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CONVEX GEOMETRY OF THE CARRYING SIMPLEX FOR THE MAY-LEONARD MAP

Stephen Baigent*

Department of Mathematics, UCL, Gower Street
London, WC1E 6BT, UK

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Abstract. We study the convex geometry of certain invariant manifolds, known as carrying simplices, for 3-species competitive Kolmogorov-type maps. We show that if all planes whose normal bundles are contained in a fixed closed and solid convex cone are rendered convex (concave) surfaces by the map, then, if there is a carrying simplex, it is a convex (concave) surface. We apply our results to the May-Leonard map.

1. **Introduction.** We consider a class of diffeomorphisms that map the first orthant of Euclidean space into itself, and that are competitive. As shown by Takáč [31], such maps genetically possess codimension-1 invariant manifolds, and no two distinct points on these manifold can be ordered (the manifold is said to be unordered). For the subclass of competitive maps we consider here there is a single codimension-1 unordered invariant manifold that attracts all nonzero orbits. M. L. Zeeman named such manifolds *carrying simplices*. In particular we study the convex geometry of the carrying simplex for the three-species May-Leonard map, a map that models growth of three interacting populations. For three dimensions, the carrying simplex is a compact surface in the first orthant which projects one-to-one onto the two-dimensional unit probability simplex. The carrying simplex thus divides the first orthant into two components: below the simplex, the component that contains the origin, and above the simplex. We will say that the carrying simplex is convex when the set below is a convex set (see below for definitions), and concave when the set above is convex. Considered as a surface, a convex carrying simplex, as just defined, is a concave surface (taking the surface normal to point above the surface) and is the graph of a concave function, and a concave carrying simplex is a convex surface.

A convex surface can be expressed as the supremum of its supporting planes, and, as we show, if each supporting plane is mapped to a new convex surface, then the image of the current surface under the map is also convex. A similar idea works for concave surfaces. We take flat surfaces formed of the convex hull of three axial points and iterate forward until the iterates converge to the carrying simplex. For

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* Corresponding author.

each surface iterate we consider the set of all tangent planes to that surface. We show that if all such tangent planes are rendered convex by the map, the next iterate is also convex. However, only a certain subset of planes are rendered convex by the map, namely those whose normal bundle belongs to a solid convex cone that depends on the specifics of the map. Convexity of the evolving surface, and of the carrying simplex, can then be established by showing that the normal bundle of each surface iterate lies in a fixed closed and solid convex cone.

2. Background.

The carrying simplex is a codimension-1 unordered invariant manifold that attracts all nonzero orbits which has been studied in the context of competitive dynamics (see definition 4.1 below). The origins of the carrying simplex for continuous time systems can be traced to Hirsch [10] and for discrete-time models de Mottoni and Schiaffino [6] and Smith [29]. It coined its name in an article by Zeeman [35] where asymptotic dynamics on the carrying simplex were used to classify 3-dimensional competitive Lotka-Volterra systems into 33 equivalence classes. Other authors have refined results for existence of the carrying simplex, and used these to unravel the long-term dynamics of competitive systems from ecology [28, 7, 14, 13, 15, 16, 25, 17].

The geometry of the carrying simplex is a newer area of research, particularly for the case of maps. Convexity of the carrying simplex for planar competitive Lotka-Volterra systems was first studied by M. L. Zeeman and E. C. Zeeman [34], and later revisited for the same model by Tineo [32] who showed that the carrying simplex was either convex or concave, dependent on the sign of a single parameter. Baigent [3] provided an alternative proof of Tineo's result via a dynamical approach based upon the graph transform. He showed that the parameter that determined convexity or concavity was proportional to the initial rate of change of curvature of the straight line joining the axial fixed points. Convexity or concavity of the carrying simplex of 3-dimensional Lotka-Volterra systems were first studied by M. L. Zeeman and E. C. Zeeman [34]. Later Baigent used the evolution equations for the 2nd fundamental form of each graph iterate in the graph transform [1] to establish examples where the carrying simplex was either convex or concave. For maps, Baigent recently established that the dichotomy between convexity or concavity of the carrying simplex carried over from the planar competitive Lotka-Volterra model to the planar discrete-time Leslie-Gower model [2].

Here we extend some of these ideas to the three-species discrete-time Leslie-Gower model in the symmetric case, which we refer to as the May-Leonard model (see equation (8) below). Figure 1 shows examples of a convex and a concave carrying simplex for the May-Leonard map.

In [3, 1, 2] confining the normal of evolving surfaces to a suitable convex cone K plays a key role, and continues to do so in the present paper since typically only planes with normals belonging to a closed and solid convex cone K are mapped to convex or concave surfaces. It is then a question of showing that the normals of the evolving surfaces remain within the cone K. Finding a suitable cone is typically not straightforward, and is sometimes (see section 12) linked to finding a cone K for which the map is $K-$competitive or equivalently that its inverse is $K-$monotone (see definitions below).

3. Preliminaries.

We take the convention that vectors are treated as column vectors and appear in boldface. Let $K \subseteq C_+ := \mathbb{R}^3_+$, where $\mathbb{R}_+ = [0, \infty)$, be a closed and solid convex cone (i.e. $\lambda K \subseteq K$ for $\lambda > 0$, $K + K \subset K$, $K \cap (-K) = \{\mathbf{0}\}$, the

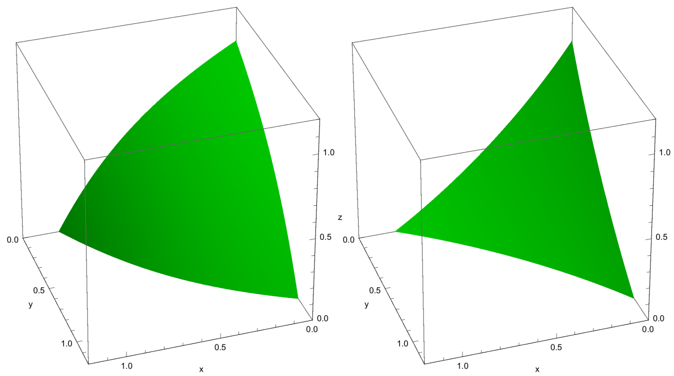

FIGURE 1. Carrying simplices for the May-Leonard model (8) with $r = 2$. Left: Convex carrying simplex for $\alpha = 3/4, \beta = 2/3$ (see example 11.2). Right: Concave carrying simplex $\alpha = 5/4, \beta = 7/6$ (see example 11.1).

interior K^0 of K is non-empty and K is closed). (For a set S, we use S^0 to denote its interior.) The cone K induces an ordering $\leq_K$ on $\mathbb{R}^3$ via $\boldsymbol{x} \leq_K \boldsymbol{y}$ if and only if $\boldsymbol{y} - \boldsymbol{x} \in K$. We also write $\boldsymbol{x} <_K \boldsymbol{y}$ if $\boldsymbol{x} \leq_K \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$ and $\boldsymbol{x} \ll_K \boldsymbol{y}$ if $\boldsymbol{y} - \boldsymbol{x} \in K^0$. Two distinct points $\boldsymbol{x}, \boldsymbol{y}$ are order-related if either $\boldsymbol{x} <_K \boldsymbol{y}$ or $\boldsymbol{y} <_K \boldsymbol{x}$, else they are unrelated. The case $K = C_+$ is the standard nonnegative cone order, and we will write $\leq, <, \ll$ for the order relations in this case. We will use $\boldsymbol{x} \cdot \boldsymbol{y}$ to denote the usual inner product on $\mathbb{R}^3$ and $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$ the Euclidean norm. The points $\boldsymbol{e}_i \in C_+$ are those unit vectors with components $(\boldsymbol{e}_i)_j = \delta_{ij}$, $i, j \in I_3 := \{1, 2, 3\}$, (where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$). $P^\star$ denotes the transpose of the (real) matrix P. By $B(\boldsymbol{x}, r)$ we mean an open ball radius $r > 0$ in $\mathbb{R}^3$ centred on $\boldsymbol{x}$.

Definition 3.1 (K-monotone map)**.** We say that a map $\boldsymbol{S} : C_+ \to C_+$ is $K-$*monotone* if $\boldsymbol{x} \leq_K \boldsymbol{y}$ implies $\boldsymbol{S}(\boldsymbol{x}) \leq_K \boldsymbol{S}(\boldsymbol{y})$.

Definition 3.2 (K-Competitive map)**.** We say that a map $\boldsymbol{T} : C_+ \to C_+$ is $K-$*competitive* if $\boldsymbol{x} \leq_K \boldsymbol{y}$ whenever $\boldsymbol{T}(\boldsymbol{x}) \leq_K \boldsymbol{T}(\boldsymbol{y})$.

This is the definition, for example, used by many other authors (e.g. [33, 12, 28, 2]), which assume that $\boldsymbol{T}$ is orientation preserving. Other authors, e.g. [30, 17] allow for $\boldsymbol{T}$ to be orientation reversing. Here our assumptions on $\boldsymbol{T}$, stated in section 4, imply that it is orientation-preserving.

When $K = C_+$ we will omit the prefix $K-$ and simply say that the map is competitive in place of C_+-competitive.

Definition 3.3 (Strongly K-competitive map)**.** We say that a map $\boldsymbol{T} : C_+ \to C_+$ is *strongly* $K-$*competitive* if $\boldsymbol{x} \ll_K \boldsymbol{y}$ whenever $\boldsymbol{T}(\boldsymbol{x}) <_K \boldsymbol{T}(\boldsymbol{y})$.

Notice that when $\boldsymbol{T} : C_+ \to \boldsymbol{T}(C_+)$ is a $K-$competitive diffeomorphism, $\boldsymbol{T}^{-1} : T(C_+) \to C_+$ is a monotone map for the order $\leq_K$ defined by the cone K, i.e. $\boldsymbol{x} \leq_K \boldsymbol{y} \Rightarrow \boldsymbol{T}^{-1}(\boldsymbol{x}) \leq_K \boldsymbol{T}^{-1}(\boldsymbol{y})$. For an open set $Y \subset \mathbb{R}^3$, when $\boldsymbol{T} \in C^1(Y)$ and $D\boldsymbol{T}$ is nonsingular on Y then $\boldsymbol{T}$ is strongly competitive on Y if $\boldsymbol{0} \ll D\boldsymbol{T}^{-1}(\boldsymbol{x})$ for $\boldsymbol{x} \in Y$. We denote by $\boldsymbol{T}^k$ the composition of $\boldsymbol{T}$ with itself k times.

Definition 3.4 (Closed order interval)**.** We set $[\boldsymbol{x}, \boldsymbol{y}] = \{\boldsymbol{a} \in C_+ : \boldsymbol{x} \leq_K \boldsymbol{a} \leq_K \boldsymbol{y}\}$.

Definition 3.5 (Unordered set)**.** A subset $X \subset \mathbb{R}^d$ is unordered if it does not contain any order-related points.

Definition 3.6 ($\boldsymbol{T}-$forward-invariant cone)**.** We say that the cone $K \subseteq C_+$ is $\boldsymbol{T}-$forward-invariant if $\boldsymbol{T}(K) \subseteq K$ (i.e. $\boldsymbol{T}(\boldsymbol{x})K \subseteq K$ for all $\boldsymbol{x} \in C_+$).

Definition 3.7 ($\boldsymbol{T}-$invariant cone)**.** We say that the cone $K \subseteq C_+$ is $\boldsymbol{T}-$invariant if $\boldsymbol{T}(K) = K$.

Definition 3.8 ($\Delta(\cdot)$)**.** For $\boldsymbol{a} \in C_+^0$ we let $\Delta(\boldsymbol{a}) = \{x \in C_+ : \boldsymbol{a} \cdot \boldsymbol{x} = 1\}$. Thus $\Delta(\boldsymbol{a})$ is the convex hull of the points $\{a_i^{-1}\boldsymbol{e}_i : i \in I_3\}$. We will use the special notation Δ_2 in place of $\Delta((1,1,1))$, the unit probability simplex, and $\Delta(\boldsymbol{q}^{-1})$ is the convex hull of $\{q_1\boldsymbol{e}_1, q_2\boldsymbol{e}_2, q_3\boldsymbol{e}_3\}$ using the notation $\boldsymbol{q}^{-1} = (1/q_1, 1/q_2, 1/q_3)$.

Definition 3.9 (Cofactor matrix)**.** Let P be a real square matrix. Then the cofactor matrix of P, denoted by $P^{\#}$ is the matrix whose i,jth element is the determinant of the matrix P obtained by removing the ith row and jth column from P. Thus when P is invertible, $P^{\#} = \det P (P^{-1})^{\star}$.

Definition 3.10 (Kolmogorov-type maps)**.** We say that $\boldsymbol{T} : C_+ \to C_+$ is a Kolmogorov-type map if $\boldsymbol{T} = (T_1, T_2, T_3)$ has $T_i(\boldsymbol{x}) = x_i f_i(\boldsymbol{x})$ for $i \in I_3$ and $f_i : C_+ \to C_+$ is at least continuous.

4. **The Carrying Simplex.** As explained in the introduction the geometrical object that we are concerned with is a codimension-1 Lipschitz invariant manifold known as the *carrying simplex* (see Figure 1 for examples).

We use the definition of a $(d-1)$-dimensional carrying simplex ($d \geq 1$ integer) provided by Hirsch [11]:

Definition 4.1 (Carrying simplex)**.** The carrying simplex is a set $\Sigma \subset \mathbb{R}_+^d \setminus \{\boldsymbol{0}\}$ that is compact, invariant and unordered, and such that for each $\boldsymbol{x} \in \mathbb{R}_+^d \setminus \{\boldsymbol{0}\}$ there is a $\boldsymbol{y} \in \Sigma$ such that $\lim_{k\to\infty} \|\boldsymbol{T}^k(\boldsymbol{x}) - \boldsymbol{T}^k(\boldsymbol{y})\| \to 0$.

The study of the carrying simplex for maps, although not referred to as the carrying simplex at the time, began with the study of evolution equations with periodic coefficients and a review of some of these results can be found in [9]. To the best of the author's knowledge, carrying simplices as defined in 4.1 have only been studied in the context of maps that are competitive for the standard cone C_+. While Hirsch's definition does not require $\boldsymbol{T}$ to be competitive, most proofs of existence of the carrying simplex assume that the map $\boldsymbol{T}$ is competitive.

For more recent existence theory for the carrying simplex for competitive maps the reader is referred to [33, 28, 15].

If Σ is continuously differentiable, then the unorderedness of Σ translates into its normal bundle being contained in C_+ [1]. It is an open question as to exactly when Σ is differentiable on its interior, but much progress has been made obtaining sufficient conditions for Σ to satisfy various smoothness properties [22, 21, 5, 4, 23, 12]. In two recent articles [24, 20] Mierczyński has shown that convex carrying simplices are C^1. We will not need to know whether Σ is smooth to establish its convexity or concavity.

SA: Standing assumptions on $\boldsymbol{T}$

1. $\boldsymbol{T} : C_+ \to C_+$ is a Kolmogorov-type diffeomorphism with $T_i(\boldsymbol{x}) = x_i f_i(\boldsymbol{x})$ where each f_i is at least C^1 smooth in a neighbourhood of C_+;

2. $\boldsymbol{f} \gg \mathbf{0}$ on C_+ and $\boldsymbol{f}(\mathbf{0}) \gg (1,1,1)$;
3. $\boldsymbol{T}$ has axial fixed points $(q_1, 0, 0)$, $(0, q_2, 0)$ and $(0, 0, q_3)$;
4. $\partial f_i / \partial x_j < 0$ for all $i, j \in I_3$ on C_+;
5. For all $\boldsymbol{x} \in [0, \boldsymbol{q}] \setminus \{\mathbf{0}\}$ the matrix $M(\boldsymbol{x})$ whose i, jth entry is $-x_i \frac{\partial \log(f_i)}{\partial x_j}$ has spectral radius less than one.

These standing assumptions that we place on our map $\boldsymbol{T} : C_+ \to C_+$ are sufficient to ensure the existence of the carrying simplex (e.g. Theorem 3.1 in [15] and see also [28]). In particular, standing assumption 5 implies that $\boldsymbol{T}$ is orientation-preserving.

Now we show that the carrying simplex Σ can be constructed from a particular sequence of images of a plane under the map $\boldsymbol{T}$. Consider the sequence of surfaces $\{\Sigma_k\}_{k=0}^{\infty}$ where $\Sigma_0 = \Delta(\boldsymbol{q}^{-1})$ (the convex hull of the axial fixed points):

$$\Sigma_k = \boldsymbol{T}^k(\Sigma_0), \;\; \Sigma_0 = \Delta(\boldsymbol{q}^{-1}), \;\; \mathcal{N}_k = \text{normal bundle of } \Sigma_k. \tag{1}$$

Note that Σ_k is unordered for each $k \in \mathbb{Z}_+$. Indeed if for some integer $k \geq 1$ there are two distinct points $\boldsymbol{x}, \boldsymbol{y} \in \Sigma_k$ such that $\boldsymbol{x}$ and $\boldsymbol{y}$ are related, then their preimages must have been related by the definition of a competitive map. Using induction, and that Σ_0 is unordered, this provides a contradiction.

$\Sigma_1 = \boldsymbol{T}(\Delta_2)$ is a simply-connected set. Since $\boldsymbol{T}$ is of Kolmogorov type, it maps the boundary ∂C_+ into itself. In particular, the edge E_{12} of Δ_2 joining $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ is mapped by $\boldsymbol{T}$ to an unordered curve connecting $\boldsymbol{q}_1, \boldsymbol{q}_2$ and lying in the plane where $z = 0$. Similarly for the other two edges of Δ_2. Hence we see that $\partial(\boldsymbol{T}(\Delta_2))$ is a closed curve in ∂C_+ that projects radially onto $\partial \Delta_2$. The radial projection onto Δ_2 of Σ_1 is a simply-connected subset of Δ_2 and $\partial \Sigma_1$ is a closed curve that projects radially onto $\partial \Delta_2$, so that Σ_1 must project radially onto Δ_2.

We conclude that Σ_1, and by induction Σ_k for all $k \geq 1$, is an ordered surface that projects radially one-to-one and onto Δ_2. Accordingly, with each Σ_k we may associate a continuous function $R_k : \Delta_2 \to \mathbb{R}$ for which $\Sigma_k = \{R_k(\boldsymbol{u})\boldsymbol{u} : \boldsymbol{u} \in \Delta_2\}$.

We will show that $\Sigma_k \to \Sigma$ uniformly in the following sense: Each Σ_k can be written as $\Sigma_k = \{R_k(\boldsymbol{u})\boldsymbol{u} : \boldsymbol{u} \in \Delta_2\}$ where $R_k : \Delta_2 \to \mathbb{R}$ is continuous and $R_k \to R^*$ uniformly where $R^* : \Delta_2 \to \mathbb{R}$ is continuous and $\Sigma = \{(R^*(\boldsymbol{u})\boldsymbol{u}, \boldsymbol{u} \in \Delta_2\}$.

Lemma 4.2. *If a surface $S \subset \mathbb{R}^3$ is unordered, then S is a Lipschitz manifold with Lipschitz constant less than or equal to $\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.*

Proof. Denote by H the plane with normal $\boldsymbol{n} = (1,1,1)/\sqrt{3}$ passing through the origin and $\pi : \mathbb{R}^3 \to H$ projection onto H along $\boldsymbol{n}$. Let $\boldsymbol{x}, \boldsymbol{y} \in S$ be distinct. Then $\boldsymbol{x} = \pi(\boldsymbol{x}) + \boldsymbol{n} \cdot (\boldsymbol{x} - \pi(\boldsymbol{x}))\, \boldsymbol{n}$ and $\boldsymbol{y} = \pi(\boldsymbol{y}) + \boldsymbol{n} \cdot (\boldsymbol{y} - \pi(\boldsymbol{y}))\, \boldsymbol{n}$. Thus $\boldsymbol{x} - \boldsymbol{y} = \pi(\boldsymbol{x}) - \pi(\boldsymbol{y}) - \boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{y})\, \boldsymbol{n}$ so that $\|\pi(\boldsymbol{x}) - \pi(\boldsymbol{y})\|_2 = \|\boldsymbol{x} - \boldsymbol{y} + ((\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{n})\, \boldsymbol{n}\|_2 \geq |\|\boldsymbol{x} - \boldsymbol{y}\|_2 - \|((\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{n})\, \boldsymbol{n}\|_2| = \|\boldsymbol{x} - \boldsymbol{y}\||_2(1 - |\cos\theta|)|$, where θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{x} - \boldsymbol{y}$. Now $\boldsymbol{x}, \boldsymbol{y}$ are unordered, so that $\boldsymbol{x} - \boldsymbol{y} \notin C_+ \cup (-C_+)$. But then $|\cos\theta| < \frac{(1,1,0)}{\sqrt{2}} \cdot \frac{(1,1,1)}{\sqrt{3}} = \sqrt{\frac{2}{3}}$. This shows that $\|\boldsymbol{x} - \boldsymbol{y}\|_2 \leq \frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}} \|\pi(\boldsymbol{x}) - \pi(\boldsymbol{y})\|_2$ for all $\boldsymbol{x}, \boldsymbol{y} \in S$. Hence S is a Lipschitz manifold with Lipschitz constant less than or equal to $L^* := \frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$. □

The following lemma was inspired by [27].

Lemma 4.3. *Let $\Theta \subset \mathbb{R}^d$ be compact and $\varphi_k : \Theta \to \mathbb{R}$ be a sequence of functions with Lipschitz constant at most L. Suppose that $\varphi_k \to \varphi$ pointwise, where φ is Lipschitz. Then $\varphi_k \to \varphi$ uniformly.*

Proof. For each $\boldsymbol{x}, \boldsymbol{y} \in \Theta$ we have $|\varphi_k(\boldsymbol{x}) - \varphi_k(\boldsymbol{y})| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|$ for all k and $|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \leq M\|\boldsymbol{x} - \boldsymbol{y}\|$. Thus for each $\epsilon > 0$, and each $\boldsymbol{x}, \boldsymbol{y} \in \Theta$,

$$\begin{aligned}|\varphi_k(\boldsymbol{x}) - \varphi(\boldsymbol{x})| &\leq |\varphi_k(\boldsymbol{x}) - \varphi_k(\boldsymbol{y})| + |\varphi_k(\boldsymbol{y}) - \varphi(\boldsymbol{y})| + |\varphi(\boldsymbol{y}) - \varphi(\boldsymbol{x})| \\ &\leq (L + M)\|\boldsymbol{x} - \boldsymbol{y}\| + |\varphi_k(\boldsymbol{y}) - \varphi(\boldsymbol{y})|.\end{aligned}$$

Since Θ is compact, given $\epsilon > 0$, Θ can be covered by a finite number, say N_ϵ, of balls $B\left(\boldsymbol{y}_i, \frac{\epsilon}{2(L+M)}\right)$, $i \in I_{N_\epsilon}$. For each $\boldsymbol{x} \in \Theta$ there is an $i \in I_{N_\epsilon}$ such that $\boldsymbol{x} \in B\left(\boldsymbol{y}_i, \frac{\epsilon}{2(L+M)}\right)$. By pointwise convergence, there is an N such that $|\varphi_k(\boldsymbol{y}_j) - \varphi(\boldsymbol{y}_j)| < \frac{\epsilon}{2}$ for $k \geq N$, for all $j \in I_{N_\epsilon}$. Hence, given $\epsilon > 0$, for all $\boldsymbol{x} \in \Theta$, there exists an N such that

$$\begin{aligned}|\varphi_k(\boldsymbol{x}) - \varphi(\boldsymbol{x})| &\leq (L + M)\|\boldsymbol{x} - \boldsymbol{y}_i\| + |\varphi_k(\boldsymbol{y}_i) - \varphi(\boldsymbol{y}_i)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for all } k \geq N.\end{aligned}$$

□

By lemma 4.2 each Σ_k is the graph of a Lipschitz function $\phi_k : \pi(\Sigma_k) \to \mathbb{R}$ with Lipschitz constant less than or equal to $L^* = \frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}$. By [19] each ϕ_k can be extended (or restricted) to a Lipschitz function $\phi_k : \Theta \to \mathbb{R}$ where $\Theta = \pi(\Sigma)$ and Σ is the carrying simplex. Σ is globally attracting and unordered, and can be represented as the graph of a Lipschitz function $\phi^* : \pi(\Sigma) \to \mathbb{R}$ and $\{\phi_k\}_{k=0}^{\infty}$ converges pointwise to ϕ^*. Hence by lemma 4.3 $\phi_k \to \phi^*$ uniformly. Finally $\max_{\boldsymbol{u} \in \Delta_2} |R_k(\boldsymbol{u}) - R^*(\boldsymbol{u})| \leq \sqrt{3} \max_{\boldsymbol{y} \in \Theta} |\phi_k(\boldsymbol{y}) - \phi^*(\boldsymbol{y})|$ so that $R_k \to R^*$ uniformly.

5. Convexity or concavity of the carrying simplex.

Next we expand upon the use of 'convex' and 'concave', such as for surfaces and sets in $\mathbb{R}^3$. We use the standard definition that a set $U \subset \mathbb{R}^3$ is convex if whenever $\boldsymbol{x}, \boldsymbol{y} \in U$ are distinct points then $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in U$ for all $t \in [0, 1]$.

Let S be a smooth, regular and connected surface in $\mathbb{R}^3$. At each point $\boldsymbol{x} \in S^0$, let $B(\boldsymbol{x}, r) \subset \mathbb{R}^3$ be any open ball radius r such that $B(\boldsymbol{x}, r)$ is divided into two disjoint components by S. Next choose one of the two unit normal vectors at $\boldsymbol{x}$, and denote this vector by $\boldsymbol{n}_+$. The choice of $\boldsymbol{n}_+$ determines an orientation of S (a normal field). We denote by $B_+(\boldsymbol{x}, r)$ the component of $B(\boldsymbol{x}, r)$ that the normal $\boldsymbol{n}_+$ points into, and $B_-(\boldsymbol{x}, r)$ the component that $-\boldsymbol{n}_+$ points into, so that $B(\boldsymbol{x}, r)$ is the disjoint union $B(\boldsymbol{x}, r) = B_+(\boldsymbol{x}, r) \cup (B(\boldsymbol{x}, r) \cap S) \cup B_-(\boldsymbol{x}, r)$.

Definition 5.1 (Convex/Concave surface)**.** We say that S is convex at $\boldsymbol{x}$ if for all sufficiently small $r > 0$ the set $B_+(\boldsymbol{x}, r)$ is convex. We say that S is convex if S is convex at each point of S. Similarly we say that S is concave at $\boldsymbol{x}$ if for all sufficiently small $r > 0$ the set $B_-(\boldsymbol{x}, r)$ is convex. We say that S is concave if S is concave at each point of S.

Here, most of the surfaces S we meet are unordered, which means that they are graphs of decreasing functions, and we choose an orientation where the normal is nonnegative. This means that when S is convex, it is the graph of a function that is convex (on each convex subset of its domain).

The following definition is in line with the definition originally given by E. C. Zeeman and M. L. Zeeman [34]. Warning: it can sometimes lead to confusion since it equivalently defines a carrying simplex Σ to be convex when the set in C_+ below Σ is convex, which is when Σ is a concave surface.

Definition 5.2 (Convex/Concave carrying simplex [34])**.** The carrying simplex Σ is said to be convex(concave) when it is a concave(convex) surface.

In Figure 1 for example, the left plot is that of a convex carrying simplex for the map (8) and the right plot is that of a concave carrying simplex for the map (8).

6. Main result.

We now come to our main theoretical result, namely the following construction of convex or concave carrying simplices in 3 dimensional space based upon a reduction to the action of the map $\boldsymbol{T}$ on planes.

Theorem 6.1 (Convex/Concave carrying simplices)**.** *Let $\boldsymbol{T} : C_+ \to \boldsymbol{T}(C_+)$ satisfy the standing assumptions SA, and let Σ denote the carrying simplex. Let $K \subset C_+^0$ be a closed and solid convex cone containing $\boldsymbol{q}^{-1}$ and such that for all $\boldsymbol{a} \in K$ the surface $\boldsymbol{T}(\Delta(\boldsymbol{a}))$ is strictly concave (strictly convex) and its normal bundle is a subset of K. Then $\Sigma = \lim_{k\to\infty} \boldsymbol{T}^k(\Delta(\boldsymbol{q}^{-1}))$ is a convex (concave) carrying simplex.*

We prove this theorem in section 7.

7. Mappings of planes to convex or concave surfaces.

Let $\phi_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{a}\cdot\boldsymbol{x} - 1$, $\boldsymbol{x} \in \mathbb{R}^3$, $\boldsymbol{a} \in \mathbb{R}^3$. The set $\phi_{\boldsymbol{a}}^{-1}(0) = \{\boldsymbol{x} \in \mathbb{R}^3 : \phi_{\boldsymbol{a}}(\boldsymbol{x}) = 0\}$ is the plane that passes through the points $a_i^{-1}\boldsymbol{e}_i$, $i \in I_3$ and $\phi_{\boldsymbol{a}}^{-1}(0) \cap C_+ = \Delta(\boldsymbol{a})$. Thus by suitable choices of $\boldsymbol{a} \in C_+$ we may generate all planes with nonnegative normals.

Under the diffeomorphism $\boldsymbol{T} : C_+ \to \boldsymbol{T}(C_+)$, the zero set of $\phi_{\boldsymbol{a}}$ in C_+ is transformed to the zero set of $L\phi_{\boldsymbol{a}} : \boldsymbol{T}(C_+) \to \mathbb{R}$ where

$$L\phi_{\boldsymbol{a}}(\boldsymbol{x}) = \boldsymbol{a}\cdot\boldsymbol{T}^{-1}(\boldsymbol{x}) - 1. \tag{2}$$

Our concern is the geometry of the level sets $(L\phi_{\boldsymbol{a}})^{-1}(0)\,(\subset \boldsymbol{T}(C_+))$ for different $\boldsymbol{a} \in C_+$, and in particular when they are convex or concave (see Figure 2). Let us

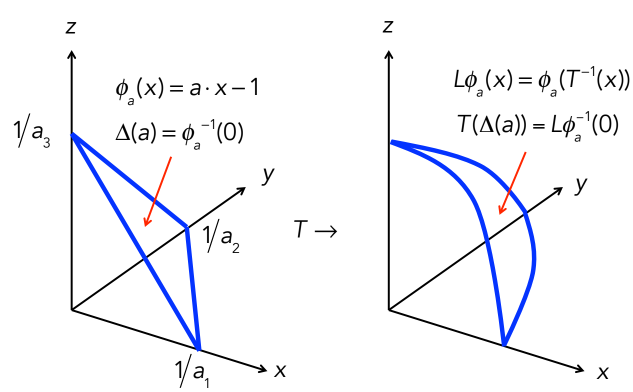

FIGURE 2. Mapping of $\Delta(\boldsymbol{a})$ by $\boldsymbol{T}$ to the new set $\boldsymbol{T}(\Delta(\boldsymbol{a}))$

consider the evolution of the normal of a surface S given implicitly as the zero set of some smooth $\phi : C_+ \to \mathbb{R}$. For $\boldsymbol{z} \in C_+$, $D\phi(\boldsymbol{z}) \in \mathcal{N}_{\boldsymbol{z}}$ (the normal bundle at $\boldsymbol{z}$), and

Lemma 7.1. *With* $\phi' = L\phi = \phi \circ \boldsymbol{T}^{-1}$,

$$D\phi'(\boldsymbol{x}) = (D\boldsymbol{T}^{-1}(\boldsymbol{x}))^\star D\phi(\boldsymbol{z}) = \frac{1}{\det D\boldsymbol{T}(\boldsymbol{z})} D\boldsymbol{T}^{\#}(\boldsymbol{z}) D\phi(\boldsymbol{z}), \quad \boldsymbol{x} = \boldsymbol{T}(\boldsymbol{z}).$$

Proof. Apply the chain rule. □

Given an open set $U \subset \mathbb{R}^3$ and a smooth $\phi : U \to \mathbb{R}$, the Gaussian curvature κ at a regular point $\boldsymbol{x} \in U$ (i.e. where $D\phi(\boldsymbol{x}) \neq \boldsymbol{0}$) can be found from the well-known formula (e.g. [8])

$$\kappa(\boldsymbol{x}) = \frac{D\phi(\boldsymbol{x}) \cdot (D^2\phi(\boldsymbol{x}))^{\#} D\phi(\boldsymbol{x})}{|D\phi(\boldsymbol{x})|^4}, \quad \boldsymbol{x} \in U. \tag{3}$$

In practice, to study the convexity or concavity of smooth surfaces given implicitly as the zero set $\phi^{-1}(0)$, at a regular point $\boldsymbol{x} \in \phi^{-1}(0)$ we can appeal to the simpler expression

$$\kappa_0(\boldsymbol{x}) = D\phi(\boldsymbol{x}) \cdot (D^2\phi(\boldsymbol{x}))^{\#} D\phi(\boldsymbol{x}), \tag{4}$$

since $\kappa_0(\boldsymbol{x})$ in (4) has the same sign as $\kappa(\boldsymbol{x})$ in (3).

7.1. **Proof of Theorem 6.1.** Let $S \subset C_+$ be a surface that projects radially one-to-one and onto Δ_2. If S is the surface $\{R(\boldsymbol{u})\boldsymbol{u} : \boldsymbol{u} \in \Delta_2\}$, then we define $S_- = \{r\boldsymbol{u} : 0 \leq r < R(\boldsymbol{u}), \boldsymbol{u} \in \Delta_2\}$ and $S_+ = \{r\boldsymbol{u} : r > R(\boldsymbol{u}), \boldsymbol{u} \in \Delta_2\}$.

Proof. We start with the case of a convex carrying simplex Σ (so that Σ_- is a convex set).

Consider the sequence (1), i.e. let $\Sigma_0 = \Delta(\boldsymbol{q}^{-1})$ and $\Sigma_k = \boldsymbol{T}^k(\Sigma_0)$, $k \in \mathbb{Z}_+$, so that $\Sigma = \lim_{k\to\infty} \Sigma_k$. By the hypothesis of the theorem, $\Sigma_1 = \boldsymbol{T}(\Sigma_0) = \boldsymbol{T}(\Delta(\boldsymbol{q}^{-1}))$ is a strictly concave surface, since $\boldsymbol{q}^{-1} \in K$. Since each $\boldsymbol{q}_i$, $i \in I_3$ is a fixed point, and Σ_1 is a strictly concave surface, Σ_1 lies on or above $\Delta(\boldsymbol{q}^{-1})$ and the intersection of C_+ with every tangent plane to Σ_1 is of the form $\Delta(\boldsymbol{a})$ for some $\boldsymbol{a} \in K$. Hence the normal bundle $\mathcal{N}_1 \subseteq K$ is such that $(\Sigma_1)_- = \bigcap_{\boldsymbol{a}\in\mathcal{N}_1}(\Delta(\boldsymbol{a}))_-$. We then have $(\Sigma_2)_- = \boldsymbol{T}(\Sigma_1)_- = \bigcap_{\boldsymbol{a}\in\mathcal{N}_1} \boldsymbol{T}(\Delta(\boldsymbol{a}))_-$ which is convex, since each $\boldsymbol{T}(\Delta(\boldsymbol{a}))$ is a strictly concave surface, and $\Sigma_2 = \boldsymbol{T}(\Sigma_1)$ is a strictly concave surface. Continuing the argument shows that each Σ_k is a strictly concave surface and by preservation of concavity in the limit (e.g. [26]), Σ is a concave surface, and therefore a convex carrying simplex.

Next, we consider the case where Σ is a concave carrying simplex. Now the $\Sigma_1 = \boldsymbol{T}(\Sigma_0) = \boldsymbol{T}(\Delta(\boldsymbol{q}^{-1}))$ is strictly convex surface, since $\boldsymbol{q}^{-1} \in K$. The set $(\Sigma_1)_+$ is convex and can be written as the intersection $(\Sigma_1)_+ = \bigcap_{\boldsymbol{a}\in\mathcal{N}_2}(\Delta(\boldsymbol{a}))_+$ where $\mathcal{N}_2 \subseteq K$ is the normal bundle of Σ_1. Then $\boldsymbol{T}((\Sigma_1)_+) = \bigcap_{\boldsymbol{a}\in\mathcal{N}_2} \boldsymbol{T}(\Delta(\boldsymbol{a}))_+$ which is convex since each $\boldsymbol{T}(\Delta(\boldsymbol{a}))_+$ is a strictly convex surface, and $\Sigma_2 = \boldsymbol{T}(\Sigma_1)$ is a strictly convex surface. As in the case of a convex carrying simplex we obtain a sequence of surfaces, but now all strictly convex, that converge to a convex surface Σ, and hence Σ is a concave carrying simplex. □

8. Putting bounds on the set of supporting planes to $\boldsymbol{T}(\Delta(\boldsymbol{a}))$.

In this section we show how the containment of the normal bundle sequence $\{\mathcal{N}_k\}_{k=0}^{\infty}$ (see (1)) in some solid convex cone $K \subset C_+^0$ of each Σ_k in (1) can be used to restrict which $\boldsymbol{a} \in C_+^0$ need to be tested to see whether $\boldsymbol{T}(\Delta(\boldsymbol{a}))$ is a convex or concave surface. In Figure 3 we highlight the key difference between the convex and concave case. In the case when Σ_k is a convex surface, tangent planes meet the boundary on or inside the order interval $[0, \boldsymbol{q}]$, whereas in when Σ_k is a concave surface they meet the boundary on or outside $[0, \boldsymbol{q}]$.

Recall from (1) that the carrying simplex Σ is obtained as the (uniform) limit $\Sigma = \lim_{k\to\infty} \boldsymbol{T}^k(\Delta(\boldsymbol{q}^{-1}))$. Suppose that each normal bundle of Σ_k, $\mathcal{N}_k$, is a subset of $K \subset C_+^0$ for $k \in \mathbb{Z}_+$. Fix some $k \geq 1$.

If $\Sigma_k = \boldsymbol{T}^k(\Delta(\boldsymbol{q}^{-1}))$ is a concave surface, then since the normal bundle $\mathcal{N}_k$ of Σ_k is positive, out of all its supporting planes, there is one which cuts the $x-$axis furthest from the origin, say x^k. Then x^k is bounded above by the maximum intercept $x_{\max}$ on the $x-$axis of all planes through each of the axial fixed points $\boldsymbol{q}_3$ and $\boldsymbol{q}_2$ whose normals lie in K. Similarly there are maximum y and z intercepts which we name $y_{\max}$ and $z_{\max}$ respectively.

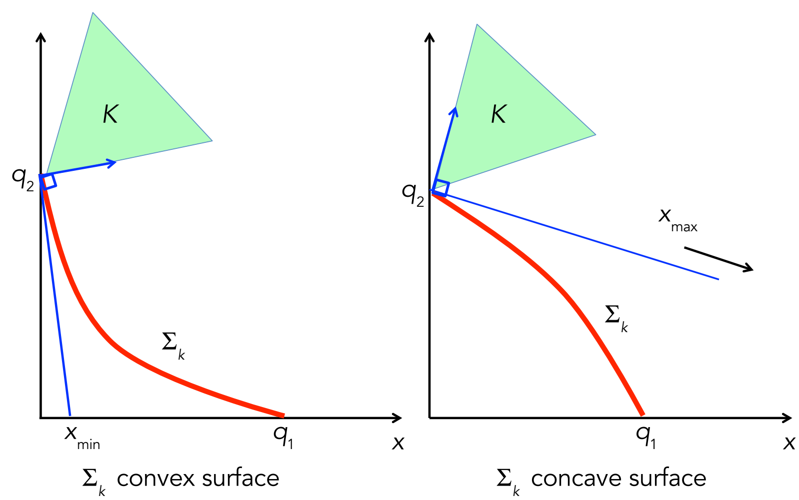

FIGURE 3. Bounds on the intersection of planes with the axes. Left figure: Convex surface, $0 < x_{\min} < x_{\max} < q_1$. Right figure: Concave surface, $q_1 < x_{\min} < x_{\max}$.

Let $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \in C_+^0$ be linearly independent and $K_{\boldsymbol{p}} = \mathbb{R}_+\boldsymbol{p}_1 + \mathbb{R}_+\boldsymbol{p}_2 + \mathbb{R}_+\boldsymbol{p}_3 \subset C_+^0$. Let P be the matrix whose ith-row is $\boldsymbol{p}_i$, and assume that the $\boldsymbol{p}_i$ are ordered so that $\det P > 0$. Then every $\boldsymbol{n} \in K_{\boldsymbol{p}}$ can be written as $\boldsymbol{n} = \lambda_1\boldsymbol{p}_1 + \lambda_2\boldsymbol{p}_2 + \lambda_3\boldsymbol{p}_3 = P\boldsymbol{\lambda}$, $\boldsymbol{\lambda} \in C_+$, so that $\boldsymbol{n} \in K_{\boldsymbol{p}}$ if and only if $P^{\#}\boldsymbol{n} \geq \boldsymbol{0}$. At $\boldsymbol{q}_3 = (0, 0, q_3)$, for $\boldsymbol{n} \in K_{\boldsymbol{p}}$ there is a tangent plane Π with normal $\boldsymbol{n}$ given by $\boldsymbol{n} \cdot \boldsymbol{x} = n_3 q_3$. The plane Π cuts the $x-$axis at the point $x^* = \frac{n_3 q_3}{n_1}$. To find $x_{\max}$ we must maximise $\frac{n_3 q_3}{n_1}$ over all $\boldsymbol{n} \in K_{\boldsymbol{p}}$, i.e. over all $\boldsymbol{n}$ such that $P^{\#}\boldsymbol{n} \geq 0$. Hence the maximum of x^* over all possible normals in $K_{\boldsymbol{p}}$ is

$$\max_{\lambda \in C_+ \setminus \{\boldsymbol{0}\}} q_3 \left(\frac{\lambda_1 p_{13} + \lambda_2 p_{23} + \lambda_3 p_{33}}{\lambda_1 p_{11} + \lambda_2 p_{21} + \lambda_3 p_{31}} \right) = \max_{i \in I_3} \frac{p_{i3}}{p_{i1}} q_3.$$

If instead we consider planes through the point $\boldsymbol{q}_2$, we obtain the same formula with 3 replaced by 2. Hence the maximum intercept value of x is

$$x_{\max} = \max\left\{ \max_{i\in I_3} \frac{p_{i3}}{p_{i1}} q_3, \max_{i\in I_3} \frac{p_{i2}}{p_{i1}} q_2 \right\}. \quad (5)$$

Reasoning in a similar way we have

$$y_{\max} = \max\left\{ \max_{i\in I_3} \frac{p_{i1}}{p_{i2}} q_1, \max_{i\in I_3} \frac{p_{i3}}{p_{i2}} q_3 \right\},\ z_{\max} = \max\left\{ \max_{i\in I_3} \frac{p_{i1}}{p_{i3}} q_1, \max_{i\in I_3} \frac{p_{i2}}{p_{i3}} q_2 \right\}. \quad (6)$$

Now consider the case where Σ_k is a convex surface, where we would now like to find the lower bound $x_{\min}$ counterpart to the $x_{\max}$ derived just above for the concave case. The upper bounds are $x_{\max} = q_1, y_{\max} = q_2, z_{\max} = q_3$.The same approach works, except now we replace maxima by minima:

$$x_{\min} = \min\left\{ \min_{i\in I_3} \frac{p_{i3}}{p_{i1}} q_3, \min_{i\in I_3} \frac{p_{i2}}{p_{i1}} q_2 \right\}, \quad (7)$$

with similar expressions for $y_{\min}$ and $z_{\min}$.

9. **Applications to the May-Leonard model.**

9.1. **The May-Leonard map.** The map that we study here is a symmetric version of the Leslie-Gower map from Ecology. We take $\boldsymbol{x} = (x, y, z) \in C_+$, $\alpha, \beta > 0$ and $\boldsymbol{T}$ to be the map

$$\boldsymbol{T}_{ML}(\boldsymbol{x}) = \left(\frac{rx}{1 + x + \alpha y + \beta z}, \frac{ry}{1 + y + \alpha x + \beta x}, \frac{rz}{1 + z + \alpha x + \beta y} \right),\quad r > 1. \quad (8)$$

In the remainder of the paper we assume with loss of generality that

$$\alpha > \beta. \quad (9)$$

As shown in [15], $\boldsymbol{T}_{ML}$ is competitive and has a carrying simplex Σ for all $\alpha, \beta > 0$.

The geometry of the planar version of (8), obtained by setting $z = 0$ and taking only the first two components of $\boldsymbol{T}_{ML}$, was studied in [2]. The planar carrying simplex is exactly the intersection of the 3-dimensional carrying simplex Σ of (8) with a coordinate plane. We denote by $\Sigma_{x=0}$ the intersection of the plane $\{x = 0\}$ with Σ, and similarly for y, z. In [2] Baigent showed that the carrying simplex $\Sigma_{z=0}$ for the planar model (obtained, for example, by setting $z = 0$ in (8) and restricting to the $xy-$plane) is either convex or concave. Specifically he showed that if $(1 + \alpha(r - 1))(1 + \beta(r - 1)) < r^2$ $(> r^2)$ the planar carrying simplex is convex (concave). Since the intersection of $z = 0$ with Σ, say $\Sigma_{z=0}$ is a planar carrying simplex, we see that a necessary condition for a Σ to be a convex (concave) carrying simplex is that $(1 + \alpha(r - 1))(1 + \beta(r - 1)) < r^2$ $(> r^2)$. Notice also that $\max\{\alpha, \beta\} < 1$ $(\min\{\alpha, \beta\} > 1)$ is a necessary condition for Σ to be a convex (concave) carrying simplex. In the sequel our study of convex and concave carrying simplices for the 3-species May-Leonard map will be exclusively for these two cases: $\max\{\alpha, \beta\} < 1$ and $\min\{\alpha, \beta\} > 1$.

The May-Leonard map $\boldsymbol{T}_{ML}$ (8) is a diffeomorphism from C_+ to $\Omega_{ML} := \boldsymbol{T}_{ML}(C_+)^0$ and $\boldsymbol{T}_{ML}^{-1} : \Omega_{ML} \to C_+$ is given by

$$\begin{aligned}\boldsymbol{T}_{ML}^{-1}(\boldsymbol{x}) = \frac{1}{R(\boldsymbol{x})}\Big(& x(r^2 + r(\alpha - 1)y + (\beta - 1)z + (1 - \alpha - \beta - \alpha\beta + \alpha^2 + \beta^2)yz), \\ & y(r^2 + r(\alpha - 1)z + (\beta - 1)x + (1 - \alpha - \beta - \alpha\beta + \alpha^2 + \beta^2)xz), \\ & z(r^2 + r(\alpha - 1)x + (\beta - 1)y + (1 - \alpha - \beta - \alpha\beta + \alpha^2 + \beta^2)xy)\Big),\ \boldsymbol{x} \in \Omega_{ML}.\end{aligned}$$

Here $R(\boldsymbol{x}) = r^3 - r^2(x+y+z) + r(1-\alpha\beta)(xy+yz+zx) + (3\alpha\beta - \alpha^3 - \beta^3 - 1)xyz$.

Equation (2) becomes, with $\boldsymbol{a} = (a,b,c) \in C^0_+$,

$$\begin{aligned} L\phi_{\boldsymbol{a}}(\boldsymbol{x}) = \frac{1}{R(\boldsymbol{x})}\Big\{ & -r^3 + r^2(a+1)x + r^2(b+1)y + r^2(c+1)z \\ & + rxy(a(\alpha-1) + b(\beta-1) + (\alpha\beta-1)) \\ & + ryz(b(\alpha-1) + c(\beta-1) + (\alpha\beta-1)) \\ & + rxz(c(\alpha-1) + a(\beta-1) + (\alpha\beta-1)) \\ & + xyz((1+\alpha+\beta+a+b+c)(1+\alpha^2+\beta^2-\alpha\beta-\alpha-\beta)\Big\},\ \boldsymbol{x} \in \Omega_{ML}. \end{aligned} \tag{10}$$

Here the expressions $1+\alpha^3+\beta^3-3\alpha\beta > 0$ and $\alpha^2+\beta^2-\alpha\beta-\alpha-\beta+1 > 0$ for all $\alpha, \beta > 0$. For the May-Leonard map (8) we are led to the study of the zero level sets of functions $\psi : \Omega_{ML} \to \mathbb{R}$ of the form

$$\psi(\boldsymbol{x}) = b_0xyz + b_1xy + b_2yz + b_3zx + c_1x + c_2y + c_3z - d, \tag{11}$$

where, setting $\alpha = A+1$ and $\beta = B+1$,

$$b_0 = (A^2 - AB + B^2)(3 + A + B + (a+b+c)) \tag{12}$$
$$b_1 = r(aA + bB + A + B + AB) \tag{13}$$
$$b_2 = r(bA + cB + A + B + AB) \tag{14}$$
$$b_3 = r(cA + aB + A + B + AB) \tag{15}$$
$$c_1 = r^2(a+1) \tag{16}$$
$$c_2 = r^2(b+1) \tag{17}$$
$$c_3 = r^2(c+1) \tag{18}$$
$$d = r^3. \tag{19}$$

On Ω_{ML} it is straightforward to calculate the gradient

$$D\psi = (b_0yz + b_1y + b_3z + c_1, b_0xz + b_1x + b_2z + c_2, b_0xy + b_2y + b_3x + c_3), \tag{20}$$

and

$$(D^2\psi)^{\#} = \begin{pmatrix} -(b_2+b_0x)^2 & (b_2+b_0x)(b_3+b_0y) & (b_2+b_0x)(b_1+b_0z) \\ (b_2+b_0x)(b_3+b_0y) & -(b_3+b_0y)^2 & (b_3+b_0y)(b_1+b_0z) \\ (b_2+b_0x)(b_1+b_0z) & (b_3+b_0y)(b_1+b_0z) & -(b_1+b_0z)^2 \end{pmatrix}.$$

Setting

$$X(\boldsymbol{x}) = b_2 + b_0x, Y(\boldsymbol{x}) = b_3 + b_0y, Z(\boldsymbol{x}) = b_1 + b_0z \tag{21}$$

(defined for $\boldsymbol{x} \in \Omega_{ML}$), we obtain

$$b_0^2\psi = XYZ + \theta_1X + \theta_2Y + \theta_3Z + 2b_1b_2b_3 - b_0b_2c_1 - b_0b_3c_2 - b_0b_1c_3 - b_0^2d \tag{22}$$

where $\theta_1 = b_0c_1 - b_1b_3$, $\theta_2 = b_0c_2 - b_1b_2$, $\theta_3 = b_0c_3 - b_2b_3$. Explicitly, the θ_i conveniently factor into two expressions that are affine in the $\boldsymbol{a} = (a,b,c)$:

$$\begin{aligned} \theta_1 &= r^2\big((A-B)a + Ab + A^2 + 2A - B\big)\big((A-B)a - Bc - (B^2+2B-A)\big) \\ \theta_2 &= r^2\big((A-B)b + Ac + A^2 + 2A - B\big)\big((A-B)b - Ba - (B^2+2B-A)\big) \\ \theta_3 &= r^2\big((A-B)c + Aa + A^2 + 2A - B\big)\big((A-B)c - Bb - (B^2+2B-A)\big). \end{aligned} \tag{23}$$

We also have for the cofactor matrix:

$$(D^2\psi)^{\#} = \begin{pmatrix} -X^2 & XY & XZ \\ XY & -Y^2 & YZ \\ XZ & YZ & -Z^2 \end{pmatrix}.$$

The positive factor of b_0^2 is immaterial for the zero set of ψ, so we may drop it from the lefthand side in equation (22), and simply work with

$$\psi = XYZ + \theta_1 X + \theta_2 Y + \theta_3 Z - \gamma, \tag{24}$$

where

$$\begin{aligned}\gamma &= b_0b_2c_1 + b_0b_3c_2 + b_0b_1c_3 + b_0^2 d - 2b_1b_2b_3 \\ &= b_1b_2b_3 + b_2\theta_1 + b_3\theta_2 + b_1\theta_3 + b_0^2 d \end{aligned} \tag{25}$$

and so

$$D\psi = (YZ + \theta_1, XZ + \theta_2, XY + \theta_3).$$

From (4) we find that

$$\begin{aligned}\kappa_0 &= 3X^2Y^2Z^2 + 2XYZ(\theta_1 X + \theta_2 Y + \theta_3 Z) \\ &\quad + 2(\theta_1\theta_2 XY + \theta_2\theta_3 YZ + \theta_1\theta_3 XZ) - \theta_1^2X^2 - \theta_2^2Y^2 - \theta_3^2Z^2 \\ &= (\theta_1 X + XYZ)^2 + (\theta_2 Y + XYZ)^2 + (\theta_3 Z + XYZ)^2 \\ &\quad -(\theta_1 X - \theta_2 Y)^2 - (\theta_1 X - \theta_3 Z)^2 - (\theta_2 Y - \theta_3 Z)^2. \end{aligned} \tag{26}$$

Restricted to $\psi(X, Y, Z) = 0$ we have, using $XYZ = \gamma - \theta_1 X - \theta_2 Y - \theta_3 Z$,

$$\begin{aligned}\kappa_0 &= (\gamma - \theta_2 Y - \theta_3 Z)^2 + (\gamma - \theta_1 X - \theta_3 Z)^2 + (\gamma - \theta_1 X - \theta_2 Y)^2 \\ &\quad -(\theta_1 X - \theta_2 Y)^2 - (\theta_1 X - \theta_3 Z)^2 - (\theta_2 Y - \theta_3 Z)^2 \\ &= (\gamma - 2\theta_1 X)(\gamma - 2\theta_2 Y) + (\gamma - 2\theta_2 Y)(\gamma - 2\theta_3 Z) + (\gamma - 2\theta_1 X)(\gamma - 2\theta_3 Z).\end{aligned}$$

From the foregoing calculations we obtain the basic result that says how the curvature of a plane $\Delta(\boldsymbol{a})$ changes under the map $\boldsymbol{T}_{ML}$. Note that $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a})) \subset \Omega_{ML}$ for each $\boldsymbol{a} \in C_+^0$.

Lemma 9.1. *Let $\boldsymbol{a} \in C_+^0$ be fixed and consider the surface $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$. At a point $\boldsymbol{x} \in \boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ the Gaussian curvature is (positively) proportional to*

$$\kappa_0(\boldsymbol{x}) = (\gamma - 2\theta_1 X)(\gamma - 2\theta_2 Y) + (\gamma - 2\theta_2 Y)(\gamma - 2\theta_3 Z) + (\gamma - 2\theta_1 X)(\gamma - 2\theta_3 Z), \tag{27}$$

where X, Y, Z are defined in terms of $\boldsymbol{x} \in \Omega_{ML}$ and $\boldsymbol{a}$ by (21), γ is defined in terms of $\boldsymbol{a}$ via (25) using (12) - (19) and θ_i in terms of $\boldsymbol{a}$ via (23) using (12) - (19).

10. **The geometry of the May-Leonard carrying simplex.** We will study the geometry of the carrying simplex of the May-Leonard map for convex and concave cases separately. The concave carrying simplex is somewhat simpler to investigate because the tangent planes to $\Sigma_k = \boldsymbol{T}_{ML}^k(\Sigma_0)$ all lie below $\Sigma_0 = \Delta(r-1, r-1, r-1)$. In the convex case, as discussed in section 8, we need to obtain bounds on the tangent planes to $\Sigma_k = \boldsymbol{T}_{ML}^k(\Sigma_0)$ which will all lie above $\Sigma_0 = \Delta(r-1, r-1, r-1)$, and so the intersection of these tangent planes with the axes is more difficult to bound. This is where the methods of section 8 become useful.

10.1. **Choosing the cone K.** Owing to the cyclic symmetry of $\boldsymbol{T}_{ML}$ in α, β we are lead to consider the following possibility for K.

Take $\boldsymbol{p}_1(s) = (s, s^2, 1)$, $\boldsymbol{p}_2(s) = (1, s, s^2)$ and $\boldsymbol{p}_3(s) = (s^2, 1, s)$, $s > 0$. Then $\boldsymbol{\alpha}_1(s) = \boldsymbol{p}_2(s) \times \boldsymbol{p}_3(s) = (1-s^3)(0, -s, 1)$, $\boldsymbol{\alpha}_2(s) = \boldsymbol{p}_3(s) \times \boldsymbol{p}_1(s) = (1-s^3)(1, 0, -s)$ and $\boldsymbol{\alpha}_3(s) = (1-s^3)(-s, 1, 0)$. Moreover $\boldsymbol{\alpha}_1(s) \cdot \boldsymbol{\alpha}_2(s) \times \boldsymbol{\alpha}_3(s) = (s^3 - 1)^4 > 0$ if $s \neq 1$. We set

$$K_{ML}(s) = \mathbb{R}_+ \boldsymbol{p}_1(s) + \mathbb{R}_+ \boldsymbol{p}_2(s) + \mathbb{R}_+ \boldsymbol{p}_3(s). \tag{28}$$

Then $K_{ML}(s)$ is a closed and solid convex cone when $s \neq 1$. $K_{ML}(0)$ is the first orthant C_+ and $K_{ML}(1)$ is the ray $\mathbb{R}_+(1,1,1)$. When $s > 1$,

$$K_{ML}(s) = \{(a, b, c) \in C_+ : a \leq sc, b \leq sa, c \leq sb\}. \tag{29}$$

whereas when $s < 1$ the inequalities in (29) are reversed.

In order to obtain some sufficient conditions for $\kappa_0 \geq 0$ in (27), our strategy will be to establish that each bracketed term is either nonnegative or nonpositive. An integral part of this strategy is to determine the signs of γ and $\theta_1, \theta_2, \theta_3$ in terms of the parameters A, B. For this we will need:

Lemma 10.1.

1. *Suppose that $A, B > 0$ and $\boldsymbol{a} \in K_{ML}\left(\frac{B}{A-B}\right)$. Then $\gamma(\boldsymbol{a}) > 0$.*
2. *Suppose that $-1 < B < A < 0$ and $\boldsymbol{a} \in K_{ML}\left(\frac{B-A}{A}\right)$. Then $\gamma(\boldsymbol{a}) < 0$.*

The proof is given in appendix B.

10.2. **Concave Carrying Simplices: The case $\min\{\alpha, \beta\} > 1$.** It is known [2] that when $\min\{\alpha, \beta\} > 1$ (and $r > 1$) the planar carrying simplices $\Sigma_{x=0}, \Sigma_{y=0}, \Sigma_{z=0}$ are all concave, so in this case we are seeking further conditions for Σ to be concave.

Since we have the standing assumption $\alpha > \beta$, in the case $\min\{\alpha, \beta\} > 1$ we have $A > B > 0$, $b_0, b_1, b_2, b_3 > 0$ and also $\boldsymbol{0} \ll (X, Y, Z)$ since $\boldsymbol{x} \in \Omega_{ML} \subset C_+$. Thus κ_0 in (27) will be positive when $\boldsymbol{a} = (a, b, c)$ is such that simultaneously $\gamma > 0$ and $\theta_1 < 0$, $\theta_2 < 0$ and $\theta_3 < 0$ (all these depend on $\boldsymbol{a}$). To establish that positive curvature leads to a concave (rather than convex) surface $\boldsymbol{T}(\Delta(\boldsymbol{a}))$ we will look at $\{z = 0\} \cap \boldsymbol{T}(\Delta(\boldsymbol{a}))$, which is a planar curve for which the convexity or concavity can easily be established (see (33)).

From (23), for $\theta_1 < 0$, $\theta_2 < 0$ and $\theta_3 < 0$ we require $\boldsymbol{a} \in C_+^0$ to satisfy

$$(A - B)a - Bc \quad < \quad B^2 + 2B - A \tag{30}$$

$$(A - B)b - Ba \quad < \quad B^2 + 2B - A \tag{31}$$

$$(A - B)c - Bb \quad < \quad B^2 + 2B - A. \tag{32}$$

We denote the set of $\boldsymbol{a} \in C_+^0$ satisfying (30), (31) and (32) by $P_>$. (The subscript $>$ is meant to distinguish this case where $A, B < 0$ which is considered later in section 10.3).

Lemma 10.2 (Characterisation of $P_>$)**.**

L1 If $2B \geq A > B$ then $P_>$ is a nonempty and unbounded convex set;
L2 If $A > 2B$ and $B^2 + 2B < A$, $P_>$ is empty;
L3 If $A > 2B$ and $B^2 + 2B > A$, $P_>$ is a nonempty and bounded convex set.

Proof. If $P_>$ is nonempty, then as the intersection of 3 open half-spaces with C_+ it is a nonempty convex set. Consider the ray $t(1,1,1)$ for $t \geq 0$. From (30), (31) and (32) $t(1,1,1) \in P_>$ if $(A - 2B)t < B^2 + 2B - A$. If $2B \geq A$ then $P_>$ contains any $t(1,1,1)$ with $t > 0$. This shows L1. On the other hand, for L2, summing (30) -

(32) we obtain $(A-2B)(a+b+c) < 3(B^2+2B-A)$, and hence $P_>$ is empty when $A > 2B$ and $B^2+2B-A < 0$. Finally consider L3. If $A > 2B$ and $B^2+2B-A > 0$, and $\boldsymbol{a} \in P_>$ then $(A-2B)(a+b+c) < 3(B^2+2B-A)$. Since $\boldsymbol{a} \in C_+^0$, $P_>$ is a bounded nonempty set (and in particular not a cone). $\square$

Now consider $\{z=0\} \cap \boldsymbol{T}(\Delta(\boldsymbol{a}))$ for $\boldsymbol{a} \in C_+^0$. This planar curve is given parametrically by

$$\Big\{\Big(\frac{rs/a_1}{1+s/a_1+\alpha(1-s)/a_2}, \frac{r(1-s)/a_2}{1+(1-s)/a_2+\beta s/a_1}, 0\Big) : s \in [0,1]\Big\}$$

and its curvature is positively proportional to

$$\frac{2a_1^3a_2^3\,(\alpha+a_2)\,(a_1+\beta)\,((\alpha-1)a_1+a_2(\beta-1)+\alpha\beta-1)}{(\alpha a_1(1-s)+a_2\,(a_1+s))^{\,3}\,(a_1(1-s)+a_2\,(a_1+\beta s))^{\,3}} \tag{33}$$

which is positive for $s \in [0,1]$ when $\min\{\alpha,\beta\} > 1$. Hence $\{z=0\} \cap \boldsymbol{T}(\Delta(\boldsymbol{a}))$ is a strictly convex curve.

From (27), lemma 10.2, and the fact that $\{z=0\} \cap \boldsymbol{T}(\Delta(\boldsymbol{a}))$ is a strictly convex curve when $\min\{\alpha,\beta\} > 1$ we obtain:

Lemma 10.3. *Suppose that $2B > A > B > 0$, $\boldsymbol{a} \in P_>$ and $\gamma(\boldsymbol{a}) > 0$. Then $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ is a strictly convex surface.*

The next lemma concerns when the cone K_{ML} is also subset of $P_>$, the set that controls the convexity of mapped planes.

Lemma 10.4. *$K_{ML}\left(\frac{B}{A-B}\right) \subseteq P_>$ when $2B > A > B > 0$.*

Proof. When $2B > A$ we have $B^2+2B-A > 0$ and we need only show that $(A-B)a - Bc \le 0$, $(A-B)b - Ba \le 0$ and $(A-B)c - Bb \le 0$ whenever $\boldsymbol{a} \in K_{ML}\left(\frac{B}{A-B}\right)$. In this instance $s > 1$ and $K_{ML}\left(\frac{B}{A-B}\right)$ is given by (29), so that if $\boldsymbol{a} \in K_{ML}\left(\frac{B}{A-B}\right)$ then $a \le cs, b \le as, c \le bs$. Then $(A-B)a - Bc \le (A-B)sc - Bc = c((A-B)s-B) = c(A-(s+1)B < c(A-2B) < 0$. The two other inequalities are established in the same manner. $\square$

Lemma 10.5. *Suppose that $\frac{1}{2}(B-1)+\frac{1}{2}\sqrt{1+6B-3B^2} > A > B > 0$. Then for $\boldsymbol{a} \in K_{ML}\left(\frac{B}{A-B}\right)$ the normal bundle of $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ is contained in $K_{ML}\left(\frac{B}{A-B}\right)$.*

Proof. Under the conditions on A, B it is easily shown that $2B > A$: We have that $\sqrt{1+6B-3B^2} > 2A-B+1 > 0$ (since $A > B$). Thus $1+6B-3B^2 > (2A-B+1)^2$ which tidies to $4(A^2+B^2-AB)+4(A-2B) < 0$. Since $A^2+B^2-AB > 0$ we must have $2B > A$. Suppose that $\boldsymbol{a} \in K_{ML}\left(\frac{B}{A-B}\right)$. Then by lemma 10.4 $\boldsymbol{a} \in P_>$ and by lemma 10.1, $\gamma(\boldsymbol{a}) > 0$. Thus by lemma 10.3 $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ is a strictly convex surface. To show that the normal bundle of $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ is a subset of $K_{ML}\left(\frac{B}{A-B}\right)$ we need only consider points on the boundary of $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$, i.e. the intersection of $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ with the boundary of C_+. Hence we are concerned with $D\psi$ on the boundary where ψ is given by (24).

Consider, for example, $\{z=0\} \cap \boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ from (24) where we have $\psi(x,y,0) = b_1xy + c_1x + c_2y - d$, so that $\{z=0\} \cap \boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ is the graph of the function $x \mapsto y(x) = \frac{d-c_1x}{b_1x+c_2}$ with x in the range $x \in [0, \frac{r}{a+1}]$. Then using that

$D\psi(x,y,0) = (b_1 y + c_1, b_1 x + c_2, b_0 xy + b_2 y + b_3 x + c_3)$ we find

$$\begin{aligned} u_1(x) &:= (\psi_x - s\psi_z)(x, y(x), 0) \\ &= b_1 y(x) + c_1 - s\left(b_0 x y(x) + b_3 x + b_2 y(x) + c_3\right) \\ &= (b_1 - sb_2) y(x) + c_1 - sc_3 - sx(b_0 y(x) + b_3) \\ &\le (b_1 - sb_2) y(x) + c_1 - sc_3. \end{aligned}$$

Our aim is to show that $u_1(x) < 0$ for $x \in [0, \frac{r}{a+1}]$. Writing

$$(a,b,c) = \lambda_1(s, s^2, 1) + \lambda_2(1, s, s^2) + \lambda_3(s^2, 1, s),\ \boldsymbol{\lambda} \in C_+ \tag{34}$$

we find that

$$b_1 - sb_2 = \frac{r(A-2B)\left(\left(A^2 - AB + B^2\right)(A\lambda_2 + B\lambda_1) + (AB + A + B)(A-B)^2\right)}{(A-B)^3},$$

and $c_1 - sc_3 = r^2(a+1-s(c+1)) = r^2(a - sc) + (1-s) < 0$ when $a \le sc$ and $s > 1$. When $2B > A > B$ on inspection we see that all coefficients in the multinomial η_1 are negative and hence $\psi_x - s\psi_z < 0$ on $\{z = 0\} \cap \boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$.

Similarly on $y = 0$ we have $u_2(x) := (\psi_x - s\psi_z)(x, 0, z(x)) = b_3(z(x) - sx) + c_1 - sc_3$. Then $u_2(x) = \frac{Q_2(x)}{b_3 x + c_3}$ where $Q_2(x) = -sb_3^2 x^2 - 2sb_3 c_3 x + db_3 + c_3(c_1 - sc_3)$. Q_2 is a concave function that takes its minimum at $x = 0$ or $x = \frac{r}{a+1}$ (or both). We find that $Q_2(0) = \frac{r^2}{c+1}\left((a+1+B)(c+1+A) - s(c+1)^2\right)$. Then with (34) and $\eta_2 = Q_2(0)/r^4$, we compute

$$\begin{aligned} \eta_2 = {} & \frac{\lambda_1\left(A^2 - AB + A + (B-2)B\right)}{A-B} + \frac{B\lambda_3\left(A^2 - AB + A + (B-2)B\right)}{(A-B)^2} \\ & + \lambda_1\lambda_2\left(\frac{B^3}{(B-A)^3} + 1\right) + \frac{B\lambda_3\lambda_2\left((A-B)^3 - B^3\right)}{(A-B)^4} + \frac{B^2\lambda_2^2\left((A-B)^3 - B^3\right)}{(A-B)^5} \\ & + \lambda_2\left(\frac{2B^3}{(B-A)^3} + \frac{(A+1)B^2}{(A-B)^2} + B + 1\right) + AB - \frac{B}{A-B} + A + B + 1. \end{aligned}$$

As shown in lemma A.1 the coefficients in this multinomial in λ are all negative when

$$0 < B < A < 1 \text{ and } A < \frac{1}{2}(B-1) + \frac{1}{2}\sqrt{1 + 6B - 3B^2}. \tag{35}$$

Similarly, $Q_2\left(\frac{r}{a+1}\right) = \frac{r^4(a+A+1)(B+c+1)\left((a+1)^2 - s(a+A+1)(B+c+1)\right)}{(a+1)^2}$. Set $\zeta_2 = (a+1)^2 - s(a+A+1)(B+c+1)$. Then

$$\begin{aligned} \zeta_2 = {} & -\frac{B^2\lambda_3\left(A^2 - A(B+1) + B(B+2)\right)}{(A-B)^3} + \frac{B^2\lambda_2\lambda_3(A-2B)\left(A^2 - AB + B^2\right)}{(A-B)^5} \\ & + \frac{B\lambda_1\lambda_2(A-2B)\left(A^2 - AB + B^2\right)}{(A-B)^4} - \frac{B\lambda_1\left(A^2 - A(B+1) + B(B+2)\right)}{(A-B)^2} \\ & + \frac{\lambda_2\left(2A^3 - A^2B(B+7) + AB^2(B+8) - B^3(B+4)\right)}{(A-B)^3} + \lambda_2^2\left(\frac{B^3}{(B-A)^3} + 1\right) \\ & - \frac{A\left(B^2 + B - 1\right) + B(B+2)}{A-B}. \end{aligned}$$

In lemma A.2 in the appendix we show that ζ_2 is negative when (35) holds.

We conclude that when (35) holds and $s = \frac{B}{A-B}$, $\psi_x - s\psi_z < 0$ on all of the boundary of $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$, and since $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ is a strictly convex surface holds also in the interior of $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$. By the permutational symmetry of $\boldsymbol{T}_{ML}$ in $\boldsymbol{x}$, (35) is also sufficient for $\psi_y < s\psi_x$ and $\psi_z < s\psi_y$ on $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$. Thus $D\psi(\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))) \subseteq K_{ML}\left(\frac{B}{A-B}\right)$ as required. □

Hence we have established:

Theorem 10.6. *Suppose that* $\frac{1}{2}(B-1)+\frac{1}{2}\sqrt{1+6B-3B^2} > A > B > 0$. *Then the carrying simplex of (8) is concave.*

Proof. By lemma 10.3 $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ is a strictly convex surface when $\boldsymbol{a} \in K_{ML}\left(\frac{B}{A-B}\right)$. Now take $K = K_{ML}\left(\frac{B}{A-B}\right)$ in Theorem 6.1. □

We give some examples in section 11.

10.3. **Convex Carrying Simplices: The case $0 < \max\{\alpha,\beta\} < 1$.** It is known [2] that when $0 < \max\{\alpha,\beta\} < 1$ the planar carrying simplices $\Sigma_{x=0}, \Sigma_{y=0}, \Sigma_{z=0}$ are all convex, so in this case we will be seeking a convex carrying simplex.

When $0 < \max\{\alpha,\beta\} < 1$, $b_1, b_2, b_3 < 0$, but b_0 remains positive. Continuing to assume that $A > B$ we also seek $\theta_i < 0$ for $i = 1,2,3$. Since now $-1 < B < A < 0$, then $A > 2B + B^2$ and $\theta_1 < 0$, $\theta_2 < 0$ and $\theta_3 < 0$ if

$$(A-B)a + Ab \quad < \quad B - 2A - A^2 \tag{36}$$
$$(A-B)b + Ac \quad < \quad B - 2A - A^2 \tag{37}$$
$$(A-B)c + Aa \quad < \quad B - 2A - A^2. \tag{38}$$

We let this solution set in C_+^0 be $P_<$.

Lemma 10.7 (Characterisation of $P_<$)**.** *Suppose* $-1 < B < A < 0$.

M1 If $B \geq A^2 + 2A$ *then* $P_<$ *is a nonempty and unbounded convex set;*
M2 If $2A < B < 2A + A^2$, $P_<$ *is a nonempty and unbounded convex set;*
M3 If $2A \geq B$ *and* $B < 2A + A^2$, $P_<$ *is empty.*

Proof. If $P_<$ is nonempty, then as the intersection of 3 open half-spaces with C_+ it is a convex set. From (36), (37) and (38) $t(1,1,1) \in P_<$ if $(2A-B)t < B - 2A - A^2$. When $B \geq A^2 + 2A$, $B > 2A$ (since $A \neq 0$) and $P_<$ contains any $t(1,1,1)$ with $t > 0$. This shows M1. On the other hand, for M2, $(2A-B)t < B - 2A - A^2 < 0$ for $t > 0$ large enough. Finally consider M3. If $2A \geq B$ then if $\boldsymbol{a} \in P_<$ we have $(A-B)c + Aa < 0$, $(A-B)b + Ac < 0$, $(A-B)c + Aa < 0$ and so $(2A-B)(a+b+c) < 0$ which is not possible for $\boldsymbol{a} \in C_+^0$ when $2A \geq B$. □

We take the cone $K_{ML}(s) = \mathbb{R}_+(s,s^2,1) + \mathbb{R}_+(1,s,s^2) + \mathbb{R}_+(s^2,1,s)$, but now with $s = \frac{B-A}{A}$ with $0 > B > 2A$ so that $s < 1$. If $\boldsymbol{a} \in K_{ML}(s)$ then $a \geq sc, b \geq sa, c \geq sb$ and there exists $\boldsymbol{\lambda} \in C_+$ such that $a = \lambda_1 s + \lambda_2 + \lambda_3 s^2$, $b = \lambda_1 s^2 + \lambda_2 s + \lambda_2 s + \lambda_3$ and $c = \lambda_1 + \lambda_2 s^2 + \lambda_3 s$.

By lemma 10.1, $\gamma(\boldsymbol{a}) < 0$ whenever $-1 < B < A < 0$ and $\boldsymbol{a} \in K_{ML}\left(\frac{B-A}{A}\right)$.

In order to use the same strategy as for the case $0 < B < A < 1$ we first need to establish that the coordinates $X, Y, Z < 0$. First we note:

Lemma 10.8. *If* $0 > 2A > 2B > 1 + A - \sqrt{1-6A-3A^2}$ *then* $B > 2A + A^2$.

Proof. We have $\sqrt{1-6A-3A^2} > 1 + A - 2B > 0$, so that squaring and rearranging $B > 2A + A^2 + B(B-A) > 2A + A^2$ since $0 > A > B$. □

Lemma 10.9. *When* $-1 < B < A < 0$ *and* $B > 2A + A^2$, $-(X,Y,Z) = -(b_0x + b_2, b_0y + b_3, b_0z + b_1) \in C_+$ *for* $\boldsymbol{x} \in [0,\boldsymbol{q}]$.

Proof. Consider $X = b_0x + b_2$. Here $b_0 > 0$, and since $A, B < 0$, $b_2 < 0$, so by section 8 we find that $0 \leq x \leq x_{\max}$. We wish to find conditions that $X < 0$ for all $0 \leq x \leq x_{\max}$. Since we are seeking convex carrying simplices, we are only interested in $\boldsymbol{a} < \boldsymbol{q}^{-1}$, i.e. $\max\{a,b,c\} < \frac{1}{r-1}$. We have

$$\begin{aligned}X =& (A^2 + B^2 - AB)(3 + a + b + c + A + B)x + r(AB + A(b+1) + B(c+1))\\ \leq&(A^2 + B^2 - AB)(3 + a + b + c + A + B)x_{\max} + r(AB + A(b+1) + B(c+1))\\ =&(A^2 + B^2 - AB)(3 + a + b + c + A + B)\left(\frac{B-A}{A}\right)(r-1)\\ &+ r(AB + A(b+1) + B(c+1)).\end{aligned}$$

Set $\sigma = \left(\frac{B-A}{A}\right)(A^2 + B^2 - AB) > 0$ so that

$$\begin{aligned}X &\leq (3 + A + B + a + b + c)(r-1)\sigma + r(AB + A(b+1) + B(c+1))\\ &= \sigma(r-1)a + ((r-1)\sigma + rA)b + ((r-1)\sigma + rB)c \\ &\quad + (r-1)\sigma(3 + A + B) + r(A + B + AB).\end{aligned} \tag{39}$$

The righthand side of (39) is linear in $\boldsymbol{a}$ and r and so is maximised at a vertex of $[0,\boldsymbol{q}^{-1}]$. In particular, since $\sigma > 0$,

$$X < ((r-1)\sigma + rA)b + ((r-1)\sigma + rB)c + (r-1)\sigma(3 + A + B) + r(A + B + AB) + \sigma.$$

Let $Y(b,c) = ((r-1)\sigma + rA)b + ((r-1)\sigma + rB)c + (r-1)\sigma(3 + A + B) + r(A + B + AB) + \sigma$ and $Y_1 = Y(0,0)$, $Y_2 = Y(\frac{1}{r-1},0)$, $Y_3 = (0,\frac{1}{r-1})$ and $Y_4 = Y(\frac{1}{r-1},\frac{1}{r-1})$. First we show that when $B > A^2 + 2A$, $Y_1 < 0$. We have $Y_1 = (r-1)\sigma(3 + A + B) + r(A + B + AB) + \sigma = (A^2 + B^2 - AB)((r-1)(3 + A + B) + 1)(A - B) - rA(AB + A + B) = (1-r)(A-B)^2(B - A^2 - 2A) + B(B-A)\left(A^2 + A + B^2 + B\right) + r\left(A^3(B+1) - A^2(B+1)^2 + AB(B(B+2) - 1) - B^3(B+2)\right)$. Now use that $-1 < B < A < 0$, $B > A^2 + 2A$ and $r > 1$ to obtain that $(A^2 + B^2 - AB)((r-1)(3 + A + B) + 1)(A - B) - rA(AB + A + B) < 0$. Then $Y_2 - Y_1 = \frac{B-A}{A}(A^2 + B^2 - AB) + \frac{rA}{r-1} < \frac{B-A}{A}(A^2 + B^2 - AB) + A = \frac{(B-A)(A^2+B^2-AB)+A^2}{A}$ and $A^2 + (B-A)(A^2 + B^2 - AB) = (B - 2A - A^2)(2A^2 + 2A^3 + A^4 + A^2B + B^2) + A^2((A+1)^4 - A)$. Now $(A+1)^4 - A$ is convex and minimised at $A = \frac{1}{2^{2/3}} - 1$ at the value $1 - \frac{3}{4\times 2^{2/3}} > 0$ and so is everywhere positive. On the other hand, $2A^2 + 2A^3 + A^4 + A^2B + B^2 = (B + A^2/2)^2 + A^2(2 + 2A + \frac{3}{4}A^2) > 0$. Hence $Y_2 < Y_1$ when $B > 2A + A^2$. $Y_3 - Y_1 = \frac{B-A}{A}(A^2 + B^2 - AB) + \frac{rB}{r-1} < 0$ when $B > 2A + A^2$ since $B < A$. Finally, $Y_4 - Y_1 = 2\frac{B-A}{A}(A^2 + B^2 - AB) + \frac{r(A+B)}{r-1} = Y_2 - Y_1 + Y_3 - Y_1 < 0$ when $B > 2A + A^2$. □

Lemma 10.10. $K_{ML}\left(\frac{B-A}{A}\right) \subseteq P_<$ *when* $-1 < B < A < 0$, $B > 2A + A^2$.

Proof. Similar to the proof of lemma 10.4 and omitted. □

Referring back to (33), we see that when $0 < \max\{\alpha,\beta\} < 1$ the curve $\{z = 0\} \cap \boldsymbol{T}(\Delta(\boldsymbol{a}))$ is a strictly concave surface and we obtain:

Lemma 10.11. *Suppose that* $-1 < B < A < 0$ *and* $A > B > 2A + A^2$, $\boldsymbol{a} \in P_<$ *and* $\gamma(\boldsymbol{a}) < 0$. *Then* $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ *is a strictly concave surface.*

Using lemmas 10.10 and 10.1 together we can show

Lemma 10.12. *Suppose* $0 > A > B > \frac{1+A-\sqrt{1-6A-3A^2}}{2}$. *Then for* $\boldsymbol{a} \in K_{ML}\left(\frac{B-A}{A}\right)$ *the normal bundle of* $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ *is contained in* $K_{ML}\left(\frac{B-A}{A}\right)$.

Proof. Suppose that $\boldsymbol{a} \in K_{ML}(s)$ with $s = \frac{B-A}{A}$. Then by lemma 10.10 $\boldsymbol{a} \in P_<$ and by lemma 10.1, $\gamma(\boldsymbol{a}) < 0$ and $a \geq sc$, $b \geq sa$, $c \geq sb$. Thus ψ is strictly concave and the boundary of $D\psi(\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a})))$ is attained at points on the boundary of $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$. Hence to show that $D\psi(\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))) \subseteq K_{ML}\left(\frac{B-A}{A}\right)$ we need only consider points on the boundary of $\boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$. Note that now $-1 < B < A < 0$ so that $b_1, b_2, b_3 < 0$.

On $\{z = 0\} \cap \boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$ we have $u_1(x) = (\psi_x - s\psi_z)(x, y(x), 0)$ as in lemma 10.5

$$u_1(x) = (b_1 - sb_2)y(x) + c_1 - sc_3 - sx(b_0 y(x) + b_3).$$

But now $A > 2B$ and so from lemma 10.5 we have $b_1 > sb_2$ and $c_1 > sc_3$. Moreover we have established that $b_0 y(x) + b_3 = Y < 0$. Hence $u_1(x) > 0$ on $\{z = 0\} \cap \boldsymbol{T}_{ML}(\Delta(\boldsymbol{a}))$.

Similarly on $y = 0$ we have $u_2(x) := (\psi_x - s\psi_z)(x, 0, z(x)) = b_3(z(x) - sx) + c_1 - sc_3$. Then $u_2(x) = \frac{Q_2(x)}{b_3 x + c_3}$ where $Q_2(x) = -sb_3^2 x^2 - 2sb_3c_3x + db_3 + c_3(c_1 - sc_3)$. Q_2 is a concave function that takes its minimum at $x = 0$ or $x = \frac{r}{a+1}$ (or both). We find that $Q_2(0) = \frac{r^2}{c+1}\left((a+1+B)(c+1+A) - s(c+1)^2\right)$. Then with (34) and $\eta_2 = Q_2(0)/r^4$, but now $s = \frac{B-A}{A}$, we compute

$$\begin{aligned}
\eta_2 = &-\frac{\lambda_3(A-B)\left(-(A+1)B + A(A+2) + B^2\right)}{A^2} + \frac{\lambda_1\lambda_2(2A-B)\left(A^2 - AB + B^2\right)}{A^3} \\
&+ \frac{\lambda_2^2(A-B)^2(2A-B)\left(A^2 - AB + B^2\right)}{A^5} + \lambda_1\left(\frac{(B-1)B}{A} + A - B + 2\right) \\
&+ \lambda_2\left(-\frac{2B^3}{A^3} + \frac{(A+7)B^2}{A^2} - \frac{(A+8)B}{A} + A + 4\right) \\
&- \frac{\lambda_3\lambda_2(A-B)(2A-B)\left(A^2 - AB + B^2\right)}{A^4} + AB - \frac{B}{A} + A + B + 2.
\end{aligned}$$

We show in lemma A.3 in the appendix that this expression is positive for all $\boldsymbol{\lambda} \in C_+$ when

$$0 > A > B > \frac{1 + A - \sqrt{1 - 6A - 3A^2}}{2}. \tag{40}$$

At $x = \frac{r}{a+1}$, we find that

$$\begin{aligned}
\zeta_2 = &\\
&\frac{\lambda_1\left(A^2 - A(B+2) + B^2 + B\right)(A-B)}{A^2} + \frac{\lambda_2\lambda_3(2A-B)\left(A^2 - AB + B^2\right)(A-B)^2}{A^5} \\
&- \frac{\lambda_1\lambda_2(2A-B)\left(A^2 - AB + B^2\right)(A-B)}{A^4} + \frac{\lambda_2^2(2A-B)\left(A^2 - AB + B^2\right)}{A^3} \\
&- \frac{\lambda_3\left(A^2 - A(B+2) + B^2 + B\right)(A-B)^2}{A^3} + A(B+1) - \frac{B(B+1)}{A} - B^2 + 2 \\
&+ \lambda_2\left(4 - \frac{(A+1)B^3}{A^3} + \frac{(2A+3)B^2}{A^2} - \frac{2(A+2)B}{A} + A\right).
\end{aligned}$$

We show in lemma A.4 in the appendix that this expression is positive for all $\boldsymbol{\lambda} \in C_+$ when (40) holds. $\square$

Hence we have established:

Theorem 10.13. *Suppose that* $0 > A > B > \frac{1+A-\sqrt{1-6A-3A^2}}{2}$. *Then the carrying simplex of (8) is convex.*

Proof. Essentially the same as Theorem 10.6 and omitted. □

11. Examples of convex or concave carrying simplices.

We now provide some specific examples of convex or concave carrying simplices.

11.1. Concave carrying simplex, $r = 2$, $\alpha = 5/4, \beta = 7/6$.

$A = \frac{1}{4}$, $B = \frac{1}{6}$ and $\frac{1}{2}(B-1)+\frac{1}{2}\sqrt{1+6B-3B^2} = \frac{1}{12}(\sqrt{69}-5) \approx 0.276 > A = 0.25$. Hence by Theorem 10.6, the carrying simplex is concave.

Concave carrying simplex, $r = 2$, $\alpha = 7/5, \beta = 4/3$. $A = \frac{2}{5}$, $B = \frac{1}{3}$ and $\frac{1}{2}(B-1)+\frac{1}{2}\sqrt{1+6B-3B^2} = \frac{1}{3}\left(\sqrt{6}-1\right) \approx 0.483 > A = 0.4$. Hence by Theorem 10.6, the carrying simplex is concave.

Concave carrying simplex, $r = 2$, $\alpha = 3/2, \beta = 7/5$. $A = \frac{1}{2}$, $B = \frac{2}{5}$ and $\frac{1}{2}(B-1)+\frac{1}{2}\sqrt{1+6B-3B^2} = \frac{1}{10}\left(\sqrt{73}-3\right) \approx 0.554 > A = 0.5$. Hence by Theorem 10.6, the carrying simplex is concave.

11.2. Convex carrying simplex, $r = 2$, $\alpha = 3/4, \beta = 2/3$.

We take $A = -\frac{1}{4}, B = -\frac{1}{3}$ Note that $A > B$, $0 > A > B > \frac{1+A-\sqrt{1-6A-3A^2}}{2} = \frac{3-\sqrt{37}}{8} \approx -0.385$. Hence by Theorem 10.13 the carrying simplex is convex.

Convex carrying simplex, $r = 2$, $\alpha = 4/5, \beta = 3/4$. Here $A = -1/5, B = -1/4$ and $A > B$, $-0 > A > B > \frac{1+A-\sqrt{1-6A-3A^2}}{2} = \frac{2-\sqrt{13}}{5} \approx -0.321$. Hence by Theorem 10.13 the carrying simplex is convex.

Convex carrying simplex, $r = 2$, $\alpha = 2/3, \beta = 7/12$. Here $A = -1/3$, $B = -5/12$ and $A > B$, $-0 > A > B > \frac{1+A-\sqrt{1-6A-3A^2}}{2} = \frac{1-\sqrt{6}}{3} \approx -0.483$. Applying Theorem 10.13 shows that the carrying simplex is convex.

The carrying simplices for these 6 examples are shown in Figures 1 and 4.

12. Conclusions and discussion.

Here we have introduced a new approach to study the convex or concave geometry of carrying simplices of competitive Kolmogorov diffeomorphisms $\boldsymbol{T}$. We have shown how the study of their convexity or concavity can be reduced to the study of the action of $\boldsymbol{T}$ on planes. Our approach has been demonstrated using the May-Leonard map as an example. The May-Leonard map has significant symmetry which has aided calculations, but the method (i.e. Theorem 6.1) can be applied to any competitive Kolmogorov diffeomorphism $\boldsymbol{T}$ of C_+ onto $\boldsymbol{T}(C_+)$ with a carrying simplex.

In the study of which maps transform planes into convex or concave surfaces we have elected to use a level-set approach which we have found convenient since it simplifies the formulae for gradients and Gaussian curvature, and does not assume a preferred coordinate direction as is necessary in representation of a surface as a graph of a function. It would be interesting to explore what new insights into the existence and smoothness of carrying simplices can be gained through a zero-set approach.

As mentioned in the introduction, but not explored in the main text, the containment all normal bundles of the sequence (1) in a closed and solid convex cone K can be established by showing that $\boldsymbol{T}$ is $K-$competitive on C_+. When $\boldsymbol{T}$ is a

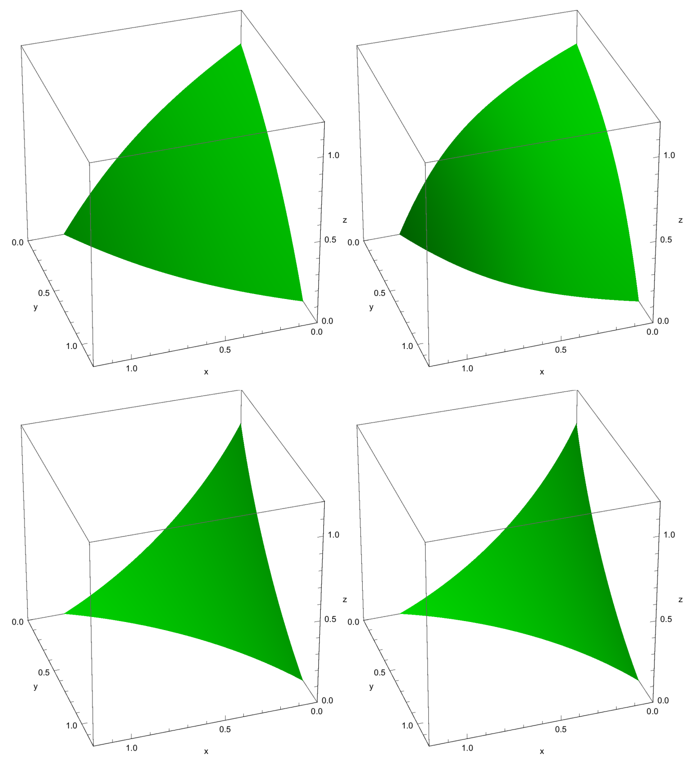

FIGURE 4. Carrying simplices for the May-Leonard model (8) with $r = 2$. Top left: $\alpha = 4/5, \beta = 3/4$. Top right: $\alpha = 2/3, \beta = 7/12$, Bottom left: $\alpha = 7/5, \beta = 4/3$. Bottom right: $\alpha = 3/2, \beta = 7/5$

$K-$competitive and orientation-preserving diffeomorphism from C_+ onto $\boldsymbol{T}(C_+)$, $\boldsymbol{T}^{-1}$ is $K-$monotone on $\boldsymbol{T}(C_+)$ and so $D(\boldsymbol{T}^{-1})(\boldsymbol{y})K \subseteq K$ for all $\boldsymbol{y} \in \boldsymbol{T}(C_+)$ (see, for example, [18]). Hence $(D\boldsymbol{T}(\boldsymbol{x}))^{-1}K \subseteq K$ for $\boldsymbol{x} \in C_+$, which implies that $D\boldsymbol{T}^{\#}K \subseteq K$. By lemma 7.1, all the normal bundles in the sequence defined by (1) are contained in K. Moreover, it is likely that theorem 6.1 can also be improved by using $K-$competitiveness to prove the existence of the carrying simplex directly.

Here our results do not collectively show that $\boldsymbol{T}_{ML}$ is $K-$competitive, but extensive computations (not shown here) suggest that $\boldsymbol{T}_{ML}$ is actually $K-$competitive on C_+ when the real parameters A, B lie in the ranges $B < A < \frac{1}{2}(B-1) + \frac{1}{2}\sqrt{1+6B-3B^2}$ with $K = K_{ML}\left(\frac{B}{A-B}\right)$, and also $A > B > \frac{1+A-\sqrt{1-6A-3A^2}}{2}$ with $K = K_{ML}\left(\frac{B-A}{A}\right)$.

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Appendix A. **Proof of lemmas.**

Lemma A.1. *When* $0 < B < A < 1$ *and* $A < \frac{1}{2}(B-1) + \frac{1}{2}\sqrt{1+6B-3B^2}$ *the function*

$$\begin{aligned} &= \frac{\lambda_1\left(A^2-AB+A+(B-2)B\right)}{A-B} + \frac{B\lambda_3\left(A^2-AB+A+(B-2)B\right)}{(A-B)^2} \\ &+ \lambda_1\lambda_2\left(\frac{B^3}{(B-A)^3}+1\right) + \frac{B\lambda_3\lambda_2\left((A-B)^3-B^3\right)}{(A-B)^4} + \frac{B^2\lambda_2^2\left((A-B)^3-B^3\right)}{(A-B)^5} \\ &+ \lambda_2\left(\frac{2B^3}{(B-A)^3}+\frac{(A+1)B^2}{(A-B)^2}+B+1\right) + AB - \frac{B}{A-B} + A + B + 1. \end{aligned}$$

is negative for all $\boldsymbol{\lambda} \in C_+$.

Proof. First, $0 < B < A < 1$ and $B < A < \frac{1}{2}(B-1) + \frac{1}{2}\sqrt{1+6B-3B^2}$, we have $B - 1 + \sqrt{1+6B-3B^2} < 4B$ (which can be checked by rearranging and squaring both sides). Hence we have $2B > A > B$. It is clear that when $2B > A > B > 0$ we have $(A-B)^3 < B^3$. Which shows that the coefficients of $\lambda_2\lambda_3$, λ_2^2 and $\lambda_1\lambda_2$ are negative. The coefficients of λ_1 and λ_3 are negative when $A^2+B^2-AB+A-2B < 0$ which simplifies to $B < A < \frac{1}{2}(B-1) + \frac{1}{2}\sqrt{1+6B-3B^2}$.

Next, the coefficient of λ_2 is

$$\begin{aligned} &-\frac{2B^3}{(A-B)^3} + \frac{(A+1)B^2}{(A-B)^2} + B + 1 = \frac{B^2}{(A-B)^3}(A^2-AB+A-B-2B) + B + 1 \\ &= \frac{B^2}{(A-B)^3}((A^2+B^2-AB+A-2B) - B^2 - B) + B + 1 \\ &< -\frac{B^2}{(A-B)^3}(B^2+B) + B + 1 = (B+1)\Big(1 - \left(\frac{B}{A-B}\right)^3\Big) < 0 \end{aligned}$$

since $2B > A$. Finally, the constant term is

$$\begin{aligned} AB - \frac{B}{A-B} + A + B + 1 &= \frac{A^2B - AB^2 + A^2 - B^2 + A - 2B}{A-B} \\ &= \frac{1}{A-B}\Big((B+1)(A^2+B^2-AB+A-2B) - B^3\Big) < 0 \end{aligned}$$

since $A^2 + B^2 - AB + A - 2B < 0$. □

Lemma A.2. *When* $0 < B < A < 1$ *and* $A < 2B$ *the function*

$$\begin{aligned} &-\frac{B^2\lambda_3\left(A^2-A(B+1)+B(B+2)\right)}{(A-B)^3} + \frac{B^2\lambda_2\lambda_3(A-2B)\left(A^2-AB+B^2\right)}{(A-B)^5} \\ &+ \frac{B\lambda_1\lambda_2(A-2B)\left(A^2-AB+B^2\right)}{(A-B)^4} - \frac{B\lambda_1\left(A^2-A(B+1)+B(B+2)\right)}{(A-B)^2} \\ &+ \frac{\lambda_2\left(2A^3-A^2B(B+7)+AB^2(B+8)-B^3(B+4)\right)}{(A-B)^3} + \lambda_2^2\left(\frac{B^3}{(B-A)^3}+1\right) \\ &- \frac{A\left(B^2+B-1\right)+B(B+2)}{A-B} \end{aligned}$$

is negative for all $\boldsymbol{\lambda} \in C_+$. *In particular, the function is positive under the conditions of lemma A.1.*

Proof. Since $2B > A > B$ it is immediate that the coefficients of $\lambda_2\lambda_3$, $\lambda_1\lambda_2$, λ_2^2 are negative. The constant term is negative when $AB^2 + AB - A + B^2 + 2B > 0$ which holds since $B > 2A > 0$. Next consider the coefficients of λ_3 and λ_1. These are negative when $A^2 - A(B+1) + B(B+2) = A^2 - AB + B^2 + 2B - A$, which also holds since in addition to $2B > A$ we also have $A^2 + B^2 - AB > 0$ (for all A, B). Lastly we consider the coefficient of λ_2 which is negative when $\tau = 2A^3 - A^2B(B+7) + AB^2(B+8) - B^3(B+4) < 0$. Setting $B = 2A + \epsilon$ where $\epsilon > 0$ we have $\tau = -12A^4 - 24A^3\epsilon - 12A^3 - 19A^2\epsilon^2 - 23A^2\epsilon - 7A\epsilon^3 - 16A\epsilon^2 - \epsilon^4 - 4\epsilon^3 < 0$. □

Lemma A.3. *When* $0 > A > B > \frac{1+A-\sqrt{1-6A-3A^2}}{2}$*, the function*

$$\begin{aligned} &-\frac{\lambda_3(A-B)\left(-(A+1)B + A(A+2) + B^2\right)}{A^2} + \frac{\lambda_1\lambda_2(2A-B)\left(A^2 - AB + B^2\right)}{A^3} \\ &+ \frac{\lambda_2^2(A-B)^2(2A-B)\left(A^2 - AB + B^2\right)}{A^5} + \lambda_1\left(\frac{(B-1)B}{A} + A - B + 2\right) \\ &- \frac{\lambda_3\lambda_2(A-B)(2A-B)\left(A^2 - AB + B^2\right)}{A^4} + AB - \frac{B}{A} + A + B + 2 \\ &+ \lambda_2\left(-\frac{2B^3}{A^3} + \frac{(A+7)B^2}{A^2} - \frac{(A+8)B}{A} + A + 4\right) \end{aligned}$$

is positive for $\boldsymbol{\lambda} \in C_+$*.*

Proof. First we note that under the conditions in the lemma $B > 2A$, so that the coefficients of λ_2^2, $\lambda_2\lambda_3$, $\lambda_1\lambda_2$ are all positive (note that $A < 0$). Moreover, when $0 > A > B > \frac{1+A-\sqrt{1-6A-3A^2}}{2}$ implies that $(A+1)B - A(A+2) - B^2 > 0$ which gives that the coefficient of λ_3 is positive. In turn, $(A+1)B - A(A+2) - B^2 > 0$ implies that $B - A(A+2) > B^2 - AB = B(B-A) > 0$ since $B < 0$ and $A > B$. The constant coefficient $AB - \frac{B}{A} + A + B + 2$ is positive when when $0 > A > B > \frac{A(A+2)}{1-A-A^2}$ as can be seen by solving for B. Furthermore, since $1 - A - A^2 > 1$ for $0 < A < -1$ we conclude that $B > A(A+2) \Rightarrow B > \frac{A(A+2)}{1-A-A^2}$ and so the constant coefficient is positive. Lastly we need to show that the coefficient of λ_2 is positive, i.e. $4 + A - ((A+8)B)/A + ((A+7)B^2)/A^2 - (2B^3)/A^3 > 0$. This is equivalent to showing that $4A^3 + A^4 - 8A^2B - A^3B + 7AB^2 + A^2B^2 - 2B^3 < 0$. By decomposition we find that

$$\begin{aligned} &4A^3 + A^4 - 8A^2B - A^3B + 7AB^2 + A^2B^2 - 2B^3 \\ &= (A-B)(B-2A)(2B - A - A^2) - A^2(A^2 + B - 2A - 2AB). \end{aligned}$$

The first term in the last expression is negative since $A - B + A^2 - B > 0$ when $A > B$ and $B < 0$, and the final term is negative when $A^2 + B - 2A - 2AB > 0$. But

$$A^2 + B - 2A - 2AB = \frac{1-2A}{1-A-A^2}(B(1-A-A^2) - A^2 - 2A) - \frac{A^3(1+A)}{1-A-A^2} > 0$$

since $B(1-A-A^2) - A^2 - 2A > 0$ and $-1 < A < 0$. □

Lemma A.4. *When $0 > A > B > -1$ and $B > 2A$, the function*

$$\frac{\lambda_1\left(A^2-A(B+2)+B^2+B\right)(A-B)}{A^2}+\frac{\lambda_2\lambda_3(2A-B)\left(A^2-AB+B^2\right)(A-B)^2}{A^5}$$
$$-\frac{\lambda_1\lambda_2(2A-B)\left(A^2-AB+B^2\right)(A-B)}{A^4}+\frac{\lambda_2^2(2A-B)\left(A^2-AB+B^2\right)}{A^3}$$
$$-\frac{\lambda_3\left(A^2-A(B+2)+B^2+B\right)(A-B)^2}{A^3}+A(B+1)-\frac{B(B+1)}{A}-B^2+2$$
$$+\lambda_2\left(-\frac{(A+1)B^3}{A^3}+\frac{(2A+3)B^2}{A^2}-\frac{2(A+2)B}{A}+A+4\right)$$

is positive for all $\boldsymbol{\lambda}\in C_+$. In particular, the function is positive under the conditions of lemma A.3.

Proof. Since $0 > A > B > 2A$, so that it is clear that the coefficients of $\lambda_2\lambda_3$, $\lambda_1\lambda_2$, λ_2^2 are positive. The coefficients of λ_1 and λ_3 are also positive since $A^2-AB+B^2+(B-2A)>0$. Also the constant term is positive when $B(B+1)-A^2B-A^2-2A+AB^2>0$. But $B(B+1)-A^2B-A^2-2A+AB^2=\frac{1}{4}(B-2A)(2(A+2)(B+1)-B(B+3))+\frac{B^3}{4}+\frac{3B^2}{4}>0$ when $A>2B$ and $0>A, B>-1$. Finally the coefficient of λ_2 is positive when $-\frac{(A+1)B^3}{A^3}+\frac{(2A+3)B^2}{A^2}-\frac{2(A+2)B}{A}+A+4>0$, or equivalently $-A^4+2A^3B-4A^3-2A^2B^2+4A^2B+AB^3-3AB^2+B^3>0$. We have

$$\begin{aligned}&-A^4+2A^3B-4A^3-2A^2B^2+4A^2B+AB^3-3AB^2+B^3\\&=\frac{1}{16}(B-2A)\left((A-B)(5A^2+3B^2-7AB-16B)+A^2(32+3A)\right)+\frac{3}{16}B^4.\end{aligned}$$

But $5A^2+3B^2-7AB-16B=5A^2+3B^2-B(7A+16)>0$ since $B<0$ and $A>-1$. ☐

Appendix B. Proof of lemma 10.1.

Proof. (i) First we prove that $\gamma(\boldsymbol{a})>0$ when $A>B>0$ and $\boldsymbol{a}\in K_{ML}\left(\frac{B}{A-B}\right)$. Using (25) and (12) - (19) we find

$$\begin{aligned}&\frac{\gamma}{r^3}=\\&(A^3-2A^2B+B^3)(a^2b+b^2c+c^2a)+(A^3-2AB^2+B^3)(b^2a+c^2b+a^2c)\\&+\Big(3(A^4+B^4-A^3B-AB^3)+5(A^3+B^3)+2A^2B^2\\&-4AB(A+B)\Big)(ab+bc+ca)+(A^3+B^3)abc\\&+\Big(2A^5-A^4B+8A^4-8A^3B+7A^3+4A^2B^2-6A^2B-AB^4\\&-8AB^3-6AB^2+2B^5+8B^4+7B^3\Big)(a+b+c)+A^6+6A^5-3A^4B\\&+12A^4-12A^3B+7A^3+6A^2B^2-6A^2B-3AB^4-12AB^3\\&-6AB^2+B^6+6B^5+12B^4+7B^3.\end{aligned}$$

We constrain $\boldsymbol{a} \in K_{ML}$ by setting $\boldsymbol{a} = \lambda_1(s, s^2, 1) + \lambda_2(1, s, s^2) + \lambda_3(s^2, 1, s)$ where $s = \frac{B}{A-B}$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$. The above expression becomes

$$\begin{aligned}
&\frac{\gamma}{r^3} = \\
&(A+B+A^2+B^2)(A^4+5A^3-A^2B^2-4A^2B+7A^2 \\
&-4AB^2-13AB+B^4+5B^3+7B^2) + \frac{B\left(A^2-BA+B^2\right)^3(\lambda_1^3+\lambda_2^3+\lambda_3^3)}{(A-B)^4} \\
&+\frac{\left(A^2-BA+B^2\right)^2(A+B)\left(A^2+2A-B\right)(\lambda_1^2+\lambda_2^2+\lambda_3^2)}{(A-B)^3} \\
&+\frac{\left(A^2-BA+B^2\right)^4(\lambda_1^2\lambda_3+\lambda_1^2\lambda_2+\lambda_1^2\lambda_2+\lambda_2^2\lambda_1+\lambda_2\lambda_3^2+\lambda_3^2\lambda_2)}{(A-B)^5} \\
&+\frac{(A^2-AB+B^2)}{(A-B)^2}\Big(2A^5-A^4B+8A^4-8A^3B+7A^3+4A^2B^2 \\
&-6A^2B-AB^4-8AB^3-6AB^2+2B^5+8B^4+7B^3\Big)(\lambda_1+\lambda_2+\lambda_3) \\
&+\frac{\left(A^2-BA+B^2\right)^2}{(A-B)^4}\Big(3A^4+(5-4B)A^3 \\
&+B(3B-5)A^2-4B^2(B+1)A+2B^3(2B+3)\Big)(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3) \\
&+\frac{\left(A^2-BA+B^2\right)^3\left(A^3+3BA^2-12B^2A+10B^3\right)\lambda_1\lambda_2\lambda_3}{(A-B)^6}.
\end{aligned} \tag{41}$$

By inspection all degree 3 terms except that of $\lambda_1\lambda_2\lambda_3$ are obviously positive. The coefficient of $\lambda_1^2+\lambda_2^2+\lambda_3^2$ is also positive since $A > B$. This leaves the requirements

$$\begin{aligned}
g_1 &= A^4+5A^3-A^2B^2-4A^2B+7A^2-4AB^2-13AB+B^4+5B^3+7B^2 > 0 \\
g_2 &= 2B^3(B+1)^2+\left(16B^2+24B+7\right)\epsilon^3+\left(14B^3+28B^2+15B\right)\epsilon^2 \\
&\quad +\left(5B^4+8B^3+3B^2\right)\epsilon+(9B+8)\epsilon^4+2\epsilon^5 \\
g_3 &= 3A^4+(5-4B)A^3+B(3B-5)A^2-4B^2(B+1)A+2B^3(2B+3) > 0 \\
g_4 &= A^3+3BA^2-12B^2A+10B^3 > 0.
\end{aligned}$$

Finally to show that each of these expressions is positive for $A > B$ we simply substitute $A = B + \epsilon$ for $\epsilon > 0$. We obtain

$$\begin{aligned}
g_1 =& B^2(B+1)^2+(4B+5)\epsilon^3+(5B^2+11B+7)\epsilon^2+B(B+1)(2B+1)\epsilon+\epsilon^4 \\
g_2 =& 2B^3(B+1)^2+B^2(B+1)(5B+3)\epsilon+(9B+8)\epsilon^4 \\
&+(8B(2B+3)+7)\epsilon^3+B(14B(B+2)+15)\epsilon^2+2\epsilon^5 \\
g_3 =& 2B^4+2B^3+\left(9B^2+10B\right)\epsilon^2+\left(2B^3+B^2\right)\epsilon+(8B+5)\epsilon^3+3\epsilon^4 \\
g_4 =& 2B^3-3B^2\epsilon+6B\epsilon^2+\epsilon^3,
\end{aligned}$$

the first 3 of which are clearly all positive. For g_4, we simply note that showing $g_4 > 0$ is equivalent to showing that $2x^3 - 3x^2 + 6x + 1 > 0$ for $x > 0$. But $2x^3 - 3x^2 + 6x + 1 = 1 + x(2x^2 - 3x + 6)$ and $2x^2 - 3x + 6$ has no real zeros and hence $g_4 > 0$.

(ii) Now consider the case where $-1 < B < A < 0$ and $\boldsymbol{a} \in K_{ML}\left(\frac{B-A}{A}\right)$. The counterpart of (41) in this case is

$$\begin{aligned}
\frac{\gamma}{r^3} &= \left(A^2+A+B^2+B\right)\Big(A^4+5A^3-A^2B^2-4A^2B+7A^2-4AB^2 \\
&\quad -13AB+B^4+5B^3+7B^2\Big)+\frac{(A-B)^2\left(A^2-AB+B^2\right)^3}{A^5}(\lambda_1^3+\lambda_2^3+\lambda_3^3) \\
&\quad +\frac{(A^2-B^2)\left(A^2-AB+B^2\right)^2\left(A-2B-B^2\right)}{A^4}(\lambda_1^2+\lambda_2^2+\lambda_3^2) \\
&\quad -\frac{(A-B)\left(A^2-AB+B^2\right)^4}{A^6}(\lambda_1\lambda_2^2+\lambda_1\lambda_3^2+\lambda_2\lambda_3^2+\lambda_2\lambda_1^2+\lambda_3\lambda_2^2+\lambda_3\lambda_1^2) \\
&\quad +\frac{(A^2-AB+B^2)}{A^2}\Big(2A^5-A^4B+8A^4-8A^3B+7A^3+4A^2B^2-6A^2B \\
&\quad -AB^4-8AB^3-6AB^2+2B^5+8B^4+7B^3\Big)(\lambda_1+\lambda_2+\lambda_3) \\
&\quad +\frac{\left(A^2-AB+B^2\right)^2}{A^4}\Big(4A^4-4A^3B+6A^3+3A^2B^2-4A^2B \\
&\quad -4AB^3-5AB^2+3B^4+5B^3\Big)(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1) \\
&\quad +\frac{\left(A^2-AB+B^2\right)^3\left(10A^3-12A^2B+3AB^2+B^3\right)}{A^6}\lambda_1\lambda_2\lambda_3.
\end{aligned}$$

Recalling that $A<0$, the coefficients of $\lambda_1^3+\lambda_2^3+\lambda_3^3$ and $\lambda_1\lambda_2^2+\lambda_1\lambda_3^2+\lambda_2\lambda_3^2+\lambda_2\lambda_1^2+\lambda_3\lambda_2^2+\lambda_3\lambda_1^2$ are obviously negative. The coefficient of $\lambda_1^2+\lambda_2^2+\lambda_3^2$ is $\frac{\left(A^2-AB+B^2\right)^2(A^2-B^2)(A-2B-B^2)}{A^4}$. Now note that for $-1<B<A<0$ we have $A^2<B^2$ and $A>2B+B^2$ so that the coefficient of $\lambda_1^2+\lambda_2^2+\lambda_3^2$ is negative. Next consider the coefficient of $\lambda_1+\lambda_2+\lambda_3$, which is negative when

$$\begin{aligned}
w_1(A,B) &= 2(A^5+B^5)+7(A^3+B^3)-8AB(A^2+B^2) \\
&\quad +8(A^4+B^4)+4A^2B^2-6AB(A+B)
\end{aligned}$$

is negative. We need to to show that the maximum of $w_1(A,B)$ over $[-1,0]^2$ is negative. This is the same as showing that the maximum of $w_1(A+u,A-u)$ is negative for all $u\in[0,A]$ and $A\in[-1,0]$. But $w_1(A+u,A-u)=2A\left(20A^2+44A+27\right)u^2+2A^3\left(2A^2+2A+1\right)+4(5A+9)u^4$. Note that $w_1(A,A)=2A^3\left(2A^2+2A+1\right)<0$ for $A\in[-1,0]$ and $2A\left(20A^2+44A+27\right)<0$, $4(5A+9)>0$ for $A\in[-1,0]$ so $w_1(A+u,A-u)$ is a convex function of u^2 and we need only show that $w_1(2A,0)<0$. But $w_1(2A,0)=4(5A+9)A^4+2\left(2A^2+2A+1\right)A^3+2\left(20A^2+44A+27\right)A^3=8A^3(7+16A+8A^2)\le 0$ for $A\in[-1,0]$, and so $w_1(A,B)<0$ for $-1<B<A<0$. The coefficient of $\lambda_1\lambda_2\lambda_3$ is negative since $10A^3-12A^2B+3AB^2+B^3=-((A-B)^2+9A^2)(A-B)+2B^3<0$. The coefficient of $\lambda_1\lambda_2+\lambda_3\lambda_2+\lambda_1\lambda_3$ is negative since when $-1<B<A<0$ we have

$$\begin{aligned}
&4A^4-4A^3B+6A^3+3A^2B^2-4A^2B-4AB^3-5AB^2+3B^4+5B^3 \\
&=A^3(B+1)+(A+1)A^2B+(A-B)^2(2(A+1)B \\
&\quad +4(A+1)A+A+3B(B+1)),
\end{aligned}$$

which is negative since $-1<B<A<0$. This leaves the constant term, which is negative since when $-1<B<A<0$ the factor $A+A^2+B+B^2=A(1+A)+$

$B(1+B) < 0$ and

$$\begin{aligned}&A^4 + 5A^3 - A^2B^2 - 4A^2B + 7A^2 - 4AB^2 - 13AB + B^4 + 5B^3 + 7B^2\\&= (A-B)^2\left(A^2 + 2(B+1)(A+B+2) + 3(A+1)\right)\\&- B(1+B)((A-B)^2 - A(A+1)) > 0.\end{aligned}$$

□

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E-mail address: steve.baigent@ucl.ac.uk