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Dissipation-consistent modelling and classification of extended plasticity formulations

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ABSTRACT

A unified classification framework for models of extended plasticity is presented. The models include well known micromorphic and strain-gradient plasticity formulations. A unified treatment is possible due to the representation of strain-gradient plasticity within an Eringen-type micromorphic framework. The classification is based on the form of the energetic and dissipative model structures and exploits the framework of dissipation-consistent modelling to elucidate the flow relation and yield condition. Models are identified as either serial or parallel. This designation is also applicable to familiar models of classical plasticity. Particular attention is paid to the rate-dependent problem arising from the choice of a smooth dissipation potential. The inability to locally determine the region of admissible stresses for the non-smooth (rate-independent) parallel models of plasticity is made clear.

1. Introduction

Classical theories of plasticity are unable to account for the experimentally-observed, size-dependent response exhibited by structures at the mesoscopic scale. Starting with the early works of Aifantis (1984, 1987), extended models of plasticity have been actively developed and analysed over the past three decades to remedy these and related deficiencies associated with the pathological localisation of softening problems. A significant proportion of extended plasticity models are members of either the gradient or micromorphic frameworks, or indeed both. Micromorphic continua are characterised by additional degrees of freedom at each continuum point (Eringen, 1999; Mindlin, 1964; Toupin, 1964). By contrast, gradient continua possess higher gradients of their primary fields. Important examples of micromorphic and gradient plasticity theories include Steinmann and Willam (1991), de Borst (1991), Grammenoudis and Tsakmakis (2001) and Fleck and Hutchinson (1997, 2001), Svedberg and Runesson (1997), Gurtin (2002), Gudmundson (2004), Gurtin and Anand (2005), Gurtin and Needleman (2005), Nix and Gao (1998), respectively. For classifications of extended models of inelasticity, the reader is referred to Kirchner and Steinmann (2005), Hirschberger and Steinmann (2009), Forest (2009).

Despite the considerable work on models of extended plasticity, open questions and challenges remain. For example, our recent work (Carstensen et al., 2017) on a small-strain theory of strain-gradient plasticity due to Gurtin and Anand

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(2005), detailed the implications of the structure of the energetic and dissipative components of the model on the nature of yield and subsequent plastic flow. An objective of the current work is to present a novel classification scheme based on the energetic and dissipative structures for models of extended plasticity. The classification scheme also clarifies important features of various models of plasticity, including the local or global nature of the region of admissible stresses. Furthermore, we significantly extend our previous work (Carstensen et al., 2017) to include the important rate-dependent case obtained when the dissipation potential is chosen to be smooth.

Models are classified here as being examples of either *serial plasticity* or *parallel plasticity*. This novel distinction is based upon the choice of the energy storage potential and the dissipation potential, and has important consequences for the determinability of the (macroscopic) stress and hence the region of admissible stresses. We show that the essence of this distinction can be explained in the familiar setting of local plasticity (i.e. in the absence of strain-gradient terms). Models of *local serial plasticity* and *local parallel plasticity* are constructed from simple rheological units and analysed for both smooth and non-smooth dissipation potentials, corresponding to the rate-dependent and rate-independent problem, respectively. The key features of the local models are shown to hold for the models of extended plasticity. Importantly, we show that the stress is fully determined from the elastic law in the local serial model. This is not however the case for the local parallel plasticity model.

The framework of generalised dissipative materials (Biot, 1965; Halphen and Nguyen, 1975; Ziegler, 1963) is exploited in this work to provide the basic procedure to determine the structure of the boundary value problem and the internal variable evolution relations (see e.g. Miehe, 2011; 2014; Miehe et al., 2014; Runesson et al., 2017). The framework of generalised dissipative materials is related to the primal formulation of plasticity (Alberty and Carstensen, 2000; Carstensen, 1999; Han and Reddy, 2010; Han et al., 1997) central to our previous contribution (Carstensen et al., 2017). Primal formulations of strain-gradient plasticity have received considerable attention recently and their variational structure analysed (see e.g. Djoko et al., 2007; Reddy, 2011a; 2011b; 2012; Reddy et al., 2008).

The proposed classification scheme is applicable to models of extended plasticity. A unified scheme is achieved by considering the chosen extended models as variants of an Eringen-type micromorphic continuum. This allows for the classification of a range of important micromorphic and strain-gradient plasticity models. The extended models are categorised as either *micromorphic serial plasticity* or *micromorphic parallel plasticity*. The proposed classification scheme provides considerably more structure than the designations *energetic* or *dissipative* (or variants thereof) used in many strain-gradient plasticity models (see e.g. Carstensen et al., 2017; Fleck and Willis, 2009a; 2009b).

The classification of micromorphic parallel plasticity encapsulates the important Gurtin and Anand (2005) model and extensions thereof. In order to elucidate the implications of the choice of the energetic and dissipative structures for the Gurtin and Anand model, we introduce three categories of micromorphic parallel plasticity: (i) the general energetic-dissipative case, (ii) the hybrid energetic-dissipative case, and (iii) the fully-dissipative case. The fully-dissipative, rate-independent case was the focus of our previous work (Carstensen et al., 2017). That work was motivated, in turn, by the recent work of Fleck et al. (2014) on the behaviour of strain-gradient models under conditions of non-proportional loading. Of particular interest in that work is the appearance of a so-called “elastic gap” in the response at a material point undergoing plastic flow due to the instantaneous halting of plastic flow on part of the boundary, or due to non-proportional loading. We clearly show here that a local treatment of the *fully-dissipative problem* is possible provided the dissipation potential is chosen to be smooth. This is a new and important result and supports the numerical findings presented in Fleck et al. (2014). The analysis of the fully-dissipative case with a non-smooth dissipation potential differs from that in Carstensen et al. (2017) as we do not resort to a spatial discretisation from the onset. Key features of the theory are elucidated using the one-dimensional example of a strip subject to shear loading.

The structure of the presentation is as follows. The general setting of dissipation-consistent modelling based upon standard dissipative materials is presented in Section 2. This abstract presentation is then applied to the familiar problem of local plasticity in Section 3. Here models of local serial plasticity and local parallel plasticity are introduced. Section 4 introduces micromorphic continua as a framework for extended models of plasticity. The models of micromorphic serial plasticity and micromorphic parallel plasticity are presented in Section 5 and Section 6, respectively. Conclusions are drawn and extensions proposed in Section 7.

Notation and results

Consider a continuum body occupying a domain Ω with boundary $\partial\Omega$. The outward unit normal to $\partial\Omega$ is denoted by $\boldsymbol{n}$. A continuum point at position $\boldsymbol{x}\in\Omega$ undergoes a displacement at time t denoted by $\boldsymbol{u}(\boldsymbol{x}, t)$. The deformations are assumed infinitesimal.

The respective scalar products of arbitrary vectors ($\boldsymbol{a}$ and $\boldsymbol{b}$), second-order tensors ($\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$) and third-order tensors ($\boldsymbol{\eta}$ and $\boldsymbol{\gamma}$) are defined by

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_i b_i\,, \qquad \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}\,, \qquad \boldsymbol{\eta} \,\vdots\, \boldsymbol{\gamma} = \eta_{ijk}\gamma_{ijk}\,,$$

where summation over repeated indices is implied. A Cartesian coordinate system is used throughout.

The generalised scalar product between two arbitrary generalised measures $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\boldsymbol{A}\circ\boldsymbol{B}$. For example, if $\boldsymbol{A} := \{\boldsymbol{\sigma}, \boldsymbol{\eta}\}$ and $\boldsymbol{B} := \{\boldsymbol{\varepsilon}, \boldsymbol{\gamma}\}$, then

$$\boldsymbol{A}\circ\boldsymbol{B} := \sigma_{ij}\varepsilon_{ij} + \eta_{ijk}\gamma_{ijk}\,.$$

Due to its prominent role in the theoretical developments that follow, the scalar product of a driving force and the conjugated set of internal variables is distinguished from the generalised scalar product, denoted by an open circle, by using a closed circle, i.e. $\bullet$.

Differentiation of an arbitrary function $f(\boldsymbol{A})$ with respect to a variable $\boldsymbol{A}$ is denoted by $f_{,\boldsymbol{A}}$. Similarly, the subdifferential (defined in Eq. (A.4)) of an arbitrary function $f(\boldsymbol{A})$ with respect to a variable $\boldsymbol{A}$ is denoted by $f_{;\boldsymbol{A}}$.

The spatial gradients of arbitrary scalar, vector, and tensor fields are respectively defined by

$$\nabla a = \frac{\partial a}{\partial \boldsymbol{x}}, \qquad \nabla \boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{x}}, \qquad \nabla \boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{x}},$$

where $[\nabla \boldsymbol{A}]_{ijk} = \partial A_{ij}/\partial x_k$. The symmetric gradient operator is defined by $\nabla^{\mathrm{sym}}\boldsymbol{a} = [1/2][\nabla \boldsymbol{a} + [\nabla \boldsymbol{a}]^{\mathrm{T}}]$.

Standard results from convex analysis required for the presentation are provided in Appendix A.

2. The general setting of dissipation-consistent modelling

The structure of the constitutive equations and the inelastic evolution relations for a broad range of inelastic physical processes can be obtained from the general setting of dissipation-consistent modelling based on the notion of standard dissipative materials (see e.g. Biot, 1965; Germain, 1973; Halphen and Nguyen, 1975; Ziegler, 1963). The general setting presented in this section provides the basis for the investigation of both local and micromorphic plasticity that follows.

2.1. The dissipation inequality

We denote by $\boldsymbol{S}$ and $\boldsymbol{E}$ generalised stress (kinetic) and strain (kinematic) measures, respectively. The set of internal (hidden) variables quantifying inelastic processes is denoted by $\boldsymbol{I}$. The generalised stress $\boldsymbol{S}$ is further additively decomposed into energetic and dissipative parts as follows

$$\boldsymbol{S} = \boldsymbol{S}^{\mathrm{en}} + \boldsymbol{S}^{\mathrm{dis}}.$$

Note that $\boldsymbol{A}^{\mathrm{en}}$ and $\boldsymbol{A}^{\mathrm{dis}}$ denote respectively the energetic and dissipative parts of an arbitrary variable $\boldsymbol{A}$.

Within the isothermal setting assumed here, the dissipation density d per unit volume of Ω is defined by

$$d := \boldsymbol{S} \circ \dot{\boldsymbol{E}} - \dot{\psi} \geq 0, \tag{1}$$

where ψ is the energy storage potential (i.e. the Helmholtz energy density) parametrised by $\boldsymbol{E}$ and $\boldsymbol{I}$, that is

$$\psi = \psi(\boldsymbol{E}, \boldsymbol{I}). \tag{2}$$

For simplicity, but without loss of generality, we assume the energy storage potential to be henceforth a quadratic function in the generalised strain $\boldsymbol{E}$.

2.2. The energy storage potential

Substitution of expression (2) for the energy storage potential ψ into the dissipation inequality (1), and employing the standard Coleman–Noll procedure (Coleman and Noll, 1963), yields the relation for the energetic stress as

$$\boldsymbol{S}^{\mathrm{en}} := \psi_{,\boldsymbol{E}} = \boldsymbol{S} - \boldsymbol{S}^{\mathrm{dis}}. \tag{3}$$

Thus ψ serves as the potential for the energetic stress $\boldsymbol{S}^{\mathrm{en}}$.

The *reduced dissipation inequality* thus follows from the dissipation inequality (1) as

$$d = \boldsymbol{S}^{\mathrm{dis}} \circ \dot{\boldsymbol{E}} - \psi_{,\boldsymbol{I}} \bullet \dot{\boldsymbol{I}} \geq 0. \tag{4}$$

From the reduced dissipation inequality, it is apparent that the energy storage potential also serves as the potential function for the energetic driving force $\boldsymbol{X}^{\mathrm{en}}$ defined by

$$\boldsymbol{X}^{\mathrm{en}} := \psi_{,\boldsymbol{I}} \qquad \text{where we define} \qquad \boldsymbol{X}^{\mathrm{dis}} := -\boldsymbol{X}^{\mathrm{en}}. \tag{5}$$

The dissipative driving force $\boldsymbol{X}^{\mathrm{dis}}$ follows from Biot's relation (Halphen and Nguyen, 1975) in Eq. $(5)_b$ as the negative of the energetic driving force $\boldsymbol{X}^{\mathrm{en}}$.

Finally, the reduced dissipation inequality (4) can be expressed as

$$d = \boldsymbol{S}^{\mathrm{dis}} \circ \dot{\boldsymbol{E}} + \boldsymbol{X}^{\mathrm{dis}} \bullet \dot{\boldsymbol{I}} \geq 0. \tag{6}$$

In this format, the reduced dissipation inequality identifies the dissipation conjugate pairings as $\boldsymbol{S}^{\mathrm{dis}} \leftrightarrow \dot{\boldsymbol{E}}$ and $\boldsymbol{X}^{\mathrm{dis}} \leftrightarrow \dot{\boldsymbol{I}}$.

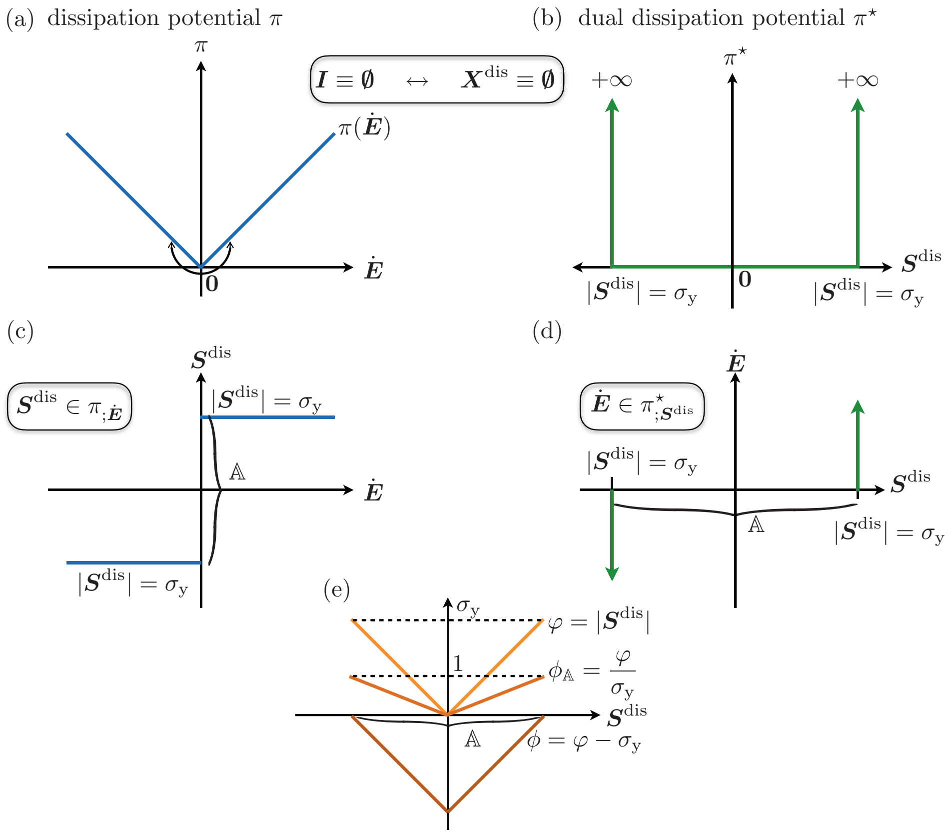

Fig. 1. A schematic of the relation between the non-smooth dissipation potential π in (a) and the corresponding dual dissipation potential $\pi^\star$ in (b), for the restricted case where $\pi = \pi(\dot{\boldsymbol{E}})$ and $\pi^\star = \pi^\star(\boldsymbol{S}^{\rm dis})$. The relation for the dissipative stress $\boldsymbol{S}^{\rm dis}$ in terms of the dissipation potential π, and the generalised strain rate $\dot{\boldsymbol{E}}$ in terms of the dual dissipation potential $\pi^\star$ are shown in (c) and (d), respectively. For this restricted case, the region of admissible stresses $\mathbb{A} = \mathbb{A}(\boldsymbol{S}^{\rm dis})$ has as its limit the yield stress $\sigma_{\rm y}$. The interior of the region of admissible stresses is denoted by $\mathrm{int}(\mathbb{A})$. The equivalent stress function φ, the yield function ϕ and the canonical yield function $\phi_{\mathbb{A}}$ are depicted in (e).

2.3. The dissipation potential

The dissipation potential is denoted by π and parametrised by $\dot{\boldsymbol{E}}$ and $\dot{\boldsymbol{I}}$, that is

$$\pi = \pi(\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}) .$$

The dissipation potential provides the required structure for the dissipative quantities $\{\boldsymbol{S}^{\rm dis}, \boldsymbol{X}^{\rm dis}\}$ in the reduced dissipation inequality (6). The dissipation potential is a *gauge* and hence satisfies the three properties listed in Eq. (A.6). Endowed with these properties, the dissipation potential characterises a *standard dissipative material*. A schematic of a non-smooth dissipation potential, for the restricted case where $\pi = \pi(\dot{\boldsymbol{E}})$, is shown in Fig. 1(a).

For a standard dissipative material, the dissipative stress $\boldsymbol{S}^{\rm dis}$ and the dissipative driving force $\boldsymbol{X}^{\rm dis}$ are in the subdifferential of the dissipation potential (see Fig. 1(c)). That is,

$$\boldsymbol{S}^{\rm dis} \in \pi_{;\dot{\boldsymbol{E}}} \qquad \text{and} \qquad \boldsymbol{X}^{\rm dis} \in \pi_{;\dot{\boldsymbol{I}}} . \tag{7}$$

2.4. The dual dissipation potential

The dual dissipation potential $\pi^\star$, the convex conjugate to π, is given by the Legendre–Fenchel transformation (defined in Eq. (A.9)) as

$$\pi^\star(\boldsymbol{S}^{\rm dis}, \boldsymbol{X}^{\rm dis}) = \sup_{\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}} \{\boldsymbol{S}^{\rm dis} \circ \dot{\boldsymbol{E}} + \boldsymbol{X}^{\rm dis} \bullet \dot{\boldsymbol{I}} - \pi(\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}})\} . \tag{8}$$

The dual dissipation potential serves as the potential for the rate of the generalised strain $\dot{\boldsymbol{E}}$ and the rate of the internal variables $\dot{\boldsymbol{I}}$. That is,

$$\dot{\boldsymbol{E}} \in \pi^\star_{;\boldsymbol{S}^{\rm dis}} \qquad \text{and} \qquad \dot{\boldsymbol{I}} \in \pi^\star_{;\boldsymbol{X}^{\rm dis}} .$$

Remark 1. If the dissipation potential π satisfies the requirements of a gauge, and the material can therefore be classified as standard dissipative, then the reduced dissipation inequality (6) is automatically satisfied. To see this, note that positively homogeneous functions (defined in Eq. (A.5)) are characterised by Euler's Theorem for homogeneous functions. This implies that

$$\boldsymbol{S}^{\mathrm{dis}} \circ \dot{\boldsymbol{E}} + \boldsymbol{X}^{\mathrm{dis}} \bullet \dot{\boldsymbol{I}} = k\pi(\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}),$$

for $k \geq 1$. The convexity of π, property (A.6a), and property (A.6b) imply that

$$0 \geq \pi(\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}) - \boldsymbol{S}^{\mathrm{dis}} \circ \dot{\boldsymbol{E}} - \boldsymbol{X}^{\mathrm{dis}} \bullet \dot{\boldsymbol{I}},$$
$$\Rightarrow d = \boldsymbol{S}^{\mathrm{dis}} \circ \dot{\boldsymbol{E}} + \boldsymbol{X}^{\mathrm{dis}} \bullet \dot{\boldsymbol{I}} = k\pi(\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}) \geq \pi(\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}) \geq 0.$$

For the choice of a non-smooth dissipation potential (the rate-independent problem), $k \equiv 1$ and the dissipation density d and the dissipation potential π coincide. This is not the case for a smooth dissipation potential (the rate-dependent problem) where, in general, $d \not\equiv \pi$. □

Remark 2. As mentioned, the rate-independent formulation corresponds to the choice of a non-smooth dissipation potential. The rate-independent problem is important as it represents the limit of the rate-dependent theory. Some key results concerning the *rate-independent* problem are now summarised.

The admissible region $\mathbb{A} = \mathbb{A}(\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}})$ is a closed convex set. The boundary of $\mathbb{A}$, denoted by $\mathrm{bdy}(\mathbb{A})$, is the level set σ_{y} of the convex function $\varphi(\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}}) = \sigma_{\mathrm{y}}$, where $\sigma_{\mathrm{y}} > 0$, and φ is termed the *equivalent stress function*. The equivalent stress function φ can be expressed as a gauge on $\mathbb{A}$, and in this form is distinguished by the notation $\phi_{\mathbb{A}}$ and termed the *canonical yield function* (see (A.5)). The yield function, denoted by ϕ, is defined by

$$\phi := \varphi - \sigma_{\mathrm{y}} \leq 0,$$

and the canonical yield function by

$$\phi_{\mathbb{A}} := \frac{\varphi}{\sigma_{\mathrm{y}}} \leq 1.$$

For a schematic depiction of the various yield functions and the equivalent stress function, for the restricted case where $\pi = \pi(\dot{\boldsymbol{E}})$, see Fig. 1(e).

For a non-smooth dissipation potential π, the dual dissipation potential $\pi^\star$ is the indicator function (see the definition in Eq. (A.7)) of the admissible region $\mathbb{A}$ of generalised stresses and driving forces, that is,

$$\pi^\star(\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}}) = I_{\mathbb{A}}(\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}}) = \begin{cases} 0 & \text{if } \{\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}}\} \in \mathbb{A} \\ +\infty & \text{otherwise} \end{cases}.$$

The support function of the region $\mathbb{A}$, denoted by $\sigma_{\mathbb{A}}$ and defined in Eq. (A.8), is given by

$$\sigma_{\mathbb{A}}(\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}) = \sup_{\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}}} \{\boldsymbol{S}^{\mathrm{dis}} \circ \dot{\boldsymbol{E}} + \boldsymbol{X}^{\mathrm{dis}} \bullet \dot{\boldsymbol{I}}\}, \quad \{\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}}\} \in \mathbb{A}.$$

The dissipation potential π is therefore the support function of the region $\mathbb{A}$.

Thus, from Eq. (A.10), the support function is conjugate to the indicator function, that is

$$I^\star_{\mathbb{A}} = \pi^{\star\star} = \sigma_{\mathbb{A}} = \pi.$$

Then, from Eq. (A.14), for $\{\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}}\} \in \mathbb{A}$ and $\boldsymbol{S}^{\mathrm{dis}} \in \pi_{;\dot{\boldsymbol{E}}}$ and $\boldsymbol{X}^{\mathrm{dis}} \in \pi_{;\dot{\boldsymbol{I}}}$, we have

$$\boldsymbol{S}^{\mathrm{dis}} \circ \dot{\boldsymbol{E}} + \boldsymbol{X}^{\mathrm{dis}} \bullet \dot{\boldsymbol{I}} = \phi_{\mathbb{A}}(\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}})\pi(\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}) = \begin{cases} \pi & \text{for } \{\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}\} \neq \boldsymbol{0} \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

From Eq. (8), the dissipation potential π and and the dual dissipation potential $\pi^\star$ are convex conjugates and possess an additive duality, whereas from Eq. (9) the dissipation potential and the canonical yield function $\phi_{\mathbb{A}}$ have a multiplicative duality.

Thus, in summary, we have

$$\{\boldsymbol{S}^{\mathrm{dis}} \in \pi_{;\dot{\boldsymbol{E}}}, \boldsymbol{X}^{\mathrm{dis}} \in \pi_{;\dot{\boldsymbol{I}}}\} \qquad \leftrightarrow \qquad \{\dot{\boldsymbol{E}} \in \pi^\star_{;\boldsymbol{S}^{\mathrm{dis}}}, \dot{\boldsymbol{I}} \in \pi^\star_{;\boldsymbol{X}^{\mathrm{dis}}}\} = N_{\mathbb{A}}(\boldsymbol{S}^{\mathrm{dis}}, \boldsymbol{X}^{\mathrm{dis}}),$$

where $N_{\mathbb{A}}$ is the normal cone to the admissible region $\mathbb{A}$ as defined in Eq. (A.2). □

Remark 3. The relation between the non-smooth dissipation potential π and the corresponding dual dissipation potential $\pi^\star$ is schematically depicted in Fig. 1(a) and (b), for the restricted case where $\pi = \pi(\dot{\boldsymbol{E}})$ and $\pi^\star = \pi^\star(\boldsymbol{S}^{\mathrm{dis}})$. The dual dissipation potential $\pi^\star$ is zero for all $\boldsymbol{S}^{\mathrm{dis}} \in \mathbb{A}$ and $+\infty$ otherwise (i.e. it is the indicator function of the admissible region $\mathbb{A}$). The subdifferential of the dissipation potential at the origin contains the set of all possible dissipative stresses interior to the admissible region $\mathbb{A}$ (i.e. the region $\mathrm{int}(\mathbb{A})$) (see Fig. 1(c)). Within the admissible region, the magnitude of the dissipative stress is bounded by the yield stress σ_{y}. The rate of the generalised strain $\dot{\boldsymbol{E}}$ lives in the subdifferential to the dual dissipation potential as shown in Fig. 1(d). □

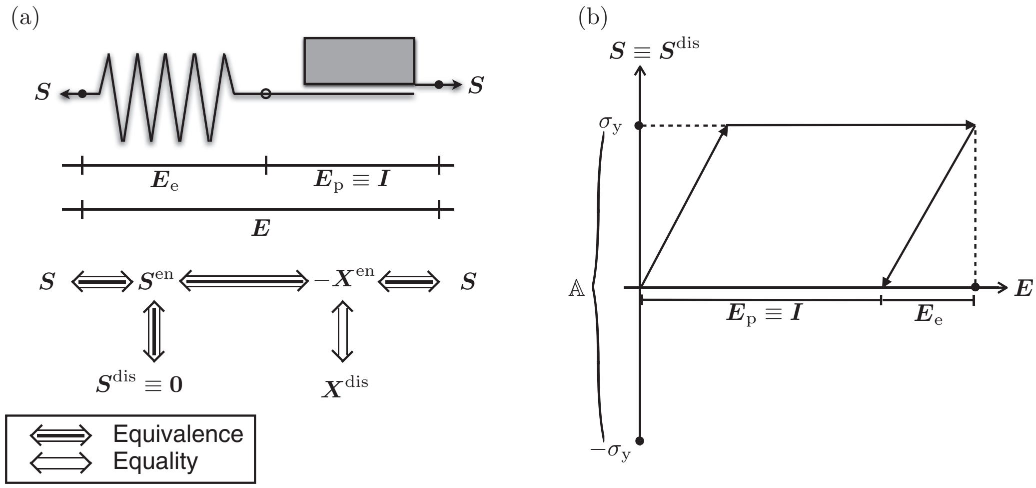

Fig. 2. The problem of local serial plasticity. A schematic of the rheological model and the relation between the stress and driving force is shown in (a). In (b), the relation between the stress and the strain is depicted.

3. Specialisation of the general setting to local plasticity

To fix the abstract ideas presented in the previous section on generalised dissipative materials and to illustrate features of the micromorphic models presented in subsequent sections, concrete examples of energetic and dissipative structures for the familiar problem of local plasticity in three dimensions are presented. The local plasticity models are constructed by combining simple one-dimensional rheological units. The resulting models are termed *local serial* and *local parallel* plasticity. The terminology serial or parallel denotes whether the energetic and dissipative rheological units act in series or in parallel to form the rheological structure. The local plasticity models could be extended by the addition of further rheological units. However such additional complexity is not required to illustrate the defining features of the more complex micromorphic models introduced in Section 4.

For the case of local plasticity, the generalised stress $\boldsymbol{S}$ is the Cauchy stress $\boldsymbol{\sigma}$, that is $\boldsymbol{S} \equiv \{\boldsymbol{\sigma}\}$, and the generalised strain $\boldsymbol{E}$ is the conventional strain $\boldsymbol{\epsilon}(\boldsymbol{u}) := \nabla^{\text{sym}}\boldsymbol{u}$, that is $\boldsymbol{E} \equiv \{\boldsymbol{\epsilon}(\boldsymbol{u})\}$.

3.1. Local serial plasticity

Consider the common rheological model of rate-independent *local serial plasticity* presented in Fig. 2(a), where an elastic spring acts in *serial* with a frictional sliding element. A stress $\boldsymbol{S}$ is applied to the system and the total strain $\boldsymbol{E}$ recorded. The stress-strain response is shown in Fig. 2(b). A certain magnitude of applied stress (the yield stress σ_y) is required to activate the frictional sliding element and thereby induce sliding (plastic flow). Once the frictional sliding element has yielded, it offers no further resistance to plastic deformation. That is, no hardening occurs. The measure of plastic deformation (sliding) is given by the plastic strain $\boldsymbol{E}_p$ which serves as the internal variable. This rheological model is the prototype for classical rate-independent elasto-perfect-plasticity.

For the case of local serial plasticity, the dissipative stress vanishes as the total stress $\boldsymbol{S}$ must be the same as the stress in the elastic spring, i.e. $\boldsymbol{S}^{\text{dis}} \equiv \boldsymbol{0}$ and hence $\boldsymbol{S} \equiv \boldsymbol{S}^{\text{en}}$ (see the schematic of the force balance relation in Fig. 2(a)). The internal variable is associated with the plastic strain, that is $\boldsymbol{I} \equiv \boldsymbol{E}_p$. The energy storage potential (2) (i.e. the energy stored in the elastic spring) is thus given by

$$\psi = \psi(\boldsymbol{E}, \boldsymbol{E}_p) .$$

A classical *constitutive choice* is to then assume that $\psi = \psi(\boldsymbol{E} - \boldsymbol{E}_p) = \psi(\boldsymbol{E}_e)$. Choosing the energy potential to be quadratic in the elastic strain $\boldsymbol{E}_e := \boldsymbol{E} - \boldsymbol{E}_p$ gives a linear relation between the stress $\boldsymbol{S}^{\text{en}}$ and the strain $\boldsymbol{E}$ in the interior of the admissible region $\text{int}(\mathbb{A}) := \{\boldsymbol{S}^{\text{en}} : \phi(\boldsymbol{S}^{\text{en}}) < 0\}$, where the yield function ϕ is given here by

$$\phi(\boldsymbol{S}^{\text{en}}) = \varphi(\boldsymbol{S}^{\text{en}}) - \sigma_y = |\boldsymbol{S}^{\text{en}}| - \sigma_y \le 0 , \tag{10}$$

and φ is the equivalent stress function. Note that in the interior of the admissible region $\dot{\boldsymbol{E}}_p \equiv \boldsymbol{0}$. The case of possible plastic flow is given when $\boldsymbol{S}^{\text{en}}$ is on the boundary of the admissible region, i.e. when $\phi(\boldsymbol{S}^{\text{en}}) \equiv 0$.

The total and energetic stresses, $\boldsymbol{S}$ and $\boldsymbol{S}^{\text{en}}$ respectively, are obtained from the elastic law and the generalised stress decomposition (3) as

$$\boldsymbol{S} \equiv \boldsymbol{S}^{\text{en}} = \psi_{,\boldsymbol{E}} .$$

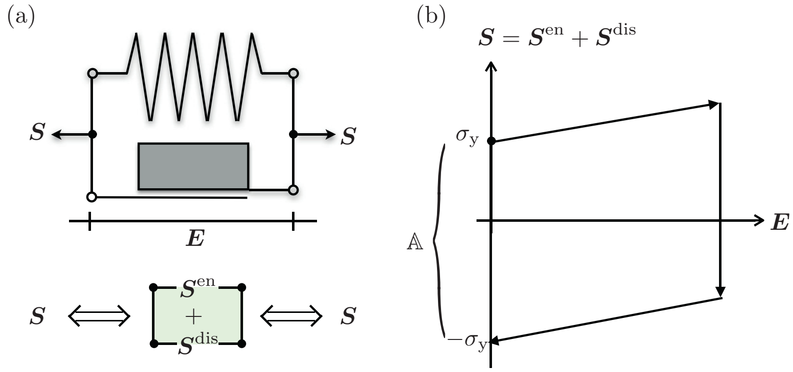

Fig. 3. The problem of local parallel plasticity. A schematic of the rheological model and the relation between the energetic and dissipative components of the stress is shown in (a). In (b), the relation between the stress $\boldsymbol{S}$ and the strain $\boldsymbol{E}$ is depicted.

From definition (5) of the energetic and dissipative driving forces, $\boldsymbol{X}^{en}$ and $\boldsymbol{X}^{dis}$ respectively, one obtains the relation between the driving force and the stress as

$$\boldsymbol{X}^{en} = \psi_{,\boldsymbol{E}_p} \equiv -\psi_{,\boldsymbol{E}} = -\boldsymbol{S}^{en} \equiv -\boldsymbol{S}\,,$$
$$\boldsymbol{X}^{dis} = -\boldsymbol{X}^{en} \equiv \boldsymbol{S} \in \pi_{;\dot{\boldsymbol{E}}_p}\,.$$

Hence the dissipative driving force $\boldsymbol{X}^{dis}$ and the stress $\boldsymbol{S} \equiv \boldsymbol{S}^{en}$ are equivalent for local serial plasticity. The dissipative driving force can be obtained from the dissipation potential (7). The reduced dissipation inequality for local serial plasticity follows from Eq. (6) as

$$d = \boldsymbol{S} \circ \dot{\boldsymbol{E}}_p = \boldsymbol{S}^{en} \circ \dot{\boldsymbol{E}}_p \geq 0\,.$$

It is important to note that the stress $\boldsymbol{S}$ is fully determined from the elastic law for the model of local serial plasticity. That is, the energetic force in the spring $\boldsymbol{S}^{en} \equiv \boldsymbol{S}$ is known throughout the loading process.

3.1.1. Prototype non-smooth and smooth dissipation potentials

The prototypical non-smooth dissipation potential (see e.g. Han and Reddy, 2010) for local serial plasticity (see Fig. 1) is given by

$$\pi\,(\dot{\boldsymbol{E}}_p) = \sigma_y |\dot{\boldsymbol{E}}_p|\,. \tag{11}$$

Note that the non-smooth dissipation potential is a positively homogeneous function of degree 1 (see Eq. (A.5)). Recall that the choice of a non-smooth dissipation potential corresponds to the rate-independent problem.

A prototypical regularised form of the dissipation potential[1] is given by

$$\pi\,(\dot{\boldsymbol{E}}_p) = \frac{\sigma_y}{1+\gamma} |\dot{\boldsymbol{E}}_p|^{1+\gamma}\,, \tag{12}$$

where the dissipation parameter $\gamma \in (0, 1]$. The choice of a smooth dissipation potential corresponds to the rate-dependent problem. For details of alternative regularisations of the dissipation potential and their associated numerical features, the reader is referred to Miehe et al. (2014), among others. For details of an alternative regularisation for strain-gradient plasticity, see Panteghini and Bardella (2016) and selected references therein. For the limit of $\gamma = 0$, the smooth (regularised) potential is equivalent to the non-smooth potential.

3.2. Local parallel plasticity

Consider the alternative rheological model of *local parallel plasticity* depicted in Fig. 3(a). In local parallel plasticity, an elastic spring and a frictional slider act in parallel to form the rheological unit. The model as depicted is rate independent. The slider will not yield until the magnitude of the applied stress $\boldsymbol{S}$ reaches the yield stress σ_y and correspondingly $\phi(\boldsymbol{S}) = 0$. Up until the point of yield, the system acts as a rigid body and the split of the stress $\boldsymbol{S}$ between the slider and the spring can not be determined. The stress-strain response for the system is shown in Fig. 3(b). Upon plastic flow, the spring extends and is responsible for the hardening-type response. This behaviour corresponds to rigid plasticity.

For local parallel plasticity, the extension of the spring and of the slider is the same and is given by the total strain $\boldsymbol{E}$. The energy storage potential is thus parametrised solely in terms of the strain $\boldsymbol{E}$, that is

$$\psi = \psi\,(\boldsymbol{E})\,.$$

[1] The results presented in this and subsequent sections hold for any of the widely-used regularisations of the non-smooth dissipation potential.

In addition the dissipative stress $\boldsymbol{S}^{\mathrm{dis}} \neq \boldsymbol{0}$, whereas the set of internal variables is empty, i.e. $\boldsymbol{I} \equiv \emptyset$. It is clear from Biot's relation in Eq. (5)$_b$ that $\boldsymbol{X}' = -\boldsymbol{X}'' \equiv \emptyset$.

The reduced dissipation inequality thus follows from Eq. (6) as

$$d = \boldsymbol{S}^{\mathrm{dis}} \circ \dot{\boldsymbol{E}} \geq 0\,.$$

The energetic and dissipative stresses, given by Eqs. (3) and(7)$_b$, take the form

$$\boldsymbol{S}^{\mathrm{en}} = \psi_{,\boldsymbol{E}} \qquad \text{and} \qquad \boldsymbol{S}^{\mathrm{dis}} \in \pi_{;\dot{\boldsymbol{E}}} \qquad \text{with} \qquad \boldsymbol{S} = \boldsymbol{S}^{\mathrm{en}} + \boldsymbol{S}^{\mathrm{dis}}\,.$$

Note, as is clear from the schematic in Fig. 3(b), $\boldsymbol{S}^{\mathrm{dis}}$ is not determinable in the admissible region for the model of local parallel plasticity. Such a response corresponds to rigid-plastic behaviour. As will be shown in Section 6 on micromorphic parallel plasticity, the identical problem of an undetermined generalised stress in the admissible region is present in both the general energetic-dissipative and the fully-dissipative cases.

3.2.1. Prototype non-smooth and smooth dissipation potentials

In a near-identical manner to local serial plasticity (see Section 3.1.1), the non-smooth dissipation potential for local parallel plasticity is given by

$$\pi(\dot{\boldsymbol{E}}) = \sigma_{\mathrm{y}} |\dot{\boldsymbol{E}}|\,.$$

Likewise, a prototypical regularised form of the dissipation potential is given by

$$\pi(\dot{\boldsymbol{E}}) = \frac{\sigma_{\mathrm{y}}}{1+\gamma} |\dot{\boldsymbol{E}}|^{1+\gamma}\,,$$

with the regularisation parameter $\gamma \in (0, 1]$.

4. Micromorphic continua as a framework for extended plasticity

Micromorphic continua are characterised by additional *micro* degrees of freedom associated with each continuum point. Gradient continua, by contrast, possess higher gradients of the primary fields. Both approaches can be applied to the problem of plasticity. For a classification of a wide range of micromorphic and strain-gradient plasticity formulations, the reader is referred to Hirschberger and Steinmann (2009), Forest (2009, 2010). The Eringen-type micromorphic framework considered here can be viewed as a penalised approximation of a Mindlin-type gradient theory (see Appendix B for details). By associating the micro degrees of freedom with measures of the plastic deformation, this choice of framework allows for a unified classification of various important micromorphic and strain-gradient plasticity models. The additional *micro* degrees of freedom in the micromorphic theory allow for the modelling of size-dependent phenomena during plastic deformation. The framework for micromorphic continua is now presented. The specialisation of the framework to models of extended plasticity is presented in Sections 5 and 6.

The conventional *macro strain* $\boldsymbol{\epsilon}(\boldsymbol{u})$ in a micromorphic continuum is defined, as in the local theory presented in Section 3, by the symmetric gradient of the displacement field $\boldsymbol{u}$, that is

$$\boldsymbol{\epsilon}(\boldsymbol{u}) := \nabla^{\mathrm{sym}} \boldsymbol{u}\,.$$

The *micro strain* and *micro double strain* of the micromorphic continuum are respectively denoted by

$$\boldsymbol{\varepsilon} \qquad \text{and} \qquad \boldsymbol{\gamma}(\boldsymbol{\varepsilon}) := \nabla \boldsymbol{\varepsilon}\,.$$

To avoid confusion between the similar notation adopted for the macro and micro strain measures, the functional dependence of the macro strain on the displacement $\boldsymbol{u}$ will always be indicated. The two-scale *relative strain* $\boldsymbol{\delta}$ is defined as the difference between the macro strain $\boldsymbol{\epsilon}(\boldsymbol{u})$ and the micro strain $\boldsymbol{\varepsilon}$, that is

$$\boldsymbol{\delta}(\boldsymbol{u}, \boldsymbol{\varepsilon}) := \boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}\,.$$

The magnitude of the relative strain quantifies the closeness of the micromorphic and the Mindlin-type gradient formulations. The generalised micro strain $\boldsymbol{E}$ is defined by

$$\boldsymbol{E} := \{\boldsymbol{\varepsilon}, \boldsymbol{\gamma}\ell\}\,,$$

where $\ell > 0$ is an internal length scale.

The internal power density $p^{\mathrm{int}}_{\mathrm{E}}$ for the two-field Eringen-type micromorphic continuum considered here is given by

$$p^{\mathrm{int}}_{\mathrm{E}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{\varepsilon}}) = \boldsymbol{\sigma} : \dot{\boldsymbol{\delta}}(\boldsymbol{u}, \boldsymbol{\varepsilon}) + \boldsymbol{\varsigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\mu} \,\vdots\, \boldsymbol{\gamma}(\dot{\boldsymbol{\varepsilon}})\,, \tag{13}$$

where the conventional *macro stress* is denoted by $\boldsymbol{\sigma}$, and the *micro stress* and *micro double stress* are denoted by $\boldsymbol{\varsigma}$ and $\boldsymbol{\mu}$, respectively. The generalised micro stress is thus defined by $\boldsymbol{S} := \{\boldsymbol{\varsigma}, \boldsymbol{\mu}/\ell\}$. The corresponding Euler–Lagrange equations are given by

$$\boldsymbol{0} = \operatorname{div} \boldsymbol{\sigma}\,, \tag{14}$$

$$\boldsymbol{\varsigma} = \boldsymbol{\sigma} + \operatorname{div} \boldsymbol{\mu} . \tag{15}$$

The macro relation (14) is the standard statement of (macro) equilibrium. The micro relation (15) is a two-scale stress balance.

The resulting dissipation inequality for the two-field Eringen-type micromorphic model reads as follows

$$d := p_{\mathrm{E}}^{\mathrm{int}} - \dot{\psi} \geq 0 .$$

In the next two sections we analyse various specialisations of the energy storage potential ψ for micromorphic plasticity models. In the spirit of the local problem presented in Section 3, these specialisations are termed *micromorphic serial plasticity* and *micromorphic parallel plasticity*.

5. Micromorphic serial plasticity

Serial models for micromorphic plasticity have been presented under different names (see e.g. Forest, 2009; Grammenoudis and Tsakmakis, 2009; Grammenoudis et al., 2009; Hirschberger and Steinmann, 2009; Sansour et al., 2010). For the case of local serial plasticity presented in Section 3.1, the energy storage potential was parametrised by the difference between the total strain $\boldsymbol{E}$ and the plastic strain $\boldsymbol{E}_{\mathrm{p}}$. By analogy, the energy storage potential for micromorphic serial plasticity is given by

$$\psi = \psi(\boldsymbol{\delta}(\boldsymbol{u}, \boldsymbol{\varepsilon}), [\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\mathrm{p}}], [\boldsymbol{\gamma}(\boldsymbol{\varepsilon}) - \boldsymbol{\gamma}_{\mathrm{p}}]) ,$$

where the plastic micro strain and plastic micro double strain are denoted by

$$\boldsymbol{\varepsilon}_{\mathrm{p}} \qquad \text{and} \qquad \boldsymbol{\gamma}_{\mathrm{p}} .$$

The generalised micro stress and plastic micro strain pairs are defined by

$$\boldsymbol{S} = \{\boldsymbol{\varsigma}, \boldsymbol{\mu}/\ell\} \qquad \text{and} \qquad \boldsymbol{E}_{\mathrm{p}} = \{\boldsymbol{\varepsilon}_{\mathrm{p}}, \boldsymbol{\gamma}_{\mathrm{p}}\ell\} \equiv \boldsymbol{I} ,$$

where the internal length scale $\ell > 0$ is introduced, as before, for dimensional consistency. The generalised inner product of $\boldsymbol{S}$ and $\dot{\boldsymbol{E}}_{\mathrm{p}}$ is thus given by

$$\boldsymbol{S} \circ \dot{\boldsymbol{E}}_{\mathrm{p}} = \boldsymbol{\varsigma} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} + \boldsymbol{\mu} \mathbin{\vdots} \dot{\boldsymbol{\gamma}}_{\mathrm{p}} .$$

The macro (energetic) stress $\boldsymbol{\sigma}$ follows as per the standard definition as

$$\boldsymbol{\sigma} = \psi_{,\boldsymbol{\delta}} . \tag{16}$$

As in the local serial model, the generalised micro stress is fully energetic, that is $\boldsymbol{S} \equiv \boldsymbol{S}^{\mathrm{en}}$ and thus $\boldsymbol{S}^{\mathrm{dis}} \equiv \emptyset$ (see Fig. 2). The micro stress and micro double stress follow as

$$\boldsymbol{\varsigma} \equiv \boldsymbol{\varsigma}^{\mathrm{en}} = \psi_{,\boldsymbol{\varepsilon}} = -\psi_{,\boldsymbol{\varepsilon}_{\mathrm{p}}} \in \pi_{;\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}}} \qquad \text{and} \qquad \boldsymbol{\mu} \equiv \boldsymbol{\mu}^{\mathrm{en}} = \psi_{,\boldsymbol{\gamma}} = -\psi_{,\boldsymbol{\gamma}_{\mathrm{p}}} \in \pi_{;\dot{\boldsymbol{\gamma}}_{\mathrm{p}}} .$$

The generalised micro stress can therefore be expressed as

$$\boldsymbol{S} \equiv \boldsymbol{S}^{\mathrm{en}} = \psi_{,\boldsymbol{E}} \equiv -\psi_{,\boldsymbol{E}_{\mathrm{p}}} \in \pi_{;\dot{\boldsymbol{E}}_{\mathrm{p}}} , \tag{17}$$

where the subdifferential of the dissipation potential (see the Definition A.4) is given in generalised form as

$$\pi_{;\dot{\boldsymbol{E}}_{\mathrm{p}}} := \left\{ \boldsymbol{S}^{\mathrm{en}} \;\middle|\; \pi(\boldsymbol{Q}) - \pi(\dot{\boldsymbol{E}}_{\mathrm{p}}) - \boldsymbol{S}^{\mathrm{en}} \circ [\boldsymbol{Q} - \dot{\boldsymbol{E}}_{\mathrm{p}}] \geq 0 , \quad \forall \boldsymbol{Q} \right\} . \tag{18}$$

The relations characterising micromorphic serial plasticity are identical in structure to those of the familiar local serial plasticity problem presented in Section 3.1. We will now show briefly that the identical conclusions on the determinability of the stress $\boldsymbol{S}$ in the admissible region for classical local plasticity hold for the generalised stress in the micromorphic case. The case of a prototypical non-smooth dissipation potential is considered first.

5.1. Prototype non-smooth dissipation potential

The prototypical non-smooth dissipation potential for local serial plasticity was given in Eq. (11). The corresponding non-smooth dissipation potential (a positively homogeneous function of degree 1) for the problem of micromorphic serial plasticity is given by

$$\pi(\dot{\boldsymbol{E}}_{\mathrm{p}}) = \sigma_{\mathrm{y}} |\dot{\boldsymbol{E}}_{\mathrm{p}}| .$$

The structure of the admissible region $\mathbb{A}$ is a consequence of the choice of the dissipation potential π. The interior of the admissible region $\mathrm{int}(\mathbb{A})$ is obtained by evaluating the subdifferential of the dissipation potential (18) at $\dot{\boldsymbol{E}}_{\mathrm{p}} \equiv \boldsymbol{0}$, which gives

$$\boldsymbol{S}^{\mathrm{en}} \circ \boldsymbol{Q} \leq \sigma_{\mathrm{y}} |\boldsymbol{Q}| , \quad \forall \boldsymbol{Q} .$$

The definition of the admissible region of $\boldsymbol{S}^{\text{en}}$ then follows from the supremum of the function $r(\boldsymbol{S}^{\text{en}};\boldsymbol{Q})$ given below over all arbitrary $\boldsymbol{Q}$, that is

$$y(\boldsymbol{S}^{\text{en}}) = \sup_{\boldsymbol{Q}} r(\boldsymbol{S}^{\text{en}};\boldsymbol{Q}) \leq 1 \qquad \text{where} \qquad r(\boldsymbol{S}^{\text{en}},\boldsymbol{Q}) := \frac{\boldsymbol{S}^{\text{en}} \circ \boldsymbol{Q}}{\sigma_{\text{y}}|\boldsymbol{Q}|}. \tag{19}$$

The problem of determining the supremum can be stated as a local optimality problem. First, define $\overline{\boldsymbol{Q}}$ by

$$\overline{\boldsymbol{Q}} = \arg\left\{\sup_{\boldsymbol{Q}} r(\boldsymbol{S}^{\text{en}};\boldsymbol{Q})\right\}.$$

Consequently, we seek the roots of

$$\left.\frac{\partial r(\boldsymbol{S}^{\text{en}};\boldsymbol{Q})}{\partial \boldsymbol{Q}}\right|_{\overline{\boldsymbol{Q}}} \doteq \boldsymbol{0}.$$

The local optimality condition follows in the format of an orthogonality condition as

$$\boldsymbol{S}^{\text{en}} - r(\boldsymbol{S}^{\text{en}};\overline{\boldsymbol{Q}})\sigma_{\text{y}}\frac{\overline{\boldsymbol{Q}}}{|\overline{\boldsymbol{Q}}|} = \boldsymbol{S}^{\text{en}} - \boldsymbol{S}^{\text{en}} \circ \left[\frac{\overline{\boldsymbol{Q}}}{|\overline{\boldsymbol{Q}}|} \otimes \frac{\overline{\boldsymbol{Q}}}{|\overline{\boldsymbol{Q}}|}\right] = \boldsymbol{0}. \tag{20}$$

The local optimality point is therefore given by $\overline{\boldsymbol{Q}}$ being coaxial to the given $\boldsymbol{S}^{\text{en}}$, that is

$$\frac{\boldsymbol{S}^{\text{en}}}{|\boldsymbol{S}^{\text{en}}|} \equiv \frac{\overline{\boldsymbol{Q}}}{|\overline{\boldsymbol{Q}}|}.$$

Thus, inserting $\overline{\boldsymbol{Q}}$ into the function $r(\boldsymbol{S}^{\text{en}};\boldsymbol{Q})$, the admissible region is defined *locally* by

$$y(\boldsymbol{S}^{\text{en}}) \equiv \frac{|\boldsymbol{S}^{\text{en}}|}{\sigma_{\text{y}}} \leq 1, \tag{21}$$

which is the canonical yield function $\phi_{\mathbb{A}} = y(\boldsymbol{S}^{\text{en}})$ of the admissible region $\mathbb{A}$. The yield function for micromorphic serial plasticity thus has the familiar local structure (c.f. Eq. (10)):

$$\phi(\boldsymbol{S}^{\text{en}}) = \varphi(\boldsymbol{S}^{\text{en}}) - \sigma_{\text{y}} = |\boldsymbol{S}^{\text{en}}| - \sigma_{\text{y}} \leq 0.$$

The structure of micromorphic serial plasticity is therefore identical to the local problem as depicted in Fig. 2.

For the case of plastic flow (i.e. $\dot{\boldsymbol{E}}_{\text{p}} \neq \boldsymbol{0}$), the flow rule follows from the smooth part of the dissipation potential π (i.e. the region away from the origin) in an identical fashion to local serial plasticity as

$$\boldsymbol{S}^{\text{en}} = \pi_{,\dot{\boldsymbol{E}}_{\text{p}}} = \sigma_{\text{y}}\frac{\dot{\boldsymbol{E}}_{\text{p}}}{|\dot{\boldsymbol{E}}_{\text{p}}|},$$

which can be inverted to obtain

$$\dot{\boldsymbol{E}}_{\text{p}} = \frac{|\dot{\boldsymbol{E}}_{\text{p}}|\boldsymbol{S}^{\text{en}}}{\sigma_{\text{y}}} =: \lambda\frac{\boldsymbol{S}^{\text{en}}}{|\boldsymbol{S}^{\text{en}}|},$$

where $\varphi(\boldsymbol{S}^{\text{en}}) = |\boldsymbol{S}^{\text{en}}| \equiv \sigma_{\text{y}}$ at yield, and the (positive) plastic multiplier is defined by $\lambda := |\dot{\boldsymbol{E}}_{\text{p}}| \geq 0$. The flow rule for the generalised stress for the case of plastic flow is a positively homogeneous function of degree 0. Hence, $\boldsymbol{S}^{\text{en}}(\dot{\boldsymbol{E}}_{\text{p}}) = \boldsymbol{S}^{\text{en}}(k\dot{\boldsymbol{E}}_{\text{p}})$ for all $k > 0$. The non-smooth dissipation potential therefore corresponds to the rate-independent problem.

Note that, as for local serial plasticity, $\boldsymbol{S} = \boldsymbol{S}^{\text{en}}$ is fully determined from the elastic law (17).

Remark 4. The definition of the admissible region for $\boldsymbol{S}^{\text{en}}$ in Eq. (19) can be obtained alternatively from the properties of the polar function given in Eq. (A.5). The support function of $\mathbb{A}$ is given by

$$\sigma_{\mathbb{A}}(\boldsymbol{Q}) = \sup_{\boldsymbol{S}^{\text{en}}}\{\boldsymbol{S}^{\text{en}} \circ \boldsymbol{Q}\} = \pi(\boldsymbol{Q}) = \sigma_{\text{y}}|\boldsymbol{Q}|.$$

Then from Eq. (A.14), for $\boldsymbol{S}^{\text{en}} \in \mathbb{A}$ and $\boldsymbol{S}^{\text{en}} \in \sigma_{\mathbb{A};\boldsymbol{Q}} = \pi_{;\boldsymbol{Q}}$, $\boldsymbol{Q} \neq \boldsymbol{0}$, we obtain

$$\begin{aligned}\boldsymbol{S}^{\text{en}} \circ \boldsymbol{Q} &= \phi_{\mathbb{A}}(\boldsymbol{S}^{\text{en}})\sigma_{\mathbb{A}}(\boldsymbol{Q}) \\ &= \phi_{\mathbb{A}}(\boldsymbol{S}^{\text{en}})\sigma_{\text{y}}|\boldsymbol{Q}|,\end{aligned}$$

where $\phi_{\mathbb{A}}(\boldsymbol{S}^{\text{en}}) \equiv y(\boldsymbol{S}^{\text{en}}) \leq 1$ is the canonical yield function. □

5.2. Prototype smooth dissipation potential

The admissible domain and the flow law for a prototypical smooth dissipation potential for the case of micromorphic serial plasticity are now determined. Consider the following regularised dissipation potential (c.f. Eq. (12)):

$$\pi(\dot{\boldsymbol{E}}_{\mathrm{p}}) = \frac{\sigma_{\mathrm{y}}}{1+\gamma}|\dot{\boldsymbol{E}}_{\mathrm{p}}|^{1+\gamma},$$

for $\gamma \in (0, 1]$. The admissible domain can be determined by evaluating the subdifferential of the dissipation potential (18) using the regularised dissipation potential evaluated at $\dot{\boldsymbol{E}}_{\mathrm{p}} \equiv \boldsymbol{0}$. This gives

$$\boldsymbol{S}^{\mathrm{en}} \circ \boldsymbol{Q} \leq \frac{\sigma_{\mathrm{y}}}{1+\gamma}|\boldsymbol{Q}|^{1+\gamma}, \quad \forall \boldsymbol{Q}.$$

The admissible domain is obtained by maximising the function $r_\gamma = r_\gamma(\boldsymbol{S}^{\mathrm{en}}; \boldsymbol{Q})$ given below over all arbitrary $\boldsymbol{Q}$ to obtain

$$y_\gamma(\boldsymbol{S}^{\mathrm{en}}) = \max_{\boldsymbol{Q}}\{r_\gamma(\boldsymbol{S}^{\mathrm{en}}; \boldsymbol{Q})\} \leq \frac{1}{1+\gamma} \qquad \text{where} \qquad r_\gamma(\boldsymbol{S}^{\mathrm{en}}; \boldsymbol{Q}) := \frac{\boldsymbol{S}^{\mathrm{en}} \circ \boldsymbol{Q}}{\sigma_{\mathrm{y}}|\boldsymbol{Q}|^{1+\gamma}}.$$

As will be shown in the subsequent remark, the only possible option for the above relation to hold true is for

$$\boldsymbol{S} \equiv \boldsymbol{S}^{\mathrm{en}} = \boldsymbol{0} \qquad \Rightarrow \qquad |\boldsymbol{S}| \equiv |\boldsymbol{S}^{\mathrm{en}}| = 0.$$

Thus there is no admissible domain for this choice of smooth dissipation potential, with the consequence that there is always plastic flow, albeit potentially negligible.

Remark 5. The maximisers $\overline{\boldsymbol{Q}}$ of $r_\gamma(\boldsymbol{S}^{\mathrm{en}}; \boldsymbol{Q})$ are given by the following local optimality problem:

$$\boldsymbol{S}^{\mathrm{en}} \doteq [1+\gamma]\left[\boldsymbol{S}^{\mathrm{en}} \circ \frac{\overline{\boldsymbol{Q}}}{|\overline{\boldsymbol{Q}}|}\right]\frac{\overline{\boldsymbol{Q}}}{|\overline{\boldsymbol{Q}}|}, \tag{22}$$

for $\gamma \in (0, 1]$. For the relation (22) to hold, it is clear that $\boldsymbol{S}^{\mathrm{en}}$ and $\overline{\boldsymbol{Q}}$ must be colinear. Thus, the only general choice of $\boldsymbol{S}^{\mathrm{en}}$ for $\gamma \in (0, 1]$ is $\boldsymbol{S}^{\mathrm{en}} = \boldsymbol{0}$. □

The flow rule then follows directly from the partial derivative of the smooth dissipation potential with respect to $\dot{\boldsymbol{E}}_{\mathrm{p}}$ for $\dot{\boldsymbol{E}}_{\mathrm{p}} \neq \boldsymbol{0}$[2]:

$$\boldsymbol{S}^{\mathrm{en}} = \sigma_{\mathrm{y}}|\dot{\boldsymbol{E}}_{\mathrm{p}}|^{\gamma}\frac{\dot{\boldsymbol{E}}_{\mathrm{p}}}{|\dot{\boldsymbol{E}}_{\mathrm{p}}|} \qquad \text{with} \qquad |\boldsymbol{S}^{\mathrm{en}}| = \sigma_{\mathrm{y}}|\dot{\boldsymbol{E}}_{\mathrm{p}}|^{\gamma}.$$

Hence,

$$\left[\frac{|\boldsymbol{S}^{\mathrm{en}}|}{\sigma_{\mathrm{y}}}\right]^{1/\gamma} = |\dot{\boldsymbol{E}}_{\mathrm{p}}|.$$

Upon rearranging,

$$\dot{\boldsymbol{E}}_{\mathrm{p}} = \left[\frac{|\boldsymbol{S}^{\mathrm{en}}|}{\sigma_{\mathrm{y}}}\right]^{1/\gamma}\frac{\boldsymbol{S}^{\mathrm{en}}}{|\boldsymbol{S}^{\mathrm{en}}|} = \lambda_\gamma\frac{\boldsymbol{S}^{\mathrm{en}}}{|\boldsymbol{S}^{\mathrm{en}}|}.$$

Regarding the determination of $\boldsymbol{S} \equiv \boldsymbol{S}^{\mathrm{en}}$, note that from the assumed smoothness of the dissipation potential π, the flow rule for the generalised stress is an alternative to the elastic law (17).

6. Micromorphic parallel plasticity

The extension of the local model of parallel plasticity discussed in Section 3.2 to the micromorphic setting is now considered. Eringen-type micromorphic plasticity, presented in Section 4 encapsulates the strain-gradient plasticity model of Gurtin and Anand (2005). In the micromorphic model, the total micro strain is the plastic strain in the Gurtin and Anand model, and $\boldsymbol{\gamma}(\boldsymbol{\varepsilon})$ is the plastic strain gradient, that is $\boldsymbol{\varepsilon} \equiv \boldsymbol{\varepsilon}^{\mathrm{p}}$ and $\boldsymbol{\gamma}(\boldsymbol{\varepsilon}) \equiv \nabla\boldsymbol{\varepsilon}^{\mathrm{p}}$. Thus, the Gurtin and Anand strain-gradient plasticity model, and variants thereof, can be viewed as micrcomorphic parallel plasticity models.

The two-scale stress balance (15) is central to the strain-gradient plasticity theories of Gurtin and co-workers (henceforth referred to as the Gurtin model where it is termed the microforce balance. The microforce balance was derived by Gurtin, first in the context of single crystal plasticity, using a virtual power balance wherein the micro stress $\boldsymbol{\varsigma}$ is power conjugate to the plastic strain rate, and the double micro stress is power conjugate to the gradient of the plastic strain rate. Thus, by examining the expression for the internal power density (13), Gurtin identifies $\boldsymbol{\varepsilon}$ as the plastic strain and $\boldsymbol{\gamma}$ as the plastic strain gradient.[3]

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[2] For the case of $\dot{\boldsymbol{E}}_{\mathrm{p}} = \boldsymbol{0}$, $\boldsymbol{S}^{\mathrm{en}}$ is determined exclusively from the elastic relation or directly using an alternative regularised form of π (see e.g. Miehe et al., 2014).

[3] In order to simplify the presentation, the consequences of the volume preserving nature of plastic flow in metals have not been accounted for here. In the gradient plasticity theory of Gurtin and Anand (2005), the macro stress appearing in Eq. (15) is replaced by its deviatoric part. As a consequence, the micro stress is symmetric and deviatoric, and the micro double stress is symmetric and deviatoric in its first two components.

Gurtin and co-workers formulate their models solely for the rate-dependent (viscoplastic) case. The work (Reddy et al., 2008) was the first to present the model in a rate-independent framework, using the tools of convex analysis to derive generalisations of the normality condition and, importantly, the notion of a global yield condition. This was further developed in Reddy (2011a,b). See also Han and Reddy (2010) for a detailed account. Much of the material on the choice of a non-smooth dissipation potential presented in the previous sections follows from these works. A major contribution of the present work is the extension to the smooth case. In Section 6, the relation between the various strain-gradient plasticity theories of Gurtin and co-workers and the more general micromorphic setting is made clear.

Three specialisations of the mircromorphic parallel plasticity model are now considered:

(i) the general energetic-dissipative case;
(ii) the hybrid energetic-dissipative case;
(iii) the fully-dissipative case.

The hybrid energetic-dissipative case and the fully-dissipative case are specialisations of the general energetic-dissipative case. By direct analogy with local parallel plasticity where the energy storage potential is parametrised by the total strain $\boldsymbol{E}$, the inelastic terms in the energy storage potential for the corresponding micromorphic problem relate to the total micro strain $\boldsymbol{\varepsilon}$ and its gradient $\boldsymbol{\gamma}(\boldsymbol{\varepsilon})$, and the two-scale relative strain $\boldsymbol{\delta}$. The energy storage potential is therefore given by

$$\psi = \psi\left(\boldsymbol{\delta}(\boldsymbol{u}, \boldsymbol{\varepsilon}), \boldsymbol{\varepsilon}, \boldsymbol{\gamma}(\boldsymbol{\varepsilon})\right).$$

The dissipation potential is now a function of the rates of the micro strain and micro double strain, that is

$$\pi = \pi\left(\dot{\boldsymbol{\varepsilon}}, \boldsymbol{\gamma}(\dot{\boldsymbol{\varepsilon}})\right).$$

The macro stress follows as for the serial problem (see Eq. (16)) as

$$\boldsymbol{\sigma} = \psi_{,\boldsymbol{\delta}}.$$

The micro stress $\boldsymbol{\varsigma}$ and micro double stress $\boldsymbol{\mu}$ are, in general, composed of energetic and dissipative components which can be determined from the energy storage and dissipation potentials as follows:

$$\boldsymbol{\varsigma} = \boldsymbol{\varsigma}^{\text{en}} + \boldsymbol{\varsigma}^{\text{dis}} \in \psi_{,\boldsymbol{\varepsilon}} + \pi_{;\dot{\boldsymbol{\varepsilon}}} \qquad \text{and} \qquad \boldsymbol{\mu} = \boldsymbol{\mu}^{\text{en}} + \boldsymbol{\mu}^{\text{dis}} \in \psi_{,\boldsymbol{\gamma}} + \pi_{;\dot{\boldsymbol{\gamma}}}. \tag{23}$$

The dissipative generalised micro stress $\boldsymbol{S}^{\text{dis}} := \boldsymbol{S} - \boldsymbol{S}^{\text{en}}$ can therefore be defined in terms of the dissipation potential by

$$\boldsymbol{S}^{\text{dis}} := \{\boldsymbol{\varsigma}^{\text{dis}}, \boldsymbol{\mu}^{\text{dis}}/\ell\} \in \pi_{;\dot{\boldsymbol{E}}},$$

where the generalised micro strain is defined by

$$\boldsymbol{E} := \{\boldsymbol{\varepsilon}, \boldsymbol{\gamma}\ell\}.$$

The three specialisations of mircromorphic parallel plasticity are now presented. These are obtained by restricting the decomposition of the micro stress $\boldsymbol{\varsigma}$ and micro double stress $\boldsymbol{\mu}$ presented in Eq. (23).

6.1. The general energetic-dissipative case

For the general energetic-dissipative case of micromorphic parallel plasticity, the energetic micro stress $\boldsymbol{S}^{\text{en}} \neq \boldsymbol{0}$ and, likewise, the dissipative micro stress $\boldsymbol{S}^{\text{dis}} \neq \boldsymbol{0}$.

The admissible region and the flow rule follow from evaluating the subdifferential of the dissipation potential which is given by

$$\pi_{;\dot{\boldsymbol{E}}} := \left\{\boldsymbol{S}^{\text{dis}} \mid \pi(\boldsymbol{Q}) - \pi(\dot{\boldsymbol{E}}) - \boldsymbol{S}^{\text{dis}} \circ [\boldsymbol{Q} - \dot{\boldsymbol{E}}] \geq 0, \quad \forall \boldsymbol{Q}\right\}.$$

As in the case of local parallel plasticity, in the admissible region the generalised dissipative stress $\boldsymbol{S}^{\text{dis}}$ is not determinable from the elastic law for a non-smooth dissipation potential. The determination of the dissipative stress contribution is similar to the fully-dissipative case discussed in Section 6.3, where further details are provided.

6.2. The hybrid energetic-dissipative case

Consider now the hybrid energetic-dissipative case where the micro stress $\boldsymbol{\varsigma} \equiv \boldsymbol{\varsigma}^{\text{dis}}$ is fully dissipative and the micro double stress $\boldsymbol{\mu} \equiv \boldsymbol{\mu}^{\text{en}}$ is fully energetic, that is:

$$\boldsymbol{\varsigma} = \boldsymbol{\varsigma}^{\text{dis}} \in \pi_{;\dot{\boldsymbol{\varepsilon}}} \qquad \text{and} \qquad \boldsymbol{\mu} = \boldsymbol{\mu}^{\text{en}} = \psi_{,\boldsymbol{\gamma}}.$$

The hybrid case has been considered by various authors including Reddy et al. (2012). For the hybrid case, the microforce balance (15) becomes

$$\boldsymbol{\varsigma}^{\text{dis}} = \boldsymbol{\sigma} + \operatorname{div} \boldsymbol{\mu}^{\text{en}},$$

and the dissipative micro stress can be determined in the admissible region as both the macro stress and the micro double stress are obtainable from the energy storage potential.

6.3. The fully-dissipative case

For the fully-dissipative problem, the generalised micro stress $\boldsymbol{S} \equiv \boldsymbol{S}^{\text{dis}}$ and hence

$$\begin{aligned}
&\boldsymbol{\varsigma} = \boldsymbol{\varsigma}^{\text{dis}} \in \pi_{;\dot{\boldsymbol{\varepsilon}}}\,, \\
&\boldsymbol{\mu} = \boldsymbol{\mu}^{\text{dis}} \in \pi_{;\dot{\boldsymbol{\gamma}}}\,, \\
&\boldsymbol{S} \equiv \boldsymbol{S}^{\text{dis}} := \{\boldsymbol{\varsigma}^{\text{dis}}, \boldsymbol{\mu}^{\text{dis}}/\ell\} \in \pi_{;\dot{\boldsymbol{E}}}\,.
\end{aligned}$$

The fully-dissipative case has been considered by various authors including Fleck et al. (2014) and Carstensen et al. (2017).

Once again, the admissible domain and the flow rule follow from evaluating the subdifferential of the dissipation potential, that is

$$\pi_{;\dot{\boldsymbol{E}}} := \left\{ \boldsymbol{S}^{\text{dis}} \mid \pi(\boldsymbol{Q}) - \pi(\dot{\boldsymbol{E}}) - \boldsymbol{S}^{\text{dis}} \circ [\boldsymbol{Q} - \dot{\boldsymbol{E}}] \geq 0\,, \quad \forall \boldsymbol{Q} \right\}.$$

As was the case for local parallel plasticity and for general energetic-dissipative micromorphic parallel plasticity, the generalised dissipative stress $\boldsymbol{S}^{\text{dis}}$ is not determinable in the admissible region for the fully-dissipative problem with a non-smooth dissipation potential.

The implications of choosing a smooth or a non-smooth dissipation potential on the structure of the admissible region and the flow rule for the fully-dissipative case are now considered. These results hold for the general energetic-dissipative case as well.

6.3.1. Smooth dissipation potential

Once again, we consider the following canonical regularised dissipation potential

$$\pi(\dot{\boldsymbol{E}}) = \frac{\sigma_{\text{y}}}{1+\gamma} |\dot{\boldsymbol{E}}|^{1+\gamma}\,,$$

where $\gamma \in (0, 1]$. The admissible region follows from evaluating the subdifferential of the dissipation potential (18) using the regularised potential π at $\dot{\boldsymbol{E}} \equiv \boldsymbol{0}$. This gives

$$\boldsymbol{S}^{\text{dis}} \circ \boldsymbol{Q} \leq \frac{\sigma_{\text{y}}}{1+\gamma} |\boldsymbol{Q}|^{1+\gamma}\,, \quad \forall \boldsymbol{Q}\,.$$

The admissible region is obtained by maximising the function $r_\gamma = r_\gamma(\boldsymbol{S}^{\text{dis}}; \boldsymbol{Q})$ over all arbitrary $\boldsymbol{Q}$ to obtain

$$y_\gamma(\boldsymbol{S}^{\text{dis}}) = \max_{\boldsymbol{Q}} r_\gamma(\boldsymbol{S}^{\text{dis}}; \boldsymbol{Q}) \leq \frac{1}{1+\gamma} \qquad \text{where} \qquad r_\gamma(\boldsymbol{S}^{\text{dis}}; \boldsymbol{Q}) := \frac{\boldsymbol{S}^{\text{dis}} \circ \boldsymbol{Q}}{\sigma_{\text{y}} |\boldsymbol{Q}|^{1+\gamma}}\,.$$

As demonstrated in Remark 5 for the smooth case of micromorphic serial plasticity, the only possible choice of the dissipative micro stress is one where

$$|\boldsymbol{S}^{\text{dis}}| = 0\,.$$

Thus there is no admissible domain for this choice of smooth dissipation potential.

The flow rule then follows directly from the partial derivative of the dissipation potential with respect to $\dot{\boldsymbol{E}}$ for $\dot{\boldsymbol{E}} \neq \boldsymbol{0}$:

$$\boldsymbol{S}^{\text{dis}} = \sigma_{\text{y}} |\dot{\boldsymbol{E}}|^{\gamma} \frac{\dot{\boldsymbol{E}}}{|\dot{\boldsymbol{E}}|}$$

or upon rearranging

$$\dot{\boldsymbol{E}} = \left[\frac{|\boldsymbol{S}^{\text{dis}}|}{\sigma_{\text{y}}} \right]^{1/\gamma} \frac{\boldsymbol{S}^{\text{dis}}}{|\boldsymbol{S}^{\text{dis}}|}\,.$$

Thus the generalised stress $\boldsymbol{S} \equiv \boldsymbol{S}^{\text{dis}}$ is determinable here from the flow rule when the dissipation potential π is smooth. Furthermore, a smooth dissipation potential allows for a *local* treatment of the flow law and the local (pointwise) determination of the admissible region (which vanishes identically).

6.3.2. Non-smooth dissipation potential

Consider the prototypical non-smooth dissipation potential

$$\pi(\dot{\boldsymbol{E}}) = \sigma_{\text{y}} |\dot{\boldsymbol{E}}|\,. \tag{24}$$

The admissible region is obtained by evaluating the subdifferential of the dissipation potential (18), with the non-smooth dissipation potential π, at $\dot{\boldsymbol{E}} \equiv \boldsymbol{0}$. For this case, the subdifferential of the dissipation potential is given by the set of generalised stresses $\boldsymbol{S}^{\text{dis}}$ satisfying

$$\boldsymbol{S}^{\text{dis}} \circ \boldsymbol{Q} \leq \sigma_{\text{y}} |\boldsymbol{Q}|\,, \quad \forall \boldsymbol{Q}\,. \tag{25}$$

As before, the definition of the admissible region follows from determining the supremum of the function $r(\boldsymbol{S}^{\text{dis}};\boldsymbol{Q})$, given below, over all arbitrary $\boldsymbol{Q}$ in the admissible region where $\dot{\boldsymbol{E}}\equiv\boldsymbol{0}$, that is find

$$y(\boldsymbol{S}^{\text{dis}})=\sup_{\boldsymbol{Q}} r(\boldsymbol{S}^{\text{dis}};\boldsymbol{Q})\leq 1 \qquad \text{with} \qquad r(\boldsymbol{S}^{\text{dis}};\boldsymbol{Q}):=\frac{\boldsymbol{S}^{\text{dis}}\circ\boldsymbol{Q}}{\sigma_{\text{y}}|\boldsymbol{Q}|}\,. \tag{26}$$

It follows directly that $\boldsymbol{S}^{\text{dis}}\leq\sigma_{\text{y}}$ with $|\boldsymbol{S}^{\text{dis}}|=\sigma_{\text{y}}$ at yield. The flow rule is given by the partial derivative of the dissipation potential with respect to $\dot{\boldsymbol{E}}$ for $\dot{\boldsymbol{E}}\neq\boldsymbol{0}$, that is

$$\boldsymbol{S}^{\text{dis}}=\sigma_{\text{y}}\frac{\dot{\boldsymbol{E}}}{|\dot{\boldsymbol{E}}|}$$

which can be inverted to obtain

$$\dot{\boldsymbol{E}}=\frac{|\dot{\boldsymbol{E}}|\boldsymbol{S}^{\text{dis}}}{\sigma_{\text{y}}}=:\lambda\frac{\boldsymbol{S}^{\text{dis}}}{|\boldsymbol{S}^{\text{dis}}|}\,,$$

with the positive plastic multiplier $\lambda\geq 0$. However, here the generalised stress $\boldsymbol{S}\equiv\boldsymbol{S}^{\text{dis}}$ is not determinable from an elastic law.

Thus, for the fully-dissipative case with a non-smooth dissipation potential, a local definition of the admissible region is not possible. The implications of the inability to locally determine the admissible region for a non-smooth dissipation potential for the fully-dissipative problem carry over to the general energetic-dissipative case with a non-smooth dissipation potential. Given the inability to locally determine the admissible region for theses cases, a possible *global* definition of the admissible domain is thus considered next.

6.4. Global dissipation potential for the fully-dissipative problem

In Section 6.3.2, it was shown that for the important case of fully-dissipative micromorphic plasticity with a non-smooth dissipation potential, a local definition of the admissible domain is not possible. In this section, a global reformulation is presented. The global relation was first presented in Reddy et al. (2008) and Reddy (2011a). The presentation here differs from that in Carstensen et al. (2017) as we do not resort to a spatial discretisation from the onset. In order to proceed, we consider first the weak statement of the microforce balance.

6.4.1. Weak form of the microforce balance

Recall that for the fully-dissipative case $\boldsymbol{\varsigma}=\boldsymbol{\varsigma}^{\text{dis}}$ and $\boldsymbol{\mu}=\boldsymbol{\mu}^{\text{dis}}$, that is $\boldsymbol{S}=\boldsymbol{S}^{\text{dis}}$.

The micro boundary conditions considered are the standard ones of either a micro-free Neumann condition $\boldsymbol{\mu}\cdot\boldsymbol{n}\doteq\boldsymbol{0}$ or a micro-hard Dirichlet condition $\boldsymbol{\varepsilon}\doteq\boldsymbol{0}$ on complementary parts of the boundary $\partial\Omega$ (for more information, see Fleck and Hutchinson (1997) and Gurtin and Needleman (2005).

The weak form of the microforce balance is obtained by testing Eq. (15) with an arbitrary micro strain $\delta\boldsymbol{\varepsilon}$ which takes zero value on the micro-hard part of the boundary, to obtain

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{x}):\delta\boldsymbol{\varepsilon}(\boldsymbol{x})\,\mathrm{d}x=\int_\Omega \boldsymbol{\varsigma}^{\text{dis}}(\boldsymbol{x}):\delta\boldsymbol{\varepsilon}(\boldsymbol{x})\,\mathrm{d}x-\int_\Omega \operatorname{div}\boldsymbol{\mu}^{\text{dis}}(\boldsymbol{x}):\delta\boldsymbol{\varepsilon}(\boldsymbol{x})\,\mathrm{d}x$$

and by employing integration by parts

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{x}):\delta\boldsymbol{\varepsilon}(\boldsymbol{x})\,\mathrm{d}x=\int_\Omega \boldsymbol{\varsigma}^{\text{dis}}(\boldsymbol{x}):\delta\boldsymbol{\varepsilon}(\boldsymbol{x})\,\mathrm{d}x+\int_\Omega \boldsymbol{\mu}^{\text{dis}}(\boldsymbol{x})\ \vdots\ \boldsymbol{\gamma}\big(\delta\boldsymbol{\varepsilon}(\boldsymbol{x})\big)\,\mathrm{d}x\,. \tag{27}$$

The generalised macro stress $\boldsymbol{\Sigma}$ is now defined as a function of position $\boldsymbol{x}$ by

$$\boldsymbol{\Sigma}(\boldsymbol{x}):=\{\boldsymbol{\sigma}(\boldsymbol{x}),\boldsymbol{0}\}\,, \tag{28}$$

and an arbitrary generalised micro strain pair by

$$\delta\boldsymbol{E}(\boldsymbol{x})=\{\delta\boldsymbol{\varepsilon}(\boldsymbol{x}),\boldsymbol{\gamma}\big(\delta\boldsymbol{\varepsilon}(\boldsymbol{x})\big)\ell\}\,.$$

The weak form of the microforce balance (27) can thus be expressed as

$$\int_\Omega \boldsymbol{\Sigma}(\boldsymbol{x})\circ\delta\boldsymbol{E}(\boldsymbol{x})\,\mathrm{d}x=\int_\Omega \boldsymbol{S}^{\text{dis}}(\boldsymbol{x})\circ\delta\boldsymbol{E}(\boldsymbol{x})\,\mathrm{d}x. \tag{29}$$

Observe that the weak form of the microforce balance allows for the exchange of the generalised dissipative micro stress $\boldsymbol{S}^{\text{dis}}$ with the generalised macro stress $\boldsymbol{\Sigma}$. The implications thereof are now presented.

6.4.2. Exchange of generalised micro and macro stresses

The global dissipation potential Π (a functional) is defined as the integral over the domain Ω of the dissipation potential, that is

$$\Pi\big(\dot{\boldsymbol{E}}\big) := \int_\Omega \pi\,(\dot{\boldsymbol{E}}(\boldsymbol{x}))\,\mathrm{d}x\,.$$

The admissible region then follows from evaluating the subdifferential of the global dissipation potential given by

$$\Pi(\boldsymbol{Q}) - \Pi(\dot{\boldsymbol{E}}) - \int_\Omega \boldsymbol{S}^{\mathrm{dis}}(\boldsymbol{x}) \circ [\boldsymbol{Q}(\boldsymbol{x}) - \dot{\boldsymbol{E}}(\boldsymbol{x})]\,\mathrm{d}x \geq 0 \qquad \forall \boldsymbol{Q}\,.$$

The generalised dissipative micro stress $\boldsymbol{S}^{\mathrm{dis}}$ and the generalised macro stress $\boldsymbol{\Sigma}$ in the subdifferential of the global dissipation potential can be exchanged by virtue of the weak form of the microforce balance (29), to give

$$\Pi(\boldsymbol{Q}) - \Pi(\dot{\boldsymbol{E}}) - \int_\Omega \boldsymbol{\Sigma}(\boldsymbol{x}) \circ [\boldsymbol{Q}(\boldsymbol{x}) - \dot{\boldsymbol{E}}(\boldsymbol{x})]\,\mathrm{d}x \geq 0\,, \quad \forall \boldsymbol{Q}\,. \tag{30}$$

6.4.3. Non-smooth global dissipation potential

Recall that a local form of the admissible region was not possible for the fully-dissipative theory described in Section 6.3.2. Consider now the global counterpart to the non-smooth local dissipation potential (24) given by

$$\Pi(\dot{\boldsymbol{E}}) := \int_\Omega \sigma_{\mathrm{y}}(\boldsymbol{x})|\dot{\boldsymbol{E}}(\boldsymbol{x})|\,\mathrm{d}x\,.$$

Note that, in contrast to the presentation in Carstensen et al. (2017), the yield stress σ_{y} is not necessarily assumed uniform.

The structure of the admissible region corresponding to the global dissipation potential can be elucidated following the procedure outlined in Section 5.1. The resulting canonical yield function is given by

$$Y(\boldsymbol{\Sigma}) = \sup_{\boldsymbol{Q}} R(\boldsymbol{\Sigma}; \boldsymbol{Q}) \leq 1 \qquad \text{where} \qquad R(\boldsymbol{\Sigma}; \boldsymbol{Q}) := \frac{\displaystyle\int_\Omega \boldsymbol{\Sigma}(\boldsymbol{x}) \circ \boldsymbol{Q}(\boldsymbol{x})\,\mathrm{d}x}{\displaystyle\int_\Omega \sigma_{\mathrm{y}}(\boldsymbol{x})|\boldsymbol{Q}(\boldsymbol{x})|\,\mathrm{d}x}\,. \tag{31}$$

The counterpart to $R(\boldsymbol{\Sigma}; \boldsymbol{Q})$ that arises when the dissipation potential is assumed to be local is given by $r(\boldsymbol{S}^{\mathrm{dis}}; \boldsymbol{Q})$ in Eq. (26). Note however that $R(\boldsymbol{\Sigma}; \boldsymbol{Q}) \neq \int_\Omega r(\boldsymbol{S}^{\mathrm{dis}}; \boldsymbol{Q})\,\mathrm{d}x$. Due to the exchange of the generalised micro and macro stresses, the numerator of $R(\boldsymbol{\Sigma}; \boldsymbol{Q})$ contains the generalised macro stress $\boldsymbol{\Sigma}$ which is determinable from an elastic law. Note that the structure of the admissible region can alternatively be obtained using an infinite-dimensional analogue of the result in (A.5) (Han and Reddy, 2010) (see also Remark 4).

The problem of finding the supremum in Eq. (31) can be expressed as a global optimality problem. Define the field $\overline{\boldsymbol{Q}}$ by

$$\overline{\boldsymbol{Q}} = \arg\left\{\sup_{\boldsymbol{Q}} R(\boldsymbol{\Sigma}; \boldsymbol{Q})\right\}.$$

The global optimality problem follows as

$$\delta R(\boldsymbol{\Sigma}; \boldsymbol{Q})\Big|_{\overline{\boldsymbol{Q}}} \doteq \boldsymbol{0}\,, \quad \forall \boldsymbol{Q}\,.$$

Thus, the global problem for the admissible region results in the variational statement

$$\int_\Omega \boldsymbol{\Sigma}(\boldsymbol{x}) \circ \delta\boldsymbol{Q}(\boldsymbol{x})\,\mathrm{d}x - R(\boldsymbol{\Sigma}; \overline{\boldsymbol{Q}}) \int_\Omega \sigma_{\mathrm{y}} \frac{\overline{\boldsymbol{Q}}(\boldsymbol{x})}{|\overline{\boldsymbol{Q}}(\boldsymbol{x})|} \circ \delta\boldsymbol{Q}(\boldsymbol{x})\,\mathrm{d}x = 0 \qquad \forall \delta\boldsymbol{Q}\,. \tag{32}$$

It is insightful to compare the global problem for the admissible region to the local orthogonality condition (20) corresponding to micromorphic serial plasticity. Eq. (32) for the admissible domain is global and nonlinear. An analytical treatment is not possible. In principle, a numerical approximation of the problem using the finite element method would be feasible. Define a trial generalised macro stress field $\boldsymbol{\Sigma}^\star$ as the solution of the macro equilibrium problem (14) obtained assuming a “frozen” plastic state. Eq. (32) could then be linearised and solved iteratively for $\overline{\boldsymbol{Q}}$. An evaluation of the functional (31) would determine if the trial stress field $\boldsymbol{\Sigma}^\star$ was indeed admissible. This form of global algorithm is not conventional and warrants further investigation.

6.4.4. An upper bound for the global admissible domain

The numerical approximation of the global admissible region sketched in the previous section is complicated and computationally expensive. Here we seek a possible estimate for the onset of plastic flow in the form of an upper bound to the global admissible region.

The numerator in the yield functional (31) can be bounded as follows

$$\int_\Omega \boldsymbol{\Sigma}(\boldsymbol{x}) \circ \boldsymbol{Q}(\boldsymbol{x})\,\mathrm{d}x \leq \int_\Omega \frac{|\boldsymbol{\Sigma}(\boldsymbol{x})|}{\sigma_{\mathrm{y}}(\boldsymbol{x})} \sigma_{\mathrm{y}}(\boldsymbol{x})|\boldsymbol{Q}(\boldsymbol{x})|\,\mathrm{d}x$$

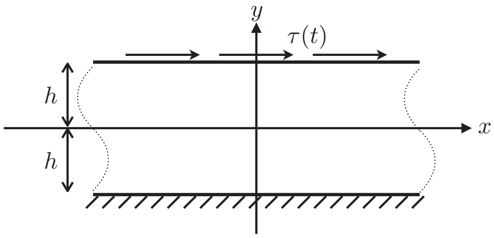

Fig. 4. One-dimensional strip subject to shear loading.

$$\leq \left\| \frac{|\boldsymbol{\Sigma}|}{\sigma_y} \right\|_{\infty,\Omega} \int_\Omega \sigma_y(\boldsymbol{x}) |\mathbf{Q}(\boldsymbol{x})| \, \mathrm{d}x \quad \forall \mathbf{Q} .$$

Hence, we define the upper bound estimate for the yield functional by

$$Y^\star := \left\| \frac{|\boldsymbol{\Sigma}|}{\sigma_y} \right\|_{\infty,\Omega} = \left\| \frac{|\boldsymbol{\sigma}|}{\sigma_y} \right\|_{\infty,\Omega} \geq Y(\boldsymbol{\Sigma}) ,$$

where $Y(\boldsymbol{\Sigma}) \leq 1$. Note that $Y^\star \geq 1$ is possible but does not imply that $Y = 1$. For the case of constant σ_y one obtains $Y^\star \equiv \| |\boldsymbol{\sigma}| \|_{\infty,\Omega} / \sigma_y$.

The utility of the upper bound would be as a check for an elastic response. If $Y^\star < 1$ then the problem is definitely elastic and a numerical approximation for the admissible domain defined in Eq. (32) need not be sought.

Note that the structure of $Y^\star$ resembles that in Eq. (21) for micromorphic serial plasticity with a non-smooth dissipation potential. However, instead of a yield evaluation based on a point-wise absolute value one must determine the maximum over the domain of the ratio of the magnitude of the determinable macro stress to the yield stress.

6.4.5. Example of a strip subject to shear

The global formulation for the fully-dissipative problem with a non-smooth dissipation potential was introduced as a local definition of the admissible domain was shown not to be possible. However, determining the admissible region for the global problem is non-trivial. An upper estimate for the yield functional was therefore derived. The tightness of the estimate is not however clear. Given these challenges, the objective here is to elucidate features of the fully-dissipative problem using the comprehensively analysed one-dimensional example of a strip subject to shear (see e.g. Anand et al., 2005) and Chiricotto et al. (2016)).

As shown in Fig. 4, the strip is of height $2h$ in the y-direction, and is subjected to an applied traction $\tau(t)$ at $y = h$. The loading is proportional. The problem is one dimensional as the response varies only in the y-direction. The lower boundary is fixed, i.e. the horizontal component of the displacement $u(-h, t) = 0$. From (macroscopic) equilibrium we have

$$\tau_{,y} = 0 ,$$

and hence the shear stress τ is spatially constant. In addition, microhard boundary conditions are assumed, that is the rate of plastic strain is zero on the boundaries. The yield stress σ_y is chosen to be constant and hardening neglected.

The generalised macro stress $\boldsymbol{\Sigma} := \{\boldsymbol{\sigma}(\boldsymbol{x}), \boldsymbol{0}\}$ (see Eq. (28)), and the arbitrary generalised rate of micro strain $\mathbf{Q}$ that appear in the canonical yield function (31) are given here by

$$\boldsymbol{\Sigma} \equiv \{\tau, 0\} \qquad \text{and} \qquad \mathbf{Q} \equiv \{q, \ell q_{,y}\} ,$$

and hence their generalised inner product is

$$\boldsymbol{\Sigma} \circ \mathbf{Q} \equiv \tau q .$$

Note that $q(-h, t) = q(h, t) = 0$ due to the prescribed microhard boundary conditions.

The function R in the canonical yield function (31) is thus given by

$$R(\boldsymbol{\Sigma}; \mathbf{Q}) \equiv \frac{\tau}{\sigma_y} \frac{\displaystyle\int_{-h}^{+h} q \, \mathrm{d}y}{\displaystyle\int_{-h}^{+h} \sqrt{|q|^2 + \ell^2 |q_{,y}|^2} \, \mathrm{d}y} =: \widetilde{\theta} \widetilde{R} , \tag{33}$$

with the dimensionless independent variable $\widetilde{\theta} := \tau / \sigma_y$ representing the ratio of the stress to the yield stress. The integral in the denominator can be bounded (see e.g. Anand et al., 2005) as follows

$$\int_{-h}^{+h} |\mathbf{Q}| \, \mathrm{d}y = \int_{-h}^{+h} \sqrt{|q|^2 + \ell^2 |q_{,y}|^2} \, \mathrm{d}y > \int_{-h}^{+h} |q| \, \mathrm{d}y . \tag{34}$$

Recall that for the classical problem (i.e. when $\ell \equiv 0$) and for micromorphic serial plasticity (see Section 5.1), a pointwise definition of the admissible region is possible. For the classical problem, the local form of R corresponding to Eq. (33) is given by

$$r \equiv \widetilde{\theta} \frac{q}{|q|} \leq 1 \qquad \Rightarrow \qquad \varphi := |\widetilde{\theta}| \leq 1 \, .$$

In the global problem $R \leq 1$. When plastic flow occurs, the canonical yield function $Y = \max_q R = 1$. Hence $\max_q \widetilde{R} < 1$ implies $\varphi > 1$ and an increase in the threshold for the onset of local yield, i.e. a strengthening effect. As the equivalent stress function φ can not exceed unity for the classical problem, the physical interpretation of the yield stress σ_{y} in the fully-dissipative theory is not obvious. This feature of the theory has been analysed in Anand et al. (2005) and Chiricotto et al. (2016).

To proceed, we follow Chiricotto et al. (2016) and define the dimensionless independent variable $\widetilde{y} := y/h$ and the dimensionless parameter $\widetilde{\ell} := \ell/h$. Thus Eq. (33) can be expressed as

$$R(\boldsymbol{\Sigma}; \mathbf{Q}) \equiv \widetilde{\theta} \frac{\displaystyle\int_{-1}^{+1} q \, \mathrm{d}\widetilde{y}}{\displaystyle\int_{-1}^{+1} \sqrt{|q|^2 + \widetilde{\ell}^2 |q_{,\widetilde{y}}|^2} \, \mathrm{d}\widetilde{y}} \, .$$

Now assume that q satisfies the normality property

$$\int_{-1}^{+1} q \, \mathrm{d}\widetilde{y} = 1 \, .$$

The implications of this assumption are discussed in detail in Chiricotto et al. (2016). The resulting canonical yield function is given by

$$Y(\boldsymbol{\Sigma}) = \sup_{\mathbf{Q}} R(\boldsymbol{\Sigma}; \mathbf{Q}) \leq 1 \qquad \text{where} \qquad R(\boldsymbol{\Sigma}; \mathbf{Q}) := \widetilde{\theta} \frac{1}{\displaystyle\int_{-1}^{+1} \sqrt{|q|^2 + \widetilde{\ell}^2 |q_{,\widetilde{y}}|^2} \, \mathrm{d}\widetilde{y}} \, , \tag{35}$$

whereby we seek the q that will provide the supremum (i.e. the least upper bound) of R. Any other admissible q will provide an upper bound to R. Problem (35) can be recast as a minimisation problem (see Anand et al. (2005) and Chiricotto et al. (2016)) where one seeks the minimum of $1/R = [\sigma_{\mathrm{y}}/\tau][1/\widetilde{R}] = [1/\widetilde{\theta}][1/\widetilde{R}]$. A key result in Chiricotto et al. (2016) following from the analysis of the minimisation problem is an expression for the relationship between $\widetilde{\theta}$ and $\widetilde{\ell}$ (see Fig. 5(b)). They clearly show the strengthening effect with increasing relative length scale. Also shown in Fig. 5(b) is the estimate obtained by Anand et al. (2005) who refer to the ratio $\sigma_{\mathrm{y}}/\widetilde{R}$ as the *actual* yield strength and to σ_{y} as the *coarse-grained* yield strength.

Computing the q that gives the maximum of R is not a trivial task even for this simplified problem. To provide further insight into the structure of the problem, we assess the strengthening for various choices of normalised q that satisfy the microhard boundary conditions.

Consider the following two families of approximations for q:

$$q_a(\widetilde{y}) = \frac{1}{|a|\sqrt{\pi}} \mathrm{e}^{-[\widetilde{y}/a]^2} \qquad \text{and} \qquad q_\alpha(\bar{y}) = \frac{1}{B} \bar{y}^{\alpha - 1} [1 - \bar{y}]^{\beta - 1} \, .$$

The function q_a is the probability density of the normal distribution where $a > 0$ is a constant that controls the sharpness. The function q_α is the probability density of the beta distribution where α, $\beta > 0$ are shape parameters. The function is defined on the domain $\bar{y} = [0, 1]$ and shifted with respect to the domain $\widetilde{y}$. B is a normalisation constant. Here we choose $\alpha = \beta$. These functions are plotted in Fig. 5(a) and (c).

The objective here is to investigate reasonable choices of q that are smooth, positive and zero on the boundary. We emphasise that these choices of q will not, in general, be the ones that give the supremum of R and hence the results that follow provide an upper bound to the canonical yield function Y in Eq. (35).

Consider the computed relationship for φ versus the relative length scale $\widetilde{\ell}$ for different choices of a in q_a and α in q_α shown in Fig. 5(b) and (d), respectively. All choices for q obviously demonstrate strengthening as they are not constant functions (they have non-zero gradients). The choice of a constant function for q is not possible due to the normalisation property and the boundary conditions.

The function q_a is a poor approximation other than for small values of $\widetilde{\ell}$. Recall that as $\widetilde{\ell} \to 0$ we recover the local theory. The choice of $a = 0.45$ shows the best agreement to the analytical result.

The function q_α is a far better approximation to the q that gives the supremum of R as the predicted strengthening compares reasonably to the analytical result of Chiricotto et al. (2016). This is not necessarily surprising as q_α was chosen to capture the key features of the computational results (obtained using a smooth dissipation potential) presented in Anand et al. (2005). The choice of $\alpha = 1.1$ and $\alpha = 1.2$ give the poorest approximations to the analytical result. The quality of approximation for the other choices of α depends on the relative length scale. For example, for $\widetilde{\ell} \lessapprox 0.25$, $\alpha = 1.8$ provides the best approximation, while for $\widetilde{\ell} = 0.5$, $\alpha = 1.4$ provides the best approximation.

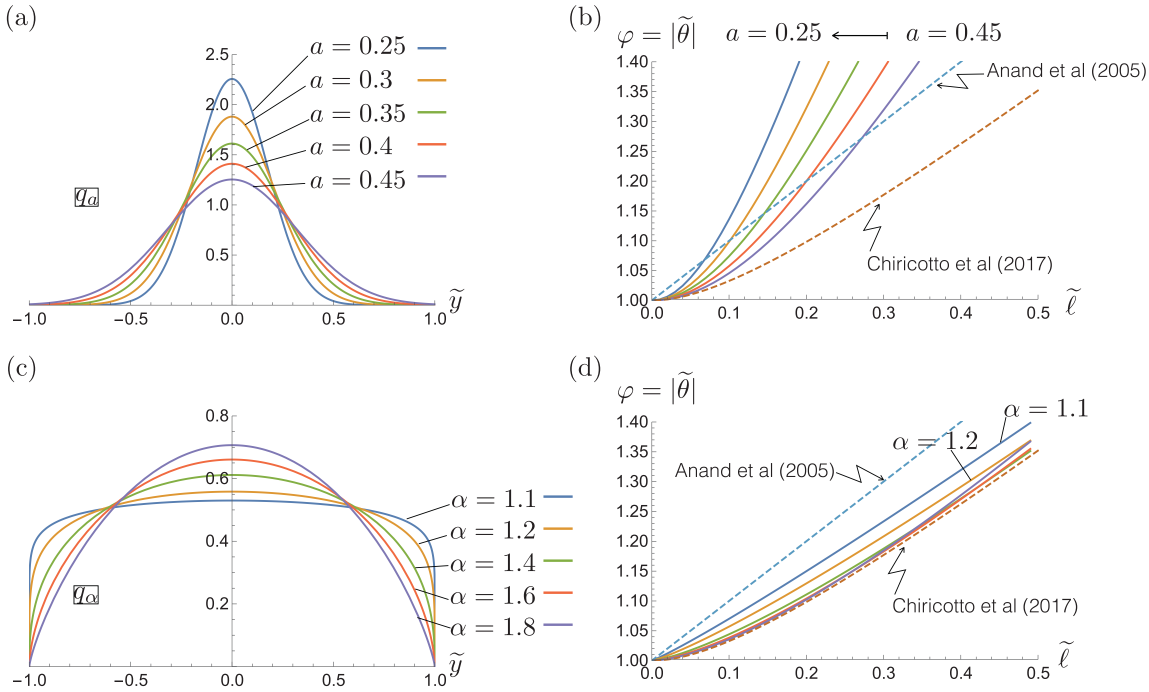

Fig. 5. Two families (q_a and q_α) of assumed forms for q for the example of a strip subject to shear are shown in (a) and (c), respectively. The strengthening effect for different relative length scales $\widetilde{\ell}$ are shown in (b) for q_a and (d) for q_α. Also shown is the analytical solution due to Chiricotto et al. (2016) and the estimate derived by Anand et al. (2005).

The maximum is achieved in Eq. (33) by the actual plastic strain increment at that time step. As shown Fig. 5(d), we have tested the expression for φ by using a (beta) function that is close in shape to the actual plastic strain increment. Furthermore, the amplitude is not relevant as the expression is homogeneous of degree 0. So, we treat it as a practical illustration. Rigorous maximization has essentially been presented in the technical analysis by Chiricotto et al. (2016).

The upper bound defined in Section 6.4.4 reduces to

$$Y^\star \equiv \frac{|\tau|}{\sigma_{\mathrm{y}}} = |\widetilde{\theta}| = \varphi \,. \tag{36}$$

It is clear from Fig. 5 that for $\widetilde{\ell} > 0$, $\varphi > 1$ and hence $Y^\star > 1$.

Remark 6. The “elastic gap”.

Passivation is the application of a microhard boundary condition on a previously microscopically unconstrained part of the boundary while the body is undergoing plastic deformation. Fleck et al. (2014) showed that passivation can induce an “elastic gap” whereby a continuum point undergoing plastic flow prior to passivation, undergoes elastic behaviour post-passivation before to returning to a plastic state. The analysis in Carstensen et al. (2017) provided further mathematical justification for the “elastic gap”.

The reason for the elastic gap is made clear by the current example. Pre-passivation, the gradient term in the denominator of Eq. (33) plays no role as the plastic strain distribution is constant and $\widetilde{R} = 1$. Thus, as for the classical problem, $\varphi = 1$. Post-passivation, the gradient term in the denominator is activated due to the imposition of the microhard boundary conditions. Following from inequality (34), $\widetilde{R} < 1$ and $\varphi \geq 1$ thereby allowing the problem to respond elastically until $Y = 1$ again and plastic flow continues. □

7. Conclusion

A unified classification framework for models of extended plasticity has been presented. Within this framework, models are classified as either serial or parallel. This classification is based on the choice of the energetic and dissipative structures. The classification has been introduced first in the familiar setting of local plasticity. Prototypical examples of non-smooth and smooth dissipation potentials were examined. These correspond to the rate-independent and rate-dependent setting, respectively. For non-smooth local parallel plasticity, an identification of the generalised stress prior to the onset of plastic flow is not possible. The key results from the local theory were shown to carry over to the extended plasticity models.

A dissipation-consistent methodology has been adopted throughout to carefully identify the structure of the constitutive relations and evolution equations. In addition, this has ensured that all models are thermodynamically consistent and can be extended to consider other dissipative mechanisms in addition to plasticity.

The classification has been extended to micromorphic models of plasticity. Three categories of micromorphic parallel plasticity have been introduced. It has been shown that for the general energetic-dissipative case and the fully-dissipative case, a local determination of the admissible region for a non-smooth dissipation potential is not possible. By contrast, the hybrid energetic-dissipative case permits a local treatment. The implications of a global non-smooth dissipation potential for fully-dissipative micromorphic plasticity has been detailed.

Future work will include the development of algorithms for the fully-dissipative, rate-independent case of micromorphic parallel plasticity. An upper bound for the global admissible domain has been proposed. However, it was not possible to comment on the sharpness of the bound and hence its utility. A careful study of the proposed bound using a smooth dissipation potential for a range of regularisation parameters, including $\gamma \to 0$, will be valuable. Such a study will also elucidate the mechanisms that underpin the appearance of an elastic gap (see Fleck et al., 2014).

The precursor to the proposed numerical investigation is recasting the various models in variational format and then as incremental variational formulations. The variational formulation does not have an associated minimisation problem, but the corresponding time-discrete incremental problem does. The basis for such an extension is given in Reddy (2011a).

An extensive classification of available models of extended plasticity using the proposed scheme will be the subject of future work.

Acknowledgments

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Appendix A. Standard results and definitions from convex analysis

Key results and definitions from convex analysis necessary for the preceding presentation are summarised here. For further details refer to Han and Reddy (2010) and the references therein.

The results are presented in an abstract setting. X denotes a finite-dimensional normed vector space. The space of linear continuous functionals on X, or dual space of X, is denoted by $X^\star$. For $\boldsymbol{x} \in X$ and $\boldsymbol{x}^\star \in X^\star$, the action of $\boldsymbol{x}^\star$ on $\boldsymbol{x}$ is defined by the scalar product

$$\boldsymbol{x}^\star \cdot \boldsymbol{x} \,.$$

When relating the forthcoming results to the main text it may be useful to make the substitution

$$\{\boldsymbol{S}^{\text{dis}}, \boldsymbol{X}^{\text{dis}}\} \to \boldsymbol{x}^\star \qquad \text{and} \qquad \{\dot{\boldsymbol{E}}, \dot{\boldsymbol{I}}\} \to \boldsymbol{x} \,.$$

A1. Convex sets and convex functions

The subset $Y \subset X$ is *convex* if

$$\text{for any } \boldsymbol{x}, \boldsymbol{y} \in Y \text{ and } 0 \leq \theta \leq 1, \quad \theta \boldsymbol{x} + [1-\theta]\boldsymbol{y} \in Y \,. \tag{A.1}$$

The *normal cone* to a convex set $Y^\star \subset X^\star$ at $\boldsymbol{x}^\star$, denoted by $N_{Y^\star}(\boldsymbol{x}^\star)$, is a *set* in X defined by

$$N_{Y^\star}(\boldsymbol{x}^\star) := \{\boldsymbol{x} \in X \;:\; \boldsymbol{x} \cdot [\boldsymbol{y}^\star - \boldsymbol{x}^\star] \leq 0 \,, \quad \forall \boldsymbol{y}^\star \in Y^\star\} \,. \tag{A.2}$$

The function f is *convex* if

$$f(\theta \boldsymbol{x} + [1-\theta]\boldsymbol{y}) \leq \theta f(\boldsymbol{x}) + [1-\theta] f(\boldsymbol{y}) \,, \quad \forall \boldsymbol{x}, \, \boldsymbol{y} \in X, \qquad \forall \theta \in [0,1] \,. \tag{A.3}$$

Given a convex function f on X, for any $\boldsymbol{x} \in X$, the *subdifferential* of f at $\boldsymbol{x}$, denoted by $f_{;\boldsymbol{x}}$, is the, possibly empty, subset of $X^\star$ defined by

$$f_{;\boldsymbol{x}} := \left\{ \boldsymbol{x}^\star \in X^\star \;:\; f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{x}^\star \cdot [\boldsymbol{y} - \boldsymbol{x}] \,, \quad \forall \boldsymbol{y} \in X \right\} . \tag{A.4}$$

Note that the notation $f_{;\boldsymbol{x}}$ for the subdifferential used here is not standard. A more conventional notation would be ∂f.

A2. Positive homogeneity

A function $f(\boldsymbol{x})$ is said to be positively homogeneous of degree k if

$$f(\alpha\boldsymbol{x}) = |\alpha|^k f(\boldsymbol{x}) \tag{A.5}$$

holds for $k > 0$, where $\alpha \in \mathbb{R}$.

A3. Gauge function

A function $g : X \to [0, \infty]$ is called a gauge if

$$g(\boldsymbol{x}) \geq 0 \quad \forall \boldsymbol{x} \in X\,, \tag{A.6a}$$

$$g(\boldsymbol{0}) = 0\,, \tag{A.6b}$$

$$g \text{ is convex and positively homogeneous.} \tag{A.6c}$$

A4. Indicator and support functions

For $S^\star \subset X^\star$, the *indicator function* $I_{S^\star}$ of $X^\star$ is defined by

$$I_{S^\star}(\boldsymbol{x}^\star) := \begin{cases} 0\,, & \boldsymbol{x}^\star \in S^\star\,, \\ +\infty\,, & \boldsymbol{x}^\star \notin S^\star\,, \end{cases} \tag{A.7}$$

and the *support function* $\sigma_{S^\star}$ is defined on X by

$$\sigma_{S^\star}(\boldsymbol{x}) := \sup_{\boldsymbol{x}^\star}\{\boldsymbol{x}^\star \cdot \boldsymbol{x} \ :\ \boldsymbol{x}^\star \in S^\star\}\,. \tag{A.8}$$

For f a function on X with values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the *Legendre–Fenchel conjugate* $f^\star$ is the function defined by

$$f^\star(\boldsymbol{x}^\star) := \sup_{\boldsymbol{x}\in X}\{\boldsymbol{x}^\star \cdot \boldsymbol{x} - f(\boldsymbol{x})\}\,, \quad \boldsymbol{x}^\star \in X^\star\,. \tag{A.9}$$

Hence the *support function is conjugate to the indicator function*, i.e.

$$I^\star_{S^\star} = \sigma_{S^\star} \qquad \longleftrightarrow \qquad I_{S^\star} = \sigma^\star_{S^\star}\,. \tag{A.10}$$

The following important result relates the support and indicator functions. Let K be a closed convex set in $X^\star$ defined by

$$K = \{\boldsymbol{x}^\star \in X^\star \ :\ \boldsymbol{x}^\star \cdot \boldsymbol{x} \leq g(\boldsymbol{x})\}\,, \tag{A.11}$$

where g is a gauge on X. Then

$$\begin{aligned} g(\boldsymbol{x}) &= \sigma_K(\boldsymbol{x})\,, \\ g^\star(\boldsymbol{x}^\star) &= I_K(\boldsymbol{x}^\star)\,, \\ K &= g_{;\boldsymbol{x}}(\boldsymbol{0})\,, \\ \boldsymbol{x}^\star \in g_{;\boldsymbol{x}} \quad &\longleftrightarrow \quad \boldsymbol{x} \in g^\star_{;\boldsymbol{x}^\star} = N_K(\boldsymbol{x}^\star)\,. \end{aligned} \tag{A.12}$$

A5. Polar functions

Let $K \subset X^\star$ be a closed convex set whose boundary, denoted $\mathrm{bdy}(K)$, is the level set c_0 of the convex function $\varphi(\boldsymbol{x}^\star)$. That is

$$K = \{\boldsymbol{x}^\star \in X^\star \ :\ \varphi(\boldsymbol{x}^\star) \leq c_0\}\,, \tag{A.13}$$

for $c_0 > 0$. The function φ can be defined so that it is a gauge on the set K, denoted by ϕ_K, where

$$K = \left\{\boldsymbol{x}^\star \ :\ \phi_K(\boldsymbol{x}^\star) \leq 1\right\}.$$

It can be shown that for $\boldsymbol{x}^\star \in K$ and $\boldsymbol{x}^\star \in \sigma_{K;\,\boldsymbol{x}}$, $\boldsymbol{x} \neq \boldsymbol{0}$ we have

$$\boldsymbol{x}^\star \cdot \boldsymbol{x} = \phi_K(\boldsymbol{x}^\star)\sigma_K(\boldsymbol{x})\,. \tag{A.14}$$

Hence ϕ_K and σ_K are polar conjugates of each other.

Appendix B. Mindlin-, Hu–Washizu- and Eringen-type theories

B1. Mindlin-type gradient continuum

The internal power density of a Mindlin-type gradient continuum (Mindlin, 1964) is given by

$$p_{\mathrm{M}}^{\mathrm{int}}(\dot{\boldsymbol{u}}) := \boldsymbol{\varsigma} : \nabla^{\mathrm{sym}}\dot{\boldsymbol{u}} + \boldsymbol{\mu} \,\vdots\, \nabla(\nabla^{\mathrm{sym}}\dot{\boldsymbol{u}}) \,, \tag{B.1}$$

where $\boldsymbol{u}$ is the displacement, $\boldsymbol{\varsigma}$ is the stress tensor, and $\boldsymbol{\mu}$ is the double stress tensor. The corresponding single-field Euler–Lagrange equation in the absence of body forces is given by

$$-\mathrm{div}\,(\boldsymbol{\varsigma} - div\,\boldsymbol{\mu}) = \boldsymbol{0} \,. \tag{B.2}$$

The Euler–Lagrange equation is a fourth-order partial differential equation in terms of the displacement field $\boldsymbol{u}$. The strain tensor $\boldsymbol{\epsilon}(\boldsymbol{u})$ is defined in terms of the symmetric displacement gradient by

$$\boldsymbol{\epsilon}(\boldsymbol{u}) := \nabla^{\mathrm{sym}}\boldsymbol{u} \,.$$

B2. Three-Field Hu–Washizu formulation

A three-field Hu-Washizu (Hu, 1955; Washizu, 1982) type-formulation for a gradient continuum is obtained by introducing an independent symmetric tensor field $\overline{\boldsymbol{\varepsilon}}$ constrained as follows

$$\overline{\boldsymbol{\varepsilon}} \doteq \nabla^{\mathrm{sym}}\boldsymbol{u} \,,$$

and whose gradient is denoted by

$$\overline{\boldsymbol{\gamma}} := \nabla\overline{\boldsymbol{\varepsilon}} \,.$$

The equality of $\overline{\boldsymbol{\varepsilon}}$ and $\nabla^{\mathrm{sym}}\boldsymbol{u}$ can be expressed in the form of a constraint by

$$\overline{\boldsymbol{\delta}}(\boldsymbol{u}, \overline{\boldsymbol{\varepsilon}}) := \nabla^{\mathrm{sym}}\boldsymbol{u} - \overline{\boldsymbol{\varepsilon}} \doteq \boldsymbol{0} \,. \tag{B.3}$$

The constrained variables are distinguished by an overbar.

The internal power density corresponding to a three-field Hu–Washizu formulation for a gradient continuum is given by

$$p_{\mathrm{HW}}^{\mathrm{int}}(\dot{\boldsymbol{u}}, \dot{\overline{\boldsymbol{\varepsilon}}}, \boldsymbol{\lambda}) := \boldsymbol{\varsigma} : \dot{\overline{\boldsymbol{\varepsilon}}} + \boldsymbol{\mu} \,\vdots\, \dot{\overline{\boldsymbol{\gamma}}} + \boldsymbol{\lambda} : \dot{\overline{\boldsymbol{\delta}}} \,, \tag{B.4}$$

where $\boldsymbol{\lambda}$ is the tensorial Lagrange multiplier to impose the constraint (B.3).

The Euler–Lagrange equations corresponding to Eq. (B.4) follow as

$$-\mathrm{div}\,\boldsymbol{\lambda} = \boldsymbol{0} \,, \tag{B.5a}$$

$$\mathrm{div}\,\boldsymbol{\mu} = \boldsymbol{\varsigma} - \boldsymbol{\lambda} \,. \tag{B.5b}$$

The Euler–Lagrange equations corresponding to a Mindlin-type gradient continuum (B.2) are recovered by substituting the expression for the Lagrange multiplier in Eq. (B.5b) into Eq. (B.5a). It is thus clear that the solution to the three-field formulation given by (B.5a) and (B.5b) is equivalent to that of the one-field formulation (B.2).

B3. Eringen-type micromorphic continua

The strict enforcement of the constraint (B.3) via the Lagrange multiplier $\boldsymbol{\lambda}$ can be relaxed via a penalisation of the constraint violation whereby

$$\boldsymbol{\lambda} \to \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\delta}) \qquad \text{with} \qquad \boldsymbol{\delta} := \nabla^{\mathrm{sym}}\boldsymbol{u} - \boldsymbol{\varepsilon} \neq \boldsymbol{0} \,.$$

As a consequence of relaxing the constraint, the internal power density becomes that of a two-field Eringen-type formulation, given by

$$p_{\mathrm{E}}^{\mathrm{int}}(\dot{\boldsymbol{u}}, \dot{\boldsymbol{\varepsilon}}) := \boldsymbol{\varsigma} : \dot{\boldsymbol{\varepsilon}} + \boldsymbol{\mu} \,\vdots\, \dot{\boldsymbol{\gamma}} + \boldsymbol{\sigma} : \dot{\boldsymbol{\delta}} \,. \tag{B.6}$$

The corresponding Euler–Lagrange equations describing an Eringen-type micromorphic continuum are given by

$$\mathrm{div}\,\boldsymbol{\sigma} = \boldsymbol{0} \,, \tag{B.7a}$$

$$\mathrm{div}\,\boldsymbol{\mu} = \boldsymbol{\varsigma} - \boldsymbol{\sigma} \,. \tag{B.7b}$$

The solution to the Eringen-type micromorphic formulation (B.7a) and B.7b) only coincides asymptotically with that of the Mindlin-type gradient formulation as $\boldsymbol{\delta} \to \boldsymbol{0}$.

References

Aifantis, E.C., 1984. On the microstructural origin of certain inelastic models. J. Eng. Mater. Technol. 106 (4), 326–330.

Aifantis, E.C., 1987. The physics of plastic deformation. Int. J. Plast. 3 (3), 211–247.

Alberty, J., Carstensen, C., 2000. Numerical analysis of time-depending primal elastoplasticity with hardening. SIAM J. Numer. Anal. 37 (4), 1271–1294.

Anand, L., Gurtin, M., Lele, S., Gething, C., 2005. A one-dimensional theory of strain-gradient plasticity: formulation, analysis, numerical results. J. Mech. Phys. Solids 53 (8), 1789–1826.

Biot, M.A., 1965. Mechanics of Incremental Deformation: Theory of Elasticity and Viscoelasticity of Initially Stressed Solids and Fluids, Including Thermodynamic Foundations and Applications to Finite Strain. John Wiley & Sons.

de Borst, R., 1991. Simulation of strain localisation: a reappraisal of the Cosserat continuum. Eng. Comput. (Swansea) 8 (4), 317–332.

Carstensen, C., 1999. Numerical analysis of the primal problem of elastoplasticity with hardening. Numer. Math. 82 (4), 577–597.

Carstensen, C., Ebobisse, F., McBride, A.T., Reddy, B.D., Steinmann, P., 2017. Some properties of the dissipative model of strain-gradient plasticity. Philos. Mag. 97 (10), 693–717.

Chiricotto, M., Giacomelli, L., Tomassetti, G., 2016. Dissipative scale effects in strain-gradient plasticity: the case of simple shear. SIAM J. Appl. Math. 76 (2), 688–704.

Coleman, B.D., Noll, W., 1963. The thermodynamics of elastic materials with heat conduction and viscosity. Arch. Ration. Mech. Anal. 13 (1), 167–178.

Djoko, J., Ebobisse, F., McBride, A., Reddy, B., 2007. A discontinuous Galerkin formulation for classical and gradient plasticity part 1: formulation and analysis. Comput. Methods Appl. Mech. Eng. 196 (37), 3881–3897.

Eringen, A.C., 1999. Microcontinuum Field Theories. Springer, New York.

Fleck, N.A., Hutchinson, J.W., 1997. Strain Gradient Plasticity. In: Advances in Applied Mechanics, 33. Elsevier, pp. 295–361.

Fleck, N.A., Hutchinson, J.W., 2001. A reformulation of strain gradient plasticity. J. Mech. Phys. Solids 49 (10), 2245–2271.

Fleck, N.A., Hutchinson, J.W., Willis, J.R., 2014. Strain gradient plasticity under non-proportional loading. Proc. R. Soc. Lond. A Math. Phys. Eng. Sci. 470, 2170.

Fleck, N.A., Willis, J.R., 2009a. A mathematical basis for strain-gradient plasticity theory. part II: tensorial plastic multiplier. J. Mech. Phys. Solids 57 (7), 1045–1057.

Fleck, N.A., Willis, J.R., 2009b. A mathematical basis for strain-gradient plasticity theory part i: scalar plastic multiplier. J. Mech. Phys. Solids 57 (1), 161–177.

Forest, S., 2009. Micromorphic approach for gradient elasticity, viscoplasticity, and damage. J. Eng. Mech. 135 (3), 117–131.

Forest, S., 2010. Some links between recent gradient thermo-elasto-plasticity theories and the thermomechanics of generalized continua. Int. J. Solids Struct. 47 (25), 3367–3376.

Germain, P., 1973. La méthode des puissances virtuelles en mécanique des milieux continus. Journal de Mécanique 12, 235–274.

Grammenoudis, P., Tsakmakis, C., 2001. Hardening rules for finite deformation micropolar plasticity: restrictions imposed by the second law of thermodynamics and the postulate of ilíushin. Contin. Mech. Thermodyn. 13 (5), 325–363.

Grammenoudis, P., Tsakmakis, C., 2009. Micromorphic continuum part i: strain and stress tensors and their associated rates. Int. J. Non Linear Mech. 44 (9), 943–956.

Grammenoudis, P., Tsakmakis, C., Hofer, D., 2009. Micromorphic continuum. part II. Int. J. Non Linear Mech. 44 (9), 957–974.

Gudmundson, P., 2004. A unified treatment of strain gradient plasticity. J. Mech. Phys. Solids 52 (6), 1379–1406.

Gurtin, M.E., 2002. A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations. J. Mech. Phys. Solids 50 (1), 5–32.

Gurtin, M.E., Anand, L., 2005. A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. part i: small deformations. J. Mech. Phys. Solids 53 (7), 1624–1649.

Gurtin, M.E., Needleman, A., 2005. Boundary conditions in small-deformation, single-crystal plasticity that account for the burgers vector. J. Mech. Phys. Solids 53 (1), 1–31.

Halphen, B., Nguyen, Q.S., 1975. Sur les matériaux standard généralisés. Journal de Mécanique 14, 39–63.

Han, W., Reddy, B.D., 2010. Plasticity. mathematical theory and numerical analysis. Interdisciplinary Applied Mathematics, second ed. Springer.

Han, W., Reddy, B.D., Schroeder, G.C., 1997. Qualitative and numerical analysis of quasi-static problems in elastoplasticity. SIAM J. Numer. Anal. 34 (1), 143–177.

Hirschberger, C.B., Steinmann, P., 2009. Classification of concepts in thermodynamically consistent generalized plasticity. J. Eng. Mech. 135 (3), 156–170.

Hu, H., 1955. On some variational principles in the theory of elasticity and the theory of plasticity. Sci. Sin. 4, 33–54.

Kirchner, N., Steinmann, P., 2005. A unifying treatise on variational principles for gradient and micromorphic continua. Philos. Mag. 85 (33–35), 3875–3895.

Miehe, C., 2011. A multi-field incremental variational framework for gradient-extended standard dissipative solids. J. Mech. Phys. Solids 59 (4), 898–923.

Miehe, C., 2014. Variational gradient plasticity at finite strains. part i: mixed potentials for the evolution and update problems of gradient-extended dissipative solids. Comput. Methods Appl. Mech. Eng. 268, 677–703.

Miehe, C., Mauthe, S., Hildebrand, F., 2014. Variational gradient plasticity at finite strains. part III: local-global updates and regularization techniques in multiplicative plasticity for single crystals. Comput. Methods Appl. Mech. Eng. 268, 735–762.

Mindlin, R.D., 1964. Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. 16, 51–78.

Nix, W.D., Gao, H., 1998. Indentation size effects in crystalline materials: a law for strain gradient plasticity. J. Mech. Phys. Solids 46 (3), 411–425.

Panteghini, A., Bardella, L., 2016. On the finite element implementation of higher-order gradient plasticity, with focus on theories based on plastic distortion incompatibility. Comput. Methods Appl. Mech. Eng. 310, 840–865.

Reddy, B.D., 2011a. The role of dissipation and defect energy in variational formulations of problems in strain-gradient plasticity. part 1: polycrystalline plasticity. Contin. Mech. Thermodyn. 23 (6), 527–549.

Reddy, B.D., 2011b. The role of dissipation and defect energy in variational formulations of problems in strain-gradient plasticity. part 2: single-crystal plasticity. Contin. Mech. Thermodyn. 23 (6), 551.

Reddy, B.D., 2012. Erratum to: the role of dissipation and defect energy in variational formulations of problems in strain-gradient plasticity. part 1: polycrystalline plasticity. Contin. Mech. Thermodyn. 24 (1). 79–79

Reddy, B.D., Ebobisse, F., McBride, A., 2008. Well-posedness of a model of strain gradient plasticity for plastically irrotational materials. Int. J. Plast. 24 (1), 55–73.

Reddy, B.D., Wieners, C., Wohlmuth, B., 2012. Finite element analysis and algorithms for single-crystal strain-gradient plasticity. Int. J. Numer. Methods Eng. 90 (6), 784–804.

Runesson, K., Ekh, M., Larsson, F., 2017. Computational homogenization of mesoscale gradient viscoplasticity. Comput. Methods Appl. Mech. Eng. 317, 927–951.

Sansour, C., Skatulla, S., Zbib, H., 2010. A formulation for the micromorphic continuum at finite inelastic strains. Int. J. Solids Struct. 47 (11), 1546–1554.

Steinmann, P., Willam, K., 1991. Localization Within the Framework of Micropolar Elasto-Plasticity. Springer, Berlin, Heidelberg, pp. 296–313.

Svedberg, T., Runesson, K., 1997. A thermodynamically consistent theory of gradient-regularized plasticity coupled to damage. Int. J. Plast. 13 (6), 669–696.

Toupin, R.A., 1964. Theories of elasticity with couple-stress. Arch. Ration. Mech. Anal. 17 (2), 85–112.

Washizu, K., 1982. Variational Methods in Elasticity and Plasticity, third ed. Pergamon Press.

Ziegler, H., 1963. Some extremum principles in irreversible thermodynamics with application to continuum mechanics. Progress in Solid Mechanics, IV. Springer Verlag, Berlin Heidelberg. chapter 2