \in \ker\gamma \cap \ker\delta$, then

$$x\gamma = x_o\gamma \neq x_o\delta = x\delta .$$

Hence $x \in D(\gamma,\delta)$ and so $x = x_o$.

Thus $\{x_o\}$ is a $(\ker\gamma \cap \ker\delta)$-class. Next, each of the sets $x_o\gamma\gamma^{-1}\setminus\{x_o\}$, $x_o\delta\delta^{-1}\setminus\{x_o\}$ is either empty or is a $(\ker\gamma \cap \ker\delta)$-class. Considering the first of these (which will be sufficient) notice first that if $z,t \in x_o\gamma\gamma^{-1}\setminus\{x_o\}$ then $z\gamma = t\gamma = x_o\gamma$ and so $(z,t) \in \ker\gamma$. Also

$$z\delta = z\gamma \qquad \text{(since } z \neq x_o\text{)}$$
$$= t\gamma$$
$$= t\delta \qquad \text{(since } t \neq x_o\text{)}$$

and so $(z,t) \in \ker \delta$. Thus $x_o\gamma\gamma^{-1} \setminus \{x_o\}$ is contained in a $(\ker \gamma \cap \ker \delta)$-class; let us call it A. Let $a \in A$. Then there exists z in $x_o\gamma\gamma^{-1} \setminus \{x_o\}$ such that $(a,z) \in \ker \gamma \cap \ker \delta$. Hence $a\gamma = z\gamma = x_o\gamma$, giving $a \in x_o\gamma\gamma^{-1}$. Moreover $a = x_o$ would give $(z,x_o) \in \ker \gamma \cap \ker \delta$ and hence

$$z\gamma = x_o\gamma \neq x_o\delta = z\delta,$$

a contradiction, since $z \notin D(\gamma,\delta) = \{x_o\}$. Thus

$$A = x_o\gamma\gamma^{-1} \setminus \{x_o\}$$

as required.

Similarly $x_o\delta\delta^{-1} \setminus \{x_o\}$ is either empty or is a $(\ker \gamma \cap \ker \delta)$-class. Since $x\gamma = x\delta$ for all x in $X \setminus (x_o\gamma\gamma^{-1} \cup x_o\delta\delta^{-1})$ the other classes are as stated in (3).

Now choose x_1 so that $x_1\alpha \neq x_1\beta$ and define

$$\overline{D(\alpha,\beta)} = \bigcup_{x \in D(\alpha,\beta)} (x\alpha\alpha^{-1} \cup x\beta\beta^{-1}).$$

This is a finite set containing x_1. Hence $Z = X \setminus \overline{D(\alpha,\beta)}$ is infinite and has the property that

$$z\alpha = z\beta \; ; \; z\alpha\alpha^{-1} = z\beta\beta^{-1},$$

for all z in Z. Let Y be a cross-section of the equivalence $\ker \alpha \cap (Z \times Z)$ ($= \ker \beta \cap (Z \times Z)$). Since $C(\alpha) \cap Z$ is infinite and since each $(\ker \alpha)$-class is finite, it follows that both Y and $Z \setminus Y$ are infinite. Also

$$y \longmapsto y\alpha \; (=y\beta)$$

is a one-to-one correspondence between Y and $Y\alpha$ ($= Y\beta = Z\alpha = Z\beta$).

Now define $\xi \in S_{\aleph_0}$ as follows. Let

$$\psi : (X/(\ker \gamma \cap \ker \delta)) \setminus \{x_o\} \longrightarrow Y$$

be a bijection. (Both sets are countably infinite). Then define

$$x_o\xi = x_1$$

$$C\xi = C\psi \quad (C \in (X/(\ker\gamma \cap \ker\delta))\setminus\{x_o\}).$$

Then $\ker\xi = \ker\gamma \cap \ker\delta$ and so by the remarks above we do have $|C(\xi)| = \aleph_0$ and $|y\xi^{-1}| < \aleph_0$ for all y in ran ξ. Since ran $\xi = Y \cup \{x_1\}$ we also have

$$|\operatorname{ran}\xi| = |\operatorname{def}\xi| = \aleph_0 .$$

and so $\xi \in S_{\aleph_0}$.

Observe now that both $x_1\alpha$ and $x_1\beta$ are not in $Y\alpha$. For if $x_1\alpha \in Y\alpha$ then $x_1\alpha = y\alpha$ for some $y \in Y$ and so $x_1 \in y\alpha\alpha^{-1}$. But $y\alpha\alpha^{-1} = y\beta\beta^{-1}$, since $y \in Z$, and so $x_1\beta = y\beta$. Now $y\alpha = y\beta$, since $y \notin D(\alpha,\beta)$. Hence $x_1\alpha = x_1\beta$, which cannot happen. Notice also that

$$Y\alpha \cap \{x_1\alpha\} \subseteq \operatorname{ran}\alpha,$$

and so $|X\setminus(Y\alpha \cup \{x_1\alpha, x_1\beta\})| = \aleph_0$. Let

$$X\setminus(Y\alpha \cup \{x_1\alpha, x_1\beta\})$$

be a disjoint union $U \cup V$, where $|U| = |V| = \aleph_0$ and let $\omega : U \longrightarrow V$ be a bijection. Define $\eta : X \longrightarrow X$ by

$$x\eta = (x\alpha^{-1} \cap Y)\,\xi^{-1}\gamma \qquad \text{if } x \in Y\alpha$$

$$(x_1\alpha)\,\eta = x_o\gamma$$

$$(x_1\beta)\,\eta = x_o\delta$$

$$u\eta = (u\omega)\,\eta = u \qquad \text{for } u \in U$$

Then $(x\alpha^{-1} \cap Y)\,\xi^{-1}\gamma$ is a single element, since $|x\alpha^{-1} \cap Y| = 1$ and $\ker\xi \subseteq \ker\gamma$. Also it is easy to verify that $\eta \in S_{\aleph_0}$, and for all $x \neq x_o$

$$x\xi\alpha\eta = x\gamma = x\delta = x\xi\beta\eta,$$

while $x_o\xi\alpha\eta = x_o\gamma$, $x_o\xi\beta\eta = x_o\delta$. That is

$$\xi\alpha\eta = \gamma \quad , \quad \xi\beta\eta = \delta,$$

giving $(\gamma,\delta) \in \rho$, as required.

LEMMA 3.25. *Let* $\alpha,\beta \in S_{\aleph_0}$ *be such that* $1 \leq |D(\alpha,\beta)| < \aleph_0$ *and let* ρ *be a congruence on* $S_{\aleph_0}$ *containing* (α,β). *Then* ρ *contains all elements* $(\gamma,\delta) \in S_{\aleph_0} \times S_{\aleph_0}$ *such that* $1 \leq |D(\gamma,\delta)| < \aleph_0$.

Proof. Let $\gamma,\delta \in S_{\aleph_0}$ and suppose that $|D(\gamma,\delta)| = n < \aleph_0$, i.e., that

$$D(\gamma,\delta) = \{a_1, a_2, \ldots\ldots, a_n\}.$$

Then, define $\gamma_o = \gamma$ and for $i = 1,2, \ldots\ldots, n$ define $\gamma_i \in S_{\aleph_0}$ by

$$a_i \gamma_i = a_i \delta$$
$$x \gamma_i = x\gamma_{i-1} \qquad (x \neq a_i)$$

Then $|D(\gamma_i, \gamma_{i+1})| = |\{a_{i+1}\}| = 1$. Also it is easily verified that $\gamma_n = \delta$. (The sequence $\gamma_1, \ldots\ldots, \gamma_n$ changes $a_i\gamma$ to $a_i\delta$ successively for $i = 1, \ldots\ldots, n$.). By the proof of the previous lemma there exist ξ_i, η_i in $S_{\aleph_0}$ $(i = 0, \ldots\ldots, n-1)$ such that

$$\xi_i\alpha\eta_i = \gamma_i \quad , \quad \xi_i\beta\eta_i = \gamma_{i+1}.$$

Hence we have a sequence

$$\gamma = \xi_o\alpha\eta_o \longrightarrow \xi_o\beta\eta_o = \gamma_1 = \xi_1\alpha\eta_1 \longrightarrow \xi_1\beta\eta_1 = \gamma_2 = \xi_2\alpha\eta_2 \longrightarrow$$
$$\ldots\ldots \longrightarrow \xi_{n-1}\beta\eta_{n-1} = \gamma_n = \delta$$

of elementary transitions connecting γ to δ and so $(\gamma,\delta) \in \rho$.

Notice that if ρ is a congruence on $S_{\aleph_0}$ such that ρ contains a pair (α,β) for which $|D(\alpha,\beta)| = \aleph_0$, then ρ is the universal congruence, by Lemmas 3.13, 3.15 and 3.20 in [16]. Hence it follows from Lemma 3.25 that

COROLLARY 3.26. *If* ρ *is a non-trivial congruence on* $S_{\aleph_0}$ *then* $\rho \supseteq \Delta_{\aleph_0}$.

It now follows from the work of Howie [16] that $\Delta_{\aleph_0}$ is the maximum non-universal congruence on $S_{\aleph_0}$. Hence if ρ is a non-trivial congruence on $S_{\aleph_0}$ then $\rho = \Delta_{\aleph_0}$, since $\rho \supseteq \Delta_{\aleph_0}$ by Corollary 3.26.

We can now state the theorem describing the lattice of congruences on $S_{\aleph_0}$.

THEOREM 3.27. *The congruence* $\Delta_{\aleph_0}$ *as described in* (3.23) *is the only proper congruence on* $S_{\aleph_0}$.

CHAPTER 4

INVERSE SEMIGROUPS GENERATED BY NILPOTENT TRANSFORMATIONS

1. INTRODUCTION

As remarked by A. H. Clifford and G. B. Preston [5, 8.1] R. Baer and F. Levi [1] presented in a paper (1932) a right cancellative, right simple semigroup which is not a group. This semigroup is the semigroup of all one-to-one mappings α of a countable set, I, say into itself with the property that $I \setminus I\alpha$ is not finite. More generally, if p and q are infinite cardinals such that $p \geq q$ we shall say that S is a *Baer-Levi semigroup of type* (p,q) *on the set* A if $|A| = p$ and if S is the semigroup of all one-to-one mappings η (combined under composition) of A into A, having the property that $|A \setminus A\eta| = q$. It then follows that if X is an infinite set of cardinality m, the *Baer-Levi semigroup* B *of type* (m,m) *on* X is defined as

$$B = \{\alpha \in \mathscr{T}(X) : C(\alpha) = \emptyset \ , \ |\operatorname{def} \alpha| = m\}.$$

The first objective of this chapter is to find a dual semigroup for B. Within $\mathscr{T}(X)$ there does not appear to be any satisfactory dual for B, but in fact $B \subseteq \mathscr{I}(X)$, the symmetric inverse semigroup on X for

$$B = \{\alpha \in \mathscr{I}(X) : \operatorname{gap} \alpha = \emptyset \ , \ |\operatorname{def} \alpha| = m\} , \qquad (4.1)$$

where $\operatorname{gap} \alpha = X \setminus \operatorname{dom} \alpha$. Within $\mathscr{I}(X)$ there is a natural dual B^* which is described in the next section.

Particular attention is given to the semigroup generated by B^*B,

$$K_m = \langle B^*B \rangle.$$

The main result of section 3, Theorem 4.17, states that K_m is the inverse semigroup generated by the nilpotent elements of $\mathscr{I}(X)$ of index 2.

Finally, in section 4 we produce an inverse and nilpotent-generated semigroup which is congruence-free.

2. PRELIMINARIES

Let X be an infinite set of cardinality $\underset{\sim}{m}$ and denote by $\mathscr{I}(X)$ the symmetric inverse semigroup on X. Let B be the Baer-Levi semigroup of type $(\underset{\sim}{m},\underset{\sim}{m})$ on X (4.1) and consider the following subset of $\mathscr{I}(X)$

$$B^{*} = \{\alpha \in \mathscr{I}(X) : |\text{gap } \alpha| = \underset{\sim}{m}, \quad \text{def } \alpha = \emptyset\}. \tag{4.2}$$

LEMMA 4.3. *Let* B^* *be the set defined in* (4.2). *Then* B^* *is a non-empty subsemigroup of* $\mathscr{I}(X)$.

Proof. If $X = Y \cup Z$ is a partition of X into two subsets both of cardinality $\underset{\sim}{m}$, then it is easy to see that B^* contains all the bijections θ from Y onto X. Thus $B^* \neq \emptyset$.

To see that B^* is a semigroup is not difficult either. Let $\alpha, \beta \in B^*$. Since

$$\text{dom } (\alpha\beta) = (\text{ran } \alpha \cap \text{dom } \beta)\, \alpha^{-1} \subseteq \text{dom } \alpha$$

it follows that $\text{gap } (\alpha\beta) \supseteq \text{gap } \alpha$, and so

$$\underset{\sim}{m} = |\text{gap } \alpha| \leqslant |\text{gap } (\alpha\beta)|.$$

Hence $|\text{gap } (\alpha\beta)| = \underset{\sim}{m}$. Also, since $\text{ran } \alpha = \text{ran } \beta = X$ we have

$$\text{ran } (\alpha\beta) = (\text{ran } \alpha \cap \text{dom } \beta)\, \beta = (\text{dom } \beta)\, \beta = X,$$

giving $\text{def } (\alpha\beta) = \emptyset$, as required.

We next prove for B^* a lemma which is the dual of [$\underset{\sim}{5}$, Lemma 8.1].

LEMMA 4.4. *Let* B^* *be the semigroup defined in* (4.2). *Then* B^* *is a left cancellative and left simple semigroup without idempotents.*

Proof. This lemma becomes obvious if we observe that $\Psi : B \longrightarrow B^*$ given by

$$\alpha\Psi = \alpha^{-1} \qquad (\alpha \in B)$$

is an anti-isomorphism. That this is so follows from (4.1) and (4.2) and from the remarks that

$$\text{gap } (\alpha^{-1}) = \text{def } \alpha \quad , \quad \text{def } \alpha^{-1} = \text{gap } (\alpha)$$
$$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}.$$

Both products BB^* and B^*B are of some interest. First, we have

LEMMA 4.5. *If* B *and* B^* *are the semigroups defined respectively by* (4.1) *and* (4.2) *then* $BB^* = \mathscr{I}(X)$.

Proof. It is obvious that $BB^* \subseteq \mathscr{I}(X)$ since both B and B^* are contained in $\mathscr{I}(X)$.

Conversely, consider $\alpha \in \mathscr{I}(X)$ with dom $\alpha = P$, ran $\alpha = Q$. Choose disjoint subsets R_1, R_2, R_3 of X such that

$$| R_1 | = | R_2 | = | R_3 | = \underset{\sim}{m}, \qquad X = R_1 \cup R_2 \cup R_3.$$

Since $| P | \leqslant \underset{\sim}{m}$, $| X \setminus P | \leqslant \underset{\sim}{m}$ there exist injections $\theta : P \longrightarrow R_1$ and $\phi : X \setminus P \longrightarrow R_2$. Define $\beta \in \mathscr{I}(X)$ by

$$x\beta = \begin{cases} x\theta & \text{if } x \in P \\ x\phi & \text{if } x \in X \setminus P \end{cases}$$

Then gap $\beta = \emptyset$, def $\beta \supseteq R_3$ and so $\beta \in B$.

Next, observe that $| X \setminus Q | \leqslant \underset{\sim}{m}$ and let $\Psi : X \setminus Q \longrightarrow R_3$ be an injection. Then define $\gamma \in \mathscr{I}(X)$ by

$$x\gamma = \begin{cases} x\theta^{-1}\alpha & \text{if } x \in P\theta \\ x\Psi^{-1} & \text{if } x \in (X \setminus Q)\Psi . \end{cases}$$

Then gap $\gamma \supseteq R_2$, ran $\gamma = X$ and so $\gamma \in B^*$.

Finally

$$\text{dom } (\beta\gamma) = (\text{ran } \beta \cap \text{dom } \gamma)\, \beta^{-1} = P = \text{dom } \alpha$$

$$\text{ran } (\beta\gamma) = (\text{ran } \beta \cap \text{dom } \gamma)\, \gamma = Q = \text{ran } \alpha,$$

and $x(\beta\gamma) = (x\theta)\gamma = [(x\theta)\,\theta^{-1}]\alpha = x\alpha$ for all $x \in P$. Thus $\alpha = \beta\gamma \in BB^*$ as required.

We now describe the product B^*B and then concentrate our attention on $K_{\underset{\sim}{m}} = \langle B^*B \rangle$.

LEMMA 4.6. *If* B *and* B^* *are the semigroups defined respectively by* (4.1) *and* (4.2) *then*

$$B^*B = \{\alpha \in \mathscr{I}(X) : |\operatorname{dom}\alpha| = |\operatorname{ran}\alpha| = |\operatorname{gap}(\alpha)| = |\operatorname{def}(\alpha)| = \underset{\sim}{m}\}.$$

Proof. Let $\alpha \in B^*$, $\beta \in B$. Then ran α = dom β = X, $|\operatorname{gap}\alpha| = |\operatorname{def}\beta| = \underset{\sim}{m}$ and $|\operatorname{dom}\alpha| = |\operatorname{ran}\beta| = \underset{\sim}{m}$. Hence

$$\operatorname{dom}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} = X\alpha^{-1} = \operatorname{dom}\alpha$$

and

$$\operatorname{ran}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta = X\beta = \operatorname{ran}\beta,$$

giving

$$\operatorname{gap}(\alpha\beta) = \operatorname{gap}\alpha, \quad \operatorname{def}(\alpha\beta) = \operatorname{def}\beta.$$

It now follows easily that

$$|\operatorname{dom}(\alpha\beta)| = |\operatorname{ran}(\alpha\beta)| = |\operatorname{gap}(\alpha\beta)| = |\operatorname{def}(\alpha\beta)| = \underset{\sim}{m}.$$

Conversely, let $\alpha \in \mathscr{I}(X)$ be such that

$$|\operatorname{dom}\alpha| = |\operatorname{ran}\alpha| = |\operatorname{gap}\alpha| = |\operatorname{def}\alpha| = \underset{\sim}{m}.$$

If $\beta : \operatorname{dom}\alpha \longrightarrow X$ is bijection then $\beta \in \mathscr{I}(X)$. Let $\theta = \beta^{-1}\alpha$. It is then easy to verify that $\alpha = \beta\theta$ and that $\beta \in B^*$, $\theta \in B$, as required.

Notice now that the set B^*B fails to be a semigroup. For if $X = Y \cup V \cup Z$ is a partition of X such that $|Y| = |V| = |Z| = \underset{\sim}{m}$ and $\alpha : Y \longrightarrow V$, $\beta : Z \longrightarrow V$ are bijections then both α and β are in B^*B. But $\alpha\beta = 0$, the empty mapping, and obviously since dom (0) = ran (0) = $\emptyset$ we have that

$$\alpha\beta = 0 \notin B^*B.$$

We will however come back to the set B^*B in section 4. There, we shall describe a Rees quotient whose non-zero elements are the singleton sets $\{\alpha\}$, where $\alpha \in B^*B$.

Before that we prove the following lemma:

LEMMA 4.7. *If* B *and* B^* *are the semigroups defined respectively by* (4.1) *and* (4.2) *then*

$$\langle B^*B\rangle = \{\alpha \in \mathscr{I}(X) : |\operatorname{gap}\alpha| = |\operatorname{def}\alpha| = \underset{\sim}{m}\}.$$

Proof. Let $K_{\underset{\sim}{m}} = \{\alpha \in \tilde{\mathscr{I}}(X) : |\text{gap } \alpha| = |\text{def } \alpha| = \underset{\sim}{m}\}$.

Notice first that $K_{\underset{\sim}{m}}$ is a subsemigroup of $\mathscr{I}(X)$. For if α and β are two elements of $K_{\underset{\sim}{m}}$ then, as we saw before, since dom $(\alpha\beta) \subseteq$ dom α and ran $(\alpha\beta) \subseteq$ ran β it follows that

$$\text{gap } \alpha \subseteq \text{gap } (\alpha\beta) \quad , \quad \text{def } \beta \subseteq \text{def } (\alpha\beta).$$

Therefore since $B^*B \subseteq K_{\underset{\sim}{m}}$ we have that $(B^*B)^2 \subseteq K_{\underset{\sim}{m}}$.

Suppose now that $\alpha \in K_{\underset{\sim}{m}}$. Then $|\text{ gap } \alpha | = | \text{ def } \alpha | = \underset{\sim}{m}$ and so we may write

$$\text{gap } \alpha = Z \cup T \quad , \quad \text{def } \alpha = P \cup Q,$$

where $Z \cap T = P \cap Q = \emptyset$ and $| Z | = | T | = | P | = | Q | = \underset{\sim}{m}$. Let $\theta : Z \longrightarrow P$ be a bijection and define

$$\beta = \alpha \cup \theta : \text{dom } \alpha \cup Z \longrightarrow \text{ran } \alpha \cup P$$

Then $| \text{ dom } \beta | = \underset{\sim}{m}$, $| \text{ gap } \beta | = | T | = \underset{\sim}{m}$; $| \text{ ran } \beta | = \underset{\sim}{m}$, $| \text{ def } \beta | = | Q | = \underset{\sim}{m}$ and so $\beta \in B^*B$.

Now define $\gamma = 1_{\text{ran } \alpha \cup Q}$. Then

$$| \text{ dom } \gamma | = | \text{ ran } \gamma | = \underset{\sim}{m} , \quad | \text{ gap } \gamma | = | \text{ def } \gamma | = | P | = \underset{\sim}{m} ,$$

and so $\gamma \in B^*B$.

Next observe that

$$(\text{ran } \beta \cap \text{dom } \gamma)\ \beta^{-1} = (\text{ran } \alpha)\ \beta^{-1} = \text{dom } \alpha,$$

$$(\text{ran } \beta \cap \text{dom } \gamma)\gamma = \text{ran } \alpha,$$

and that for all x in dom α, $x\beta\gamma = x\alpha$. Thus $\alpha = \beta\gamma \in (B^*B)^2$, as required

3. AN INVERSE AND NILPOTENT-GENERATED SEMIGROUP

Notice that the empty mapping, which we shall denote by "0", belongs to $K_{\underset{\sim}{m}}$. In fact $0 \in \mathscr{I}(X)$ and we have

$$\text{def } 0 = \text{gap } 0 = X.$$

Observe also that for each $\alpha \in K_{\underset{\sim}{m}}$ there exists $\alpha^{-1} \in \mathscr{I}(X)$ and

$$\text{gap } \alpha^{-1} = \text{def } \alpha , \quad \text{def } \alpha^{-1} = \text{gap } \alpha.$$

Hence, $|\text{gap}\ \alpha| = |\text{def}\ \alpha| = \underset{\sim}{m}$ gives

$$|\text{gap}\ \alpha^{-1}| = |\text{def}\ \alpha^{-1}| = \underset{\sim}{m}$$

and so $\alpha^{-1} \in K_{\underset{\sim}{m}}$. Thus we have

LEMMA 4.8. *$K_{\underset{\sim}{m}}$ is an inverse subsemigroup of $\mathscr{I}(X)$ containing a zero-element.*

We now recall that in a semigroup S with zero, an element *s* is said to be *nilpotent* if $s^n = 0$ for some $n \geq 1$. If $s^n = 0$ but $s^{n-1} \neq 0$ we say that s is *nilpotent of index* n. Thus, in particular, if we say that α is a *nilpotent* element of $\mathscr{I}(X)$ *of index* 2 we mean that $\alpha \neq 0$ and $\alpha^2 = 0$. This is obviously equivalent to the statement

$$\text{dom}\ \alpha \neq \emptyset \quad \text{and} \quad \text{dom}\ \alpha \cap \text{ran}\ \alpha = \emptyset.$$

Hence it is clear that the set of nilpotents of $\mathscr{I}(X)$ of index 2 is non-empty. In fact, if $X = U \cup V$ is a partition of X and $|U| = |V| = \underset{\sim}{m}$ then any bijection $\theta : U \longrightarrow V$ is a nilpotent element of $\mathscr{I}(X)$ of index 2. Write

$$N^{(2)} = \{\alpha \in \mathscr{I}(X) : \alpha \neq 0 \ \text{and}\ \alpha^2 = 0\}.$$

LEMMA 4.9. $N^{(2)} \subset K_{\underset{\sim}{m}}$.

Proof. Let $\alpha \in N^{(2)}$. Then

$$\text{dom}\ \alpha \neq \emptyset \ \text{and dom}\ \alpha \cap \text{ran}\ \alpha = \emptyset$$

and so

$$\text{ran}\ \alpha \subseteq \text{gap}\ \alpha, \quad \text{dom}\ \alpha \subseteq \text{def}\ \alpha. \tag{4.10}$$

If we suppose by way of contradiction that $\alpha \notin K_{\underset{\sim}{m}}$ then either (i) $|\text{gap}\ \alpha| < \underset{\sim}{m}$ or (ii) $|\text{def}\ \alpha| < \underset{\sim}{m}$ (or both). In case (i), $|\text{gap}\ \alpha| < \underset{\sim}{m}$ implies that $|\text{dom}\ \alpha| = \underset{\sim}{m}$ and since α is one-to-one it would follow that

$$|\text{ran}\ \alpha| = \underset{\sim}{m}. \tag{4.11}$$

But (4.10) gives

$$|\operatorname{ran}\alpha| \leqslant |\operatorname{gap}\alpha| < \underset{\sim}{m}$$

according with our supposition (i) and so,

$$|\operatorname{ran}\alpha| < \underset{\sim}{m},$$

contradicting (4.11).

A similar argument, interchanging dom α and gap α with ran α and def α shows that case (ii) leads to a contradiction too. Hence $\alpha \in K_{\underset{\sim}{m}}$, as required.

LEMMA 4.12. *Let* $\alpha \in K_{\underset{\sim}{m}}$. *Then* α *can be expressed as a product of two elements of* $N^{(2)}$ *if and only if*

$$|\operatorname{gap}\alpha \cap \operatorname{def}\alpha| = \underset{\sim}{m}$$

Proof. Let $\alpha \in K_{\underset{\sim}{m}}$. We suppose first that $|\operatorname{gap}\alpha \cap \operatorname{def}\alpha| = \underset{\sim}{m}$. Let

$$\operatorname{gap}\alpha \cap \operatorname{def}\alpha = Y \cup U, \tag{4.13}$$

where $|Y| = |U| = \underset{\sim}{m}$ and $Y \cap U = \emptyset$.

Now define $\beta : \operatorname{dom}\alpha \longrightarrow Y$ to be a injection. Then $\beta \in \mathscr{I}(X)$ and $\operatorname{dom}\beta \neq \emptyset$; also, since

$$\operatorname{dom}\beta \cap \operatorname{ran}\beta \subseteq \operatorname{dom}\alpha \cap Y ,$$

it follows from (4.13) that

$$\operatorname{dom}\beta \cap \operatorname{ran}\beta = \emptyset.$$

Thus $\beta \in N^{(2)}$.

Now let $\gamma = \beta^{-1}\alpha$. Then

$$\operatorname{dom}\gamma = (\operatorname{ran}\beta^{-1} \cap \operatorname{dom}\alpha)\beta = \operatorname{ran}\beta,$$

(since $\operatorname{ran}\beta^{-1} = \operatorname{dom}\beta = \operatorname{dom}\alpha$) and similarly

$$\operatorname{ran}\gamma = \operatorname{ran}\alpha.$$

In fact,

$$\beta\gamma = \beta\beta^{-1}\alpha = \alpha,$$

since $\beta\beta^{-1} = 1_{\mathrm{dom}\,\alpha}$. Also $\gamma \in N^{(2)}$, since dom $\gamma \neq \emptyset$ and

$$\mathrm{dom}\,\gamma \cap \mathrm{ran}\,\gamma = \mathrm{ran}\,\beta \cap \mathrm{ran}\,\alpha = \emptyset.$$

Thus $\alpha = \beta\,\gamma \in (N^{(2)})^2$, as required .

To complete the proof of Lemma 4.12 let $\alpha \in K_{\underset{\sim}{m}}$ and suppose that $\alpha = \beta\gamma$, where β and γ are elements of $N^{(2)}$. We have to consider two cases (i) $|\mathrm{ran}\,\alpha| = \underset{\sim}{m}$ and (ii) $0 < |\mathrm{ran}\,\alpha| < \underset{\sim}{m}$. If $|\mathrm{ran}\,\alpha| = \underset{\sim}{m}$ then

$$\underset{\sim}{m} = |\ \mathrm{ran}\,\alpha\ | = |\ (\mathrm{ran}\,\beta \cap \mathrm{dom}\,\gamma)\beta\ | = |\ \mathrm{ran}\,\beta \cap \mathrm{dom}\,\gamma|. \quad (4.14)$$

Now

$$\mathrm{dom}\,\alpha \subseteq \mathrm{dom}\,\beta\ , \qquad \mathrm{ran}\,\alpha \subseteq \mathrm{ran}\,\gamma,$$

$$\mathrm{dom}\,\beta \cap \mathrm{ran}\,\beta = \mathrm{dom}\,\gamma \cap \mathrm{ran}\,\gamma = \emptyset.$$

Hence

$$(\mathrm{ran}\,\beta \cap \mathrm{dom}\,\gamma) \cap \mathrm{dom}\,\alpha \subseteq \mathrm{ran}\,\beta \cap \mathrm{dom}\,\beta = \emptyset,$$

$$(\mathrm{ran}\,\beta \cap \mathrm{dom}\,\gamma) \cap \mathrm{ran}\,\alpha \subseteq \mathrm{dom}\,\gamma \cap \mathrm{ran}\,\gamma = \emptyset,$$

and so

$$\mathrm{ran}\,\beta \cap \mathrm{dom}\,\gamma \subseteq \mathrm{gap}\,\alpha,$$

$$\mathrm{ran}\,\beta \cap \mathrm{dom}\,\gamma \subseteq \mathrm{def}\,\alpha.$$

From (4.14) it now follows that

$$|\ \mathrm{gap}\,\alpha \cap \mathrm{def}\,\alpha\ | = \underset{\sim}{m}.$$

In the case where $0 < |\mathrm{ran}\,\alpha| < \underset{\sim}{m}$ we have $|\mathrm{dom}\,\alpha| = |\mathrm{ran}\,\alpha| < \underset{\sim}{m}$ and hence $|\mathrm{dom}\,\alpha \cup \mathrm{ran}\,\alpha| < \underset{\sim}{m}$, giving

$$|\mathrm{gap}\,\alpha \cap \mathrm{def}\,\alpha| = |X \setminus (\mathrm{dom}\,\alpha \cup \mathrm{ran}\,\alpha)| = \underset{\sim}{m}.$$

Lemma 4.12 is now proved.

We are now left with the case in which $\alpha \in K_{\underset{\sim}{m}}$ is such that

$$|\mathrm{gap}\,\alpha \cap \mathrm{def}\,\alpha| < \underset{\sim}{m}. \quad (4.15)$$

Let $\alpha \in K_{\underset{\sim}{m}} \setminus N^{(2)}$ be such that α satisfies (4.15).

It follows from the previous lemma that α cannot be expressed as a product of two elements of $N^{(2)}$. Hence if $\alpha \in \langle N^{(2)} \rangle$ at all, then a minimum number of three elements of $N^{(2)}$ is required. In fact we have

LEMMA 4.16. *Let* α *be an element of* $K_{\underset{\sim}{m}}$ *such that* $|\text{gap}\,\alpha \cap \text{def}\,\alpha| < \underset{\sim}{m}$. *Then there exist* η_1, η_2, η_3 *in* $N^{(2)}$ *such that* $\alpha = \eta_1\eta_2\eta_3$.

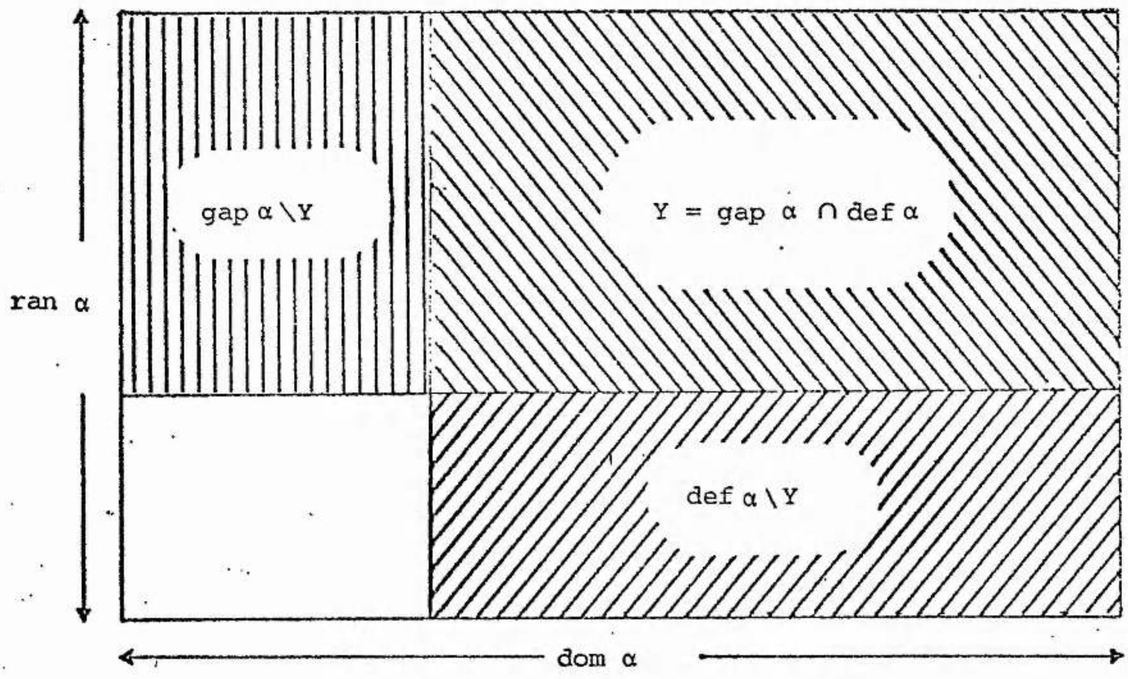
Proof. Take $\alpha \in K_{\underset{\sim}{m}}$ and suppose that $|\text{gap}\,\alpha \cap \text{def}\,\alpha| < \underset{\sim}{m}$. Hence, since $\alpha \in K_{\underset{\sim}{m}}$ it follows that

$$|\text{gap}\,\alpha| = |\text{def}\,\alpha| = \underset{\sim}{m}.$$

Thus, if $Y = \text{gap}\,\alpha \cap \text{def}\,\alpha$ then

$$|\text{def}\,\alpha \setminus Y| = |\text{gap}\,\alpha \setminus Y| = \underset{\sim}{m}.$$

We have the following diagram:



Let $\eta_1 : \text{dom}\,\alpha \longrightarrow \text{gap}\,\alpha \setminus Y$, $\eta_2 : \text{gap}\,\alpha \setminus Y \longrightarrow \text{def}\,\alpha \setminus Y$ be bijections. Clearly both η_1 and η_2 belong to $N^{(2)}$.

Now let $\eta_3 = \eta_2^{-1}\eta_1^{-1}\alpha$. Then $\alpha = \eta_1\eta_2\eta_3$, since $\eta_1\eta_2\eta_2^{-1}\eta_1^{-1} = 1_{\text{dom}\,\alpha}$. Also $\text{dom}\,\eta_3 = \text{def}\,\alpha \setminus Y \neq \emptyset$ and

$$\text{dom}\,\eta_3 \cap \text{ran}\,\eta_3 = (\text{def}\,\alpha \setminus Y) \cap \text{ran}\,\alpha = \emptyset,$$

giving $\eta_3 \in N^{(2)}$. This completes the proof of lemma 4.16.

We can now state the main result of this section.

THEOREM 4.17. *Let* $K_{\underset{\sim}{m}}$ *be as defined in Lemma 4.7. Then* $K_{\underset{\sim}{m}}$ *is the inverse semigroup generated by the nilpotent elements of* $\mathscr{I}(X)$ *of index* 2. *Moreover, if* $\langle N^{(2)} \rangle$ *denotes the semigroup generated by* $N^{(2)}$ *we have*

$$\langle N^{(2)}\rangle = K_m = [N^{(2)}]^2 \cup [N^{(2)}]^3 .$$

4. A CLASS OF INVERSE NILPOTENT-GENERATED AND CONGRUENCE-FREE SEMIGROUPS

We begin with the following lemma:

LEMMA 4.18. *For each* $k \leq m$ *the set*

$$P_k = \{\alpha \in K_m : |\operatorname{dom} \alpha| (= |\operatorname{ran} \alpha|) < k\}$$

is a proper ideal of K_m.

Proof. Let $k \leq m$. Observe first that P_k contains the zero of K_m and so $P_k \neq \emptyset$. Also, it is clear that $P_k \underset{\neq}{\subset} K_m$. For if $X = Y \cup Z$ is a partition of X such that $|Y| = |Z| = m$ and $\theta : Y \longrightarrow Z$ is a bijection then $\theta \in K_m \setminus P_k$.

Now take $\alpha \in K_m$ and $\theta \in P_k$. We are required to show that both $\alpha\theta$ and $\theta\alpha$ belong to P_k. Since $\operatorname{dom}(\theta\alpha) \subseteq \operatorname{dom}\theta$ and $\operatorname{ran}(\alpha\theta) \subseteq \operatorname{ran}\theta$ it follows that

$$|\operatorname{dom}(\theta\alpha)| \leq |\operatorname{dom}\theta| < k, \quad |\operatorname{ran}(\alpha\theta)| \leq |\operatorname{ran}\theta| < k,$$

and hence that $\theta\alpha \in P_k$, $\alpha\theta \in P_k$.

Notice that

$$k_1 < k_2 \leq m \Rightarrow P_{k_1} \subset P_{k_2}$$

Thus, P_m is maximal among the ideals P_k $(k \leq m)$.
The Rees congruence associated with P_m is defined by

$$\rho = (P_m \times P_m) \cup 1_{K_m},$$

i.e. $(\alpha,\beta) \in \rho$ if and only if *either* $\alpha = \beta$ *or* α and β are such that

$$|\operatorname{dom}\alpha| (= |\operatorname{ran}\alpha|) < m, \quad |\operatorname{dom}\beta| (= |\operatorname{ran}\beta|) < m.$$

Hence the quotient semigroup is

$$\begin{aligned} L_m = K_m / P_m &= \{P_m\} \cup \{\{\alpha\} : \alpha \in K_m \setminus P_m\} \\ &= (K_m \setminus P_m) \cup \{0\} \\ &= \{\alpha \in K_m : |\operatorname{dom}\alpha| (= |\operatorname{ran}\alpha|) = m\} \cup \{0\}. \qquad (4.19) \end{aligned}$$

By Lemma 4.6 we then have

$$L_{\underset{\sim}{m}} = B^*B \cup \{0\},$$

where all the products not falling in B^*B are zero.

We proceed now to explore the properties of the semigroup $L_{\underset{\sim}{m}}$.

LEMMA 4.20. $L_{\underset{\sim}{m}}$ *as defined in* (4.19) *is a* o-*bisimple, inverse and nilpotent-generated semigroup.*

Proof. Since $K_{\underset{\sim}{m}}$ is an inverse and nilpotent-generated semigroup (Theorem 4.17) and since these properties are inherited by homomorphic images, it follows that $L_{\underset{\sim}{m}}$ is an inverse and nilpotent-generated semigroup.

We now have to show that $L_{\underset{\sim}{m}}$ is o-bisimple. Since $K_{\underset{\sim}{m}}$ is an inverse (and hence regular) subsemigroup of $\mathscr{I}(X)$, it follows [12, II, Prop. 4.5] that if $\alpha,\beta \in K_{\underset{\sim}{m}} \setminus P_{\underset{\sim}{m}}$ then $\alpha \mathscr{R} \beta$ in $K_{\underset{\sim}{m}}$ if and only if $\alpha \mathscr{R} \beta$ in $\mathscr{I}(X)$. Similarly, $\alpha \mathscr{L} \beta$ in $K_{\underset{\sim}{m}}$ if and only if $\alpha \mathscr{L} \beta$ in $\mathscr{I}(X)$. Since every element in a regular semigroup is $\mathscr{D}$-equivalent to an idempotent [12, II, Prop 3.2] the o-bisimplicity of $L_{\underset{\sim}{m}}$ will follow if we show that $(\varepsilon,\eta) \in \mathscr{D}$ for every pair of idempotents ε,η in $K_{\underset{\sim}{m}} \setminus P_{\underset{\sim}{m}}$.

Accordingly, let ε,η be two non-zero idempotents in $K_{\underset{\sim}{m}} \setminus P_{\underset{\sim}{m}}$. Then it follows from [12, V, Prop. 1.9] that $\varepsilon = 1_A$ and $\eta = 1_B$, where A and B are subsets of X satisfying

$$|A| = |X \setminus A| = |B| = |X \setminus B| = \underset{\sim}{m}. \tag{4.21}$$

Now, let $\alpha : A \longrightarrow B$ be a bijection. Then clearly, by (4.21) $\alpha \in L_{\underset{\sim}{m}}$. Also,

$$\alpha\alpha^{-1} = 1_A = \varepsilon \quad , \quad \alpha^{-1}\alpha = 1_B = \eta.$$

Hence it follows from [12, II, Prop. 3.6] that $\varepsilon \mathscr{D} \eta$, as required.

Since $L_{\underset{\sim}{m}}$ is o-bisimple and $\mathscr{D} \subseteq \mathscr{J}$ [12, II, 1.4] it follows that $\mathscr{D} = \mathscr{J}$ in $L_{\underset{\sim}{m}}$ and so $K_{\underset{\sim}{m}} \setminus P_{\underset{\sim}{m}}$ and $\{0\}$ are the only $\mathscr{J}$-classes in $L_{\underset{\sim}{m}}$. The semigroup $L_{\underset{\sim}{m}}$ is a principal factor of $K_{\underset{\sim}{m}}$ [12, III, section 1] .

The semigroup L_m is not congruence-free. It follows from the work of Liber [19] that for each $p \leq m$ the relation

$$\delta_p = \{(\alpha,\beta) \in K_m \times K_m : |(\alpha\backslash\beta) \cup (\beta\setminus\alpha)| < p\}$$

is a congruence on K_m. In using this notation we are regarding α and β as subsets of $X \times X$ in the usual way. If P_m^o denotes the Rees congruence on K_m whose quotient semigroup is $L_m = K_m/P_m$ (where P_m is the ideal defined in Lemma 4.18) then it is easy to see that

$$P_m^o \subseteq \delta_m.$$

Hence it follows from [12, I. Theorem 5.6] that

$$\hat{\delta}_m = \delta_m/P_m^o = \{(\bar{\alpha},\bar{\beta}) \in L_m \times L_m : (\alpha,\beta) \in \delta_m\}$$

is a congruence on L_m, where $\bar{\alpha}$ denotes the congruence class containing α. It is not hard to see that

$$\hat{\delta}_m = \{(\bar{\alpha},\bar{\beta}) \in L_m\setminus\{0\} \times L_m\setminus\{0\} : (\alpha,\beta) \in \delta_m\} \cup \{(0,0)\}. \qquad (4.22)$$

For if $\alpha \in K_m$ and $\beta \in K_m \setminus P_m$ are such that $(\alpha,\beta) \in \delta_m$ then $|\text{dom } \beta| = |\text{ran } \beta| = m$ and $|\text{dom } \alpha \setminus \text{dom } \beta| < m$, $|\text{dom } \beta \setminus \text{dom } \alpha| < m$;

$$|D(\alpha,\beta)| < m.$$

Hence $|(\text{dom } \alpha \cap \text{dom } \beta) \setminus D(\alpha,\beta)| = m$ and so

$$|\text{dom } \alpha| = |\text{ran } \alpha| = m,$$

giving $\alpha \in K_m \setminus P_m$. We have shown that if $\bar{\alpha},\bar{\beta}$ in L_m are such that $(\alpha,\beta) \in \Delta_m$ then either both α,β are in $K_m \setminus P_m$ or they are both in P_m.

Having defined the congruence $\hat{\delta}_m$ in L_m it is reasonable to ask whether or not the inverse semigroup $L_m^* = L_m/\hat{\delta}_m$ is congruence-free.

Some of the properties of $L_{\underset{\sim}{m}}^{*}$ we already know. It is o-bisimple, inverse and nilpotent-generated - these being properties that it inherits as a homomorphic image of $L_{\underset{\sim}{m}}$. We shall prove that it is also congruence-free.

It is known [32, 39] that a regular o- simple semigroup S is congruence-free if and only if the congruence

$$\sigma = \{(a,b) \in S \times S : (\forall s,t \in S^1)\ sat = 0 \iff sbt = 0\}$$

is trivial. Applying this to $L_{\underset{\sim}{m}}^{*}$, we see that what we are required to show is that if $\alpha,\beta \in L_{\underset{\sim}{m}} \setminus \{0\}$ are such that

$$\lambda\alpha\nu = 0 \iff \lambda\beta\nu = 0$$

for all $\lambda,\nu \in L_{\underset{\sim}{m}}$ then $(\alpha,\beta) \in \delta_{\underset{\sim}{m}}$.

Accordingly, let us suppose that α,β in $L_{\underset{\sim}{m}}$ are such that $(\alpha,\beta) \notin \hat{\delta}_{\underset{\sim}{m}}$ and $\alpha,\beta \neq 0$. Notice that it follows from (4.22) that

$$\hat{\delta}_{\underset{\sim}{m}} = \{(\bar{\alpha},\bar{\beta}) \in L_{\underset{\sim}{m}} \times L_{\underset{\sim}{m}} : |\operatorname{dom}\alpha \setminus \operatorname{dom}\beta| + |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha| + |D(\alpha,\beta)| < \underset{\sim}{m}\},$$

where

$$D(\alpha,\beta) = \{x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta : x\alpha \neq x\beta\}$$

Hence, if $(\alpha,\beta) \notin \delta_{m}$ then at least one of the three cardinals $|D(\alpha,\beta)|$, $|\operatorname{dom}\alpha \setminus \operatorname{dom}\beta|$ and $|\operatorname{dom}\beta \setminus \operatorname{dom}\alpha|$ must be $\underset{\sim}{m}$.

We suppose that $|D(\alpha,\beta)| = \underset{\sim}{m}$. Our aim is to find λ and ν both in $L_{\underset{\sim}{m}}$ such that

$$\lambda\alpha\nu \neq 0 \quad \text{and } \lambda\beta\nu = 0.$$

To do this we proceed as follows.

By [20, lemma 2], there exists a subset Y of D (α,β) such that $|Y| = \underset{\sim}{m}$ and $Y\alpha \cap Y\beta = \emptyset$. Let $Y = Z \cup V$ be a partition of Y where $|Z| = |V| = \underset{\sim}{m}$. Then since α and β are both one-to-one we have

$$|Z\alpha| = |V\alpha| = |V\beta| = |Z\beta| = \underset{\sim}{m}. \tag{4.23}$$

Let $\lambda : Z \longrightarrow V$ be a bijection. Then,

$$|\text{dom }\lambda| = |\text{gap }\lambda| = |\text{def }\lambda| = |\text{ran }\lambda| = \underset{\sim}{m},$$

giving $\lambda \in L_{\underset{\sim}{m}} \setminus \{0\}$. Since ran $\lambda = V \subset Y$ and $Y\alpha \cap Y\beta = \emptyset$ it follows that

$$\text{ran}(\lambda\alpha) \cap \text{ran }(\lambda\beta) = \emptyset.$$

We certainly have by (4.23)

$$|\text{ran }(\lambda\alpha)| = |V\alpha| = |V\beta| = |\text{ran }(\lambda\beta)| = \underset{\sim}{m}.$$

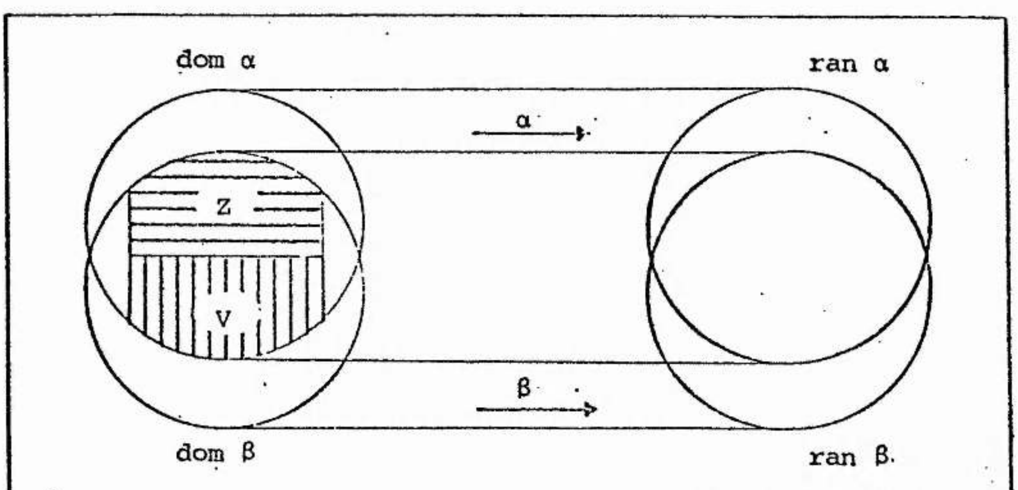
Now define $\nu : V\alpha \longrightarrow V\beta$ to be a bijection. Then

$$|\text{dom }\nu| = |\text{ran }\nu| = \underset{\sim}{m}.$$

Also, gap $\nu \supseteq Z\alpha$ and def $\nu \supseteq Z\beta$ for otherwise, since α and β are both one-to-one it would follow that $Z \cap V \neq \emptyset$, which contradicts our hypothesis. Hence (4.23) gives

$$|\text{gap }\nu| = |\text{def }\nu| = \underset{\sim}{m},$$

and so $\nu \in L_{\underset{\sim}{m}} \setminus \{0\}$.



Clearly, ran $\lambda = V \subset$ dom α and $V\alpha$ = dom ν together give ran $(\lambda\alpha\nu) \neq \emptyset$.
Moreover,

$$|\text{dom } (\lambda\alpha\nu)| = |\text{ran}(\lambda\alpha\nu)| = |(V\alpha)\nu| = |V\beta| = \underset{\sim}{m}.$$

The same does not happen with the mapping $\lambda\beta\nu$. We have ran $\lambda = V \subset$ dom β but

$$V\beta \cap \text{dom } \nu = V\beta \cap V\alpha = \emptyset,$$

and so

$$|\text{dom } (\lambda\beta\nu)| = |\text{ran } (\lambda\beta\nu)| = \underset{\sim}{0} < \underset{\sim}{m}.$$

Hence in the case where $|D(\alpha,\beta)| = \underset{\sim}{m}$ we found λ and ν both in $L_{\underset{\sim}{m}}$ such that

$$\lambda\alpha\beta \neq 0 \quad \text{and } \lambda\beta\nu = 0,$$

as required.

The remaining cases in which either $|\text{dom } \alpha \setminus \text{dom } \beta| = \underset{\sim}{m}$ or $|\text{dom } \beta \setminus \text{dom } \alpha| = \underset{\sim}{m}$ (or both) are identical to each other.

Let us therefore take α,β in $K_{\underset{\sim}{m}} \setminus P_{\underset{\sim}{m}}$ and suppose that

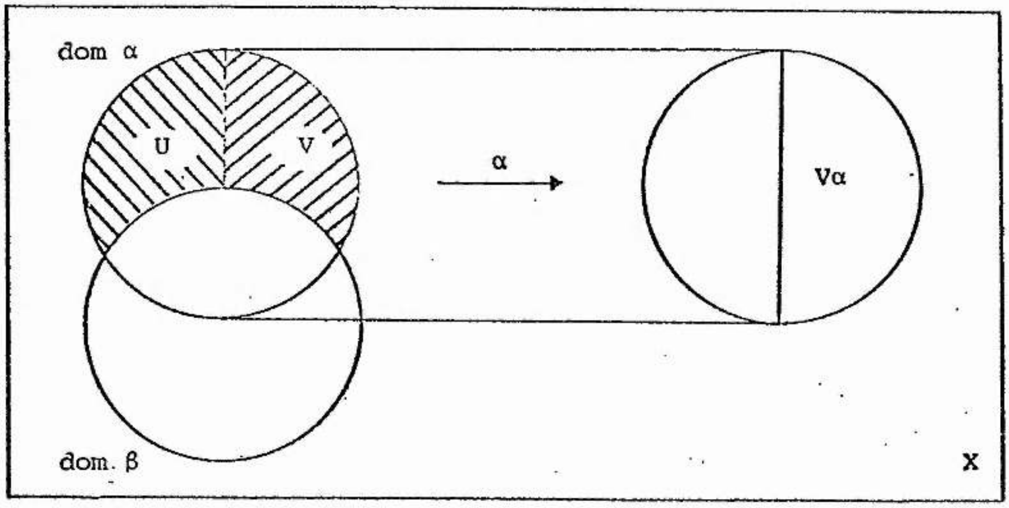
$$|\text{dom } \alpha \setminus \text{dom } \beta| = \underset{\sim}{m}.$$

Let dom $\alpha \setminus$ dom $\beta = U \cup V$ be a partition of dom $\alpha \setminus$ dom β, where $|U| = |V| = \underset{\sim}{m}$. We define $\lambda : U \longrightarrow V$ to be a bijection and we define ν to be the identity map of $V\alpha$, i.e., $\nu = 1_{V\alpha}$. It is clear that $\lambda \in L_{\underset{\sim}{m}} \setminus \{0\}$. Also since α is one-to-one, $|V| = |U| = \underset{\sim}{m}$ give $|U\alpha| = |V\alpha| = \underset{\sim}{m}$ and since

$$\text{def } \nu = \text{gap } \nu \supset U\alpha$$

it now follows that $\nu \in L_{\underset{\sim}{m}} \setminus \{0\}$. We have the following Venn diagram.

[See overleaf for diagram.]



Now

$$\text{dom }(\lambda\alpha\nu) = U, \quad \text{ran }(\lambda\alpha\nu) = V\alpha$$

and so $\lambda\alpha\nu \in L_{\underset{\sim}{m}} \setminus \{0\}$. On the other hand,

$$\text{ran }\lambda \cap \text{dom }\beta = V \cap \text{dom }\beta = \emptyset,$$

and so $\lambda\beta = 0$, giving $\lambda\beta\nu = 0$. Thus, and as in the previous case, we have defined two maps λ and ν both in $L_{\underset{\sim}{m}}$ for which

$$\lambda\alpha\nu \neq 0 \text{ and } \lambda\beta\nu = 0.$$

As already mentioned, this implies that $L_{\underset{\sim}{m}}^*$ is congruence-free.

We can now state the following theorem.

THEOREM 4.24. *Let* X *be a set with infinite cardinal* $\underset{\sim}{m}$ *and let* $L_{\underset{\sim}{m}}' = (K_{\underset{\sim}{m}} \setminus P_{\underset{\sim}{m}}) \cup \{0\}$ *be the semigroup defined in* (4.19). *Let* $\hat{\delta}_{\underset{\sim}{m}}$ *be the congruence defined in* $L_{\underset{\sim}{m}}$ *by* (4.22) *and denote* $L_{\underset{\sim}{m}}/\hat{\delta}_{\underset{\sim}{m}}$ *by* $L_{\underset{\sim}{m}}^*$. *Then* $L_{\underset{\sim}{m}}^*$ *is a congruence-free, o-bisimple, inverse and nilpotent-generated semigroup.*

CHAPTER 5

EMBEDDINGS AND CARDINALITIES

1. INTRODUCTION

When a new subsemigroup of $\mathcal{T}(X)$ or $\mathcal{I}(X)$ is introduced and described a relation between this particular semigroup and an arbitrary semigroup (or inverse semigroup) is frequently obtained, usually in the form of an embedding theorem. The process of embedding tends to be similar for many different cases, even though the semigroups may differ in their properties.

In 1963 [35] Šutov showed that an arbitrary semigroup can always be embedded in a congruence-free semigroup, using two different methods. Later, in 1972, Munn [24] provided a variant of one of these methods to establish another form of embedding result. His method makes use of the full transformation semigroup $\mathcal{T}(X)$ and is based on Mal'cev's theory of congruences on $\mathcal{T}(X)$.

More recently, using Bruck-Reilly extensions, A. Clement and F. Pastijn [3] provided a way of embedding an infinite o-bisimple semigroup into a bisimple semigroup of the same infinite cardinality.

In this chapter a series of two embedding theorems is provided. Both theorems stated in section 2 and section 3 are closely related to the work presented in Chapter 2 and Chapter 4 of this thesis.

Finally, in section 4, we investigate the cardinalities of some of the semigroups introduced in this thesis.

2. EMBEDDING A SEMIGROUP IN A O-BISIMPLE, CONGRUENCE-FREE IDEMPOTENT-GENERATED SEMIGROUP

In 1979 [9] T. E. Hall showed that every semigroup is embeddable in a bisimple, idempotent-generated congruence-free semigroup. Two years later [16] Howie achieved the same result using a different method altogether. Following Howie's method and applying the results presented in Chapter 2 of this thesis, we now show how to embed any semigroup in a different idempotent-generated, congruence-free semigroup.

Let S be a semigroup with $1 < |S| \leq \underset{\sim}{m}$, where $\underset{\sim}{m}$ is an infinite cardinal number. Let Y and Z be mutually disjoint sets of cardinality $\underset{\sim}{m}$. Then

$$X = S^1 \times (Y \cup Z)$$

is a set of cardinality $\underset{\sim}{m}$. For each $a \in S$ define ρ_a in $\mathcal{T}(X)$ by

$$(s,y)\,\rho_a = (sa,y)\,, \qquad s \in S^1,\ y \in Y$$
$$(s,z)\,\rho_a = (1,z)\quad. \qquad s \in S^1,\ z \in Z$$

It is very easy to verify that for all a,b in S

$$\rho_a \cdot \rho_b = \rho_{ab} \quad , \qquad \rho_a = \rho_b \Rightarrow \quad a = b\ ,$$

and so the mapping $\Phi : a \longmapsto \rho_a$ embeds S in $\mathcal{T}(X)$.

Moreover, since for each $a \in S$

$C(\rho_a) \supseteq S^1 \times Z$ and def $\rho_a \supseteq (S^1 \setminus \{1\}) \times Z$, it follows that

$$|\,C(\rho_a)\,| = |\,\text{def}\ \rho_a\,| = |\,Z\,| = \underset{\sim}{m},$$

and so $\rho_a \in \underset{\sim}{Q}_{\underset{\sim}{m}}$. Also, ran $\rho_a \supseteq \{1\} \times Z$ giving

$$|\,\text{ran}\ \rho_a\,| = \underset{\sim}{m}.$$

Hence $\rho_a \in P_{\underset{\sim}{m}}$, where $P_{\underset{\sim}{m}}$ is the semigroup introduced in Chapter 2 (2.2). Therefore, Φ embeds S in $P_{\underset{\sim}{m}}$.

We now recall (Theorem 2.11) that

$$P_{\underset{\sim}{m}}^{\,*} = P_{\underset{\sim}{m}} / \hat{\Delta}_{\underset{\sim}{m}},$$

where $\hat{\Delta}_{\underset{\sim}{m}}$ is the congruence defined by

$$\hat{\Delta}_{\underset{\sim}{m}} = \{(\alpha,\beta) \in J_{\underset{\sim}{m}} \times J_{\underset{\sim}{m}} : \max\left(|\,D(\alpha,\beta)\alpha\,|\ ,\ |\,D(\alpha,\beta)\beta\,| < \underset{\sim}{m}\right\} \cup \{(0,0)\}$$

We shall prove that if $a,b \in S$ are such that $a \neq b$ then $(\rho_a, \rho_b) \notin \hat{\Delta}_{\underset{\sim}{m}}$. To see this notice that since Φ is an embedding and $a \neq b$ we have that $\rho_a \neq \rho_b$ and so

$$D(\rho_a, \rho_b) = \overline{S} \times Y,$$

where $\overline{S} = \{s \in S^1 : sa \neq sb\} \neq \emptyset$. Hence it is clear that

$$D(\rho_a, \rho_b)\rho_a = \overline{S}a \times Y$$

and that

$$D(\rho_a, \rho_b)\rho_b = \overline{S}b \times Y.$$

Thus it follows that

$$|D(\rho_a, \rho_b)\rho_a| = |D(\rho_a, \rho_b)\rho_b| = |Y| = \underset{\sim}{m},$$

and so $(\rho_a, \rho_b) \notin \hat{\Delta}_{\underset{\sim}{m}}$. Therefore, the composition

$$S \xrightarrow{\ \Phi\ } P_{\underset{\sim}{m}} \xrightarrow{\ \hat{\Delta}^{\natural}_{\underset{\sim}{m}}\ } P_{\underset{\sim}{m}}^*$$

is an embedding of S in $P_{\underset{\sim}{m}}^*$. It now follows from Theorem 2.11 that

THEOREM 5.1 *If* $\underset{\sim}{m}$ *is an infinite cardinal number, then the o-bisimple, idempotent-generated, congruence-free semigroup* $P_{\underset{\sim}{m}}^*$ *contains an isomorphic copy of every semigroup of order not exceeding* $\underset{\sim}{m}$.

3. EMBEDDING AN INVERSE SEMIGROUP IN A CONGRUENCE-FREE INVERSE NILPOTENT-GENERATED SEMIGROUP

In section 4 of Chapter 4 a o-bisimple, inverse, nilpotent-generated congruence-free semigroup is described. Recall that if $\underset{\sim}{m}$ is an infinite cardinal number and if $K_{\underset{\sim}{m}}$ is the subsemigroup of $\mathscr{I}(X)$ defined by

$$K_{\underset{\sim}{m}} = \{\alpha \in \mathscr{I}(X) : |\text{gap } \alpha| = |\text{def } \alpha| = \underset{\sim}{m}\},$$

then

$$L_{\underset{\sim}{m}} = \{\alpha \in K_{\underset{\sim}{m}} : |\text{dom } \alpha| = |\text{ran } \alpha| = \underset{\sim}{m}\} \cup \{0\}$$

is a o-bisimple, inverse, nilpotent-generated semigroup which is not congruence-free. However, the semigroup

$$L_{\underset{\sim}{m}}^{*} = L_{\underset{\sim}{m}} / \hat{\delta}_{\underset{\sim}{m}} ,$$

where $\hat{\delta}_{\underset{\sim}{m}}$ is the congruence defined in $L_{\underset{\sim}{m}}$ by

$$\hat{\delta}_{\underset{\sim}{m}} = \{(\bar{\alpha},\bar{\beta}) \in L_{\underset{\sim}{m}} \times L_{\underset{\sim}{m}} : |(\alpha\setminus\beta) \cup (\beta\setminus\alpha)| < \underset{\sim}{m}\}, \qquad (5.2)$$

has the same properties as $L_{\underset{\sim}{m}}$ and it is congruence-free.

We shall show in this section that every inverse semigroup of cardinality not greater than $\underset{\sim}{m}$ is embeddable in $L_{\underset{\sim}{m}}^{*}$.

Let S be an inverse semigroup such that $|S| > \underset{\sim}{1}$. Let $\underset{\sim}{m}$ be an infinite cardinal number such that $\underset{\sim}{m} \geq |S|$, and define

$$X = S \times (Z \cup W) ,$$

where $|Z| = |W| = \underset{\sim}{m}$, and $Z \cap W = \emptyset$. Then $|X| = \underset{\sim}{m}$. For all $a \in S$ define ϕ_a by

$$\operatorname{dom} \phi_a = Saa^{-1} \times Z$$
$$(saa^{-1}, z)\,\phi_a = (sa, z) ,$$

where $s \in S$ and $z \in Z$. Since the Vagner-Preston representation is faithful is follows that $\phi_a \in \mathscr{I}(X)$ and that the map $\Phi : a \longmapsto \phi_a$ embeds S in $\mathscr{I}(X)$.

Notice now that

$$|\operatorname{dom} \phi_a| \geq \underset{\sim}{1} \times |Z| = \underset{\sim}{m}$$

and so

$$|\operatorname{dom} \phi_a| = |\operatorname{ran} \phi_a| = \underset{\sim}{m}.$$

Also, since $S \times W \subseteq \operatorname{def} \phi_a$ and $S \times W \subseteq \operatorname{gap} \phi_a$ it follows that

$$|\operatorname{gap} \phi_a| = |\operatorname{def} \phi_a| = \underset{\sim}{m}.$$

Hence $\phi_a \in L_{\underset{\sim}{m}} \setminus \{0\}$. Thus we have

LEMMA 5.3. *The map* $\Phi : a \longmapsto \phi_a$ *embeds* S *in* $L_{\underset{\sim}{m}}$.

It also follows from the fact that the Vagner-Preston representation $a \mapsto \rho_a$ is faithful that

$$(\rho_a \setminus \rho_b) \cup (\rho_b \setminus \rho_a) \neq \emptyset,$$

for all $a,b \in S$ such that $a \neq b$. Hence there exists

$$(x,y) \in (\rho_a \setminus \rho_b) \cup (\rho_b \setminus \rho_a)$$

and so, for each $z \in Z$,

$$((x,z), (y,z)) \in (\phi_a \setminus \phi_b) \cup (\phi_b \setminus \phi_a),$$

giving

$$|(\phi_a \setminus \phi_b) \cup (\phi_b \setminus \phi_a)| = |Z| = \underset{\sim}{m}.$$

Therefore, by (5.2), $(\phi_a, \phi_b) \notin \widehat{\delta}_{\underset{\sim}{m}}$. Hence we have that the composition

$$S \xrightarrow{\ \Phi\ } L_{\underset{\sim}{m}} \xrightarrow{\ \widehat{\delta}^{\natural}_{\underset{\sim}{m}}\ } L_{\underset{\sim}{m}}^{*}$$

is an embedding of S in $L_{\underset{\sim}{m}}^{*}$. Thus we have

THEOREM 5.4. *Every inverse semigroup is embeddable in a o-bisimple, inverse, nilpotent-generated congruence-free semigroup.*

Recently [18] H. Leemans and F. Pastijn described a way of embedding an infinite inverse semigroup of cardinality $\underset{\sim}{m}$ in a bisimple, congruence-free inverse semigroup of the same cardinality.

They also provide a method of embedding every finite inverse semigroup in a bisimple, congruence-free inverse semigroup of cardinality $\aleph_0$

4. CARDINALITIES

In this thesis some semigroups of particular interest were introduced and investigated, namely the semigroups $P_{\underset{\sim}{m}}$, $K_{\underset{\sim}{m}}$ and $L_{\underset{\sim}{m}}$ ($\underset{\sim}{m}$ being an arbitrary infinite cardinal).

One of the many questions that arose as the above semigroups were studied was to know how "big" they were. Our guess was that they all would have the same cardinality $2^{\underset{\sim}{m}}$. That this is the case will be shown in this final section. We shall also prove that

$$|S_{\underset{\sim}{m}}^{*}| = |P_{\underset{\sim}{m}}^{*}| = |K_{\underset{\sim}{m}}^{*}| = |L_{\underset{\sim}{m}}^{*}| = 2^{\underset{\sim}{m}}.$$

We start by reminding ourselves of a well-known result in set theory.

LEMMA 5.5. *In a set* X *of cardinality* $\underset{\sim}{m}$ *there are* $2^{\underset{\sim}{m}}$ *subsets* A *such that* $|A| = |X \setminus A| = \underset{\sim}{m}$

Proof. Let X be an infinite set such that $|X| = \underset{\sim}{m}$.

Let

$$X = P \cup Q \cup R$$

be a disjoint union where $|P| = |Q| = |R| = \underset{\sim}{m}$. Notice that for any subset C of Q, we have that

$$|P \cup C| = |R \cup (Q \setminus C)| = \underset{\sim}{m},$$

i.e.

$$|P \cup C| = |X \setminus (P \cup C)| = \underset{\sim}{m}.$$

Hence lemma 5.5 follows, for there are $2^{\underset{\sim}{m}}$ subsets C of Q.

Recall now (see Chapter 1) that

$$Q_{\underset{\sim}{m}} = \{\alpha \in \mathscr{T}(X) : |C(\alpha)| = |S(\alpha)| = |\text{def } \alpha| = \underset{\sim}{m}\},$$

and that

$$S_{\underset{\sim}{m}} = \{\alpha \in Q_{\underset{\sim}{m}} : (\forall y \in \text{ran } \alpha)\ |y\alpha^{-1}| < \underset{\sim}{m},\ |\text{ran } \alpha| = \underset{\sim}{m}\}.$$

Since

$$P_{\underset{\sim}{m}} = \{\alpha \in Q_{\underset{\sim}{m}} : |\text{ran } \alpha| = \underset{\sim}{m}\} \cup \{0\},$$

it follows that

$$|S_{\underset{\sim}{m}}| \leq |P_{\underset{\sim}{m}}| \leq |Q_{\underset{\sim}{m}}| \leq |\mathscr{T}(X)|. \tag{5.6}$$

It is well known that $|\mathscr{T}(X)| = 2^{\underset{\sim}{m}}$. Hence, if we show that $|S_{\underset{\sim}{m}}| = 2^{\underset{\sim}{m}}$ it will follow from (5.6) that

$$|S_{\underset{\sim}{m}}| = |P_{\underset{\sim}{m}}| = |Q_{\underset{\sim}{m}}| = |\mathscr{T}(X)| = 2^{\underset{\sim}{m}}.$$

Our next lemma is as follows:

LEMMA 5.7. $|S_{\underset{\sim}{m}}| = 2^{\underset{\sim}{m}}$.

Proof. Define

$$T = \{A \subseteq X : |A| = |X \setminus A| = \underset{\sim}{m}\}. \tag{5.8}$$

We already know by lemma 5.5 that $|T| = 2^{m}$. Hence lemma 5.7 will follow if we produce an injection θ from T into S_m.

Take $A \in T$ and let

$$X = \bigcup_{i \in I} X_i$$

be a disjoint union, where $|I| = m$ and $|X_i| = 2$ for all $i \in I$. Let $f: I \longrightarrow A$ be a bijection and define α in $\mathscr{T}(X)$ by

$$X_i \alpha = if \qquad (i \in I)$$

Hence it is clear that $\text{ran}\, \alpha = A$ and so, since $A \in T$ it follows that

$$|\text{ran}\, \alpha| = |\text{def}\, \alpha| = m.$$

Also $|C(\alpha)| = |X| = m$ and

$$|y\alpha^{-1}| = 2 < m,$$

for all $y \in \text{ran}\, \alpha$. Therefore $\alpha \in S_m$. It is obvious that the map $A \longmapsto \alpha$ maps T into S_m in a one-to-one manner. Hence,

$$2^{m} = |T| \leqslant |S_m| \leqslant |\mathscr{T}(X)| = 2^{m},$$

giving $|S_m| = |2^{m}$, as required.

LEMMA 5.9. $|L_m| = |K_m| = 2^{m}$.

Proof. Let us remind ourselves first that

$$L_m = \{\alpha \in K_m : |\text{dom}\, \alpha| (= |\text{ran}\, \alpha|) = m\} \cup \{0\},$$

where K_m is the subsemigroup of $\mathscr{I}(X)$ defined by

$$K_m = \{\alpha \in \mathscr{I}(X) : |\text{gap}\, \alpha| = |\text{def}\, \alpha| = m\}.$$

Now take $A \in T$, where T is the set defined in (5.8). Then $|A| = |X \setminus A| = m$. Let $f : A \longrightarrow X \setminus A$ be a bijection. Hence it follows that $f \in L_m \setminus \{0\}$. Also it is obvious that the map $A \longrightarrow f$ is an injection from T into L_m. Therefore

$$2^{m} = |T| \leqslant |L_m|.$$

Observe now that

$$2^{m} = |T| \leqslant |L_m| \leqslant |K_m| \leqslant |\mathscr{I}(X)|,$$

and so, since $| \mathscr{I}(X) | = 2^{\underset{\sim}{m}}$, it follows that

$$|L_{\underset{\sim}{m}}| = |K_{\underset{\sim}{m}}| = 2^{\underset{\sim}{m}}.$$

Next recall that

$$S_{\underset{\sim}{m}}^{*} = S_{\underset{\sim}{m}}/\Delta_{\underset{\sim}{m}},$$

where $\Delta_{\underset{\sim}{m}}$ is the congruence defined in $S_{\underset{\sim}{m}}$ by

$$\Delta_{\underset{\sim}{m}} = \{(\alpha,\beta) \in S_{\underset{\sim}{m}} \times S_{\underset{\sim}{m}} \; ; \; \max (| D(\alpha,\beta)\alpha | , | D(\alpha,\beta)\beta |) < \underset{\sim}{m}\}.$$

Notice that if $\underset{\sim}{m}$ is not a regular cardinal then $S_{\underset{\sim}{m}}$ is just a set and $\Delta_{\underset{\sim}{m}}$ is an equivalence relation defined on it.

We have the following lemma:

LEMMA 5.10 $| S_{\underset{\sim}{m}}^{*} | = 2^{\underset{\sim}{m}}$.

Proof. Consider $\alpha \in S_{\underset{\sim}{m}}$ and denote by $[\alpha]$ the $\Delta_{\underset{\sim}{m}}$-class containing α. For each $\underset{\sim}{k} < \underset{\sim}{m}$ define

$$\mathscr{S}_{\underset{\sim}{k}} = \{Z \subseteq X : | Z | = \underset{\sim}{k}\},$$

and for each Z in $\mathscr{S}_{\underset{\sim}{k}}$ define

$$B(Z,\alpha) = \{\beta \in S_{\underset{\sim}{m}} : D(\alpha,\beta) = Z\}.$$

Let

$$A_{\underset{\sim}{k}} = \bigcup_{Z \in \mathscr{S}_{\underset{\sim}{k}}} B(Z,\alpha) \tag{5.11}$$

Then

$$[\alpha] = \bigcup_{\underset{\sim}{k} < \underset{\sim}{m}} A_{\underset{\sim}{k}} . \tag{5.12}$$

Now if we consider $| B(Z,\alpha) |$ we see that for each of the $\underset{\sim}{k}$ elements z of Z there are $\underset{\sim}{m}$ choices for $z\beta$. Hence

$$| B(Z,\alpha) | = \underset{\sim}{m}^{\underset{\sim}{k}} = \underset{\sim}{m}.$$

Next, let us think of $| \mathscr{S}_{\underset{\sim}{k}} |$. In forming a set Z in $\mathscr{S}_{\underset{\sim}{k}}$ each of the $\underset{\sim}{k}$ elements of Z can be selected in $\underset{\sim}{m}$ ways. Therefore

$$| \mathscr{S}_{\underset{\sim}{k}} | = \underset{\sim}{m}^{\underset{\sim}{k}} = \underset{\sim}{m}.$$

Hence by (5.11) $| A_{\underset{\sim}{k}} | = \underset{\sim}{m}^2 = \underset{\sim}{m}$ $(\underset{\sim}{k} < \underset{\sim}{m})$, and then by (5.12) it follows that $| [\alpha] | = \underset{\sim}{m}$.

Now $S_{\underset{\sim}{m}}$ is the union over $S_{\underset{\sim}{m}}^{*}$ of all the sets $[\alpha]$. Hence, since $|S_{\underset{\sim}{m}}| = 2^{\underset{\sim}{m}}$ it follows that $2^{\underset{\sim}{m}} = |S_{\underset{\sim}{m}}^{*}|\,\underset{\sim}{m}$ and so

$$|S_{\underset{\sim}{m}}^{*}| = 2^{\underset{\sim}{m}},$$

as required.

Notice next that since $S_{\underset{\sim}{m}}^{*} \subset P_{\underset{\sim}{m}}^{*}$ we also have that

$$2^{\underset{\sim}{m}} \leqslant |P_{\underset{\sim}{m}}^{*}| \leqslant |P_{\underset{\sim}{m}}| = 2^{\underset{\sim}{m}}$$

and hence

$$|P_{\underset{\sim}{m}}^{*}| = 2^{\underset{\sim}{m}}$$

Finally recall that

$$L_{\underset{\sim}{m}}^{*} = L_{\underset{\sim}{m}} / \hat{\delta}_{\underset{\sim}{m}},$$

where $\hat{\delta}_{\underset{\sim}{m}}$ is the congruence defined in $L_{\underset{\sim}{m}}$ by

$$\hat{\delta}_{\underset{\sim}{m}} = \{(\bar{\alpha},\bar{\beta}) \in L_{\underset{\sim}{m}} \times L_{\underset{\sim}{m}} : |(\alpha \setminus \beta) \cup (\beta \setminus \alpha)| < \underset{\sim}{m}\}.$$

A similar argument to that used in the proof of lemma 5.10 allows us to obtain next lemma:

LEMMA 5.13. $|L_{\underset{\sim}{m}}^{*}| = 2^{\underset{\sim}{m}}$.

Proof. Let $\alpha \in L_{\underset{\sim}{m}} \setminus \{0\}$ and denote by $[\alpha]$ the $\hat{\delta}_{\underset{\sim}{m}}$-class containing α. For each $\underset{\sim}{k} < \underset{\sim}{m}$ define

$$\mathscr{P}_{\underset{\sim}{k}} = \{Z \subseteq X : |Z| = \underset{\sim}{k}\},$$

and for each Z in $\mathscr{P}_{\underset{\sim}{k}}$ define

$B(Z,\alpha) = \{\beta \in K_{\underset{\sim}{m}} : |(\text{dom }\alpha \setminus \text{dom }\beta) \cup (\text{dom }\beta \setminus \text{dom }\alpha)| < \underset{\sim}{m}, \text{ and } D(\alpha,\beta) = Z\}$,

where $D(\alpha,\beta) = \{\underline{x} \in \text{dom }\alpha \cap \text{dom }\beta : x\alpha \neq x\beta\}$. Hence if

$$A_{\underset{\sim}{k}} = \bigcup_{Z \in \mathscr{P}_{\underset{\sim}{k}}} B(Z,\alpha) \tag{5.14}$$

we have that

$$\alpha = \bigcup_{\underset{\sim}{k} < \underset{\sim}{m}} A_{\underset{\sim}{k}} \tag{5.15}$$

In order to find $|B(Z,\alpha)|$ we have to find out first how many choices we have for dom β and then, once dom β is defined, how many choices we have for $\underline{z}\beta$, for each of the $\underset{\sim}{k}$ elements $\underline{z}$ of Z.

We first show that

$$|\{C \subseteq \text{dom }\alpha : |C| < \underset{\sim}{m}\}| = \underset{\sim}{m}.$$

For each $\underset{\sim}{q} < \underset{\sim}{m}$, let $A_{\underset{\sim}{q}} = \{C \subseteq \text{dom }\alpha : |C| = \underset{\sim}{q}\}$. In forming a subset C in $A_{\underset{\sim}{q}}$, each of the $\underset{\sim}{q}$ elements of C can be selected in $\underset{\sim}{m}$ ways. Hence

$$|A_{\underset{\sim}{q}}| = \underset{\sim}{m}^{\underset{\sim}{q}} = \underset{\sim}{m},$$

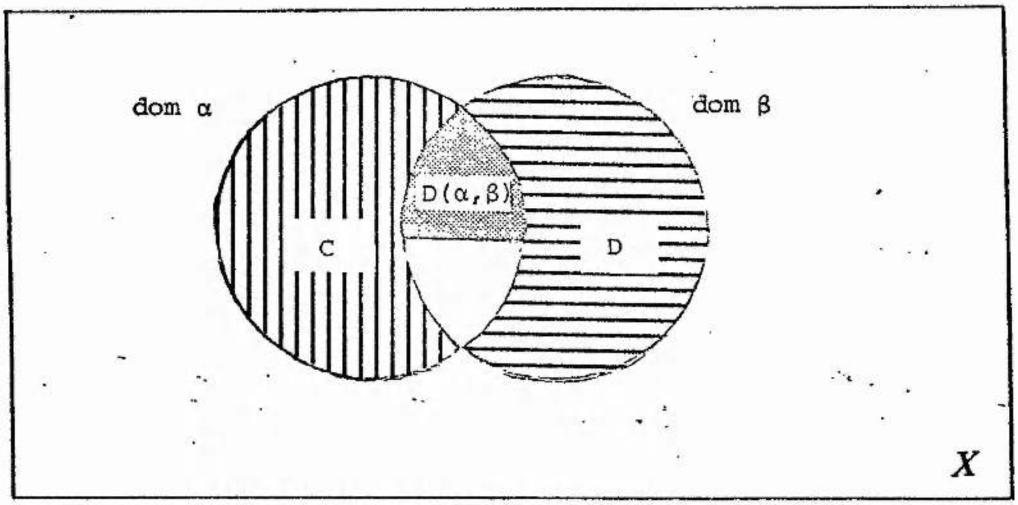
for each $\underset{\sim}{q} < \underset{\sim}{m}$, and so

$$|\{C \subseteq \text{dom }\alpha : |C| < \underset{\sim}{m}\}| = |\bigcup_{\underset{\sim}{q} < \underset{\sim}{m}} A_{\underset{\sim}{q}}| = \underset{\sim}{m}^2 = \underset{\sim}{m}. \tag{5.16}$$

Similarly

$$|\{A \subset \text{gap }\alpha : |A| < \underset{\sim}{m}\}| = \underset{\sim}{m}. \tag{5.17}$$

We now have the following diagram:



Notice that dom β = (dom $\alpha \cup$ D) $\setminus$ C, that is, dom β is determined by the choice of C and D, a subset of dom α and a subset of gap α, respectively, both of cardinality less than $\underset{\sim}{m}$. Hence since by (5.16) and (5.17) there are just $\underset{\sim}{m}$ choices for each of C and D, there are $\underset{\sim}{m}$ choices for dom β.

Finally, for each of the $\underset{\sim}{k}$ elements z of Z there are $\underset{\sim}{m}$ choices for zβ. Therefore, for each choice of dom β there are $\underset{\sim}{m}^{\underset{\sim}{k}}$ choices for $\underline{z}\beta$, for each z in Z. Hence we have

$$|B(Z,\alpha)| = \underset{\sim}{m}.\underset{\sim}{m}^{\underset{\sim}{k}} = \underset{\sim}{m}^{\underset{\sim}{k}+1} = \underset{\sim}{m}.$$

We now think of $|\mathscr{P}_{\underset{\sim}{k}}|$. In forming a set Z in $\mathscr{P}_{\underset{\sim}{k}}$ each of the $\underset{\sim}{k}$ elements of Z can be selected in $\underset{\sim}{m}$ ways. Thus

$$|\mathscr{P}_{\underset{\sim}{k}}| = \underset{\sim}{m}^{\underset{\sim}{k}} = \underset{\sim}{m}.$$

Hence by (5.14) $|A_{\underset{\sim}{k}}| = \underset{\sim}{m}^2 = \underset{\sim}{m}$ ($\underset{\sim}{k} < \underset{\sim}{m}$) and then from (5.15) it follows that $|[\alpha]| = \underset{\sim}{m}$.

Next, since $L_{\underset{\sim}{m}}$ is the union over $L^*_{\underset{\sim}{m}}$ of all the classes [α] together with the zero class and since $|L_{\underset{\sim}{m}}| = 2^{\underset{\sim}{m}}$ we have that $|L^*_{\underset{\sim}{m}}|\underset{\sim}{m} = 2^{\underset{\sim}{m}}$. Hence

$$|L_{\underset{\sim}{m}}^*| = 2^{\underset{\sim}{m}}.$$

Notice now that $|L_{\underset{\sim}{m}}^*| \leqslant |K_{\underset{\sim}{m}}^*|$ gives $|K_{\underset{\sim}{m}}^*| \geqslant 2^{\underset{\sim}{m}}$ and so

$$|K_{\underset{\sim}{m}}^*| = 2^{\underset{\sim}{m}}.$$

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