COMPUTING WITH SIMPLE GROUPS : MAXIMAL SUBGROUPS AND PRESENTATIONS

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BY

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ABSTRACT

For the non-abelian simple groups G of order up to 10^6, excluding the groups PSL(2,q), q > 9, the presentations in terms of an involution a and an element b of minimal order (with respect to a) such that G=<a,b> are well known. The presentations are complete in the sense that any pair (x,y) of generators of G satisfying $x^2=y^m=1$, with m minimal, will satisfy the defining relations of just one presentation in the list. There are 106 such presentations.

Using a computer, we give generators for each maximal subgroup of the groups G. For each presentation of G, the generators of maximal subgroups are given as words in the group generators. Similarly generators for a Sylow p-subgroup of G, for each p, are given. For each group G, we give a representative for each conjugacy class of the group as a word in the group generators.

Minimal presentations for each Sylow p-subgroup of the groups G, and for most of the maximal subgroups of G are constructed. To obtain such presentations, the Schur multipliers of the underlying groups are calculated.

The same tasks are carried out for those groups PSL(2,q) of order less than 10^6 which are included in the "ATLAS of finite groups" (J H Conway et al., Clarendon Press, Oxford, 1985). For these groups we consider a presentation on two generators x, y with $x^2=y^3=1$.

A finite group G is said to be efficient if it has a presentation on d generators and d+rank(M(G)) relations (for some d) where M(G) is the Schur multiplier of G. We show that the simple groups J_1, PSU(3,5) and M_{22} are efficient. We also give efficient presentations for the direct products $\hat{A}_5 x A_6$, $\hat{A}_5 x \hat{A}_6$, $A_6 x A_6$, $A_6 x A_7$ where $\hat{H}$ denotes the covering group of H.

DECLARATIONS

(a) I hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for a higher degree.

Signed Date 3 Oct 1988

(b) I was admitted to the Faculty of Science of the University of St. Andrews under Ordinance General No 12 on 1/10/85 and as a candidate for the degree of Ph.D. on 1/10/86

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I hereby certify that the candidate has fulfilled the conditions of the Resolutions and Regulations appropriate to the degree of Ph.D.

Signature of Supervisor Date

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Preface

I am highly indebted to Dr E.F. Robertson, under whose supervison this work has been carried out, for his invaluable guidance, and instilling an inspiration for the interesting topic of group presentations.

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This thesis is dedicated to my parents whose constant encouragment and love has inspired me in pursuing my studies abroad.

CONTENTS

Notation

(n,m)	Greatest common divisor of integers n, m
$\langle g \rangle$	Cyclic group generated by element g
$\lvert X \rvert$	Cardinality of the set X
$X \setminus Y$	Set of elements in X but not in Y (where $Y \subseteq X$)
$H \leq G$	H is a subgroup of G
$H \triangleleft G$	H is a normal subgroup of G
G/K	Quotient group of G by K (where $K \triangleleft G$)
Hg	Right cosets of H in G containing g (where $H \leq G$, $g \in G$)
$\lvert G:H \rvert$	Index of the subgroup H in the group G
$G_1 \cong G_2$	G_1 is isomorphic to G_2
$\mathrm{Core}_G(H)$	Core of H in G
$\mathrm{St}_G(X)$	Stabilizer of X in G
$\mathrm{fix}(x)$	Fixed points of x (where x is a permutation)
H^G	Normal closure of H in G (where $H \leq G$)
$C_G(x)$	Centralizer of x in G (where $x \in G$)
$N_G(H)$	Normalizer of H in G (where $H \leq G$)
$Z(G)$	Centre of G
$\mathrm{Aut}\, G$	Group of automorphisms of G
K^g	Conjugate of K by g (where $K \leq G$, $g \in G$)
x^y	$y^{-1}xy$ (where $x, y \in G$)
$[x,y]$	Commutator $x^{-1}y^{-1}xy$
G'	Derived group of G
$\mathrm{Ker}\, \phi$	Kernel of homomorphism ϕ
$\Phi(G)$	Frattini subgroup of G
$M(G)$	Schur multiplier of G
$d(G)$	Minimal number of generators of G
$\mathrm{def}\, G$	Deficiency of G
$\mathrm{rank}\, A$	Rank of A (where A is a finite abelian group)
$\langle X \rangle$	Subgroup of G generated by X ($\subseteq G$)
$\langle x_1, x_2, \ldots, x_n \rangle$	$\langle \{x_1, x_2, \ldots, x_n\} \rangle$
X^{-1}	$\{x^{-1} : x \in X\}$ (where $X \subseteq G$)
$\langle X \mid R \rangle$	Group on generators X with relators R
$F(X)$	Free group on X

$F(x_1,...,x_n)$	$F(\{x_1,...,x_n\})$
$l(w)$	Length of the word $w=w(x_1,...,x_n)$ in $F(x_1,...,x_n)$
$H \otimes K$	Tensor product of H and K
D_{2n}	Dihedral group of order 2n ($n \geq 3$)
Q_8	Quaternion group
H: K	Semi-direct product of H by K
H x K	Direct product of H and K
H o K	Central product of H and K

0.Introduction

There are 56 isomorphism classes of non-abelian simple groups of order less than 10^6. Of the classical groups these include groups PSL(2,q), the projective special linear groups of dimension n over GF(q), the unitary groups PSU(3,q) of dimension 3 over GF(q^2) for q=3, 4, 5, and symplectic groups PSp(4,q) of dimension 4 over GF(q) for q=3 and 4, and finally the alternating groups A_5, A_6, A_7, A_8 and A_9. Of the non-classical groups there is the first Suzuki group Sz(8), three Mathieu groups M_{11}, M_{12} and M_{22}, and two 'Sporadic' groups, the first Janko group of order 175560 and the Hall-Janko group of order 604800.

Twenty eight of them are PSL(2,p), p a prime, p=5, ... , 113, and eleven of them are PSL(2,p^n), p a prime, n > 1. The remaining seventeen groups listed by their orders are:

A_7	$2520=2^3.3^2.5.7$
PSL(3,3)	$5616=2^4.3^3.13$
PSU(3,3)	$6048=2^5.3^3.7$
M_{11}	$7920=2^4.3^2.5.11$
$A_8 \cong$ PSL(4,2)	$20160=2^6.3^2.5.7$
PSL(3,4)	$20160=2^6.3^2.5.7$
Sp(4,3) $\cong$ PSU(4,2)	$25920=2^6.3^4.5$
Sz(8)	$29120=2^6.5.7.13$
PSU(3,4)	$62400=2^6.3.5^2.13$
M_{12}	$95040=2^6.3^3.5.11$
PSU(3,5)	$126000=2^4.3^2.5^3.7$
Janko group	$175560=2^3.3.5.7.11.19$
A_9	$181440=2^6.3^4.5.7$
PSL(3,5)	$372000=2^5.3.5^3.31$
M_{22}	$443520=2^7.3^2.5.7.11$
Hall-Janko group	$604800=2^7.3^3.5^2.7$
PSp(4,4)	$979200=2^8.3^2.5^2.17$

These non-abelian simple groups, including the following simple groups

$A_5 \cong$ PSL(2,5) $\cong$ PSL(2,4)	$60=2^2.3.5$
PSL(3,2) $\cong$ PSL(2,7)	$168=2^3.3.7$
$A_6 \cong$ PSL(2,9)	$360=2^3.3^2.5$,

have been investigated from various points of view using a computer. Character tables, maximal subgroups, and intersections of maximal subgroups are given in [34], [21] and [33]. According to an old conjecture all finite simple groups require at most two generators. It is shown in [35] that all twenty simple groups G listed above can be generated by two generators x, y with x an involution.

A pair (a,b) is said to be a *minimal* (2,m,n) *generating pair for* G *with respect to* x if

(i) $\langle a,b\rangle = G$,

(ii) $a = x\theta$ for some automorphism θ of G,

(iii) if $\langle a, c\rangle = G$ then $|\langle c\rangle| \geq |\langle b\rangle| = m$,

(vi) $|\langle ab\rangle| = n$.

A minimal generating pair with respect to x is a minimal generating pair with respect to any element of x^{AutG}. There are 106 such minimal generating pairs tabulated in [35] as permutations of minimal degree for these G, including conjugacy classes of each group and their cycle types and the orders of centralizers of elements.

Presentations satisfied by these permutation generators are given by Cannon, McKay and Young [15] for those groups G with $|G| < 10^5$ and by Campbell and Robertson [6] in the case $10^5 < |G| < 10^6$. In [7] the authors improve the results of [16] by determining the redundant relations of the presentations and by giving exactly two words in a, b for a subgroup of G of minimal index.

For each prime p, we give generators of a Sylow p-subgroup for each of the above groups. For each minimal generating pair the generators of Sylow p-subgroups are given as words in the group generators . For each prime p, a minimal presentation on these generators is constructed. All Sylow 2-subgroups of the above simple groups have order at most 64 with the exception of the Sylow 2-subgroups of M_{22}, J_2 and PSp(4,4) which have orders 128, 128 and 256 respectively. We use the lists given in [22] and [39] to identify each of the 2-groups whose orders are less than or equal 64. We also, for each prime p, calculate the number of Sylow p-subgroups for each simple group G.

Similarly generators of each maximal subgroup are given, as words in the group generators, for each minimal generating pair of the groups. In [21] some maximal subgroups of the simple groups G, $|G| < 10^6$, are missing and there are also some errors in the structure of certain maximal subgroups of the groups. We use the information given in the "ATLAS of finite groups" [17] about the maximal subgroups of G in order to find generators for each maximal subgroup. It is seen that all of the maximal subgroups can be generated by two generators x, y (mostly with x an involution) with the exception of a maximal subgroup of A_7 of order 72 which requires at least three generators. The Schur multiplier of each maximal subgroup which was not already known is calculated. We then give an efficient presentation for each of the obtained maximal subgroups with two exceptions.

For each group G, representatives of the conjugacy classes of G are given in the group generators a, b with (a,b) the first minimal generating pair for G appearing in [35].

The same tasks are carried out for the groups PSL(2,q) of order at most 10^6 which appear in the ATLAS. For these groups we take a single generating pair (a,b) of minimal degree with $a^2=b^3=1$. There are twelve of them which are listed below by their orders:

PSL(2,8) ≅ SL(2,8)	$504=2^3.3^2.7$
PSL(2,11)	$660=2^2.3.5.11$
PSL(2,13)	$1092=2^2.3.7.13$
PSL(2,17)	$2448=2^4.3^2.17$
PSL(2,19)	$3420=2^2.3^2.5.19$
PSL(2,16) ≅ SL(2,16)	$4080=2^4.3.5.17$
PSL(2,23)	$6072=2^3.3.11.23$
PSL(2,25)	$7800=2^3.3.5^2.13$
PSL(2,27)	$9828=2^2.3^3.7.13$
PSL(2,29)	$12180=2^2.3.5.7.29$
PSL(2,31)	$14880=2^5.3.5.31$
PSL(2,32) ≅ SL(2,32)	$32736=2^5.3.11.31$

Presentations for the groups PSL(2,p), p a prime, are well known, see for example [2]. Information on presentations of the groups PSL(2,p^n) is taken from [5], [10].

In Chapter 1 we introduce some definitions and results to be used in later chapters. In Chapter 2 we describe, concisely, the Todd-Coxeter coset enumeration algorithm and some of its applications and include a discussion on computer programs used in obtaining our results. The group theory system CAYLEY [15] is briefly described. It contributes to the solution of some problems relevant to the last two chapters. Chapter 3 contains some computational methods that have been used in connection with the results of Chapter 4 and Chapter 5.

Chapter 4 contains the results on representatives of the conjugacy classes, Sylow p-subgroups, and maximal subgroups of the simple groups G=<a,b> where (a,b) is the first minimal generating pair appearing in [35]. We added our results on representatives of the conjugacy classes, Sylow p-subgroups and maximal subgroups to the file SIMGPS.TLB, a CAYLEY file of finite simple groups [8], which will be introduced in the Appendix.

Chapter 5 is concerned with the problem of efficiency of certain finite simple groups and direct products. The deficiency of a finite presentation $\langle X \mid R \rangle$ is $|X|-|R|$, and the deficiency, defG, of a group G is the maximum of this number taken over all finite presentations of G. A finite group G is said to be efficient if rank M(G)=−defG, where M(G) is the Schur multiplier of G . The simple groups of order less than 10^6 are known to be efficient with the exception of PSU(3,5), J_1, PSL(3,5), M_{22} and PSP(4,4) (see 5.1). We show that PSU(3,5), J_1 and M_{22} are efficient. We also give efficient presentations for the perfect groups $\hat{A}_5 x A_6$, $\hat{A}_5 x \hat{A}_6$, $A_6 x A_6$ and $A_5 x A_7$ where $\hat{G}$ denotes the covering group of G.

The question concerning the occurence of the non-abelian simple groups as composition factors of finite groups of deficiency zero is posed by D. L. Johnson and E. F. Robertson in [28]. Those simple groups whose covering groups are efficient have this property. We show that the simple group PSU(4,2) of order 25920 can occur as a composition factor of a deficiency zero group of order 155520.

1.Definitions and preliminary results

In this chapter we introduce some terminology and notation from a few areas of group theory to be used in later chapters. Results are introduced when they are needed. The last section of the chapter contains a brief discussion of presentations of groups and includes a deeper result on the Schur multiplier and stem extension of a finite group which is needed in trying to find efficient presentations in Chapter 4 and Chapter 5.

1.1 Elementary group theory

If H is a subgroup of a group G by the *core* of H in G we shall mean the subgroup

$$\mathrm{Core}_G(H)=\cap H^x .$$

For a subset S of G, the *normal closure* of S in G is defined to be the intersection of all normal subgroups of G which contain S, and is denoted by S^G.

1.1.1. Let $H \leq G$, S a subset of G. Then

(i) $\mathrm{Core}_G(H)$ is the largest normal subgroup of G that is contained in H.

(ii) S^G is the unique smallest normal subgroup of G containing S.

(ii) $S^G=\langle s^g \mid g\in G, s\in S\rangle$.

The *centre* of a group G is defined by

$$Z(G)=\{x\in G : x^g=x \text{ for all } g\in G\}.$$

For every non-empty subset S of a group G, the *normalizer* of S in G is defined by

$$N_G(S)=\{x\in G: S^x=S\}.$$

A subgroup H of a group G is called *self-normalizing* if $N_G(H)=H$. We say that a subset K of G normalizes a subgroup H if for each $k\in K$ we have $H^k=H$.

The *centralizer* of S in G is defined by

$$C_G(S)=\{x\in G: s^x=s \text{ for all } s\in S\}.$$

When there is no confusion over the group, we shall denote the centralizer and the normalizer of S by simply C(S) and N(S) respectively.

1.1.2. Let $H \leq G$. Then N(H) is the largest subgroup of G in which H is normal, and whenever $H \triangleleft K \leq G$, $K \leq N_G(H)$.

1.1.3. Let G be a finite group and $H \leq G$. Then

(i) $N_G(H)=\{g\in G: H \leq H^g\}=\{g\in G: H^g \leq H\}$.

(ii) $|G: N_G(H)|=|\{H^g: g\in G\}|$.

Let x, y be elements of a group G. By the *commutator* [x,y] of x and y we shall mean the element $x^{-1}y^{-1}xy=x^{-1}x^y$.

If A, B are subgroups of a group G then we shall use the notation $[A,B]=\langle [a,b]:a\in A,b\in B\rangle$. In particular, we call [G,G] the *derived group* of G. We shall denote it by G'. We extend the notation recursively and define $G^{(n)}=[G^{(n-1)},G^{(n-1)}]$ for n >1, where $G^{(1)}=G'$. Define $G^{(0)}=G$. A group G is called *perfect* if G=G'.

1.1.4. Let G be a group and $H \leq G$. Then

(i) $H^{(n)} \leq G^{(n)}$.

(ii) $G' \leq H$ if and only if $H \lhd G$ and G/H is abelian.

Let p be a prime. A group G is called a *p-group* if every element of G has order p^n for some $n \geq 1$. A p-subgroup P of a finite group G is called a *Sylow p-subgroup* of G if P is not properly contained in any p-subgroup of G.

1.1.5. (Sylow). Let G be a finite group and let P be a Sylow p-subgroup of G. Then

(i) Every Sylow p-subgroup is conjugate to P;

(ii) G has 1+kp Sylow p-subgroups, for some k;

(iii) the number $n_p(G)$ of Sylow p-subgroups divides |G|.

1.1.6. Every p-subgroup of a finite group G is contained in a Sylow p-subgroup.

1.1.7. If G is a finite group and if S_p is a Sylow p-subgroup of G of order p^m, then S_p is the only subgroup of G of order p^m lying in $N(S_p)$.

1.1.8. Let G be a finite p-group of order p^n and H be a proper subgroup of G. Then H is properly contained in N(H), hence, if H is a subgroup of order p^{n-1}, then $H \lhd G$.

By a *maximal subgroup* of a group G we shall mean a subgroup H such that $H\neq G$ and for every subgroup K with $H \leq K \leq G$ either K=H or K=G.

The *Frattini subgroup* of a group G is defined to be the intersection of the maximal subgroups of G. $\Phi(G)$ denotes the Frattini subgroup of G.

1.1.9. If X is a subset of G with $G=\langle X, \Phi(G) \rangle$, then $G=\langle X \rangle$.

If $N_0, N_1, ..., N_k$ are subgroups of G such that $1=N_0 \lhd N_1 \lhd N_2 ... \lhd N_k=G$, then we say that the subgroups N_i form a *subnormal series* of G. A subnormal series in which each of the quotients N_i/N_{i-1} is a non-trivial simple group is called a *composition series*. The factors in any composition series are called *composition factors* of G.

A group is said to be *soluble* if it has a subnormal series $1=N_0 \lhd N_1 \lhd N_2 ... \lhd N_k=G$ in which each of the factors N_i/N_{i-1} is abelian.

A group is said to be *nilpotent* if it has a normal series $G=H_0 \geq H_1 \geq ... \geq H_n=1$ such that H_{i-1}/H_i is contained in the centre of G/H_i for each i.

Suppose that G is any group. Let $\Gamma_1=G$ and define recursively $\Gamma_i=[\Gamma_{i-1},G]$, $i \geq 2$. Then the series $G=\Gamma_1 \geq \Gamma_2 \geq \Gamma_3 \geq ...$ is called the *lower central series* of G.

1.1.10. A finite p-group is nilpotent.

1.1.11. A group G is nilpotent if and only if $\Gamma_{c+1}=1$ for some $c \geq 0$.

The integer c of 1.1.11 is called the *class* of the nilpotent group G.

A subgroup H is said to be *invariant* under $\theta \in$ Aut G if $\theta(H) \leq H$. A subgroup H of G is said to be a *characteristic* subgroup if it is invariant under all elements of Aut G.

A group is said to be *characteristically simple* if the only characteristic subgroups of G are 1 and G.

1.1.12. Let G be a non-trivial finite group. Then G is characteristically simple if and only if G is a direct product of finitely many isomorphic copies of a simple group.

1.2 Permutation groups

A *permutation group* on A is a subgroup of S_A, the symmetric group on A. When $A=\{1, 2,..., n\}$ we write S_n for S_A and call it the *symmetric group of degree* n.

If a, b$\in$A we say that $a \sim b$ if and only if there exists $\pi \in G$ with $a\pi=b$. We see easily that $\sim$ is an equivalence relation and we call the equivalence classes the *orbits* of G. For $a \in A$, the orbit containing a is called the *orbit of* a : it is the subset $aG=\{a\pi : \pi \in G\}$ of A. We say that G is *transitive* if, given any pair of elements a, b of A, there exists a permutation π in G such that $a\pi=b$. Thus G is transitive if and only if there is exactly one orbit, namely A itself. Otherwise the group is called *intransitive*.

Let m be a positive integer. We say that G is m-*transitive* if given any two ordered subsets $\{a_1, a_2,..., a_m\}$ and $\{b_1, b_2,..., b_m\}$ of A of size m, then there exists $\pi \in G$ with $a_i\pi=b_i$ (i=1,2,...,m). If $m \geq 2$ then m-transitivity implies (m-1)-transitivity. Note that 1-transitivity is simply transitivity.

If X is a non-empty subset of A the *stabilizer* of X in G, $St_G(X)$, is the set of permutations in G that leave fixed every element of X. Of course $St_G(x)$ stands for $St_G(\{x\})$.

1.2.1. Let G be a permutation group on A and $H \leq G$. Then

(i) $| G : St_G(a) | = |aG|$ for all $a \in A$;

(ii) If H is transitive then so is H^g for all $g \in G$.

1.2.2 Let G be an m-transitive permutation group on A, and suppose that X and Y are subsets of A of cardinality m. Then $St_G(X)$ and $St_G(Y)$ are conjugate in G.

Let G be a permutation group of degree n, we can express every permutation $\pi \in G$ as a product of disjoint cycles in an essentially unique manner, thus

$$\pi = \gamma_1 \gamma_2 \dots \gamma_r \qquad (*)$$

where $\gamma_1, \gamma_2, \dots, \gamma_r$ are disjoint cycles involving

$$m_1, m_2, \dots, m_r \qquad (**)$$

objects. For convenience we retain cycles of unit length so that all n objects are listed in the product (*). The integers (**) are called the *cycle type* of π. Now let π contain e_1 cycles of length n_1, e_2 cycles of length n_2, ..., e_s cycles of length n_s, where

$$n_1 < n_2, \dots < n_s, \ n=m_1+m_2+..+m_r=e_1n_1+n_2e_2+\dots+n_se_s.$$

Then the cycle type of π may be displayed by the pattern

$$n_1^{e1} n_2^{e2} \dots n_s^{es}$$

For example the cycle type of (2,3)(4,5)(7,8,9)(10,11,12)(13,15,17) of degree 17 is $1^4 2^2 3^3$.

1.2.3. Conjugate elements in a permutation group have the same cycle type.

Let G be a group and A a non-empty set. Let ρ be any homomorphism from G to S_A. Such a function is called a *permutation representation* of G on A. Let $\rho : G \to S_A$ be a permutation representation of G on A. The cardinality of A is known as the *degree* of the representation. Next ρ is called *faithful* if Ker $\rho = 1$, so that G is isomorphic to a permutation group on A. Also ρ is said to be *transitive* if Im ρ is a transitive permutation group.

1.2.4. Let H be a subgroup of G and let R be the set of all right cosets of H. For each g in G define $g\rho \in S_R$ by $g\rho : Hx \to Hxg$. Then $\rho : G \to S_R$ is a transitive permutation representation of G on R with kernel $Core_G(H)$.

1.3 Product of groups

If $G_1, G_2, ..., G_n$ are groups then by their *direct product* $G_1 x G_2 x ... x G_n$ we shall mean the group whose underlying set is the set of product of G_i's and whose group operation is defined by

$$(g_1, ..., g_n)(g_1', ..., g_n') = (g_1 g_1', ..., g_n g_n').$$

1.3.1. Let $G = G_1 x G_2 x ... x G_n$ be the direct product of the the groups $G_1, ..., G_n$. Then there are subgroups $H_1, ..., H_n$ of G such that $H_i \cong G_i$ for each i and

(i) $H_i \lhd G$ for i=1, ..., n.

(ii) $H_1 H_2 ... H_n = G$.

(iii) $H_i \cap H_1 ... H_{i-1} H_{i+1} ... H_n = (1)$ for i=1, ..., n.

1.3.2 (Basis theorem for abelian groups). Every finitely generated abelian group G can be written as a direct product of cyclic groups :

$$G = B_1 x ... x B_n$$

where $|B_i| = b_i$ (i=1, ..., r) and $b_{i+1} \mid b_i$ while $B_{r+1}, ..., B_n$ are infinite cyclic.

When n=r, the group G is finite and $|G| = b_1 b_2 ... b_n$; in this case the number r is called the *rank* of G and is denoted by rank G.

An abelian group of exponent p, p a prime, is a direct product of cyclic groups. Such groups are called *elementary abelian* groups.

Let $H \lhd G$. We say that G *splits over* H if there is a subgroup K of G such that G=HK and $H \cap K = (1)$. Any such subgroup K is said to be a *complement to* H *in* G.

Let H and K be groups and let $\varphi : K \to \text{Aut } H$, described by $k \to \varphi_k$, be a group homomorphism. Then by the *semi-direct product of* H *by* K *with action* φ we shall mean the set of ordered pairs (k, h), $k \in K$, $h \in H$, with operation

$$(k, h)(k', h') = (kk', \varphi_{k'}(h) h').$$

We shall denote this by $H x_{\varphi} K$ or $H : \varphi K$ (or simply by H : K).

1.3.3.

(i) Suppose that $G = H \times_{\varphi} K$. Then G splits over H, and K is a complement to H in G.

(ii) Let $H \lhd G$. Suppose that G splits over H, and let K be a complement to H in G. Define $\varphi : K \to \text{Aut } H$ by $k \to \varphi_k$, where $\varphi_k(h) = h^k$. Then $G \cong H x_{\varphi} K$.

Let H and K be two groups with common central subgroup C. Formally we have injections $\varphi : C \rightarrow H$ and $\psi : C \rightarrow K$ of the abstract group C into the centres of H and K. The *central product of* H *and* K *over* C is defined to be the quotient of HxK by the set of ordered pairs of the form ($\varphi(c)$, $\psi(c)$) ($c \in C$). When it is omitted from the notation, C is usually understood to take the largest possible value. Central products may have more than two factors. We shall denote this group by $H \circ_C K$ (or simply by $H \circ K$).

1.3.4.

(i) Suppose that $G = H \circ_C K$. Then there are normal subgroups G_1 and G_2 of G with $G_1 \cong H$ and $G_2 \cong K$ such that $G=G_1G_2$, $[G_1, G_2]=1$ and $G_1 \cap G_2=C$.

(ii) Suppose that $H \lhd G$, $K \lhd G$, and that $G=HK$, $[H, K]=1$, $H \cap K=C$, where C is a central subgroup of G . Then $G \cong H \circ_C K$.

A finite p-group G is called *extra-special* if G' and Z(G) coincide and have order p; for example Q_8 and D_8 are extra-special groups.

The structure of extra-special p-groups is provided by the following theorem .

1.3.5. Let G be an extra-special p-group.Then there exists $n \geq 1$ such that $|G|=p^{2n+1}$ and G is a central product of n non-abelian groups of order p^3 over Z(G).

1.3.6. Let G be an extra-special group of order 2^{2n+1}. Then G is a central product of D_8's or a central product of D_8's and a single Q_8.

1.4 Special and projective special linear groups

Let GL(n,q) denote the group of all n x n non-singular matrices with entries in the field F_q of q elements. The group GL(n,q) is known as the *general linear group* (in dimension n); its subgroup consisting of matrices of determinant 1 is called the *special linear group* and is denoted by SL(n,q). The centre of GL(n,q) consists of the scalar matrices and the corresponding factor group PGL(n,q) is called the *projective linear group*. Finally, the image PSL(n,q) of SL(n,q) in PGL(n,q) is called the *projective special linear group*. The centre of SL(n, q) is a cyclic group of order (n,q-1).

1.4.1 (see [25]).

(i) With the exception of PSL(2,2) and PSL(2,3), all the groups PSL(n, q) are simple.

(ii) With the exception of SL(2,3) and SL(2,2), all the groups SL(n,q) are perfect .

(iii) The following isomorphisms hold :

(a) $PSL(3,2) \cong SL(3,2) \cong PSL(2,7)$,

(b) $PSL(2,4) \cong SL(2,4) \cong PSL(2,5) \cong A_5$,

(c) $PSL(4,2) \cong SL(4,2) \cong A_8$,

(d) $PSL(2,2) \cong SL(2,2) \cong S_3$,

(e) $PSL(2,3) \cong A_4$ and $PSL(2,9) \cong A_6$.

(iv) $|GL(n,q)|=(q-1)N$, $|SL(n,q)|=|PGL(n,q)|=N$, $|PSL(n,q)|=N/d$, where $N=q^{n(n-1)/2}(q^n-1)(q^{n-1}-1)...(q^2-1)$ and $d=(q-1,n)$.

(v) $PSL(2,q)$, $q=p^n$, has $q+1$ Sylow p-subgroups.

(vi) $PSL(2,q)$, with $q \geq 5$, has a subgroup of minimal index $d=q+1$ with the following exceptions :

(a) $q=5$	$d=5$
(b) $q=7$	$d=7$
(c) $q=9$	$d=6$
(d) $q=11$	$d=11$.

1.5 Presentations of groups; Schur multiplier

Suppose that

X is a set ,

$F=F(X)$ is the free group on X,

R is a subset of F,

$N=R^F$ is the normal closure of R in F, and $G=F/N$.

Then, we write $G=\langle X \mid R \rangle$ and call this a *presentation* of G. The elements of X are called *generators* and those of R *relators*. A group G is called *finitely presented* if it has such a presentation with both X and R finite sets.

We shall only be concerned with finite presentations and assume all our presentations will be finite. The importance of presentations is stated in the following theorem

1.5.1. Every group has a presentation, and every finite group is finitely presented.

It is convenient to work with relations rather than relators. A (defining) relation is obtained from the corresponding relator by setting it equal to 1. Conversely if w_1 and w_2 are words in the group generators, the relation $w_1=w_2$ yields the relator $w_1^{-1}w_2$, for example. We then write G as

$$G=\langle X \mid r_i=1, i=1, 2, ..., m \rangle.$$

1.5.2 (Von Dyck). If R, S are subsets of the free group F on a set X with $R \subseteq S$, then there is an epimorphism $\theta : \langle X \mid R \rangle \rightarrow \langle X \mid S \rangle$ which fixes X elementwise. The kernel of θ is just the normal closure of $S \setminus R$ as a subset of $\langle X \mid R \rangle$.

1.5.3 (Substitution Test). Suppose we are given a presentation $G= < X \mid R >$, a group H and a mapping $\theta : X \rightarrow H$. Then θ extends to a homomorphism $\theta' : G \rightarrow H$ if and only if, for all $x \in X$ and $r \in R$, the result of substituting $x\theta$ for x in r yields the identity of H.

1.5.4 If $G=< X \mid R >$ and $H=< Y \mid S >$ are two presentations, then the direct product GxH has the presentation

$$< X, Y \mid R, S, [X, Y] >,$$

where [X, Y] denotes the set of commutators $\{ [x, y] : x \in X, y \in Y \}$.

1.5.5. If $G=< X \mid R >$, then $G/G'=< X \mid R, C >$, where $X= \{ x_1, ..., x_r \}$, $C=\{ [x_i,x_j] : 1 \leq i < j \leq r \}$.

Given a presentation $G=< X \mid R >$, each *Tietze transformation* T_i (i=1,2,3,4) transforms G into the presentation $< X' \mid R' >$ according to the following definitions:

T_1, adding a relator : $X'=X$, $R'=R \cup \{r\}$, where $r \in R^F \backslash R$;

T_2, removing a relator : $X'=X$, $R'=R \backslash \{r\}$, where $r \in R \cap (R \backslash \{r\})^F$;

T_3, adding a generator : $X'=X \cup \{y\}$, $R'= R \cup \{y^{-1}w\}$, where $y \notin X$ and $w \in F(X)$;

T_4, removing a generator : $X'=X \backslash \{y\}$, $R'=R \backslash \{y^{-1}w\}$, where $y \in X$, $w \in F(X \backslash \{y\})$ and $y^{-1}w$ is the only element of R involving y.

1.5.6 (see [27], p 38). Given any two finite presentations

$$< X \mid R(X)=1 >, < Y \mid S(Y)=1 >$$

for a group G, one can be transformed into the other by means of a finite sequence of Tietze transformations.

The *deficiency* of a finite presentation $< X \mid R >$ of a group is the number of elements in X less the number of elements in R. The *deficiency* of a finitely presented group G is defined by

$$\text{def } G= \max \{ \mid X \mid - \mid R \mid : \text{all finite presentations } < X \mid R > \text{ of } G >.$$

1.5.7. Any group defined by a finite presentation with positive deficiency is necessairly infinite. Therefore, if a finitely presented group G is finite, the deficiency of G is less than or equal to zero.

The Schur multiplier M(G) of a finite group $G=< X \mid R >$ is defined by

$$\frac{F' \cap R^F}{[F, R^F]},$$

where R^F is the normal closure of R in F=F(X).

1.5.8 (Schur) (see [25]).

(i) M(G) is a finite abelian group and is independent of the presentation ;

(ii) rank $M(G) \leq$ -def G.

A finite group is said to be *efficient* if rank M(G)= -def G.

A group is called *metacyclic* if it has a normal subgroup H such that both H and G/H are cyclic. A group G is said to be *split metacyclic* if it has a cyclic normal subgroup with a cyclic complement.

1.5.9. Every non-abelian split metacyclic group G with a cyclic normal subgroup H of prime order p has trivial Schur multiplier. A deficiency zero presentation for G is of the form

$$G=< x, y \mid y^n=x^p, [y, x^{-t}]=x >,$$

for some integer t, where $n=|G/H|$.

1.5.10. Let G be a non-abelian group of order p^3, p an odd prime. Then

$$M(G)=\begin{cases} 1 & \text{if G is of exponent } p^2 \\ C_p x C_p & \text{if G is of exponent p.} \end{cases}$$

Suppose first that G is of exponent p^2. Then G is metacyclic and has the presentation

$$G=< a, b \mid a^{p^2}=b^p, a^b=a^{1+p} >.$$

Suppose now that G is of exponent p. Then G has the presentation

$$G=< a,b \mid a^p=b^p=1, [a,b]^a=[a,b]=[a,b]^b >.$$

(Note that G is always extra-special.)

1.5.11 (see [29]).

(i) (Schur 1911)

$$M(A_n)=\begin{cases} 1 & \text{if } n=3 \\ C_2 & \text{if } n \notin \{6, 7\} \text{ and } n \geq 4 \\ C_6 & \text{if } n \in \{6, 7\}; \end{cases}$$

$$M(S_n)=\begin{cases} 1 & \text{if } n \le 3 \\ C_2 & \text{if } n \ge 4 . \end{cases}$$

(ii) For all $n \ge 1$

$$M(D_{2n})=\begin{cases} 1 & \text{if } n \text{ is odd} \\ C_2 & \text{if } n \text{ is even} . \end{cases}$$

(iii) (Steinberg 1961, 1967)

The Schur multiplier of SL(n,q) is trivial with the following exceptions :

(a) If G is SL(2,4), SL(3,2), SL(3,3), or SL(4,2), then $M(G)=C_2$;

(b) If G is SL(2,9) then $M(G)=C_3$;

(c) If G is SL(3,4) then $M(G)=C_4 x C_4$.

(iv) If PSL(2,q) is not one of the following groups in (d), (e) below, then $M(PSL(2,q))=C_m$ where m=(q-1,2).

(d) $M(PSL(2,4))=C_2$;

(e) $M(PSL(2,9))=C_6$.

The *tensor product* of two groups is an abelian group defined as follows. For cyclic groups we have

$$C_l \otimes C_m = C_{(l,m)}.$$

For finite abelian groups, we use the Basis theorem (1.3.2) and the universal properties

$$G \otimes H = H \otimes G, \quad G \otimes (H \times K)=(G \otimes H) \times (G \otimes K).$$

Finally, all finite groups are covered by defining

$$G \otimes H=G/G' \otimes H/H'.$$

1.5.12 (Schur 1907) (see [29], p 37). For any finite groups G and H,

$$M(G \times H) \cong M(G) \times M(H) \times (G \otimes H).$$

Note. Suppose $G=N : \varphi K$. A theorem of Tahara (1972) asserts that M(K) is a direct factor of M(G) (see the theorem 2.2.5 of [29]). In certain cases, one can use this fact to determine rank M(G). This would be useful for seeking efficient presentations for certain semi-direct products whose multipliers cannot be calculated using a computer.

Suppose that G is a finite group. The group H is called a *stem extension* of G if there is an $A \le H$ with $H/A \cong G$ and $A \le Z(H) \cap H'$. A stem extension with $A \cong M(G)$ is called a *covering group* of G.

The next important result is taken from a paper by Campbell and Robertson [3].

1.5.13. A stem extension H of a group G is a homomorphic image of some covering group of G.

Proof. Since H is a stem extension of G, there is a subgroup A of H with $A \le Z(H) \cap H'$ and $H/A \cong G$. We shall prove that $A \le \Phi(H)$. Let $D=H' \cap Z(H)$ and assume $D \nleq \Phi(H)$. Then there is a maximal subgroup K of H with $D \nleq K$. Now $K \le DK$ and so DK=H, by maximality of K. If $h \in H$, h=dk with $d \in D$ and $k \in K$. Thus

$$h^{-1}Kh=k^{-1}d^{-1}Kdk=k^{-1}Kk=K$$

since $d \in Z(H)$. Hence $K \lhd H$ and H/K has prime order. Therefore, $H' \le K$ by 1.1.4 (ii) and $D \le H' \le K$, gives the required contradiction.

Let $G= F/R$, $F=< f_1, f_2, ..., f_n >$, $G=<g_1, g_2, ..., g_n>$ with $g_i=f_i\varphi$, $\varphi : F \to G$ the canonical epimorphism. If ψ is the canonical epimorphism from $H \to G$, then $\exists h_i \in H$ with $h_i\psi=g_i$. Then $H=< h_1, h_2, ..., h_n, A >=< h_1, h_2, ..., h_n >$, since $A \le \Phi(H)$ and $\Phi(H)$ can be omitted form any generating set, by 1.1.9. This gives rise to an epimorphism $\sigma : F \to H$ with $f_i\sigma=h_i$ such that the following diagram is commutative:

$$\begin{array}{ccc} F & \xrightarrow{\sigma} & H \\ & {\scriptstyle\varphi}\searrow & \downarrow {\scriptstyle\psi} \\ & & G \end{array}$$

We shall now prove that $R\sigma=A$. Let $r \in R$, then $r\varphi=1$. But $r\varphi=r\sigma\psi$ and so $r\sigma \in \mathrm{Ker}\psi=A$. That is, $R\sigma \subseteq A$. If $a \in A$, then $a\psi=1$. $\exists f \in F$ with $f\sigma=a$ and so $f\varphi=f\sigma\psi=a\psi=1$. Hence $f \in R$ and so $a \in R\sigma$. That is, $A \subseteq R\sigma$. Also $[R, F]\sigma=[R\sigma, F\sigma]=[A, H]=1$, since $A \le Z(H)$. Hence σ induces an epimorphism $\bar{\sigma} : F/[R, F] \to H$. Let $\bar{F}=F/[R, F]$ and $\bar{R}=R/[R,F]$, so that $M=M(G)=\bar{F}' \cap \bar{R}$. Now,

$$M\bar{\sigma}=(F' \cap R)\sigma=F'\sigma=F'\sigma \cap R\sigma=H' \cap A=A.$$

Since $R\sigma=A$, we have $\bar{R}\bar{\sigma}=M\bar{\sigma}$, so if $r \in R$, there is an m in M with $m\bar{\sigma}=r\bar{\sigma}$, i.e. $(rm^{-1})\bar{\sigma}=1$ giving $rm^{-1} \in \mathrm{Ker}\,\bar{\sigma}$. Let $N=\mathrm{Ker}\,\bar{\sigma}$, we have $r= (rm^{-1}).m \in NM$, so $\bar{R} \subseteq NM$. But $\mathrm{Ker}\,\bar{\sigma}=N \subseteq \bar{R}$, $\bar{F}' \cap \bar{R}=M \subseteq \bar{R}$, which implies that $NM \subseteq \bar{R}$ and hence $NM=\bar{R}$.

Now

$$\frac{N}{N \cap M} \cong \frac{NM}{M} \cong \frac{\bar{R}}{\bar{F}' \cap \bar{R}} \cong \frac{R}{F' \cap R} \cong \frac{RF'}{F'} \le \frac{F}{F'}.$$

Since F/F' is free abelian, $N/(N \cap M)$ is also free abelian and so $N \cap M$ is a direct factor of N,

$$N= (N \cap M)xE,$$

for some $E \le N$. However, $\bar{R}=ExM$; since $\bar{R}=NM=((N \cap M) xE)M$ which shows that $\bar{R}=EM$. Also $E \cap M \subseteq N \cap M$, $E \cap M \subseteq E$ so that $E \cap M \subseteq (N \cap M) \cap E=1$.

Now $\bar{F}/E$ is a covering group of G.

(i) $\frac{\bar{F}/E}{\bar{R}/E} \cong \frac{\bar{F}}{\bar{R}} \cong \frac{F}{R} \cong G$

(ii) $\frac{\bar{R}}{E} = \frac{ExM}{E} \cong M$

(iii) $M \leq F'$ since $\frac{F' \cap R}{[F, R]} \leq \frac{F'}{[F, R]}$. Since $\bar{R}$=EM then $\frac{R}{E} \leq (\frac{\bar{F}}{E})'$.

As $[\bar{R}, \bar{F}] = 1$, we have $\bar{R}/E \leq Z(\bar{F}/E)$.

Finally, H is a homomorphic image of the covering group $\bar{F}/E$. For $E \leq N = \text{Ker}\bar{\sigma}$, $\bar{\sigma} : \bar{F} \to H$ induces an epimorphism $\bar{\bar{\sigma}} : \bar{F}/E \to H$.

1.5.14 (Schur) (see [3]). With the notation in 1.5.13 if $(| G/G'|, | M(G) |)=1$, then G has a unique covering group.

In the case where G is perfect then G has a unique covering group (by 1.5.14) which we denote by G.

1.5.15. (see [43]). If G is a finite perfect group, then M(G)=1.

1.5.16. Let G_1, G_2 be finite perfect groups. Then $G_1 \hat{x} G_2 \cong \hat{G}_1 x \hat{G}_2$.

If G is a group, then d(G) will denote the minimal number of generators of G. The minimal number m such that G has a presentation

$$G = \langle x_1, x_2, ..., x_n \mid r_1, r_2, ..., r_m \rangle \qquad (*)$$

will be denoted by r (G).

Suppose that G has a presentation (*) then the presentation is said to be *minimal* if n=d(G) and m=r (G).

1.5.17. Let G be a finite p-group and set d=d(G). Then $G/\Phi(G)$ is elementary abelian group of order p^d.

1.5.18 (Wamsley, Sag) (see [39]). Let G be a 2-group of order at most 2^6. Then G has a minimal presentation.

2. Coset Enumeration Algorithm; Computer Programs

Because of the importance of the coset enumeration algorithm and because of the fundamental role it plays in our methods for obtaining generators for subgroups of simple groups we shall devote the first section of this chapter to a brief discussion of the coset enumeration algorithm and its applications.

We then give some details of a coset enumeration program and list a number of programs to be used in the following chapters. These are a Reidemeister-Schreier rewriting program, a modified Todd-Coxeter program, a Tietze transformation program, and a nilpotent quotient program.

We shall describe two specially written programs which help find generators and presentations for subgroups of permutation groups with the aid of a coset enumeration and a Tietze transformation program.

The chapter contains some computational methods used in connection with a coset enumeration program. We shall describe a method which enables us, in favourable circumstances, to obtain generators for a Sylow p-subgroup of a group with generators and relations. Some examples will be given to illustrate these methods.

The last section of the chapter contains a brief description of CAYLEY, a group theory system [15], to be used in connection with the results of following chapters.

2.1 Coset enumeration algorithm

The Todd-Coxeter coset enumeration is a method for evaluating the index of a subgroup H in a finitely presented group G where H is finitely generated by words in the group generators. The coset enumeration method is not in fact an algorithm for , given that $|G:H|$ is finite, there is no bound on the number of steps needed to find the index, even if the index is 1.

We give here a short description of the method; a comprehensive discussion is given in [36].

Description of the method

Suppose that G is any finitely presented group given by

$$< g_1, g_2, \ldots, g_n \mid r_1 = r_2 = \ldots = r_m = 1 >$$

where each r_i is a word in the g_i's. We shall simply write $G = <X|R>$ with $X = \{g_1, g_2, \ldots, g_n\}$, $R = \{r_1, r_2, \ldots, r_m\}$. Suppose further that H is a finite index subgroup of G generated by $h_i = h_i(g_1, g_2, \ldots, g_n)$, $i = 1, 2, \ldots, t$.

The Todd-Coxeter method described below attempts to find the index of H in G by enumerating the cosets of H in G. The procedure is based on two simple facts :

(i) $Hh_i = H$ $\quad (1 \leq i \leq t)$.

(ii) For any coset Hg we have $Hgr_i = Hg$ for $i = 1, 2, \ldots, m$.

For each relator r_i, written as a product of g_i's with powers ± 1, we draw a rectangular table having $l(r_i)+1$ columns, where $l(w)$ is the length of w in the free group $F(g_1, g_2, \ldots, g_n)$, and at least as many rows as the index of H in G. This table will be called the *relation table* of r_i and looks like :

	g_{i1}	g_{i2} ⋯	g_{ij}	

where $r_i = g_{i1} g_{i2} \ldots g_{ij}$ with $g_{ik} \in X \cup X^{-1}$ (by X^{-1} we mean the set $\{g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1}\}$).

We also draw a table with a single row for each subgroup generator h_i again written out fully as word in the g_i's with powers ± 1. This table is called the *subgroup table* of h_i.

We now use positive integers to denote right cosets of H in G. The subgroup H itself is denoted by 1. We shall fill our tables by entering the positive integers as follows: We first note that the entry

	g	
k		l

will mean that the coset k multiplied by g from the right yields the coset l. By the above remarks (i) and (ii) , we begin by entering the integer 1 in the first and last places of the first row of each table, the remaining places in the first rows being as yet empty. We then consider an empty space next to some 1 and fill it with the integer 2. Suppose the situation to be as in the following diagram

	g_{i1}	
1		2

We record this definition $2 = 1g_{i1}$ (and / or $2g_{i1}^{-1} = 1$) in a table called the *coset table*. (A coset table is headed by the $g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_n, g_n^{-1}$ and has as many rows as

|G:H| .) Clearly, the integers 1, 2 correspond to cosets H, Hg_{i1}. Now we put a 2 in the first and last places of the second row of each relation table again using the above remark (ii). We then try to put this information and its consequences into the tables wherever possible. This process is known as *scanning*. Having made sure that no more spaces can be filled in this way, we enter the integer 3 in an empty space that is adjacent to a filled space. This is a new definition of the form 'ig=3' where $g \in X \cup X^{-1}$. We record this in our coset table and scan again. Similarly we introduce the integers 4, 5, ... We continue in this way until our tables are completed. Then the number of rows in each relation table is equal to |G:H|. We note that our definitions of new integers will not suffice to complete the tables. In fact we need more information of the form l=kg than is contained in our definitions of new integers and such information is obtainable from the fact that the lines in the subgroup tables or relation tables close up. Whenever this happens, information of this kind is obtained and is called a *deduction*. When such a deduction is made three possibilities can occur : either

(1) the places of both kg and lg^{-1} in the coset table are still empty. In this case we fill the integer l into the place of kg and k into the place of lg^{-1} in the coset table and also put this information into all other relevant places in the other tables. Here we have obtained a new information; or

(2) the place kg in the coset table is already filled by the integer l (and hence the place lg^{-1} by the integer k). In this case we have no new information; or

(3) at least one of the places in the coset table is filled with an integer different from that given by the deduction. In this case we conclude that we have given integers a and b with a < b, say, to the same coset. This is called *coincidence* (or *coset collapse*). When a coincidence occurs we replace b by a in all our tables. This may lead to further coincidences which are dealt with in the same way.

The following theorem is proved in [36].

2.1.1. When a Todd-Coxeter method terminates, it determines the index |G:H|.

Example 1. Find the index of $H=\langle xyz^{-1}, zyx^{-1}z^{-1}\rangle$ in

$$G=\langle x,y,z \mid xy^{-1}zyzx^{-1}z=zx^{-1}yxz^{-1}y=yz^{-1}xzxy^{-1}x=1\rangle.$$

We apply the Todd-Coxeter method in order to determine |G:H|. The subgroup and relation tables are

	x	y	z^{-1}
1	2	3	1

	z	y	x^{-1}	z^{-1}
1	3	4	3	1

(— denotes definition)

(= denotes new information)

	x	y^{-1}	z	y	z	x^{-1}	z
1	2						1
2	7						2
3	4	3	8		2	1	3
4							4
5	6	1	3	4 —	7 =	2	5
6							6
7						4	7
8					4	3	8

	z	x^{-1}	y	x	y	z^{-1}	y
1	3						1
2	5						2
3	8				5	2	3
4	7	2	3	4 —	8 =	3	4
5							5
6			1	2	3	1	6
7							7
8					7	4	8

	y	z^{-1}	x	z	x	y^{-1}	x
1	6						1
2	3	1	2 —	5 —	6 =	1	2
3	4						3
4	8	3 =	1	3	4	3	4
5							5
6						5	6
7					3	2	7
8							8

And the coset table at present is

	x	x^{-1}	y	y^{-1}	z	z^{-1}
1	2		6		3	
2	7	1	3		5	
3	4		4	2	8	1
4		3	8	3	7	
5	6					2
6		5		1		
7		2				4
8				4		3

Now a collapse has occured and we have 3x=1, 3x=4. We conclude that 1=4; and hence 6=8, 3=7. we replace 4 by 1, 8 by 6, and 7 by 3 throughout and continue scanning. As there are no further collapses, our relation tables now appears as follows:

	x	y^{-1}	z	y	z	x^{-1}	z
1	2			2	5	6	1
2	3	2	5				2
3	1	3	6		2	1	3
5	6	1	3	1	3	2	5
6		6	1	6	1	3	6

	z	x^{-1}	y	x	y	z^{-1}	y
1	3	2	3	1	6	3	1
2	5						2
3	6	5			5	2	3
5							5
6	= 1	3	1	2	3	1	6

	y	z^{-1}	x	z	x	y^{-1}	x
1	6	3	1	3	1	3	1
2	3	1	2	5	6	1	2
3	1	6		2	3	2	3
5							5
6						5	6

Defining 7=6y, the third row of the first table gives 7z=2 and completion of the new row of the same table gives 1y=2. Again we have a coincidence, for 1y=6. This leads to further collapses: 3=7, and 5=1. Now our relation tables complete with 3 rows and the coset table looks like:

	x	x^{-1}	y	y^{-1}	z	z^{-1}
1	2	3	2	3	3	2
2	3	1	3	1	1	3
3	1	2	1	2	2	1

So $|G: H|=3$.

Information obtainable from a coset table

Suppose we are given a finite presentation $G=\langle X|R\rangle$ and a subgroup $H=\langle Y\rangle$. Assume that $X=\{g_1,g_2,..,g_r\}$, $Y=\{h_1,h_2,...,h_s\}$ and each g_i (or g_i^{-1}) appears in some member of R or Y. Suppose that the Todd-Coxeter process completes consistently and gives $|G:H|=n$. So our coset table contains the information

$$ig_k=j \qquad (1\leq i\leq n,\ 1\leq k\leq r)$$

for various values of j.

We first make a useful definition to be used in the remainder of this discussion. Suppose we have a word in the group generators and their inverses

$$w=g_{i1}g_{i2}\cdots g_{it},$$

where $g_{ij}\in X\cup X^{-1}$. By *tracing* the word w from coset number k through the coset table we shall mean the product of k by the word w by successively looking up the coset numbers kg_{i1}, $(kg_{i1})g_{i2}$,....

Now we are ready to look at some basic information obtainable from a coset table for G modulo H.

(i) Permutation representation of a group

By consistency of our coset table we have

$$i \neq i' \Rightarrow ig_k \neq i'g_k \qquad \text{for any } k \ (1 \leq k \leq r)$$

and so there is a mapping $\theta : X \rightarrow S_n$, the symmetric group on $\{1,2,...,n\}$. Since for each $r \in R$, $ir=i$ ($1 \leq i \leq n$), θ extends to a homomorphism $\rho : G \rightarrow S_n$ (by the substitution test (1.5.3)). We now have the following result:

2.1.2 (see [27], p 103). With the above notation and assumption the mapping ρ is just the permutation representation of G on the right cosets of H (of degree n).

We note that the kernel of the mapping ρ defined in this way is $Core_G(H)$, by 1.2.4, and so ρ is not necessarily faithful. But if G is a simple group with a subgroup of finite index d , then ρ is a faithful permutation representation of degree d. Also by taking H=<1>, we always find a faithful representation of degree |G| (when G is a finite group). However, this permutation representation is of large degree and is not suitable to deal with. In the next section we shall describe a method to help solve this problem.

Referring to example 1, we have the following permutation representation of G on the right cosets of H

$$\rho : G \rightarrow S_3$$

$$x \rightarrow (1,2,3), \qquad y \rightarrow (1,2,3), \qquad z \rightarrow (1,3,2).$$

(ii) Determination of coset representatives

Our complete coset table for G modulo H enables us to find a set of coset representatives for H in G.

The table consists of n-1 definitions; the j^{th} of which has the form

$$ix=j+1, \qquad 1 \leq j \leq n\text{-}1,$$

with $i \leq j$ and $x \in X \cup X^{-1}$. We use these equations to define inductively a set of words $w_1,..., w_n \in F(X)$ as follows. We put $w_1=1$ and for $1 < j \leq n$, $w_{j+1}=w_ix$. Then

$w_1,\dots,w_n$ form a set of (Schreier) coset representatives for H in G.

Returning to example 1 we have 2=1x, 3=2x and so $\{1,x,x^2\}$ is a set of coset representatives for H in G.

For each coset representative w, it is also possible to determine the least positive number q for which $1w^q=1$. Such a number q is called the *order* of coset l=1w. In particular if H=<1>, one can determine all elements of a finite group G as words in the group generators and their inverses accompanied by their orders.

(iii) Index of $<H,g>$, $g\in G$

Suppose $g\in G$ is given by a word in the group generators g_i and their inverses. The coset number k of Hg can be found by tracing the word g from coset number 1 through the coset table. If in our table we apply the additional coincidence k=1, the coset table shrinks and gives the coset table of G modulo <H,g>. The number of rows in the new coset table will the determine the index $|G:<H,g>|$.

(iv) Normalizing cosets of H *and the normalizer of* H *in* G

Suppose $g\in G\backslash H$, we shall say that the coset Hg is a *normalizing coset* if it is contained in the normalizer $N_G(H)$. This is equivalent to $H^g=H$. Now

$$H^g=H \Leftrightarrow g^{-1}Hg \le H\ ,\ H \le g^{-1}Hg.$$

These two conditions are satisfied if and only if $Hg^{-1}h_i=Hg^{-1}$ and $Hgh_i=Hg$, respectively, for all $h_i\in Y$. (We note that one of these conditions is sufficient when G is finite, by 1.1.3.) Now suppose that l and k are coset numbers of Hg and Hg^{-1} respectively. The above two conditions become

$$lh_i=l\ ,\ kh_i=k \qquad (1\le i\le s)$$

Now the test for the normalizing cosets of H can be performed by tracing the h_i's from the coset numbers l, k through the coset table. If $\{Hw_1,\dots,Hw_j\}$ is a complete set of normalizing cosets, then

$$N_G(H)=<H,w_1,\dots,w_j>.$$

Now the procedure described in (iii) enables us to determine $|G:N(H)|$.

Note. Suppose that Hg is a normalizing coset of H of order q. Then $|<H,g>:H|=q$ since $g\in N(H)$ and hence the index $|G:<H,g>|$ is reduced by a factor of q, that is

$$|G:<H,g>|=|G:H|/q.$$

(v) Normal closure

We have

$$H^G=< g_i^{-1}h_jg_i \mid 1 \le i \le r,\ 1 \le j \le s >.$$

If we now apply the procedure described in (iii) in turn to all words $h=g_i^{-1}h_jg_i$ we shall get the coset table of G modulo H^G. The procedure can be used as a test for normality of H when G is finite. If $|G:H^G|=n$ then $H^G=H$ and hence $H \lhd G$.

Example 2. The group G given by $<x,y \mid x^2=(xy)^4=1,\ (xy^2)^2=y^2x>$ has order 36. Find the normalizer in G of $H=<xy^2>$.

A coset table of G modulo H looks like:

	x	y	y^{-1}
1	3	4	2
2	5	1	3
3	1	2	4
4	8	3	1
5	2	7	6
6	7	5	11
7	6	11	5
8	4	12	9
9	12	8	10
10	11	9	12
11	10	6	7
12	9	10	8

We now must look for all cosets k satisfying

$$k\,xy^2=k \qquad (\,2 \le k \le 12\,)$$

The only cosets satisfying the above conditions are listed in the following table :

coset number	coset representative	order
3	x	2
6	$y^{-1}xy^{-1}$	3
7	$y^{-1}xy$	2
9	yxy^{-1}	2
12	yxy	3

Adding the above coset representatives to the $<xy^2>$, in turn, and using (iii), we get $|G:N_G(H)|=2$. By eliminating redundant generators of $N_G(H)$ we find

$$N_G(H)=<x,y^2,x^y>.$$

2.2 Computer programs

A Todd-Coxeter Program

Over the past thirty years the Todd-coxeter algorithim has been implemented in different ways. The most important difference between them is in the sequence in which cosets are defined.

In a first method, known as HLT method, coset numbers are defined with the primary aim to close at least one line of some relation table in order to get at least one deduction as soon as possible. This method tends to define more redundant cosets.

In a second method, known as the Felsch method, the vacant places in the coset table are filled line by line and after each definition a scan of all relations is made. This method makes few redundant definitions, but the many scans tends to be time-consuming.

A method that to some extent combines the merits of the two previous ones is the so called 'Lookahead' method in which periods of definitions a' la HLT alternates with periods of intensive scan à la Felsch.

At the University of St. Andrews we have a modern implementation of the Todd-Coxeter algorithm, one by G. Havas and W.A. Alford (the Canberra version) mounted on a Digital VAX-11/785 and more recently on a Micro VAX-II. In this implementation cosets may be enumerated both by the 'Lookahead' method and the 'Felsch' method. The program also provides a package for manipulation of partial and complete coset tables. All procedures described in 2.1 are carried out by the program.

This version also contains an implementation of the Reidemeister-Schreier algorithm, added by E. F. Robertson, which enables the user to produce defining relations for a subgroup H of finite index in favourable circumstances, given defining relations for a group G containing H. The Todd-Coxeter algorithm is first applied to obtain a coset table of G modulo H, then Reidemeister-Schreier rewriting process , [27] , uses this table to rewrite G-relators as H-relators. At this stage we have a presentation for H involving a large number of generators and relators so a number heuristics are applied so as to reduce the number of generators and simplify the resulting relators whenever possible (Havas, [23]).

Our package contains a program so called the 'Tietze Transformation Program' which was originally designed to improve the simplification stage of the above Reidemeister-Schreier program (see below). The package also contains a program to produce a presentation for the subgroup on the original subgroup generators. This is based on a modification of the Todd-Coxeter algorithim and has been implemented by D.G. Arrell and E. F. Robertson ([1]). It is worth saying that the Tietze Transformation program also forms part of this program.

A Tietze Transformation Program

This program [24], which carries out Tietze transformations (1.5), accepts a group presentation as input with the relations arranged into ascending order (the canonical form in [23]), and then uses the three following principles in simplifying the presentation:

(i) All relators of length 1 and non-involutory relators of length 2 are used to eliminate a number of generators.

(ii) A relator is chosen (either by the user or the program) and the other relators are searched for (sub-)strings of the chosen relator. If the matching (sub-)string has length greater than half the length of the relator chosen for the search, the appropraite substitution is made. (Substring searching therefore is an attempt to minimise the length of the relators of a presentation.)

(iii) Eliminate a generator using a relator with length greater than 2.

The program has a strategy for eliminating generators which allows one to prevent certain generators being eliminated. This enables us to direct the Tietze Transformation Program more positively towards obtaining a presentation on a selected subset of generators.

The program has been improved by E. F. Robertson and now can be directed by the user towards finding relators of certain types. This is important because some types of relators are of more use in determining group structure than others (see, [38]).

We note that the program had already been improved by E.F. Robertson by the addition of certain subroutines to help find presentations for certain finite simple groups in [6]. We shall use a similar method described in 3 of this paper to obtain presentations for some permutation groups generated by two elements x and y with x an involution.

A Nilpotent Quotient Program

Given a presentation for a nilpotent group, the program constructs $\Gamma_i=[\Gamma_i,G]$ for i=2,..., where

$$G=\Gamma_1 \geq \Gamma_2 \geq \ldots$$

is the lower central series for G. In particular, this provides a method for determining the order of p-groups which does not depend very heavily on the value of p.

From now on we shall use the following abbreviations for the above programs :

TC:	Todd-Coxeter program;
RS:	Reidemeister-Schreier program;
SUBGP:	Modified Todd-Coxeter program;
TTRANS:	Tietze Transformation program;
NQ:	Nipotent quotient program.

PERM, PERMGP

The program PERM which we shall describe below was originally written, by E.F. Robertson, to solve the specific problem concerning the minimal (2,m,n) presentations for certain simple groups G, $10^5 < |G| < 10^6$, see [6]. We improved the program to solve, in addition, our particular problem which will be described in 3.3.

Suppose we are given two permutations a, b of degree d with $a^2=1$ and $b^n=1$ (n>2), which generate a finite permutation group G ($\leq S_d$). We wish to find relations between these permutations to define the group. We may take the following steps:

(1) Generate a set of relations $r_1(a,b)=1,\dots,r_n(a,b)=1$ which hold in G.

(2) Remove most of the redundant relations. Suppose we are left with the relations $r_{i1}(a,b)=\cdots=r_{ij}(a,b)=1$.

(3) Take $H=\langle a,b \mid r_{i1}(a,b)=\cdots=r_{ij}(a,b)=1 \rangle$ and check that whether or not $H \cong G$. If this is so, eliminate further redundant relations; otherwise

(4) Find more relations; go to (2).

The important step in this procedure is in fact to produce a set of relations between the permutations a, b. Then we may use certain functions of TTRANS to eliminate a large number of redundant relations. Then TC can be used to show whether our relations are sufficient to define G (for details see [6]).

Relations between a, b are in the form $w(a,b)^l$, where w(a,b) is a word in the free group F(a,b) and l is the order of the corresponding permutation in G. Since $a^2=1$, we just need to produce all words of the form

$$ab^{n1}ab^{n2}\dots ab^{nk} \qquad (1 \leq n_i \leq n-1) \qquad (*)$$

up to a given length 2s. Such words can be generated by writing all non-negative integers $m \leq (n-1)^s-1$ to base n-1 with the aim to find a set of integers m_i with $1 \leq m_i \leq n-2$ and form a word of the type (*) simply by taking $n_i=m_i+1$.

We are now ready to describe the major actions of the PERM program. Suppose we have input two permutations a, b of degree d with $a^2=b^n=1$ (n > 2) and a word w in a, b to the program. Then PERM is able to

(1) compute the value, as a permutation, of w and its fixed points as well as its cycle

decomposition and order;

(2) produce a set of words $w_1, w_2, ..., w_k$ in a, b up to a preset length such that each ww_j has order less than a specific integer chosen by the user;

(3) produce a set of words $w_1, w_2, ..., w_k$ in a, b of a given order, again up to a preset length, such that either $fix(w) \cap fix(w_i) \neq \Phi$ $(1 \le i \le k)$ or $fix(w) \cap fix(w_i) = \Phi$ $(1 \le i \le k)$.

(4) produce a set of words $w_1, w_2, ..., w_k$ of a specific order with ww_i, $[w, w_i]$ (or/and ww_i^2) having a given order provided the group has such elements.

The PERMGP program which is described below was basically written in connection with the following particular problem.

Again suppose that we are given a permutation group G ($\le S_d$) generated by two permutations a and b with $a^2=b^n=1$ ($n > 2$). Suppose, further, that $w_1(a,b), ..., w_n(a,b)$ are words in the free group F(a,b) whose values as permutations generate a subgroup H of G of fairly small size (say, of order less than 10,000 with $d \le 100$). We wish to find a new set of generators for H as words in a, b having fewer elements possibly of shorter lengths than those of the w_i's. The program consists of two main parts: (1) It generates all elements of H as permutations, (2) generates all possible words in a,b (up to a preset length) and checks whether or not their values belong to H.(The program has also some other facilities which will be mentioned below.)

For the first part of the program we implemented an algorithm due to Felsch and Neubuser ([19]). In this method the elements of H are generated by repeated application of the algorithm which generates the elements of the subgroup $L=\langle K,x \rangle$; $K \le H$, $x \in H$, where the elements of subgroup K are already known. The details of this algorithm are as follows:

(i) $L := K \cup Kx \cup ... \cup Kx^{m-1}$, where x^m is the first power of x that lies in K.

(ii) $s := |K|m$. Denote the elements of L by $l_1, ..., l_s$. $j := 0$.

(iii) $j := j+1$. If $j > s$, finish. Otherwise, $y_j := x\, l_j$ and if $y_j \in L$, go to (iii). Else

(iv) Suppose $y_j^{m_j}$ is the first power of y_j that lies in L

$L := L \cup Ly_i \cup ... \cup Ly_j^{m_j-1}$; $s := s+|K|(m_j-1)$. Go to (iii).

Having stored all the elements of H as permutations in the array $|H| \times d$, a similar technique to that in PERM produces first all possible words in a, b (up to a preset length) of the form

$$a^{\varepsilon}b^{\alpha}ab^{\beta}...ab^{\gamma}a^{\varepsilon'}$$

where ε, $\varepsilon' \in \{0,1\}$, α, β,..., γ less than n, and then checks their membership in H. We now have a list of words in a, b whose values (as permutations) are in H. Some of these may form an optimal generating set for H. Having selected a set of random words of shorter lenghts, the first part of the program can now be used to check whether they generate H.

We shall widely use this program to simplify our generators for subgroups of the simple groups G, $|G| < 10^6$, in Chapter 4. (For an application of the program, see example 2 of 3.2.) We note that some of these generators, particularly for Sylow p-subgroups, come originally from the coset enumeration program as coset representatives of some subgroups of G and usually are words of fairly long length.

The program also enables the user to find all elements of the intersection of two subgroups H, K of G and allows generators for $H \cap K$ to be found (in the original group generators). This might be useful when we wish to find out whether a subgroup K is a complement to a normal subgroup H in G by showing that their intersection is trivial.

The program also checks the transitivity of subgroups of G. Since the transitivity of a subgroup is preserved by conjugation, this may help us, in certain cases, to show that two isomorphic subgroups of a permutation group are not conjugate.

2.3 Some computational methods

We now look at some computational methods in connection with a Todd-Coxeter program. We shall apply these methods in the last two chapters. One important method is ' finding a Sylow p-subgroup' of a finite group with generators and relations. We shall basically use this technique in obtaining Sylow p-subgroups of simple groups.

In what follows we suppose that a presentation $\langle X \mid R \rangle$ is given for a group G.

(i) Simplifying the coset enumeration

Sometimes, in order to simplify the coset enumeration, we may add some extra relations which hold in G to the presentation $\langle X | R \rangle$. Such relations may be obtained algebraically or computationally. For example, suppose w_1, w_2 are words in the group generators and that $|G:\langle w_1 \rangle| = n$. If on adding the generator w_2 to $\langle w_1 \rangle$ we still get $|G:\langle w_1, w_2 \rangle| = n$, then $w_2 \in \langle w_1 \rangle$ and hence $[w_1, w_2] = 1$. This relation, in certain cases, makes the coset enumeration easier and may save a great deal of time.

(ii) Test for normality

In 2.1 (v) we describe how a coset table of G modulo a subgroup H can be used to test whether H is normal. However, there is another way to check the normality of H using a coset enumeration program. Suppose $H=<h_1,h_2,..,h_r>$, where h_i's are words in the group generators of G and $|G:H|=n$. If on adding the relations $h_1=1$, $h_2=1..$, $h_r=1$ to the $<X \mid R>$ we obtain the index m, then $|G:H^G|=m$ (by Von Dyck's Theorem (1.5.2)). Now, if m=n then $H \lhd G$.

(iii) Faithful permutation representation

Referring to 2.1.1, Ker $\rho=\cap H^g$, and so ρ is faithful whenever H is not normal and contains no proper normal subgroup of G. Sometimes it is possible to find out a fairly large non-normal subgroup containing no proper normal subgroup. When this is not possible (which is most of the time) we may resort to the following technique:

(1) Take a word w of G of sufficiently large order m

(2) Check the normality of $<w>$. If $<w> \lhd G$ go to (i), else

(3) Suppose $d_1,d_2,...,d_s$ are all the distinct proper divisors of m, we test each of the subgroups $<w^{d1}>$, $<w^{d2}>$,..., $<w^{dn}>$ for normality . If any is normal we return to (i).

(iv) Determination of group structure

The following method is often successful in determining the structure of a group.

Suppose that S is a set words in the group generators and their inverses and that $H=<X|R\cup S>$. Then there is a homomorphism ϕ from $G \rightarrow H$ with Ker $\phi=<S>^G$, the normal closure of $<S>$. So G=Ker ϕ.H, that is, G is an extension of Ker ϕ by H. Sometimes this may be expressed as the direct product or a semi-direct product of Ker ϕ and a subgroup K of G isomorphic to H when G is finite. In each case one needs to show that Ker $\phi \cap K=(1)$. This can be done as follows: Suppose we have two subgroups K_1 and K_2 of G and that the elements of K_1, say, can be generated independently. If on adding each of the elements of K_1 (except the identity) , as words in the same group generators, to the subgroup generators of K_1, the index $\mid G:K_2 \mid$ alters, then $K_1 \cap K_2=(1)$.

(v) The Commutator subgroup [H,K]

If $H=<h_1,...,h_s>$, $K=<k_1,...,k_t>$ are normal subgroups of G, then the commutator subgroup [H,K] can be constructed as the normal closure of the subgroup

$$<[h_i,k_j] \mid 1 \le i \le s, 1 \le j \le t>.$$

In particular, this observation provides us a method of computing a set of generators for the derived group [G,G].

(vi) Determination of a generating set for a Sylow p-subgroup

As all the Sylow subgroups belonging to a single prime p are conjugate, it is only necessary to find a set of generators for a single Sylow p-subgroup for each prime p dividing the order of G. We construct a Sylow p-subgroup by successively extending a p-subgroup by elements of order some power of p.

Suppose G is of order $p^n.q$, where p does not divide q. Suppose further we have found a set of generators for a p-subgroup H of G of order p^m with $m < n$ and that the coset enumeration algorithm has successfully enumerated the cosets of H in G. We require to find generators in the group generators for a Sylow p-subgroup of G containing H. The subgroup H is contained in some Sylow p-subgroup of G, say P. By 1.1.8, there is an $h \in N_P(H)$ which is not in H. Obviously Hh has order p^l for some l ($\leq n$). We now use 2.1 (iii) to determine the index of <H, h> in G. According to the note made in 2.1 (vi), we will have

$$|G:\langle H,h\rangle|=p^{n-m-l}.q.$$

This shows that <H, h> is a p-subgroup of G and that H is a proper subgroup of <H, h>. Now if n=m+l, we have obtained generators in the group generators for a Sylow p-subgroup of G. Otherwise, by taking H:=<H,h> and continuing this way we eventually reach a Sylow p-subgroup P_1 of G (not necessarily P) containing H. Clearly P_1 is the only Sylow p-subgroup of G lying inside $N_G(P_1)$.

The method will be more efficient if we can start from a sufficiently large p-subroup of G for which the index |G:H| is determined by the machine. In particular, H can be taken as a cyclic subgroup of order p^α ($\alpha \geq 1$). These cyclic subgroups can be found either by looking directly at the presentation of G or by enumerating cosets of random cyclic subgroups <w> of G, where w is a word in the group generators, in the hope of finding an element of order p^β for some β. We also note that, sometimes, we are able to take H as a non-cyclic p-subgroup provided we have given a faithful permutation representation of G together with a knowledge of maximal subgroups of G. We shall describe this method in the next chapter.

We now give a couple of examples to illustrate the above methods.

Eaxmple 3. Let

$$G=\langle x,y,z \mid x^2=y^2=z^2=(xy)^3=(yz)^3=(xz)^4=(xyxz)^3=1\rangle.$$

Then TC shows that |G|=96. By adding the relation xz=1 to the presentation of G we get

$$H=\langle x,y \mid x^2=y^2=(xy)^3=1\rangle$$

which is S_3, the symmetric group. Also we have

$$\langle xz\rangle^G=\langle xz,yxzy\rangle$$

which is isomorphic to C_4xC_4 because xz and yxzy are each of order 4 and [xz,yxzy]=1. Now, we let K=<x,y> then $K \cong H$ and the non-trivial elements of K are x, y, xy, yx, $(xy)^2$. The method described in 2.3 (iv) shows that $H \cap \langle xz \rangle^G=(1)$ and so $G \cong (C_4xC_4):S_3$.

Example 4. Let G be the group given in example 2. Find generators for a Sylow 3-subgroup of G.

Using TC, |G|=36 and $|\langle xy^2 \rangle|=3$. Returning to example 2, we observe that Hyxy is a normalizing coset of order 3. Therefore a Sylow 3-subgroup may be generated by xy^2 and yxy.

2.4 CAYLEY

CAYLEY is a high level programming language designed to support convenient and efficient computation within groups and other structures that arise naturally in the study of groups [15].

CAYLEY allows the user to compute with groups of the following types.

(1) Groups whose elements are permutations;

(2) Groups whose elements are matrices over the integers;

(3) Groups whose elements are matrices over finite fields;

(4) Groups given by defining relations.

The kind of computations that can be performed with CAYLEY include :

(1) Calculations with the element of a group;

(2) Calculations with sets of group elements;

(3) The determination of the order, conjugacy classes, normal subgroups, subgroup lattices and automorphism group of a finite group;

(4) Calculation with subgroups and quotient groups of a group;

(5) The construction of homomorphisms between pairs of groups.

The language is designed both for batch and interactive use. No previous knowledge of computers and programming language is assumed. The computer problems are normally stored in the machine together with sufficient information about CAYLEY to solve them.

The following example is a simple CAYLEY program to list the right cosets of the subgroup H in the permutation group G generated by the permutation a=(2,3)(4,5) and b=(1,4,3,2)(5,6), where $H=\langle a^b,b^a \rangle$.

```
> " CAYLEY program to compute the right cosets of subgroup H in group G " ;
>g : perm (6) ;
>g. generators: a=(2,3)(4,5), b=(1,4,3,2)(5,6) ;
>h = <a^b, b^a> ;
>s = [ ] ;
>i = 0 ;
>FOR EACH x IN g DO
>  IF x IN s THEN LOOP ; ELSE
>     n = h*x ;
>     i = i+1 ;
>     PRINT ' coset ', i, n ;
>     s = s JOIN n ;
>  END;
> END;
```

(' > ' is the CAYLEY prompt sign.)

We will not waste space by printing the answer here but proceed to discuss the steps in the program. Each *statement* is terminated by a semicolon. The first statement is a comment enclosed within double quotes (") which is ignored by CAYLEY. The statement *g : perm (6) ;* declares that the symbol *g* is to denote a permutation group of degree 6. *g. generators : a=(2,3)(4,5), b=(1,4,3,2)(5,6) ;* specifies the generating permutations for G. We now define H to be the subgroup generated by a^b, b^a which is introduced by *h=<a^b, b^a>;* (the up-arrow (^) denotes exponentiation, while the asterisk (*) denotes multiplication). The remaining statements compute and print the cosets of H. The cosets are found by running through the elements of G until one is discovered which does not lie in a previously computed coset. A new coset is formed by multiplying H on the right by this element. A set S is used to denote those elements of G that appear in the already known cosets of H.

Standard functions

The power of CAYLEY as a problem solving tool in group theory derives largely from the provision of an extensive library of group theory routines. The library routines appear in the language as *standard functions* . For example, if G is a finite group, and H≤G, the standard function *normalizer* when applied to G and H, as in the expression

normalizer (G, H) ;

will find the normalizer in G of H, and return it as the value of the function. The programmer may use the expression *normalizer (G, H) ;* when a group is allowed. The contents of the library of standard function in CAYLEY are outlined in [15].

We shall mainly use the following standard functions in connection with our results in Chapter 4 and Chapter 5.

(a) Finitely presented groups

(i) Construction of defining relations from a faithful representation (the library routine is based on Cannon's algorithm [13]).

(ii) Construction of a covering group of a group.

(b) Permutation groups

Order of a group G; conjugacy classes and centralizers of elements; normalizer $N_G(H)$, centralizer $C_G(H)$ for given $H \leq G$; $H \cap K$, test for conjugacy for given H, K. (The library routines are based on the notions of a base and strong generating set of a permutation group on a finite set A, introduced by Sims [42].)

Library files

A programmer using CAYLEY interactively will often find it necessary to execute some particular sequences of statements many times over. This is made easier by the use of a *library file* which enables the user to store program segments in such a way that they can be called up and executed at any time during the run.

A library file is a file provided by the operating system of the host machine and is created externally to CAYLEY using a text editor. The file contains text organised into blocks known as *library blocks*, each of which can be called up independently. Since CAYLEY provides no facilities for text editing or storage, the library file plays a crucial role in that it allows the programmer to use the editor provided by the operating system for these tasks. It is recommended to use library files for CAYLEY programs longer than than a few lines since it is very easy to make typing errors if these are entered directly.

We conclude this section with a particular CAYLEY program (taken from [14]) to be used in Chapter 4.

A CAYLEY Program : SUBGPTEST

Suppose that it is necessary to determine whether a given finite group G contains a subgroup isomorphic to the 2-generator group H. One approach is to look for elements in G which satisfy a set of defining relations for H and which generate a group of order $|H|$. Suppose that $H=\langle x,y\rangle$, where $|\langle x\rangle|=n$, $|\langle y\rangle|=m$. It is sufficient to take a set of representatives for the classes of elements of order n of G as the possible images of x and the set of all elements of order m of G as possible images of y. We also observe that if x, y satisfy a given set of defining relations for H then so does (x,y^a) for all $a \in C(x)$. Thus the potential images of the second generator can be further reduced by taking representatives for the orbits of elements of order m under the action of the centralizer C(x).

Given below is a CAYLEY program which implement these ideas. Assuming that H and G are set up (variables ***h*** and *g* , respectively), where H is a two generator group, the program will set the Boolean variables *isom* to the variable *true* , if H is isomorphic to a subgroup of G, and *false* otherwise. The following standard functions are used in the program :

order (H):	order of the group H;
relations (H):	defining relations for the group H which are satisfied by the generators of H;
class (G,x):	conjugacy class of the element x in the group G;
centralizer (G, x):	centralizer of the element x in the group G;
satisfy (Q,W):	given a sequence Q of n elements belonging to a group G and a set of words W on the generators of the n-generator group H, returns the Boolean value *true* if every member of W is the identity of H under the substitution H.i → Q[i] (i=1,2,..,n);
setrep (R)	an arbitrary element from the set R.

```
"determine whether the 2-generator group h is isomorphic to some subgroup of the
group g"
" it is convenient to define certain variables as follows:
        horder      the order of h
        n1          the order of the first generator of h
        n2          the order of the second generator of h
        imgs1       representatives of those g-classes whose elements have order n1
        imgs2       the union of those g-classes whose elements have order n2
        hrels       a set of defining relations for h"

> isom = false ;
> horder = order (h) ;
> n1 = order (h.1) ;
> n2 = order (h.2) ;
> hrels = relations (h) ;
> imgs1 = null ;
```

```
> imgs2 = null ;
> FOR EACH x IN classes (g) DO
>     IF order (x) EQ n1
>         THEN imgs1 = imgs1 JOIN [x] ;
>     END;
>     IF order (x) EQ n2
>         THEN imgs2 = imgs2 JOIN class (g,x);
>     END;
> END;
> "run through a set of representatives from those classes of order n1"
> FOR EACH x IN imgs1 DO
>     centzr = centralizer (g,x) ;
>     notdone = imgs2 ;
> "run through the classes of order n2"
> WHILE notdone NE null DO
>     y=setrep (notdone) ;
> "check whether the obvious map is a homomorphism from h onto <x,y>"
> IF satisfy (SEQ(x,y), hrels)
>     THEN
>         IF order (<x,y>) EQ horder
>             THEN
>                 isom = true ;
>                 BREAK ;
>     END ;
> END ;
> "delete from notdone the orbit of y under conjugation action by centzr"
>     notdone = notdone - class (centzr, y) ;
> END; " images of second generator"
> IF isom
>     THEN BREAK ;
> END ;
> END ; " images of first generator"
> IF isom
>     THEN
>         PRINT ' h is isomorphic to the subgroup of g generated by ', x, y ;
>     ELSE
>         PRINT ' h is not isomorphic to any subgroup of g' ;
> END ;
```

Example 5. We take

$$G=<a,b> \cong S_5, \quad a=(3,4), b=(1,2,3,4,5).$$

Then SUBGPTEST shows that G has no subgroup isomorphic to Q_8; while it gives c=(3,4), d=(1,4,5,3) which generate a subgroup isomorphic to D_8 showing that the Sylow 2-subgroup of S_5 is D_8.

3.Methods

Throughout this chapter, G denotes a finite simple group with a minimal generating pair (a,b).

In this chapter we describe a range of methods which can be used to determine representatives for the conjugacy classes of G, generators for a Sylow p-subgroup of G, and generators for each maximal subgroup of G, all as words in a, b.

In addition, we shall describe some computational techniques for determining the Schur multiplier of a small group with generators and relations. Some methods are given which can be used to try to find an efficient presentation for a finite group.

3.1 Conjugacy classes

Suppose x is an element of a group K. The conjugacy class containing x, which is denoted by (x), consists precisely of all those elements y of K which are conjugate to x; that is $y=k^{-1}xk$ for some $k\in K$. The number of elements of K in (x) is the index in K of the centralizer of x. Clearly all elements in the conjugacy class (x) have the same order. Now suppose that (x), (y) are two distinct classes of K containing elements of the same order. We say that (x), (y) belong to the *same family* if $\langle x\rangle$, $\langle y\rangle$ are conjugate in K.

Let $k\in K$, n the order of k, and p a prime divisor of n. Then by the *p-power* of k we mean the element k^p. The *p'-part* of k is defined by $k^{\lambda q}$, where q is the highest power of p dividing n, n=qr, and $\lambda q+\mu r=1$. For example, if k has order 60 then the 2'-part, 3'-part, 5'-part of k are k^{16}, k^{21}, k^{25} respectively.

Now the information given in the ATLAS about the conjugacy classes of G enables us to give a representative for each conjugacy class of G. This is about the order of the centralizer in G of a typical element of each class, the classes into which p-power and p'-part of a typical element of each class fall (see also 4.1).

To obtain such representatives we may resort to the following techniques:

(i) Cycle types

Suppose w_1, w_2 are words in a, b of order n with the different cycle types in the permutation representation of G. Then w_1, w_2 belong to distinct classes of elements of order n of G. Such words are easily produced by PERM.

(ii) Centralizers orders

Suppose G has at least two conjugacy classes C_i, C_j of elements of order n with the same cycle type in the permutation representation of G, but with the different order of centralizer c_i, c_j. Then the centralizer orders of a number of random words of order n, produced by PERM, are examined by the standard function

centralizer (G,x)

of CAYLEY in the hope of finding two elements $w_i(a,b)$, $w_j(a,b)$ that have centralizer orders c_i, c_j.

(iii) Non-conjugate elements

If two distinct classes C_i and C_j of order n have the same length, we may use the standard function

conjugate(G,x,y)

of CAYLEY in an attempt to show that the logical value of the function is *false* for two randomly chosen words w_i, w_j of order n. If this is so, we have found representatives for each of the classes C_i, C_j.

(iv) Representatives of the classes belonging to the same family

Suppose $C_i, C_{i+1}, \ldots, C_{i+m-1}$ are classes of elements of order n which belong to the same family and that a representative for C_i has been found. Let $C_i=(w)$. Then representatives for each of $C_{i+1},\ldots, C_{i+m-1}$ may be obtained by raising w to a suitable power which is coprime to n. These powers have been specified in the ATLAS for each G, see 4.1. (It is also possible to determine these powers using (iii) above .)

Now suppose that $C_i=(w)$, $C_{i+1}=(w^{\alpha})$, ..., $C_{i+m-1}=(w^{\delta})$, and p is a prime dividing n. Then the p-powers of w, w^{α}, ..., w^{δ} fall into some classes $C'_i, C'_{i+1}, \ldots$, again members of the same family, consisting of elements of order n/p (clearly, this family has at most m members). Suppose now $w^p \in C'_t$ for some t. This can now be used to determine representatives for the rest of members of $\{C'_i\}$ in a similar way to that described for $\{C_i\}$.

Finally we note that w must be chosen in such a way that its p'-part , for each prime p, falls into the correct class which has been specified by the information given in the ATLAS about the p'-part of a typical element of each class (see, 4.1).

As an illustration we give below a representative for each conjugacy class of G=PSU(3,3). We shall follow the notation given in 4.1 for the class names.

Examle 1. We take G to be PSU(3,3) with the presentation given on page 85. Invoking the first four columns of table I:

(1) G has a single conjugacy class of involutions and hence the element a can simply be chosen as a representative of this class which is shown by 2A.

(2) There are two classes of elements of order 3 denoted by 3A, 3B with different lengths, namely $|G|/108$, $|G|/9$. In G the element b has order 6. We therefore consider $x=b^2$ and find $|C(x)|=108$. This indicates that $x\in 3A$. We now search for an element y of order 3 for which $|C(y)|=9$. It is found that the element $y=[a,b]$ of order 3 has this property. Therefore, $y\in 3B$. Here x, y have the same cycle type in the permutation representation of G.

(3) G has three conjugacy classes of elements of order 4 namely 4A, 4B, and 4C where 4A, 4B are members of the same family. Thus it is enough to determine a representative for each of 4A, 4C. To do this, we look at the conjugacy classes of order 8. G has two such classes, namely 8A, 8B (again members of the same family), and note that elements in 8A are not conjugate to their inverses. By the information given about 2-powers of elements in 8A, 8B, we have

$$x^2\in 4A \qquad \text{if } x\in 8A,$$

$$x^2\in 4B \qquad \text{if } x\in 8B.$$

By PERM, we find that $u=abab^{-2}$ has order 8, and so this is taken as a representative for the class 8A. Now $v=b^2ab^{-1}a$ determines a representative for the class 8B. Therefore, u^2 and v^2 are representatives for the classes 4A, 4B respectively.

It remains to find a representative for the class 4C. Using PERM, we observe that ab^3 is an element of order 4 with the cycle type $2^2 4^6$ and hence can be regarded as a representative for the class 4C because its cycle type is different from those of 4AB.

(4) G has a single class of elements of order 6 and b clearly belongs to this class.

(5) There are two classes of elements of order 7 belonging to the same family. Since ab is of order 7 in G, we have $ab\in 7A$, and thus $b^{-1}a\in 7B$ by table I.

(6) Finally we give representatives for each of the classes 12A, 12B (same family). Again using PERM we observe that $z=ab^2$ is an element of order 12. So either $z\in 12A$ or $z\in 12B$. But the 3-power of an element in 12A must lie in 4B. A simple calculation with CAYLEY shows that z^3 is, in fact, conjugate to u^2 in G confirming that $z\in 12B$.

Now we have $b^{-2}a \in 12A$.

Now, these representatives have been found consistently in the sense that for each representative x of order n the p-power and p'-part of x (p, a prime dividing n) fall exactly into the classes specified in our table I. For example the 2'-part, 3'-part of ab^2, namely $(ab^2)^4$, $(ab^2)^9$ are in 3A, 4B respectively.

Note. In the above example we were able to start with a representative x for the class 12A and by the information given in the table about p-powers of x determine representatives for each of the classes 6A, 4B just by taking x^2, x^3. Likewise, the information about the p'-part of x could have been used to determine representatives for the classes 3A, 4A by taking x^4, x^9. Although this technique is often useful, it usually increases the length of the resulting representatives (as words in a, b). Having found such kinds of words we are still able to use (iii) in the hope of finding shorter words for the desired classes.

3.2 Sylow p-subgroups

In 2.3 (vi) we explained how a coset enumeration program can be used to determine a generating set for a Sylow p-subgroup of a finite group K with generators and relations. The efficiency of the method is heavily based on the order of K and also the amenability of the presentation of K to computation with implementations of the coset enumeration algorithm. For instance, if K is a moderately large group with a small Sylow p-subgroup P, the method often fails to find a generating set for P.

To overcome this problem we may resort to the following technique:

(1) Look for a small subgroup H of K containing a Sylow p-subgroup of K for which the method described in 2.3 (vi) is applicable. Suppose
$K=\langle k_1,k_2,\dots,k_n \mid r_1=r_2=\dots=r_m=1\rangle$, $H=\langle h_1,h_2,\dots,h_s\rangle$, where

$$h_i=h_i(k_1,k_2,\dots,k_n) \quad (1 \le i \le s) \qquad (*)$$

(2) Find a presentation for H on the generators h_1, h_2, ..., h_s;

(3) Apply the method desribed in 2.3 (vi) in order to find a set of generators

$$x_1=x_1(h_1,h_2,\dots,h_s),\dots, x_t=x_t(h_1,h_2,\dots,h_s) \qquad (**)$$

for a Sylow p-subgroup of H;

(4) Substitute (*) for h_1, h_2, ..., h_s in (**).

The resulting generators x_i's are now words in the original group generators $k_1,k_2,\dots,k_n$ and obviously generate a Sylow p-subgroup of G.

The technique works quite well for the finite simple groups G as we have a considerable knowledge about their subgroups as well as a faithful permutation representation of sufficiently small degree for each G which allows one to compute with G using Sims' powerful methods [41], [42].

The information about the subgroups of G enables us to determine generators in a, b for a relatively small subgroup H of G having a Sylow p-subgroup of G. We are now able to use either Cannon's algorithm [13] or the method described in 3 of [6], due to E. F. Robertson, to construct a presentation for H on its generators.

We note that the only problem with implementing this method for the simple groups G is that the resulting generators x_i's at the stage (4) for a Sylow p-subgroup usually become very long. To overcome this problem, one may use the PERMGP program to simplify these generators. We are also able, in certain cases, to simplify the generators algebraically using the relations of G. The simplification techniques will be explained in chapter 4 as and when they are needed to be applied.

Notes.

(1) For a simple group G for which generators of a Sylow p-subgroup are sought, we may find various subgroups containing a Sylow p-subgroup of G. The appropriate ones among them as our starting point are normally those with smaller orders and possibly shorter generators as words in a, b. Also for such subgroups one may find various sets of generators for a Sylow p-subgroup. Again those with generators of shorter lengths are eligible to be taken for our second stage.

(2) The Felsch method using our Todd-Coxeter program usually gives shorter normalizing coset representatives than the Lookahead method.

Example 2. The simple group G=SL(2,32) has order $2^5.3.11.31$ and presentation

$$\langle a,b \mid a^2=b^3=(ab)^{31}=((ab)^3(ab^{-1})^7)^2=1\rangle.$$

Find generators for a Sylow 2-subgroup of G.

Although TC can be used here to construct a Sylow 2-subgroup of G by successively extending $\langle a\rangle$ by 2-elements, we proceed to apply the above method in order to illustrate how it works in practice.

By 1.4.1 (vi), G has a subgroup of minimal index 33 - of order $2^5.31$- which obviously contains a Sylow 2-subgroup of G. Suppose we have constructed the following permutations a, b of degree 33 which generate G and satisfy the presentation of G :

a=(1,2)(3,16)(4,17)(5,18)(6,19)(7,20)(8,21)(9,22)(10,23)(11,24)(12,25)
(13,26)(14,27)(15,28)(29,33)(30,32),

b=(1,3,2)(4,14,27)(5,24,10)(6,8,23)(7,33,32)(9,13,31)(11,20,18)
(12,17,22)(15,26,16)(19,29,28)(21,30,25).

Now PERM enables us to find the generators

$$x=a,\ y=bab^{-1}(ab)^3ab^{-1} \qquad (*)$$

for the stabilizer of a point in the permutation representation of G which is a subgroup of index 33 (see (i) of 3.3). A presentation for $H=\langle x,y\rangle$ on x, y is

$$H=\langle x,y \mid x^2=y^{31}=1,\ xy^2(xy)^3=y^5x\rangle.$$

Now using TC we can easily find the generators

$$x,\ y^{-1}xy,\ yxy^{-1},\ y^{-2}xy^2,\ y^2xy^{-2} \qquad (**)$$

for a Sylow 2-subgroup of H which is an elementary abelian group of order 32. Substituting (*) into (**) gives the generators

$$w_1=a,\ w_2=a^r,\ w_3=r^s,\ w_4=a^t,\ w_5=a^q,$$

where $r=bab^{-1}(ab)^3ab^{-1}$, $s=b(ab^{-1})^3abab^{-1}$, $t=(bab^{-1}(ab)^3ab^{-1})^2$, $q=(b(ab^{-1})^3abab^{-1})^2$ for a Sylow p-subgroup P of G. The words $w_1,\ldots, w_5$ have length 1, 23, 23, 45, 45 in the free group F(a,b) respectively and the total length of the words is 135. We now input these words to PERMGP with a, b the above permutations and obtain the new generators :

$$w_1'=a$$
$$w_2'=bab^{-1}(ab)^7ab^{-1}ab$$
$$w_3'=b(ab^{-1})^3(ab)^3(ab^{-1})^3ab$$
$$w_4'=b(ab^{-1})^3(abab^{-1})^2(ab)^3ab^{-1}$$
$$w_5'=b(ab^{-1}ab)^2(ab)^4(ab^{-1}ab)^2$$

for P. Here the words $w_1',\ldots,w_5'$ have length 1, 21, 21, 23, 25 respectively and the total length is now 91. A further simplification is possible by conjugating by b the words $w_1',\ldots,w_5'$ which reduces the total length of the newly obtained words to 87.

3.3 Maximal subgroups

The following will be repeatedly used in obtaining generators for a maximal subgroup of G.

3.3.1 (Finkelstein [20]). Any maximal subgroup of a simple group S is the normalizer in S of a characteristically simple group L (i.e. a direct product of isomorphic simple groups (1.4.2)).

A maximal subgroup M of S is called a *p-local subgroup* if L is an elementary abelian p-group.

3.3.2. If S is a simple group, and M is a maximal subgroup of S, and L is a normal subgroup of M, then $M=N_S(L)$.

Proof. By 1.1.2, $M \leq N_S(L) \leq S$. Since M is maximal in S, either $M=N_S(L)$ or $S=N_S(L)$. But $N_S(L) \neq S$ for S is simple. Hence $M=N_S(L)$.

Let K be a finite 2-generator group and x, y $\in$ K. We call (x,y) a *generating pair* for K if K is generated by x and y. We say that (x,y) is of *type* (l,m,n;q) if $|\langle x\rangle|=l$, $|\langle y\rangle|=m$, $|\langle xy\rangle|=n$ and $|\langle [x,y]\rangle|=q$.

Now suppose that H is a 2-generator subgroup of a simple group G with a generating pair (x,y) of type (l,m,n;q). Then x, y, xy, and [x,y] fall into some classes C_α, C_β, C_γ, C_δ of elements of order l, m, n, q of G. Following the ATLAS, let us denote these classes by l X, mY, nZ, qU respectively , where X, Y, Z, U are the letter-names of the classes introduced in 4.1. With this notation and assumption, we call (x,y) a generating pair of *type* (l X,mY,nZ;qU).

We are now ready to describe some methods which help us to find generators in a, b for each maximal subgroup of G. We first note that a complete list of maximal subgroups of G has been given in the ATLAS together with some information about them such as their orders, indices, structures, and specifications. The specification of a maximal subgroup M of G gives information about the conjugacy classes of a normal subgroup N of M whose normalizer in G is M (by 3.3.2).

Throughout this discussion we assume that the group G is given by generators a, b, which may either be abstract generators with a set of defining relations or elements from a faithful permutation representation on the cosets of a subgroup of minimal index d. Let G act on A={1,2,...,d}.

(i) The stabilizer of a point

By 1.2.4, G is a transitive permutation group of degree d and thus the stabilizer of a point has index d in G using 1.2.1. That is, $St_G(x)$ is a subgroup of minimal index d for each x. Providing St(x) is a 2-generator group, we may use PERM in order to find the words $w_1(a,b)$ and $w_2(a,b)$ with the property that w_1, w_2 fix only the point x$\in$A with w_1, w_2, w_1w_2 having specified orders l, m, n. If l, m, n are correctly chosen such words almost always generate St(x). The integers l, m, n are usually obtained from the information given about the structure of the subgroup.

(ii) 2-generator maximal subgroups with presentation

Suppose M is a maximal subgroup of G for which a 2-generator presentation is known. Let

$$M=\langle x,y \mid r_1=r_2=...=r_s=1\rangle \qquad (*)$$

Suppose that x, y, xy, [x,y] have order l, m, n, q respectively. Then M is a factor group of the group

$$\langle x, y \mid x^l=y^m=(xy)^n=[x,y]^q=1\rangle \qquad (**)$$

It is now constructed by picking elements x, y from the appropriate conjugacy classes C_α, C_β of order l, m of G such that xy, [x,y] lie in the appropriate classes C_γ, C_δ of order n, q.

The type of (x,y) is usually obtainable from the specification of the maxinaml subgroup. To find such a generating pair for M we may begin with a random element x of G lying in C_α and use PERM in the hope of finding small random sets of words y with (x,y) of type (l,m,n;q). Using the representatives of the conjugacy classes of G we are now able to obtain a set of generating pairs (x,y) with $y\in C_\beta$, $xy\in C_\gamma$, $[x,y]\in C_\delta$. As often happens, x and y generate M and satisfy the presentation (*).

We note that if the classes into which x, y, xy, [x,y] fall are unspecified we then try to find a generating pair (x,y) for a subgroup isomorphic to M with x, y satisfying the presentation (**). We emphasise that the obtained subgroup need not be maximal, that is it may happen that G has two non-conjugate subgroups H_1, H_2 isomorphic to M with H_1 maximal in G but H_2 not.

We also note that, sometimes, the obtained generators x, y for M with (x,y) of type (l,m,n;q) are words in a, b of fairly long lengths. To find neater generators for M we may use the presentation (*) in order to obtain a new generating pair for M which we hope to lead to shorter generators for M.

(iii) Maximal subgroups with a 2-generator normal subgroup

Suppose M is a maximal subgroup of G having a non-trivial normal subgroup N with a known 2-generator presentation. By 3.3.2, $M=N_G(N)$. Now a similar method to that in (ii) is used to obtain a generating pair for N. Then TC enables us to determine $N_G(N)$.

Note that for a given maximal subgroup M we may have more than one non-trivial normal subgroup which is generated by two elements. In practice those with larger orders are prefered to be taken as our starting point. This clearly facilitates the enumeration of cosets N for the determination of $N_G(N)$.

If the generators of N are in unspecified classes of G, we then examine a set of random subgroups of G isomorphic to N in the hope of finding a subgroup whose normalizer in G is isomorphic to M.

(iv) p-local subgroups

Suppose M is a p-local subgroup of G. Then $M=N_G(E)$ where E is an elementary abelian group of order p^k (p, a prime dividing the order of G). Now suppose we have found generators for a Sylow p-subgroup P of G. Employing the presentation of P we are able to obtain a set of elementary abelian subgroups of order p^k. These subgroups are now examined by TC to determine a subgroup K isomorphic to E with $N_G(K)$ of order $|M|$. The hope is that $N_G(K) \cong M$. We note that in certain cases the information given about the classes into which the elements of E fall usually helps us to choose an elementary abelian subgroup whose normalizer in G leads to a maximal subgroup of G isomorphic to M.

(v) Non-local subgroups

Suppose M is a non-local subgroup of G. Then $M=N_G(K)$, where K is a direct product of isomorphic non-abelian simple groups S. Using a known presentation of S, we may find a presentation for K. Now if K is a 2-generator group (which is the case for our simple groups G, $|G| <10^6$), then a similar method to that of in (ii) can be used to determine generators for M.

(vi) structure constants technique

A further source of information about the subgroups of G which are generated by two elements x and y with x an involution is the structure constants for the classes of involutions of G.

Suppose C_i, C_j, C_k are conjugacy classes of G. Then *structure constants* [34], combinatorially, are the number a_{ijk} of ordered pairs (x,y) such that

$$xy=z \qquad (x\in C_i, y\in C_j, z\in C_k)$$

for fixed z.

The tables of [34] give the a_{ijk} for all classes C_i of involutions. We observe that the number of solutions $(x,y)\in G\times G$ to $xy=z$ for x, y belonging to fixed classes of G and z a fixed element cannot be less than the number of solutions in HxH of the same equation. If there is no contradiction we try to find a pair of elements of G which satisfy a known presentation of H.

(vii) Non-conjugate isomorphic maximal subgroups

In certain cases G has more than one conjugacy class of isomorphic subgroups. The following might help us to obtain generators in a, b for each of them using the methods (i)-(vi).

(1) Suppose that G has two non-conjugate subgroups of minimal index d. Suppose further that $H_1=St_G(r)$, where, $r\in A$. Then for each s in A, H_1 is conjugate to $St_G(s)$, by 1.2.2. We therefore look for a subgroup H_2 of index d with $|fix(x)\cap fix(y)|=0$, for each x, y in G , $x\neq y$.

(2) Suppose that G has two non-conjugate isomorphic subgroups T, S whose elements of order l, say, lie in two distinct classes of G. If T is a 2-generator group with a known presentation, then using (ii) we may obtain a generating pair for each of T, S. Sometimes $N_G(T)$, $N_G(S)$ remain non-conjugate in G giving two non-conjugate maximal subgroups of G.

(3) Finally, if G has two non-conjugate isomorphic subgroups T, S with elements lying in the same classes of G, then pairs of random subgroups S isomorphic to T are examined by the standard function

conjugate (G, T,S)

of CAYLEY in the hope of finding two non-conjugate subgroups T, S of G.

Suppose now we have found generators $x_1(a,b)$, $x_2(a,b)$, ..., $x_n(a,b)$ for a maximal subgroup M of G. We then try to combine x_i's differently in order to reduce the number of generators of M. If M is a 2-generator group, an attempt is made to show that M can be generated by two of its elements x, y with x an involution.

Having found two generators $w_1(x_1,...x_n)$, $w_2(x_1,...,x_n)$ for M, we are now able to determine the classes into which these generators and their product, commutator, etc. fall. We then use PERM again to find new generators in a and b, possibly of shorter lengths, for a conjugate of M. It has been found experimentally that if the orders of w_1, w_2 are sufficiently small, then almost always PERM will produce a generating set for a conjugate of M.

In addition, if w_1 is an involution, a presentation for M on w_1, w_2 can be found using the efficient method of [6].

Tests for maximality of subgroups

Suppose that M_1, M_2, .., M_n are maximal subgroups of G, arranged in order of decreasing order. Suppose further that we have found a generating set for each of the maximal subgroups $M_1, M_2, ..., M_{k-1}$ ($1 < k \leq n$) of G and a generating set for a subgroup H of G of order $|M_k|$. If none of the M_i's have subgroups isomorphic to H, then H is clearly maximal. We now suppose that the only maximal subgroups among the M_i's which have a subgroup isomorphic to H are $M_{i1}, M_{i2}, ..., M_{ij}$. If for each t=1,...,j, M_{it} and H contain elements of a specific order from distinct classes of G, then H cannot be a subgroup of any conjugate in G of M_{it} and thus is maximal in G.

Note. Various techniques can be used to show whether a finite group K has a subgroup isomorphic to a given group H. These will be explained in chapter 4 as and when they are needed to be applied. Alternatively, the CAYLEY program SUBGPTEST (2.4) can always be used to check whether or not a certain finite group (of small order) has any subgroup isomorphic to a given 2-generator group H.

Example 3. Determine generators for each maximal subgroup of the group $G=A_8$ with presentation

$$\langle a,b \mid a^2=b^4=(ab)^{15}=(ab^2)^4=(ab)^5ab^2ab(ab^{-1})^2(ab)^2ab^{-1}(ab)^7ab^{-1}=1\rangle$$

where (a,b) is a minimal generating pair of the group with

a=(1,4) (2,7) (3,5) (6,8),
b=(3,4)(5,6,7,8).

In order to be able to give generators for each of the maximal subgroups of G we invoke the table II given on page 105 which lists the six conjugacy classes of maximal subgroups H_i (i=1,...,6) of G. We proceed as follows:

(i) H_1 structure: A_7

This is the stabilizer of a point in the permutation representation of G and can be constructed using 3.3 (i) . The simple group A_7 has presentation

$$\langle x,y \mid x^2=y^4=(xy)^7=[x,y]^5=(xyxy^2xy^{-1})^3=1\rangle$$

and thus is generated by two elements x, y of G of order 2, 4 whose product has order 7. The group G has two conjugacy classes of involutions with representatives a, b^2. Since the element a moves all eight points of A={1,2,...,8}, we take $x=b^2$ which fixes 4 points of A, namely 1, 2, 3, 4. Using PERM, we find the word $y=(bab)^{aba}$ of order 4 which fixes the points 1, 7 and its product with x has order 7. In fact $H_1=\langle x,y\rangle=St_G(1)$ with x, y satisfying the above presentation for A_7.

(ii)-(iii) H_2, H_3 structure: 2^3:PSL(2,7) (2-local subgroup)

G has two non-conjugate maximal subgroups of structure 2^3:PSL(2,7) both being the normalizer in G of an elementary abelian subgroup of order 8 whose involutions are in 2A. Suppose we have found the generators

$$x=a,\ y=a^b,\ z=b^2ab^2 \qquad (*)$$

for a Sylow 2-subgroup P of G with presentation

$$\langle x,y,z \mid x^2=y^2=z^2=(xz)^2=(xy)^4=(yz)^4=(xyz)^4=1\rangle$$

on the generators x, y, z. Using this presentation we may obtain the elementary abelian subgroup E ($\leq$ P) generated by x, x^y, x^{yz} of order 8 whose generators when written in terms of a, b using (*) generate an elementary abelian subgroup of G with involutions in 2A. Let us call this subgroup, also, by E. Then

$$E=\langle a, a^t, a^s\rangle$$

where $t=a^b$, $s=(b^{-1}a)^2b^2$. Now TC enables us to determine $N_G(E)$. We find $N(E)=\langle u,v\rangle$, where $u=a^b$, $v=b(ab^{-1})^5(ab)^2(ba)^2(b^{-1}a)^2b^{-1}$, with $|\langle u,v\rangle|=1344$. Since 1344 does not divide the order of H_1, $\langle u,v\rangle$ is maximal in G. Now an easy calculation with CAYLEY using the class information of G shows that $v\in 7A$, $uv\in 4B$, $[u,v]\in 4A$. Thus (u, uv) is a generating pair of type (2A,4B,7A; 4A) for a maximal subgroup of G of order 1344. This helps us to seek a neater generating pair for a subgroup of this order. PERM easily gives the generating pair (a^b, b^a) for a maximal subgroup H_2 of G isomorphic to $\langle u,v\rangle$. Similarly we find the generating pair (bab^{-1},b^a) of the above type for a maximal subgroup H_3 ($\cong H_2$) which is not conjugate to H_2.

(iv) H_4 structure: S_6 (non-local)

S_6 has presentation

$$\langle x, y \mid x^2=y^6=(xy)^5=[x,y]^3=[x,y^2]^2=1\rangle$$

and thus is generated by two elements x and y with (x,y) of type (2,6,5;3). We start with x=a which is in 2A and find, by PERM, $y=(ba)^2b$ in 6A with $xy\in 5A$, $[x,y]\in 3B$. Now x and y generate a subgroup H_4 of order 720 and satisfy the above presentation for S_6. We note that $H_4=N_G(N)$, where $N=\langle [x,y^2], xy^3xy\rangle$. Letting $r=[x,y^2]$, $s=xy^3xy$, we see that (r,s) is a generating pair of type (2B,4B,5A;4B) for an A_6 subgroup of G. Again H_4 is maximal in G for its order does not divide the order of H_i (i=1, 2, 3).

(v) H_5 structure: 2^4:(S_3xS_3) (2-local)

This is the normalizer in G of an elementary abelian subgroup of G of order 16. Returning to the Sylow 2-subgroup P of G, it is found that P has only one elementary abelian subgroup of order 16 which is generated by

$g=xyzyxyz$, $h=xzyxyzy$, $k=xyzyxzy$, $l=xyxyzyz$.

Substitution of (*) for x, y, z in these generators, gives an elementary abelian subgroup K of G of order 16 whose fifteen cyclic subgroups number 9 containing the class 2A and 6 containing the class 2B. A coset enumeration using TC verifies that $N_G(K)=\langle u,v\rangle$ where $u=(b^2a)^2$, $v= b^2ab^{-1}ab(ab^{-1})^3abab^{-1}$ with $|G:\langle u,v\rangle|=35$. Since $|G:H_i| \nmid 35$ (i= 1, 2, 4), $\langle u,v\rangle$ is maximal in G. Again it is easy to check that $u\in 2B$, $v\in 6B$, $uv\in 6B$ and $[u,v]\in 6A$. We may now use PERM to obtain the neater generators $x=(ab^2)^2$, $y=(ab)^3ab^{-1}$ with (x,y) of type (2B,6B,6B;6A) for a maximal subgroup of H_5 of index 35.

(vi) H_6 structure: $(A_5x3):2$

The structure of H_6 shows that $H_6=N_G(H)$, where H is a subgroup of G isomorphic to A_5x3. The group A_5x3 has presentation

$$\langle R, S, T \mid R^2=S^3=(RS)^5=T^3=[R,T]=[S,T]=1\rangle,$$

and can be generated by R and ST with (R, ST) of type (2,3,15;5). Now the information given in [34] about the structure constants for the class 2A shows that the equation xy=z with $x\in 2A$, $y\in 3AB$, $z\in 15AB$ has no solution in G. Also such an equation cannot have any solution with $x\in 2B$, $y\in 3A$, $z\in 15AB$. Therefore, a generating pair for a subgroup isomorphic to $3xA_5$ must be of type (2B,3B,15AB). Armed with this knowledge, we find by PERM $x=b^2$, $y=b^{-1}ab(ab^{-1})^3(ab)^2b$ which generate a $3xA_5$ subgroup of G. Now using TC, we get $N_G(\langle x,y\rangle)=\langle b,y\rangle=\langle b, ab(ab^{-1})^3aba\rangle$ which has order 360. Let $z=ab(ab^{-1})^3aba$, $H_6=\langle b,z\rangle$. Then H_6 is a maximal subgroup of G since $|H_6|$ does not divide $|H_i|$ for i=1, 2, 3, 5, and that H_6, having an element of order 15, cannot be a subgroup of H_4.

We note that bz^{-2} has order 2 and thus H_6 can be generated by two elements x'and y' with x' an involution. This can be used to find the new generators $x'=b^2$, $y'=(ab)^2ab^2ab$ for a maximal subgroup of G of order 360.

3.4 The Schur multiplier

Suppose that G is a finite group given by

$$G=\langle x_1,x_2,...,x_n \mid r_1= ...=r_m=1 \rangle$$

where each r_i is a word in x_i's. The standard function *darstellungsgruppe* of CAYLEY when applied to G, as in the expression *darstellungsgruppe(G)*

will find a covering group C of G. The group C is created as a finitely presented group of the form

$$<x_1, \dots , x_n, z_1, \dots , z_t \mid r_1=r_1', \dots, r_m=r_m', [x_i,z_j]=1, [z_k,z_l]=1 \quad (1 \le i \le n, 1 \le j \le t, 1 \le k < l \le t)>$$

where each r_i' is a word in the z_j's.

Now the Schur multiplier M(G) of G may be computed using the following facts:

(i) $C/M(G) \cong G$,

(ii) $M(G) \le Z(C) \cap C'$.

By (i) we first try to determine $|M(G)|$. Then (ii) will enable us to find a generating set for M(G). As often happens these generators for M(G) are among the z_i's which are central in C. After determining their orders, it is an easy task to write M(G) as a direct product of cyclic subgroups.

We note that the presentation found for C is often unpleasant so that the function "darstellungsgruppe" is limited in its usefulness.

We also note that if the presentation for G is badly chosen, the function results in a complicated presentation for C. It has been experimentally found that for a fixed number of generators for G, the number of generators for C increases by increasing the number of relations of G. This means that if we choose a presentation for G with a small number of relations, we will almost always arrive at a somewhat simpler presentation for C.

Example 4. We begin with a group where the answer is known in advance, and find the Schur multiplier of the group $D_{12}xA_4$.

We start with the presentation

$$<r,s,t,u \mid r^2=s^6=(rs)^2=t^2=u^3=(tu)^3=[r,t]=[r,u]=[s,t]=[s,u]=1>$$

and construct the following new presentation

$$<x, y \mid x^2=y^6=(xy)^6=(xy^3)^2=((xy)^2(xy^{-1})^2)^2= 1>$$

for G on x=rt, y=su using SUBGP. Now a covering group C for G using this presentation is found to have presentation

$$<x,y,z,u,v \mid x^2=y^6zu=(xy)^6u=(xy^3)^2=((xy)^2(xy^{-1})^2)^2v^{-1}=1, [x,z]=[x,u]=[x,v]= [y,z]=[y,u]=[y,v]=[z,u]=[z,v]=[u,v]=1>.$$

Eliminating z, u, and v gives

$$C=<x,y|x^2=(xy^3)^2=[x,y^6]=[x,(xy)^6]=[y,(xy)^6]=[x,((xy)^2(xy^{-1})^2)^2]= [y,((xy)^2(xy^{-1})^2)^2]=1>.$$

By enumerating cosets of <x> using TC, we find that $|C|=576$ and thus $|M(G)|=4$. It is now easy to check that $C'=<[x,y],[x,y^2],[x,y^3]>$ and that $X=(xy)^6$, $Y=((xy)^2(xy^{-1})^2)^2$ are each central elements of C of order 2 belonging to C'. That is, $M(G)=C_2xC_2$ as expected.

3.5 Efficient presentations

Suppose G is a finite group and rank M(G)=d. In order to be able to obtain an efficient presentation for G one needs to show that G has a presentation on n generators and n+d relations. We describe here a method similar to that used by Campbell and Robertson in [4] which attempts to reduce a given presentation of a group K to an efficient one.

We start with a presentation of K having as small a number of generators and relations as possible. Suppose

$$K=\langle X \mid R\rangle,$$

$X=\{x_1,...,x_s\}$, $R=\{r_1^{n_1},...,r_t^{n_t}\}$ where $r_i=r_i(x_1,...,x_s)$ and n_i's are positive integers. We now try to reduce the number t of relations by considering a stem extension H of K which we hope to be isomorphic to K. To do this, we take two words $r_k^{n_k}$ and $r_j^{n_j}$ from the set R in such a way that r_k and r_j generate the group H defined by

$$H=\langle X \mid r_k^{n_k} r_j^{-n_j}, R\backslash\{r_k^{n_k}, r_j^{n_j}\}\rangle.$$

Then the element $r_k^{n_k}$ is in Z(H) because it commutes both with r_k and r_j. Now if, in addition, $r^{n_k}\in H'$ then

$$r_k^{n_k} \in Z(H) \cap H'$$

and thus H will be a stem extension of K, that is, a finite group of order at most $|M(K)||K|$ having K as a homomorphic image. It may happen that $|H|=|K|$, i.e. $H \cong K$. If this is so, we have found a new presentation of K on d generators and t-1 relations. This process of reduction is continued until an efficient presentation for K is obtained.

We note that in constructing a stem extension H of K the condition that r_k and r_j generate H is a self-imposed condition to guarantee that the element $r_k^{n_k}$ is central in H. In certain situations, it is possible to combine two arbitrary elements of R and yet obtain a stem extension of K.

Suppose now we have constructed a stem extension H of K. Then the following lemma taken from [4] allows us to determine when H is actually K itself using a coset enumeration program to determine the indices of certain cyclic subgroups of H rather than enumerating the cosets of the trivial subgroup which, in certain cases, exceeds the storage capacity of the machine.

3.5.1 Suppose H is a stem extension of K, $h \in H$, and φ the homomorphism from H to K. Let $r=|\langle h\varphi\rangle|$, $m=|M(K)|$. If $(m,r)=1$. Then

(i) $|H|=r\,|H:\langle h^m\rangle|$;

(ii) if $|H:\langle h^p\rangle|=|H:\langle h\rangle|=|K:\langle h\varphi\rangle|$ for each prime p dividing m, then $H \cong K$.

Proof. (i) Let $N=\mathrm{Ker}\varphi$, $s=|\langle h\rangle|$. Then $s=rq$ where $q \mid |N|$. But $|N|$ divides m, by 1.5.13, so $q \mid m$. Now $|\langle h^m\rangle|=s/(s,m)=r$ for $(m,r)=1$.

(ii) It follows that $|\langle h^p\rangle|=|\langle h\rangle|=s$, for each prime p dividing m. Hence $(s,m)=1$and so $q=1$. That is, h and $h\varphi$ have the same order. Therefore, $H \cong K$.

Let us carry out the foregoing technique for the group S_7, the symmetric group.

Example 5. Prove that S_7 is efficient.

The symmetric group S_7 has order 5040 and Schur multiplier the cyclic group of order 2 by 1.5.11 (i). To prove S_7 efficient we look for a 2-generator 3-relation presentation. We begin with the following presentation of S_7 given in [18] :

$$K=\langle x,y \mid x^2=y^7=(xy)^6=(xy^2xy^{-2})^2=(xy^3xy^{-3})^2=1\rangle.$$

Let H_1 be the group obtained by combining two relations as follows:

$$H_1=\langle x,y \mid x^2=y^7=(xy^3xy^{-3})^2=1,\ (xy)^6=(xy^2xy^{-2})^2\rangle$$

It is easy to check that H_1 is generated by xy and xy^2xy^{-2} so $(xy)^6$ is a central element of H_1. Next we see that $H_1'=\langle [x,y],[x,y^{-1}],[x,y^2],[x,y^3],[x,y^{-3}]\rangle$ and that $(xy)^6 \in H_1'$. Therefore $(xy)^6$ is in $Z(H_1)\cap H_1'$ showing that H_1 is a stem extension of K so H_1 is either S_7 or its covering group by 1.5.13. TC verifies that $|H_1:\langle y\rangle|=720$, i.e. $H_1 \cong S_7$. Now we define

$$H_2=\langle x,y \mid x^2=y^7,\ (xy^3xy^{-3})^2=1,\ (xy)^6=(xy^2xy^{-2})^2\rangle.$$

A similar verification shows that H_2 is a stem extension of H_1 ($\cong S_7$). Using TC we have $|H_2:\langle y^2\rangle|=720$. It follows that $H_2 \cong S_7$ by 3.5.1 (i).

Unfortunately, the aim of our reduction method cannot always be acheived. As often happens, on combining the realtions of a given presentation of a group K using the above method we arrive at larger groups –even infinite groups– having K as a homomorphic image. We therefore need to examine several presentations of K in the hope of finding a suitable one to start with.

Since an exhaustive search for such presentations is not practicable, a step forward might be to try to find a small set of presentations for K to be examined by the reduction method. For 2-generator groups of small orders this could be done using a method suggested by Kenne [31]. In this method a small set of generating paris (x,y) for K are chosen and a presentation on x and y is constructed using Cannon's algorithim [13]. Then those presentaions with fewer relations are taken to be modified.

In chapter 4 where we examine the efficiency of maximal subgroups of simple groups G, $|G|<10^6$, we will employ this technique. We note that since we will be dealing with permutation groups of small order, the method will work successfully.

We conclude this section with a CAYLEY program, similar to that in [31], which attemps to find generating pairs for A_4xA_5

```
> g : perm(9);
> g.generators: a=(1,2)(3,4), b=(1,2,4), c=(5,7)(8,9),d=(5,8,6) ;
> cl=classes (g);
>  FOR i=2 TO length (cl) DO
>    FOR j=i+1 TO length (cl) DO
>      h=< cl[i], cl[j] >;
>      IF order (h) EQ order (g)
>         THEN print i, j,  relations (h) ;
>      END;
>    END;
> END;
```

It is found that class representatives x=(2,4,3)(5,9,7,8,6) and y=(1,2,4)(5,6,7) generate A_4xA_5 and yield a presentation

$$<x,y \mid x^{15}=y^3=(xy)^2=(x^3y^{-1})^3=1>,$$

which is an efficient presentation for A_4xA_5 since $M(A_4xA_5)=C_2xC_2$ by 1.5.12.

Note. Suppose H is a subgroup of G generated by two words x=x(a,b), y=y(a,b). By considering the permutation representation of G, we may use a similar program to find a set $\mathcal{P}$ of presentations for H. Suppose that $P\in\mathcal{P}$ reduces to an efficient presentation for H. Now we require words $w_1(a,b)$, $w_2(a,b)$ which generate H and satisfy the presentation P for H. To do so, we first determine the classes of G into which the new generators of H (as permutations) fall ; then PERM can be used to produce two words in x, y having the desired property.

4.The Results

In this chapter we shall make use of the methods described in chapter 3 in order to give generators in a, b for each maximal subgroup of the twenty non-abelian simple groups G listed on page 1, where (a,b) is the first minimal generating pair of G appearing in [35]. For each group G, generators in a, b of a Sylow p-subgroup, for each prime p, are also given. Similarly we give generators in a, b for Sylow p-subgroups and maximal subgroups of the family of PSL(2,q) listed on page 3, where (a,b) is a minimal generating pair of the groups satisfying the relations $a^2=b^3=1$. We also give a representative for each conjugacy class of the groups as a word in a, b.

More details about Sylow p-subgroups and maximal subgroups including their defining relations, multipliers, etc. are given. We note that a complete list of generators for Sylow p-subgroups and maximal subgroups for each distinct minimal generating pair of each group G is given in the CAYLEY file SIMGPS.TLB (see Appendix).

Throughout this chapter, G denotes a non-abelian simple group with a minimal generating pair (a,b) mentioned above.

4.1 The tables

For each of the 32 simple groups G, we record its name, order, minimal degree d as a permutation group, multiplier, and a presentation on a, b. (Presentations for the twenty non-abelian simple groups are taken from [16], [6].) When G is a PSL(2,q), q > 9, these are followed by two permutations of degree d satisfying the presentation . We then record two tables under 'conjugacy classes of elements of G' and 'conjugacy classes of maximal subgroups of G' , mainly taken from the ATLAS, to be used in obtaining our results on subgroups of G.

The first table contains the following information :

(1) Class name

The first column gives the class names of conjugacy classes of G. The conjugacy classes that contain elements of order n are named nA, nB, nC,... e.g., class 15D is the fourth class of order 15. For the first class C of each family we give the name of the class (order followed by letter-name) in full. Each succeeding class of the same family is identified by its class name by applying the algebraic conjugacy operator *k (or **k,

**, *). The algebraic conjugacy operators are defined on classes as follows:

(nX)*k	contains the kth powers of elements of nX;
(nX)**k	contains the (-k)th powers of elements of nX;
(nX)**	contains the inverses of elements of nX;
(nX)*	is the class other than nX containing elements of order n that are powers of elements of nX, when this class is unique.

(It is to be understood that k is prime to n.)

(2) Centralizer orders

The second column gives the order of the centralizer in G of a typical element of each class.

(3) p.power

The third column gives the letter-names of the classes that contain the powers g^p, g^q, g^r,... of g, where $p < q < r <$... are the distinct prime divisors of the order of g. Thus if g has order 60, an entry ABC means that $g^2 \in 30A$, $g^3 \in 20B$, $g^5 \in 12C$.

(4) p'-part

The fourth column gives the letter-names of the classes containing the p'-part of g, also in increasing order of the primes p dividing n. For example, if g has order 60, an entry BAB means that $g^{16} \in 15B$, $g^{21} \in 20A$, $g^{25} \in 12B$.

(5) Representatives of classes

The fifth column gives a representative as a word in a, b for each of the classes.

(6) Cycle type

The sixth column indicates the cycle type of each of the representatives in the permutation representation of G.

The second table contains information about the order, index, structure, specification, and the multiplier of each maximal subgroup H_i of G. The *specification* of maximal subgroups gives information which might help us to locate a copy of H_i inside G. A specification expresses H_i as the normalizer in G of some element or subgroup (see 3.3 for details). This is indicated by writing N(...), where the parentheses contain information about the conjugacy classes of the group being normalized.

The class of a cyclic subgroup is indicated by its order, followed by the letter-names of its generators, subscripts are used to count cyclic subgroups in a larger group, and superscripts indicate direct powers of groups. Thus the symbols below would be used for the normalizers in G of elements or groups of the indicated forms :

N(2A):	an involution in G, of class 2A.
N(3A):	a group of order 3, with each generator in class 3A.

N(5AB):	a group of order 5, containing both classes 5A and 5B.
$N(2A^2)$:	a four-group, whose involutions are in class 2A.
$N(3^3)=N(3AB_4C_3D_6)$	an elementary abelian group of order 27, whose 13 cyclic subgroups number 4 containing both classes 3A and 3B, 3 containing 3C only, 6 containing 3D only .
$N(2^6)$:	an elementary abelian group of order 2^6, unspecified class.
N(2A,3B,5CD):	an A_5, containing elements of classes 2A, 3B, 5C, 5D.
$N(2A,3B,5CD)^2$:	the direct product of two such A_5 groups.
N(2A,2C,3A,3B,...):	a group containing elements of the indicated classes, among others.

The type of the group being normalized, when not obvious, can usually be deduced from the structure information about H_i.

4.2 Notation for group structure

We follow the ATLAS' notation to indicate the structure of maximal subgroups excepting linear groups which will be denoted by GL(n,q), PGL(n,q), etc. (after Van der Waerden). The notations C_n (cyclic), D_n (dihedral), Q_n (quaternion) are often used for groups of order n with the indicated structures. The notations S_n, A_n will denote the symmetric and alternating groups of degree n respectively. We also use the following notation :

AxB denotes the direct product of A and B.

A.B (or AB) denotes any group having a normal subgroup of structure A, for which the corresponding factor group has structure B.

A:B indicates a case of A.B which is a semi-direct product (or split extension).

$A{\cdot}B$ indicates any case of A.B which is not a split extension.

In our structure symbols for complicated groups, we used the abbreviations:

n indicating a cyclic group of order n.

A^n for the direct product of n groups of structure A .

p^n where p is prime, indicates the elementary abelian group of that order. Thus the two types of groups of order four become 4 (the cyclic group), and 2^2 (the four-group).

p^{n+m} indicates a case of $p^n.p^m$, and so on.

p^{1+2n} or p_+^{1+2n} or p_-^{1+2n} is used for the particular case of an extra-special group. For each prime p and positive n, there are just two types of extra-special groups, which are central products of n non-abelian groups of order p^3 (see, 1.3.5). For odd p, the subscript is + or − according as the group has exponent p or p^2.

For p=2, it is + or − according as the central product has an even or odd number of quaternionic factors.

4.3 Notation for generating pairs

We use the following notation in conjunction with generating pairs for a group (subgroup) defined in 3.3 :

(l,m,n)	indicates type of a generating pair (x,y) for a group (subgroup): $|\langle x\rangle|^l=|\langle y\rangle|^m=|\langle xy\rangle|^n=1$, x, y in unspecified classes.
(l,m,n;k)	indicates a case of (l,m,n) in which the commutator [x,y] has order k.
(l,m,n/k)	indicates a case of (l,m,n) in which the element xy^2 has order k.
(l,m,n;k,q)	indicates a case of (l,m,n;k) in which xy^2 has order q.
(2A,3B,10B;5E)	indicates type of a generating pair (x,y) for a subgroup H of a group K with $x\in 2A$, $y\in 3B$, etc.(2A, 3B,... are classes of K).
(2A,3B,10;5E)	indicates type of a generating pair for H where K has only one family of conjugacy classes of elements of order 10.
(2A,3B,10ABC)	indicates type of a generating pair (x,y) for H where xy is either in 10A, 10B, or 10C (same family).
#(2A,3B,5D)	denotes the number of distinct generating pairs (x,y) for A_5 subgroups with $x\in 2A$, $y\in 3B$, $xy\in 5D$.

4.4 The Results

As was mentioned earlier, each maximal subgroup of G can be generated by two elements of G with the exception of a maximal subgroup of A_7 of order 72. For purposes of finding such generating pairs for each maximal subgroup M of G in its generators the conjugacy classes of G matter. For this reason we shall determine not only a generating pair (x,y) for M but its type indicating the classes into which x, y, xy, and [x,y] (and /or a specific element of G) belong. This would be useful when we are given an arbitrary generating pair (u,v) for G (not necessarily minimal) and wish to find two generators in u,v for a maximal subgroup of G. We therefore obtain representatives of conjugacy classes of G in a, b and include in table I (as stated in 4.2) followed by their cycle types when a, b are regarded as permutations. One word of warning- for certain simple groups G, some of the class names in [35] are slightly different from those in the ATLAS. Accordingly, the cycle types given in table I, in a few cases, may differ from those in [35].

For each group G, we shall fully indicate how the methods described in chapter 3 are used in obtaining generators of Sylow p-subgroups, maximal subgroups of G and their minimal presentations. To obtain a minimal presentation for each maximal subgroup we shall first compute its Schur multiplier and record this in the sixth column of our second table.

For the rest of the chapter, H_i will denote a maximal subgroup of G and P_p, p prime, a Sylow p-subgroup of G. The notation $n_p(G)$ will be used to denote the number of Sylow p-subgroups of G. All Sylow 2-subgroups whose order are at most 64 will be identified using the lists given in [22], [39] for groups of order 2^n, $n \le 6$. In [39] an individual group P is given a designation such as 64,30, $(12^2,12^2)$ where 64 is the order of the group, 30 means it is number 30 of the groups of order 64 in [22], 12^2 is the terminology used in [22] to designate the abelian group $C_2xC_4xC_4$. The first 12^2 means that $P/P'=C_2xC_2xC_4$ and the second 12^2 means that the multiplier of P is $C_2xC_4xC_4$. We shall use both the notations used in [22] and [39] in identifying our Sylow 2-subgroups.

We note that H_1, the stabilizer of a point in the permutation representation of G, is taken from [7], [6] unless otherwise stated. We shall also determine intersections of all pairs of non-conjugate isomorphic maximal subgroups in G.

Data concerning the minimal generating pairs for the twenty simple groups are taken from the CAYLEY library of finite simple groups [8]. For the ten PSL groups we shall consider a presentation of the form $G=\langle a,b \mid a^2=b^3=(ab)^k=((ab)^i(ab^{-1})^j)^2=1\rangle$ and construct a permutation representation for G of minimal degree.

Finally we note that the permutation representation and the presentation of G will alternatively be used in this chapter. A use of the program PERM will mean that we deal with the permutation representation of G while that of the TC will indicate that the abstract group G (with generators and relations) is under our consideration.

$A_5 \cong PSL(2,5) \cong PSL(2,4)$

order=60=2^2.3.5   d=5   mult=2

G=< a,b | $a^2=b^3=(ab)^5=1$>

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	60			1	1^5
2A	4	A	A	a	$1^1 2^2$
3A	3	A	A	b	$1^2 3^1$
5A	5	A	A	ab	5^1
B*	5	A	A	$(ab)^2$	5^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	12	5	A_4	N($2A^2$)	2
H_2	10	6	D_{10}	N(5AB)	1
H_3	6	10	S_3	N(3A)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup   *order=4*

The Sylow 2-subgroup of G is the Klein 4-group since G has no element of order 4. Using this fact we look for, by PERM, a generating pair (x,y) of type (2,2,2;1). We find the generators x=a and y=$[b,a]^2b^{-1}$ for a Sylow 2-subgroup P_2 of G.

$n_2(G)$=| G:N(P_2) |=5.

(ii) Sylow 3-subgroup   *order=3*

P_3=<b>=C(b), $n_3(G)$=| G: N(P_3) |=10.

(iii) Sylow 5-subgroup   *order=5*

P_5=<ab>=C(ab), $n_5(G)$=| G: N(P_5) |=6.

II.Maximal subgroups

(i) structure: A_4

H_1=< a^b, b^a>.

Taking x=a^b and y=b^a we see that x and y satisfy the presentation

$$<x,y \mid x^2=y^3=(xy)^3=1>.$$

The multiplier of A_4 is C_2 and the above presentation shows that A_4 is efficient.

(ii) structure: D_{10}

We use PERM to find the generators $x=a^{ba}$ and $y=bab$ of type (2,5,2). Then x, y generate a subgroup H_2 of G of order 10 and satisfy

$$<x,y \mid x^2=y^5=(xy)^2=1>.$$

H_2 is maximal in G since $10 \nmid 12$.

(iii) structure S_3

A generating pair for S_3 is of type (2,3,2). PERM now gives x=a and $y=bab^{-1}(ab)^2$ with x, y, and xy having order 2, 3, and 2 respectively. $H_3=<x, y>$ is a maximal subgroup of G for H_1 ($\cong A_4$) has no subgroup isomorphic to S_3 and 6 does not divide 10.

$M(S_3)=1$ and we have

$$S_3=<x,y \mid x^2y^3=(xy)^2=1>.$$

$PSL(2,7) \cong PSL(3,2)$

order=168=2^3.3.7 d=7 mult=2

$G=\langle a,b \mid a^2=b^3=(ab)^7=[a,b]^4=1\rangle$

conjugacy classes of elements of G

class	$\lvert c(x)\rvert$	p-power	p'-part	representative	cycle type
1A	168			1	1^7
2A	8	A	A	a	$1^3 2^2$
3A	3	A	A	b	$1^1 3^2$
4A	4	A	A	[a,b]	$1^1 2^1 4^1$
7A	7	A	A	ab	7^1
B**	7	A	A	$b^{-1}a$	7^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	24	7	S_4	$N(2A^2)$	2
H_2	24	7	S_4	$N(2A^2)$	2
H_3	21	8	7:3	N(7AB)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=8

Since PSL(2,7) has a S_4 subgroup, its Sylow 2-subgroup is D_8. Taking x=a of order 2 PERM finds $y=a^b$ with xy of order 4 and hence x, y generate a Sylow 2-subgroup P_2 of G.

$n_2(G)=\lvert G:N(P_2)\rvert=21.$

(ii) Sylow 3-subgroup order=3

$P_3=\langle b\rangle=C(b)$, $n_2(G)=\lvert G:N(P_3)\rvert=28.$

(iii) Sylow 7-subgroup order=7

$P_7=\langle ab\rangle=C(ab)$, $n_7(G)=\lvert G:N(P_7)\rvert=8.$

II. Maximal subgroups

(i)-(ii) structure: S_4

G has two conjugacy classes of maximal subgroups of the structure S_4. A generating pair for S_4 is of type (2,3,4). Starting from x=a of order 2 we look for, by PERM, generating pairs (x, y_1) and (x, y_2) of the above type with $|fix(x) \cap fix(y_1)|=1$ and $|fix(x) \cap fix(y_2)|=0$ respectively. We find $y_1=b^{ab}$ and $y_2=b^t$ where $t=ab^{-1}$. For i=1,2, the elements x and y_i generate a subgroup H_i of minimal index 7 and satisfy

$$S_4=\langle x,y \mid x^2=y^3=(xy)^4=1\rangle.$$

H_1 is not conjugate to H_2, and has intersection S_3 with H_2 in G.

$M(S_4)=C_2$, by 1.5.11(ii), and the above presentation provides an efficient presentation for S_4.

(iii) structure: *7:3*

This is the normalizer in G of a cyclic subgroup of G whose generator lies in 7AB. Taking u=ab, TC finds $N_G(\langle u\rangle)=\langle u,v\rangle$ where $v=b^s$ with $s=abab^{-1}$. The pair (u,v) has type (7,3,3) and is a generating pair for a split metacyclic group of order 21. PERM now enables us to give neater generators $x=b^a$ and $y=b^{ab}$ of type (3,3,7) for a maximal subgroup H_2 of G of order 21. The generators x and y satisfy the deficiency zero presentation :

$$\langle x,y \mid x^3=1, (yx)^2=xy \rangle.$$

It is easy to show that xy generates a cyclic normal subgroup of $\langle x, y\rangle$ of order 7 with $\langle xy\rangle \cap \langle x\rangle=(1)$, i.e. $H_2 \cong 7:3$.

$A_6 \cong PSL(2,9)$

order=360=$2^3.3^2.5$ d=6 mult=6

G=< a,b | $a^2=b^4=(ab)^5=(ab^2)^5=1$>

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	360			1	1^6
2A	8	A	A	a	$1^2 2^2$
3A	9	A	A	$ab^2ab^{-1}ab$	$1^3 3^1$
3B	9	A	A	$abab^{-1}ab^2$	3^2
4A	4	A	A	b	$2^1 4^1$
5A	5	A	A	ab^2	$1^1 5^1$
B*	5	A	A	ab	$1^1 5^1$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	60	6	A_5	N(2A,3A,5A)	2
H_2	60	6	A_5	N(2A,3B,5A)	2
H_3	36	10	3^2:4	$N(3^2) \cong N(3A_2B_2)$	3
H_4	24	15	S_4	$N(2A^2)$	2
H_5	24	15	S_4	$N(2A^2)$	2

I. Sylow p-subgroups

(i) Sylow 2-subgroup order=8

By PERM, we find the generators x=a and y=a^b for a Sylow 2-subgroup P_2 of G isomorphic to D_8.

$n_2(G)=|G:N(P_2)|=45$.

(ii) Sylow 3-subgroup order=9

G has no element of order 9 and so the Sylow 3-subgroup of G is $C_3 x C_3$. A generating pair for such a group is (3,3,3;1). PERM can now be used to give the generators $x=ab^{-1}abab^2$ and $y=bab^{-1}(ab)^2$ for a Sylow 3-subgroup P_3 of G.

$n_3(G)=|G:N(P_3)|=10$.

(iii) Sylow 5-subgroup order=5

P_5=<ab>=c(ab), $n_5(G)=36$.

II.Maximal subgroups

(i)-(ii) structure: A_5

By table II, G has two non-conjugate A_5 subgroups with generating pairs (x_1,y_1) and (x_2,y_2) of types (2A,3A,5A) and (2A,3B,5A) respectively. We take $x_1=x_2=a$ and find, by PERM, $y_1=ab^2ab^{-1}ab$ and $y_2=abab^{-1}ab^2$ with $y_1\in 3A$ and $y_2\in 3B$ where xy_i has order 5 (i=1, 2). Taking $H_1=<x_1,y_1>$ and $H_2=<x_2,y_2>$ we observe that $H_1\cong H_2 \cong A_5$ and that H_1 is not conjugate to H_2 because H_1 is the stabilizer of a point in the permutation representation of G while y_2 being an element of H_2 fixes no points (see the cycle type of y_2 as a representative for the class 3B).

H_1 has intersection D_{10} with H_2 in G .

(ii) structure: $3^2{:}4$

This is the normalizer in G of a Sylow 3-subgroup. We take the generators x and y of P_3 given in (I) and find , by TC, that $N_G(<x,y>)=<y,b>$ with $|<y,b>|=36$ where $y=bab^{-1}(ab)^2$. This leads to the generators $r=a^b$ and $s=b^a$ for a maximal subgroup H_3 of G of order 36 since $6\nmid 10$. A presentation for H_3 on r and s may be given by

$$H=<r,s \mid r^2=s^4=(rs^2)^3=(rs)^4=1>.$$

Taking $N=<rs^2, srs>$ and $M=<s>$ we see that N is normal in H with the structure C_3xC_3 and that $N\cap M=(1)$. This shows that H is a soluble group of structure $3^2{:}4$.

We now show that $M(H)=C_3$. A covering group C for H is given by

$$<a_1,a_2,a_3,a_4 \mid a_1{}^2a_3=a_2{}^4=(a_1a_2{}^2)^3a_3=(a_1a_2)^4a_4{}^{-1}=[a_1,a_3]=[a_1,a_4]=[a_2,a_3]=[a_2,a_4]=[a_3,a_4]=1>.$$

Next, TC verifies that $|C|=108$ and hence $M(H)=C_3$. We have $M(H)\cong< a_4>\cong C_3$. An efficient presentation for H is now obtained form the above presentation for H_3:

$$<r,s \mid r^2=(rs)^4=(rs^2)^2rs^{-2}=1 >.$$

(iv)-(v) structure: S_4

G has two non-conjugate S_4 subgroups H_4 and H_5. The symmetric group S_4 is generated by two of its elements of order 2 and 3 whose product has order 4. By the information given in [34] about the structure constants for 2A we see that both #(2A,3A,4A) and #(2A,3B,4A) are non-zero and therefore H_4 and H_5 have generating pairs (x_4,y_4) and (x_5,y_5) of types (2A,3A,4A) and (2A,3B,4A) respectively (notice that S_4 has only one conjugacy class of elements of order 3). PERM now gives $x_4=a$, $y_4=bab(ab^2)^2$ and $x_5=a$, $y_5=(b^2a)^2bab$. H_4 and H_5 are maximal in G because $|G:H_i| \nmid 15$ (i=1,2,3); and their intersection in G is a cyclic group of order 2.

PSL(2,8)≅SL(2,8)

order=504=$2^3.3^2.7$ d=9 mult=1

G=< a,b |$a^2=b^3=(ab)^9=((ab)^3(ab^{-1})^4)^2=1$>

a=/1,3,2,5,4,7,6,9,8/, b=/8,1,4,7,6,9,3,2,5/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	504			1	1^9
2A	8	A	A	a	$1^1 2^4$
3A	9	A	A	b	3^3
7A	7	A	A	[a,b]	$1^2 7^1$
B*2	7	A	A	$[a,b]^2$	$1^2 7^1$
C*4	7	A	A	$[a,b]^4$	$1^2 7^1$
9A	9	A	A	ab	9^1
B*2	9	A	A	$(ab)^2$	9^1
C*4	9	A	A	$(ab)^4$	9^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	56	9	2^3:7	N($2A^3$)	1
H_2	18	28	D_{18}	N(3A)	1
H_3	14	36	D_{14}	N(7ABC)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup *order=8*

The Sylow 2-subgroup of G is an elementary abelian group of order 8. Thus G has the Klein 4-group as a subgroup. By PERM we find the generators $x=a^t$, $y=a^s$, where $t=b^{-1}ab^{-1}$ and $s=(ba)^2b$ for such a subgroup. Now TC can be used to extend <x,y> to a Sylow 2-subgroup P_2 of G. We obtain P_2=<x,y,z>, where $z=(ab)^3(ab^{-1})^4$.

$n_2(G)=|G:N(P_2)|=9$.

(ii) Sylow 3-subgroup *order=3*

P_3=<b>=C(b), $n_3(G)=|G:N(P_3)|=28$.

(iii) Sylow 7-subgroup *order=7*

P_7=<[a,b]>=C([a,b]), $n_7(G)=|G:N(P_7)|=36$.

II. Maximal subgroups

(i) structure: $2^3{:}7$

This is the stabilizer of a point in the permutation representation of G. Using this fact PERM finds $x=a$, $y=bab^{-1}(ab)^2$ for a subgroup H_1 of minimal index 9.

A presentation for H_1 on x, y is

$$\langle x, y \mid x^2=y^7=xyxy^{-3}xy^2=1 \rangle.$$

Let $N=\langle x,x^y, y^{-2}xy^2\rangle$. Then $\langle x,y\rangle$ is a split extension of N ($\cong 2^3$) by $\langle y\rangle$, that is $H_1 \cong 2^3{:}7$. By combining the relations in this presentation we are able to give the following deficiency zero presentation for the soluble group H_1

$$\langle x,y \mid x^2y^7=1,\ xy^2xy=y^3x\rangle.$$

This shows that $M(H_1)=1$, by 1.5.8 (ii).

(ii) structure: D_{18}

We simply find, by PERM, the generators $x=a$, $y=a^{bab}$ for a dihedral subgroup H_2 of G of order 18. Here (x,y) is of type (2,2,9).

(iii) structure: D_{14}

Similarly PERM produces the elements $x=a$, $y= a^b$ for a dihedral subgroup H_3 of G of order 14. Now the maximality of H_3 follows from the fact that H_1 has no dihedral subgroups of order 14 since $|H_3'|=7$, $|H_1'|=8$.

PSL(2,11)

order=660=2^2.3.5.11 d=11 mult=2

G=<a,b | $a^2=b^3=(ab)^{11}=((ab)^3(ab^{-1})^3)^2=1$>

a=/2,1,4,3,5,6,8,7,9,11,10/, b=/1,7,2,6,4,5,3,10,8,9,11/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	660			1	1^{11}
2A	12	A	A	a	$1^3 2^4$
3A	6	A	A	b	$1^2 3^3$
5A	5	A	A	[a,b]	$1^1 5^2$
B*	5	A	A	$[a,b]^2$	$1^1 5^2$
6A	6	A	A	$(ab)^3ab^{-1}$	$2^1 3^1 6^1$
11A	11	A	A	ab	11^1
B**	11	A	A	$b^{-1}a$	11^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	60	11	A_5	N(2A,3A,5AB)	2
H_2	60	11	A_5	N(2A,3A,5AB)	2
H_3	55	12	11:5	N(11AB)	1
H_4	12	55	D_{12}	N(2A),N(3A)	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=4

Clearly the Sylow 2-subgroup of G is C_2xC_2. PERM now gives x=a, y= a^t, where t= $(ba)^2b$, for a Sylow 2-subgroup P_2 of G.

$n_2(G)=|G: N(P_2)|=55$.

(ii) Sylow 3-subgroup order=3

P_3=<b>, $n_3(G)=|G: N(P_3)|=55$.

(iii) Sylow 5-subgroup order=5

P_5=<[a,b]>=C([a,b]), $n_5(G)=|G: N(P_5)|=66$.

(iv) Sylow 11-subgroup order=11

P_{11}=<ab>=C(ab), $n_{11}(G)=|G: N(P_{11})|=12$.

II. Maximal subgroups

(i)-(ii) structure: A_5

G has two conjugacy classes of maximal subgroups of the structure A_5. We therefore seek two generating pairs (x_1,y_1), (x_2,y_2) of type (2,3,5) with $|fix(x_1)\cap fix(y_1)|=0$, $|fix(x_2)\cap fix(y_2)|=1$. PERM simply gives $x_1=x_2=a$, $y_1=b^{ab}$, $y_2=b^t$, where $t=ab^{-1}$. Clearly $H_1=\langle x_1,y_1\rangle$, $H_2=\langle x_2,y_2\rangle$ remain non-conjugate in G. Moreover H_1 has intersection D_{10} with H_2.

(iii) structure: $11:5$

By table II, this is the normalizer in G of a cyclic subgroup of G whose generator lies in 11AB. Take x=ab. Then TC verifies that $N_G(\langle x\rangle)=\langle x,y\rangle$ where $y=bab(abab^{-1})^2$. The subgroup $H_3=\langle x,y\rangle$ of order 55 is obviously maximal in G. A deficiency zero presentation for H_3 on x, y may now be given by

$$\langle x,y \mid x^{11}=y^5,\ y^{-1}x^3y=x^4\rangle$$

(Note that H_3 is a split metacyclic group of structure 11:5, by 1.5.9).

(iv) structure: D_{12}

Using PERM we find the generators x=a, $y=a^{bab}$ for a dihedral subgroup H_4 of G with (x,y) of type (2,2,6). That H_4 is maximal in G follows immediately from the fact A_5 cannot have D_{12} as a subgroup.

PSL(2,13)

order=1092=2^2.3.7.13 d=14 mult=2

$G=\langle a,b \mid a^2=b^3=(ab)^{13}=((ab)^3(ab^{-1})^{-5})^2=1\rangle$

a=/2,1,4,3,6,5,13,8,10,9,11,14,7,12/, b=/3,1,2,9,4,8,6,7,5,12,10,11,13,14/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	1092				1^{14}
2A	12	A	A	a	$1^2 2^6$
3A	6	A	A	b	$1^2 3^4$
6A	6	AA	AA	$(ab)^2ab^{-1}$	$1^2 6^2$
7A	7	A	A	[a,b]	7^2
B*2	7	A	A	$[a,b]^2$	7^2
C*4	7	A	A	$[a,b]^4$	7^2
13A	13	A	A	ab	$1^1 13^1$
B*	13	A	A	$(ab)^2$	$1^1 13^1$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	78	14	13:6	N(13AB)	1
H_2	14	78	D_{14}	N(17ABC)	1
H_3	12	91	D_{12}	N(2A),N(3A)	2
H_4	12	91	A_4	N($2A^2$)	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=4

Using PERM, we obtain the generators x=a, y=$b[a,b]^3$ for a Sylow 2-subgroup P_2 of G.

$n_2(G)=|G:N(P_2)|=91$.

(ii) Sylow 3-subgroup order=3

$P_3=\langle b\rangle$, $n_3(G)=|G:N(P_3)|=91$.

(iii) Sylow 7-subgroup order=7

$P_7=\langle [a,b]\rangle=C([a,b])$, $n_7(G)=|G:N(P_7)|=78$.

(iv) Sylow 13-subgroup order=13

$P_{13}=\langle ab\rangle=C(ab)$, $n_{13}(G)=|G:N(P_{13})|=14$.

II. Maximal subgroups

(i) structure: *13:6*

By PERM we find that $St_G(11)=<a, b^{ab}>$. Let $y=b^{ab}$, $u=[a,y]$, $v=ay$. Then u,v generate $H_1=St_G(11)$ and satisfy the deficiency zero presentation

$$<u, v \mid u^{13}=v^6, v^{-1}u^3v=u^4>.$$

This is a split metacyclic group of structure 13:6, by 1.5.9.

(ii) structure: D_{14}

PERM simply gives the generators x=a, $y=a^b$ for a dihedral subgroup H_2 of G of order 14 with (x,y) is of type (2,2,7). The subgroup H_2 is maximal in G for $|H_1|$ is not divisible by 14.

(iii) structure: D_{12}

Similarly we obtain the elements x=a, $y=bab(abab^{-1})^2$ which generate a maximal subgroup H_3 of G isomorphic to D_{12}.

(iv) structure: A_4

That an A_4 subgroup has a generating pair of type (2,3,3) helps us to find the generators x=a, $y= b(ab)^3(ab^{-1})^3$ for a maximal subgroup H_4 of this structure.

PSL(2,17)

order=2448=$2^4.3^2.17$ d=18   mult=2

$G=\langle a,b \mid a^2=b^3=(ab)^{17}=((ab)^3(ab^{-1})^5)^2=1\rangle$

a=/2,1,4,3,6,5,8,7,9,11,10,13,12,15,14,16,18,17/,

b=/3,1,2,5,13,7,14,9,10,8,18,11,4,6,16,17,15,12/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	2448			1	1^{18}
2A	16	A	A	a	$1^2 2^8$
3A	9	A	A	b	3^6
4A	8	A	A	$(ab)^2(ab^{-1})^2$	$1^2 4^4$
8A	8	A	A	$(ab)^3ab^{-1}$	$1^2 8^2$
B*	8	A	A	$((ab)^3ab^{-1})^3$	$1^2 8^2$
9A	9	A	A	[a,b]	9^2
B*2	9	A	A	$[a,b]^2$	9^2
C*4	9	A	A	$[a,b]^4$	9^2
17A	17	A	A	ab	$1^1 17^1$
B*	17	A	A	$(ab)^3$	$1^1 17^1$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	136	18	17:8	N(17AB)	1
H_2	24	102	S_4	$N(2A^2)$	2
H_3	24	102	S_4	$N(2A^2)$	2
H_4	18	136	D_{18}	N(3A)	1
H_5	16	153	D_{16}	N(2A)	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup   order=16

By table II, the Sylow 2-subgroup of G is D_{16}. Using PERM we may obtain the generators x=a, $y=a^t$, where $t=bab^{-1}(ab)^2$, for a Sylow 2-subgroup P_2 of G. The elements x, y satisfy the presentation $\langle x, y \mid x^2=y^2=(xy)^8=1\rangle$.

$n_2(G)=|G:N(P_2)|=153$.

(ii) Sylow 3-subgroup   order=9

$P_3=\langle[a,b]\rangle=C([a,b])$, $n_3(G)=|G:N(P_3)|=136$.

(iii) Sylow 17-subgroup *order=17*

$P_{17}=\langle ab\rangle=C(ab)$, $n_{17}(G)=|G:N(P_{17})|=18$.

II.Maximal subgroups

(i) Structure *17:8*

This is the stabilizer of a point in the permutation representation of G. Starting with x=a which fixes the points 9, 16 PERM finds $y=baa^{bab}$ with fix(y)={4,9} and $|\langle x,y\rangle|=136$, that is $St_G(9)=\langle x,y\rangle$. Let z=[x,y].Then $H_1=\langle x,y\rangle$ splits over $\langle z\rangle$ and $\langle y\rangle$ is a complement to $\langle z\rangle$ in H_1. Now using 1.5.9 we are able to give the following deficiency zero presentation for the split metacyclic group H_1

$$\langle y,z \mid y^8=z^{17},\ y^{-1}z^6y=z^5\rangle.$$

(ii)-(iii) structure: S_4

G has two conjugacy classes of maximal subgroups of the structure S_4 with representatives H_2 and H_3. As we remarked earlier S_4 has a generating pair of type (2,3,4). Using this fact we obtain the generators $x_2=a$, $y_2=(ab)^3(abab^{-1})^2$ for H_2 and $x_3=a$, $y_3=(ab)^2ab^{-1}(ab)^3ab^{-1}$ for H_3.

The subgroup H_2 has intersection S_3 with H_3 in G.

(iv) structure: D_{18}

It is easily found that the elements x=a and $y=a^b$ generate a maximal subgroup H_4 of G isomorphic to D_{18} using PERM.

(v) structure: D_{16}

This is simply a Sylow 2-subgroup of G. Thus we may take $H_5=P_2$.

A_7

order=2520=$2^3.3^2.5.7$ d=7 mult=6

G=< a,b l $a^2=b^4=(ab)^7=[a,b]^5=(abab^2ab^{-1})^3=1$>

conjugacy classes of elements of G

class	lc(x)l	p-power	p'-part	representative	cycle type
1A	2520			1	1^7
2A	24	A	A	a	$1^3 2^2$
3A	36	A	A	$(ab^2)^2$	$1^4 3^1$
3B	9	A	A	$abab^2ab^{-1}$	$1^1 3^2$
4A	4	A	A	b	$1^1 2^1 4^1$
5A	5	A	A	[a,b]	$1^2 5^1$
6A	12	AA	AA	ab^2	$2^2 3^1$
7A	7	A	A	ab	7^1
B**	7	A	A	$b^{-1}a$	7^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	360	7	A_6	N(2A,3A,3B,4A,5A)	6
H_2	168	15	PSL(2,7)	N(2A,3B,4A,7AB)	2
H_3	168	15	PSL(2,7)	N(2A,3B,4A,7AB)	2
H_4	120	21	S_5	N(2A,3A,5A)	2
H_5	72	35	$(A_4x3):2$	N(3A),N($2A^2$)	6

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=8

Since A_7 has A_6 as a subgroup of odd index, the Sylow 2-subgroup of G is isomorphic to D_8. Using PERM we obtain the generators $x=b^2ab^2$ and $y=a^{ba}$ for a Sylow 2-subgroup P_2 of G.

$n_2(G)$=|G:N(P_2)|=315.

(ii) Slow 3-subgroup order=9

It is easy to see that the Sylow 3-subgroup of G is C_3xC_3. Again we use PERM to find the generators $x=(ab^2)^2$ and $y=(bab)^2$ with (x,y) of type (3,3,3;1) for a Sylow 3-subgroup P_3 of G. We note that P_3=C(x).

$n_3(G)$=|G:N(P_3)|=70.

(iii) Sylow 5-subgroup *order=5*

$P_5=<[a,b]>=C([a,b])$, $n_5(G)=|G:N(P_5)|=126$.

(iv) Sylow 7-subgroup *order=7*

$P_7=<ab>=C(ab)$, $n_7(G)=|G:N(P_7)|=120$.

II. Maximal subgroups

(i) structure: *A6*

$H_1=<a^{ba}, babab^{-1}>$ in 3.1 of [16].

(ii)-(iii) structure: *PSL(2,7)*

G has two conjugacy classes of maximal subgroups with representatives H_2 and H_3 of the structure PSL(2,7). The information given in table II about these maximal subgroups of G shows that both H_2 and H_3 are constructed by picking elements x and y from the classes 2A and 3B such that the product xy and the commutator [x,y] lie in the classes 7AB and 4A respectively. We find, by PERM, the generators $x_2=a^b$, $y_2=(ba)^2$ for H_2 and $x_3=bab^{-1}$, $y_3=(ab)^2$ for H_3 with (x_i,x_iy_i) of type (2A,3B,7AB;4) (i=2,3) and H_2, H_3 remaining non-conjugate in G.

H_2 and H_3 are maximal in G because $7 \nmid 15$; and that $H_2 \cap H_3 \cong S_4$.

(iv) structure: S_5

S_5 has presentation

$$<R,T \mid R^5=T^6=(RT)^2=(R^2T^2)^2=1>,$$

and is generated by elements of order 5 and 6 whose product has order 2. We take x=a of order 2 and obtain, by PERM, $y=(bab)^s$, where $s=ab^2$, of order 6 such that $|<x>|=5$ and $|<x, y>|=120$. Taking R=xy and $T=y^{-1}$ we see that R, T satisfy the above presentation for S_5.

$H_5=<x, y>$ is a maximal subgroup of G since S_5, having an element of order 6, is not embeddable in A_6; and also $|H_5|$ does not divide $|H_i|$ (i=2, 3).

S_5 has multiplier C_2 with the following efficient presentation

$$<R,T \mid R^5=(RT)^2, T^6=(R^2T^2)^2=1 >.$$

(v) structure: *$(A_4x3):2$*

The maximal subgroup of G of the above structure is the normalizer in G of a subgroup K ($\leq$ G) isomorphic to A_4x3. K has the following presentation :

$$<A,B,C \mid A^3=B^2=C^3=(BC)^3=[A,B]=[A,C]=1>,$$

and can be generated by AB and C with (AB, C) of type (6,3,3;2).Starting from $x=ab^2$ of order 6 PERM finds $y=abab^2ab^{-1}$ of order 3 such that $|<xy>|=3$ and $|<[x,y]>|=2$. Now x and y generate a subgroup of order 36 with $N_G(<x,y>)=<x,y,a>$, using TC. This leads to the generators $r=b^2$, $s=a^b$, and $t=ab^2a$ for a subgroup H_5 of order 72. A specially written CAYLEY program shows that <r,s,t> cannot be generated by two of its elements and thus $d(H_5)=3$. A presentation for H_5 on r, s, and t is :

$$L=<r,s,t \mid r^2=s^2=t^2=(rt)^3=(st)^3=(rtsrs)^2=(rtrs)^3=1>.$$

Taking $N=<rt, rs>$ we observe that $N \triangleleft L$, $N \cap <r>=(1)$ and $N\cong<x,y \mid x^3=(xy)^3=(xy^{-1})^3=[x,y^2]=1>$. This shows that $H_5 \cong N{:}C_2$. On the other hand N is the direct product of $<y^3,x>$ ($\cong A_4$) and $<y^2>$ ($\cong C_3$) proving that $H_5 \cong (A_4x3){:}2$.

The maximality of H_5 is now assured on noting that $|G{:}H_i|\dagger 35$ (i=2,3,4) and that H_1 ($\cong A_6$) has no element of order 6 whereas ts is an element of order 6 in H_5.

A covering group C for H_5 is given by

$$<a_1,a_2,a_3,a_4,a_5,a_6,a_7 \mid a_1^2a_4=a_2^2a_5=a_3^2=(a_1a_3)^3a_4= (a_2a_3)^3a_5=(a_1a_3a_2a_1a_2)^2a_6^{-1}$$
$$=(a_1a_3a_1a_2)^3a_7^{-1}=1, [a_i,a_j]=1 \ (1 \leq i \leq 7, 4 \leq j \leq 7, i < j) >.$$

Then $|C|=432$ showing that the multiplier of H_5 is C_6.

Unfortunately we failed to determine whether H_5 is efficient.

PSL(2,19)

order=3420=$2^2.3^2.5.19$ d=20 mult=2

G=< a,b | $a^2=b^3=(ab)^{19}=((ab)^3(ab^{-1})^{-7})^2=1$ >

a=/2,1,4,3,6,5,15,9,8,11,10,13,12,20,7,17,16,19,18,14/

b=/3,1,2,10,4,8,6,7,9,5,14,11,13,12,19,15,20,17,16,18/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	3420			1	1^{20}
2A	20	A	A	a	2^{10}
3A	9	A	A	b	$1^2 3^6$
5A	10	A	A	$(ab)^3ab^{-1}$	5^4
B*	10	A	A	$(ab)^2ab^{-1}$	5^4
9A	9	A	A	[a,b]	$1^2 9^2$
B*2	9	A	A	$[a,b]^2$	$1^2 9^2$
C*4	9	A	A	$[a,b]^4$	$1^2 9^2$
10A	10	AA	BA	$(ab)^2(ab^{-1})^2$	10^2
B*	10	BA	AA	$((ab)^2(ab^{-1})^2)^3$	10^2
19A	19	A	A	ab	$1^1 19^1$
B**	19	A	A	$b^{-1}a$	$1^1 19^1$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	171	20	19:9	N(19AB)	1
H_2	60	57	A_5	N(2A,3A,5AB)	2
H_3	60	57	A_5	N(2A,3A,5AB)	2
H_4	20	171	D_{20}	N(2A),N(5AB)	2
H_5	18	190	D_{18}	N(3A)	1

I.Sylow p-subgroup

(i) Sylow 2-subgroup order=4

The Sylow 2-subgroup of G is C_2xC_2 and thus is generated by two elements x, y of G with (x,y) of type (2,2,2;1). It is now found, by PERM, that x=a, y=a^t, where t=$b(ab^{-1})^2ab$. Put P_2=<x,y>.

$n_2(G)=|G: N(P_2)|=285$.

(ii) Sylow 3-subgroup order=9

P_3=<[a,b]>=C([a,b]), $n_3(G)=|G:N(P_3)|=190$.

(iii) Sylow 5-subgroup *order=5*

$P_5=<(ab)^3ab^{-1}>=C((ab)^3ab^{-1})$, $n_5(G)=|G:N(P_5)|=171$.

(iv) Sylow 19-subgroup *order=19*

$P_{19}=<ab>=C(ab)$, $n_{19}(G)=|G:N(P_{19})|=20$.

II.Maximal subgroups

(i) structure: *19:9*

By table II, this is the normalizer in G of a cyclic subgroup of G whose generator lies in 19AB. Taking x=ab, TC gives $N_G(<x>)=<x,y>$, where $y=bab((ab)^2ab^{-1})^2$. The elements x, y generate a split metacyclic group H_1 of order 171 and satisfy the deficiency zero presentation

$$<x, y \mid x^{19}=y^9, y^{-1}x^4y=x^5>.$$

(ii)-(iii) structure: A_5

G has two non-conjugate A_5 subgroups H_2 and H_3 which are maximal in G by table II. We now seek generating pairs (x_2,y_2), (x_3,y_3) of type (2,3,5) for H_2, H_3. PERM enables us to find $x_2=x_3=a$, $y_2=b^{ab}$, $y_3=b^t$, where $t=ab^{-1}$.

H_2 has intersection C_2 with H_3 in G.

(iv) structure: D_{20}

Using PERM we simply find the generators x=a, $y= a^t$, where $t=(ab)^2$, for a subgroup H_4 of G isomorphic to D_{20}.This subgroup is indeed maximal in G since A_5 has no elements of order 10.

(v) structure: D_{18}

Similarly we obtain the generators x=a, $y=a^b$ for a maximal subgroup H_5 of G isomorphic to D_{18}.

PSL(2,16)≅SL(2,16)

order=4080=2^4.3.5.17 d=17 mult=1

G=<a,b | $a^2=b^3=(ab)^{15}=((ab)^3(ab^{-1})^5)^2=1$>

a=/2,1,4,3,6,5,8,7,10,9,12,11,14,13,15,17,16/

b=/3,1,2,5,7,6,4,9,15,16,10,14,12,13,8,11,17/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	4080			1	1^{17}
2A	16	A	A	a	$1^1 2^8$
3A	15	A	A	b	$1^2 3^5$
5A	15	A	A	$(ab)^6$	$1^2 5^3$
B*	15	A	A	$(ab)^3$	$1^2 5^3$
15A	15	BA	AA	ab	$1^2 15^1$
B*4	15	BA	AA	$(ab)^4$	$1^2 15^1$
C*2	15	AA	BA	$(ab)^2$	$1^2 15^1$
D*8	15	AA	BA	$(ab)^8$	$1^2 15^1$
17A	17	A	A	[a,b]	17^1
B*4	17	A	A	$[a,b]^4$	17^1
C*2	17	A	A	$[a,b]^2$	17^1
D*8	17	A	A	$[a,b]^8$	17^1
E*6	17	A	A	$[a,b]^6$	17^1
F*7	17	A	A	$[a,b]^7$	17^1
G*5	17	A	A	$[a,b]^5$	17^1
H*3	17	A	A	$[a,b]^3$	17^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	240	17	2^4:15	N($2A^4$)	1
H_2	60	68	A_5	N(2A,3A,5AB)	2
H_3	34	120	D_{34}	N(17A-H)	1
H_4	30	136	D_{30}	N(3A),N(5AB)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=16

The Sylow 2-subgroup of G is clearly $C_2xC_2xC_2xC_2$ by table I. Using PERM we first find that $x=a^b$, $y= ab^{-1}(ab)^5(ab^{-1})^2$ generate a subgroup of order 4. Then on successively extending <x,y> by 2-elements we obtain, by TC, $z= ba(b(ab^{-1})^2aba)^2$, $u=(ab)^4(ab^{-1})^4(ab)^3a$ with x,y,z and u generating a subgroup P_2 of order 16 as required.

$n_2(G)=|G:N(P_2)|=17$.

(ii) Sylow 3-subgroup *order=3*

$P_3=<b>$, $n_3(G)=|G:N(P_3)|=136$.

(iii) Sylow 5-subroup *order=5*

$P_5=<(ab)^3>$, $n_5(G)=|G:N(P_5)|=136$.

(iv) Sylow 17-subgroup *order=17*

$P_{17}=<[a,b]>=C([a,b])$, $n_{17}(G)=|G:N(P_{17})|=120$.

II.Maximal subgroups

(i) Structure: $2^4{:}15$

Starting with x=a we find $y=(bab)^t$, where $t=(ab)^2$, with x, y generating a subgroup H_1 of minimal index 17. In fact $H_1=St_G(15)$. A presentation for H_1 on x, y is

$$<x,y \mid x^2=y^{15}=1, xy^3xy=y^4x>.$$

Let $N=<x,x^y,yxy^{-1}, y^{-2}xy^2>$. Then N is a normal subgroup of $<x,y>$ isomorphic to 2^4 and thus $<x,y>$ is a split extension of N by $<y>$, that is $H_1 \cong 2^4{:}15$. By combining the relations $x^2=y^{15}=1$ into the single relation $x^2y^{15}=1$, we get a deficiency zero presentation for the soluble group H_1.

(ii) structure: A_5

We look for a generating pair (x,y) of type (2,3,5) for an A_5 subgroup. PERM finds x=a, $y=b^t$, where $t=(ab)^2$. Now H_1 can not have A_5 as a subgroup since A_5 is not soluble. This shows that $H_2=<x,y>$ is actually maximal in G.

(iii) structure: D_{34}

PERM gives the generators x=a, $y= a^b$, with (x,y) of type (2,2,17), for a subgroup H_3 of G isomorphic to D_{34}.

(iv) structure: D_{30}

Similarly we find x=a, $y= a^t$, where $t=b^{-1}(ab)^2$ for a dihedral subgroup D_{30} of G which is maximal in G since $|H_4'| =15$, $|H_1'|=16$ and hence H_4 is not embeddable in H_1, also H_2 ($\cong A_5$) cannot have H_4 ($\cong D_{30}$) as a subgroup.

PSL(3,3)

order=5616=$2^4.3^3.13$ d=13 mult=1

$G=< a,b \mid a^2=b^3=(ab)^{13}= ((ab)^4ab^{-1})^2(ab)^2(ab^{-1})^2ab(ab^{-1})^2(ab)^2ab^{-1}=1>$

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	5616			1	1^{13}
2A	48	A	A	a	$1^5 2^4$
3A	54	A	A	$((ab)^2(ab^{-1})^2)^2$	$1^4 3^3$
3B	9	A	A	b	$1^1 3^4$
4A	8	A	A	[a,b]	$1^1 2^2 4^2$
6A	6	A	A	$(ab)^2(ab^{-1})^2$	$1^2 2^1 3^1 6^1$
8A	8	A	A	$(ab)^2ab^{-1}$	$1^1 4^1 8^1$
B**	8	A	A	$b(ab^{-1})^2a$	$1^1 4^1 8^1$
13A	13	A	A	ab	13^1
B**	13	A	A	$b^{-1}a$	13^1
C*5	13	A	A	$(ab)^5$	13^1
D*8	13	A	A	$(b^{-1}a)^5$	13^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	432	13	$3^2:2S_4$	$N(3A^2)$	1
H_2	432	13	$3^2:2S_4$	$N(3A^2)$	1
H_3	39	144	13:3	N(13ABCD)	1
H_4	24	234	S_4	$N(2A^2)$	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=16

We take $x=(ab)^2ab^{-1}$ of order 8 and use TC to find an element y in $N_G(<x>)$ of order 2 such that $y \notin <x>$. We obtain $y=(b^{-1}ababa)^2b$. Conjugation of x and y by $b^{-1}a$ leads to the generators $r=(ab)^4a$ and $s=(ba)^2b^{-1}(ab)^2$ for a Sylow 2-subgroup P_2 of G which satisfy the presentation

$$<r,s \mid s^2=1, [s,r]r^2=1>.$$

This is the semi-dihedral group of order 16 and isomorphic to $16,13(1^2,0) \cong \Gamma_3a_2$. We shall denote this group by <-2,4 | 2> after Coxeter and Moser ([18], p 134).

$n_2(G)=| G:N(P_2)|=351$.

(ii) Sylow 3-subgroup *order=27*

Starting with $x=((ab)^2(ab^{-1})^2)^2$ of order 3 TC finds $z=ab((ab)^2ab^{-1})^2$ in $N_G(<x>)$ with $<x,z>$ of order 9. Again we use TC to extend $<x, z>$ by a 3-element. We obtain $y=abab^{-1}(ab)^3ab^{-1}ab$ in $N_G(<x,z>)$ of order 3 with $|<x,y,z>|=27$. Now z can be omitted from the generating set {x,y,z} and we have the generators x and y for a Sylow 3-subgroup P_3 of G.Next x and y satisfy the presentation :

$$<x,y \mid x^3=y^3=(xy)^3=(xy^{-1})^3=1>,$$

which is an extra-special group of order 27 of exponent 3 (i.e. 3_+^{1+2}).

$n_3(G)=|G:N(P_3)|=52.$

(iii) Sylow 13-subgroup *order=13*

$P_{13}=<ab>=C(ab)$, $n_{13}(G)=|G:N(P_{13})|=144.$

II.Maximal subgroups

(i)-(ii) structure: $3^2{:}2S_4$

$H_1=<a^b, b^a>.$

Taking $x_1=a^b$ and $y_1=b^a$ we see that (x_1,y_1) is a generating pair of type (2,3B,8) for a subgroup of minimal index 13 with the property that x_1 and y_1 fix one, and only one, point in the permutation representation of G.

We now use PERM to look for a generating pair (x_2, y_2) of the same type with $|fix(x_2)\cap fix(y_2)|=0$. (We note that #(2,3A,8)=0.) Starting from $x_2=a$ PERM finds $y_2=b^{-1}(ab)^2$ and TC shows that $H_2=<x_2, y_2>$ has minimal index 13 in G.

H_1 and H_2 are not conjugate in G; and their intersection is a group of order 48 with structure GL(2,3).

The pair (x_2,y_2^{-1}) satisfies the following presentation, given in [7], for H_1 on the generators x_1, y_1

$$H=<x,y \mid x^2=y^3=(xy)^8=((xy)^2xy^{-1})^2(xy(xy^{-1})^2)^2=1>.$$

Taking $N=<(xyxyxy^{-1})^2, (y^{-1}xyxyx)^2>$ and $M=<y^{-1}(xy)^3, (xyxy^{-1})^2>$ we see that N is a normal subgroup of H isomorphic to C_3xC_3 and that $N\cap M=(1)$. This shows that $H \cong N{:}M$. We now show that $M \cong 2S_4$. A presentation for M on its generators is $<r,s|s^3=(rs)^2=(r^3s^{-1})^2=1>$. The subgroup $<r^4>$ of order 2 is normal in $<r,s>$ and we have $<r,s>/<r^4> \cong S_4$. So $H \cong 3^2{:}2S_4$.

As has been remarked in [7], H is the Hessian group of order 216 extended by

C_2. To see this we take $T=y$ and $S=(xyxy^{-1})^2$ and observe that T and S generate a subgroup of H of index 2 and satisfy the following presentation for the Hessian group given in [18]

$$\langle T,S \mid T^3=S^3=(TS)^4=1, (TST)^2S=S(TST)^2\rangle.$$

Next we shall see that M(H)=1. A covering group C for H is

$$\langle x,y,z,u \mid x^2=y^3z^{-1}=(xy)^8z^{-3}=((xy)^2xy^{-1})^2(xy(xy^{-1})^2)^2u^{-1}=[x,z]=[x,u]=[y,z]=[y,u]=[z,u]=1\rangle.$$

TC shows that |C|=432 proving that M(H)=1. A deficiency zero presentation for H can now be obtained from the above presentation for H :

$$\langle x,y \mid x^2y^3=(xy)^3(xy^{-1}xy)^2x^{-1}yxy^{-1}x^{-1}y(x^{-1}y^{-1})^3x^{-1}y=1\rangle.$$

(iii) structure *13:3*

This is the normalizer in G of a cyclic subgroup $\langle x\rangle$ of G with x in 13ABC. Taking x=ab we find, by TC, $N_G(\langle x\rangle)=\langle x,y\rangle$ where $y=b(ab^{-1})^5ab(ab^{-1})^2abab^{-1}$. The elements x and y generate a split metacyclic group of order 39 as required. That (x,y) has type (13,3B,3B) enables PERM to give the neater generators $u=b^{aba}$ and $v=b^t$, where $t=(ab^{-1})^2$, for a maximal subgroup H_3 of G of order 39 since 39 does not divide 432. We note that all elements of order 3 in H_3 lie in 3B and hence G has no maximal subgroups of order 39 having 3-elements in 3A.

By 1.5.9, it is easy to see that u and w=uv satisfy the deficiency zero presentation:

$$\langle u,w \mid u^3=w^{13}, [u,w^{-5}]=w\rangle.$$

(iv) structure: S_4

A generating pair for S_4 is of type (2,3,4). The information given in [34] about structure constants for the class 2A shows that #(2,3A,4)=0. So a generating pair for a S_4 subgroup in G would be of type (2,3B,4). PERM now is able to give the generators x=a and $y=b^{-1}(ab^{-1}(ab)^4)^2$ for a subgroup H_4 isomorphic to S_4. For maximality of H_4 we may use SUBGPTEST in order to show that H_1 of structure $3^2{:}2S_4$ has no S_4 subgroups.

PSU(3,3)

order=6048=$2^5.3^3.7$ d=28 mult=1

G=< a,b l $a^2=b^6=(ab)^7=(ab^2)^3(ab^{-2})^3=(abab^{-2})^3ab(ab^{-1})^2=1$ >

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	6048			1	1^{28}
2A	96	A	A	a	$1^4 2^{12}$
3A	108	A	A	b^2	$1^1 3^9$
3B	9	A	A	[a,b]	$1^1 3^9$
4A	96	A	A	$(abab^{-2})^2$	$1^4 4^6$
B**	96	A	A	$(b^2ab^{-1}a)^2$	$1^4 4^6$
4C	16	A	A	ab^3	$2^2 4^6$
6A	12	AA	AA	b	$1^1 3^1 6^4$
7A	7	A	A	ab	7^4
B**	7	A	A	$b^{-1}a$	7^4
8A	8	A	A	$abab^{-2}$	$1^2 2^1 8^3$
B**	8	B	A	$b^2ab^{-1}a$	$1^2 2^1 8^3$
12A	12	AB	AA	$b^{-2}a$	$1^1 3^1 12^2$
B**	12	AA	AB	ab^2	$1^1 3^1 12^2$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	216	28	3_+^{1+2}:8	N(3A)	1
H_2	168	36	PSL(2,7)	N(2A,3B,4C,7AB)	2
H_3	96	63	$4{\cdot}S_4$	N(2A)	1
H_4	96	63	$4^2{:}S_3$	$N(2A^2)$	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=32

We take $z=(ab)^2ab^{-2}$ of order 8 and apply our method 2.3 (vi) to extend <z> to a Sylow 2-subgroup of G. TC finds $u=b^2abab^{-1}ab^2ab^{-1}ab$ in $N_G(z)$ with |<z, u>|=16 and $v=b^2ab^{-2}a$ in N_G(<z, u>) with |<z, u, v>|=32. Now it is easy to see that |<z,v>|=32. Conjugating zv and v^{-1} by ab leads to the generators x=a and $y=bab^{-2}ab$ for a Sylow 2-subgroup P_2 of G.

A presentation for P_2 on the generators x,y is

$$<x,y \mid x^2=y^4=[y,x,y]=1>,$$

which is isomorphic to $32,31,(1^2,1)\cong\Gamma_3e$.

$n_2(G)=|G:N(P_2)|=189$.

(ii) Sylow 3-subroups *order=27*

The information given in table I about the conjugacy classes of G shows that the Sylow 3-subgroup of G is a 3-group of exponent 3. So it is an extra-special group of structure 3_+^{1+2}. Using this fact we seek, by PERM, a generating pair (x,y) of type (3,3,3;3). We find the generators x=[a, b] and $y=abab^3ab^{-1}$ for a Sylow 3-subgroup P_3 of G which satisfy $<x,y \mid x^3=y^3=(xy)^3=(xy^{-1})^3>$.

$n_3(G)=|G:N(P_3)|=28$.

(iii) Sylow 7-subgroup *order=7*

$P_7=<ab>=C(ab)$, $n_7(G)=|G:N(P_7)|=288$.

II.Maximal subgroups

(i) structure $3_+^{1+2}:8$

$H_1=< a^b, (ab)^3b^3 >$.

Putting $x=a^b$ and $y=(ab)^3b^3$, a presentation for H_1 on x and y is given by

$$H=<x,y \mid x^2=y^8=(xy^4)^3=[x,y^{-1}][x,y^2]=1>,$$

(see [7]). On taking $N=<xy^4, [y^2,x]>$ we see that N is normal in H, $N \cong 3_+^{1+2}$, and $N \cap <y>=(1)$. This shows that H is a semi-direct product of 3_+^{1+2} by C_8.

As was remarked in [7], H is not isomorphic to the Hessian group as stated in [21].

A covering group C for H is

$$<a_1,a_2,a_3,a_4 \mid a_1^2a_3=a_2^8=(a_1a_2^4)^3a_3=[a_1,a_2^{-1}][a_1,a_2^2]a_4^{-1}=[a_1,a_3]= [a_1,a_4]=[a_2,a_3] =[a_2,a_4]=[a_3,a_4]=1>.$$

Using TC we have that |C|=216 giving M(H)=1. Applying the method described in 3.5, we find that r=y and $s=(xy)^2x$ generate H_1 and satisfy the following deficiency zero presentation for H_1

$$<r,s \mid r^3=srs^2, s^2r^2s=rs^{-1}r>.$$

(ii) structre *PSL(2,7)*

By table II, the maximal subgroup of G with structure PSL(2,7) has a generating pair (x,y) of type (2,3B,7;4). Starting with x=a of order 2 PERM finds $y=ab^3abab^{-1}$ in 3B such that $|<xy>|=7$ and $|<[x,y]>|=4$. Now x and y generate a subgroup H_2 of order 168 which is isomorphic to PSL(2,7). We note that #(2,3A,7AB)=0 and therefore G has no PSL(2,7) subgroups whose 3-elements lie in the class 3A of G.

H_2 is maximal in G since PSL(2,7) is not embeddable in H_1.

(iii) structure $4{\cdot}S_4$

By table II, this is the normalizer in G of an involution. TC finds $N_G(a)=<u,v>$ where $u=bab^{-1}ab$ and $v=(b^2a)^2(ba)^2b^{-1}$. Conjugation by b^{-2} of u^{-1} and $(vu)^{-1}$ leads to the generators $x=[a,b^2]$ and $y=(ba)^2b^3$ for a maximal subgroup H_3 of G of order 96.

A presentation for H_3 on the generators x and y is

$$K=<x,y \mid x^4=y^8=(xy)^3=[x,y^2]=1>.$$

Next, it is easy to check that $<y^2>$ is normal in K and that the factor $K/<y^2>\cong S_4$. This proves that K is an extension of C_4 by S_4. Notice that K cannot split over C_4 because it has no S_4 subgroup, by SUBGPTEST. So $K \cong 4{\cdot}S_4$.

The group K has trivial multiplier. To see this we first construct the following covering group C for K

$$<a_1,a_2,a_3,a_4 \mid a_1^4a_4^{-1}=(a_1a_2)^3a_4^{-1}=[a_1,a_2^2]a_3^{-1}=a_2^8a_4^{-1}=[a_1,a_3]=[a_1,a_4]=[a_2,a_3]$$
$$=[a_2,a_4]=[a_3,a_4]=1>,$$

then $|C|=96$ and so $M(K)=1$.

A deficiency zero presentation for H_3 can now be obtained from the above presentation for H_3 by modifying its relations as follows :

$$<x,y \mid x^4=(xy)^3,\ x^4[x,y^2]=y^8>.$$

(iv) structure $4^2{:}S_3$

This is the normalizer in G of a C_4xC_4 subgroup, by 3.3.2. We take $u=ab^3$ of order 4 and, by PERM, find $v=(bab^2a)^2ba$ with $<u,v> \cong C_4xC_4$. By TC, we then have $N_G(<u,v>)=<a,u,w>$, where $w=(ba)^2(b^{-1}a)^2$, and $|<a,u,w>|=|<a,wu>|=96$. This gives the generators $x=a$ and $y=(ba)^2b^{-1}ab^2$ for a maximal subgroup H_4 of order 96 which has the structure $4^2{:}S_3$ and is not isomorphic to H_3. To see this we first construct the following presentation for H_4 on its generators x and y

$$L=<x,y \mid x^2=y^3=(xy)^8=[x,y]^3=1>.$$

Next, L is a semi-direct product of $N=<(xy)^2,(yx)^2>$ and $M=<x,x^y>$ with $N \cong 4^2$ and $M\cong S_3$. That H_3 and H_4 are not isomorphic follows from the fact that H_3 has elements of order 6 and 12 while H_4 has no such elements.

It is also worth mentioning that each of H_3 and H_4 has a single conjugacy class of elements of order 3 and that such elements in H_3 are all in 3A whereas those of H_4 are in 3B.

Note. We may now use PERM to find the neater generators $x=a^b$ and $y=[a, b^{-1}]$, with x and y satisfing the above presentation, for a maximal subgroup of G isomorphic to H_4.

Finally a covering group C for L is

$$<a_1,a_2,a_3,a_4 \mid a_1^2=a_2^3a_4^{-1}=(a_1a_2a_1a_2^{-1})^3a_3^{-1}=(a_1a_2)^8a_4^{-3}=[a_1,a_3]=[a_1,a_4]=[a_2,a_3]=[a_2,a_4]=[a_3,a_4]=1>.$$

TC then verifies that $|C|=192$ giving $M(L)=C_2$. In fact we have $M(L) \cong Z(C) \cap C'=<a_3>$. An efficient presentation for L can now given by

$$<x,y \mid x^2=y^3=1, [x,y]^3=(xy)^8>.$$

PSL(2,23)

order=6072=2^3.3.11.23 d=24 mult=2

G=<a,b | $a^2=b^3=(ab)^{11}=((ab)^3(ab^{-1})^{-3})^2=1$>

a=/2,1,4,3,6,5,15,9,8,11,10,22,14,13,7,17,16,19,18,21,20,12,24,23/,

b=/3,1,2,14,4,8,6,7,23,9,13,11,12,5,20,15,19,17,18,16,24,21,10,22/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	6072			1	1^{24}
2A	24	A	A	a	2^{12}
3A	12	A	A	b	3^8
4A	12	A	A	[a,b]	4^6
6A	12	AA	AA	$(ab)^4(ab^{-1})^2$	6^4
11A	11	A	A	ab	$1^2 11^2$
B*3	11	A	A	$(ab)^3$	$1^2 11^2$
C*2	11	A	A	$(ab)^2$	$1^2 11^2$
D*5	11	A	A	$(ab)^5$	$1^2 11^2$
E*4	11	A	A	$(ab)^4$	$1^2 11^2$
12A	12	AA	AA	$(ab)^2(ab^{-1})^2$	12^2
B*	12	AA	AA	$((ab)^2(ab^{-1})^2)^5$	12^2
23A	23	A	A	$(ab)^2ab^{-1}$	$1^1 23^1$
B**	23	A	A	$b(ab^{-1})^2a$	$1^1 23^1$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	253	24	23:11	N(23AB)	1
H_2	24	253	S_4	N($2A^2$)	2
H_3	24	253	S_4	N($2A^2$)	2
H_4	24	253	D_{24}	N(2A),N(3A)	2
H_5	22	276	D_{22}	N(11ABCDE)	1

I. Sylow p-subgroups

(i) Sylow 2-subgroup order=8

Since G has D_{24} as a subgroup of odd index, the Sylow 2-subgroup of G is isomorphic to D_8 ($\leq D_{24}$). Using this fact we find the generating pair (a,a^b) for a Sylow 2-subgroup P_2 of G.

$n_2(G)=|G:N(P_2)|=759$.

(ii) Sylow 3-subgroup *order=3*

$P_3=<b>$, $n_3(G)=|G:N(P_3)|=253$.

(iii) Sylow 11-subgroup *order=11*

$P_{11}=<ab>=C(ab)$, $n_{11}(G)=|G:N(P_{11})|=276$.

(iv) Sylow 23-subgroup *order=23*

$P_{23}=<(ab)^2ab^{-1}>=C((ab)^2ab^{-1})$, $n_{23}(G)=|G:N(P_{23})|=24$.

II.Maximal subgroups

(i) structure: *23:11*

Taking $x=b^{-1}a(ba)^2$ in 23A, gives $N(<x>)=<x,y>$, where $y=(ab)^3ab^{-1}$. The elements x, y generate a subgroup H_1 of minimal index 24 and satisfy the deficiency zero presentation

$$<x,y \mid x^{23}=y^{11}, y^{-1}x^2y=x>$$

which is a split metacylic group of structure 23:11.

(ii)-(iii) structure: S_4

Using PERM we find the pairs (x_2,y_2), (x_3,y_3), where $x_2=x_3=a$, $y_2=(ba)^4b^{-1}(ab)^2$, $y_3=(ba)^2b^{-1}(ab)^4$ for two non-conjugate S_4 subgroups of G. The subgroups $H_2=<x_2,y_2>$ and $H_3=<x_3,y_3>$ are maximal in G and their intersection is C_2.

(iv) structure: D_{24}

By PERM we simply find the generators x=a and y= a^{bab} for a maximal subgroup H_4 of G isomorphic to D_{24}. The elements x, y satisfy $<x,y \mid x^2=y^2=(xy)^{12}=1>$.

(v) structure: D_{22}

Similarly we obtain the generators x=a and $y=(ba)^2b^{-1}(ab)^2$ for a maximal subgroup H_5 of G isomorphic to D_{22}.

PSL(2,25)

order=7800=$2^3.3.5^2.13$ d=26   mult=2

G=<a,b| $a^2=b^3=(ab)^{13}=((ab)^3(ab^{-1})^4)^2=1$>

a=/6,3,2,5,4,1,8,7,10,9,12,11,14,13,16,15,17,19,18,21,20,23,22,24,26,25/,

b=/5,1,21,3,2,19,6,22,8,20,10,18,12,17,14,16,15,13,7,11,4,9,25,23,24,26/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	7800			1	1^{26}
2A	24	A	A	a	$1^2 2^{12}$
3A	12	A	A	b	$1^2 3^8$
4A	12	A	A	$(ab)^2(ab^{-1})^2$	$1^2 4^6$
5A	25	A	A	$(ab^{-1}ab)^2abab^{-1}$	$1^1 5^2$
5B	25	A	A	$ab^{-1}ab(abab^{-1})^2$	$1^1 5^2$
6A	12	AA	AA	$(ab)^2ab^{-1}$	$1^2 6^4$
12A	12	AA	AA	[a,b]	$1^2 12^2$
B*	12	AA	AA	$[a,b]^5$	$1^2 12^2$
13A	13	A	A	ab	13^2
B*5	13	A	A	$(ab)^5$	13^2
C*4	13	A	A	$(ab)^4$	13^2
D*6	13	A	A	$(ab)^6$	13^2
E*3	13	A	A	$(ab)^3$	13^2
F*2	13	A	A	$(ab)^2$	13^2

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	300	26	5^2:12	$N(5A^2)$	1
H_2	120	65	S_5		2
H_3	120	65	S_5		2
H_4	26	300	D_{26}	N(13A-F)	1
H_5	24	325	D_{24}	N(2A),N(3A)	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup   order=8

The Sylow 2-subgroup of G is D_8. Now PERM finds the generators x=a, y=a^{bab} for a Sylow 2-subgroup P_2 of G isomorphic to D_8. P_2 is a self-normalizing subgroup of G and we have that

$n_2(G)=|G:N(P_2)|=975.$

(ii) Sylow 3-subgroup *order=3*

$P_3=\langle b\rangle$, $n_3(G)=|G:N(P_3)|=325$.

(iii) Sylow 5-subgroup *order=25*

Clearly the Sylow 5-subgroup of G is C_5xC_5. Starting with $x=(ab^{-1}ab)^2abab^{-1}$ of order 5 PERM finds $y=(ab)^2(ab^{-1})^2abab^{-1}$ with x, y generating a Sylow 5-subgroup of G.

$P_5=\langle x,y\rangle$, $n_5(G)=|G:N(P_5)|=26$.

(iv) Sylow 13-subgroup *order=13*

$P_{13}=\langle ab\rangle$, $n_{13}(G)=|G:N(P_{13})|=300$.

II.Maximal subgroups

(i) structure: $5^2{:}12$

Taking x=a, we find $y=b^{-1}ab(ab^{-1})^2(ab)^4$ with $\langle x,y\rangle=St_G(17)$. The elements x, y generate a subgroup H_1 of minimal index 26 and satisfy the deficiency zero presentation

$$\langle x,y \mid x^2=(xy^2)^3, (xy)^3xy^4=yx\rangle.$$

Let $N=\langle[x,y],[x,y^{-1}]\rangle$. Then $\langle x,y\rangle$ is an split extension of N ($\cong 5^2$) by $\langle y\rangle$ ($\cong C_{12}$) which shows that H_1 is a soluble group.

(ii)-(iii) structure: S_5

G has two conjugacy classes of maximal subgroups of the structure S_5 with representatives H_2, H_3. The group S_5 has presentation $\langle x,y \mid x^2=y^4=(xy)^5=[x,y]^3\rangle$ and thus a generating pair of type (2,4,5;3). Using PERM we may now obtain the generators $x_2=a$, $y_2=((ab)^2ab^{-1})^2ab$ for H_2 and $x_3=a$, $y_3=ab(ab^{-1}(ab)^2)^2$ for H_3.(The subgroups H_3 and H_4 remain non-conjugate in G because their elements of order 5 are in 5A, 5B respectively.) We note that H_3 has intersection C_6 with H_4 in G.

(iv) structure: D_{26}

It is easily found that x=a, y= a^t, where $t=b(ab)^2$, generate a maximal subgroup H_4 of G isomorphic to D_{26}.

(v) structure: D_{24}

Similarly we obtain the generating pair (a,a^b) for a subgroup H_5 isomorphic to D_{24}. This subgroup having an element of order 12 is not embeddable in H_i (i=2,3) and so is maximal.

M_{11}

order=7920=$2^4.3^2.5.11$ d=11 mult=1

presentation:

$G=< a,b \mid a^2=b^4=(ab)^{11}=(ab^2)^6=(ab)^2(ab^{-1})^2abab^{-1}ab^2abab^{-1}=1>$

conjugacy classes of elements of G

class	$\|c(x)\|$	p-power	p'-part	representative	cycle type
1A	7920			1	11^1
2A	48	A	A	a	$1^3 2^4$
3A	18	A	A	$(ab^2)^2$	$1^2 3^3$
4A	8	A	A	b	$1^3 4^2$
5A	5	A	A	$abab^{-1}ab^2$	$1^1 5^2$
6A	6	AA	AA	ab^2	$2^1 3^1 6^1$
8A	8	A	A	$(ab)^4 b$	$1^1 2^1 8^1$
B**	8	A	A	$b^{-1}(b^{-1}a)^4$	$1^1 2^1 8^1$
11A	11	A	A	ab	11^1
B**	11	A	A	$b^{-1}a$	11^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	720	11	$M_{10} \cong A_6 \cdot 2$		3
H_2	660	12	PSL(2,11)		2
H_3	144	55	$M_9{:}2 \cong 3^2{:}Q_8.2$	$N(3A^2)$	1
H_4	120	66	S_5	N(2A,3A,5A)	2
H_5	48	165	$M_8{:}S_3 \cong 2 \cdot S_4$	N(2A)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=16

We shall employ the method 2.3 (vi) to extend the cyclic subgroup <b> of order 4 to a Sylow 2-subgroup of G by 2-elements. TC finds $u=(ab^2)^2ab(ab^2)^2a$ of order 2 in $N_G(<b>)$ with $|<b,u>|=8$ and $v=(ab^2)^2a$ in $N_G(<b,u>)$ with $|<b,u,v>|=16$. Eliminating the redundant generator u and conjugating b and v by a we get the generators $x=b^2ab^2$ and $y=b^a$ for a Sylow 2-subgroup P_2 of G. The pair (x,y) satisfy the presentation

$<x,y \mid x^2=1, (xy)^3=yx>$, and we have $P_2 \cong <-2,4 \mid 2>$.

$n_2(G)=|G:N(P_2)|=495$.

(ii) Sylow 3-subgroup *order=9*

G has no element of order 9 and so the Sylow 3-subgroup of G is C_3xC_3. Starting with $x=[a,b]^2$ of order 3 PERM finds $y=abab^2ab^{-1}$ of order 3 with $|\langle xy\rangle|=3$ and $[x,y]=1$. Now x and y generate a Sylow 3-subgroup P_3 of G.

$n_3(G)=|G:N(P_3)|=55$.

(iii) Sylow 5-subgroup *order=5*

$P_5=\langle abab^{-1}ab^2\rangle=C(abab^{-1}ab^2)$, $|G:N(P_5)|=396$.

(iv) Sylow 11-subgroup *order=11*

$P_{11}=\langle ab\rangle=C(ab)$, $|G:N(P_{11})|=144$.

I.Maximal subgroups

(i) Structure: $M_{10} \cong A_6{\cdot}2$

$H_1=\langle a,b^t\rangle$, where $t=(ab)^2$.

A presentation for H_1 on $x=a$ and $y=b^t$, where $t=(ab)^2$, is

$$\langle x,y \mid x^2=y^4=(xy^2)^5=(xy^{-1}xy^2)^4y^{-1}xy=1\rangle$$

(see [7]). We now put $N=\langle x^y, (xy)^2\rangle$. Then N being isomorphic to A_6 has index 2 in $\langle x, y\rangle$. So $\langle x,y\rangle$ ($\cong M_{10}$) is an extension of A_6 by C_2.That M_{10} cannot split over A_6 follows immediately from the fact that both groups have an equal number of involutions.

A covering group C for H_1 is

$$\langle x,y \mid (xy^2)^5=x^4,\ (xy^{-1}xy^2)^4y^{-1}xy^{85}=[x,y^4]=[y,x^2]=1\rangle.$$

By TC, we have $|C|=2160$ and hence $M(H_1)=C_3$. Now a deficiency -1 presentation for M_{10} may be given by

$$\langle x, y \mid x^2y^4=(xy^{-1}xy^2)^4y^{-1}xy=1,\ (xy^2)^5=y^4\rangle.$$

Before going ahead, we make the following notes on M_{10}, M_9, and M_8 subgroups in M_{11} (as a permutation group acting on 11 objects) to be used in (iii) and (v).

Notes The permutation group $G=\langle a, b\rangle$, where

$$a=/10,8,11,4,7,6,5,2,9,1,3/,\quad b=/4,11,3,7,5,1,6,8,2,9,10/,$$

is (sharply) 4-transitive on $A=\{1,2,....,11\}$ and has order 11.10.9.8=7920. We have

(1) M_{10} is the stabilizer of a single point in G and has order 10.9.8=720. By 1.2.2 we can assume without loss of generality that $M_{10}= St_G(11)$;

(2) M_9 is the stabilizer of two points in G and has order 9.8=72. It is also the stabilizer of a point in M_{10} (as a 3-transitive group on $A\setminus\{11\}$). Again we may assume that $M_9=St_G(\{10,11\})$;

(3) M_8 is the stabilizer of three points in G and has order 8. It can also be regarded as the stabilizer of a point in M_9 (as a 2-transitive group on $A\setminus\{10,11\}$).

These facts can now be used to determine the structure of M_9 and M_8. As was seen in (i), $M_{10}=\langle x,y\rangle$ where x=a, $y=(b^{-1}a)^2b(ab)^2$. We now check that $\langle x,y\rangle=St_G(4)$. Next we see, by PERM, that $M_9=\langle y,z\rangle$ where $z=xy^{-1}(xy)^2x$. We have here $\langle y,z\rangle=St_G(\{1,4\})=St_H(1)$, where $H=\langle x, y\rangle$. Now a presentation for M_9 on its generators can be constructed as follows :

$$\langle y, z \mid y^4=z^4=(y^2z^2)^3=y^2z^2yzyz^{-1}=1 \rangle.$$

Finally, on setting $u=z^2yz^{-1}$ we observe that $M_8=\langle y,u\rangle=St_G(\{1,3,4\})=St_K(3)$, where $K=\langle y,z\rangle$. A presentation for M_8 is

$$\langle y,u \mid y^2=u^2,\ u^4=1,\ u^{-1}=yuy^{-1}\rangle,$$

which is the quaternion group Q_8.

(ii) Structure: PSL(2,11)

PSL(2,11) has presentation

$$\langle x,y \mid x^2=y^3=(xy)^{11}=((xy)^3(xy^{-1})^3)^2=1\rangle,$$

and thus is generated by elements of order 2 and 3 whose product has order 11. We begin with x=a which is the representative for the only class of involutions in G and find, by PERM, $y=b^{-1}(ab)^2b$ of order 11 with xy of order 3. TC now verifies that $|G:\langle x,y\rangle|=12$. This proves that x and y generate a maximal subgroup H_3 of G of the structure PSL(2,11). Next, (x,xy) satisfies the above presentation for PSL(2,11).

(iii) Structure: $M_9{:}2\cong 3^2{:}Q_8.2$

This is the normalizer in G of an M_9 subgroup by 3.3.2. As we remarked above, M_9 is generated by two elements r and s with (r,s) of type (4,4,4) and $|fix(r) \cap fix(s)|= 2$. We take r=b and find, by PERM, $s=ab^{-1}abab^2ab(ab^{-1})^2a$ with r and s satisfying the above conditions. Next r and s generate a subgroup of order 72 and satisfy the presentation for M_9. TC is now used to find $N_G(\langle r,s\rangle)=\langle r,t\rangle$, where $t=[a,b]^2a$, with $|G:\langle r,t\rangle|=55$. Conjugation by ab^{-1} of r and t gives the generators $x=a^u$ and $y=b^v$, where $u=b^{-1}a$ and $v=ab^{-1}$, for a subgroup H_3 of G of order 144. The subgroup H_3, having an element of order 6 (namely xy^2), cannot be a subgroup of H_1 and hence is a maximal subgroup of G.

A presentation for H_3 on x and y is

$$K=\langle x,y \mid x^2=y^4=(xy)^2(xy^2)^2xyxy^{-1}=1 \rangle.$$

On taking $N=\langle (xy^2)^2, yxy^2xy\rangle$ and $M=\langle x^y, y^x\rangle$, we have $N \lhd K$, $N\cong C_3xC_3$, $M \cong \langle A,B \mid A^2=(AB)^3AB^{-1}=1\rangle$, and $N\cap M=(1)$. Next we observe that $\langle B,ABA\rangle$ has index 2 in $\langle A,B\rangle$ and is isomorphic to Q_8. Therefore $K \cong 3^2{:}Q_8.2$. (we note that $M\cong P_2$.)

K has the following covering group

$$\langle x,y,z \mid x^2=y^4z=(xy)^2(xy^2)^2xyxy^{-1}z=[x,z]=[y,z]=1\rangle$$

of order 144 showing that $M(K)=1$. A deficiency zero presentation for K may now be given by

$$\langle x,y \mid x^2y^4=(x^{-1}y)^2(xy^2)^2xyxy^{-1}=1\rangle.$$

(iv) Structure: S_5

S_5 has presentation

$$\langle x, y \mid x^4=y^6=(xy)^2=(x^{-1}y)^3=1\rangle$$

(see[18], p137) and thus a generating pair of type (2,4,6). PERM simply gives $x=b^2$ of order 2 and $y=b^a$ of order 4 with $|\langle xy\rangle|=6$. Now x and y generate a subgroup H_5 of order 120 and satisfy the above presentation for S_5. H_5 is maximal in G since $|H_4|$ does not divide $|H_i|$ (i=2, 3) and that S_5, having an element of order 6, is not embeddable in M_{10}.

(v) structure: $M_8{:}S_3 \cong 2{\cdot}S_4$

This is the normalizer in G of an M_8 subgroup, by 3.3.2. A generating pair (r,s) for M_8 ($\cong Q_8$) in G is of type (4,4,4) with $|fix(r) \cap fix(s)|=3$. On taking r=b, which fixes exactly three points in the permutation representation of G, PERM gives $s=(ab^2)^2ab(ab^2)^2a$ of order 4 with the property that s fixes the same points which are left fixed by r and that $|\langle rs\rangle|=4$. It is now easy to check that $\langle r,s\rangle\cong Q_8$ and $N_G(\langle r,s\rangle)=\langle t,q\rangle$, where $t=ab^2ab^2a$ and $q=ab(ab^{-1})^5abab^2ab((ab^{-1})^2ab)^2ab^{-1}aba$, with $|\langle t,q\rangle|=48$. Next we observe that (t,q^2) is a generating pair for $\langle t,q\rangle$ of type (2,3,8;6). This allows PERM to find the generators x=a and $y=bab^{-1}(ab)^2$ for a subgroup H_5 of G of order 48.

It remains to show that H_5 is, in fact, a maximal subgroup of G. To see this we first construct the following presentation for H_5 on its generators x and y :

$$L=\langle x,y \mid x^2=y^3=(xyxyxy^{-1})^2=1\rangle,$$

and then observe that $|H_5'|=24$, $|H_3'|=36$ which shows that H_5 is not embeddable in H_3, by 1.1.4 (i). That H_5 is not isomorphic to any subgroup of H_1 easily follows from the fact that H_1 has no elements of order 6 while $[x,y]$ in L ($\cong H_5$) is an element of order 6.

On setting $N=\langle (xy)^2, y[x,y]\rangle$ and $M=\langle x^y, y^x\rangle$ one observes that $N \lhd L$, $N \cong Q_8$, $M \cong S_3$ and $N \cap M=(1)$.Thus $L \cong M_8:S_3$. On the other hand $\langle (xy)^4\rangle$ is a normal subgroup of L of order 2 and $L/\langle (xy)^4\rangle \cong S_4$. So L is an extension C_2 by S_4. However, L has no S_4 subgroup using SUBGPTEST. This shows that $L \cong 2{\cdot}S_4$.

A covering group C for L has presentation

$$\langle x,y,z \mid x^2=y^3z^{-1}=(xyxyxy^{-1})^2z^{-1}=[x,z]=[y,z]=1\rangle$$

with $|C|=48$, that is $M(L)=1$. A deficiency zero presentation for L is

$$\langle x,y \mid x^2y^3=(xyxyx^{-1}y^{-1})^2=1\rangle.$$

PSL(2,27)

order=9828=$2^2.3^3.7.13$ d=28 mult=2

G=< a, b | $a^2=b^3=(ab)^{13}=((ab)^3(ab^{-1})^3)^2=1$>

a=/2,1,4,3,6,5,8,7,10,9,12,11,14,13,16,15,18,17,20,19,22,21,24,23,26,25,28,27/

b=/1,20,2,25,4,15,6,14,8,17,10,16,12,9,7,13,11,21,18,3,19,24,22,23,5,28,26,27/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	9828			1	1^{28}
2A	28	A	A	a	2^{14}
3A	27	A	A	b	$1^1 3^9$
B**	27	A	A	b^{-1}	$1^1 3^9$
7A	14	A	A	$[a,b]^6$	7^4
B*3	14	A	A	$[a,b]^{-4}$	7^4
C*2	14	A	A	$[a,b]^2$	7^4
13A	13	A	A	ab	$1^2 13^2$
B*3	13	A	A	$(ab)^3$	$1^2 13^2$
C*4	13	A	A	$(ab)^4$	$1^2 13^2$
D*5	13	A	A	(ab)5	$1^2 13^2$
E*2	13	A	A	$(ab)^2$	$1^2 13^2$
F*6	13	A	A	$(ab)^6$	$1^2 13^2$
14A	14	CA	AA	[a,b]	14^2
B*3	14	AA	BA	$[a,b]^3$	14^2
C*5	14	BA	CA	$[a,b]^5$	14^2

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	351	28	3^3:13	N($3AB^3$)	1
H_2	28	351	D_{28}	N(2A),N(7ABC)	2
H_3	26	378	D_{26}	N(13A-F)	1
H_4	12	819	A_4	N($2A^2$)	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=4

Using PERM we obtain the generators x=a, y= a^t, where t=$(ba)^2$b, for a Sylow 2-subgroup P_2 of G isomorphic to C_2xC_2.

$n_2(G)=|G:N(P_2)|=819$.

(ii) Sylow 3-subgroup order=27

The Sylow 3-subgroup of G is an elementary abelian group of order 27 and thus has C_3xC_3 as a subgroup. Starting from $x=b^a$ we find $y=(ba)^3(b^{-1}a)^2(ba)^2b$ with $<x,y> \cong C_3xC_3$. Now TC can be used to extend $<x,y>$ to the Sylow 3-subgroup $P_3=<x,y,z>$, where $z=(ba)^4(b^{-1}aba)^2b^{-1}(ab)^4$, of G.

$n_3(G)=|G:N(P_3)|=28$.

(iii) Sylow 7-subgroup *order=7*

$P_7=<[a,b]^2>$, $n_7(G)=|G:N(P_7)|=351$.

(vi) Sylow 13-subgroup *order=13*

$P_{13}=<ab>=C(ab)$, $n_{13}(G)=|G:N(P_{13})|=378$.

II. Maximal subgroups

(i) structure: $3^3{:}13$

We simply find that $St_G(1)=<b,ab^{-1}(ab)^3ab^{-1}aba>$. Set $x=b$, $y=ab^{-1}(ab)^3ab^{-1}aba$. Then x, y generate a subgroup H_1 of minimal index 28 and satisfy the presentation

$$<x,y \mid y^{13}=[x,y][x^{-1},y]=1,\ y^2xyx=xy^3>.$$

Now let $N=<x,[x,y],[x,y^{-1}]>$. Then $<x,y>$ is an split extension of N ($\cong 3^3$) by $<y>$. That is, $H_1 \cong 3^3{:}31$. By combining the first two relations of the presentation into the single relation $[x,y][x^{-1},y]y^{13}=1$ we are able to show that the soluble group H_1 has deficiency zero.

(ii) structure: D_{28}

PERM gives the generating pair (a,a^b) for a maximal subgroup H_2 of G isomorphic to D_{28}.

(iii) structure: D_{26}

Similarly we find $H_3=<a, a^{bab}>$ with $H_3 \cong D_{26}$.

(iv) structure: A_4

That A_4 has a generating pair of type (2,3,3) helps us to find the generators $x=a$, $y=b^t$, where $t=ab^{-1}(ab)^2$, for a maximal subgroup H_4 of G isomorphic to A_4.

PSL(2,29)

order=12180=2^2.3.5.7.29   d=30   mult=2

G=<a,b | $a^2=b^3=(ab)^{15}=((ab)^3(ab^{-1})^{-5})^2=1$>

a=/10,3,2,5,4,7,6,9,8,1,12,11,14,13,16,15,18,17,20,19,22,21,24,23,26,25,28,27,29,30/,

b=/2,11,28,3,6,26,30,7,29,9,1,19,12,18,14,24,16,15,13,27,20,25,22,17,23,5,21,4,10,8/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	12180		1		1^{30}
2A	28	A	A	a	$1^2 2^{14}$
3A	15	A	A	b	3^{10}
5A	15	A	A	$(ab)^6$	5^6
B*	15	A	A	$(ab)^3$	5^6
7A	14	A	A	$[a,b]^3$	$1^2 7^4$
B*2	14	A	A	[a,b]	$1^2 7^4$
C*3	14	A	A	$[a,b]^2$	$1^2 7^4$
14A	14	BA	AA	$(ab)^4 ab^{-1}$	$1^2 14^2$
B*5	14	CA	BA	$((ab)^5(ab^{-1})^2)^3$	$1^2 14^2$
C*3	14	AA	CA	$(ab)^5(ab^{-1})^2$	$1^2 14^2$
15A	15	BA	AA	ab	15^2
B*2	15	AA	BA	$(ab)^2$	15^2
C*4	15	BA	AA	$(ab)^4$	15^2
D*7	15	AA	BA	$(ab)^7$	15^2
29A	29	A	A	$(ab)^2(ab^{-1})^2$	$1^1 29^1$
B*	29	A	A	$((ab)^2(ab^{-1})^2)^2$	$1^1 29^1$

conjugacy classes of maximal subgroups

group	order	index	structure	specification	mult
H_1	406	30	29:14	N(29AB)	1
H_2	60	203	A_5	N(2A,3A,5AB)	2
H_3	60	203	A_5	N(2A,3A,5AB)	2
H_4	30	406	D_{30}	N(3A),N(5AB)	1
H_5	28	435	D_{28}	N(2A),N(7ABC)	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup   order=4

The elements x=a, $y=(bab^{-1}a)^3b$ generate a Sylow 2-subgroup P_2 of G which is the Klein 4-group.

$n_2(G)=|G: N(P_2)|=1015$.

(ii) Sylow 3-subgroup *order=3*

$P_3=<b>$, $n_3(G)=|G:N(P_3)|=406$.

(iii) Sylow 5-subgroup *order=5*

$P_5=<(ab)^3>$, $n_5(G)=|G:N(P_5)|=406$.

(vi) Sylow 7-subgroup *order=7*

$P_7=<[a,b]>$, $n_7(G)=|G:N(P_7)|=435$.

(v) Sylow 29-subgroup *order=29*

$P_{29}=<(ab)^2(ab^{-1})^2>=C((ab)^2(ab^{-1})^2)$, $n_{29}(G)=|G:N(P_{29})|=30$.

II.Maximal subgroups

(i) structure: *29:14*

This is the normalizer in G of an element in 29AB by table II. Taking $x=(ab)^2(ab^{-1})^2$, we find that $N(x)=<x,y>$, where $y=ab^{-1}ab(ab^{-1})^2(ab)^3$, with $<x,y>$ of order 406 using TC. The pair (x,y) satisfies the deficiency zero presentation

$$<x,y \mid x^{29}=y^{14}, y^{-1}x^{17}y=x^{18}>$$

for the split metacyclic group $H_1=<x,y>$.

(ii)-(iii) structure: A_5

By table II, G has two non-conjugate A_5 subgroups H_2, H_3 both having generating pairs of type (2A,3A,5AB). Using PERM we may obtain the generating pairs $(a^b, (ba)^5)$ and $((ab)^2(ab^{-1})^3(ab)^2a,b)$ for H_2, H_3. We have $H_2 \cap H_3=D_{10}$.

(iv) structure: D_{30}

We find $H_5=<a, (ab)^2(abab^{-1})^2>$ isomorphic to D_{30}. This subgroup is maximal in G for A_5 cannot have D_{30} as a subgroup.

(v) structure: D_{28}

Similarly we obtain the generators $x=a$, $y=b(abab^{-1})^2ab^{-1}ab$ for a dihedral subgroup of order 28 which is obviously maximal in G.

PSL(2,31)

order=14880=2^5.3.5.31 d=32 mult=2

$G=\langle a,b \mid a^2=b^3=(ab)^{15}=((ab)^4(ab^{-1})^6)^2=1\rangle$

a=/2,1,4,3,6,5,8,7,10,9,12,11,14,13,16,15,18,17,20,19,22,21,24,23,26,25,28,27,30,29,32,31/,

b=/20,21,2,4,1,7,18,9,19,11,16,13,17,25,14,10,12,6,8,5,3,24,22,23,15,31,26,30,28,29,27,32/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle-type
1A	14880			1	1^{32}
2A	32	A	A	a	2^{16}
3A	15	A	A	b	$1^2 3^{10}$
4A	16	A	A	$[a,b]^2$	4^8
5A	15	A	A	$(ab)^6$	$1^2 5^6$
B*	15	A	A	$(ab)^3$	$1^2 5^6$
8A	16	A	A	$[a,b]^3$	8^4
B*	16	A	A	[a,b]	8^4
15A	15	BA	AA	ab	$1^2 15^2$
B*2	15	AA	BA	$(ab)^2$	$1^2 15^2$
C*4	15	BA	AA	$(ab)^4$	$1^2 15^2$
D*7	15	AA	BA	$(ab)^7$	$1^2 15^2$
16A	16	A	A	$(ab)^5ab^{-1}$	16^2
B*3	16	B	A	$((ab)^5ab^{-1})^3$	16^2
C*7	16	A	A	$((ab)^5ab^{-1})^7$	16^2
D*5	16	B	A	$((ab)^5ab^{-1})^5$	16^2
31A	31	A	A	$(ab^{-1}ab)^2(ab)^3ab^{-1}$	$1^1 31^1$
B**	31	A	A	$b(ab^{-1})^3(ab^{-1}ab)^2a$	$1^1 31^1$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	465	32	31:15	N(31AB)	1
H_2	60	248	A_5	N(2A,3A,5AB)	2
H_3	60	248	A_5	N(2A,3A,5AB)	2
H_4	32	465	D_{32}	N(2A)	2
H_5	30	496	D_{30}	N(3A),N(5AB)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=32

By table II, the Sylow 2-subgroup of G is D_{32}. Using this fact we may obtain, by PERM, the generators x=a, $y=b(abab^{-1})^3ab^{-1}$ for a Sylow 2-subgroup P_2 of G.

$n_2(G)=|G:N(P_2)|=465$.

(ii) Sylow 3-subgroup *order=3*

$P_3=\langle b\rangle$, $n_3(G)=|G:N(P_3)|=496$.

(iii) Sylow 5-subgroup *order=5*

$P_5=\langle (ab)^3\rangle$, $n_5(G)=|G:N(P_5)|=496$.

(iv) Sylow 31-subgroup *order=31*

$P_{31}=\langle (ab^{-1}ab)^2(ab)^3ab^{-1}\rangle=C((ab^{-1}ab)^2(ab)^3ab^{-1}\rangle$, $n_{31}(G)=|G:N(P_{31})|=32$.

II.Maximal subgroups

(i) structure: *31:15*

This is the stabilizer of a point in the permutation representation of G. Taking $x=ab$ we find $y=ba^t$, where $t=(ba)^2$, with $|fix(x)\cap fix(y)|=1$ and $|\langle x,y\rangle|=465$. Now let $u=y^{-1}x^4$. Then x, u generate $H_1=\langle x,y\rangle$ and satisfy the following deficiency zero presentation $\langle x, u \mid x^{15}=u^{31}, x^{-1}u^4x=u^5\rangle$ for the split metacyclic group H_1.

(ii)-(iii) structure: A_5

Using PERM, we find the generating pairs (a, b^{ab}), $((ab)^5, ((ba)^2bab^{-1}a)^2)$ for two non-conjugate A_5 subgroups H_1, H_2 of G. We note that H_1 has trivial intersection with H_2.

(iv) structure: D_{32}

This is simply a Sylow 2-subgroup of G. Thus we may take $H_4=P_2$.

(v) structure: D_{30}

Similarly we find $H_5=\langle a, a^{bab}\rangle$ isomorphic to D_{30}.

A_8

order=20160=$2^6.3^2.5.7$ d=8   mult=2

G=< a,b | $a^2=b^4=(ab)^{15}=(ab^2)^4=(ab)^5ab^2ab(ab^{-1})^2(ab)^2ab^{-1}(ab)^7ab^{-1}=1$>

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	20160			1	1^8
2A	192	A	A	a	2^4
2B	96	A	A	b^2	1^42^2
3A	180	A	A	$(ab)^5$	1^53^1
3B	18	A	A	$(abab^2ab^{-1})^2$	1^23^2
4A	16	A	A	[a,b]	4^2
4B	8	B	A	b	$1^22^14^1$
5A	15	A	A	$(ab)^3$	1^35^1
6A	12	AB	AB	$abab^2$	$1^12^23^1$
6B	6	BA	BA	$abab^2ab^{-1}$	2^16^1
7A	7	A	A	$(ab)^2ab^{-1}$	1^17^1
B**	7	A	A	$b(ab^{-1})^2a$	1^17^1
15A	15	AA	AA	ab	3^15^1
B**	15	AA	AA	$b^{-1}a$	3^15^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	2520	8	A_7		6
H_2	1344	15	2^3:PSL(2,7)	$N(2A^3)$	2x2
H_3	1344	15	2^3:PSL(2,7)	$N(2A^3)$	2x2
H_4	720	28	S_6		2
H_5	576	35	$2^4:(S_3xS_3)$	$N(2^4) \cong N(2A_9B_6)$	2x2
H_6	360	56	$(A_5x3):2$	N(3A),N(2B,3A,5A)	2

I.Sylow p-subgroups

(i) Sylow 2-subgroup   order=64

In G the commutator [a, b] has order 4 and so <a, a^b > is a D_8 subgroup of G. Using the method 2.3 (vi), we find u=$(ba)^6b^{-1}(ab)^6ab^2$ in N_G(< a, a^b>) with <a,a^b,u> of order 16 and v=$b^2(ab)^2(ab^2)^2$ in N_G(<a,a^b,u>) with <a,a^b,u,v> of order 32. This subgroup can now be extended by b^2ab^2 to a subgroup of G of order 64. Removing the redundant generators u and v gives a Sylow 2-subgroup P_2 of G. On putting x=a, y=a^b, and z=b^2ab^2, we find the following presentation for P_2 on x, y, z :

<x, y, z | $x^2=y^2=z^2=(xz)^2=(xy)^4=(yz)^4=(xyz)^4=1$ >.

This group is isomorphic to the 2-group $64,259,(1^3,1^4) \cong \Gamma_{25}a_1$ with presentation

$<a_1, a_2, a_3 \mid a_1{}^2=a_2{}^2=a_3{}^2=1, [a_1,a_2]^2=[a_1, a_3]^2=[a_1,a_2]=1, [a_1,a_2,a_3]=[a_1,a_3,a_2] >$.

This is immediate on setting $a_1=y$, $a_2=x$, and $a_3=z$.

$n_2(G)=| G : N(P_2) |=315$.

(ii) Sylow 3-subgroup order=9

It is clear that the Sylow 3-subgroup of G is C_3xC_3. Now PERM gives $x=(abab^2)^2$, $y=(bab^2a)^2$ for a Sylow 3-subgroup P_3 of G.

$n_3(G)=| G : N(P_3) |=280$.

(iii) Sylow 5-subgroup order=5

$P_5=< (ab)^3 >$, $n_5(G)=| G : N(P_5) | =336$.

(iv) Sylow 7-subgroup order=7

$P_7=< (ab)^2ab^{-1} >=C((ab)^2ab^{-1})$, $n_7(G)=| G : N(P_7) |=960$.

II.Maximal subgroups

In example 3 of Ch. 3 we gave generators for the six conjugacy classes of maximal subgroups of G. We proceed here to give presentations for these subgroups satisfied by their corresponding generators . Their multipliers are calculated when they are needed.

(i) structure : A_7

$H_1=< b^2, (bab)^{aba} >$ in 4.1 of [16].

(ii)-(iii) structure $2^3 : PSL(2,7)$

$H_2=< a^b, b^a >$, $H_3=< bab^{-1}, b^a >$.

Let $x_2=a^b$, $y_2=b^a$ and $x_3=bab^{-1}$, $y_3=b^a$. Then the pairs (x_2,y_2) and $(x_3, y_3{}^{-1})$ satisfy the presentation

$$H=< x ,y \mid x^2=y^4=(xy)^7=[x,y]^4=((xy)^3yxy^{-1})^3=1 >.$$

Taking $N=< (xy^2)^3, (yxy)^3, (yx)^2(y^2x)^2y^{-1}xy^2xy >$ and $M=< x^t, xy>$, where $t=x^y$, shows that $N \cong 2^3$, $M \cong PSL(2,7)$, $N \lhd H$, and $N \cap M=(1)$. So that $H \cong 2^3 : PSL(2,7)$.

The group H is a perfect group and hence has a unique covering group C, by 1.5.14. In order to find a presentation for C we first observe that

$$H \cong \langle x, y \mid x^2 = ((xy)^3yxy^{-1})^3 = (xyxy^{-1})^4y^4 = (xy)^7y^{-4} = 1 \rangle.$$

Then C may be given by

$$C = \langle x, y \mid ((xy)^3yxy^{-1})^3 = (xyxy^{-1})^8y^8x^{-4}, (xy)^6x^{-5}y^{-3} = [y,x^2] = [x,(xyxy^{-1})^4y^4] = [y, (xyxy^{-1})^4] = 1 \rangle.$$

C has order 5376 proving that $|M(H)|=4$. Now it is easy to check that $M(H) \cong \langle x^2,(xyxy^{-1})^4y^4 \rangle \cong C_2xC_2$. This fact together with the latter presentation of H proves that H is efficient.

It is of interest to note that there are exactly two isomorphism classes of perfect groups G_1 and G_2 of order 1344 with $G_1 \cong 2^3$: PSL(2,7) and $G_2 \cong 2^3 \cdot PSL(2,7)$ (the normal subgroup has no complement but supplement) as stated in [40]. Our maximal subgroup H_2 ($\cong H_3$) is clearly isomorphic to G_1.

(iv) structure : S_6

$H_4 = \langle a, (ba)^2b \rangle$.

A presentation for H_4 on $x=a$ and $y=(ba)^2b$ is

$$\langle x,y \mid x^2 = y^6 = (xy)^5 = [x,y]^3 = [x,y^2]^2 = 1 \rangle.$$

Next $M(S_6)=C_2$, by 1.5.11 (i) and an efficient presentation for S_6 is given by

$$\langle x ,y \mid x^2 = (xy)^5, (xyxy^{-1})^3 = 1, (xy^2xy^{-2})^2y^6 = 1 \rangle.$$

(v) structure $2^4 : (S_3xS_3)$

$H_5 = \langle (ab^2)^2, (ab)^3ab^{-1} \rangle$.

Let $x=(ab^2)^2$ and $y=(ab)^3ab^{-1}$. Then x, y satisfy the presentation

$$K = \langle x , y \mid x^2 = y^6 = (xy^2)^4 = (xyxy^3xy^{-2})^2 = (xyxy^{-1}xy^3)^2 = 1 \rangle.$$

Taking $N = \langle (xy^2)^2,(y^2x)^2,(yxy)^2,(xy)^2(yx)^2 \rangle$ and $M=\langle r,s \rangle$, where $r=y^x$ and $s=xyxy^2xy^{-2}$, shows that $N \triangleleft K$, $N \cap M=(1)$, $N \cong 2^4$, and $M \cong \langle r,s \mid r^6=s^6=(rs)^2=(rs^{-1})^2=1 \rangle$. So $K \cong 2^4 : M$. Next M is the direct product of $\langle r^2, s^3 \rangle$ ($\cong S_3$) and $\langle r^3, s^2 \rangle$ ($\cong S_3$) indicating that $K \cong 2^4 : (S_3xS_3)$.

Using the above presentation for H_5, we may construct the presentation

$$C = \langle a_1,a_2,a_3,a_4,a_5 \mid a_1^2 = a_2^6a_3 = (a_1a_2^2)^4a_3 = (a_1a_2a_1a_2^3a_1a_2^{-2})^2a_4^{-1} = (a_1a_2a_1a_2^{-1}a_1a_2^3)^2a_5^{-1} = 1, [a_i,a_j] = 1\ (1 \le i \le 5, 3 \le j \le 5, i < j) \rangle$$

for a covering group of H_5. Now $|C|=2304$ and so $|M(H_5)|=4$. Indeed, $M(H_5) \cong \langle a_3,a_5 \rangle \cong C_2xC_2$. Combining the first and second relation of K gives the following efficient presentation for H_4 :

$$< x, y \mid x^2y^6=(xy^2)^4=(xyxy^3xy^{-2})^2=(xyxy^{-1}xy^3)^2=1 >.$$

(vi) structure: $(A_5x3) : 2$

$H_6=< b^2, (ab)^2ab^2ab >$.

We set $x=b^2$ and $y=(ab)^2ab^2ab$ and find the presentation

$$L=< x, y \mid x^2=y^6=(xy^3)^3=(xy^2)^4=1 >$$

for H_5 on the generators x and y. The group L is a semi-direct product of $N=<(xy)^2,(yx)^2>$ and $< x >$. Putting $t=(xy)^2$ and $s=(yx)^2$ we find

$$N \cong < t, s \mid (ts)^2=(ts^{-1})^3=t^3s^2t^{-2}s^2=1 >.$$

We now take $M_1=< t^3, s^3>$ and $M_2=< t^5 >$. Then $M_1 \cong A_5$, $M_2 \cong C_3$ and $N \cong <t,s> \cong M_1xM_2$. Thus $L \cong (A_5x3) : 2$.

A covering group of H_6 has presentation :

$$C=<a_1,a_2,a_3,a_4 \mid a_1{}^2a_3a_4{}^{-3}=a_2{}^6a_4=(a_1a_2{}^3)^3a_3a_4{}^{-3}=(a_1a_2{}^2)^4a_4{}^{-5}=[a_1,a_3]=[a_1,a_4]=[a_2,a_3]=[a_2,a_4]=[a_3,a_4]=1 >.$$

C having order 720 gives $M(H_6)=C_2$. We have $M(H_6) \cong < a_4 > \cong C_2$. The group H_6 is efficient for

$$H_6 \cong <x,y \mid x^2=(xy^2)^4=1, (xy^3)^3=y^6 >.$$

PSL(3,4)

order=20160=$2^6.3^2.5.7$ d=21 mult=4x12

G=< a,b | $a^2=b^4=(ab)^7=(ab^2)^5=(b(ab)^3)^7= b(ab)^3b^2(ab)^3a(b(b(ab)^3)^2bab)^2=1$>

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	20160			1	1^{21}
2A	64	A	A	a	$1^5 2^8$
3A	9	A	A	$ab^2(abab^{-1})^2$	$1^3 3^6$
4A	16	A	A	b	$1^1 2^2 4^2$
4B	16	A	A	$abab^2ab^{-1}$	$1^1 2^2 4^2$
4C	16	A	A	$abab^{-1}ab^2$	$1^1 2^2 4^2$
5A	5	A	A	ab^2	$1^1 5^4$
B*	5	A	A	$(ab^2)^2$	$1^1 5^4$
7A	7	A	A	ab	7^3
B**	7	A	A	$b^{-1}a$	7^3

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	960	21	$2^4{:}A_5$	$N(2A^4)$	2x4x4
H_2	960	21	$2^4{:}A_5$	$N(2A^4)$	2x4x4
H_3	360	56	A_6	N(2A,3A,3A,4A,5AB)	6
H_4	360	56	A_6	N(2A,3A,3A,4B,5AB)	6
H_5	360	56	A_6	N(2A,3A,3A,4C,5AB)	6
H_6	168	120	PSL(2,7)	N(2A,3A,4A,7AB)	2
H_7	168	120	PSL(2,7)	N(2A,3A,4B,7AB)	2
H_8	168	120	PSL(2,7)	N(2A,3A,4C,7AB)	2
H_9	72	280	$3^2{:}Q_8$	$N(3A^2)$	3

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=64

By table II, we see that G has a subgroup of the structure C_4xC_4 with a generating pair (r,s) of type (4A,4A,4A;1). Taking r=b, PERM gives $s=ab^2(ab)^2ab^{-1}(ab)^2ab^2a$ with $<r,s>\cong C_4xC_4$. Now TC is used to find $t=(ab^{-1})^2(ab^2)^2(ab)^2a$ in $N_G(<r,s>)$ with $|<r,s,t>|=32$ and $q=(ab^2ab^{-1})^2(abab^2)^2a$ in $N_G(<r,s,t>)$ with $|<r,s,t,q>|=64$. Conjugation by ab^2 of r, s, t, and q gives the generators $x=b^c$, $y=b^d(ba)^2$, $z=a^e$, and $u=a^f$, where $c=ab^2$, $d=ab^{-1}a$, $e=b^2abab^{-1}$, and $f=bab^2aba$, for a Sylow 2-subgroup P_2 of G. A presentation for P_2 on x, y, z, and u is

$$<x,y,z,u \mid z^2=u^2=[x,y]=(yu)^2=(xyz)^2=(yzu)^2=1, [y,z]=x^2, (xu)^2=y^2>.$$

This group is isomorphic to the group $64,183,(1^4,1^22^2) \cong \Gamma_{13}a_1$ with the presentation

$$<a_1,a_2,a_3,a_4 \mid a_3^2=a_4^2=1,[a_3,a_1]a_1^2=[a_1,a_4]a_2^{-2}a_1^2=1$$
$$[a_2,a_3]a_2^2a_1^{-2}=[a_4,a_2]a_2^2=[a_1,a_2]=[a_3,a_4]=1>,$$

by the mapping $a_1 \to xy$, $a_2 \to y$, $a_3 \to z$, and $a_4 \to uy$.

$n_2(G)=|G:N(P_2)|=105$

(ii) Sylow 3-subgroup order=9

Clearly the Sylow 3-subgroup of G is C_3xC_3. Starting with $x=ab^2[a,b]^2$, PERM finds $y=b[a,b]^2ba$ with (x,y) of type (3,3,3;1). Now x and y generate a Sylow 3-subgroup P_3 of G.

$n_3(G)=|G:N(P_3)|=280$

(iii) Sylow 5-subgroup order=5

$P_5=<ab^2>=C(ab^2)$, $n_5(G)=|G:N(P_5)|=2016$

(iv) Sylow 7-subgroup order=7

$P_7=<ab>=C(ab)$, $n_7(G)=|G:N(P_7)|=960$

II.Maximal subgroups

(i)-(ii) Structure $2^4{:}A_5$

G has two non-conjugate isomorphic subgroups H_1 and H_2 of minimal index 21. In [7] the generators $x_1=b^a$ and $y_1=a^tab$, where $t=bab^2$, are given for H_1 with $|fix(x_1)\cap fix(y_1)|=1$ and x_1, y_1 satisfying the presentation

$$H=<x,y \mid x^4=y^5=(xy)^3=(xy^{-1})^3=(x^2y)^5=(x^2y^2xy^{-2})^2=1>.$$

We now take $x_2=b^a$ and find, by PERM, $y_2=baa^s$, where $s=b^{-1}ab^2$, with (x_2,y_2) of type (4,5,3) and $|fix(x_2)\cap fix(y_2)|=0$. Let $H_2=<x_2,y_2>$. Then H_2 has index 21 in G and is not conjugate to H_1. The generators x_2 and y_2 satisfy the above presentation for H_1. The subgroup H_1 has $2^4.A_4$ intersection with H_2.

We now take $N=<x^2,y^{-1}x^2y, yx^2y^{-1},y^{-2}x^2y^2>$, $M=<xy,y^{-1}xy^2xy^{-2}>$ and observe that $N \triangleleft H$, $N \cong 2^4$, $M \cong A_5$, and $N\cap M=(1)$. This shows that H is a semi-direct product of 2^4 by A_5. We also note that $H/H'=1$ and hence H is a perfect group of order 960.

We shall show that $M(H) \cong C_2xC_4xC_4$. Since the above presentation for H leads to an unpleasant presentation for a covering group of H, we use the method described in 3.5 to obtain a new presentation for H with fewer relations. It is easily seen that H can be generated by two of its elements x and y with $|<x>|=|<y>|=3$ and $|<xy>|=5$. By PERM, we find the generators $x=(ab^{-1}ab)^2bab$ and $y=abab^{-1}ab^2abab^{-1}$ with (x,y) of type (3,3,5) for the stabilizer of a point in the permutation representation of G.Now a presentation for <x, y> on x and y is

$$<x,y \mid x^3=y^3=(xy)^5=(xy^{-1})^5=[x,y]^3=1>.$$

Now a covering group C for <x,y> is

$$<a_1,a_2,a_3,a_4,a_5 \mid a_1^3a_3^{-1}a_4^{-5}=a_2^3a_4^{-1}=(a_1a_2)^5a_3^{-2}a_4^{-10}=(a_1a_2^{-1})^5a_4^{-7}=[a_1,a_2]^3a_5^{-1}=1,$$
$$[a_i,a_j]=1\ (1 \leq i \leq 5, 3 \leq j \leq 5, i < j)>.$$

Eliminating the redundant generators a_3, a_4, and a_5 we get

$$C=<a_1,a_2 \mid (a_1a_2)^5=a_1^6,(a_1a_2^{-1})^5=a_2^{21}, [a_1,a_2^3]=[a_1^3,a_2]=[a_1,[a_1,a_2]^3]=[a_2,[a_1,a_2]^3]=1>.$$

Next, TC verifies that $|C|=30720$ and hence $|M(H)|=32$. In fact $M(H) \cong <a_1^3,a_2^3,[a_1,a_2]^3> \cong C_2xC_4xC_4$.

(iii)-(v) structure: A_6

By table II, G has three conjugacy classes of A_6 subgroups with representatives H_3, H_4, and H_5 whose elements of order 4 lie in the classes 4A, 4B, and 4C respectively (notice that A_6 has only one conjugacy class of order 4). The group A_6 has a generating pair of type (2,4,5/5) and this allows PERM to seek generating pairs (x_3,y_3), (x_4,y_4), and (x_5,y_5) of type (2,4A,5/5), (2,4B,5/5), and (2,4C,5/5) for maximal subgroups H_3, H_4, and H_5 respectively. We find $x_3=b^2$, $y_3=(ab)^2ab^{-1}a$, $x_4=a$, $y_4=bab^{-1}(ab)^2$, $x_5=a$, $y_5=(ba)^2b^{-1}ab$.

The subgroups H_3, H_4, and H_5 are mutually non-conjugate, and we have $H_3 \cap H_i \cong D_{10}$ (i=4, 5) , $H_4 \cap H_5 \cong 3^2.2$.

(vi)-(viii) Structure: *PSL(2,7)*

Analogous to (iii)-(v) we obtain the generators $x_6=b^2$, $y_6=abab^{-1}aba$ for H_6, $x_7=a$, $y_7=(ba)^2b^{-1}ab(ab^2)^2$ for H_7 and $x_8=a$, $y_8=(b^2a)^2bab^{-1}(ab)^2$ for H_9. For i=6,7,8, (x_i,y_i) is a generating pair of type (2,3,7;4) for a PSL(2,7) subgroup with $[x_6,y_6] \in 4A$, $[x_7,y_7] \in 4B$, and $[x_8,y_8] \in 4C$.

Again H_6 and H_7 and H_8 are mutually non-conjugate; and we have $H_i \cap H_j = S_3$ $(6 \le i, j \le 8,\ i \ne j)$.

(ix) structure: $3^2{:}Q_8$

This is the normalizer in G of a Sylow 3-subgroup, by table II. We take $P_3=\langle x,y\rangle$, where $x=ab^2[a,b]^2$ and $y=b[a,b]^2ba$. Then TC gives $N_G(P_3)=\langle z,u\rangle$, where $z=b^a$ and $u=(b^2a)^2bab^{-1}(ab^2)^2ab^{-1}ab^2abab^2$, with $|\langle z, u\rangle|=72$. It is easily checked that (z,u) is of type (4A,4C,4B;2). This helps us to find, by PERM, the generators $v=b$ and $w=abab^2ab^{-1}(ab)^2a$ for a subgroup H_9 of G of order 72 with (v,w) of the same type. We now proceed to show that H_9 is a maximal subgroup of G with the structure $3^2{:}Q_8$. We first construct the following presentation for H_9 on its generators

$$K=\langle v,w \mid v^4=1,\ v^2wv=wvw^2, wvw=vw^2v^2 \rangle.$$

Next, on setting $N=\langle vwv^{-1}w,\ vw^2v\rangle$, $M=\langle w^{-1}vw,\ vwv^{-1}\rangle$ one observes that $N \cong 3^2$, $M \cong Q_8$, $N \lhd K$ and $N \cap M=(1)$ showing that $K \cong 3^2{:}Q_8$. Now the maximality of H_9 follows from the fact that the Sylow 2-subgroup of A_6 is D_8 while that of K is Q_8.

A covering group for K may be given by

$$C=\langle v,w \mid [w,v^4]=1,\ v^2wv=wvw^2, wvw=vw^2v^2 \rangle.$$

Then $|C|=216$ showing that $M(K)=C_3$. In fact we have $M(K) \cong \langle v^4\rangle$. So H_9 having a 2-generator 3-relation presentation is efficient. Eliminating the first relation of C gives a deficiency zero presentation for a covering group of H_9.

$Sp(4,3) \cong PSU(4,2)$

order=25920=$2^6.3^4.5$ d=27 mult=2

G=<a,b | $a^2=b^4=(ab)^9=[a,b]^5= ab^{-1}ab^2ab^{-1}(ab)^2ab^2abab^2(ab^{-1})^2(ab)^4=1$>

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	25920			1	1^{27}
2A	576	A	A	b^2	$1^3 2^{12}$
2B	96	A	A	a	$1^7 2^{10}$
3A	648	A	A	$(ab)^3$	3^9
B**	648	A	A	$(b^{-1}a)^3$	3^9
3C	108	A	A	$(ababab^{-1})^2$	$1^9 3^6$
3D	54	A	A	$(ab^2)^2$	3^9
4A	48	A	A	b	$1^3 4^6$
4B	8	B	A	$ab(ab^2)^2$	$1^2 2^3 4^5$
5A	5	A	A	[a,b]	$1^2 5^5$
6A	72	BA	AA	$(b^2ab^{-1}a)^2$	$3^1 6^4$
B**	72	AA	BA	$(abab^2)^2$	$3^1 6^4$
6C	36	CA	CA	$(ab)^2ab^{-1}$	$1^3 2^3 6^3$
D**	36	CA	CA	$b(ab^{-1})^2a$	$1^3 2^3 6^3$
6E	18	DA	DA	ab^2	$3^1 6^4$
6F	12	CB	CB	$(abab^2)^2ab$	$1^1 2^4 3^2 6^2$
9A	9	A	A	ab	9^3
B**	9	B	A	$b^{-1}a$	9^3
12A	12	BA	AA	$abab^2$	$3^1 12^2$
B**	12	AA	BA	$b^2ab^{-1}a$	$3^1 12^2$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	960	27	$2^4:A_5$	$N(2^4) \cong N(2A_5B_{10})$	2x2
H_2	720	36	S_6	N(2B,3C,3D,4B,5A)	2
H_3	648	40	$3_+^{1+2}:2A_4$	N(3AB)	2
H_4	648	40	$3^3:S_4$	$N(3^3)=N(3AB_4C_3D_6)$	1
H_5	576	45	$2\cdot(A_4xA_4).2$	N(2A)	2

I.Sylow p-subgroup

(i) Sylow 2-subgroup order=64

By the information given in [34] about the structure constants for the classes of involutions, we see that G has D_8 as a subgroup. By PERM we find $x=a^b$ and $y=ab^2a$ with <x, y> $\cong D_8$. On successively extending <x, y> by 2-elements, we obtain, by TC,

$z=a^t$, where $t=bab^2$, with x, y, and z generating a Sylow 2-subgroup P_2 of G. A presentation for P_2 on x, y, and z is

$$<x,y,z \mid x^2=y^2=z^2=(xy)^4=(yz)^4=(xyzy)^2=(xyzxz)^2=1>.$$

This group is isomorphic to the 2-group $64,259(1^3,1^4) \cong \Gamma_{25}a_1$ with the presentation

$$<a_1,a_2,a_3 \mid a_1{}^2=a_2{}^2=a_3{}^2=1, [a_1,a_2]^2=[a_1,a_3]^2=[a_2,a_3]=1, [a_1,a_2,a_3]=[a_1,a_3,a_2]>,$$

by the mapping $a_1 \to y$, $a_2 \to x$, $a_3 \to xyzy$.

$n_2(G)=|G:N(P_2)|=135$.

(ii) Sylow 3-subgroup order=81

Beginning from $x=ab^2(ab)^2$ of order 9, TC gives $z=b^2(abab^{-1})^2abab^2(ab^{-1})^3abab^{-1}ab^2ab^{-1}$ in $N_G(<x>)\setminus<x>$ with $|<x,z>|=27$. Now $N_G(<x,z>)=<x, y>$, where $y=bab^2ab$, with x and y generating a Sylow 3-subgroup P_3 of G. A presentation for P_3 on x and y is given by

$$<x,y \mid y^3=(xy)^3=(x^2y)^2x^{-1}y=1>. (*)$$

It may be worth remarking that there are exactly fifteen isomorphism classes of p-groups of order p^4, $(p > 2)$, (see Burnside, Theory of groups of finite order, p 145) and that P_3 is isomorphic to the group

$$<P,Q,R \mid P^{p^2}=Q^p=1, P^Q=P^{1+p}, P^R=PQ, Q^R=Q, R^P=1>,$$

for p=3 (It is number (xi) in the list, p 145). To see this one may put $x=P$ and $y=QR^{-1}$.

It should also be emphasised that the presentation (*) for P_3 is minimal for rank $M(P_3)=1$ as we shall see now. A covering group for P_3 may be given by

$$<x,y \mid (xy)^3=y^3, (x^2y)^2x^{-1}y=[x,y^3]=1>$$

which has order 243 and so $M(P_3)=C_3$.

$n_3(G)=|G:N(P_3)|=166$.

(iii) Sylow 5-subgroup order=5

$P_5=<[a,b]>=C([a,b])$, $|G:N(P_5)|=1296$.

II. Maximal subgroups

(i) structure $2^4{:}A_5$

$H_1=< a, b^2ab(babab)^2 >$.

Putting $x=a$ and $y=b^2ab(babab)^2$, a presentation for H_1 on x and y is

$$H=<x,y \mid x^2=y^5=(xy^2)^5=(xyxy^2)^4=((xy)^2xy^{-1}xy^2)^2=1>$$

(see [7]). We now set $N=\langle (xyxy^2)^2,(xy^2xy)^2,(yxy^2x)^2,(yxyxy)^2\rangle$, $M=\langle yxy^{-1},y^x\rangle$. Then $N \lhd H$, $N\cong 2^4$, $M=A_5$, and $N\cap M=(1)$ showing that H is a semi-direct product of 2^4 by A_5.

Since $H/H'=1$, H is a perfect group of order 960. As was seen earlier, the simple group PSL(3,4) has also a (maximal) subgroup K of the same order with K being both perfect and isomorphic to a semi-direct product of 2^4 by A_5. In fact there are two isomorphism classes of perfect groups of order 960, see [40]. On computing the Schur multiplier of H we shall see that H and K are really representatives for these classes.

In order to find a somewhat simpler covering group for H to be more amenable to computation with TC we first replace the third and fifth relation of H by the single relation $xy^{-1}xy^2xy^{-1}xyxy^{-1}xy^{-2}xy^2=1$. Then a covering group for H can be given by

$$C=\langle x,y,z \mid y^5=z^2,\ (xyxy^2)^4=x^4z^5,\ xy^{-1}xy^2xy^{-1}xyxy^{-1}xy^{-2}xy^2=x^6,\ [x,z]=[y,z]=[y,x^2]=[z,x^2]=1\rangle.$$

Next $|C|=3840$ and hence $|M(H)|=4$. Now it is easy to check that $M(H)\cong\langle x^2,z\rangle\cong C_2\times C_2$. This fact together with the latter presentation for H, proves that H is efficient.

(ii) structure S_6

S_6 has presentation

$$\langle x,y \mid x^2=y^6=(xy)^5=[x,y]^3=[x,y^2]^2=1\rangle$$

(see [18]) and thus is generated by two of its elements of order 2 and 6 whose product and commutator have order 5 and 3. Starting from $x=a$,which is in 2B, PERM gives $y=bab^2(ab)^2ab^{-1}ab^2$ with x and y generating a subgroup H_2 of G and satisfying the above presentation for S_6.

We note that S_6 has three conjugacy classes of involutions with lengths 15, 45, and 15. These classes in H_2 have representatives belonging to the classes 2A, 2B, and 2B respectively. This shows that a S_6 subgroup in G has involutions both in 2A and 2B. We may use this fact to give the generating pair $(b^2, (abab^2)^2ab)$ of type (2A,6,5;3) for a S_6 subgroup conjugate to H_2.

Next $M(S_6)=C_2$, by 1.5.11 (i) and an efficient presentation for S_6 is given by

$$\langle x,y \mid x^2=(xy)^5,\ (xyxy^{-1})^3=1,(xy^2xy^{-2})^2y^6=1\rangle.$$

(iii) structure $3_+^{1+2}{:}2A_4$

This is the normalizer in G of a cyclic subgroup of G whose generator lies in 3AB. We take $r=(ab)^3$, which is in 3A, and use TC to determine $N_G(\langle r\rangle)$. We find

$N_G(r)=<s,t>$, where $s=b(ab^{-1})^2ab^2$, $t=ab^{-1}(ab)^5ab^{-1}$ and $|G:<s,t>|=40$. Since $|G:H_i| \nmid 40$ (i=1, 2), $<s,t>$ is a maximal subgroup of G of order 648. It is now easy to chck that (s,t) is of type (3B,6E,4A;4A). As elements of order 3 are not conjugate to their inverses, (s^{-1},t^{-1}) will have the type (3A,6E,4A;4A), using table I. This enables PERM to give the neater generators x=b and $y=(ab)^2a$ with (x, y) having the type (4A,6E,3A;4A) and x, y generating a maximal subgroup H_3 of order 648.

It remains to show that H_3 is a semi-direct product of 3_+^{1+2} by $2A_4$. To see this, we first construct the following presentation for H_3 on x and y :

$$K=< x, y \mid x^4=(xy)^3=(x^2y^3)^3=x^2y^2xy^{-2}xyxy^{-1}=1 >.$$

Next we put $N=<y^3x^2, y^2x^2y>$, $M=<x, yx^{-1}y>$.Then N is a normal subgroup of K, with the structure 3_+^{1+2}, which intersects M trivially. That is, $K \cong 3_+^{1+2}: M$. The subgroup M is now found to have a presentation on its generators x and $z=yxy^{-1}$ with relations $x^4=(xz^{-1})^3=(xz)^6=x^2z^{-3}=1$. We have $<x^2> \triangleleft <x,z>$ and the factor $<x,z>/<x^2>$ is isomorphic to A_4 showing that $M \cong 2A_4$. Therefore $K \cong 3_+^{1+2}: 2A_4$.

A covering group for K has presentation

$$<a_1,a_2,a_3,a_4 \mid a_1^4a_3=(a_1a_2)^3a_3=(a_1^2a_2^3)^3a_3^2=a_1^2a_2^2a_1a_2^{-2}a_1a_2a_1a_2^{-1}a_4^{-1}=$$
$$[a_1,a_3]=[a_1,a_4]=[a_2,a_3]=[a_2,a_4]=[a_3,a_4]=1>.$$

Then C has order 648 proving that M(K)=1.

Our reduction method fails here to direct the above presentation of K to a deficiency zero presentation. We therefore try to find a new 2-generator presentation for K having at most 3 relations. Using the method described in 3.5, we find that $r=y^3x$ and s=xy generate the group K and satisfy the presentation

$$<r,s \mid s^3=r^3s(r^{-1}s)^2=r^2srsr^{-1}s^{-1}r^{-1}srs=1 >.$$

Now $s^3=1$ is easily shown to be redundant and hence K has deficiency zero.

(iv) structure $3^3: S_4$

By table II, this is the normalizer in G of an elementary abelian group P ($\leq$ G) whose 13 cyclic subgroups number 4 containing both classes 3A and 3B, 3 containing 3C only , and 6 containing 3D. Reverting to the Sylow 3-subgroup P_3 given in (I) one can easily check that $P=<y, y^x, x^3>$ has this property. Substituting $ab^2(ab)^2$, bab^2ab for x , y and using TC we get $N_G(P)=<z, u>$, where $z=b^{-1}abab^2(ab)^2$ and $u=(b^2a)^2bab$. The elements z and u generate a maximal subgroup of G of order 648. The pair (uz^{-1},uzu^{-1}) being of type (6E,3D,4B;9) is clearly a generating pair for the same subgroup. Using this fact we are now able to find, by PERM, the generators $v=(b^2a)^2$ and w=bab for a maximal subgroup H_4 of G of order 648 with (v, w) of type (3D,6E,4B;9).

We must show that H_4 has structure $3^3{:}S_4$ and that it is not isomorphic to H_3. We begin with the following presentation for H_4 on its generators :

$$L=\langle v, w \mid v^3=w^6=(vw)^4=(vw^{-1})^4=(vw^2)^3=1\rangle.$$

Now take $N=\langle w^2, v^{-1}w^2v, vwvw^{-1}(vw)^2 \rangle$ and $M=\langle w^3, v \rangle$. Then $N\cong 3^3$ and $M\cong S_4$ with $N\cap M=(1)$, $N\lhd L$. So $L \cong 3^3{:}S_4$. That H_3 and H_4 are not isomorphic follows simply from the fact that H_4 has an element of order 12 while H_3 has no such element.

Finally we compute the Schur multiplier of H_4 in the following way. We first combine the second and the fourth relations of L in order to find a new presentation for L with four relations. We have

$$L=\langle v, w \mid v^3=(vw)^4=(vw^2)^3=1, (vw^{-1})^4w^6=1\rangle.$$

Then a covering group C for L is found to have a presentation on generators a_1, a_2, a_3, a_4 with relations $a_1^3a_3^{-1}a_4^2=(a_1a_2)^4a_3^{-1}a_4^2=(a_1a_2^2)^3a_4=(a_1a_2^{-1})^4a_2^6a_4^2=1$, $[a_1,a_3]=[a_1,a_4]= [a_2,a_3]=[a_2,a_4]=[a_3,a_4]=1$.

The group C has order 1296 and hence $M(L)=C_2$. In fact we have that $M(L) \cong Z(C) \cap C'=\langle a_3\rangle \cong C_2$. Now an efficient presentation for L is obtainable from the latter presentation for L as follows:

$$\langle v, w \mid v^3=(vw)^4, v^3=(vw^2)^3, (vw^{-1})^4w^6=1 \rangle.$$

(v) structure: $2\cdot(A_4xA_4).2$

By table II, this is the normalizer in G of an involution $x\in 2A$. We take $x=b^2$ and, by TC, find $N_G(b^2)=\langle r, s \rangle$, where $r=(ab^{-1})^5(ab)^3bab^{-1}ab^2(ab^{-1})^2a$,

$s=ab^2(ab^{-1})^3(ab)^2ab^2(ab^{-1})^2aba$, with $| G: \langle r, s \rangle |=45$. Since $| G : H_i | \nmid 45$ $(i=1,2,3,4)$, $\langle r, s \rangle$ is a maximal subgroup of G of order 576. Next we see that (r,s) is of type (4B,6D,3B;3D). By an argument similar to that in (iii), (r^{-1}, s^{-1}) is a generating pair of type (4B,6C,3A;3D) for a maximal subgroup of G of order 576. Using this fact PERM produces the generators $x=(ab)^3$ and $y=bab^2abab^2$ for a maximal subgroup H_4 conjugate to $\langle r,s\rangle$ with (x,y) of type (3A,4B,6C;3D). Then x and y satisfy the presentation

$$H=\langle x,y \mid x^3=y^4=(xy)^6=1, [x,y]=[y,x^{-1}] \rangle.$$

Using this presentation for H_4 we shall show that $H_4 \cong 2\cdot(A_4xA_4).2$.The element xy^2 has order 12 in H and $(xy^2)^6$ generates a cyclic normal subgroup of H of order 2. On adding $(xy^2)^6=1$ to the presentation of H we have

$$K=\langle x,y \mid x^3=y^4=(xy)^6=(xy^2)^6=1, [x,y]=[y,x^{-1}] \rangle,$$

with $| K |=288$. Then the subgroup N generated by $r=xy^2$ and $s=yxy$ has index 2 in K.

Using SUBGP, we may obtain the following symmetric presentation for N on $u=rs$, $v=rs^2$

$$\langle u,v \mid u^3=v^3=[u,v]^2=(uv)^6=1\rangle.$$

Now we let $L_1=\langle u, u^v\rangle$ and $L_2=\langle (uv)^2, (vu)^2\rangle$ and observe that L_1, L_2 are each normal in $\langle u, v\rangle$ with $L_1\cap L_2=(1)$ and $L_1 \cong L_2 \cong A_4$. Thus $N \cong A_4 x A_4$ and $K \cong 2.(A_4 x A_4)$. This proves that H ($\cong H_5$) is an extension of C_2 by $(A_4 x A_4).2$. Next SUBGPTEST enables us to show that H_5 has no subgroup isomorphic to K.Therefore $H_5 \cong 2\cdot(A_4 x A_4).2$. Meanwhile, we have found an efficient presentation for $A_4 x A_4$.

We now show that $M(H_5)=C_2$. Using the presentation for H_5 we find the following presentation for a covering group C of H_5 :

$$\langle x,y,z,u \mid x^3=y^4z=(xy)^6z=1, [x,y]=[y,x^{-1}]u, [x,z]=[x,u]=[y,z]=[y,u]=[z,u]=1\rangle.$$

Then $|C|=1152$ and we have $M(H) \cong \langle z\rangle=Z(C) \cap C'$, where $|\langle z\rangle|=2$. Now an efficient presentation for H_5 may be given by :

$$\langle x, y \mid x^3y^4=1, (xy)^6=x^3, [x,y]=x^3[y,x^{-1}]\rangle.$$

<u>Note</u>. By taking $r=ab^{-1}ab^2abab^{-1}ab$ and $s=r^{-1}(abab^2)^2ab$ we see that r and s generate a subgroup of G conjugate to H_5 with r, s satisfying the following deficiency zero presentation for a covering group of H_5

$$\langle r,s \mid r^2s=srs^3r, rsrs^{-2}r=srs\rangle.$$

SZ(8)

order=29120=2^6.5.7.13   d=65   mult=2x2

G=< a,b | $a^2=b^4=(ab)^5=[a,b]^7=(ab^2)^{13}= ab^{-1}(ab^2)^2(ab^{-1}abab^2)^2ab^2ab(ab^2)^4=1$>

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	29120			1	1^{65}
2A	64	A	A	a	$1^1 2^{32}$
4A	16	A	A	b	$1^1 4^{16}$
B**	16	A	A	b^{-1}	$1^1 4^{16}$
5A	5	A	A	ab	5^{13}
7A	7	A	A	[a,b]	$1^2 7^9$
B*2	7	A	A	$[a,b]^2$	$1^2 7^9$
C*4	7	A	A	$[a,b]^4$	$1^2 7^9$
13A	13	A	A	ab^2	13^5
B*3	13	A	A	$(ab^2)^3$	13^5
C*9	13	A	A	$(ab^2)^{-4}$	13^5

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	448	65	2^{3+3}:7	N($2A^3$)	2x2
H_2	52	560	13:4	N(13ABC)	1
H_3	20	1456	5:4	N(5A)	1
H_4	14	2080	D_{14}	N(7ABC)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup   order=64

In [7] the generators $r=b^{ab}$ and $s=(ba)^2(b^2a)^2$ are given for a subgroup H of G of minimal index 65 which, clearly, contains a Sylow 2-subgroup P_2 of G. We shall employ the method described in 3.2 to give generators for P_2 by determining generators for a Sylow 2-subgroup of H. We begin with the following presentation for H on r, s :

$$<r,s \mid r^4=s^7=(rs^{-2})^7=[r^2, srs^{-1}]=rsr^{-1}s^{-3}rs^2=1 > \qquad (\dagger)$$

(see [7]). Using TC we find the generators r, $s^{-1}rs$, and srs^{-1} for a Sylow 2-subgroup of H. Substituting b^{ab}, $(ba)^2(b^2a)^2$ for r, s and using PERMGP we obtain P_2=<x,y,z> where $x=b^{ab}$, $y=b^2abab^{-1}abab^2ab(ab^{-1})^2$ and $z=ab^{-1}abab^2(abab^{-1})^2$. By the information given in table I about the orders of centralizers of involutions we are now able to check that $P_2=C(x^2)$.

Next, a presentation for P_2 on x, y, and z may be given by

$$<x ,y, z \mid xyzxzy=xzyzx^{-1}y=xzy^{-1}x^{-1}zy=x^{-1}yzx^{-1}zy=xyz^{-1}xz^{-1}y=1 >$$

which is isomorphic to the 2-group $64,153,(1^3,1^2) \cong \Gamma_9e$ with the presentation

$<a_1, a_2, a_3 \mid a_1^2a_2^2a_3^{-2}=1, [a_2,a_1][a_2,a_3]a_1^{-2}=1, [a_3,a_1][a_2,a_3]a_2^2=1,$

$$[a_2,a_3,a_1]=[a_2,a_3,a_2]=1>.$$

To see this, we may consider the mapping $a_1 \to x$, $a_2 \to z$, $a_3 \to yz$ which yields an isomorphism between these two groups.

$n_2(G)=| G : N(P_2) |=65.$

(ii) Sylow 5-subgroup *order=5*

$P_5=< ab >=C(ab)$, $n_5(G)=| G : N(P_5) |=1456.$

(iii) Sylow 7-subgroup *order=7*

$P_7=< [a,b] >=C([a,b])$, $n_7(G)= | G : N(P_7) |=2080.$

(iv) Sylow 13-subgroup *order=13*

$P_{13}=< ab^2 >=C(ab^2)$, $n_{13}(G)=| G: N(P_{13}) |=560.$

II. Maximal subgroups

(i) structure: $2^{3+3} : 7$

$H_1=< b^{ab}, (ba)^2(b^2a)^2 >.$

Let $x=b^{ab}$ and $y=(ba)^2(b^2a)^2$. Then x and y satisfy the presentation (†) given in (I) for H_1, by the change of (r, s)= (x, y). We now proceed to show that H_1 is a semi-direct product of 2^{3+3} by C_7. By the foregoing discussion x, $t=y^{-1}xy$ and $q=yxy^{-1}$ form generators for a Sylow 2-subgroup N of H which is, in fact, normal and has trivial intersection with <y>. This shows that $H \cong N : 7$. Next $< x^2, t^2, q^2 >$ is a nomal subgroup of N with the structure 2^3 and we have $N/< x^2, t^2, q^2 > \cong 2^3$. Therefore $H \cong 2^{3+3} : 7$.

A covering group C for H_1 is found to have presentation on generators a_1, a_2, a_3, a_4, a_5 with relations $a_1^4a_3^{-1}a_5^4=a_2^7=(a_1a_2^{-2})^7a_3^{-2}a_5^7=[a_1^2, a_2a_1a_2^{-1}]a_4^{-1}=1$, $a_1a_2a_1^{-1}a_2^{-3}a_1a_2^2=1$, $[a_i, a_j]=1$ $(1 \leq i \leq 5, 3 \leq j \leq 5, i < j)$.

Then $| C |=1792$ and $M(H) \cong Z(C) \cap C'=< a_4, a_5 > \cong C_2xC_2$. Now by combining the first and second relations of H we find the following deficiency -2 presentation for H_1.

$$<x,y \mid x^4y^7=(xy^{-2})^7=[x^2, yxy^{-1}]=xy^2xyx^{-1}y^{-3}=1 >.$$

(ii) structure: $13:4$

By table II, this is the normalizer in G of a cyclic subgroup of G whose generator lies in 13ABC. We take $z=ab^2$ in 13A and find, by TC, $N_G(<z>)=<z,u>$ where $u=b^{-1}(ab^2)^2ab^{-1}ab(ab^{-1})^2ab(ab^2ab^{-1})^2$. Then the pair (z,u) of type (13,4,4) is a generating pair for a maximal subgroup of G of order 52. It is now easy to check that u^2 and zu generate $< z, u >$ and that (u^2, zu) is of type (2,4B,4A;13). This helps us to give, by PERM, the generators $x=a^b$ and $y=a[b^2, aba]$ for a maximal subgroup H_2 of G of order 52. The generators x and y satisfy the deficiency zero presentation

$$<x, y \mid x^2=xy^2[x, y]^2=1 >.$$

This is a split metacyclic group of structure $13:4$. In fact $<x,y> \cong <[x,y]>:<y>$.

(iii) structure: $5:4$

By a similar method to that of in (ii), we first find $N_G(<ab>)=<ab,ab^{-1}(ab^2)^2(ab^2abab^{-1}ab)^2b>$ which is a maximal subgroup of G of order 20. Then on setting $r=ab$, $s= ab^{-1}(ab^2)^2(ab^2abab^{-1}ab)^2b$, we observe that s^2 and rs generate $<r,s>$ and that (s^2, rs) is of type (2,4B,4A;5). Using this fact, PERM gives the generators $x=(b^2)^{aba}$ and $y= b^{abb}$ with (x,y) of type (2,4A,4B;5). Then $H_3 =<x,y>$ is a maximal subgroup of G of order 20 with the presentation

$$K=< x, y \mid x^2=xy^2xyxy^{-1}=1>.$$

K is a split metacyclic group of structure $5:4$ for $K \cong < [x,y] > : <y>$ in which the commutator [x,y] has order 5 in K.

(iv) structure: D_{14}

We simply find , by PERM, the subgroup generators $x=a$ and $y=bab^{-1}$ for a dihedral subgroup of G of order 14. We shall now show that $H_4= < x, y >$ is maximal in G by proving that H_1 has no D_{14} subgroups. Using the presentation (†), we see that $|H'|=64$. On the other hand D_{14} has C_7 as its derived group proving that D_{14} is not embeddable in H_1, by 1.1.4 (i).

PSL(2,32) ≅ SL(2,32)

order=32736=2^5.3.11.31 d=33 mult=1

$G=\langle a,b \mid a^2=b^3=(ab)^{31}=((ab)^3(ab^{-1})^7)^2=1\rangle$

a=/2,1,16,17,18,19,20,21,22,23,24,25,26,27,28,3,4,5,6,7,8,9,10,11,12,13,14,15,33,32,31,30,29/,

b=/3,1,2,14,24,8,33,23,13,5,20,17,31,27,26,15,22,11,29,18,30,12,6,10,21,16,4,19,28,25,9,7,32/

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	32736			1	1^{33}
2A	32	A	A	a	$1^1 2^{16}$
3A	33	A	A	b	3^{11}
11A	33	A	A	$((ab)^2ab^{-1})^2$	11^3
B*2	33	A	A	$((ab)^2ab^{-1})^4$	11^3
C*4	33	A	A	$((ab)^2ab^{-1})^{-3}$	11^3
D*3	33	A	A	$((ab)^2ab^{-1})^{-5}$	11^3
E*5	33	A	A	$((ab)^2ab^{-1})^{-1}$	11^3
31A	31	A	A	ab	$1^2 31^1$
B*2	31	A	A	$(ab)^2$	$1^2 31^1$
C*4	31	A	A	$(ab)^4$	$1^2 31^1$
D*8	31	A	A	$(ab)^8$	$1^2 31^1$
E*15	31	A	A	$(ab)^{15}$	$1^2 31^1$
F*5	31	A	A	$(ab)^5$	$1^2 31^1$
G*10	31	A	A	$(ab)^{10}$	$1^2 31^1$
H*11	31	A	A	$(ab)^{11}$	$1^2 31^1$
I*9	31	A	A	$(ab)^9$	$1^2 31^1$
J*13	31	A	A	$(ab)^{13}$	$1^2 31^1$
K*6	31	A	A	$(ab)^6$	$1^2 31^1$
L*12	31	A	A	$(ab)^{12}$	$1^2 31^1$
M*7	31	A	A	$(ab)^7$	$1^2 31^1$
N*14	31	A	A	$(ab)^{14}$	$1^2 31^1$
O*3	31	A	A	$(ab)^3$	$1^2 31^1$
33A	33	DA	AA	$(ab)^4ab^{-1}$	33^1
B*2	33	EA	BA	$(ab)^6ab^{-1}$	33^1
C*4	33	AA	CA	$(ab)^4(ab^{-1})^2$	33^1
D*8	33	BA	DA	$((ab)^4(ab^{-1})^2)^2$	33^1
E*16	33	CA	EA	$(ab)^5(ab^{-1})^3$	33^1
F*10	33	DA	AA	$((ab)^6ab^{-1})^5$	33^1
G*13	33	EA	BA	$(ab)^9(ab^{-1})^4$	33^1
H*7	33	AA	CA	$((ab)^4ab^{-1})^7$	33^1
I*14	33	BA	DA	$(ab)^5(ab^{-1})^4$	33^1
J*5	33	CA	EA	$(ab)^3(ab^{-1})^3$	33^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	992	33	$2^5{:}31$	$N(2A^5)$	1
H_2	66	496	D_{66}	N(3A),N(11A-F)	1
H_3	62	528	D_{62}	N(31A-O)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup *order=32*

In example 2 of 3.2 we gave the generators $x=a^b$, $y=ab^{-1}(ab)^7(ab^{-1})^2$, $z=(ab^{-1})^3(ab)^3(ab^{-1})^4$, $u=(ab^{-1})^3(abab^{-1})^2(ab)^3a$, $v=(ab^{-1}ab)^2(ab)^3(abab^{-1})^2ab^{-1}$ for a Sylow 2-subgroup P_2 of G which is an elementary abelian group of order 32. By information given in table I about the order of centralizer of involutions we can now check that $C(x)=P_2$.

$n_2(G)=|G:N(P_2)|=33$.

(ii) Sylow 3-subgroup *order=3*

$P_3=\langle b\rangle$, $n_3(G)=|G:N(P_3)|=496$.

(iii) Sylow 11-subgroup *order=11*

$P_{11}=\langle (ab)^2abab^{-1}\rangle$, $n_{11}(G)=|G:N(P_{11})|=496$.

(iv) Sylow 31-subgroup *order=31*

$P_{31}=\langle ab\rangle=C(ab)$, $n_{31}(G)=|G:N(P_{31})|=528$.

II.Maximal subgroups

(i) structure: $2^5{:}31$

By PERM we find the generators $x=a^{ba}$ and $y=(bab)^{ab}$ for the stabilizer of a point in the permutation representation of G. The elements x, y generate a subgroup H_1 of index 33 and satisfy the presentation

$$\langle x, y \mid x^2=y^{31}=1,\ xy^2(xy)^3=y^5x\rangle.$$

Let $N=\langle [x,y],[x,y^{-1}],[x,y^2],[x,y^{-2}],[x,y^3]\rangle$. Then N is a normal subgroup of $\langle x,y\rangle$ isomorphic to 2^5 and thus $\langle x,y\rangle$ is an split extension of N by $\langle y\rangle$, that is H_1 has structure $2^5{:}31$. Now substitution of a^{ba}, $(bab)^{ab}$ for x,y into the generators of N gives a subgroup of G of the structure 2^5 whose normalizer in G is exactly H_1 as our second table indicates. By combining the relations of the above presentation we obtain a deficiency zero presentation for the soluble group H_1

$$\langle x,y \mid x^2y^{31}=1,\ y^2(xy)^3=xy^5x\rangle.$$

(ii) structure: D_{66}

By PERM we find the generators $x=a$, $y=a^t$, where $t=(ba)^2b$, for a maximal subgroup H_2 of G isomorphic to D_{66}.

(iii) structure: D_{62}

Similarly we find $x=a$, $y=a^b$ for a subgroup H_3 of G isomorphic to D_{62}. To prove H_3 maximal we observe that $|H_3'|=31$, $|H_1'|=32$, that is H_3 is not embeddable in H_1.

PSU(3,4)

order=62400=$2^6.3.5^2.13$ d=65 mult=1

$G=\langle a,b \mid a^2=b^3=(ab)^{13}= ab(ab^{-1})^2(ab)^2ab^{-1}ab(ab^{-1})^4abab^{-1}(ab)^4ab^{-1}=1\rangle$

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	62400			1	1^{65}
2A	320	A	A	a	$1^1 2^{32}$
3A	15	A	A	b	$1^2 3^{21}$
4A	16	A	A	$(ab)^4(ab^{-1})^2$	$1^1 4^{16}$
5A	300	A	A	$((ab)^2ab^{-1})^6$	$1^5 5^{12}$
B**	300	A	A	$((ab)^2ab^{-1})^4$	$1^5 5^{12}$
C*2	300	A	A	$((ab)^2ab^{-1})^2$	$1^5 5^{12}$
D*3	300	A	A	$(b(ab^{-1})^2a)^2$	$1^5 5^{12}$
5E	25	A	A	[a,b]	5^{13}
F*	25	A	A	$[a,b]^2$	5^{13}
10A	20	CA	AA	$(ab)^2ab^{-1}$	$1^1 2^2 10^6$
B**	20	DA	BA	$b(ab^{-1})^2a$	$1^1 2^2 10^6$
C*7	20	BA	CA	$((ab)^2ab^{-1})^{-3}$	$1^1 2^2 10^6$
D*3	20	AA	DA	$((ab)^2ab^{-1})^3$	$1^1 2^2 10^6$
13A	13	A	A	ab	13^5
B**	13	A	A	$b^{-1}a$	13^5
C*5	13	A	A	$(ab)^5$	13^5
D*8	13	A	A	$(b^{-1}a)^5$	13^5
15A	15	DA	AA	$(ab)^3(ab^{-1})^2$	$1^2 3^1 15^4$
B**	15	CA	BA	$(ba)^2(b^{-1}a)^3$	$1^2 3^1 15^4$
C*2	15	AA	CA	$((ab)^3(ab^{-1})^2)^2$	$1^2 3^1 15^4$
D*8	15	BA	DA	$((ab)^3(ab^{-1})^2)^8$	$1^2 3^1 15^4$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	960	65	$2^{2+4}:15$	$N(2A^2)$	1
H_2	300	208	$5xA_5$	N(5ABCD)	2
H_3	150	416	$5^2:S_3$	$N(5^2)$	1
H_4	39	1600	13:3	N(13ABCD)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=64

We shall use our method described in 3.2 to give generators for a Sylow 2-subgroup of G by determining generators for a Sylow 2-subgroup of $\langle b,(ab(ab^{-1})^2)^2ab^{-1}aba\rangle$ which is a subgroup of G of minimal index 65. We set

$$r=b,\ s=(ab(ab^{-1})^2)^2ab^{-1}aba. \qquad (*)$$

Then r and s satisfy the presentation

$$H = \langle r, s \mid (rs)^2 s (r^{-1}s)^2 = rsr^2 s^{-1} r s^{-4} r^{-1} s^{-1} = 1 \rangle \qquad (\dagger)$$

as stated in [7]. Using TC, we may arrive at the generators

$$[r, s], [s, r^{-1}], [r^{-1}, s^{-1}], [r, s^{-1}] \qquad (**)$$

for a Sylow 2-subgroup of H. Substituting (*) into (**) and using PERMGP, we find the generators $x=(ab^{-1})^2(ab)^4$, $y=b(ab^{-1})^2(ab)^3a$, $z=((ab)^2ab^{-1})^4ab^{-1}ab$, $u=b((ab)^2ab^{-1})^4ab^{-1}a$ for a Sylow 2-subgroup P_2 of G.

A presentation for P_2 on these generators may be given by

$$\langle x, y, z, u \mid [x,y]=[z,u]= xzx^{-1}z=xuxu^{-1}=yuy^{-1}u=1, y^2=z^2, x^2=(yz)^2, x^2=y^2u^2 \rangle.$$

This group is isomorphic to the 2-group $64,187,(1^4,1^4) \cong \Gamma_{13}a_5$ with presentation

$$\langle a_1, a_2, a_3, a_4 \mid a_1^2a_2^2a_4^{-2}=a_2^2a_3^{-2}=[a_3,a_1]a_1^2=[a_1,a_4]a_2^{-2}a_1^2=[a_2,a_3]a_2^2a_1^{-2}= [a_4,a_2]a_2^2=[a_1,a_2]=[a_3,a_4]=1\rangle.$$

This is immediate on setting $a_1=u$, $a_2=z$, $a_3=y$, and $a_4=x$.

$n_2(G)=|G : N(P_2)|=65.$

(ii) Sylow 3-subgroup *order=3*

$P_3=\langle b \rangle$, $n_3(G)=|G : N(P_3)|=2080.$

(iii) Sylow 5-subgroup *order=25*

Since G has no element of order 25, the Sylow 5-subgroup of G is $C_5 x C_5$. Now PERM can be used to give generators x and y with (x, y) of type (5,5,5;1). We find $x=(ab^{-1}abab)^2$ and $y=(bab^{-1}aba)^2$ for a Sylow 5-subgroup P_5 of G.

$n_5(G)=|G : N(P_5)|=416.$

(iv) Sylow 13-subgroup *order=13*

$P_{13}=\langle ab \rangle$, $n_{13}(G)=|G : N(P_{13})|=1600.$

II.Maximal subgroups

(i) structure: $2^{2+4} : 15$

$H_1= \langle b, (ab(ab^{-1})^2)^2ab^{-1}aba \rangle.$

Setting $x= b$, $y=(ab(ab^{-1})^2)^2ab^{-1}aba$, we have the deficiency zero presentation ($\dagger$) for H_1 by the change of (r, s)=(x,y). We take $c=[x,y]$, $d=[y,x^{-1}]$, $e=[x^{-1},y^{-1}]$, and $f=[x,y^{-1}]$ and observe that $N=\langle c, d, e, f \rangle$ is a normal subgroup of $\langle x, y, z, u \rangle$ with presentation

$$\langle c, d, e, f \mid c^4=c^2f^{-2}=[c,d]=[e,f]=cec^{-1}e=dfdf^{-1}=d^2e^{-2}=1, c^2=(de)^2, cfcf^{-1}=d^2 \rangle.$$

Now it is easy to verify that $\langle c^2, d^2 \rangle \lhd N$ and $N/\langle c^2,d^2\rangle \cong 2^4$. Therefore $N \cong 2^{2+4}$

for $<c^2,d^2> \cong 2^2$. Finally on taking $M=<xy>$ we see that $H_1 \cong N : M \cong 2^{2+4} : 15$.

$M(H_1)=1$, by 1.5.8 (ii).

(ii) structure: $5xA_5$

This is the normalizer in G of an A_5 subgroup. By information given in [34] about the structure constants for the class 2A, # (2,3,5ABCD)=0 and hence a generating pair for an A_5 subgroup in G has type (2,3,5EF). We find, by PERM, x=a and $z=b(ab^{-1})^2(ab)^3ab^{-1}$ with $<x, z> \cong A_5$. Next, TC finds $N_G(<x, z>)= < x, y >$, where $y=bab^{-1}ab(ab^{-1})^2(ab)^2$, with $| G : <x, y > |=208$. This shows that x and y generate a maximal subgroup H_2 of G of order 300. Here (x,y) is of type (2,5D,5E;3) and the 5-elements in < x, y > are both in 5ABCD and 5EF.

A presentation for H_2 on x and y is

$$K=< x, y \mid x^2=y^5=(xy)^5=[x,y]^3=([x,y][x,y^{-2}])^2=1 >.$$

It is now easy to check that $K=NxM$ where $N=<[x,y], [x,y^{-1}]> \cong A_5$ and $M=<(xy^2)^3> \cong C_5$.

By 1.5.12, $M(5xA_5)=C_2$; and an efficient presentation for H_2 is obtainable from the above presentation for H_2 as follows :

$$< x, y \mid x^2y^5=(xy)^5=xyx^{-1}y^{-1}xy^3x^{-1}y^{-2}xy^{-1}x^{-1}yxy^2x^{-1}y^2=1>.$$

(iii) structure $5^2 : S_3$

This is the normalizer in G of a Sylow 5-subgroup of G, by 3.3.2. Returning to the Sylow 5-subgroup P_5 of G , we find, by TC, $N_G(P_5)=< b, t >$ where $t=a^{ba}$ with $| N(P_5) |=150$. Conjugation by a of b and t gives the generators $x=a^b$ and $y=b^a$ for a subgroup H_3 of G of order 150. In fact H_3 is maximal in G as we shall see now.To prove H_3 maximal we just need to show that H_3 of structure $5^2:S_3$ cannot be isomorphic to any subgroup of H_2. First, we construct the following presentation for H_3 on x and y:

$$L=< x, y \mid x^2=y^3=(xy)^{10}=[x,y]^3=1>.$$

Then $L \cong <(xy)^2,(yx)^2>:<x,[x,y]> \cong 5^2:S_3$ and we have $| L' |=75$. On the other hand, referring to (ii), $| K' |=60$. Thus L cannot be embeddable in K, by 1.1.4 (i).

In order to compute $M(H_3)$, we first consider the following presentation for a covering group C of H_3 obtained from the above presentation :

$< a_1, a_2, a_3, a_4 \mid a_1{}^2=a_2{}^3a_4=(a_1a_2a_1a_2{}^{-1})^3a_3{}^{-1}=(a_1a_2)^{10}a_4{}^3=[a_1,a_3]=[a_1,a_4]=[a_2,a_3]=$

$[a_2,a_4]=[a_3,a_4]=1 >.$

Then $\mid C \mid=150$ showing that $M(H_3)=1$. A deficiency zero presentation for H_3 may now be presented as :

$$< x, y \mid x^2y^3= (x^{-1}y)^6xy^{-2}(xy^{-1})^2xy^{-2}xy^{-1}xy^2 =1>.$$

(iv) structure: *13 : 3*

This is the normalizer in G of a cyclic subgroup of G whose generator lies in 13ABCD, by table II. Using TC, we find $N_G(<ab>)=< ab, u >$, where $u=b(ab^{-1})^3((ab)^2ab^{-1})^3(ab)^3ab^{-1}$, with $\mid<ab, u >\mid=39$. Setting z=ab, we observe that (u,zu) is of type (3,3,3;13). This allows PERM to look for a generating pair (x,y) of this type for a maximal subgroup H_4 of G of order 39. We obtain $x=b^r$ and $y=b^s$ where $r=abab^{-1}$ and $s=ab^{-1}aba$. Putting v=[x,y] gives the following deficiency zero presentation for the metacyclic group H_4 of order 39 :

$$< x, v \mid x^3=v^{13}, [x, v^{-5}]=v >.$$

M_{12}

order=95040=$2^6.3^3.5.11$ d=12 mult=2

$G=\langle a,b \mid a^2=b^3=(ab)^{10}=[a,b]^6=((ab)^4ab^{-1}abab^{-1})^3=1\rangle$

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	95040			1	1^{12}
2A	240	A	A	$(ab)^5 \sim [a,b]^3$	2^6
2B	192	A	A	a	1^42^4
3A	54	A	A	$[a,b]^2$	1^33^3
3B	36	A	A	b	3^4
4A	32	A	A	$((ab^{-1})^2(ab)^3ab^{-1}ab)^2$	2^24^2
4B	32	A	A	$((ab)^3(ab^{-1})^2abab^{-1})^2$	1^44^2
5A	10	A	A	$(ab)^2$	1^25^2
6A	12	BA	BA	$(ab)^4(ab^{-1})^2$	6^2
6B	6	AB	AB	[a,b]	$1^12^13^16^1$
8A	8	A	A	$(ab^{-1})^2(ab)^3ab^{-1}ab$	4^18^1
8B	8	B	A	$(ab)^3(ab^{-1})^2abab^{-1}$	$1^22^18^1$
10A	10	AA	AA	ab	2^110^1
11A	11	A	A	$(ab)^2ab^{-1}$	1^111^1
B**	11	A	A	$ba(b^{-1}a)^2$	1^111^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	7920	12	M_{11}		1
H_2	7920	12	M11		1
H_3	1440	66	$M_{10}:2 \cong A_6 \cdot 2^2$	N(2B,3A,3A,4B,5A)	2
H_4	1440	66	$M_{10}:2 \cong A_6 \cdot 2^2$	N(2B,3A,3A,4A,5A)	2
H_5	660	144	PSL(2,11)	N(2A,3B,5A,6A,11AB)	2
H_6	432	220	$M_9:S_3 \cong 3^2:2S_4$	N($3A^2$)	1
H_7	432	220	$M_9:S_3 \cong 3^2:2S_4$	N($3A^2$)	1
H_8	240	396	$2xS_5$	N(2A),N(2B,3B,5A)	2x2
H_9	192	495	$M_8.S_4 \cong 2_+^{1+4}.S_3$	N(2B)	2x2
H_{10}	192	495	$4^2:D_{12}$	N($2B^2$)	2x2
H_{11}	72	1320	A_4xS_3	N($2A^2$),N(3B)	2

I. Sylow p-subgroups

(i) *Sylow 2-subgroup* *order=64*

Since M_{11} is embeddable in M_{12}, G has a subgroup isomorphic to $\langle 2,4 \mid 2\rangle$ with a generating pair of type (8,4,2;4). Starting with $x=(ab)^3ab^{-1}ab(ab^{-1})^2$ of order 8,

PERM finds $y=(bab^{-1}a)^2b^{-1}ab(abab^{-1})^2$ of order 2 with $\langle x,y\rangle \cong \langle -2,4 \mid 2\rangle$. A Sylow 2-subgroup of G is now constructed by successively extending $\langle x,y\rangle$ by 2-elements. Applying our method 2.3 (vi), TC finds
$u=(ba)^2b^{-1}(ab)^2(ab^{-1})^5ab(abab^{-1})^2ab(abab^{-1})^2(ab)^2$ with $|\langle x,y,u\rangle|=64$. PERMGP now gives $z=(ab^{-1})^2ab(ab^{-1})^4(ab)^3a$ with x, y, and z generating a Sylow 2-subgroup P_2 of G.

A presentation for P_2 on x , y , and z is

$$\langle x, y, z \mid y^2=z^2=(xz)^4=(xyz)^2=(xzy)^2=(xzyz)^2=1\rangle.$$

This group is isomorphic to the 2-group $64,261,(1^3,1^3) \cong \Gamma_{13}a_5$ with the presentation

$$\langle a_1, a_2, a_3 \mid a_1^4=a_2^2=a_3^2=[a_1,a_2]a_1^2=[a_2,a_3]=[a_1,a_3,a_1]=1\rangle,$$

by the mapping $a_1 \to xy$, $a_2 \to z$, $a_3 \to xzy$.

$n_2(G)=|G: N(P_2)|=1485$.

(ii) Sylow 3-subgroup order=27

The information about the conjugacy classes of G shows that the Sylow 3-subgroup of M_{12} is a 3-group of exponent 3. So it is an extra-special group isomorphic to 3_+^{1+2}. Using this fact, we look for a generating pair (x,y) of type (3,3,3;3). PERM gives $x=b^t$, $y=b^{ta}$ where $t=(ab)^2$ .Then x and y generate a subgroup P_3 of order 27 and x, y satisfy the presentation $\langle x,y \mid x^3=y^3=(xy)^3=(xy^{-1})^3=1\rangle$.

$n_3(G)=|G:N(P_3)|=880$.

(iii) Sylow 5-subgroup order=5

$P_5=\langle (ab)^2\rangle$, $n_5(G)=|G:N(P_5)|=2376$.

(iv) Sylow 11-subgroup order=11

$P_{11}=\langle (ab)^2ab^{-1}\rangle=C((ab)^2ab^{-1})$, $n_{11}(G)=|G:N(P_{11})|=1728$.

II.Maximal subgroups

(i)-(ii) structure : M_{11}

$H_1=\langle a, ab^{-1}(ab^{-1}(ab)^3)^2ab\rangle$ in 7.1 of [16].

Setting $x_1=a$, $y_1=ab^{-1}(ab^{-1}(ab)^3)^2ab$ we observe that (x_1, y_1) is a generating pair of type (2,4,11) for an M_{11} subgroup with the property that x_1, y_1 fix one, and only one, point in the permutation representation of G.

We now seek a generating pair (x_2,y_2) of the same type with y_2 moving the points which are left fixed by x_2. Start with $x_2=a$, which fixes 4 points, PERM finds

$y_2=b((ab)^3ab^{-1})^2ab^{-1}a$. TC verifies that $H_2=<x_2, y_2>$ has index 12 in G and that x_2, y_2 satisfy 7.2 of [16].

H_1 and H_2 are not conjugate in G. We note that M_{11} has a single conjugacy class of elements of order 4 and that $y_1 \in 4B$, $y_2 \in 4A$. This specifies each of the maximal subgroups H_1, H_2 in G. Finally, $H_1 \cap H_2$ is a simple group of order 660 proving that the intersection of these maximal subgrops in G is isomorphic to PSL(2,11).

(iii)-(iv) structure : $M_{10}:2 \cong A_6 \cdot 2^2$

G has two conjugacy classes of maximal subgroups with representatives H_3 and H_4 of the structure $M_{10}:2$ ($\cong A_6 \cdot 2^2$) both being the normalizer in G of an M_{10} (A_6) subgroup, by 3.3.2. In fact G has two conjugacy classes of M_{10} (A_6) subgroups with representatives K_1 , K_2 and we have $H_3 \cong N_G(K_1)$, $H_4 \cong N_G(K_2)$.

We begin with H_3 which contains elements in the class 4B of G, by table II. A generating pair (x,y) for K_1 is of type (2,4,8) where x and y fix two, and only two, points in the permutation representation of G. Clearly x must be chosen from 2B, for elements belonging to 2A fix no points. So we take x=a which fixes 4 points. PERM gives $y=ab^{-1}(ab)^2ab^{-1}ab(ab^{-1})^3(ab)^4$ with $y \in 4B$ and $xy \in 8B$. TC verifies that $|G:<x,y>|=132$ showing that $|<x,y>|=720$. Now x and y satisfy the presentation for M_{10} given earlier. Next we find, by TC, that $N_G(<x,y>)=<y,z>$, where $z=b(ab)^4$ with $|G:<y,z>|=66$. So <y,z> is a maximal subgroup of M_{12}, for $12 \nmid 66$.

It is easy to check that (yz^5,z) is a generating pair for <y, z> of type (2B,10,8B;4B). This helps to obtain neater generators x_3, y_3 for a maximal subgroup H_3 of G conjugate to <y, z> with x_3 being an involution. By PERM we obtain $x_3=a^{ba}$, $y_3=(ba)^2(b^{-1}a)^2b$ of the above type with x_3, y_3 satisfying the presentation :

$$H=<x,y \mid x^2=y^{10}=(xyxy^5)^2=xyxy^2xy^{-2}xy^3(xy^4)^2=1>.$$

Taking N=<r, s>, where $r=xyxy^5$ and $s=(xy)^2$, we see that $N \triangleleft H$ and $N \cong A_6$ with $H/N=<x,y \mid x^2=y^2=(xy)^2=1> \cong C_2xC_2$. This shows that H is an extension of A_6 by C_2xC_2.

Substituting x_3, y_3 for x, y in r, s we observe that $N_G(< r, s >)=H_3$. It also easy to verify that (r, s) is a generating pair for an A_6 subgroup of type (2B,4B,5A/5A).

Similarly, generators for H_4 may be given by $x_4=a^t$, where $t=b^{-1}a$, and

$y_4=b(ab^{-1})^2(ab)^2$. Here (x_4,y_4) is of type (2B,10,8A;4A) with x_4, y_4 satisfying the above presentation for H_3. We note that $H_4=N_G(K_2)$, where K_2 is an M_{10} (A_6) subgroup with a generating pair (x,y) of type (2B,4A,8A) ((2B,4A,5A/5A)). (Notice that $|fix(x)\cap fix(y)|=0$.)

Finally, we see that H_3 is an intransitive permutation subgroup of G acting on 12 objects with orbit lengths 2, 10 while H_4 is a transitive subgroup of G proving that H_3 and H_4 are not conjugate in G.

It is worth noting that the maximal subgroups of G of the above structure have exactly two conjugacy classes of elements of order 8 and that these elements in H_3 fall into the class 8B of G while those of H_4 fall into 8A. This also proves that H_3 and H_4 are not conjugate in G.

The intersection of H_3 and H_4 is a group of order 40 with the structure 2x(5 : 4).

We now determine the multiplier of H. We first see, by TC, that

$$H \cong \langle x,y \mid x^2=xyxy^2xy^{-2}xy^3(xy^4)^2=1,(xyxy^5)^2=y^{10} \rangle.$$

Then a covering group C for H is

$$\langle x,y \mid x^2=xyxy^5xyxy^{-5}=[x,yxy^2xy^{-2}xy^3(xy^4)^2]=[y,xyxy^2xy^{-2}xy^3xy^4x]=1\rangle.$$

TC verifies that $|C|=2880$ and hence $M(H)=C_2$. So H is efficient .

(v) structure : PSL(2,11)

A generating pair (x,y) for a PSL(2,11) is of type (2,3,11). The specification of a maximal subgroup of G of the structure PSL(2,11) indicates that $x \in$ 2A and $y \in$ 3B (notice that PSL(2,11) has a single conjugacy class of elements of each of orders 2 and 3). We take $x=(ab)^5$ and by PERM find $y=ab^{-1}(ab)^2$ with y of order 11, $xy \in$ 3B, and $|\langle x,y\rangle|=660$. Then x, xy satisfy the presentation for PSL(2,11).

Now $H_5=\langle x, y \rangle$ is a maximal subgroup of G, for H_5 has a single conjugacy class of involutions with a representative belonging to the class 2A of G while involutions in each of H_1 and H_2 fall into the class 2B. This shows that H_5 cannot be a subgroup of any conjugate in G of $H_1$and H_2.

We note that PSL(2,11) subgroups in G with generating pairs of type (2B,3,11) are not maximal in G.

(vi)-(vii) structure : $M_9{:}S_3 \cong 3^2{:}\,2S_4$

G has two non-conjugate maximal subgroups with representatives H_6 and H_7 of the structure $M_9{:}S_3$. In fact $H_6=N_G(L_1)$ and $H_7=N_G(L_2)$ where L_i is a subgroup of K_i

isomorphic to M_9, for i=1,2.

As we mentioned earlier M_9 is the stabilizer of two points in M_{11} (as a 4-transitive permutation group acting on 11 objects) of order 72 with a generating pair of type (4,4,4;2). We shall take L_1 to be the stabilizer of three points in the permutation representation of G. Starting from $x=((ab)^3(ab-1)^2abab^{-1})^2$ of order 4 which fixes 4 points (see the cycle type of x as a representative for the class 4B), PERM gives $y=((ab)^3(ab^{-1})^2)^2b^{-1}$ of order 4 with (x,y) of the above type and $|\ fix(x) \cap fix(y)\ |=3$. Now y, x generate a subgroup L_1 ($\cong M_9$) of order 72 and satisfy the presentation given for M_9 (see M_{11}).

Now by TC, we obtain $N_G(< x, y >)=<y, z, u>$, where $z=a^{bab}$, $u=b^{-1}(ab)^4$ with $|G:< y, z, u >|=220$. It is then easy to check that (z,uy^{-1}) is a generating pair for $<y,z,u>$ of the type (2B,3B,8B;6B). Using this fact PERM produces the generators $x_6=a^r$ and $y_6=b^s$, where $r=a^b$ and $s=(b^{-1})^a$, for a subgroup H_6 of G of order 432. H_6 is maximal in G, for $|\ G:H_i\ | \nmid 220$, for i=1,2,3,4,5.

H_6 is an intransitive subgroup of the permutation group G with orbit lengths 3, 9. And x_6, y_6 satisfy the presentation given for the maximal subgroup H_1 of PSL(3,3).

Similarly, PERM gives the generators $x_7=a^t$ and $y_7=b^q$, where $t=bab^{-1}$ and $q=b^a$ for a subgroup H_7 of order 432 with (x_7, y_7) of the type (2B,3B,8A;6B). Here H_7 is a transitive subgroup of G of degree 12 and hence cannot be conjugate to H_6.

We note that $r=(x_7y_7)^2$, $s=(y_7x_7)^2$ generate a normal subgroup L_2 of H_7 which is isomorphic to M_9 and that $N_G(L_2)=H_7$. Here (r, s) is of the type (4A,4A,4A) and so $|fix(x) \cap fix(y)|=0$. ($<r, s>$ is, in fact, a subgroup of $K_2{}^g$, for some g in G.) Furthermore, H_6 has intersection S_3 with H_7 in G.

(viii) structure: $2xS_5$

A presentation for $2xS_5$ is given by

$$<R,S,T \mid R^2=S^6=T^5=(TS)^2=(T^2S^2)^2=[R,S]=[R,T]=1>.$$

It is easy to see that $<R,S,T>$ is generated by S^3, RT with (S^3, RT) of type (2,10,4;3) .

In order to give generators for a maximal subgroup of G of the structure $2xS_5$ we may use PERM to look for a generating pair of the above type with $[x, y] \in 3B$. Such a generating pair for $2xS_5$ subgroups (if any), certainly, generate a maximal subgroup of G of this structure, for none of H_1, H_2, H_3, and H_4 have elements of order 3 belonging to the class 3A (each of them has only one conjugacy class of order 3). By PERM we obtain $x=(ab)^5$, $y=(bab)^u$, where $u=abab^{-1}$, with (x,y) of type (2A,10,4B;3B) and $|\langle x,y\rangle|=240$. Then x, y satisfy the presentation

$$\langle x,y \mid x^2=y^{10}=(xy)^4=[x,y]^3=[x,y^5]=1\rangle.$$

Taking $N=\langle x, y^2\rangle$ and $M=\langle y^5\rangle$ we see that N and M are normal in $\langle x, y\rangle$ and $N\cap M=(1)$. It is also seen that $N \cong S_5$, and so $\langle x, y\rangle \cong 2xS_5$.

We note that $2xS_5$ is not embeddable in M_{11}, for $2xS_5$ has an element of order 10 while M_{11} does not have such an element and also SUBGPTEST shows that $2xS_5$ is not isomorphic to any subgroup of H, where H is a group of the structure M_{10}: 2 discussed in (iii)-(iv).

By 1.5.12, $M(2xS_5)=C_2xC_2$ and an efficient presentation for $2xS_5$ can be obtained from the above presenatation as follows

$$\langle x,y \mid x^2y^{10}=(xy)^4=[x,y]^3=xy^5xy^{-5}=1\rangle.$$

(ix) structure: $M_8.S_4 \cong 2_+^{1+4}.S_3$

This is the normalizer in G of an M_8 subgroup. An M_8 ($\cong Q_8$) subgroup of G is the stabilizer of 4 points in the permutation representation of G. PERM finds the generators $z=((ab)^3(ab^{-1})^2abab^{-1})^2$, $u=(ab)^3(ab^{-1})^2(ab)^3(ab^{-1}ab)^2a$, for an M_8 subgroup of G .

Using TC we find that $N_G(\langle z, u\rangle)=\langle v, w\rangle$, where $v=(ab)^3(ab^{-1})^2abab^{-1}$ and $w=bab^{-1}ab(ab^{-1})^2ab(ab^{-1})^4(ab)^2$ with $|\langle v, w\rangle|=192$. It is seen that (w, v) is of type (3A,8B,8A/3A). Now PERM can be used to give neater generators x, y in a, b with (x,y) of the above type for a subgroup of G of order 192. We find $x=[a,b]^2$, $y=bab^{-1}(ab)^4(ab^{-1})^2a$. The elements x,y generate a subgrop H_9 of order 192 and satisfy the presentation

$$K=\langle x,y \mid x^3=(xy^{-1})^4=(xy^2)^3=(xyxy^{-1})^2=1\rangle.$$

We take $N=\langle y^2, x^{-1}y^2x\rangle$ and see that $N \lhd K$ with $N \cong Q_8$.

Now $K/N=\langle x,y \mid x^3=y^2=(xy)^4=1\rangle \cong S_4$. So $K \cong Q_8.S_4$. It is also easy to check that $K\cong 2^{1+4}.S_3$

Since $|G:H_i| \nmid 495$ for i=1,2,...,8, we conclude that H_9 is a maximal subgroup of G of structure $Q_8.S_4$.

A covering group C for K is

$\langle a_1,a_2,a_3,a_4 \mid a_1{}^3a_3{}^{-1}a_4{}^{-2}=(a_1a_2{}^{-1})^4a_3{}^{-1}a_4{}^{-2}=(a_1a_2{}^2)^3a_3{}^{-1}a_4{}^{-3}=1,\ (a_1a_2a_1a_2{}^{-1})^2a_4{}^{-3}=1,$
$[a_i, a_j]=1\ (1\le i\le 4, 3\le j\le 4, i<j)\rangle$.

Then $|C|=768$ showing that $|M(K)|=4$. In fact $M(K) \cong <a_3,a_4>$ which is isomorphic to C_2xC_2. So H is efficient.

(x) structure: $4^2{:}D_{12}$

H_{10} is the normalizer in G of a subgroup with the structure C_4xC_4. Referring to the Sylow 2-subgroup S_2 given in (I), we take $r=x^2$, $s=xz$. Then $<r, s>=C_4xC_4$ and TC gives $N_G(<r, s>)=<t, q>$, where $t=b(abab^{-1})^2(ab^{-1})^2a$, $q=b(ab^{-1}ab)^2(ab)^3(ab^{-1})^2abab^{-1}a$ with $|<t,q>|=192$. It is checked that (t, q) is of type (8A,6A,2A;3B). Using this fact, PERM gives $x=(ab)^5$, $y=(ab(ab^{-1})^2)^2ab^{-1}(ab)^2ab^{-1}$ of type (2A,6A,8A;3B).The generators x, y satisfy the presentation

$$L=<x,y \mid x^2=y^6=(xy)^8=(xyxy^{-2})^2=(xyxyxy^{-1})^2=1>.$$

Letting $N=< (xy)^2, y^{-1}(xy)^2y >$ and $M=<x, (yx)^2y^{-1}>$, we observe that $N \triangleleft L$, $N \cap M=(1)$ and that $N \cong 4^2$, $M \cong D_{12}$ proving that $L \cong 4^2{:}D_{12}$.

A similar argument given for maximality of H_9 guarantees that H_{10} is a maximal subgroup of G. We note that H_9, H_{10} are not isomorphic since $L/L' \cong 2^2$ and $K/K' \cong 2$.

A covering group C for L is

$$<a_1,a_2,a_3,a_4,a_5 \mid a_1^2=a_2^6a_3a_5^{-3}=(a_1a_2a_1a_2^{-2})^2a_5=(a_1a_2)^8a_4^{-1}a_5^{-4}= (a_1a_2a_1a_2a_1a_2^{-1})^2=1, [a_i,a_j]=1 \ (1 \le i \le 5, 3 \le j \le 5, i < j) >.$$

Then $|C|=768$, and hence $|M(L)|=4$. We have $M(L) \cong <a_3, a_5>$. An efficient presentation for L is now given by

$$<x,y \mid x^2=y^6=(xyxy^{-2})^2=1, (xy)^8=((xy)^2xy^{-1})^2>.$$

(xi) structure: A_4xS_3

A_4xS_3 has the following presentation :

$$<R,S,T,U \mid R^2=S^3=(RS)^3=T^2=U^3=(TU)^2=[R,T]=[R,U]=[S,T]=[S,U]=1>.$$

It is easy to check that (RT,SU) is a generating pair for A_4xS_3 of type (2,3,6;6). In an attempt to give a generating pair (x,y) for a maximal subgroup of G of the structure A_4xS_3, we find that (x,y) must be of type (2A,3B,6B;6A) (see below). PERM then gives $x=(ab)^5$, $y=b^t$, where $t=ab^{-1}abab^{-1}$, with (x,y) of above type and x, y generate a subgroup H_{11} of order 72 and satisfy the presentation

$$<x,y \mid x^2=y^3=(xy)^6=[x,y^{-1}]^2[x,y]^2=1>.$$

It is seen that $<x,y>$ is the direct product of $N=<(xy)^2,(yx)^2>$ and $M=<(xy)^3,[x,y]^2>$ with $N \cong A_4$ and $M \cong S_3$.

It remains to show that H_{11} is maximal in G. To see this, we observe that all elements of order 3 in H_i, for i=1,2,3,4, fall into the class 3A of G while H_{11} contains elements of order 3 lying in the class 3B. This shows that H_{11} cannot be a subgroup of any conjugate in G of H_i, for i=1,2,3,4. Finally all elements of order 6 in H_j, for j=6,7, are in 6B while H_{11} has elements of order 6 in 6A.

Note. We note that A_4xS_3 is not embeddable in M_{11} because the Sylow 2-subgroup of A_4xS_3 is an elementary abelian group of order 8 and that of M_{11} is $<-2,4 \mid 2>$ which is a 2-group of order 16 containing no subgroups of the structure $C_2xC_2xC_2$.

The multiplier of A_4xS_3 is simply computed by 1.5.12. We have $M(A_4xS_3)=C_2$. An efficient presentation for A_4xS_3 is now given by

$$<x,y \mid x^2y^3=(xy)^6=[x,y^{-1}]^2xy^{-1}(xy)^2yxy=1>.$$

PSU(3,5)

order=126000=$2^4.3^2.5^3.7$ d=50   mult=3

$G=\langle a,b \mid a^2=b^4=(ab)^{10}=(abab^2)^7=[a,(ba)^2b^{-1}ab^2ab^{-1}(ab)^2]=$
$((ab^{-1})^2(ab)^3)^2b(ab^{-1})^2(ab)^2ab^{-1}=1 \rangle$

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	126000			1	1^{50}
2A	240	A	A	a	$1^{10}2^{20}$
3A	36	A	A	$(ab^2)^2$	$1^5 3^{15}$
4A	8	A	A	b	$1^2 2^4 4^{10}$
5A	250	A	A	$(ab)^2$	5^{10}
5B	25	A	A	$(ab^{-1})^2(ab)^2ab^2$	$1^5 5^9$
5C	25	A	A	$(ab)^2(ab^{-1})^2ab^2$	5^{10}
5D	25	A	A	$(ab)^2(ab^2)^2ab^{-1}$	5^{10}
6A	12	AA	AA	$ab^2 \sim [a,b]$	$1^1 2^2 3^3 6^6$
7A	7	A	A	$abab^2$	$1^1 7^7$
B**	7	A	A	$b^2ab^{-1}a$	$1^1 7^7$
8A	8	A	A	$(ab)^2ab^2$	$2^1 4^2 8^5$
B**	8	A	A	$b^2(ab^{-1})^2a$	$2^1 4^2 8^5$
10A	10	AA	AA	ab	$5^2 10^4$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	2520	50	A_7	N(...5B,...)	6
H_2	2520	50	A_7	N(...5C,...)	6
H_3	2520	50	A_7	N(...5D,...)	6
H_4	1000	126	$5_+^{1+2}:8$	N(5A)	1
H_5	720	175	$M_{10} \cong A_6{\cdot}2$	N(...5B,...)	3
H_6	720	175	$M_{10} \cong A_6{\cdot}2$	N(...5C,...)	3
H_7	720	175	$M_{10} \cong A_6{\cdot}2$	N(...5D,...)	3
H_8	240	525	$2S_5$	N(2A)	1

I.Sylow p-subgroup

(i) Sylow 2-subgroup   order=16

By table II, G has M_{10} as a subgroup of odd index and hence their Sylow 2-subgroups are isomorphic . However, the Sylow 2-subgroup of M_{10} is isomorphic to that of M_{11}, namely $\langle 2, 4 \mid 2 \rangle$, since $\mid M_{11}:M_{10} \mid$ is odd. As we remarked earlier, the 2-group $\langle 2,4 \mid 2 \rangle$ has a generating pair (r,s) of type (2,8,4/2). This suggests that we look for, using PERM, such a generating pair for a Sylow 2-subgroup of G. Starting from $r=a$ we find $s=ab^2abab^2$ with $|\langle r, s\rangle|=16$. This leads to the generators $x=b^2ab^2$ and $y=b^a$ for a Sylow 2-subgroup P_2 of G. The pair (x,y) then satisfies the presentation

$$\langle x,y \mid x^2=1, (xy)^3=yx\rangle.$$

$n_2(G)=|G:N(P_2)|=7875.$

(ii) Sylow 3-subgroup order=9

Clearly the Sylow 3-subgroup of G is C_3xC_3. Using PERM, we find $x=(ab^2)^2$, $y=((ba)^4b)^2$ with (x,y) of type (3,3,3;1). Then x and y generate a Sylow 3-subgroup P_3 of G.

$n_3(G)=|G:N(P_3)|=1750.$

(iii) Sylow 5-subgroup order=125

By table I, we can easily see that the Sylow 5-subgroup of G is an extra-special group of exponent 5. Hence G has a C_5xC_5 subgroup. Starting with $r=(ab)^2$ of order 5, PERM finds $s=bab(ab^{-1})^2abab^2$ with $\langle r,s\rangle\cong C_5xC_5$. Now TC gives $t=b^2(ab^2ab^{-1})^3ab^2abab^2(ab^{-1})^2$ in $N_G(\langle r,s\rangle)$ with $|\langle r,s,t\rangle|=125$. Putting these generators in PERMGP gives $q=bab^{-1}ab^2abab^2ab^{-1}aba$ with $|\langle s,q\rangle|=125$. Conjugation by b of s and q gives the generators $x=ab(ab^{-1})^2abab^{-1}$ and $y=ab^{-1}ab^2abab^2ab^{-1}(ab)^2$ for a Sylow 5-subgroup P_5 of G. Next x and y satisfy the presentation

$$\langle x,y \mid x^5=y^5=1, [x,y]^x=[x,y]=[x,y]^y\rangle.$$

An easy computation shows that $M(P_5)=C_5xC_5$ and so the above presentation is minimal.

$n_5(G)=|G:N(P_5)|=126.$

(iv) Sylow 7-subgroup order=7

$P_7=\langle abab^2\rangle=C(abab^2)$, $n_7(G)=|G:N(P_7)|=6000.$

II.Maximal subgroups

(i)-(iii) structure: A_7

By table II, G has three conjugacy classes of A_7 subgroups with representatives H_1, H_2, and H_3 whose elements of order 5 lie in the classes 5B, 5C, and 5D respectively (notice that A_7 has only one conjugacy class of elements of order 5.) We now exploit the fact that A_7 has a generating pair of type (2,4,7;5) in order to find generators for each of H_1, H_2, and H_3. Using PERM, we find the generating pairs

(bab^{-1}, b^a), (a^b, b^a), and $((ab^2)^3, (ba)^4 b^{-1} ab^2)$ for the maximal subgroups H_1, H_2, and H_3 respectively. Each pair now satisfies the presentation 4.1 of [16].

The subgroups H_1, H_2, and H_3 are mutually non-conjugate in G because their 5-elements belong to distinct classes of elements of order 5 of G. Note that H_1 having an element in 5B is the stabilizer of a point in the permutation representation of G while 5-elements in each of H_2 and H_3 fix no points.

Next, H_1 has intersection PSL(2,7) with H_2 and $S_3 x S_4$ with H_3. Also $H_2 \cap H_3 = S_3 x S_4$.

(iv) structure: $5_+^{1+2} : 8$

This is the normalizer in G of a Sylow 5-subgroup of G. Using TC, we find $N_G(P_5) = \langle P_5, z \rangle$ of index 126, where $z = (ab)^2 ab^2 ab^{-1}$ and $P_5 = \langle x, y \rangle$ in (I). Now it is easy to check that $x = ab(ab^{-1})^2 abab^{-1}$ and z suffice to generate $\langle P_5, z \rangle$ which is a maximal subgroup of G of order 1000 for $|G:H_i| \nmid 126$ (i=1,2,3). Next we see that $((xz^2)^2, z)$ is a generating pair of type (2,8,8;5) for $\langle x, z \rangle$. This helps us to obtain, by PERM, the generators $u=a$, $v = b(ab)^2(ab^2)^2$ for a maximal subgroup H_4 of G with u an involution.

The pair (u,v) then satisfies the presentation

$$H = \langle u, v \mid u^2=1, [u,v^2]=v^4 u, (uv)^2 v (uv)^3 = (vu)^2 \rangle.$$

On taking $N = \langle [u,v], [u,v^2] \rangle$, $M = \langle v \rangle$, we observe that $N \lhd H$, $N \cap M = (1)$ with $N \cong 5_+^{1+2}$ and $M \cong C_8$. So $H \cong 5_+^{1+2} : 8$.

A covering group C for H may be given by

$$\langle u, v, w \mid u^2 = w, [u,v^2] = v^4 u, (uv)^2 v (uv)^3 = (vu)^2, [u,w]=[v,w]=1 \rangle.$$

Then $|C|=1000$ giving M(H)=1. Despite considerable efforts we failed to determine whether H is efficient.

(v)-(vii) structure: $M_{10} \cong A_6{\cdot}2$

By table II, the three conjugacy classes of maximal subgroups of the structure M_{10} are distinguished by their elements of order 5 which fall into distinct classes of G. As was mentioned earlier an M_{10} subgroup has a generating pair of type (2,4,8/5). Using this fact and that M_{10} has only one conjugacy class of elements of order 5, we seek generating pairs (x_i, y_i) (i=6,7,8) of the above type with the property that $x_5 y_5^2 \in 5B$, $x_6 y_6^2 \in 5C$, and $x_7 y_7^2 \in 5D$. We find, by PERM, $x_5 = a^b$, $y_5 = ab^{-1}(ab)^3 ab^{-1}$, $x_6 = bab^{-1}$, $y_6 = b^{-1}(ab)^3 ab^{-1} a$, $x_7 = b(ab^2)^2 ab$, $y_7 = b^a$. Then $H_i = \langle x_i, y_i \rangle$ is a maximal subgroup of G of the structure M_{10}, for i=5,6,7. And the pairs (x_i,y_i) satisfy the prsentation of M_{10} (see, M_{11}). Next we can check that $H_5 \cap H_6 \cong S_3$, $H_i \cap H_7 \cong <-2,4 \mid 2>$ for i=5, 6.

(viii) structure: $2S_5$

By information given in [33] about the intersection of maximal subgroups, we see that a maximal subgroup M of the above structure is an extension of SL(2,5) by C_2. This helps us to give generators for M by determining the normalizer in G of a SL(2,5) subgroup. The group SL(2,5) has presentation $<c,d \mid c^3=d^5=(cd)^2>$ and can be generated by c^2 and d^2 with (c^2,d^2) of type (3,5,5;6). Using this fact, PERM finds $r=(ab^2)^2$ and $s=(ba)^2$ for a SL(2,5) subgroup. Now, $N_G(<r,s>)=<s,t>$, where $t=ab^2$, with $|<r,s>|=240$. This leads us to the generators $x=a$, $y=(ab)^2$ for a subgroup H_8 of G of order 240.

We shall now show that H_8 is maximal in G with structure $2S_5$. We begin with the following presentation for H_8 on its generators :

$$K=< x,y \mid x^2=y^5=[x,y^2]^3=(xy^{-1}xy^2xy^2)^2=1 >.$$

Now $(xy^2)^4$ of order 2 generates a normal subgroup of K with $K/<(xy)^4> \cong S_5$. So $K \cong 2S_5$. Next, K having an element of order 6 cannot be isomorphic to any subgroup of M_{10}. This proves that H_8 is maximal in G.

We now show that $M(H_8)=1$. To see this we first find the following presentation for a covering group of G, using the presentation K given for H_8 :

$$C=< a_1,a_2,a_3,a_4 \mid a_1^2=a_2^5a_4=[a_1,a_2^2]^3a_3^{-1}=(a_1a_2^{-1}a_1a_2^2a_1a_2^2)^2a_4=$$
$$[a_1,a_3]=[a_1,a_4]=[a_2,a_3]=[a_2,a_4]=[a_3,a_4]=1>.$$

Then C has order 240 and therefore $M(H_8)=1$.Now by modifying the relations in the presentation for K we are able to give a deficiency zero presentation for H_8 as follows :

$$<x,y \mid x^2y^5=y^4xy^{-1}xy^2x^{-1}y^2xyxy^{-2}x^{-1}y^2xy^{-2}x^{-1}=1>.$$

J_1

order=175560=2^3.3.5.7.11.19 d=266 mult=1

$G=\langle a,b \mid a^2=b^3=(ab)^7=[a,b]^{10}=[a,b^{-1}(ab)^2]^6=1\rangle$

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-prart	representative	cycle type
1A	175560			1	1^{266}
2A	120	A	A	a	$1^{10}2^{128}$
3A	30	A	A	b	$1^5 3^{87}$
5A	30	A	A	$[a,b]^6$	$1^6 5^{52}$
B*	30	A	A	$[a,b]^2$	$1^6 5^{52}$
6A	6	AA	AA	$(ab)^2(ab^{-1})^2abab^{-1}$	$1^1 2^2 3^3 6^{42}$
7A	7	A	A	ab	7^{38}
10A	10	BA	AA	[a,b]	$2^3 5^2 10^{25}$
B*	10	AA	BA	$[a,b]^3$	$2^3 5^2 10^{25}$
11A	11	A	A	$(ababab^{-1})^2ab^{-1}$	$1^2 11^{24}$
15A	15	BA	AA	$ab(abab^{-1})^2$	$3^2 5^1 15^{17}$
B*	15	AA	BA	$(ab(abab^{-1})^2)^2$	$3^2 5^1 15^{17}$
19A	19	A	A	$(ab)^2(ab^{-1})^2$	19^{14}
B*2	19	A	A	$((ab)^2(ab^{-1})^2)^2$	19^{14}
C*4	19	A	A	$((ab)^2(ab^{-1})^2)^4$	19^{14}

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	660	266	PSL(2,11)		2
H_2	168	1045	2^3:7:3	$N(2A^3)$	1
H_3	120	1463	$2xA_5$	N(2A)	2
H_4	114	1540	19:6	N(19ABC)	1
H_5	110	1596	11:10	N(11A)	1
H_6	60	2926	D_6xD_{10}	N(3A),N(5AB)	2
H_7	42	4180	7:6	N(7A)	1

I.Sylow p-subgroups

(i) Sylow 2-subgroup , order=8

By table II, it is easy to see that G has $2xA_4$ as a subgroup of odd index. The group $2xA_4$ has presentation

$$H=\langle u,v \mid u^2=v^3=(uv)^6=[u,v]^2=1\rangle$$

(see [18]), and thus is generated by two of its elements u, v with (u,v) of type (2,3,6;2). Using PERM, we find

$$u=a\ ,\ v=a(ab(ab^{-1})^2)^2((ab)^2ab^{-1})^3 \qquad (*)$$

with $|\langle u,v\rangle|=24$ and u, v satisfying the above presentation. Now generators for a Sylow 2-subgroup of G are constructed by determining generators for a Sylow 2-subgroup of H, using the method described in 3.2. By TC we obtain the following generators for a Sylow 2-subgroup of H :

$$u,\ v^{-1}uv,\ vuv^{-1}. \qquad (**)$$

Substituting (*) into (**) and conjugating the resulting words by $b(ab^{-1}ab^{-1}ab)^2a$ gives the generators $x=a^r$, $y=a^s$, and $z=a^t$, where $r=b((ab^{-1})^2ab)^2a$, $s=b((ab^{-1})^2ab)^2ab$, and $t=b((ab^{-1})^2ab)^2ab^{-1}$ for a Sylow 2-subgroup P_2 of G. Then x, y, z are each elements of order 2 and any two of them generate a subgroup of order 4. So $\langle x,y,z\rangle \cong C_2xC_2xC_2$.

$n_2(G)=|G:N(P_2)|= 1045$.

(ii) Sylow 3-subgroup *order=3*

$P_3=\langle b\rangle$, $n_3(G)=|G:N(P_3)|=2926$.

(ii) Sylow 5-subgroup *order=5*

$P_5=\langle [a,b]^2\rangle$, $n_5(G)=|G:N(P_5)|=2926$.

(ii) Sylow 7-subgroup *order=7*

$P_7=\langle ab\rangle=C(ab)$, $n_7(G)=|G:N(P_7)|=4180$.

(ii) Sylow 11-subgroup *order=11*

$P_{11}=\langle (ababab^{-1})^2ab^{-1}\rangle=C((ababab^{-1})^2ab^{-1})$, $n_{11}(G)=|G:N(P_{11})|=1596$.

(ii) Sylow 19-subgroup *order=19*

$P_{19}=\langle (ab)^2(ab^{-1})^2\rangle=C((ab)^2(ab^{-1})^2)$, $n_{19}(G)=|G:N(P_{19})|=1540$.

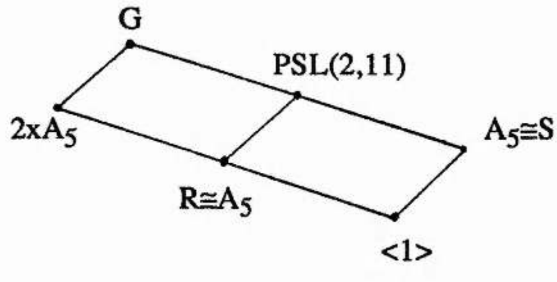
II.Maximal subgroups

(i) Structure: *PSL(2,11)*

$H_1=\langle a, b^t\rangle$, where $t=ab(ab^{-1})^2$.

Let $x=a$, $y=b^t$. Then the elements x, y satisfy the presentation for PSL(2,11) given earlier.

Note. The group PSL(2,11) has precisely two non-conjugate classes of A_5 subgroups. If R, S are representatives of these classes then $R \cong \langle x,y^{xy}\rangle$ and $S \cong \langle x, yxyxy^{-1}\rangle$. Next, R and S remain non-conjugate in G. It is also easy to check that S is a self-normalizing subgroup of G and $N_G(R) \cong 2xA_5$ as we shall see in (iii) below.In fact there are only two distinct classes of A_5 subgroups in G, (see lemma 8.2 of [26]). The following Hasse diagram illustrates the subgroups R and S in G :



(ii) structure: $2^3:7:3$

In [26] the following presentation is constructed for the above maximal subgroup:

$K=\langle \mu,\nu, t_1,t_2,t_3 \mid \mu^3=\nu^7=1, \nu^\mu=\nu^2, t_1^\nu=t_2, t_2^\nu=t_3, t_3^\nu=t_1t_3, t_1^\mu=t_1, t_2^\mu=t_3, t_3^\mu=t_1t_2t_3\rangle$,

where $\langle t_1,t_2,t_3\rangle$ is a Sylow 2-subgroup of K of structure $C_2xC_2xC_2$, and $\langle\mu,\nu\rangle \cong$ 7:3. We now set $x=\mu$ and $y=\nu\mu t_1t_2t_3$ and observe that $K=\langle x,y\rangle$. Then SUBGP constructs the presentation

$$\langle x,y \mid x^3=y^3=1, [xyx,y^{-1}]=y^{-1}x\rangle$$

for K on the generators x, y. This shows that (x,y) is a generating pair of type (3,3,6;7,7) for a subgroup of G isomorphic to K. Using this fact PERM finds the generators

$$x=b^{ab},\ y=ab^{-1}(abab^{-1})^2((ab)^2(ab^{-1})^2)^2 \qquad (\dagger)$$

for a maximal subgroup H_2 of order 168 with x, y satisfying the above presentation. On taking $X=(xy)^3$, $Y=(yx)^3$, $Z=Y^x$, we see that $P=\langle X,Y,Z\rangle$ ($\cong C_2xC_2xC_2$) is a normal subgroup of $\langle x,y\rangle$. By substituting (†) into X, Y, Z we have that $N_G(P)=\langle x,y\rangle$, that is H_3 is the normalizer in G of a Sylow 2-subgroup of G as our second table indicates. On the other hand, by letting $M=\langle x^y,xy^{-1}\rangle$ one can show that $M\cap N=(1)$ with the subgroup $M\cong\langle c,d \mid c^3=[c,d]d=1\rangle$. This can now be used to prove that M is a split metacyclic group of structure 7 : 3 (indeed, $M \cong \langle d\rangle:\langle c\rangle$). Hence $H_2 \cong N:M \cong 2^3:7:3$.

It is easily seen that $y^3=1$ is a redundant relation giving a 2-generator 2-relation presentation for the soluble group H_2. This, in turn, proves that $M(H_2)=1$, by 1.5.8 (ii).

(iii) structure: $2xA_5$

$2xA_5$ has presentation $\langle u,v,w \mid u^2=v^2=w^3=(vw)^5=[u,v]=[u,w]=1\rangle$ and can be generated by the elements uv and w. We may now use SUBGP to construct the presentation

$$L=\langle x,y \mid x^2=y^3=((xy)^3xy^{-1})^2=1\rangle$$

for $2xA_5$ on x=uv and y=w. In L the product xy and the commutator [x,y] have order 10 and 5 respectively.This suggests looking for a generating pair of type (2,3,10;5) for a maximal subgroup H_3 isomorphic to $2xA_5$. PERM simply finds the generators $x=a^t$, $y=b^s$, where $t=[b^{-1},a]$, $s=[a,b^{-1}]$, for H_3 with x, y satisfying the above presentation .

Let $N=\langle (xy)^2,(yx)^2\rangle$. Then N is a normal subgroup of L isomorphic to A_5 with $N_G(N)=\langle x,y\rangle$. It is clear that $N \cong R$ (see the note made in (i)).

By 1.5.12, $M(2xA_5)=C_2$ and thus $2xA_5$ is efficient.

(iv) structure: *19:6*

This is the normalizer in G of a cyclic subgroup $\langle x\rangle$ of G with $x\in$ 19ABC, by table II. We begin with $r=(ab)^2(ab^{-1})^2$ which is in 19A. Then TC gives $N_G(\langle r\rangle)=\langle r,s\rangle$, where $s=(b^{-1}a)^3b(ab(ab^{-1})^2)^2ab(abab^{-1})^2ab^{-1}aba$, with $|\langle r,s\rangle|=114$ and $|\langle s\rangle|=|\langle rs\rangle|=6$. Now it is checked that $x=s^3$ and $y=(rs)^2$ generate $\langle r,s\rangle$ with (x,y) of type (2,3,6;19). This enables us to give, by PERM, the neater generators $x=a^t$, where $t=(ba)^2b^{-1}a$, and $y=(bab^{-1}a)^2b^{-1}ab$ with x, y generating a split metacyclic group H_4 of structure 19:6. To see this we set X=[x,y] and Y=xy. Then $\langle X,Y\rangle$ has order 114 with presentation

$$H_4\cong\langle X,Y \mid X^{19}=Y^6,\ Y^{-1}X^8Y=X^7\rangle,$$

which shows that $H_4 \cong 19:6$, by 1.5.9.

(v) structure: *11:10*

By table II, this is the normalizer in G of a Sylow 11-subgroup of G. Setting $r=(ababab^{-1})^2ab^{-1}$, TC gives $s=b^{-1}(ab^{-1}ab)^2(abab^{-1})^2(ab^{-1}ab)^2ab(ab^{-1})^2ab$ such that $N_G(\langle r\rangle)=\langle r,s\rangle$. The elements r, s generate a subgroup of order 110 with $|\langle r\rangle|=11$ and $|\langle s\rangle|=|\langle rs\rangle|=10$. It is now easy to check that $(s^5,(rs)^2)$ of type (2,5,10,11) is a generating pair for $\langle r,s\rangle$. Using this fact we may find the generators $x=a^t$ and $y=(ba)^2b^q$, where $t=(ba)^2$ and $q=abab^{-1}ab$ for a subgroup H_5 of order 110. The subgroup H_5 having an element of order 10 is not embeddable in PSL(2,11), and so it is a maximal subgroup of G.Next on putting u=[x,y], we have

$$H_5\cong\langle u,y \mid u^{11}=y^{10},\ y^{-1}u^3y=u^2\rangle.$$

This shows that $H_5 \cong 11:10$.

(vi) structure: D_6xD_{10}

The group D_6xD_{10} has presentation

$$<z,u,v,w \mid z^2=u^2=(zu)^3=v^2=w^2=(vw)^5=[z,v]=[z,w]=[u,v]=[u,w]=1>,$$

and can be generated by x=zv, y=uzw. Using SUBGP, we arrive at the following presentation for D_6xD_{10} :

$$<x,y \mid x^2=y^6=(xy)^{10}=(xy^2)^2=1> \qquad (\dagger)$$

Thus a D_6xD_{10} subgroup in G has a generating pair of type (2,6,10/2). Now PERM gives the generators x=a, $y=a[b^{-1},a]^4[b,a]^2b$ for a subgroup H_6 of order 60 with x,y satisfying the above presentation. That H_6 is maximal in G follows immediately from the fact that H_6 has an element of order 15 while neither of H_1 and H_3 has such an element.

Finally, $M(D_6xD_{10}) = C_2$, by 1.5.12. A deficiency -1 presentation for D_6xD_{10} is found by combining two of the relations in (†) as follows:

$$<x,y \mid x^2=(xy)^{10}=xy^2xy^{-4}=1>.$$

(vii) structure: *7:6*

This is the normalizer in G of a cyclic subgroup of G whose generator lies in 7A. We take r=ab and find, by TC, $N_G(<r>)=<r,s>$ where $s=b(ab^{-1}(ab)^2)^3ab^{-1}ab((ab^{-1})^2ab)^2(abab^{-1})^2$ wiyh $|<r,s>|=42$. Now we exploit the fact that $(s^3,(rs)^2)$ is a generating pair of type (2,3,6;7) for $<r,s>$ in order to give generators in a, b of shorter lengths for a subgroup isomorphic to $<r,s>$. By PERM, we find x=a and $y=bab^{-1}(ab^{-1}ab)^2$ which generate a subgroup H_7 of order 42. Setting u=[x,y] and v=xy, we have

$$H_7 \cong <u,v \mid u^7=v^6,\ v^{-1}u^2v=u^3 >$$

which is a split metacyclic group of structure 7:6.

Now SUBGPTEST ensures that H_2 of structure 2^3:7:3 has no subgroup isomorphic to H_7 proving that H_7 is maximal in G.

A_9

order=181440=$2^6.3^4.5.7$ d= 9 mult=2

$G=\langle a,b \mid a^2=b^4=(ab)^9=[a,b]^4=[a,bab^2]^3=[a,b^2(ab)^2]^2 = ((ab)^2(ba)^3b^{-1})^3=1\rangle$

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	181440			1	1^9
2A	480	A	A	a	$1^5 2^2$
2B	192	A	A	b^2	$1^1 2^4$
3A	1080	A	A	$(ab^2)^4$	$1^6 3^1$
3B	81	A	A	$(ab)^3$	3^3
3C	54	A	A	$(ababab^2)^2$	$1^3 3^2$
4A	24	A	A	[a,b]	$1^3 2^1 4^1$
4B	16	A	A	b	$1^1 4^2$
5A	60	A	A	$(ab)^2(ab^2)^3$	$1^4 5^1$
6A	24	AA	AA	$(ab^2)^2$	$1^2 2^2 3^1$
6B	6	CB	CB	$(ab)^2ab^2$	$1^1 2^1 6^1$
7A	7	A	A	$(ab)^3(ab^{-1})^2$	$1^2 7^1$
9A	9	B	A	ab	9^1
9B	9	B	A	$abab^2$	9^1
10A	20	AA	AA	$(ab^2ab)^2abab^2$	$2^2 5^1$
12A	12	AA	AA	ab^2	$2^1 3^1 4^1$
15A	15	AA	AA	$(ab)^2ab^{-1}$	$1^1 3^1 5^1$
B**	15	AA	AA	$b(ab^{-1})^2a$	$1^1 3^1 5^1$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	20160	9	A_8		2
H_2	5040	36	S_7		2
H_3	2160	84	$(A_6 x3):2$	N(3A)	2
H_4	1512	120	SL(2,8):3	N(2B,3B,7A,9A)	1
H_5	1512	120	SL(2,8):3	N(2B,3B,7A,9B)	1
H_6	1440	126	$(A_5 xA_4):2$	$N(2A^2)$,N(2A,3A,5A)	2x2
H_7	648	280	$3^3:S_4$	$N(3^3)=N(3A_3B_4C_6)$	2
H_8	216	840	$3^2:2A_4$	$N(3B^2)$	3

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=64

The Sylow 2-subgroup of A_9 is isomorphic to that of A_8 because A_9 has A_8 as a subgroup of odd index. Returning to the Sylow 2-subgroup $P_2=\langle x,y,z\rangle$ of A_8, we see that x and yz generate a subgroup of order 32 with (x,yz) of type (2,4,4;2). Using this fact we find, by PERM, $r=a$, $s=(ba)^3b^{-1}(ab)^4b$ with $|\langle r,s\rangle|=32$. Now TC gives $t=b^{-1}ab$ in $N_G(\langle r,s\rangle)$ with $|\langle r,s,t\rangle|=64$. Using PERMGP we obtain the generators a, a^b, and $(b^2)^q$, where $q=(ab^{-1})^3$, for a Sylow 2-subgroup P_2 of G.

$n_2(G)=|G:N(P_2)|=2835$.

(ii) Sylow 3-subgroup order=81

By a similar method to that used for the Sylow 3-subgroup of PSU(4,2), we proceed to extend the cyclic subgroup generated by x=ab to a Sylow 3-subgroup P_3 of G. We find $P_3=\langle x,y\rangle$ where $y=ab^{-1}(ab)^2ab^2abab^{-1}(ab^2)^2$. The pair (x^{-1},yx^4) satisfies the presentation given for the Sylow 3-subgroup of PSU(4,2).

By information given in table I, we can easily check that $C((ab)^3)=P_3$.

$n_3(G)=|G:N(P_3)|=1120$.

(iii) Sylow 5-subgroup order=5

$P_5=\langle (ab)^2(ab^2)^3\rangle$, $n_5(G)=|G:N(P_5)|=756$.

(iv) Sylow 7-subgroup order=7

$P_7=\langle (ab)^3(ab^{-1})^2\rangle=C((ab)^3(ab^{-1})^2)$, $n_7(G)=|G:N(P_7)|=4320$.

II.Maximal subgroups

(i) structure: A_8

$H_1=\langle bab^{-1}, b^a\rangle$ in 8.2 of [16].

(ii) structure: S_7

S_7 has presentation

$$\langle T,S \mid T^2=S^7=(TS)^6=[T,S^2]^2=[T,S^3]^2=1\rangle$$

(see [18]) and thus is generated by elements T, S of order 2 and 7 whose product has order 6. This fact together with $|\langle[T,S]\rangle|=3$ helps us to look for a generating pair of type (2,7,6;3) for a S_7 subgroup. PERM finds the generators $x=a^b$, $y=(ab^{-1})^2(ab)^2ab^{-1}$ satisfying the above presenation for S_7 with $|\langle x,y\rangle|=5040$. Next $H_2=\langle x,y\rangle$ is a maximal subgroup of G for A_8 cannot have S_7 as a subgroup (see maximal subgroups of A_8).

$M(S_7)=C_2$, and an efficient presentation for S_7 may be given by :

$$\langle T,S \mid T^2S^7=1,\ (TS)^6=(TS^2TS^{-2})^2,\ (TS^3TS^{-3})^2=1\rangle.$$

(iii) structure: $(A_6x3):2$

By 3.3.2, this is the normalizer in G of a subgroup isomorphic to A_6x3. The group A_6x3 has presentation $<c,d,e \mid c^3=d^2=e^4=(de)^5=(de^2)^5=[c,d]=[c,e]=1>$ and can be generated by d, cd with (d,ce) of type (2,12,15;4). Starting with r=a , which is in 2A, PERM finds $s=bab(ab^2)^2$ of order 12 such that $|<rs>|=15$ and $|<[r,s]>|=4$. Now r, s generate a subgroup of order 1080 with $N_G(<r,s>)=<r,s,t>$ where $t=(b^2a)^2b^2$. The subgroup <r,s,t> has index 84 in G and can be generated by $st=bab^{-1}$ and $rs=(ab)^2(ab^2)^2$. Since $|G:H_i| \nmid 84$ (i=1,2), $H_3=<st,rs>$ is maximal in G. Here the pair (st,rs) is of type (2A,15,6B;3C).

Put x=st, y=strs. Then x, y satisfy the presentation

$$H=<x,y \mid x^2=y^6=[x,y]^3=[x,y^2]^2=((xy)^4y)^2=(xy(xy^2)^2)^3=1>.$$

Now take $N=<[x,y^2],xy>$. Then N has index 2 in H and presentation on generators $u=[x,y^2]$, v=xy with relations $u^2=[u,v^5]=1$, $uv^4=(vu)^4$, $uv^3=(v^2u)^3$. Next $<u,v^3>$ is a normal subgroup of <u,v> isomorphic to A_6 and has trivial intersection with $<(uv^2)^4>$ of order 3. This shows that $H \cong N:2 \cong (A_6x3):2$ as required.

$M(H_3)\cong C_2$. To see this we first observe that

$$H_3\cong< x,y \mid x^2=(xy^2xy^{-2})^2y^6=((xy)^4y)^2y^{-6}=1,\ (xyxy^{-1})^3=(xy(xy^2)^2)^3>.$$

Then a covering group C for H_3 can be presented as

$$C=<x,y \mid x^2(xyxy^{-1})^3=(xy(xy^2)^2)^3,\ x^4(xy^2xy^{-2})^2y^{12}=((xy)^4y)^2=1,$$
$$[x^2,y]=[x,(xy^2xy^{-2})^2y^6]=[y,(xy^2xy^{-2})^2]=1>.$$

TC now verifies that $|C|=4320$ proving that $M(H_3)=C_2$.

We failed to find an efficient presentation for H_3 from the above presentation by combining relations differently. Therefore we try to find a new presentation for H_3 having a few number of relations. Employing the method described in 3.5, we arrive at the generators $u=xy^3$ and $v=(yx)^2y^3$, where x, y are the generators of H_3, which generate H_3 and satisfy the following deficiency -1 presentation

$$<u,v \mid v^4=1,\ u^2v^2u^2v=vu^2,\ [u,v]u[v,u^{-1}]=v^2>.$$

(iv)-(v) structure: $SL(2,8):3$

G has two non-conjugate maximal subgroups R, S of structure SL(2,8):3 both being the normalizer in G of a SL(2,8) subgroup. In fact G has two non-conjugate SL(2,8) subgroups K, L whose elements of order 9 lie in the classes 9A and 9B respectively and we have $R \cong N_G(K)$, $S \cong N_G(L)$.

We begin with R whose SL(2,8) subgroup contains elements in the class 9A, by table II. The group SL(2,8) has three conjugacy classes of elements of order 9 (same family); and can be generated by elements k_1, k_2 with (k_1,k_2) of type (2,3,9;7). Using PERM, we find the generators $k_1=b^2$, $k_2=(ab^2ab)^3$ for a SL(2,8) subgroup K whose elements of order 9 belong to the class 9A. Now TC gives $N_G(K)=\langle K,k\rangle$, where $k=ab^2(ab^{-1}ab)^2$, with $|G:\langle K,k\rangle|=120$. The subgroup $R=\langle K,k\rangle$ is a maximal subgroup in G since $|G:H_i| \nmid 120$ (i=1,2,3).

Next we check that (k_1,k_2k^{-1}) is a generating pair of type (2B,6B,6B;9A) for R. This helps us to obtain the generators $x_4=b^2$ and $y_4=(ab)^2(ba)^2(b^{-1}a)^2ba$ for the maximal subgroup H_4 of G isomorphic to R.

A presentation for H_4 on x_4, y_4 may be given by

$$\langle x,y \mid x^2=y^6=(xy^2)^6=(xy^2xy^3)^3=(xy^2xy^3xy)^3=1\rangle.$$

We take $N=\langle x,yxy^2\rangle$, $M=\langle y^2\rangle$. Then $N\cap M=(1)$ with $N \cong SL(2,8)$ showing that $H_4 \cong SL(2,8){:}3$. Now $N_4=\langle x_4,y_4x_4y_4^2\rangle$ is a SL(2,8) subgroup of G whose elements of order 9 are in 9A with $N_G(N_4)=H_4$.

Similarly, we find the generators $x_5=b^2$ and $y_5=bab^{-1}ab^2a$ for a maximal subgroup of G isomorphic to S as we shall see now. Here $N_5=\langle x_5,y_5x_5y_5^2\rangle$ is a SL(2,8) subgroup of G with elements of order 9 in 9B (i.e. isomorphic to S) and that $N_G(N_5)=H_5$. Moreover, we note that N_4 and N_5 are non-conjugate in G. The pair $((x_5y_5)^3,y_5x_5)$ is a generating pair of type (2B,6B,6B;9B) and satisfies the above presentation.

That H_4, H_5 remain non-conjugate in G follows from the fact that elements of order 9 in H_4 are all in 9A while those of H_5 are in 9B. Finally H_4 has intersection S_3xC_3 with H_5.

The group H_4 (H_5) is isomorphic to the deficiency zero group $G(l,m,n)$, for $l=2$, $m=3$, $n=-2$, studied by C.M. Campbell (see [28]). To see this we take $u=y_5$, $v=x_5y_5^2x_5y_5^{-1}$. Then u, v generate H_5 and satisfy the following deficiency zero presentation

$$\langle u,v \mid uv^3=vu^{-2}vu,\ vu^3=uv^{-2}uv\rangle.$$

This shows that $M(H_4)=1$.

(vi) structure: $(A_4xA_5){:}2$

By 3.3.2, this is the normalizer in G of an A_4xA_5 subgroup. The group A_4xA_5 has presentation $\langle r,s,t,q \mid r^2=s^3=(rs)^3=t^2=q^3=(tq)^5=[r,t]=[r,q]=[s,t]=[s,q]=1\rangle$ and can be generated by rt and sq. That (rt,sq) is of type (2,3,15) allows us to look for, by

PERM, a generating pair (c,d) for an A_5xA_4 subgroup. We find the generators $c=b^2$ and $d=(ba)^2(b^2a)^2$. TC now gives $N_G(<c,d>)=<d,e>$, where $e=(ab^2)^2a$, with $<d,e>$ of index 126. It is then easy to check that (e,d^2e) is a generating pair of type (2A,4B,15;3C) for $<d,e>$. Using this enables us to obtain the nice generators $x=a^b$, $y=b^a$ for a subgroup H_6 of index 126. H_6 having an element of order 12 (e.g., xy^2) cannot be isomorphic to any subgroup of H_1 ($\cong A_8$). This fact together with $|G:H_i| \nmid 126$ (i=2,3,4,5) proves that H_6 is maximal in G.

The pair (x,y) satisfies the following presentation for H_6 :

$$<x,y \mid x^2=y^4=(xy)^{15}=[x,y]^3=((xy)^4xy^2xy^{-1})^2=1> \qquad (\dagger)$$

Now let $N=<y^2,xy>$. Then N has index 2 in $<x,y>$ and trivial intersection with $<x>$. Thus $<x,y> \cong N: 2$. On taking $u=y^2$, $v=xy$ we find the presentation $<u,v \mid u^2=v^{15}=(uv^2)^3=uv^3uvuv^3uv^{-4}>$ for N which is easily shown to be the direct product of $<v^3,(v^3)^u>$($\cong A_5$) and $<(uv^3)^3,v^5>$($\cong A_4$). Hence $H_6 \cong (A_4xA_5):2$.

To construct a covering for H_6 we first combine the third and fifth relations of (†) into the single relation $(xy)^{15}=((xy)^4xy^2xy^{-1})^2$. Then the obtained group is isomorphic to H_5 with a covering group C :

$$<x,y \mid x^2y^{16}= (xy)^{15}((xy)^4xy^2xy^{-1})^{-2}\ y^8=[x,y^4]=[x,(xyxy^{-1})^3]=[y,(xyxy^{-1})^3]=1>.$$

C has order 5760 and thus $|M(H_6)|=4$. We have $M(H_5) \cong <y^4,(xyxy^{-1})^3> \cong C_2xC_2$. The latter presentation of H_6 now shows that H_6 is efficient.

(vii) structure: $3^3{:}S_4$

By information given in table II, this is the normalizer in G of an elementary abelian group P ($\leq$G) of order 27 whose 13 cyclic subgroups number 3 containing class 3A, 4 containing 3B, and 6 containing 3C. Considering the Sylow 3-subgroup P_3 of G, the only subgroup of P_3 with this property is found to be generated by x^2yx^2, $x^{-2}y$, yx, where x, y are the generators of P_3 given in (I). Substituting ab, $ab^{-1}(ab)^2ab^2abab^{-1}(ab^2)^2$ for x, y in these generators gives a subgroup P whose normalizer in G is <ab, z>, where $z=(b^{-1}aba)^2b^2(ab^{-1})^2(ab^2ab^{-1})^2(ab^2)^2ab(ab^{-1})^2$, with <ab,z> of order 648. Next, we check that $((abz)^3,z^5)$ is a generating pair of type (2B,12,3B;9A) for <ab,z>. This allows PERM to give the generators $x=b^2$, $y=b(abab^2)^2(ab)^2$ for a maximal subgroup H_7 of order 648.

A presentation for H_7 on x, y is

$$<x,y \mid x^2=y^3=(xy)^{12}=((xy)^3(xy^{-1})^2)^2=1>.$$

Let $N=<[x,y]^3,(xy)^4,(yx)^4>$, $M=<x^{yx},y^{xy}>$. Then <x,y> splits over N and M is a complement to N in <x,y> with $M \cong S_4$ and $N \cong 3^3$. This means that $H_7 \cong 3^3{:}S_4$.

A covering group C for H_7 may be given by

$$<a_1,a_2,a_3,a_4 \mid a_1^2=a_2^3a_4^{-1}=(a_1a_2)^{12}a_3^{-1}a_4^{-4}=((a_1a_2)^3(a_1a_2^{-1})^2)^2a_4^{-1}=[a_1,a_3]=$$
$$[a_1,a_4]=[a_2,a_3]=[a_2,a_4]=[a_3,a_4]=1>.$$

Then $|C|=1296$ and so $M(H_7) \cong C_2$; and an efficient presentation for H_7 is

$$<x, y \mid x^2y^3=(xy)^{12}=((xy)^3(xy^{-1})^2)^2=1>.$$

It is worth noting that H_7 is not isomorphic to the maximal subgroup of PSU(4,2) of structure $3^3{:}S_4$ since their multipliers are different.

(viii) structure: $3^2{:}A_4$

The table of [21] shows that the maximal subgroup with the above structure is the Hessian group. The Hessian group has a generating pair of type (3,3,4;4). On searching for such a pair by PERM we find that $t=(ab)^3$, $s=(ab)^2(ba)^3b^{-1}$ generate a subgroup H_8 of order 216 and satisfy the presentation of the Hessian group

$$<T,S \mid T^3=S^3=(TS)^4=1,\ (TST)^2S=S(TST)^2>.$$

It remains to shows that H_8 is maximal in G. Firstly, H_8 is not embeddable in H_7 because $|H_8'|$ (=72) does not divide $|H_7'|$ (=324). Secondly, H_8 has a single conjugacy class of elements of order 4 with elements in 4B whereas such elements in H_3 are all in 4A; so H_3 cannot be a subgroup of any conjugate in G of H_3. Finally, the index of H_8 in G is not divisible by $|G:H_i|$ (i=1,2). These prove that H_8 is a maximal subgroup of G.

Using the above presentation for the Hessian group we are able to construct the following presentation for a covering group of H_8

$$C=<T,S,U,V \mid T^3U=S^3=(TS)^4U=1,(TST)^2S=VS(TST)^2,[T,U]=[T,V]=[S,U]=$$
$$[S,V]=[U,V]=1>.$$

Then C having order 648 gives $|M(H_8)|=C_3$. Now an efficient presentation for H_8 is given as follows :

$$<T,S \mid T^3=1,\ (TS)^4=S^3,\ (TST)^2S^{-2}=S(TST)^2>.$$

PSL(3,5)

order=372000=$2^5.3.5^3.31$ d= 31 mult=1

$G=\langle a,b \mid a^2=b^3=(ab)^{24}=[a,b]^6=((ab)^6(ab^{-1})^2)^3=((ab)^3(abab^{-1})^2)^5=$

$((ab)^5(ab^{-1}ab)^2(ab)^2(abab^{-1})^2)^2=1\rangle$

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	372000			1	1^{31}
2A	480	A	A	a	$1^7 2^{12}$
3A	24	A	A	b	$1^1 3^{10}$
4A	480	A	A	$(ab)^6$	$1^7 4^6$
B**	480	A	A	$(b^{-1}a)^6$	$1^7 4^6$
4C	16	A	A	$(ab)^2(ab^{-1})^2abab^{-1}$	$1^3 2^4 4^6$
5A	500	A	A	$((ab)^2(ab^{-1})^2)^2$	$1^6 5^5$
5B	25	A	A	$(ab)^3(abab^{-1})^2$	$1^1 5^6$
6A	24	AA	AA	$[a,b]$	$1^1 3^2 6^4$
8A	24	A	A	$(ab)^3$	$1^1 2^3 8^3$
B**	24	B	A	$(b^{-1}a)^3$	$1^1 2^3 8^3$
10A	20	AA	AA	$(ab)^2(ab^{-1})^2$	$1^1 2^2 5^1 10^2$
12A	24	AB	AA	$(b^{-1}a)^2$	$1^1 3^2 12^2$
B**	24	AA	AB	$(ab)^2$	$1^1 3^2 12^2$
20A	20	AA	AA	$(ba)^2(b^{-1}a)^5$	$1^2 4^1 5^1 20^1$
B**	20	AB	AB	$(ab)^5(ab^{-1})^2$	$1^2 4^1 5^1 20^1$
24A	24	AB	AA	$(ab)^7$	$1^1 6^1 24^1$
B**	24	BA	AB	$(b^{-1}a)^7$	$1^1 6^1 24^1$
C**7	24	AB	AA	$b^{-1}a$	$1^1 6^1 24^1$
D*7	24	BA	AB	ab	$1^1 6^1 24^1$
31A	31	A	A	$(ab)^2ab^{-1}$	31^1
B**	31	A	A	$b(ab^{-1})^2a$	31^1
C*2	31	A	A	$((ab)^2ab^{-1})^2$	31^1
D**2	31	A	A	$(b(ab^{-1})^2a)^2$	31^1
E*4	31	A	A	$((ab)^2ab^{-1})^4$	31^1
F**4	31	A	A	$(b(ab^{-1})^2a)^4$	31^1
G*8	31	A	A	$((ab)^2ab^{-1})^8$	31^1
H**8	31	A	A	$(b(ab^{-1})^2a)^8$	31^1
I*16	31	A	A	$((ab)^2ab^{-1})^{-15}$	31^1
J*15	31	A	A	$((ab)^2ab^{-1})^{15}$	31^1

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	12000	31	5^2:GL(2,5)	N($5A^2$)	2
H_2	12000	31	5^2:GL(2,5)	N($5A^2$)	2
H_3	120	3100	S_5	N(2A,3A,5B)	2
H_4	96	3875	4^2:S_3	N($2A^2$)	2
H_5	93	4000	31:3	N(31ABCDEFGHIJ)	1

I.Sylow p-subgroups

(i) structure: *order=32*

By table II, G has GL(2,5) as a subgroup of odd index and thus its Sylow 2-subgroup is isomorphic to that of GL(2,5). Let u, v,w be the matrices over GF(5)

$$u=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, \quad v=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \quad w=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}.$$

Then u, v, and w generate GL(2,5) (see 7.5 of [18]). Taking $r=w^2$, $s=uvw$ gives $<r,s>=<u,v,w>$ with

$$GL(2,5) \cong <r,s \mid r^2=(rs)^4=1, rs^2=(s^4r)^2>. \qquad (*)$$

Using the method 2.3 (vi), TC finds the generators $x=(rs)^2$, $y=s^3$ for a Sylow 2-subgroup of GL(2,5). Having found the generators x, y we observe that (x, xy) is of type (2,4,8;4). Now PERM simply gives $x=a^t$, where $t=(ba)^2b^{-1}$, and $y=(ab)^6$ for a Sylow 2-subgroup P_2 of G. It is now easy to check that P_2 is isomorphic to the 2-group

Γ_3e.

$n_2(G)=|G:N(P_2)|= 11625$.

(ii) Sylow 3-subgroup *order=3*

$P_3=<b>$, $n_3(G)=|G:N(P_3)|=7750$.

(iii) Sylow 5-subgroup *order=125*

By table I, it is readily seen that the Sylow 5-subgroup of G is an extra-special group of exponent 5 and thus is generated by elements x, y with (x,y) of type (5,5,5;5). Starting with $x=((ab)^2(ab^{-1})^2)^2$ PERM gives $y=(ab)^3(abab^{-1})^2$ with $P_5=<x,y>$ of order 125.

$n_5(G)=|G:N(P_5)|=186$.

(iv) Sylow 31-subgroup *order=31*

$P_{31}=<(ab)^2ab^{-1}>=C((ab)^2ab^{-1})$, $n_{31}(G)=|G:N(P_{31})|=4000$.

II.Maximal subgroups

(i)-(ii) structure: *5^2:GL(2,5)*

$H_1=<a,(bab)^{ab}>$.

Let $x_1=a$, $y_1=(bab)^{ab}$. Then a presentation for H_1 on x_1, y_1 is

$$H=<x,y \mid x^2=(xy^3)^4=[x,y]^3=(xy^5xy^{-1})^3=(xy^3xy^2xy^{-1})^2=xy^2xyxy^2xy^{-2}xy^{-1}xyxy^3=1>.$$

We now take $N=\langle((xy)^3xy^{-1})^2,(y^{-1}(xy)^3x)^2\rangle$, $M=\langle x,y^5xy^2\rangle$ and observe that $N \triangleleft H$, $N\cap M=(1)$ with $N \cong C_5xC_5$. This proves that H is a split extension of C_5xC_5 by M. Next the generators $r=x$, $s=y^5xy^2$ of M satisfy the presentation (*) given in (I) showing that $M\cong GL(2,5)$. Thus $H \cong 5^2{:}GL(2,5)$. On substituting x_1, y_1 for x, y in the generators of M we find a subgroup of G of structure C_5xC_5 whose normalizer in G is precisely H_1 as stated in our second table.

A covering group C of H_1 is found to have a presentation on generators x, y with relations

$x^4(xy^3)^4=(xy^3xy^2xy^{-1})^2$, $x^{-5}y^2xyxy^2xy^{-2}xy^{-1}xyxy^3=1$, $[y,x^2]=[x,(xy^3)^4]=1$,
$[y,(xy^3)^4]=[x,(xyxy^{-1})^3]=[y,(xyxy^{-1})^3]=[x,(xy^5xy^{-1})^3]=[y,(xy^5xy^{-1})^3]=1$,
$[(xy^3)^4,(xyxy^{-1})^3]=[(xy^3)^4,(xy^5xy^{-1})^3]=1$.

TC now verifies that $|C|=24000$. So $M(H_1)=C_2$. To prove H_1 efficient we look for a deficiency -1 presentation using the method described in 3.5. It is found that $R=y_1^{-4}x_1y_1^{-1}$, $S=x_1$ generate H_1 and satisfy the presentation

$$\langle R,S \mid S^2=R^5SR^4SR^{-1}S(RS)^2=R^2(RSR^{-2}S)^2R^{-1}SR(RS)^2=1\rangle.$$

We now proceed to give generators x_2, y_2 for (ii). It is easy to check that (x_1,y_1) is a generating pair for H_1 of type (2,24,24;3) with $|\,fix(x_1)\cap fix(y_1)| =1$. PERM is then able to produce the words $x_2=a$, $y_2=(bab)^t$, where $t=ab^{-1}$, with (x_2,y_2) of the same type and $|fix(x_2)\cap fix(y_2)|=0$. The elements x_2, y_2 generate a subgroup H_2 of G of order 12000 which is obviously not conjugate to H_1. Next (x_2,y_2^{-1}) satisfies the above presentation for H_1.

The intersection of H_1 and H_2 is GL(2,5).

(iii) structure: S_5

As was remarked earlier S_5 is generated by two elements T and S with (T,S) of type (6,5,2;2). Using PERM, we find the generators $x=a^{ba}$ and $y=b^{-1}(ab)^4$ for a subgroup H_3 isomorphic to S_5. Here (x,y) is of type (2,6,5B;2). We note that by the information given in [34] about the structure constants for 2A, #(2,6,5A)=0. This indicates that G has no S_5 subgroup with elements of order 5 in 5A. Now we prove that H_3 is a maximal subgroup of G in the following way. The derived group H_1' of H_1 has index 4 and may be generated by $z=[x_1,y_1]$, $u=[x_1,y_1^{-1}]$. A presentation for H_1' on z, u is found to have relations $z^3=(zu)^3zu^{-1}zuz^{-1}uzu^{-1}=(zu^{-1})^3z^{-1}u^{-1}zuzu^{-1}z^{-1}u^{-1}=1$. A simple check using TC shows that $v=zuzu^{-1}$, $w=(uz)^3uz^{-1}(uz)^3$ generate a Sylow 2-subgroup of H_1' and that $v^2=w^2$, $vwv=w$ from which we imply that H_1' has Q_8 as a Sylow

2-subgroup. On the other hand the Sylow 2-subgroup of S_5' (=A_5) is the Klein 4-group. Thus H_1' cannot have S_5' as a subgroup for the quaternion group Q_8 has no $C_2 x C_2$ subgroups. This shows that H_1 has no S_5 subgroups.

It may be worth mentioning that there is only one perfect group K of order 3000; $K \cong N \cdot A_5$, $N = \langle r,s,t \mid r^5=s^5=t^2=[r,s]=1, [r,t]=r^3, [s,t]=s^3 \rangle$, see [40]. The group H_1' being perfect of order 3000 is clearly isomorphic to K. A somewhat easy computation shows that the covering group of K has order 15000 which proves that $M(K)=C_5$. Thus K having a 2-generator 3-relation presentation is efficient.

(iv) structure: $4^2:S_3$

This is the normalizer in G of a $C_4 x C_4$ subgroup. By PERM we first find $r=(ab)^6$, $s=(ba)^2(bab^{-1}a)^2(ba)^3(b^{-1}a)^2$ such that $\langle r,s \rangle \cong C_4 x C_4$.Then TC gives $N_G(\langle r,s \rangle) = \langle z,u \rangle$, where $z=(ab)^3$, $u=b^{-1}((ab^{-1})^2ab)^2ab^{-1}((ab)^2(ab^{-1})^5)^2(abab^{-1})^3(ab)^2(ab^{-1})^4abab^{-1}$, with $|\langle z,u \rangle|=96$. It is now found that (zu^{-1},u) is a generating pair of type (2,3,8;3) for $\langle z,u \rangle$ which helps us to find the generators $x=a^{ba}$, $y=(ba)^4(b^{-1}a)^3bab$ for a subgroup H_4 of G of order 96. The pair (x,y) satisfies the presentation $\langle x,y \mid x^2=y^3=(xy)^8=[x,y]^3=1 \rangle$. The maximality of H_4 follows on noting that $|H_4'|$ (=48) does not divide $|H_1'|$ (=3000).

(The multiplier of the same group was calculated earlier.)

(v) structure: *31:3*

By table II, this is the normalizer in G of a cyclic subgroup of G whose generator t lies in 31A-J. We take $t=(ab)^2ab^{-1}$, which is in 31A, and find $N_G(\langle t \rangle)=\langle t,q \rangle$where $q=b^{-1}(ab^{-1})^6(ab)^2(ab^{-1}ab)^3ab(ab^{-1})^3(ab)^2ab^{-1}ab$ with $|\langle t,q \rangle|=39$. The subgroup $\langle t,q \rangle$ of G is maximal since 39 does not divide $|H_i|$ (i=1,2,3,4). That (q,tq) is of type (3,3,31;31) enables us to obtain the neater generators $x=(abab^{-1})^2$ and $y=b(ab)^3a^g$, where $g=(ba)^2$, for a maximal subgroup H_5 of G isomorphic to $\langle t,q \rangle$. Putting u=[x,y], we obtain the following presentation for the split metacyclic group H_5

$$\langle u,y \mid u^{31}=y^3, y^{-1}u^8y=u^9 \rangle.$$

M_{22}

order=443520=$2^7.3^2.5.7.11$ d=22 mult=12

$G=\langle a,b \mid a^2=b^4=(ab)^{11}=(ab^2)^6=(abab^2)^8=[a,b^2abab^2]^2=$

$((ab)^2(ab^2)^2)^3=((ab)^2(ab^{-1})^2ab^2)^3=1\rangle$

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	443520			1	1^{22}
2A	384	A	A	a	$1^6 2^8$
3A	36	A	A	$(ab^2)^2$	$1^4 3^6$
4A	32	A	A	b	$1^2 2^2 4^4$
4B	16	A	A	$ab(ab^2)^2$	$1^2 2^2 4^4$
5A	5	A	A	[a,b]	$1^2 5^4$
6A	12	AA	AA	ab^2	$2^2 3^2 6^2$
7A	7	A	A	$(ab)^2ab^2ab^{-1}$	$1^1 7^3$
B**	7	A	A	$bab^2(ab^{-1})^2a$	$1^1 7^3$
8A	8	A	A	$abab^2$	$2^1 4^1 8^2$
11A	11	A	A	ab	11^2
B**	11	A	A	$b^{-1}a$	11^2

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	20160	22	M_{21}=PSL(3,4)		4x12
H_2	5760	77	$2^4:A_6$	$N(2A^4)$	2x12
H_3	2520	176	A_7		6
H_4	2520	176	A_7		6
H_5	1920	231	$2^4:S_5$	$N(2A^4)$	2x4
H_6	1344	330	2^3:PSL(2,7)	$N(2A^3)$	2x2
H_7	720	616	$M_{10}\cong A_6\cdot 2$		3
H_8	660	672	PSL(2,11)		2

I.Sylow p-subgroups

(i) Sylow 2-subgroup order=128

By table I, the centralizer in G of an involution is a subgroup of order 384 and thus contains a Sylow 2-subgroup of G. Using the permutation representation of G it is found, by CAYLEY, that

$$h=(1,17,16,21)(3,9,15,6)(4,10,22,12)(5,19)(7,14,11,18)(8,20),$$
$$g=(1,20,21)(2,3)(4,14,12,9,5,6)(7,19,11,22,15,10)(8,17,16)(13,18)$$

generate C(a). A simple check now shows that (hg^2,g) is a generating pair of type (2,6,8;6) for C(a) which allows PERM to obtain the generators $r=a$, $s=b(ab^{-1})^2ab$ for a subgroup H of G of order 384. A presentation for H on r, s is

$$\langle r,s \mid r^2=(rs)^6=(rs^2)^2(rs^{-2})^2=(rs)^2s^2(rsrs^{-1})^2=1\rangle.$$

Using TC we find the generators

$$rsr,\ s^2,\ srs \qquad (*)$$

for a Sylow 2-subgroup of H. Substituting a, $b(ab^{-1})^2ab$ for r, s in (*) and using PERMGP gives the generators $ab(ab^{-1})^2aba$, $(ab^2)^3a$, $ab(ab^2ab^{-1})^2ab(ba)^2$ from which we may arrive at the generators $x=(b^2)^t$, where $t=ab^{-1}$, $y=(b^2a)^2(bab^2a)^2$, $z=b^2(ab)^2a$ for a Sylow 2-subgroup P_2 of G.

A presentation for P_2 on these generators may be given by

$$\langle x,y,z \mid x^2=(xy)^2=y^4=(yz)^2=1,\ xz^3x=z,\ zy^2z=yz^2y\rangle.$$

Again using the permutation representation of G, CAYLEY simply shows that $P_2/\Phi(P_2) \cong 2^3$ and thus $d(P_2)=3$, by 1.5.17. Next we shall see that rank $M(P_2)=3$ which, together with $d(P_2)=3$, will prove that the above presentation for P_2 is minimal. For this purpose, we first construct the following presentation for a covering group of P_2 :

$$C=\langle a_1,a_2,a_3,a_4,a_5,a_6 \mid a_1{}^2a_5{}^{-1}=(a_1a_2)^2=a_2{}^4a_4a_5{}^2=(a_2a_3)^2a_5=a_1a_3{}^{-1}a_1a_3{}^3=1,$$
$$a_2{}^2a_3a_2{}^{-1}a_3{}^{-2}a_2{}^{-1}a_3a_6{}^{-1}=1,\ [a_i,a_j]=1\ (1\le i\le 6,\ 4\le j\le 6,\ i<j)\rangle.$$

Now NQ can be used to determine the order of the 2-group C. The group C being a nilpotent group is of class 5 and order 2^{11}. Then $M(P_2) \cong \langle a_4,a_5,a_6\rangle \cong C_2 x C_4 x C_2$.

$n_2(G)=|G:N(P_2)| =3465$.

(ii) Sylow 3-subgroup order=9

A simple use of PERM gives the generators $x=(ab^2)^2$, $y=b(ab^2ab^{-1})^2(ab^2ab)^3$ for a Sylow 3-subgroup of G.

$n_3(G)=|G:N(P_3)|=6160$.

(iii) Sylow 5-subgroup order=5

$P_5=\langle[a,b]\rangle=C([a,b])$, $n_5(G)=|G:N(P_5)|=22176$.

(iv) Sylow 7-subgroup *order=7*

$P_7=\langle (ab)^2ab^2ab^{-1}\rangle=C((ab)^2ab^2ab^{-1})$, $n_7(G)=|G:N(P_7)|=21120$.

(v) Sylow 11-subgroup *order=11*

$P_{11}=\langle ab\rangle=C(ab)$, $n_{11}(G)=|G:N(P_{11})|=8064$.

II.Maximal subgroups

(i) structure: $M_{21}=PSL(3,4)$

$H_1=\langle a, b^t\rangle$ where $t=ab^2ab$, in 9.1 of [16].

(ii) structure: $2^4{:}A_6$

This is the normalizer in G of an elementary abelian group E ($\leq G$) of order 16. Returning to the Sylow 2-subgroup P_2 of G given in (I), after some experimenting, we find $E=\langle y^2z^2,[z,y],zy,y^2z^{-1}y\rangle$. Now TC verifies that $N_G(E)=\langle a,a^b,b^2ab^2\rangle$ with $|G:N(E)|=77$ from which we obtain the generators u=a and $v=b^2(ab)^2$ for a maximal subgroup H_2 of G of index 77. The pair (u,v) has type (2,5,8;4) and satisfies the presentation :

$$H=\langle u,v \mid u^2=v^5=(uv^2)^5=[u,v]^4=(uvuv^{-1}uv^2)^3=1\rangle.$$

Take $N=\langle (uv)^4,(vu)^4,v^{-1}(uv)^4v,v(vu)^4v^{-1}\rangle$, $M=\langle u,v^2uvuv^2\rangle$. Then $N \lhd H$, $N\cap M=(1)$ with $N\cong 2^4$, $M\cong A_6$ showing that $H\cong 2^4{:}A_6$.

We now determine the multiplier of H_2. To do this, we first see that

$$H_2\cong\langle u,v \mid u^2=v^5=(uv^2)^5=1,\ (uvuv^{-1})^4=(uvuv^{-1}uv^2)^3\rangle.$$

Then a covering group C for H_2 is found to have the following presentation :

$$\langle u,v \mid (uv^2)^5=v^5u^4,\ (uvuv^{-1})^4=(uvuv^{-1}uv^2)^3,\ [v,u^2]=[u,v^5]=1\rangle.$$

TC is now used to verify that $|C|=138240$. This gives $|M(H_2)|=24$. We have $M(H_2)\cong \langle u^2,v^5\rangle\cong C_2xC_{12}$. This, in turn, shows that H_2 is efficient.

(iii)-(iv) structure: A_7

The group A_7 has a generating pair of type (2,4,7;5). Using this fact we find, by PERM, the generating pairs (x_3,y_3) and (x_4,y_4) , where $x_3=x_4=ab^2a$, $y_3=b^{-1}ab^2abab^{-1}$, $y_4=b^{-1}abab^2ab^{-1}$, for two non-conjuagate A_7 subgroups H_3 and H_4 of G. The subgroup

H_3 has intersection $(S_3xS_4)+$ with H_4 in G, where (K)+ denotes the subgroup of even permutations in the permutation group K. That H_i (i=3,4) is maximal in G follows from the fact that A_7 has an element of order 6 while H_1 ($\cong$ PSL(3,4)) has no such element.

(v) structure: $2^4{:}S_5$

Similar to that in (ii) we search for an elementary abelian group E ($\leq$ G) of order 16 whose normalizer in G has index 231. Considering the Sylow 2-subgroup P_2 of G we find $E=\langle xy,y^2,(y^{-1}z)^2,z^4\rangle$ for which $N_G(E)=\langle r,s,t\rangle$, where $r=a^{ba}$, $s=(ab^2)^2$, $t=(ab^2ab)^4$, with $|G:N(E)|=231$. Now it is seen that $(r, s(tr)^2)$ is a generating pair of type (2,5,6;4) for N(E). Then PERM is able to produce the words $u=(b^2)^h$, $v=(babab^2)^k$, where $h=ab^{-1}a$, $k=abab^{-1}$, for a subgroup H_5 of index 231. A simple check using SUBGPTEST shows that H_5 is not isomorphic to any subgroup of H_2. That is, H_5 is a maximal subgroup of G.

The generators u, v satisfy the presentation

$$K=\langle u,v \mid u^2=v^5=(uv)^6=(uvuv^2)^3=([u,v][u,v^2])^2=1\rangle.$$

Put $N=\langle (uv^2)^4, (v^2u)^4,(vuv)^4,v^{-1}(uv^2)^4v\rangle$, $M=\langle v,uv^{-1}uvu\rangle$. Then $N \lhd K$, $N\cap M=(1)$ with $N \cong 2^4$, $M \cong S_5$. Therefore $K \cong 2^4{:}S_5$.

To calculate $M(H_5)$ we first see that

$$H_5 \cong \langle u,v \mid u^2=v^5=([u,v][u,v^2])^2=1,\ (uv)^6=(uvuv^2)^3\rangle.$$

Then a covering group for H_5 may be given by

$$C=\langle a_1,a_2,a_3,a_4 \mid a_1^2=a_2^5a_3^2=(a_1a_2)^6(a_1a_2a_1a_2^2)^{-3}a_3^{-1}=([a_1,a_2][a_1,a_2^2])^2a_4^{-1}=1$$
$$[a_i,a_j]=1\ (\ 1\leq i \leq 4,\ 3 \leq j \leq 4,\ i < j\ )\rangle.$$

C has order 15360 and so $|M(H_5)|=8$. Next it is easily verified that $M(H_5) \cong \langle a_3,a_4\rangle \cong C_4xC_2$. Thus, H_5 having a 2-generator 4-relation presentation is efficient.

(vi) structure: $2^3{:}PSL(2,7)$

This is the normalizer in G of an elementary abelian group E ($\leq$ G) of order 8. On examining the normalizer in G of randomly chosen subgroups of P_2 of the structure 2^3 we found $E=\langle x,y^2,zxz\rangle$ with $|G:N_G(E)|=330$. We have, in fact, $N(E)=\langle r,s\rangle$ where $r=a^b$, $s=(ab^2ab^{-1})^3(ab^{-1})^2$. That $(r,s(rs)^2)$ is a generating pair of type (2,4,7/6) for $\langle r,s\rangle$ can be used to give, by PERM, the generators $u=b^2$, $v=b^{aba}$ for a subgroup H_6 of

G of index 330. The maximality of H_6 now follows on noting that H_1 ($\cong$ PSL(3,4)) has no element of order 6 whereas uv^2 is an element of order 6 in H_6.

The pair (u,v) satisfies the presentation given for the maximal subgroup H_2 of A_8.

(vii) structure: M_{10}

As was remarked, M_{10} has a generating pair of type (2,4,8/5). Using this fact PERM finds x=a, $y=(ba)^2b^2(ab)^2ab^{-1}$ with x, y generating a subgroup H_7 of order 720 and satisfying the presentation for M_{10} given earlier. Now H_7 having an element of order 8 is not embeddable in H_1($\cong$PSL(3,4)). Also SUBGPTEST ensures that H_2 has no subgroup isomorphic to H_7. These facts prove the maximality of H_7.

(viii) structure: PSL(2,11)

The group PSL(2,11) is generated by two of its elements of order 2 and 3 whose product has order 11. PERM finds x=a, y= $b(ab^{-1})^2(ab)^3$ for a maximal subgroup H_8 isomorphic to PSL(2,11). The pair (x,y) then satisfies the presentation of PSL(2,11).

J_2

order=604800=$2^7.3^3.5^2.7$ d=100 mult=2

G=<a,b | $a^2=b^5=(ab)^6=[a,b]^5=(ab^2)^7=[a,b^2]^4=[a,bab^{-2}]^3=1$>

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	604800			1	1^{100}
2A	1920	A	A	a	$1^{20}2^{40}$
2B	240	A	A	$(ab)^3$	2^{50}
3A	1080	A	A	$(ab(ab^2)^2)^4$	$1^{10}3^{30}$
3B	36	A	A	$(ab)^2$	$1^4 3^{32}$
4A	96	A	A	$[a,b^2]$	$1^8 2^6 4^{20}$
5A	300	A	A	$[a,b]^2$	5^{20}
BB	300	A	A	$[a,b]$	5^{20}
5C	50	A	A	$(ab^2ab^{-1})^2$	5^{20}
D*	50	A	A	$(ab^2ab^{-1})^4$	5^{20}
6A	24	AA	AA	$(ab(ab^2)^2)^2$	$1^2 2^4 3^6 6^{12}$
6B	12	BB	BB	ab	$2^2 6^{16}$
7A	7	A	A	ab^2	$1^2 7^{14}$
8A	8	A	A	$(ab)^2ab^2ab^{-1}$	$1^2 2^3 4^3 8^{10}$
10A	20	BB	AB	$abab^2$	10^{10}
B*	20	AB	BB	$(abab^2)^3$	10^{10}
10C	10	DA	CA	$(ab^2ab^{-1})^3$	$5^4 10^8$
D*	10	CA	DA	ab^2ab^{-1}	$5^4 10^8$
12A	12	AA	AA	$ab(ab^2)^2$	$1^2 3^2 4^2 6^2 12^6$
15A	15	BA	AA	$(abab^2ab^{-2})^2$	$5^2 15^6$
B*	15	AA	BA	$abab^2ab^{-2}$	$5^2 15^6$

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	6048	100	PSU(3,3)		1
H_2	2160	280	3·PGL(2,9)	N(3A)	2
H_3	1920	315	2_-^{1+4}:A_5	N(2A)	2
H_4	1152	525	2^{2+4}:$(3xS_3)$	N($2A^2$)	2
H_5	720	840	A_4xA_5	N($2B^2$),N(2A,3B,5AB)	2x2
H_6	600	1008	A_5xD_{10}	N(5AB),N(2B,3A,5CD)	2
H_7	336	1800	PSL(2,7):2	N(2A,3B,4A,7A)	2
H_8	300	2016	5^2:D_{12}	N(5^2)=N($5AB_3CD_3$)	2
H_9	60	10080	A_5	N(2B,3B,5CD)	2

I. Sylow p-subgroups

(i) Sylow 2-subgroup order=128

Since G has PSU(3,3) as a subgroup, it contains a 2-subgroup isomorphic to the

Sylow 2-subgroup of PSU(3,3). Invoking the presentation of the Sylow 2-subgroup of PSU(3,3), we see that such a 2-subgroup has a generating pair of type (2,8,4;4). Now PERM simply finds $r=a$, $s=b^2ab(ab^{-1})^2ab$ with $<r,s>$ isomorphic to the Sylow 2-subgroup of PSU(3,3). Now TC can be used to extend $<r,s>$ to a Sylow 2-subgroup of G. We find $t=b^{-1}(ab)^3b$ of order 2 in $N(<r,s>)$ with $|<r,s,t>|=64$ and $q=b^{-1}(ab^{-2}ab)^2ab^{-1}abab^{-2}abab^{-1}$ in $N(<r,s,t>)$ with $|<r,s,t,q>|=128$. Removing the redundant generator s and conjugating by b^{-1} the elements r, t, q gives the generators $x=bab^{-1}$, $y=(ab)^3$, $z=(ab^{-2}ab)^2ab^{-1}(abab^{-2})^2$ for a Sylow 2-subgroup P_2 of G. A presentation on x, y, z for P_2 may now be given by

$$<x,y,z \mid x^2=y^2=z^2=(yz)^2=(xy)^3(zx)^3zy=(xy)^2xzxyzxzyxz=1>.$$

We now proceed to show that the presentation is actually minimal. We first observe that $P_2/\Phi(P_2)$ is an elementary abelian group of order 8 and so $d(P_2)=3$. Then a covering group of P_2 is found to have the presentation: $C=<a_1,a_2,a_3,a_4,a_5,a_6 \mid a_1{}^2=a_2{}^2=a_3{}^2a_4=(a_2a_3)^2=(a_1a_2)^3(a_3a_1)^3a_3a_2a_5{}^{-1}=1,$

$(a_1a_2)^2a_1a_3a_1a_2a_3a_1a_3a_2a_1a_3a_6{}^{-1}=1, [a_i,b_j]=1 \ (1 \leq i \leq 6, 4 \leq j \leq 6, i < j >$,

which is easily shown, by NQ, to have order 2^{10} with nilpotency class 4 . This helps us to obtain $M(P_2) \cong <a_4,a_5,a_6> \cong C_2xC_2xC_2$.

$n_2(G)=|G:N(P_2)| =1575.$

(ii) Sylow 3-subgroup *order=27*

The fact that G has PSU(3,3) as a subgroup of an index not divisible by 3 allows us to determine a Sylow 3-subgroup of G using the method described in 3.2. We begin with the following generators of a PSU(3,3) subgroup

$$A=a, \quad B=(b^{-2}ab^2a)^2bab \qquad (*)$$

with A, B satisfying the presentation 6.2 of [16], see (II) below. However, a Sylow 3-subgroup of PSU(3,3) can be generated by

$$r=AB^3, \quad s=(BA)^2B^{-1}(AB)^2 \qquad (**)$$

Substitution of (*) in (**) determines generators for a Sylow 3-subgroup of G which may be simplified by an application of PERMGP. In so doing, we find the generators $x=a^ta$, $y=aa^q$, where $t=b^{-2}abab^{-1}$, $q=b^{-1}(ab^{-2})^2$, for a Sylow 3-subgroup of G.

$n_3(G)=|G:N(P_3)|=2800.$

(iii) Sylow 5-subgroup *order=25*

Obviously the Sylow 5-subgroup of G is C_5xC_5 and thus has a generating pair of type (5,5,5;1). Using PERM, we obtain the generators $x=abab^{-1}$, $y=b^2ab^{-2}(ab)^2ab^{-2}ab$ for a Sylow 5-subgroup of G.

$n_5(G)=|G:N(P_5)|=2016.$

(iv) Sylow 7-subgroup *order=7*

$P_7=< ab^2 >=C(ab^2),$ $n_7(G)=|G:N(P_7)|=14400.$

II.Maximal subgroups

(i) structure: *PSU(3,3)*

$H_1=<a, (b^{-2}ab^2a)^2bab>$ in 6.2 of [16].

(ii) structure: *3·PGL(2,9)*

By table II, the above maximal subgroup of G is the normalizer in G of a cyclic subgroup <x> of G with $x\in 3A$. Taking $r=(ab(ab^2)^2)^4$, which is in 3A, we find, by TC, that $N_G(<r>)=<a,s>$, where $s=b^2abab^{-2}abab^2ab^{-2}ab^2ab^{-1}ab^{-2}ab^2ab$, with $|<a,s>|=2160$. A simple check now shows that (a,s) is of type (2A,10A,10A;4) which enables us to give the generators $x=a^t$, $y=b^2ab^{-1}ab$, where $t=b^{-1}a$, for a maximal subgroup H_2 of G using PERM. The pair (x,y) satisfies the presentation

$$H=<x,y \mid x^2=y^{10}=(xy^2)^5=(xyxy^2xy^{-2})^2=((y^3x)^2y^{-1}x)^2=1>.$$

Note. We note that y^5, xy^4 generate H which when written in terms of a , b gives a generating pair of type (2B,3B,10AB;4) for the same maximal subgroup of G. Using this, we may find the generators $x'=(ab)^3$, $y'=bab^2ab(ab^2)^3$ for a subgroup of G isomorphic to H_2. For a given minimal generating pair (a,b) for G, this is, sometimes, useful when the class 2B has elements (as words in a, b) of shorter lengh than those of 2A and we wish to give neater generators for the maximal subgroup of G of order 2160.

Take $N=<(xy^{-1}xy^3)^4>$. Then N ($\cong C_3$) is normal in H and

$$H/N=<x,y \mid x^2=y^{10}=(xy^2)^5=(xy^{-1}xy^3)^4=(xyxy^2xy^{-2})^2=((y^3x)^2y^{-1}x)^2=1>.$$

We shall now show that the factor H/N of order 720 is, in fact, isomorphic to PGL(2,9) which proves that H is an extension of C_3 by PGL(2,9). A simple calculation with CAYLEY shows that

$$g=\begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix}, \qquad h=\begin{bmatrix} \omega^4 & 1 \\ \omega^4 & 0 \end{bmatrix}$$

(where ω is a primitive element of $GF(3^2)$) generate GL(2,9) and that $Z(<g,h>)=<k>$, where

$$k=\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}.$$

Now $\langle g,h\rangle/\langle k\rangle$ is PGL(2,9) and has the following presentation :

$$\langle t,q \mid q^3=[t,q]^4=(t^2q^{-1}tq^{-1})^2=1, [q,t][q,t^2]=t^{-2}\rangle.$$

Next, the mapping $x\rightarrow[t,q]^2$, $y\rightarrow qt^3$ yields an isomorphism between H/N and PGL(2,9). At this stage, SUBGPTEST can be used to show that G has no PGL(2,9) subgroups. Therefore, H_2 cannot split over C_3, that is, $H_2 \cong 3\cdot PGL(2,9)$.

To compute $M(H_2)$, we first reduce the above presentation of H_2 to the following 2-generator 4-relation presentation

$$H_2 \cong \langle x,y \mid x^2=(xyxy^2xy^{-2})^2=((y^3x)^2y^{-1}x)^2=1, y^{10}=(xy^2)^5\rangle.$$

Then a covering group using this presentation can be constructed as follows

$$C=\langle x,y,z,u \mid x^2u=(xyxy^2xy^{-2})^2u^3=((y^3x)^2y^{-1}x)^2z^{-1}u^3=1, y^{10}(xy^2)^{-5}u^{-2}=1,$$
$$[x,z]=[x,u]=[y,z]=[y,u]=[z,u]=1\rangle.$$

C has order 4320 and so $M(H_2)=C_2$.

We failed to reduce the above presentation for H_2 to a 2-generator 3-relation presentation and therefore resorted to the method described in 3.5 which enabled us to construct the following deficiency -1 presentation for H_2

$$\langle X,Y \mid X^2=(XY)^8=XY[X,Y]XY^4=1\rangle,$$

where $X=y^5$, $Y=(xy)^2xy^4x$ with x, y generators of H_2 given above.

We finally note that by combining the first and the second relation of the presentation we may obtain a deficiency zero presentation for a covering group of H_2.

(iii) structure: $2_-^{1+4}:A_5$

This is the normalizer in G of a 2-subgroup P ($\leq$ G) of the strcture 2_-^{1+4}. Considering the Sylow 2-subgroup P_2 of G given in (I), we find $P=\langle r,s,t,q\rangle$, where

r=xyzxzxy, s=xyxyzxz, t=yxzxyxz, q=xzxyxyz,

with $N_G(P)=\langle(ab)^3,b^2ab^{-1}ab^2(ab)^2a\rangle$ of order 1920. (We note that P is the only subgroup of P_2 of order 32 with the property that $|N(P)|=1920$.) Put $X=(ab)^3$ and $Y=b^2ab^{-1}ab^2(ab)^2a$. Then (X,Y) is a generating pair of type (2B,5D,5C/6A) for a maximal subgroup of H_3 of order 1920 because $|H_3| \nmid |H_i|$ (i=1,2); and satisfies the presentation :

$$\langle X,Y \mid X^2=Y^5=(XY)^5=(XY^2)^6=([X,Y][X,Y^{-2}])^2=1\rangle.$$

It is important to note that X, Y^2 generate H_3 and that the pair (X,Y^2) has type

(2B,5CD,6A;10CD). Using this fact we may find, by PERM, the words $u=(ab)^3$ and $v=(b^2)^k$, where $k=ab^{-1}ab^2$, for a maximal subgroup of G isomorphic to H_3. A new presentation for H_3 using these generators is now found to have four relations rather than five relations as obtained earlier :

$$<u,v \mid u^2=v^5=(uv^2)^5=((uv)^2uv^{-1}uv^2)^2=1> \quad (\dagger)$$

This presentation will ease the computing of the multiplier of H_3 .We are now going to show that H_3 is a split extension of 2_-^{1+4} by A_5. Let $N=<c,d,e,f>$, where $c=(uv)^3$, $d=(vu)^3$, $e=v^{-1}(uv)^3v$, $f=v(vu)^3v^{-1}$. Then N is a normal subgroup of $<u,v>$ of order 32 which has trivial intersection with $M=<(uv)^2, uv^2>$. An easy check shows that $M \cong A_5$ and thus we have $H_3 \cong N{:}A_5$. To show $N \cong 2_-^{1+4}$, we first construct the following presentation for N on c, d, e, f

$$<c,d,e,f \mid c^2=d^2=e^2=f^2=cdcede=cdcfdf=cdeced=cdfcfd=cecfef=1>.$$

Then $N_1=<def,cdfe>$ and $N_2=<fe,fd>$ are each normal in N with $[N_1,N_2]=1$, $N_1 \cap N_2=Z(N)=<(cd)^2> \cong C_2$, $N_1 \cong D_8$, $N_2 \cong Q_8$. Therefore, N is a central product of N_1 and N_2, i.e. $N= D_8 \circ Q_8$ ($= 2_-^{1+4}$).

It may be worth remarking that there are only two perfect groups G_1 and G_2 having normal subgroups of type $D_8 \circ Q_8$ with factor groups A_5 , see [40] . In fact $G_1 \cong (D_8 \circ Q_8){:}A_5$, $G_2 \cong (D_8 \circ Q_8) \cdot A_5$ (the normal subgroup has no complement but supplement). The group H_3 being perfect of the structure $(D_8 \circ Q_8){:}A_5$ is clearly isomorphic to G_1. We shall show here that H_3 is efficient.

To prove H_3 efficient we combine the relations of (†) as follows :

$$<u,v \mid u^2=1,\ (uv^2)^5=v^5,\ ((uv)^2uv^{-1}uv^2)^2=v^5 >.$$

The group obtained is now easily shown to be isomophic to H_3 using TC. Therefore, $M(H_3)$ is cyclic, by 1.5.8 (ii). But, $M(H_3)$ has $M(A_5)$ as a direct factor (see the note made in 1.6). This shows that rank $M(H_3)=1$. So H_3 is efficient. By constructing the covering group C of H_3 we may see that $M(H_3)$ is exactly $M(A_5)$. We have that

$$C= <u,v \mid (uv^2)^5=u^4v^5,\ ((uv)^2uv^{-1}uv^2)^2=u^8v^5,\ [v,u^2]=1>.$$

$|C|=3840$ giving $M(H_3)=C_2$ as required.

(iv) structure: $2^{2+4}{:}(3xS_3)$

This is the normalizer in G of a subgroup P of structure 2^{2+4}. Again considering the Sylow 2-subgroup P_2 of G we find $P=<xyxyz,xyzxy,xyxz,yxzx>$. Substituting bab^{-1}, $(ab)^3$, $(ab^{-2}ab)^2ab^{-1}(abab^{-2})^2$ in P for x,y, z and using TC, we find $N(P)=<r,s,t>$

where $r=bab^{-1}$, $s=b^2ab^{-1}ab^{-2}(ab^{-1})^2abab^{-1}(ab)^2$, $t=bab^{-2}(ab^{-2}ab^2)^2abab^{-1}ab^2a$, with $|N(P)|=1152$. It is easily checked that (rst, tr) is a generating pair of type (2A,6A,6B;3B) for N(P) which allows PERM to produce the generators $u=a$, $v=b^2ab^{-1}ab^2ab^{-2}ab^2$ of type (2A,6B,6A;3B) for a maximal subgroup H_4 of G of order 1152.The pair (u,v) satisfies the presentation

$$K=\langle u,v \mid u^2=v^6=(uv)^6=[u,v]^3=(uvuv^2)^3=[uvu,v^{-2}]^2=1\rangle.$$

Let $N=\langle v^3,uv^3u,(v^{-1}u)^2(vu)^2,uv^{-1}uv^3uvu\rangle$, $M=\langle u,v^2uv^3uv^2\rangle$. Then $N\lhd K$, $N\cap M=(1)$.

So $K \cong N{:}M$. In what follows we show that $N \cong 2^{2+4}$ and $M \cong 3xS_3$. Take $r=v^3$, $s=uv^3u$, $t=(v^{-1}u)^2(vu)^2$, $q=uv^{-1}uv^3uvu$. Then a presentation for N on r, s, t, q is

$$\langle r,s,t,q \mid r^2=s^2=t^4=q^2=(rq)^2=(rs)^2t^2=rstrst^{-1}=rtrqtq=stqsqt=1\rangle.$$

Now $\langle t^2,(st)^2\rangle$ of the structure C_2xC_2 is normal in $\langle r,s,t,q\rangle$ and the factor $\langle r,s,t,q\rangle / \langle t^2,(st)^2\rangle$ is a 2-group of order 16 having elements of order 2 only. Therefore, $N \cong 2^{2+4}$. Next, on taking $w=v^2uv^3uv^2$ we find the presentation $\langle u,w \mid u^2=w^3=(uw)^2(uw^{-1})^2=1\rangle$ for M on the generators u, w which is easily shown to be the direct product of $\langle u,u^w\rangle$ ($\cong S_3$) and $\langle (uw)^2\rangle$ ($\cong C_3$) as required.

A covering group for H_4 using the above presentation of K may be given by

$$C=\langle a_1,a_2,a_3,a_4,a_5,a_6 \mid a_1^2=a_2^6a_3a_5=(a_1a_2)^6a_5=(a_1a_2a_1a_2^{-1})^3a_4^{-1}=(a_1a_2a_1a_2^2)^3a_5 =[a_1a_2a_1,a_2^{-2}]^2a_6^{-1}=1,\ [a_i,a_j]=1\ (1 \le i \le 6,\ 3 \le j \le 6,\ i < j\ )\rangle.$$

$|C|=2304$ and so $M(H_4)=C_2$.

To prove H_4 efficient we look for a deficiency -1 presentation. Our reduction method fails here to reduce the presentation of K to such a presentation. We therefore apply the method described in 3.5. In this way we may arrive at the generators $X=uvuv^3$, $Y=v^3u$ for H_4 on which a deficiency -1 presentation is obtained as follows:

$$\langle X,Y \mid (XY^2)^3=1,\ X^3Y^3=YX^3,\ X^2YXYX^4=YXY\rangle.$$

(v) structure: A_4xA_5

By table II, this is the normalizer in G of an A_5 subgroup with a generating pair (r,s) of type (2A,3B,5AB). Using PERM, we find $r=a$, $s=ab(ab^2)^2ab^{-2}ab^2$. Now TC gives $N_G(\langle r,s\rangle)=\langle t,q\rangle$, where $t=b^{-1}ab^2ab^{-2}(ab^{-1})^2(ab^2)^2ab^{-2}$, $q=(ab^2)^2ab^{-1}ab(ab^{-2})^2ab^{-1}ab^2ab$, with $|\langle t,q\rangle|=720$. Now an easy check shows that $(tqt^{-1}q^{-2}, t)$ is a generating pair of type (2B,3B,15AB,10AB) for $\langle t,q\rangle$. Using this fact PERM gives the generators $x=(ab)^3$, $y=b^2ab^{-2}ab^2(ab^{-2})^2ab^2$ for a subgroup H_5 of G of order 720. We must show that H_5 is a maximal subgroup of G of structure A_4xA_5. To see this, we first construct the following presentation for H_5 on x, y :

$$<x, y \mid x^2=y^3=((xy)^2xy^{-1})^3=((xy)^3(xy^{-1})^3)^3=1>.$$

Then $<x,y>$ is the direct product of $K_1=<[x,y]^2,[x,y^{-1}]^2>$ and $K_2=<(xy)^5,(yx)^5>$ with $K_1 \cong A_5$ and $K_2 \cong A_4$. Now a simple computation shows that the Sylow 2-subgroup of H_2 is D_{16} while that of H_5 is $C_2xC_2xC_2xC_2$ proving that H_5 is not embeddable in H_2. So H_5 is maximal.

$M(A_4xA_5)=C_2xC_2$, by 1.5.12, and the above presentation having deficiency -2 proves that A_4xA_5 is efficient.

(vi) structure: A_5xD_{10}

This maximal subgroup is the normalizer in G of an A_5 subgroup with a generating pair of type (2B,3A,5CD). Now PERM is able to produce the words $z=(ab)^3$, $u=(ab(ab^2)^2)^4$ with (z,u) of type (2B,3A,5D). Using TC we get $N_G(<z,u>)=<z,v,w>$, where $v=(ab^{-1})^2(ab)^2(ab^2)^2ab^{-2}$ and $w=b^{-2}ab^2(abab^{-1})^2ab^{-2}ab^2$, with $|<z,v,w>|=600$. The elements zw, zv^{-1} generate $<z,v,w>$ with (zw,zv^{-1}) of type (2B,5AB,6A;5CD) which allows PERM to give the generating pair (x,y), $x=(ab)^3$, $y=b^2(ab^{-1})^2(ab^2)^2$, of the same type for a subgroup H_6 of order 600. H_6 is, in fact, maximal in G since $|H_6|$ does not divide $|H_i|$ (i=1,2,3,4,5).

A presentation for H_6 on the generators x, y is

$$<x, y \mid x^2=y^5=(xy)^6=(xy^2xyxy^{-2})^2=1>.$$

Now $<x,y>$ is the direct product of $L_1=<(xy)^2,(yx)^2>$ and $L_2=<(xy)^3,(yx)^3>$ with $L_1 \cong A_5$ and $L_2 \cong D_{10}$.

$M(A_5xD_{10})=C_2$, by 1.5.12, and a deficiency -1 presentation for H_6 can be obtained from the above presentation :

$$<x,y \mid x^2y^5=(xy^2xyxy^{-2})^2=1,\ (xy)^6=y^5>.$$

(vii) structure: *PSL(2,7):2*

This is the normalizer in G of a PSL(2,7) subgroup with a generating pair (u,v) of type (2A,3B,7A;4A) by table II. Now PERM gives u=a and $v=(ab)^2$ with $<u,v> \cong$ PSL(2,7). We then have $N_G(<u,v>)=<u,z>$, where $z=b^{-2}abab^{-2}abab^{-1}abab^{-2}abab^2$, with $|<u,z>|=336$. An easy check shows that $(u, (zu)^2z^3)$ has type (2A,6B,6B;4A) which helps us to obtain, by PERM, the generators $x=a^{ba}$, $y=bab^{-2}$ for a subgroup H_7 of order 336. The maximality of H_7 now follows on noting that H_1 ($\cong$PSU(3,3)) cannot have a subgroup of order 336 (see maximal subgroups of PSU(3,3)).

The pair (x,y) satisfies the presentation

$$<x,y \mid x^2=y^6=(xy)^6=(xyxy^{-2}xy^2)^2=((xy)^2xy^{-2})^3=1>.$$

We now let $N=<x^y,(yx)^2>$. Then N ($\cong$ PSL(2,7)) has index 2 in $<x,y>$ and trivial intersection with $<y^3>$, i.e. $<x,y>$ is an split extension of PSL(2,7) by C_2.

A covering group C for H_7 may be presented as

$$<a_1,a_2,a_3,a_4,a_5,a_6 \mid a_1^2a_5=a_2^6a_3a_4=(a_1a_2)^6a_4a_5^3=(a_1a_2a_1a_2^{-2}a_1a_2^2)^2a_5^3=1,$$
$$((a_1a_2)^2a_1a_2^{-2})^3a_5^4=1, [a_i,a_j]=1 \ (1 \le i \le 6, 3 \le j \le 6, i<j)>.$$

C has order 672 and so $M(H_7)=C_2$.

Using the method described in 3.5, we constructed the following deficiency -1 presentation for H_7 on the generators $X=y^3$, $Y=(xy)^2yx$:

$$<X, Y \mid X^2Y^8=(XYXY^{-1})^2=(XY^3)^3=1>.$$

(viii) structure: $5^2{:}D_{12}$

By table II, this is the normalizer in G of a Sylow 5-subgroup. Taking P_5 given in (I), we find $N(P_5)=<a,z>$, where $z=b^{-2}ab^{-2}(ab^{-1}ab^{-2}ab^2)^2ab^2ab^{-1}(ab^{-2})^2ab^2(ab)^2ab^{-1}ab^2$, That the pair (a,z) has type (2A,10AB,6B;3B) can be used to obtain, by PERM, the generators $x=a$, $y=b^2ab^{-1}ab^{-2}ab^2ab$ for a subgroup H_8 of G of order 300. We now prove that this subgroup is, in fact , maximal in G by showing that H_6 of order 600 has no subgroups isomorphic to H_8. To see this, we calculate the normal subgroup lattice of H_6 using CAYLEY. It is found that H_6 has a unique normal subgroup of order 300 with presentation $<u,v \mid u^5=(uv)^5=uvu^{-1}vuv^{-1}=1>$. This group has no element of order 6 while in H_8 the element xy is of order 6.

A presentation for H_8 on x, y may be given by

$$<x,y \mid x^2=y^{10}=(xy)^6=((xy)^2xy^{-1})^2=(xy^2xy^{-1})^2=1>.$$

Now it is easy to verify that $<x,y>$ is a split extension of $<y^2,xy^2x>$ ($\cong 5^2$) and $<xy,yxy^{-1}>$ ($\cong D_{12}$).

To get at $M(H_8)$ we construct the following presentation for a covering group C of G obtained from the above presentation for H_8 :

$$<a_1,a_2,a_3,a_4,a_5 \mid a_1^2=a_2^{10}a_3a_4^5=((a_1a_2)^2a_1a_2^{-1})^2a_4=(a_1a_2^2a_1a_2^{-1})^2=(a_1a_2)^6a_5^{-1}=1,$$
$$[a_i,a_j]=1 \ (1 \le i \le 5, \ 3 \le j \le 5, i<j)>.$$

Then $|C|=600$, so $M(H_8)=C_2$. An efficient presentation for H_8 may now be given by

$$<x,y \mid x^2y^{10}=((xy)^2xy^{-1})^2=1, (xy^2xy^{-1})^2=(xy)^6>.$$

(ix) structure: A_5

By table II, this is an A_5 subgroup with a generating pair of type (2B,3B,5CD). Using PERM we obtain the generators $x=(ab)^3$, $y=(b^2a)^2b^{-1}ab^2ab^{-2}$ for a subgroup H_9 ($\cong A_5$) with (x,y) of the above type which is maximal in G according to the following comments:

(1) H_8 of order 300 has no A_5 subgroups because $|H_8'|=75$ is not divisible by $|A_5'|=60$.

(2) H_6 and H_9 have each a single conjugacy class of elements of order 3 and that H_6 contains elements of order 3 from the class 3A only while H_9 contains such elements from 3B.

(3) H_5 has two conjugacy classes of elements of order 5 and these elements are all in the class 5AB whereas elements of order 5 in H_8 are all in 5CD.

(4) H_3 contains elements of order 3 from 3A only (one class) while such elements in H_9 are all in 3B.

(5) H_2 contains elements of order 5 from 5AB only (two classes) while such elements in H_9 are all in 5CD.

From (1) it follows that H_9 is not embeddable in H_8 and each of (2), (3), (4), (5) shows that none of H_i (i=2,3,5,6,8) have an A_5 subgroup conjugate to H_9. These prove that H_9 is maximal.

Note. As we saw in (v), (vi), and (ix), G had three non-conjugate A_5 subgroups with generating pairs of type (2A,3B,5AB), (2B,3A,5CD), (2B,3B,5CD). In fact there are exactly three non-conjugate A_5 subgroups in G (see the proposition 2.3 of [20]). If R, S, T are representatives of these classes then $N(R) \cong H_5 \cong A_5 x A_4$, $N(S) \cong H_6 \cong A_5 x D_{10}$, $N(T) \cong H_9 \cong A_5$. We also note that by information given in [33] about the intersections of maximal subgroups H_2, H_3 contain A_5 subgroups of type R and S respectively.

PSP(4,4)

order=979200=$2^8.3^2.5^2.17$ d=85 mult=1

G=< a,b | $a^2=b^5=[a,b]^5=(ab^2)^{17}=[a,b^2]^2=((ab)^2ab^{-2})^4=1$>

conjugacy classes of elements of G

class	|c(x)|	p-power	p'-part	representative	cycle type
1A	979200			1	1^{85}
2A	3840	A	A	a	$1^{21}2^{32}$
2B	3840	A	A	$((ab)^4ab^{-1})^3$	1^52^{40}
2C	256	A	A	$[a,b^2]$	1^52^{40}
3A	180	A	A	$((ab^{-1}ab)^2b)^2$	1^73^{26}
3B	180	A	A	$(ab)^5$	$1^{10}3^{25}$
4A	32	C	A	$(ab)^3(ab^2)^2$	$1^12^24^{20}$
4B	32	C	A	$(ab)^2ab^{-2}$	$1^12^24^{20}$
5A	300	A	A	[a,b]	1^55^{16}
B*	300	A	A	$[a,b]^2$	1^55^{16}
5C	300	A	A	$(ab)^3$	5^{17}
D*	300	A	A	$(ab)^6$	5^{17}
5E	25	A	A	b	5^{17}
6A	12	AA	AA	$(ab^{-1}ab)^2b$	$1^32^23^66^{10}$
6B	12	BB	BB	$(ab)^4ab^{-1}$	$1^22^43^16^{12}$
10A	20	BA	AA	$ab(ab^2)^2$	$1^12^25^410^6$
B*	20	AA	BA	$(ab(ab^2)^2)^3$	$1^12^25^410^6$
10C	20	DB	CB	$(ab)^2(ab^2)^2$	5^110^8
D*	20	CB	DB	$((ab)^2(ab^2)^2)^3$	5^110^8
15A	15	BA	AA	$((ab)^3bab^{-1})^2$	$1^23^15^115^5$
B*	15	AA	BA	$(ab)^3bab^{-1}$	$1^23^15^115^5$
15C	15	DB	CB	$(ab)^2$	5^215^5
D*	15	CB	DB	ab	5^215^5
17A	17	A	A	ab^2	17^5
B*2	17	A	A	$(ab^2)^2$	17^5
C*3	17	A	A	$(ab^2)^3$	17^5
D*6	17	A	A	$(ab^2)^6$	17^5

conjugacy classes of maximal subgroups of G

group	order	index	structure	specification	mult
H_1	11520	85	$2^6:(3xA_5)$	$N(2A^2),N(2C^4)$	2
H_2	11520	85	$2^6:(3xA_5)$	$N(2B^2),N(2C^4)$	2
H_3	8160	120	SL(2,16):2	N(2C,3A,5AB,...)	1
H_4	8160	120	SL(2,16):2	N(2C,3B,5CD,...)	1
H_5	7200	136	$(A_5xA_5):2$	$N(2B,3B,5CD)^2$	2
H_6	7200	136	$(A_5xA_5):2$	$N(2A,3A,5AB)^2$	2
H_7	20	1360	S_6	N(2C,3A,3B,4B,5E)	2

I. Sylow p-subgroups

(i) Sylow 2-subgroup order=256

By table II, G has a subgroup of minimal index 85 which contains a Sylow 2-subgroup of G. Such a subgroup can be generated by

$$r=a, \quad s=(ab)^3ab^{-1}ab \qquad (*)$$

(see II below). By constructing the following presentation for $<r,s>$ on r, s we shall give generators for a Sylow 2-subgroup of G using the method described in 3.2.

$$<r,s \mid r^2=s^6=(rs)^{15}=[r,s]^5=(rs^3)^4=(rs^{-1}rsrs^3)^2=(rs^{-1}rs^2rs)^3=1>.$$

Now an easy computation using TC shows that

$$srs^{-1}, \quad s^3, \quad (s^3)^r, \quad r^{[s,r]} \qquad (**)$$

generate a Sylow 2-subgroup of $<r,s>$. Substituting (*) into (**) gives the generators $x=a^t$, $y=((ab)^3ab^{-1}ab)^3$, $z=((ba)^3b^{-1}aba)^3$, $u=a^q$, where $t=b^{-1}ab(ab^{-1})^3a$, $q=[(ba)^3b^{-1}ab,a]$, for a Sylow 2-subgroup P_2 of G.

A presentation for P_2 on x, y, z, u may be given by

$$<x,y,z,u \mid x^2=y^2=z^2=u^2=(xu)^2=(yz)^2=(xy)^4=(xz)^4=(yu)^4=(zu)^4=(xyxz)^2=(xzuz)^2$$
$$=(yuzu)^2=(xyuy)^2=1>.$$

We now prove that the above presentation is minimal by showing that $d(P_2)=4$ and $\text{rank}M(P_2)=10$. First a simple calculation with CAYLEY verifies that the factor $P_2/\Phi(P_2)$ is an elementary abelian group of order 16 and thus $d(P_2)=4$, by 1.5.17. Next a covering group C for P_2 is found to have a presentation on fourteen generators 1,2,3...,14 –chosen for simplicity– with 99 relators :

$1^2\,9\,11$, $2^2\,8\,13$, $3^2\,6$, $4^2\,5$, $(1\,2)^2\,9\,11$, $(2\,3)^2\,8\,13$, $(1\,2)^4\,7^{-1}\,8^2\,9^2\,11^2\,13^2$, $(1\,2\,1\,3)^2\,9^2\,11^2\,13$, $(1\,2\,4\,2)^2\,11\,13^2$, $(1\,3)^4\,10^{-1}\,11^2$, $(1\,3\,4\,3)^2$, $(2\,4)^4\,12^{-1}\,13^2$, $(2\,4\,3\,4)^2$, $(3\,4))^4\,14^{-1}$, $[i,j]$ $(1 \le i \le 14, 5 \le j \le 14, i < j)$.

NQ verifies that C is of class 3 and order 2^{18}. Hence $|M(P_2)|=2^{10}$. It is not hard now to check that

$$C'=<[1,2], [1,3], [1,4], [2,3], [2,4], [3,4], [1,2\,3], [14,2\,3], [1\,2,3\,4], [1\,3,2\,4]>,$$

and that the subgroup H generated by the elements 5,6,7...,14 is contained in $Z(C)\cap C'$. Since $|C/H|=2^{10}$, $M(P_2) \cong H$. The elements 5,6,7,...,14 are each of order two and thus $M(P_2)$ is the the direct product of ten copies of C_2 as required.

$n_2(G)=|G:P_2|=425.$

(ii) Sylow 3-subgroup order=9

In order to find generators for a Sylow 3-subgroup of G we exploit the fact that G has a subgroup of order 180 containing a Sylow 3-subgroup of G by table I. This subgroup is in fact the centralizer in G of an element belonging to the class 3A (3B). A similar method to that used for the Sylow 2-subgroup of M_{22} works here to determine generators for $C((ab)^5)$. Having found generators for $C((ab)^5)$ (as permutations) we are able to show that it contains a subgroup of order 36 with a generating pair of type (3,3,3 ; 2). Using this fact PERM gives the generators $u=(ab)^5$, $v=b(ab^{-1})^2(ab)^3ab^{-2}$ for a subgroup of order 36 with presentation

$$K=\langle u,v \mid u^3=v^3=(uv)^3=[u,v]^2=1\rangle.$$

Now $uv^{-1}u$, v generate a Sylow 3-subroup of K. Substituting $(ab)^5$, $b(ab^{-1})^2(ab)^3ab^{-2}$ for u, v in these generators leads us to the generators $x=b(ab^{-1})^2(ab)^3ab^{-2}$, $y=(ab)^4ab^{-2}(ab^{-1})^3(ab)^2ab^{-1}(ab)^5$ for a Sylow 3-subgroup P_3 of G.

$n_3(G)=|G:P_3|=13600.$

(iii) Sylow 5-subgroup order=25

By table I, $C((ab)^3)$ is a subgroup of G of order 300 which contains a Sylow 5-subgroup of G. Similar to that in (ii) we are able to find the following presentation for $C((ab)^3)$:

$$L=\langle u,v \mid u^2=v^5=(uv^2)^5=(uv)^3(uv^{-1})^3=1\rangle.$$

In L the element v has order 15 and thus a subgroup of G isomorphic to L has a generating pair of type (2,5,15/5). PERM now gives the generators

$$u=a, \quad v=b(ab^{-1})^2ab^{-2} \qquad (*)$$

for a subgroup of order 300 with u, v satisfying the above presentation. However, a Sylow 5-subgroup of L can be generated by

$$uvu, \quad vuv. \qquad (**)$$

Substituting (*) in (**) gives the generators $x=ab^2(ab)^2ab^{-1}a$, $y=b^2(ab)^2ab^{-1}ab^2(ab)^2ab^{-1}$ for a Sylow 5-subgroup P_5 of G.

$n_5(G)=|G:N(P_5)|=4896.$

(iv) Sylow 17-subgroup order=17

$P_{17}=\langle ab^2\rangle=C(ab^2)$, $n_{17}(G)=|G:P_{17}|=14400.$

II.Maximal subgroups

(i) -(ii) structure: $2^6{:}(3xA_5)$

In [6] the generators $u=b(ba)^3b^{-1}a$, $v=(ab^{-1}ab)^2b$ are given for the stabilizer of a point in the permutation representation of G with (u,v) of type (15,6,6;5). We use these generators to show that $\langle u,v\rangle$ can be generated by two of its elements x, y with x an involution either in 2A or 2C. This would be particularly useful in obtaining generators for a Sylow 2-subgroup of G using a subgroup isomorphic to <u,v>. To achieve this we take $r=v^3$, $s=uv$. Then r, s generate <u,v> with (r,s) of type (2A,6B,15AB;5AB). Now PERM simply gives r=a, $s=(ab)^3ab^{-1}ab$ for the stabilizer of a point in the permutation representation of G. After some experimenting we found that $r'=uvuv^2uv^3$, s'=uv generate <u,v> and that (r',s') has type (2C,6B,15AB). Starting with $x_1=((ab)^2ab^{-2})^2$, which is in 2C, PERM finds $y_1=(ab^{-1}ab)^2bab^{-1}ab^2ab^{-1}$ in 6B, with $x_1y_1\in$ 15B, $|fix(x_1)\cap fix(y_1)|=1$, and x_1, y_1 generating a subgroup H_1 of minimal index 85.

We now take $x_2=x_1$ and search for a generating pair (x_2,y_2) of type (2,6,15) with y_2 moving the points which are left fixed by x_2 such that $|G:\langle x_2,y_2\rangle|=85$. PERM gives $y_2=(ab)^5(ab^2)^2$. Here (x_2,y_2) is of type (2C,6A,15CD). Clearly H_1 and H_2 are not conjugate in G.We note that the group H_i has four conjugacy classes of elements of order 15 and that H_1 contains elements of order 15 from 15AB only while H_2 contains such elements from 15CD. This also proves that these subgroups remain non-conjugate in G.

Furthermore, H_1 has $2^4.3^2$ intersection with H_2.

The pairs (x_1,y_1) and (x_2,y_2) satisfy the presentations

$$\langle x,y \mid x^2=y^6=(xy)^{15}=(xy^3)^4=(xyxy^{-1}xy^3)^2=(xy^{-1}xyxy^2)^3=((xy)^3xy^3)^5=(xy^{-1}xy^3xy)^2=((xy^2)^2xy^{-2})^3=1\rangle,$$

$$\langle x, y \mid x^2=y^6=(xy^3)^4=(xyxy^3xy^{-1})^2=(xy^2xy^{-2}xy^3)^2=(xyxy^2)^5=(xy^2xyxy^{-1})^3=1\rangle$$

respectively. These two presentations are related by the change of $(x,y)\leftrightarrow(x,y^3xy^{-1}x)$.

We now consider the latter presentation and try to show that H_i is a split extension of 2^6 by $3xA_5$. Let $N=\langle y^3,(y^3)^x,(y^3)^{xy},(y^3)^c,(y^3)^{xyx},(y^3)^d\rangle$, where $c=xy^{-1}$, $d=xyxy^{-1}$, and $M=\langle xy, yxy^{-1}xy^3\rangle$. Then N is an elementary abelian group of order 64 and normal in <x,y> with $N\cap M=(1)$, i.e. $\langle x,y\rangle \cong 2^6{:}M$. Next, it is found that M has a presentation $\langle g,h \mid (gh^{-1})^3=h^5=1, hg^2h=ghg\rangle$ on its generators h=xy, $g=yxy^{-1}xy^3$ which is easily shown to be the direct product of $\langle h,h^g\rangle$ ($\cong A_5$) and $\langle gh^{-1}\rangle$ ($\cong C_3$).

Note. Taking $u_1=x_1y_1{}^2x_1y_1{}^3x_1y_1{}^{-2}$, $v_1=[x_1,y_1{}^{-2}][x_1,y_1{}^{-1}][x_1,y_1{}^{-2}]$ gives a subgroup of the structure C_2xC_2 whose involutions lie in 2A. Now a calculation with CAYLEY using the permutation representation of G shows that $S=N_G(\langle u_1,v_1\rangle)$ is the stabilizer of a point in the permutation representation of G and thus S is conjugate to H_1. A similar technique can be used to show that H_2 is conjugate to the normalizer in G of a C_2xC_2 subgroup whose involutions are in 2B (see table II for the specifications of H_1 and H_2).

We shall show that $M(H_i)=C_2$ (i=1,2). To see this we first proceed to find a new presentation for H_i because the above presentations give unpleasant ones for the covering groups of H_i. In an attempt to obtain an appropriate presentation for H_i we arrived at the generators $x=b^{-2}abab^{-1}ab^2a$, $y=(ab)^5$ of a subgroup of minimal index 85 (the stabilizer of a point) having a presentation on x, y with 5 relations :

$$\langle x,y \mid x^5=y^3=(xy)^6=(yx^{-1}yxyx^{-2})^2=x^2y(xy^{-1})^3xyxy^{-1}x^{-1}y^{-1}=1\rangle.$$

Now using this presentaion a covering group C can be given by

$$\langle x,y \mid y^3=x^5(xy)^{-6}=[y,x^5]=[x,r_1]=[y,r_1]=[x,r_2]=[y,r_2]=[r_1,r_2]=1\rangle$$

where $r_1=(yx^{-1}yxyx^{-2})^2$, $r_2=x^2y(xy^{-1})^3xyxy^{-1}x^{-1}y^{-1}$. A coset enumeration using TC verifies that $|C:\langle y\rangle|=7680$ and thus $M(H_i)=C_2$. Finally using the method described in 3.5 we were able to construct the following deficiency -1 presentation for H_i :

$$\langle X,Y \mid X^3Y^5=X^2YXY(X^{-1}Y)^2(XY^{-1})^2=(Y^{-1}XYXY^{-1}X^{-1}YX^{-1})^2X^3=1\rangle.$$

(We note that $X=y_1{}^2$, $Y=(x_1y_1{}^{-1})^3$ generate H_1 and satisfy the the above presentation.)

(iii)-(iv) structure: *PSL(2,16):2*

G has two conjugacy classes of maximal subgroups of structures PSL(2,16):2. If R, S are representatives of these classes then $R=N_G(K)$, $S=N_G(L)$ where K, L are two non-conjugate PSL(2,16) subgroups of G. The group PSL(2,16) has a single class of both involutions and elements of order 3 with a generating pair of type (2,3,15;17). Our second table indicates that K and L are distinguished by their elements of order 3 which are in 3A and 3B respectively (their involutions are in 2C). Using this fact we find , by PERM, the generators $x=ab^2ab^{-2}$, $y=b^{-2}abab^{-1}ab^2(ab^{-1}ab)^2bab$ for K and $u=((ab)^2ab^{-2})^2$, $v=(ba)^5$ for L. Clearly K, L are non-conjugate in G; and we have $R=N(K)=\langle K,a\rangle$, $S=N(L)=\langle L,b^2ab(abab^{-1})^2\rangle$ with R, S of index 120, using TC. A simple check now shows that elements of order 5 of R are all in 5AB and those of S in 5CD proving that the maximal subgroups R, S remain non-conjugate in G. Having found these representatives for maximal subgroups of types (iii) and (iv) we proceed to obtain neater generating pairs for R, S in order to find simpler generators and presentations for these classes of maximal subgroups. To do this we first see that $R=\langle x,ay\rangle$ and $S=\langle u,b^2ab(abab^{-1})^2\rangle$. Let $z=ay$, $w=b^2ab(abab^{-1})^2$. Then it can easily be shown that $(x,(xz^2)^2xz)$, $(u,(uw)^2w)$ are generating pairs of types (2C,4B,6A;17) and (2C,4B,6B;15CD) for R, S respectively. Now by PERM we may obtain the generators $x_3=((ab)^2ab^{-2})^2$, $y_3=(ba)^3b^2$ for a maximal subgroup H_3 conjugate to R and $x_4=x_3$, $y_4=ab^{-1}(ab)^3$ for a maximal subgroup H_4 conjugate to S (see below). Furthermore, H_3 has intersection 17:4 with H_4 in G.

The pairs (x_3,y_3) and (x_4,x_4) satisfy the following presentations P1 and P2 :

P1 $\langle x,y \mid x^2=y^4=(xy)^6=(xyxy^{-1}(xy^2)^2)^2=(xyxy^2xy^{-1})^3=1\rangle$

P2 $\langle x, y \mid x^2=y^4=(xy)^6=((xy^{-1})^2(xy)^2(xy^2)^2)^2=(xy^{-1}xy(xy^2)^2)^3=(xy^2(xy^{-1}xy)^2)^3=1\rangle.$

Let $N=\langle (xy)^2,(yx)^2\rangle$. Then N has index 2 in both groups presented by P1, P2 with N isomorphic to PSL(2,16). Next the groups split over N, and $\langle (xy)^3\rangle$ is a complement to N in both cases. Notice that on setting $N_1=\langle (x_3y_3)^2,(y_3x_3)^2\rangle$, $N_2=\langle (x_4y_4)^2,(y_4x_4)^2\rangle$ we find two PSL(2,16) subgroups of G conjugate to K, L respectively.

Finally we note that P1 and P2 are related by $(x,y) \leftrightarrow (x,y(xy^2)^2(xy^{-1})^2x)$.

Using the presentation P1 a covering group C for H_i can be constructed as follows :

$$\langle x,y \mid (xy)^6=x^6y^4,\ (xyxy^2xy^{-1})^3=x^8,\ [x,y^4]=[y,x^2]=[x,\ (xyxy^{-1}(xy^2)^2)^2]= [y,\ (xyxy^{-1}(xy^2)^2)^2]=1\rangle.$$

C has order 8160 and so H_i has trivial multiplier. To find a deficiency zero presentation for H_i we take $X=x_3$, $Y=(x_3y_3)^2y_3$ where Y , XY have order 6. Then PERM finds the relation $(XY^2)^2(YX)^3(Y^{-1}X)^2Y^2XY^3XY^{-2}=1$ which together with relations $X^2=1$, $Y^6=1$ defines the group H_3. Combining the two last relations of this presentation gives

$$H=\langle X,Y \mid X^2Y^6=(XY^2)^2(YX)^3(Y^{-1}X)^2Y^2XY^3XY^{-2}=1\rangle$$

which is, in fact, isomorphic to H_3 as we shall now see . It is easy to check that the derived group H' can be generated by $[X,Y]$, $[X,Y^2]$ and that Y^6 is in H'. Now the centrality of Y^6 in H shows that H is a stem extension of H_3 and thus $H \cong H_3$ by (1.6.13), (1.6.14).

(v)-(vi) structure: $(A_5xA_5):2$

G has two conjugacy classes of maximal subgroups with representatives H_5 and H_6 of structure (A_5xA_5):2 both being the normalizer in G of A_5xA_5 subgroups. In fact G has two non-conjugate A_5xA_5 subgroups S, T where the direct factors of S have generating pairs of type (2B,3B,5CD) and those of T of type (2A,3A,5AB). We then have $H_5 \cong N_G(S)$, $H_6 \cong N_G(T)$. First we show that $d(A_5xA_5)=2$ in order to determine generators (by PERM) for each of S, T. The group A_5xA_5 has presentation

$$\langle x,y,z,u \mid x^2=y^3=(xy)^5=z^2=u^3=(zu)^5=[x,z]=[x,u]=[y,z]=[z,u]=1\rangle$$

and can be generated by xz, yuzuy using TC. The pair (xz,yuzuy) is of type (2,15,5;15). After some experimenting we found that $S=\langle c,d\rangle$, where $c=((ab)^2ab^{-2})^2$, $d=b^{-2}(ab)^2$, with (c,d) of type (2C,5AB,15CD;15CD). TC verifies that $N(S)=\langle S,a^b\rangle$. The subgroup N(S) having order 7200 is actually a maximal subgroup of G of type (v). We now let $r=dca^b$. Then it can be checked that the pair (c, rcr^3cr) is a generating pair of type (2C,4A,6A) for N(S) which enables us to obtain the generators $x_5=[b^2,a]$, $y_5=(b^2a)^2(ba)^3$ for a maximal subgroup H_3 conjugate to N(S) (see below).

Similarly we find the generators $x_6=((ba)^2b^{-2}a)^2$, $y_6=(ab)^3(ab^2)^2$ for a maximal subgroup H_6 conjugate to N(T) with (x_6,y_6) of type (2C,4A,6B). That H_5 and H_6 are not conjugate in G follows from the fact that H_5 contains elements of order 15 from 15CD only while such elements in H_6 are all in 15AB. Furthermore, H_5 has intersection $3^2.Q_8$ with H_6.

A presentation for H_5 on the generators x_5, y_5 is

$$<x,y \mid x^2=y^4=(xy)^6=((xy)^2xy^{-1}xyxy^2)^3=(xyxy^2xy(xy^{-1})^2)^3=((xy)^2y(xy^{-1}xy)^2)^2=1>.$$

Let $N=<x,(xyxy^2)^2>$. Then N has index 2 in $<x,y>$ with $N\cap<(xy)^3>=(1)$, i.e. H_5 is a split extension of N by C_2. Next, it is found that N has a presentation on X=x, $Y=(xyxy^2)^2$ with relations $X^2=1, Y^5=1$, $(XY)^{15}=1$, $(XYXY^2)^6=1$, $(XYXYXY^2XY^{-1})^2=1$, $(XYXYXY^{-1}XY^2)^2=1$. Taking

$$N_1=<(XYXY^2)^3, (XYXY^{-2})^2>, \qquad N_2=<(XYXYXY^{-1})^5, (YXY^2X)^2>$$

gives $N \cong N_1xN_2$ with $N_1 \cong N_2 \cong A_5$. So $H_5 \cong (A_5xA_5):2$. Finally we note that on substituting x_5, y_5 for x, y in the generators of N_1, N_2 we find two A_5 subgroups whose generating pairs are of type (2B,3B,5CD), i.e. $N_1xN_2 \cong S$ as claimed already.

The pair (x_6,y_6) satisfies the presentation

$$<x,y \mid x^2=y^4=(xy)^6=(xy^2)^5=[x,y]^6=(xyxy^2)^6=1>.$$

Similarly we can show that $H_6 \cong M:2$ where $M \cong A_5xA_5 \cong T$.

We note that the above presentations for H_5 and H_6 are related by the change of $(x,y) \leftrightarrow (x,y(xy^2)^3)$.

A covering group C for H_6 may be given by

$$<x,y \mid (xy)^6=x^6y^4, (xy^2)^5=x^4, [x,y^4]=[y,x^2]=[x,r_1]=[x,r_2]=[y,r_1]=[y,r_2]=[r_1,r_2]=1>$$

where $r_1=(xyxy^{-1})^6$, $r_2=(xyxy^2)^6$. Then $|C|=14400$ and so $M(H_i)=C_2$ (i=5, 6). To prove H_i efficient we looked for a 2-generator 3-relation presentation. Using the method described in 3.5 we arrived at the presentation:

$$<v,w \mid v^4=w^5=(vw^{-1})^6=(vwvw^{-1})^3=[v,w][v,w^{-1}]=1>$$

for H_6 on the new generators $v=y_6$, $w=(x_6y_6x_6y_6^2x_6y_6^{-1})^3$ from which the following deficiency -1 presentation was obtained

$$<v,w \mid (vw^{-1})^6w^5= (vwvw^{-1})^3w^5=[v,w][v,w^{-1}]w^5=1>.$$

(vii) structure: S_6

As was remarked earlier S_6 has a generating pair of type (2,6,5;3). In an attempt to obtain a generating pair (x,y) of this type for a S_6 subgroup PERM finds x=a, $y=(ba)^4b^{-1}$. The elements x, y then generate a subgroup H_7 of G of order 720 with (x,y) satisfying the presentation $\langle x,y \mid x^2=y^6=(xy)^5=[x,y]^3=[x,y^2]^2=1\rangle$ given in [18] for S_6. It remains to show that H_7 is maximal in G. An easy calculation first shows that $|H_i'|=3840$, for i=1,2, and $|H_7'|=360$. Therefore, H_7 is not embeddable in H_i (i=1,2). We shall prove that H_7 is not also embeddable in H_i, for i=5,6. To see this, we consider the presentation given in (v) for H_5. It can easily be shown that $xyxy^{-1}$, xy^2xy^{-2} generate the derived group of H_5. Now using RS we may obtain the following presentation for H_5'

$$\langle r,s,t \mid r^2=s^2=t^2=(rst)^3=((rs)^2trt)^3=(rs(rt)^2st)^3=(rs)^3r(ts)^2trt(st)^2=1\rangle.$$

This group has no elements of order 4 while H_7' ($\cong A_6$) contains elements of order 4. Hence H_5' has no subgroup isomorphic to H_7' which , in turn, shows that H_5 is not embeddable in H_7.

5. Efficient presentations for certain simple groups and direct products

In this chapter we give efficient presentations for the simple groups PSU(3,5), J_1 and M_{22} which were not previously known to be efficient. We also give efficient presentations for the perfect groups $\hat{A}_5 x A_6$, $\hat{A}_5 x \hat{A}_6$, $A_6 x A_6$ and $A_5 x A_7$ including a new neat presentation for $\hat{A}_5 x \hat{A}_5$. It will be shown that the simple group PSU(4,2) occurs as a composition factor of a deficiency zero group of order 155520.

5.1 Introduction

In this section we discuss shortly the problem of finding efficient presentations for the twenty simple groups listed on page 1. A comprehensive discussion including the recent progress in investigating the efficiency of finite simple groups and related groups will appear in a forthcoming survey, Campbell, Robertson and Williams [11].

In 1983 Campbell and Robertson commenced the investigation of the efficiency of all the thirteen simple groups of order up to 10^5 excluding PSL(2,p^n), $n \geq 2$. They were successful in finding efficient presentations for all but M_{11} and PSU(3,3), see [4]. This work used as a starting point the minimal permutation generators given in [35] and the presentations satisfied by these permutations [16]. The method employed to make these presentations efficient was explained in 3.5 of this thesis. Three years later P.E. Kenne showed that M_{11} and PSU(3,3) are also efficient [32]. To obtain efficient presentations for these groups he used a method which we described in 3.5.

In [37] E.F. Robertson suggested extending the work of [4] to the remaining seven simple groups PSU(3,5), J_1, A_9, PSL(3,5), M_{22}, J_2 and PSp(4,4). A result of himself and Campbell [6] showed that that $\hat{J}_2$, and consequently J_2, are efficient.

We will show that the three simple groups PSU(3,5), J_1 and M_{22} are efficient. Taking these results together with the efficient presentation for A_9, [9], among the twenty simple groups up to 10^6 only the efficiency of PSL(3,5) and PSp(4,4) remain undecided. Note that each of these groups has trivial multiplier and an efficient presentation for each of them requires an equal number of generators and relations.

It is also worth noting that, at the present time, all the simple groups of order up to

10^6 with the above exceptions have been shown to be efficient [11].

5.2 Method

Suppose G is a moderately large "concrete" group, say a permutation or matrix group, generated by two of its elements x and y with x an involution. In order to be able to obtain an efficient presentation for G we may use methods similar to those of Kenne but since we are dealing with a group of large order we are unable to follow his method of computing presentations on many randomly chosen pairs of generators using Cannon's algorithm. We therefore try to choose a set of generating pairs (x,y) for G with x an involution and y having a specified order. This allows us a more efficient method of finding relations and also simplifies the final stages of reduction to an efficient presentation. In particular if G is a finite simple group $< 10^6$, such generating pairs always exist. Now PERM (or a similar matrix program) can be used to find a presentation for one of the groups G that we are examining. We then attempt to reduce the number of relations using the method described in 3.5.

Note.Suppose G is a finite permutation group. The following CAYLEY program tries to determine whether G has a generating pair (x,y) with x an involution and y, xy having specified orders n_1 and n_2. Whenever G is a large group the program can be modifed so that a small set of random elements y of a specific order for which G=<x,y> are sought.

```
> gorder = order (g) ;
> s = null ;
> t = null ;
> FOR EACH x IN classes (g) DO
>         IF order (x) EQ 2 THEN
>            s = s JOIN [x] ;
>         END;
>         IF order (x) EQ n1 THEN
>            t = t JOIN class (g,x)
>          END ;
> END ;
> FOR EACH x IN s DO
> FOR EACH y IN t DO
>          IF order (x*y) EQ n2 THEN
>          IF order (< x, y > ) EQ gorder THEN
>      PRINT x , y ;
>      BREAK ;
```

```
>    END ;
>  END ;
> END;
> END;
```

The following examples illustrate how the method works in practice.

Example 1. Find a deficiency zero presentation for M_{11}.

The Mathieu group M_{11} has a unique pair of minimal generating permutations a and b of degree 11:

$$a=(1,10)(2,8)(3,11)(5,7),\ b=(1,4,7,6)(2,11,10,9).$$

We choose an involution x and an element y of order 5 with $\langle x,y\rangle=M_{11}$. Take x=a and $y=abab^{-1}ab^2$ where y is a representative of the single conjugacy class of elements of order 5 chosen so that the product xy has order 11. PERM gives

$$M_{11}=\langle x,y \mid x^2=y^5=(xy)^{11}=xy^{-1}xyxy^{-2}xy^2xyxy^{-2}xy^{-2}xy^2xy^{-2}=1\rangle$$

and $(xy)^{11}=1$ is easily shown to be redundant using TC. Now

$$H=\langle x,y \mid x^2y^5=1,\ [x^{-1},y^{-2}][x,y][x,y^2][x^{-1},y^{-1}]=y^2xy\rangle$$

is a stem extension of M_{11} since H is perfect and x^2, being both a power of x and y, is central. But $M(M_{11})=1$ so $x^2=1$ and $H=M_{11}$.

Example 2. Find an efficient presentation for A_5xA_5.

The group A_5 is generated by the permutations (1,3)(4,5) and (1,4,2) so $A_5xA_5=\langle a,b,c,d\rangle$ where

$$a=(1,3)(4,5),\ b=(1,4,2),\ c=(6,8)(9,10),\ d=(6,9,7).$$

We choose the involution x=ac and look for an element y of order 5 with $\langle x,y\rangle=A_5xA_5$. Using CAYLEY, we find y=(1,3,4,5,2)(6,9,8,10,7). Now PERM gives

$$A_5xA_5=\langle x,y \mid x^2=y^5=(xy)^{15}=(xy^{-1})^2(xy)^4(xy^{-1})^2xy^2=1\rangle$$

which proves that A_5xA_5 is efficient for $M(A_5xA_5)=C_2xC_2$.

<u>Note</u>. The above presentations for M_{11} and A_5xA_5 are quite different presentations to those found by Kenne in [32] and [30]. In particular, the latter presentation can be reduced to a deficiency zero presentation for $\hat{A}_5x\hat{A}_5$ as follows:

Let

$$G=\langle x,y \mid x^2y^5=1,\ (x^{-1}y^{-1})^2(xy)^4(x^{-1}y^{-1})^2xy^2=1\rangle.$$

Then G is a perfect group of order 14400. Now a simple check using TC verifies that the relations $[x,(xy)^{15}]=[y,(xy)^{15}]=1$ hold in G. Therefore, $(xy)^{15}$ is a central element of G. But, $G/<x^2,(xy)^{15}> \cong A_5xA_5$ so G is a stem extension of A_5xA_5. Hence G is a homomorphic image of $\hat{A_5xA_5}$. Now, since G and $\hat{A_5xA_5}$ have the same order, $G \cong \hat{A}_5x\hat{A}_5$.

Note that on adding the relation $(xy)^{15}=1$ to the presentation for G we find an efficient presentation for $\hat{A_5xA_5}$. Also, adding the relation $y^5=1$ to the same presentation gives a perfect group of order 7200 which is denoted by P_4 in [40] (in fact, P_4 is the factor group $(\hat{A}_5x\hat{A}_5)/C_2$).

5.3 Efficiency of PSU(3,5), J_1 and M_{22}.

In this section we give 2-generator 3-relation presentations for the simple group PSU(3,5) of order 126000 and the Mathieu group M_{22} of order 443520. We also give a 2-generator 2-relation presentation for the Janko group J_1 order 175560. This proves that these three simple groups are efficient.

5.3.1 PSU(3,5) is efficient.

Proof. Since M(PSU(3,5)) is cyclic of order 3 we seek a 2-generator 3-relation presentation. We start with the unique minimal generating pair (a, b) with $a^2=b^4=1$ given in [6]. PSU(3,5) has 4 conjugacy classes of order 5 and, after some experimenting, we choose $y=(ab)^2ab^{-1}(ab)^2$ of order 5 and the involution x=a so that xy has order 10.

PERM finds the presentation

$$PSU(3,5)=<x,y \mid x^2=y^5=(xy)^{10}=[x,y^2]^4=r=1>$$

where $r=xy^{-2}xy^2x(yxy^{-2}xyxy)^2$. Coset enumeration shows that

$$PSU(3,5)=<x,y \mid x^2=(xy)^{10}=r=1, [x,y^2]^4=y^5>.$$

Let H be the stem extension of PSU(3,5)

$$H=<x,y \mid x^2=(xy)^{10}, r=1, [x,y^2]^4=y^5>.$$

Now, coset enumeration shows that $|H:<(xy)^3>|=12600$ so, by 3.5.1 (i), we see that $H \cong PSU(3,5)$. This gives an efficient presentation as required.

5.3.2 The Janko group J_1 has deficiency zero.

Proof. $M(J_1)$ is trivial so every stem extension of J_1 is isomorphic to J_1. We seek a 2-generator 2-relation presentation starting from the presentation 15.20 for J_1 given in [6]:

$$J_1 = \langle a,b \mid a^2=b^3=r^6=q^7=s^2=1 \rangle$$

where $r=abab^{-1}$, $s=(ab)^3(ab^{-1})^3abab^{-1}$ and $q=ab^{-1}(ab)^2$.

Let

$$H = \langle a,b \mid a^2=b^3=q^7,\ r^6=s^2 \rangle.$$

Now coset enumeration verifies that $H=\langle r,s \rangle$ so r^6 is central in H and thus H, being perfect, is a stem extension of J_1. Hence $H \cong J_1$.

Now rewrite $r^6=s^2$ as $u=1$ where

$$u=ab^{-1}ab(abab^{-1})^3ab^{-1}ab(ab^{-1})^3abab^{-1}(ab)^3$$

and consider

$$K=\langle a,b \mid a^2=b^3=1,\ u=q^7 \rangle.$$

Coset enumeration shows, with considerable difficulty, that $|K:\langle b \rangle|=58520$ so $K \cong J_1$.

Finally let

$$L=\langle a,b \mid a^2b^3=1,\ (a^{-1}b^{-1}(ab)^2)^4(a^{-1}b)^2a^{-1}b^{-1}(ab)^3ab^{-1}(ab)^2(a^{-1}b^{-1})^2=1 \rangle.$$

But L is perfect, a^2 is central in L and $L/\langle a^2 \rangle \cong K \cong J_1$. Hence L is a stem extension of J_1 so $L \cong J_1$.

5.3.3 The Mathieu group M_{22} is efficient.

Proof. $M(M_{22})$ is cyclic of order 12 so we seek a 2-generator 3-relation presentation. Start with the minimal generating pair of permutations given in 18.3 of [35] but rather than use the presentation 18.3 of [6] we use PERM to give the presentation

$$M_{22}=\langle a,b \mid a^2=b^4=(ab)^{11}=s^7=r=1 \rangle$$

where $s=abab^2$ and $r=(ab)^2(ab^{-1})^2ab^2(ab)^2ab^{-1}ab(ab^2)^2$. Now coset enumeration shows that

$$G=\langle a,b \mid a^2=(ab)^{11}=1,\ s^7=b^4,\ r=b^4 \rangle$$

is isomorphic to M_{22}. Now let H be the stem extension of G given by

$$H=\langle a,b \mid a^2=(ab)^{11},\ s^7=b^4,\ r=b^4 \rangle.$$

Coset enumeration verifies that

$$|H:\langle ab \rangle|=|H:\langle (ab)^2 \rangle|=|H:\langle (ab)^3 \rangle|=|G:\langle ab \rangle|,$$

so, by 3.5.1 (ii), we obtain $H \cong M_{22}$ as required.

5.4. Direct products

Questions concerning the efficiency of direct products are posed by Wiegold in [43]. In particular, he asked whether $A_5 x A_5$ is efficient and conjectured that $\hat{A}_5 x \hat{A}_5$ is not efficient. Kenne [30] showed that $A_5 x A_5$ and $A_5 x \hat{A}_5$ are efficient while Campbell et al. [10] proved that $\hat{A}_5 x \hat{A}_5$ has deficiency zero.

We investigate here the efficiency of $\hat{A}_5 x A_6$, $\hat{A}_5 x \hat{A}_6$, $A_6 x A_6$ and $A_5 x A_7$ and show that all these groups are efficient.

5.4.1. $A_6 x A_6$ is efficient.

Proof. The group $G=A_6 x A_6$ has order 129600 and multiplier $C_6 x C_6$. Let

a=(2,3)(4,5)

b=(1,4,3,2)(5,6)

c=(8,9)(10,11)

d=(7,10,9,8)(11,12).

Then a, b, c and d generate G. We take x=ac and look for an element y of order 5 with x and y generating G. A calculation with CAYLEY gives y=(1,6,3,4,5)(7,10,12,11,9) of order 5 such that G=<x,y>. Now PERM gives

$G=\langle x,y \mid x^2=y^5=(xy)^{20}=((xy)^2(xy^2)^3)^2=xy^{-1}(xy^2)^2xy^{-2}xy^{-1}(xy^{-2})^2xy^{-1}(xy)^2(xy^2)^3=1\rangle$.

Let

$$H=\langle x,y \mid x^2y^5=1,\ (xy)^{20}=y^5, ((xy)^2(xy^2)^3)^2=1,\ (xy^2)^2xy^{-2}xy^{-1}(xy^{-2})^2xy^{-1}(xy)^2(xy^2)^3xy^{-1}y^5=1\rangle.$$

Now, coset enumeration shows that

$$|H:\langle y\rangle|=|H:\langle y^2\rangle|=|H:\langle y^3\rangle|=25920.$$

Thus $H \cong G$ by 3.5.1 (ii).

5.4.2 $A_5 x A_7$ is efficient

Proof. The group $G=A_5 x A_7$ has order 151200 and Schur multiplier $C_2 x C_6$. A_7 is generated by the permutations (1,5)(6,7) and (1,2,3,4)(5,6). Now let

a=(1,3)(4,5)

b=(1,4,2)

c=(6,10)(11,12)

d=(6,7,8,9)(10,11).

We now take x=ac and by a similar method to that used in (5.4.1) find

$$y=(1,4,2,3,5)(6,12,9,7,8)$$

of order 5 such that $G=\langle x,y\rangle$. PERM gives

$$G=\langle x,y \mid x^2=y^5=((xy)^2xy^{-2})^3=((xy)^3xy^{-1})^3=r=1\rangle$$

where $r=xy^{-1}xyxy^{-2}xy^2xy^{-2}xyxy^{-1}xy^{-2}(xy)^3xy^{-2}$.

Let

$$H=\langle x,y \mid x^2=y^5,\ x^2=(xyxyxy^{-2})^3,\ ((xy)^3xy^{-1})^3=1,\ ry^5=1\rangle.$$

Then H is a stem extension of G and we have

$$|H:\langle y\rangle|=|H:\langle y^2\rangle|=|H:\langle y^3\rangle|=30240.$$

Therefore, $H \cong G$, by 3.5.1(ii).

5.4.3. $\hat{A}_5x\hat{A}_6$ has deficiency zero.

Proof. We begin by constructing a presentation for $G=A_5xA_6$. Since $A_5=\langle(1,3)(4,5),(1,4,2)\rangle$ and $A_6=\langle(2,3)(4,5),\ (1,4,3,2)(5,6)\rangle$, $G=\langle a,b,c,d\rangle$ where

$a=(1,3)(4,5)$

$b=(1,4,2)$

$c=(7,8)(9,10)$

$d=(6,9,8,7)(10,11)$.

We take $x=ac=(1,3)(4,5)(7,8)(9,10)$ and search for an element y of order 5 with $G=\langle x,y\rangle$ using CAYLEY. We find $y=(1,5,4,2,3)(6,11,8,9,10)$. Now PERM gives a 2-generator 4-relation presentation for G on x and y :

$$G \cong \langle x,y \mid x^2=y^5=(xyxy^2)^4=(xy^{-1})^2xy(xy^2)^2xy^{-1}xy(xy^{-2})^2xyxy^2=1\rangle$$

which shows that A_5xA_6 is efficient. Next, removing the relation $(xyxy^2)^4=1$ from the presentation of G gives a perfect group of order 129600. Now, let

$$H=\langle x,y \mid x^2y^5=1,\ (x^{-1}y^{-1})^2xy(xy^2)^2x^{-1}y^{-1}xy(x^{-1}y^{-2})^2xyxy^2=1\rangle.$$

Then a moderately difficult coset enumeration showed (by adding the trivial relation $[x^2,y]=1$ to the presentation of H) that H is a group of order 259200 and that $[x,(xyxy^2)^4]=1$, $[y,(xyxy^2)^4]=1$. Now $H/\langle x^2,(xyxy^2)^4\rangle\cong G$ showing that H is a stem extension of G ($\cong A_5xA_6$). Since H and $\hat{A}_5x\hat{A}_6$ have the same order, $H \cong \hat{A}_5x\hat{A}_6$ by 1.5.13.

We now derive the following consequence for $\hat{A}_5xA_6$.

5.4.4. $\hat{A}_5xA_6$ is efficient.

Proof. On adding the relation $(xyxy^2)^4=1$ to the presentation for H, we arrive at a group K of order 43200. Let

$$N=\langle (xy)^3,(yx)^3\rangle,\ \ M=\langle (xy)^5,(yx)^5\rangle.$$

Then $|K:N|=120$, $|K:M|=360$, $N \lhd K$, $M \lhd K$ and $N\cap M=(1)$. Now it is easily checked that the factor groups K/N, K/M are perfect groups of order 360, 120 respectively. Since the only perfect groups of these orders are A_6 and $\hat{A}_5$ ([40]), $K \cong A_6 x \hat{A}_5$.

Notes.

(1) We failed to find a 2-generator 3-relation presentation for $A_5 x \hat{A}_6$ using the above presentation of H. It is probable that the group obtained by adding the relation $x^2=1$ to the presentation of H is isomorphic to $A_5 x \hat{A}_6$.

(2) In [12] Campbell, Robertson and Williams give the following efficient presentation for $A_5 x A_6 \cong SL(2,2^2) x PSL(2,3^2)$:

$$\langle a,b \mid a^2=b^5=(abab^2ab^{-1})^3=1,\ (ab^2)^5=(abab^{-1}ab^2)^3\rangle.$$

5.5 The group G

The question as to which non-abelian simple groups are composition factors of finite groups of deficiency zero was posed by D.L. Johnson and E.F. Robertson in the survey [28]. Clearly those non-abelian simple groups whose covering groups are efficient each naturally occurs as a composition factor of a finite deficiency zero group. In 1980, the efficiency of $SL(2,p)=\hat{PSL}(2,p)$, p prime, [2] showed that there are an infinite number of such simple groups. Since then an attempt was made to show that the covering groups of simple groups of order up to 10^6 are efficient. The survey [11] discusses the current status of the efficiency problem for these groups.

We show below that the simple group PSU(4,2) is a composition factor of a finite (non-perfect) group G of deficiency zero.

5.5.1 There exists a deficiency zero group G of structure C_6 PSU(4,2).

We begin with the generating permutations a, b given in 10.1 of [35]. Let x=a, and $y=abab^2abab^{-1}ab^2$. Then x and y generate G, and using PERM, we find that they satisfy the presentation

$$\langle x,y \mid x^2=y^5=(xyxy^2)^2(xy^2xy)^2xy^{-1}\rangle.$$

Now, we define

$$G=\langle x,y \mid x^2y^5=1,\ (xyxy^2)^2(xy^2x^{-1}y)^2xy^{-1}=1\rangle.$$

Coset enumeration shows that $|G|=155520$ and $|G:\langle y\rangle|=5184$. Thus y has order 30 so $G\cong C_6 PSU(4,2)$.

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APPENDIX

We give below some details of the file SIMGPS.TLB, a Cayley file of finite simple groups, which contains our results on conjugacy classes, maximal subgroups and Sylow subgroups of the non-abelian simple groups G, $|G|<10^6$. The file will be distributed world-wide with the group theory system CAYLEY.

What follows are the libraries INFO, CONTENTS, EXAMPLE and NOTATION of the file.

A library of simple groups : permutations, presentations conjugacy classes, maximal subgroups and Sylow subgroups

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I. DESCRIPTION OF THE LIBRARY

The first version of this file is described in [9]. This revised version includes data on conjugacy classes, maximal subgroups and Sylow subgroups computed by A R Jamali and described in detail in [7].

The non-abelian simple groups G with $|G| < 10^6$, excluding the groups $PSL(2,p^n)$, have been studied using computing techniques in a series of papers. These groups can each be generated by two elements a and b with a an involution. The pair (x,y) is a minimal generating pair for G with respect to the involution a if x and y generate G, if x = af for some automorphism f of G and if y has order less than or equal to the order of z for any z in G such that x and z generate G. All minimal generating pairs are given in [8]. For those groups G with $| G | < 10^5$, presentations on each of the minimal generating pairs are given by Cannon, McKay and Young in [5]. In [3] presentations are given for those G with $10^5 < | G | < 10^6$. This file contains each pair of minimal permutation generators for each of the groups (taken from [8]) together with a presentation of the group satisfied by those permutations (taken from [3] or [4]).There are 106 such generating pairs and accompanying presentations which are numbered as in

[3], [4], [5] and [8] (see also Contents of this file). We have, however, deliberately inserted redundant relations in certain presentations to facilitate coset enumeration and these insertions are noted.

We list the conjugacy classes and maximal subgroups of each group and input generators for each maximal subgroup in terms of the particular minimal generating pair. We list the structure of the Schur multiplier of each maximal subgroup. For the first version of each group only we input a representative of each conjugacy class. For each of the versions we input generators for a representative of each non-cyclic Sylow p-subgroup. Note that cyclic Sylow p-subgroups are not input. The data concerning conjugacy classes, maximal subgroups and Sylow subgroups is taken from [7]. The notation used for conjugacy classes and the structure of maximal subgroups follows fairly closely the notation of the Atlas of Finite Groups [6]. The notation is sumarized in the Notation library of this file.

Neat presentations for the groups PSL(2,p) and SL(2,p), p a prime, are well known, see for example [1]. We include those groups PSL(2,p^n), n >1, of order less than 10^6 which are included in the Atlas [6]. For these groups PSL(2,p^n) we give only one generating pair of permutations and include the same data as for version 1 of the other simple groups. Information on presentations of these groups PSL(2,p^n) is taken from [2] and [7] while other data is taken from [7].

An example of the data contained in each library is given in the library EXAMPLE of this file.

References

1. C M Campbell and E F Robertson, A deficiency zero presentation for SL(2,p), Bull. London Math. Soc. 12 (1980), 17-20.
2. C M Campbell and E F Robertson, On a class of groups related to SL(2,2^n), Proc. Durham Symposium on Computational Group Theory 1982, Academic Press, London (1984), 43-49.
3. C M Campbell and E F Robertson, Presentations for simple groups G, $10^5 < |G| < 10^6$, Comm. Alg. 12 (1984), 2643-2663.
4. C M Campbell and E F Robertson, On the simple groups of order less than 10^5, Groups - Korea 1983, Lecture Notes in Mathematics 1098, Springer, Berlin Heidelberg New York Tokyo (1984), 15-20.
5. J J Cannon, J McKay and K-C Young, The non-abelian simple groups G, $|G| < 10^5$-presentations, Comm. Alg. 7 (1979), 1397-1406.
6. J H Conway, R T Curtis, S P Norton, R A Parker and R A Wilson, Atlas of Finite Groups : Maximal Subgroups and Ordinary Characters for Simple Groups (Clarendon Press, Oxford, 1985).

7. A Jamali, Computing with simple groups : maximal subgroups and presentations, Ph D Thesis, University of St Andrews, 1988.
8. J McKay and K-C Young, The non-abelian simple groups G, |G| <10^6- minimal generating pairs, Math. Comp. 33 (1979), 812-814.
9. E F Robertson and C M Campbell, A library of finite simple groups of order less than one million, Cayley Bulletin 3 (1987), 70-73.

II. CONTENTS

This file contains at least one library for each non-abelian simple group of order less than one million except those groups PSL(2,p^n) which are not contained in the `Atlas of finite groups` . See INFO for details of what these libraries contain.

1. For the groups other than PSL(2,p^n) there is one library for each (essentially) distinct minimal generating pair. The following is a list of these libraries:

A5V1

A6V1

A7V1 A7V2

A8V1 A8V2

A9V1 A9V3 A9V4

J1V1 J1V3 J1V5 J1V7 J1V8 J1V10 J1V12 J1V14 J1V15 J1V16
J1V18 J1V19 J1V20 J1V22 J1V24 J1V26 J1V27 J1V28 J1V29 J1V31
J1V32

J2V1 J2V2 J2V3 J2V4 J2V5 J2V6 J2V8 J2V9 J2V11 J2V13
J2V15 J2V17 J2V18 J2V20 J2V21 J2V22

M11V1

M12V1 M12V2 M12V3

M22V1 M22V2 M22V3 M22V5 M22V6

PSL27V1

PSL33V1 PSL33V2

PSL34V1

PSL35V1 PSL35V2 PSL35V3 PSL35V4 PSL35V5 PSL35V6 PSL35V7

PSP44V1 PSP44V2 PSP44V3 PSP44V4 PSP44V6 PSP44V8 PSP44V9
PSP44V11 PSP44V13 PSP44V15 PSP44V17 PSP44V18 PSP44V19 PSP44V20
PSP44V22 PSP44V23 PSP44V24 PSP44V26 PSP44V27 PSP44V28 PSP44V29
PSP44V31

PSU33V1 PSU33V2

PSU34V1 PSU34V2

PSU35V1

PSU42V1 PSU42V2 PSU42V3 PSU42V5 PSU42V7 PSU42V8 PSU42V9

SZ8V1 SZ8V3 SZ8V5 SZ8V7 SZ8V9 SZ8V11 SZ8V13 SZ8V15

For each of the above, the permutation version is named as above, with the V replaced by a P. E.g. the permutation version of SZ8V7 will be SZ8P7. The finitely presented version will have the V replaced by an F. E.g. the finitely presented version of SZ8V7 will be SZ8F7.

2. For the groups PSL(2,p^n) there is one library for each group of order less than one million which appears in the `Atlas of finite groups` as follows:

SL28 PSL211 PSL213 PSL217 PSL219 SL216 PSL223

PSL225 PSL227 PSL229 PSL231 SL232

For each of the above,the permutation version is named as above witha P1 appended, e.g. SL28 is represented as SL28P1.The finitely presented version has an F1 appended,e.g. SL28 is represented as
SL28F1.

III. EXAMPLE

The following is an example of the data input for version 1 of each of the groups. Note that versions other than 1 do not input representatives of the conjugacy classes. We have chosen PSU33V1 as the example with its two libraries PSU33P1 and PSU33F1.

```
LIBRARY PSU33P1;
GP:PERM(28);
GP.GENERATORS:A=/21,2,8,15,5,22,16,3,28,27,20,19,23,14,4,7,26,25,12,11,1,6
,13,24,18,17,10,9/,B=/3,5,16,22,23,18,10,4,9,14,7,17,6,12,26,24,11,13,1,25,20,
15,21,27,2,28,19,8/;
GP.RELATIONS:A^2=B^6=(A*B)^7=(A*B^2)^3*(A*B^-2)^3=(A*B*A*B^-2)^3*A
*B*(A*B^-1)^2=1;
PRINT'PSU33V1 as permutation group is GP';
PRINT'with maximal subgroups MPi and sylow subgroups SPi';
PRINT'For information about the conjugacy classes and the';
PRINT'maximal subgroups of PSU(3,3) type "CLASSINFO;" and "MAXINFO;"';
procedure classinfo;
print '  class :  1A    2A    3A    3B   4A    B**   4C   6A   7A ';
print '  |c(x)| : 6048  96    108   9    96    96    16   12   7  ';
print ' ---------------------------------------------------------------------------';
print '           B**   8A    B**   12A  B** ';
print '           7     8     8     12   12 ';
end;
procedure maxinfo;
print '  group   order   index   structure    specification      mult ';
print '  -------  ------  ------  -----------  --------------     -----';
print '  MP1     216     28      3^(1+2):8    N(3A)              1 ';
print '  MP2     168     36      PSL(2,7)     N(2A,3B,4C,7AB)    2 ';
print '  MP3     96      63      4"S4         N(2A)              1 ';
print '  MP4     96      63      4^2:S3       N(2A^2)            2 ';
end;
C2A=A;
C3A=B^2;
C3B=(A,B);
C4A=(A*B*A*B^-2)^2;
C4B=(B^2*A*B^-1*A)^2;
```

```
C4C=A*B^3;
C6A=B;
C7A=A*B;
C7B=B^-1*A;
C8A=A*B*A*B^-2;
C8B=B^2*A*B^-1*A;
C12A=B^-2*A;
C12B=A*B^2;
MP1=<A^B,(A*B)^3*B^3>;
MP2=<A^B,B*(B*A)^2>;
MP3=<A*B^2*A*B^-2,B^3*(A*B)^2>;
MP4=<A^B,A*B*A*B^-1>;
SP2=< A,B*A*B^-2*A*B>;
SP3=< (A,B),A*B*A*B^3*A*B^-1>;
FINISH;

LIBRARY PSU33F1
G:FREE(X,Y);
G.RELATIONS:X^2=Y^6=(X*Y)^7=(X*Y^2)^3*(X*Y^-2)^3=(X*Y*X*Y^-2)^3*X
*Y*(X*Y^-1)^2
=1;
PRINT'';
PRINT'PSU33V1 as finitely presented group is G';
PRINT'with maximal subgroups Mi and sylow subgroups Si';
PRINT'For information about the conjugacy classes and the';
PRINT'maximal subgroups of PSU(3,3) type "CLASSINFO;" and "MAXINFO;"';
procedure classinfo;
print '  class :  1A    2A    3A    3B   4A   B**   4C   6A   7A ';
print '  |c(x)| : 6048  96    108   9    96   96    16   12   7  ';
print '  ----------------------------------------------------------------';
print '           B**   8A    B**   12A  B** ';
print '           7     8     8     12   12  ';
end;
procedure maxinfo;
print '  group   order   index   structure    specification      mult ';
print '  -------  ------  ------  -----------  ---------------   ----- ';
print '  M1      216     28      3^(1+2):8    N(3A)              1    ';
print '  M2      168     36      PSL(2,7)     N(2A,3B,4C,7AB)    2    ';
print '  M3      96      63      4"S4         N(2A)              1    ';
```

```
print '    M4        96        63        4^2:S3        N(2A^2)             2  ';
end;
C2A=X;
C3A=Y^2;
C3B=(X,Y);
C4A=(X*Y*X*Y^-2)^2;
C4B=(Y^2*X*Y^-1*X)^2;
C4C=X*Y^3;
C6A=Y;
C7A=X*Y;
C7B=Y^-1*X;
C8A=X*Y*X*Y^-2;
C8B=Y^2*X*Y^-1*X;
C12A=Y^-2*X;
C12B=X*Y^2;
M1=<X^Y,(X*Y)^3*Y^3>;
M2=<X^Y,Y*(Y*X)^2>;
M3=<X*Y^2*X*Y^-2,Y^3*(X*Y)^2>;
M4=<X^Y,X*Y*X*Y^-1>;
S2=< X,Y*X*Y^-2*X*Y>;
S3=< (X,Y),X*Y*X*Y^3*X*Y^-1>;
FINISH;
```

IV. NOTATION

After a library for a simple group G has been read into Cayley with a command such as LIBRARY PSU33P2; or LIBRARY PSU33F2; data on conjugacy classes and maximal subgroups may be displayed by typing CLASSINFO; and MAXINFO;. Notice that in all cases the data input for the corresponding permutation version and the finitely presented version of a group agree in the following sense. Substituting the generating permutations of the permutation version into the words given in the finitely presented version for conjugacy class representatives and subgroup generators will always yield the permutations given in the permutation version.

Conjugacy classes

The conjugacy classes that contain elements of order n are named nA, nB, nC, .. etc. Class names of the form Y*k, Y**k, Y**, Y* give the additional information that the class can be obtained from the most recently named class nY by applying the algebraic conjugacy operator of k th powers, -k th powers, inverses, other powers respectively. For each class the order of the centralizer in G of a typical element x of the class is given. A representative of the class 3A say, is input to Cayley as C3A, so PRINT C3A; will display a representative of the class. Note that the representatives are input to Cayley for version 1 of each group only. If the class named B** follows the main class 4A, say, then the class is input to Cayley as C4B so PRINT C4B; prints a representative.

Maximal subgroups

The information regarding the structure of a subgroup is given in a notation similar to the Atlas.

m	denotes a cyclic group of order m
m^n	denotes a direct product of n cyclic groups of order m
A x B	is the direct product of groups A and B
A.B or AB	is a group with normal subgroup A and quotient B
A:B	is a case of A.B where the extension splits
A"B	is a case of A.B where the extension does not split
p^(m+n)	is a case of (p^m).(p^n)

The specification of H locates a copy of H inside G. For example N(2A) is the normalizer of an involution in 2A; N(5AB) is the normalizer of a group of order 5 containing elements of classes 5A and 5B; N(3^3)=N(3AB4 C3 D6) is the normalizer of an elementary abelian group of order 27 whose 13 cyclic subgroups number 4 containing both classes 3A and 3B, 3 containing 3C only, 6 containing 3D only; N(2A,2C,3A,3B,...) is the normalizer of a group containing elements in the indicated classes among others. Within Cayley the maximal subgroups are named M1, M2, M3,... for a finitely presented group and MP1, MP2, MP3,... within a permutation group.

Sylow subgroups

If the prime p divides |G| then generators for a Sylow p-subgroup are input to Cayley provided that the Sylow subgroup is not cyclic. Hence if the library for the permutation group PSU42P5, say, has been read into Cayley a Sylow 2-subgroup is input as SP2 while if the finitely presented group PSU42F5, say, has been read into Cayley a Sylow 2-subgroup is input as S2.